Chapter 2
Newtonian Fluid Dynamics as a Mathematical – Physical Science

There are two mathematical–physical descriptions of fluid dynamics.[1] The first of them is a microscopic description, from the Boltzmann equation for the (one-particle) distribution function $f(\mathrm{t}, \mathrm{x}; \xi)$:

$$\partial f/\partial \mathrm{t} + \xi.\nabla f = (1/\mathrm{Kn})Q(f;f),$$

where $f(\mathrm{t}, \mathrm{x}; \xi)$ is precisely the density probability of finding a molecule at the space-position $\mathbf{x}$ (at the time t), with the velocity ξ. The parameter Kn in the Boltzmann equation is the Knudsen number, which is the ratio of the mean free path (between collisions of molecules – a microscopic reference length, l°) and a typical (macroscopic) reference length, L°, of the classical continuum theory, which is the ratio of Mach (M) and Reynolds (Re) numbers:

$$\mathrm{Kn} = \mathrm{M}/\mathrm{Re},$$

where $\mathrm{M} = \mathrm{U}^\circ/\mathrm{c}^\circ$ is the constant Mach (dimensionless) number, based on the reference macroscopic velocity U° and the speed of sound c°, that characterizes the compressibility effect. The parameter $\mathrm{Re} = \mathrm{U}^\circ\mathrm{L}^\circ/\nu^\circ$ is the constant Reynolds (dimensionless) number that characterizes the viscosity (via the kinematic viscosity coefficient ν°) effect.

[1] In my 2001 review paper [29] – written by a fluid dynamicist for fluid dynamicists – the curious reader can find a contribution concerning the many theoretical mathematical investigations of Navier–Stokes–Fourier problems. My intent was to extract from the huge literature the basic results, ideas, and goals of this currently wide activity and to present the results to the readers of *Applied Mechanical Review*. I am sure that rigorous mathematicians will find in this paper many shortcomings, non-rigorous formulations, and so on. I think, however, that such a paper will stimulate further thinking by engineers and applied scientists, including some exchange of opinions, and so on, and that it is therefore needed. The distance between theoretical mathematicians and applied mathematicians and engineers has become too large! I hope that both old and new investigators interested in Newtonian fluid flow problems might learn much from it.

R. Kh. Zeytounian, *Navier–Stokes–Fourier Equations*,
DOI 10.1007/978-3-642-20746-4_2,

The right-hand side of the above Boltzmann equation, the collision operator Q, is typical of kinetic theory of gases in that it preserves mass, momentum, and, namely:

$$\int \psi Q d\xi, = 0,$$

where $\psi = (1, \xi_1, \xi_2, \xi_3, |\xi|^2)$ is a five-component (the so-called collisional invariants) vector. For a modern exposition of the kinetic theory of gases (dilute gas) see Cercignani, Illner, and Pulvirenty, 1994 [30].

The second continuum description is linked with the macroscopic length scale (which is the real scale for applications in fluid flows), and is governed by the three conservation equations of classical continuum mechanics: principle of conservation of mass (assuming that the fluid possesses a density function $\rho(\mathrm{t}, \mathbf{x})$), principle of conservation of linear momentum (adopting the stress principle of Cauchy; see Sect. 2.3.1), and the conservation of energy (we postulate that the total energy of a volume – the sum of its kinetic energy and its internal energy – is conserved; see Sect. 2.3.3).

The resulting three equations of continuum mechanics, which proceeds on the assumption that a fluid is practically continuous and homogeneous in structure (see, Serrin, 1959 [31]) are:

$$\mathrm{D}\rho/\mathrm{Dt} + \rho\nabla.\ \mathrm{u} = 0,$$

$$\rho\mathrm{D}\mathbf{u}/\mathrm{Dt} = \rho \boldsymbol{f} + \nabla.\boldsymbol{T},$$

$$\rho\mathrm{D}E/\mathrm{Dt} = \boldsymbol{T} : \boldsymbol{D} - \mathrm{div}\boldsymbol{q},$$

where $\mathbf{u}$ is the velocity vector, $\boldsymbol{f}$ is the extraneous force per unit mass (a known function of position $\mathbf{x}$ and time t), $\boldsymbol{T}$ is the Cauchy stress tensor, E is the specific internal energy, $\boldsymbol{q}$ is the heat flux vector, and the term $\boldsymbol{T} : \boldsymbol{D}$, in the energy equation for E, is a “dissipation” term involving the interaction of stress and deformation (second-order tensor $\boldsymbol{D}$). We observe that $\boldsymbol{T} : \boldsymbol{D}$ stands for the scalar product $T^{\mathrm{ij}}D_{\mathrm{ij}}$ of two second-order tensors (dyadics), and T^{ij} and D_{ij} are, respectively, the components of $\boldsymbol{T}$ and $\boldsymbol{D}$.

The problem of the derivation of the fluid dynamic equations (derived from above three continuum mechanics equations, with Cauchy stress tensor $\boldsymbol{T}$ and the heat flux vector $\boldsymbol{q}$, due to Navier–Stokes and Fourier constitutive equations – see Sects. 2.3.2 and 2.3.3) from the Boltzman equation for small Knudsen numbers ($\mathrm{Kn} \downarrow 0$) is shortly expounded in Sect. 5 of our review paper [29], where the reader can find various pertinent references concerning this fluid dynamics limit of kinetic equations, initiated by Hilbert in 1912.

2.1 From Newton to Euler

Sir Issac Newton, English mathematician and physicist, was the greatest single influence on theoretical physics until Einstein. In his major treatise, *Philosophia Naturalis Principia Mathematica* (1687) [32] he presented a mathematical description of the laws of mechanics and gravitation, and applied this theory to explain planetary and lunar motions. In the Second Law we read: "The body moves in such a way that at each moment the product of its acceleration vector by the density is equal to the sum of certain other vectors, called forces, which are determined by the motion taking place." That is:

$$\rho \, \mathrm{D}\mathbf{u}/\mathrm{Dt} = \rho\mathbf{g} - \text{internal force per unit volume.} \tag{2.1}$$

A second part of Newton's *Principia* is related to the conservation of mass: "To each small solid body can be assigned a positive number *m*, invariant in time, called its mass." That is:

$$\mathrm{D}/\mathrm{Dt}\left\{\int_V \rho dV\right\} = 0. \tag{2.2}$$

In (2.1), $\mathbf{u}$ is the velocity vector, $\mathbf{g}$ is the gravitational force per unit mass, and ρ is the density. The (Cartesian) components of the nabla, ∇, operator, in material (or substantial) derivative

$$\mathrm{D}/\mathrm{Dt} = \partial/\partial \mathrm{t} + \mathbf{u}.\nabla,$$

are $\partial/\partial \mathrm{x_i}, \ \mathrm{i} = 1, 2, 3$, where the time is denoted by t, and $\mathbf{x} = (\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3)$ is the position vector.

In (2.2), dV is a volume element in the neighbourhood of the point P, and to this volume element will be assigned a mass ρdV.

We observe that D/Dt is related to the Euler rule of differentiation, and t, $\mathbf{x}$ is the Euler time–space variable. To express (2.2) in the form of a differential equation, the differentiation indicated in this equation is carried out by transforming the integral suitably. In this case we derive the so-called equation of continuity (this derivation is, in fact, due to Euler in 1755 [33]):

$$\mathrm{D}\rho/\mathrm{Dt} + \rho\nabla.\mathbf{u} = 0. \tag{2.3}$$

This (compressible) equation of continuity (2.3) remains unaltered when viscosity is admitted.

2.1.1 Eulerian Elastic Fluid

In reality, fluid dynamics was first envisaged as a systematic mathematical–physical science in Johann Bernoulli's *Hydrodraulics* (1737) [34], in Daniel

Bernoulli's *Hydrodynamica* (1738) [35], and also in D'Alembert *Traité de l'Équilibre et du mouvement des fluides* (1744) [36]. However, the fundamental ideas expounded in these books were formulated mathematically as partial differential equations in an epochal paper by Euler (1755) [33] which firmly established him as the founder of rational fluid dynamics. Nevertheless, he considered only non-viscous (inviscid) fluid flows with the pressure a function only of the density (a so-called isentropic/barotropic fluid flow – the fluid being an elastic fluid). We observe that an inviscid fluid is one in which it is assumed that the internal force acting on any surface element *dS*, at which two elements of the fluid are in contact, acts in a direction normal to the surface element. At each point *P* (with coordinates x_i, i = 1, 2, 3) the stress, or internal force per unit area, is independent of the orientation (direction of the normal) of *dS*, and the value of this stress is called the pressure, p, at the point *P*. Therefore, the internal force per unit volume, appearing in Newton's equation (2.1), has x_i – component $(-\partial p/\partial x_i)$, i = 1, 2, 3. As a consequence, for an inviscid (non-viscous–Eulerian) fluid we determine, from (2.1), the classical Euler equation of motion (momentum equation):

$$\rho D\mathbf{u}/Dt = \rho\mathbf{g} - \nabla p. \tag{2.4}$$

Equations 2.3 and 2.4, which express Newton's principles for the motion of an inviscid fluid, are usually referred to as the Eulerian fluid flow (compressible) equations, and include one vector equation (2.4) and one scalar equation (2.3) for **u**, ρ and p (five unknowns).

It follows that one more equation is needed in order that a solution of the system of Euler equations be uniquely determined for given initial and boundary conditions. According to Euler, if we add to Eq. 2.3 and 2.4 the following specifying equation:

$$p = p(\rho) \tag{2.5}$$

which gives the relation between the pressure and the density, we shall have five equations (a closed system) which include all the theory of the motion of fluids.

By this formulation, Euler believed (255 years ago!) that he had reduced fluid dynamics, in principle, to a mathematical–physical science; but it is crucial to note that, in fact, Eq. 2.5 is not an equation of state, but specifies only the particular type of motion (so-called barotropic) under consideration, and in this case the fluid is just called an elastic fluid.

In my book (Zeytounian, 2002) [37], the reader can find a theory and applications of non-viscous fluid flows, and in the next chapter, devoted to a discussion of various general models derived from Navier–Stokes–Fourier equations, we obtain, for large Reynolds number Re $\gg$ 1 – as a vanishing viscosity limit – the full unsteady Euler compressible non-viscous adiabatic and baroclinic equations for a thermally perfect gas (a trivariate fluid). (Concerning the NS–F equations see Sect. 2.3.)

These general models very often form the basis of various chapters in fluid dynamics treatises. It is obvious, therefore, that these treatises may be organized through some models which are best obtained by asymptotic modelling. As an example we mention the case of inviscid flows which are often considered as a model, used from the outset, and need to be embedded in the more general model of slightly (vanishing) viscous (laminar) or with slight friction (turbulent) flow, to which asymptotic modelling is applied. Incompressible flows are seldom considered as flow at small Mach numbers – which may lead to almost nonsensical conclusions, as when one deals with incompressible aerodynamics, because phenomena such as sound produced by quite low-speed flow cannot be understood other than by low-Mach-number (hyposonic) aerodynamics.

2.1.2 From Adiabaticity to Isochoricity

In many cases the specification of the type of flow is given in thermodynamic terms. The most common (and rather naive) assumption in the study of compressible fluids is that no heat output or input occurs for any particle. In this case, heat transfer by radiation, chemical processes, and heat conduction between neighbouring particles are excluded, and the fluid flow is called adiabatic.

In order to translate either assumption into a specifying equation, the First Law of Thermodynamics must be used, which gives the relation between heat input and the mechanical variables [J. R. von Mayer (1842)]. If the total heat input from all sources, per unit of time and mass, is zero, the First Law for an inviscid fluid can be written in the following form:

$$C_v DT/Dt + pD/Dt(1/\rho) = 0, \tag{2.6}$$

where C_v is the specific heat of the fluid at constant volume. The first term in (2.6) represents the part of the heat input expended for the increase in temperature T, and the second term corresponds to the work done by expansion. It is well known, also from thermodynamics, that for each type of matter a certain relation exists among the three (thermodynamic) variables, pressure p, density ρ, and temperature T:

$$f(p, \rho, T) = 0, \tag{2.7}$$

Thus the temperature can be computed when p and ρ are known. Naturally, the equation of state (2.7) is not a specifying equation, since it implies temperature as a new variable. Finally, Eqs. 2.3, 2.4, and 2.6, together with (2.7), form a closed system of six equations for the six unknowns: **u,** p, ρ, and T.

For a thermally perfect gas (naturally, a perfect gas is not necessarily inviscid), the equation of state (2.7) is explicit:

$$p = R\rho T, \tag{2.8}$$

where R is a constant depending upon the particular perfect gas. From (2.8) it follows that for a perfect gas the condition p/ρ = const implies a fluid flow at constant temperature, or isothermal flow.

The specific entropy S of a perfect gas is then given by:

$$S = [\mathrm{R}/(\gamma - 1)]\log(\mathrm{p}/\rho^{\gamma}) + \mathrm{const}, \tag{2.9}$$

where γ is a constant, having the value 1.40 for dry air. Thus the motion of a perfect gas with the condition $p/\rho^{\gamma} = \mathrm{const}$, as a specifying equation, is isentropic (constant entropy motion or, since $\gamma > 1$, polytropic). The equation of state for a perfect gas in equilibrium, connected with the names of Boyle (see, Birch (1744) [38]), Mariotte, Gay Lussac, and Charles, has been widely known since 1800.

In precisely the modern form, it was used freely by Euler, but did not appear again in the hydrodynamical literature until used by Kirchhoff in his paper of 1868 [39].

In some presentations, no distinction is made between the term "perfect gas" and "ideal gas". Here the term "perfect gas" is defined precisely by the equation of state (2.8). The term "Eulerian fluid flow" is used for an inviscid (non-viscous) and non-heat-conducting flow, governed by the system of Eqs. 2.3, 2.4, and 2.6, with (2.7). According to (2.6) and (2.7) this Eulerian fluid flow is a baroclinic and adiabatic fluid flow. In Eq. 2.6 an expression for C_v in term of the variables T, p, and ρ is needed, but for a perfect gas, where the equation of state is (2.8), it is generally assumed that $C_v = R/(\gamma - 1)$ is a constant – R being the usual gas constant.

As a consequence, we derive, for such a perfect gas with constants C_v and C_p (= γC_v), specific heats, the following conservation equation for specific entropy in the case of a thermally perfect gas:

$$\mathrm{D}/\mathrm{Dt}\ [\log(\mathrm{p}/\rho^{\gamma}) = 0 \Rightarrow\ \mathrm{D}S/\mathrm{Dt}\ = 0. \tag{2.10}$$

Equation 2.10, however, holds only for an adiabatic flow of a perfect inviscid gas, when the entropy is constant for each particle but varies from particle to particle. Generally, a thermally perfect inviscid gas in adiabatic flow does not necessarily behave like an elastic fluid.

If we assume that in (2.10),

$$\gamma \text{ tends to infinity (incompressible limit case)}, \tag{2.11a}$$

such that R = O(1), and that in such a case,

$$\mathrm{C_v}\ \textit{tends to zero}\ \text{but}\ \mathrm{Cp} \equiv\ \mathrm{R}\ =\ \mathrm{O}(1), \tag{2.11b}$$

then we derive, again, from (in place of) (2.10) the following evolution equation (conservation law) for density (isochoricity):

$$\mathrm{D}\rho/\mathrm{Dt} = 0 \Rightarrow \nabla \cdot \mathbf{u} = 0, \tag{2.11c}$$

As a consequence, for a Eulerian incompressible but non-homogeneous (isochoric) fluid flow we obtain the following system of three equations for the velocity **u**, pressure p, and density ρ:

$$\rho\, D\mathbf{u}/Dt = \rho\mathbf{g} - \nabla p, \tag{2.12a}$$

$$D\rho/Dt = 0, \tag{2.12b}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{2.12c}$$

This isochoric system of three Eqs. 2.12a–2.12c is very well investigated in Yih's 1980 book [4]. The above "incompressible limiting process" (2.11a,2.11b) presents the possibility of taking into account some compressible (second-order) effects of the order $O(1/\gamma)$ – which is the case, for instance, in the theory of lee waves downstream of a mountain!

2.2 Navier Viscous Incompressible, Constant Density Equations

The equation of motion of a viscous and incompressible homogeneous (with a constant density) fluid flow was first obtained by Navier in 1821 [40] and later by Poisson in 1831 [41]. The necessity of such a viscous equation (in place of the above Euler equation (2.4) was strongly linked with the d'Alembert theorem (paradox?): "An object moving with constant velocity $U\infty$ in a potential field (from Bernoulli equation and Lagrange theorem in an incompressible fluid the velocity-potential Φ must satisfy Laplace's equation) does not feel any force – neither drag nor lift."

Obviously, this result is in sharp contrast with experience! For instance, an aircraft could not fly. Suppose that, initially, the aircraft and the fluid (air) are both at rest, then the aircraft begins to move. Since vorticity cannot be produced (Lagrange – permanence of irrotational flow), the potential flow around the aircraft cannot produce any lift, so that flight is impossible. Such a paradox can be avoided if vorticity is present.

However, the problem remains of understanding how vorticity can be created in the system. The conservation of vorticity in an inviscid (incompressible) fluid, while reasonably far from the obstacle, is too drastic near the boundary of this obstacle!

A more accurate description of the interaction among the particles of the fluid and the obstacle leads us to introduce the Navier (viscous and incompressible) equation, which is a correction to the Euler (incompressible and non-viscous) equation of motion.

Such a new (Navier) equation can explain the effects, such as vorticity production, which are relevant near the boundary. This Navier equation has the following rather simple form:

$$\mathrm{D}\mathbf{u}_{\mathrm{N}}/\mathrm{Dt} \ + (1/\rho_{\circ})\nabla \mathrm{p} + \mathbf{g} = \nu_{\circ}\Delta\mathbf{u}_{\mathrm{N}}, \tag{2.13a}$$

where $\nu_{\circ}$ is the constant kinematic viscosity, and $\Delta \equiv \nabla^{\mathbf{2}}$ is the Laplace operator for the Navier velocity vector $\mathbf{u}_{\mathrm{N}}$. The companion to (2.13a), equation of continuity is simply:

$$\nabla.\mathbf{u}_{\mathrm{N}} = \mathrm{O}. \tag{2.13b}$$

Because ν° multiplies the derivative of highest order in Navier equation (2.13a), it cannot be inferred that the solutions of (2.13a), for very small values of ν°, reduces to Euler, below (2.14a), equation for an incompressible homogeneous and non-viscous fluid flow, with $\mathbf{u}_{\mathrm{E}}$ as velocity vector:

$$\mathrm{D}\mathbf{u}_{\mathrm{E}}/\mathrm{Dt} \ + (1/\rho_{\circ})\nabla \mathrm{p} + \mathbf{g} = \mathrm{O}. \tag{2.14a}$$

with

$$\nabla.\mathbf{u}_{\mathrm{E}} = \mathrm{O}. \tag{2.14b}$$

It is important (in particular, in the framework of our RAMA) to observe that the passage from compressible flow to incompressible flow, which filters the acoustic fast waves, is a strongly singular limit.

The reader can find in our 2006 monograph [13], devoted to hyposonic flow theory, various facets of the unsteady very slow flows at low Mach number, which are strongly related to a category of fluid flow problems, called “hyposonic”, when $M << 1$.

In the above Navier incompressible viscous equation (2.13a) and also in the Euler incompressible non-viscous equation (2.14a), the term $(1/\rho_{\circ})\nabla \mathrm{p}$ is not an unknown quantity of the initial value problem. In fact, $\nabla(\mathrm{p}/\rho_{\circ})$ is the force term acting on the particles of fluid allowing them to move as freely as possible, but in a way compatible with the incompressibility constraint (2.13b) or (2.14b): $\nabla.\mathbf{u}_{\mathrm{N}} = 0$.

Note particularly that for a Eulerian incompressible flow, $\mathrm{D}\mathbf{u}_{\mathrm{E}}/\mathrm{Dt} = 0$ admits solutions violating the condition: $\nabla\boldsymbol{.}\ \mathbf{u}_{\mathrm{E}} = 0$ at $\mathrm{t} > 0$, even if the velocity divergence vanishes at $\mathrm{t} = 0$. The pressure term in the above incompressible equations (2.13a) and (2.14a) is not an unknown quantity, because it can be determined when we have found the velocity field $\mathbf{u}_{\mathrm{N}}$ or $\mathbf{u}_{\mathrm{E}}$ – for instance, taking the divergence of the Euler equation (2.14a), we obtain a Poisson (elliptic) equation:

$$\Delta \mathrm{p} \ = -\rho_{\circ}\{\nabla\,.\,[(\mathbf{u}_{\mathrm{E}}\,.\,\nabla)\mathbf{u}_{\mathrm{E}}] + \nabla.\mathbf{g}\},$$

and, knowing $\mathbf{u}_{\mathrm{E}}$ and external force $\mathbf{g}$, we can find p by solving a Poisson equation with a Neumann boundary condition,

$$\partial \mathrm{p}/\partial \mathrm{n} \ = -\rho_{\circ}\{\nabla\,.\,[(\mathbf{u}_{\mathrm{E}}\,.\,\nabla)\mathbf{u}_{\mathrm{E}} + \mathbf{g}]\,.\,\mathbf{n},$$

in a domain with a boundary (after that the Euler equation (2.14a) is projected on the outward unit normal **n**).

As a consequence of the above, it is sufficient to consider the Navier incompressible equation in terms of vorticity $\omega_N (= \nabla \wedge \mathbf{u}_N)$ assuming that **g** is conservative:

$$D\omega_N/Dt = (\omega_N \,.\, \nabla)\mathbf{u}_N + \nu_\circ \Delta \omega_N, \tag{2.15a}$$

with

$$\nabla . \mathbf{u}_N = 0. \tag{2.15b}$$

Obviously, when we assume that **g** is conservative, any potential flow,

$$\mathbf{u} = \nabla \Phi,$$

trivially satisfies the Navier equation (2.13a) in term of the vorticity ω_N!

However, to obtain a well-set boundary value problem, for a fixed $\nu_\circ > 0$, one must also (according to Stokes) replace the slip boundary condition on a (stationary) boundary of a fluid flow domain, for $\mathbf{u}_E$ in Euler non-viscous equation (2.14a):

$$\mathbf{u}_E . \mathbf{n} = 0, \tag{2.16a}$$

by the more stringent condition of no-slip boundary condition on a stationary boundary:

$$\mathbf{u}_N = 0, \tag{2.16b}$$

for $\mathbf{u}_N$ in Navier (2.13a).

Concerning this no-slip boundary condition (2.16b), it is interesting to note that in his 1904 lecture to the ICM, Prandtl stated:

"The physical processes in the boundary-layer (BL – Grenzschicht) between fluid and solid body can be calculated in a sufficiently satisfactory way if it is assumed that the fluid adheres to the walls, so that the total velocity there is zero – or equal to the velocity of the body. If the viscosity is very small and the path of the fluid along the wall not too long, the velocity will have again its usual value very near to the wall (outside the thin transition layer). In the transition layer (Übergangsschicht) the sharp changes of velocity, in spite of the small viscosity coefficient, produce noticeable effects."

Prandtl not only mentions the existence and nature of the thin boundary-layer and its connection with frictional drag, but derives heuristically the boundary-layer (so-called Prandtl) equations valid in a thin viscous layer close to the wall of the solid body. These BL Prandtl equations, however, are not valid near the time $t = 0$, where the initial data are given in the case of an initial-boundary value problem.

Prandtl – curiously – did not have any idea concerning this singular nature of his discovered BL equations in unsteady compressible case!

But it is also necessary to not overlook the important investigations of Lanchester (1907) in England, concerning the nature of the boundary-layer and explanation of separation (independently of Prandtl). (For a detailed discussion concerning the initial and boundary conditions, see Sect. 2.4.)

In Chap. 5, as a consequence of the singular nature of BL compressible equations, the unsteady full NS–F equations for large Reynolds number are analyzed in detail. For this, it is necessary to consider five regions and the related matching conditions.

2.3 Navier–Stokes–Fourier Equations for Viscous Compressible and Heat-Conducting Fluid Flow

According to Truesdell, in "*The Mechanical Foundations of Elasticity and Fluid Dynamics*" (1966) [42, p. 2]:

"Classical fluid dynamics describes the flow of media altogether without springiness of form, so that when released from all deforming forces except a hydrostatic pressure, they retain their present shapes; it is a partially linear theory, in which a uniformly doubled rate of deformation if dynamically possible would lead to doubled viscous forces."

2.3.1 The Cauchy Stress Principle

The derivation of the equation of motion for the velocity vector **u,** for real (viscous compressible and heat conducting) fluids is based on the following stress principle of Cauchy, 1828 [43]:

"Upon any imagined closed surface S (with outward normal $\boldsymbol{n}$ to S) there exists a distribution of stress vector $\Sigma(\mathbf{x},\ \mathrm{t};\mathbf{n}) = \mathbf{n}.T$, where $\boldsymbol{T}$ is the stress tensor, whose resultant and moment are equivalent to those of the actual forces of material continuity exerted by the material outside S upon that inside."

This statement of Cauchy's principle is due to Truesdell's paper of 1952 [44]; and as Truesdell remarks (1953) [45], the above well-known Cauchy principle

> … has the simplicity of genius. Its profound originality can be grasped only when ones realizes that a whole century of brilliant geometers had created very special elastic problems in very complicated and sometimes incorrect ways without ever hitting upon this basic idea, which immediately became the foundation of the mechanics of distributed matter.

As a consequence of his stress principle, Cauchy obtained a general equation of motion – the simple and elegant Eq. 2.17 below – which is valid for any fluid, and indeed for any continuous medium, regardless of the form which the stress tensor $\boldsymbol{T}$ may take.

$$\mathrm{D}\mathbf{u}/\mathrm{Dt} \; = \rho \boldsymbol{f} + \nabla . \boldsymbol{T}. \tag{2.17}$$

We observe that the above equation of motion, discovered by Cauchy in 1828, can be derived easily according to the principle of conservation of linear momentum: "The rate of change of linear momentum of a material volume V equals the resultant force on the volume."

The necessity for a clear-cut statement of the postulates on which continuum mechanics rests was pointed out by Felix Klein and David Hilbert, but the first axiomatic presentation is due to G. Hamel (1908) [46]. With $\Sigma(\mathbf{x}, \mathrm{t}; \mathbf{n}) = \mathbf{n} \,.\boldsymbol{T}$, for the stress vector Σ, the above principle is expressed by the statement:

$$\mathrm{D}/\mathrm{Dt}\left(\int_{v} (\rho \mathrm{D}\mathbf{u}/\mathrm{Dt}\right) dv = \int_{v} \rho \boldsymbol{f} \, dv + \int_{v} \mathrm{div} \boldsymbol{T} dv, \tag{2.18a}$$

applying the divergence theorem. Since v (a fixed volume) is arbitrary, we obtain the Eq. 2.17.

We observe also that the stress forces are in local equilibrium, and it is postulated that the stress tensor is symmetric:

$$T^{\mathrm{ij}} = T^{\mathrm{ji}}. \tag{2.18b}$$

2.3.2 Navier–Stokes Constitutive Equations: The Cauchy–Poisson Law

In our 2004 book *Theory and Applications of Viscous Fluid Flows* [47], in Sect. 1.4 of Chap. 2, the reader can find a detailed account of the constitutive equation of a viscous (*à la* Navier–Stokes) classical fluid, mainly inspired by Serrin (1959) [31].

Here we present only a short comment. A first important moment in the history of N–S constitutive equations is Stokes' idea (1845) [48] of "fluidity" which can be stated as four postulates:

1. $\boldsymbol{T} = F(\boldsymbol{D})$ and $\boldsymbol{D} = \boldsymbol{D}(\mathbf{u})$.
2. $\boldsymbol{T}$ does not depend explicitly on the position vector $\mathbf{x}$ (spatial homogeneity).
3. There is no preferred direction in space (isotropy).
4. When $\boldsymbol{D}(\mathbf{u}) = 0$, then $\boldsymbol{T} = -\,\mathrm{p}\boldsymbol{I}$ (Eulerian non-viscous fluid flow).

A medium whose constitutive equation (via stress tensor $\boldsymbol{T}$, which define or delimit the type of medium subject to study) satisfies these above four postulates is called a Stokesian fluid.

With the above four postulates, according to matrix algebra, and if we add the condition that the components of **T** be linear in the components of $\boldsymbol{D}(\mathbf{u})$, we deduce (Cauchy–Poisson law):[2]

$$\boldsymbol{T} = -(p + \lambda\,\mathrm{div}\mathbf{u})\boldsymbol{I} + 2\mu\boldsymbol{D}(\mathbf{u}). \tag{2.19}$$

The coefficients λ and μ (of viscosity) being of scalar functions of the thermodynamic state (considered in Sect. 2.3.3) and $\boldsymbol{I}$ is the unit tensor, with δ_{ij} (the so-called Kronecker symbol with $\delta_{kk} = 1$ and $\delta_{ij} = 0$ if $i \equiv j$) as components.

Indeed, the fully general expression is Poisson's (1831) relation [41] – but the name of Poisson is rarely quoted today. In Cauchy, (1828), [43], the term $-p\boldsymbol{I}$, in (2.19), is absent.

Dynamical (Navier) equation (2.13a) equivalent to those resulting from (2.19) when $\mathrm{div}\mathbf{u} = 0$ and, $\mu = \text{const}$, is due to Navier (1821) [40], and Saint–Venant (1843) [49], proposed (2.19) in the special case when $\mu = \mu_0 = \text{const}$ and $\mu_v \equiv \lambda + (2/3)\mu = 0$, which is the so-called (1845) "Stokes relation" [48].

This is the simple and elegant constitutive equation (2.19) for the viscous NS motion, discovered by Cauchy in 1828. It is valid for any fluid, and indeed for any continuous medium, regardless of the form which the stress tensor $\boldsymbol{T}$ may take.

The coefficients λ and μ (of viscosity) are scalar functions of the thermodynamic state (considered in the next Section).

Concerning the long controversy regarding the Stokes relation:

$$3\mu_v = 0, \tag{2.20a}$$

$3\mu_v$ being the bulk viscosity, in the classical theory of viscous fluids; see Truesdell (1966) [42].

The viscosities coefficients (shear/dynamic and bulk) and the thermal conductivity k (see Sect. 2.3.3, Fourier's law (2.30)) are known functions, subject to the thermodynamic restriction (Clausius–Duhem inequalities):

$$\mu \geq 0,\ \kappa \geq 0 \text{ and } \mu_v \geq 0. \tag{2.20b}$$

[2] For a perfect (absence of viscosity) fluid, the pressure has already appeared as a dynamical variable in Euler Eq. 2.4. Characteristic of the discipline of gas dynamics is the postulate that the thermodynamic pressure, introduced via functional relations among the state variables (see Sect. 2.3.3), is equal to this dynamical pressure. When the deformation $\boldsymbol{D}(\mathbf{u}) = 0$, for a perfect fluid, p is the thermodynamic pressure when the fluid is compressible, while p is simply an independent dynamical variable otherwise. For an incompressible perfect or viscous fluid (Navier, see Sect. 2.2) p is not an unknown quantity, because it can be determined when we have found the velocity field **u**. In some works (see [31]) a mean pressure, $p^* = -(1/3)\mathrm{Trace}\,\boldsymbol{T}$ is defined, and we have the following relation: $p - p^* = [\lambda + (2/3)\mu]\mathrm{div}\mathbf{u}$.

The dynamical equation (in components form, with indices, $i = 1, 2$ and 3) resulting from (2.17) with (2.19) is the Navier–Stokes (compressible) equation for the component u_i of the velocity **u**:

$$\rho\, Du_i/Dt + \partial p/\partial x_i = \partial/\partial x_j[\mu(\partial u_i/\partial x_j + \partial u_j/\partial x_i)] + \partial/\partial x_i[\lambda(\partial u_k/\partial x_k)], \tag{2.21}$$

since for the components of deformation tensor $\boldsymbol{D}(\mathbf{u})$ we have:

$$2D_{ij} = \partial u_i/\partial x_j + \partial u_j/\partial x_i. \tag{2.22}$$

A Stokesian fluid whose constitutive, NS compressible, equation is given by (2.19) is called a Newtonian fluid.

In Saint–Venant (1843) (and Stokes (1845)), the resulting dynamical equation, for u_i, in place of (2.21), is:

$$\rho Du_i/Dt + \partial p/\partial x_i = \mu_0\{\Delta u_i - (1/3)\partial/\partial x_i\{D\log\rho/Dt\}, \tag{2.23a}$$

if we take into account the equation of continuity:

$$\partial u_k/\partial x_k = -D\log\rho/Dt, \tag{2.23b}$$

and the Stokes relation (2.20a).

For an incompressible homogeneous fluid, again we derive the Navier dynamical equation (2.13a), with $\nu_0 \equiv \mu_0/\rho_0 = \text{const}$.

Equation 2.23a (where μ_0 is a constant), with the specifying equation $p = p(\rho)$ and continuity equation (2.23b), forms a closed system of equations – the so-called Navier–Stokes (compressible N–S equations) for u_i, p, and ρ, and governs a barotropic(?) viscous and compressible fluid flow, without an energy equation for the temperature T – which unfortunately do not have any physical (fluid dynamics) signification.

Slightly more complete general Navier–Stokes compressible N–S equations, for the unknowns u_i, p, and ρ, are obtained from (2.21), with again the specifying equation $p = p(\rho)$ and continuity equation (2.23b), if we assume that viscosities μ and λ do not depend on temperature T and are known functions of the density ρ only.

These above N–S equations do not emerge via a RAM Approach from the full unsteady NS–F equations and, in fact, do not have in reality any interest for fluid dynamicians!

We also obtain a simplified model of compressible viscous fluid flow if we assume in addition, instead of $p = p(\rho)$, that the pressure is identically constant in the fluid flow (isobaric fluid flow).

In this case, then, we arrive at the so-called Burger's model equations (as in Kazhikov (1994) [50]).

$$\mathrm{D}\log\rho/\mathrm{Dt} + \partial u_k/\partial x_k = 0, \tag{2.24a}$$

$$\begin{aligned}\rho\, \mathrm{D}u_i/\mathrm{Dt} &= \partial/\partial x_j[\mu(\rho)(\partial u_i/\partial x_j + \partial u_j/\partial x_i)] \\ &\quad + \partial/\partial x_i\,[\lambda(\rho)(\partial u_k/\partial x_k)],\end{aligned} \tag{2.24b}$$

which is a closed system of two equations for the velocity components u_i and the density ρ, when we assume that the viscosity coefficients, λ and μ, are a function only of ρ.

A final important remark concerning the above N–S compressible viscous barotropic system equations, (2.23a) and (2.23b) with $p = p(\rho)$, or Eqs. 2.21 and 2.23b with $p = p(\rho)$, for the unknowns u_i, p and ρ, which are mainly used by applied mathematicians in their rigorous mathematical analyses (see, for instance, P. L. Lions (1998) [51]) does not have any physical reality, mainly because just viscosity always generates entropy (baroclinity).

For this, in particular, the various rigorous mathematical results concerning the so called "incompressible limit", related to the limiting process $M \downarrow 0$, in the framework of above, compressible barotropic ($p = p(\rho)$) and viscous, two systems, seems (to me) very questionable.

In the short paper by Leray (1994) [52], this question is pertinently discussed.

2.3.3 Thermodynamics and Energy Equation via Fourier Constitutive Equation

But in general (in reality) the coefficients of viscosity are assigned or empirical functions of the positive variables ρ (density) and especially T (temperature), which both are present also in equation of state (2.7) or (2.8) for a trivariate realistic fluid.

Indeed, Euler and Lagrange not only failed to include viscosity effects in their equations of motion, forcing them adopt corresponding simplified (slip) boundary condition (2.16a), but also oversimplified their equation of state.

In real fluids, the pressure, p, is a function of two variables, ρ and T (for a trivariate fluid in a baroclinic motion, see the equation of state (2.7)). Obviously, again, it is necessary to associate with N–S compressible equations (2.21), (2.23b), and (2.7), for u_i, ρ, p and T, an energy equation, if we want to obtain a closed system of equations for our six unknown functions. For this, some thermodynamical assumptions are required.

For the real fluid flows – compressible, viscous, and heat-conducting – the mechanical energy is converted into heat by viscosity, and the heat of compression is diffused by heat conduction.

Here we consider only an homogeneous fluid when the local equation of state is according to the basic postulate of Gibbs (1875) [53]; see Truesdell (1952) [44], and Serrin (1959) [31]:

$$E = \mathrm{E}(\rho, S), \tag{2.25}$$

where E is the specific internal energy and S is the specific entropy.

In this case the temperature T and the thermodynamic pressure p are defined by the two relations:

$$\mathrm{T} \;= \partial E/\partial S \text{ and } \mathrm{p} \;= -\partial E/\partial(1/\rho). \tag{2.26}$$

Now, for any compressible fluid, by differentiating (2.25) along any curve on the energy surface (characterizing the fluid) we obtain:

$$\mathrm{D}E/\mathrm{Dt} \;=\; \mathrm{T\ D}S/\mathrm{dt} - \mathrm{p\ D}(1/\rho)\mathrm{Dt}. \tag{2.27}$$

But, for any homogeneous medium in motion, the conservation of energy is expressed by the equation of C. Neumann (1894):

$$\rho\,\mathrm{D}E/\mathrm{Dt} + \mathrm{p}\ (\partial \mathrm{u_k}/\partial \mathrm{x_k}) = -\partial \mathrm{q_i}/\partial \mathrm{x_i} + [2\mu\,\mathrm{D_{ij}} + \lambda\,\mathrm{D_{kk}}\ \delta_{\mathrm{ij}}]\mathrm{D_{ij}}, \tag{2.28}$$

where the q_i are the Cartesian components of the heat flux vector **q**.

For the special case of a non-viscous incompressible fluid, the energy equation was given by Fourier (1833) [54], and for small motions of a viscous perfect gas by Kirchhoff (1868) [39], and in this case we also have $\mathrm{p} = \mathrm{R}\rho\mathrm{T}$, where R is the constitutive constant of the viscous thermally perfect gas.

We observe also, that for a medium suffering deformation the two Eqs. 2.27 and 2.28 express different and independent assumptions: the former, the existence of an energy surface, characterizing the fluid, and the latter, that mechanical and thermal energy are interconvertible. Indeed the First Law, Second Law, and so on, of thermodynamics is rather misleading terminology. (For a history of the origin of thermodynamics, see H. Poincaré (1892) [55], and Truesdell and Muncaster (1980) [56].)

Finally, from (2.27) it follows that in place of (2.28) we can write the following equation for the specific entropy:

$$\rho\mathrm{TDS/Dt} \;= -\partial \mathrm{q_i}/\partial \mathrm{x_i} + (2\mu\,\mathrm{D_{ij}} + \lambda \mathrm{D_{kk}}\ \delta_{\mathrm{ij}})\mathrm{D_{ij}}. \tag{2.29a}$$

In particular, if the heat flux (via the vector **q**) rises solely from thermal conduction, then according to Fourier's law gives:

$$\mathrm{q_i} = -k\,(\partial \mathrm{T}/\partial \mathrm{x_i}), \tag{2.30}$$

where k is the thermal conductivity and q_i the components of **q**.

With (2.30) we obtain from (2.29a) the usual form of the energy equation:

$$\rho\mathrm{TDS/Dt} \;= -\partial(k(\partial \mathrm{T}/\partial \mathrm{x_k}))/\partial \mathrm{x_k} + (2\mu \mathrm{D_{ij}} + \lambda \mathrm{D_{kk}}\delta_{\mathrm{ij}})\mathrm{D_{ij}}. \tag{2.29b}$$

For an adiabatic ($q_i = 0$) and non-viscous (inviscid) fluid with $\mu = 0$ and $\lambda = 0$:

$$\mathrm{D}S/\mathrm{Dt} = 0. \tag{2.31}$$

Therefore, if a non-viscous homogeneous fluid be in continuous motion devoid of heat flux, then the entropy of each particle remains constant. In particular, if the motion be steady, then the entropy is constant along each streamline.

But in general, the real motion is not isentropic – S is different of a constant in the each point of the flow and in time – but constant for each particle (along the trajectory) for a Eulerian fluid flow.

But even if the flow is isentropic, isentropicity in fluid remains valid only up to the first shock front encountered by the particles, after which it may well fail its isentropicity property.

2.3.4 Navier–Stokes–Fourier (NS–F) Equations

The three equations – continuity (2.23b), N–S for compressible motion (2.21), and energy (2.28) – with the two state relations

$$\mathrm{p} = \mathrm{R}\rho\ \mathrm{T} \text{ and } E = \mathrm{C_v T}, \tag{2.32a, b}$$

valid for a thermally perfect gas with constants specific heats, constitute the so-called Navier–Stokes–Fourier (NS–F) equations for a compressible, viscous and heat-conducting Newtonian fluid.

In this case it is assumed that the three constitutive (dissipative) coefficients are functions of ρ and T:

$$\lambda = \lambda(\rho,\ \mathrm{T}), \tag{2.33a}$$

and

$$\mu = \mu(\rho,\ \mathrm{T}) \tag{2.33b}$$

in (2.21) and (2.28), and

$$\mathrm{k} = \mathrm{k}(\rho,\ \mathrm{T}) \tag{2.33c}$$

in (2.30).

The compact form of these NS–F equations is:

$$\mathrm{D}\rho/\mathrm{Dt} + \nabla \cdot \mathbf{u} = 0, \tag{2.34a}$$

$$\rho\ \mathrm{D}\mathbf{u}/\mathrm{Dt} + \nabla \mathrm{p} + \rho g\mathbf{k} = \nabla . \Pi, \tag{2.34b}$$

$$\rho C_v DT/Dt \ + \ p \ \nabla \cdot \mathbf{u} = \nabla \cdot [k \ \nabla T] + \Phi, \tag{2.34c}$$

where Φ is the viscous dissipation function and gravity $\mathbf{g} = -g\mathbf{k}$ acts in the negative x_3 direction. In (2.34b, 2.34c):

$$\mathit{\Pi} = \lambda(\nabla \cdot \mathbf{u})I + 2\mu\mathbf{D}(\mathbf{u}) \tag{2.35a}$$

and

$$\Phi = 2\mu\text{Trace }[(\mathbf{D}(\mathbf{u})^2] + \lambda(\nabla \cdot \mathbf{u})^2, \tag{2.35b}$$

where:

$$\text{Trace }[(\mathbf{D}(\mathbf{u}))^2] = \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{u}) = (1/4)[\partial u_i/\partial x_j + \partial u_j/\partial x_i]^2, \tag{2.36}$$

and $\mathbf{D}(\mathbf{u})$ is the rate-of-deformation tensor.

2.4 Initial and Boundary Conditions

Obviously, there has always been considerable interest in initial-boundary value problems for various systems of partial differential equations arising in Newtonian fluid dynamics.

This interest of fluid dynamicians stems primarily from efforts to create useful computational models of various processes for the purposes of simulation, prediction, and the detailed study of various fluid flow phenomena.

Naturally, the initial-boundary value problems for fluid dynamics equations should have been carefully investigate – but unfortunately, rigorous proof of the existence and uniqueness of solutions of these well-posed fluid dynamics problems requires very difficult mathematical investigation.

While initial-boundary value problems for these systems of equations are not easy to analyze, mathematical tools useful for such problems can be found in the works of Kreiss (1970, 1974) [57, 58], Belov and Yanenko (1971) [59], Oliger and Sundström (1978) [60], Majda (1984) [61], and Kreiss and Lorenz (1989) [62].

The solveability of these problems (a fundamental problem in rigorous mathematical theory of NS–F equations) is discussed in Chap. 8 of our *Theory and Applications of Viscous Fluid Flows* (2004) [47], and here we note only that the baroclinic and barotropic Eulerian equations are both symmetrical, hyperbolic systems, but isochoric and incompressible equations are not hyperbolic. This has a profound influence on the well-posedness of initial-boundary value problems for these systems of partial differential equations.

2.4.1 The Problem of Initial Conditions

The above NS–F unsteady equations (2.34a–2.34c) contain a total of five times derivatives for the components u_i of the velocity $\mathbf{u}$, density ρ, and temperature T. As a consequence, if we want to resolve a pure initial value, or Cauchy, problem (in the L^2 norm, for example), then it is necessary to have a complete set of initial conditions (data) for $\mathbf{u}$, ρ, and T:

$$t = 0 : \mathbf{u} = \mathbf{u}^{\circ}(\mathbf{x}), \rho = \rho^{\circ}(\mathbf{x}),\ T = T^{\circ}(\mathbf{x}), \tag{2.37}$$

where $\rho^{\circ}(\mathbf{x}) > 0$ and $T^{\circ}(\mathbf{x}) > 0$.

Moreover, when we consider a free-boundary problem or an unsteady flow in a bounded container, with a boundary depending on time, an initial condition for the (moving) boundary $\partial\Omega(t)$ has to be specified.

For an NS compressible but barotropic flow for the velocity $\mathbf{u}$ and density ρ, governed by Eqs. 2.21 and 2.23b with $p = p(\rho)$, as initial conditions we assume:

$$t = 0 : \mathbf{u} = \mathbf{u}_b{}^{\circ}(\mathbf{x}) \text{ and } \rho = \rho_b{}^{\circ}(\mathbf{x}). \tag{2.38a}$$

For the isochoric Euler equations (2.21) it is necessary to impose also

$$t = 0 : \mathbf{u} = \mathbf{u}_i{}^{\circ}(\mathbf{x}) \text{ and } \rho = \rho_i{}^{\circ}(\mathbf{x}). \tag{2.38b}$$

If the flow is continuous, when $\rho^{\circ}(\mathbf{x}) = \text{const}$, we have an incompressible flow (Eqs. 2.22 or 2.23b), and it is sufficient to assume only an initial condition for the velocity u:

$$t = 0 : \mathbf{u} = \mathbf{u}_i{}^{\circ}(\mathbf{x}). \tag{2.38c}$$

It is important to note that for both isochoric and incompressible divergence, free flows, it is necessary that

$$\text{the boundary integral} \int \mathbf{u} \cdot \mathbf{n}\, d\Omega \text{ vanish} \tag{2.39a}$$

and

$$\nabla \cdot \mathbf{u}_i{}^{\circ} = 0. \tag{2.39b}$$

Naturally, this last condition has no analogue for compressible (baroclinic or barotropic) flows because of the occurrence of the term $\partial\rho/\partial t$ in the continuity equation (2.23b).

Obviously, for the Laplace equation, which governs an incompressible, irrotational Eulerian unsteady flow (for example, waves on water), we do not have the possibility of imposing any initial conditions! But this Laplace equation is very appropriate for the investigation of waves on (incompressible) water, and in this case it is necessary to consider a free-boundary problem; that is, a problem for which the fluid (water) is not contained in a given domain but can move freely.

Usually, for this Laplace elliptic equation, one boundary condition is given (on the contour line containing the fluid) – but in the case when the boundary is known!

Two unsteady, dynamic and kinematic, conditions are needed (and also two initial conditions) at the free surface (interface) $x_3 = \eta(t, x_1, x_2)$, because the surface position $\eta(t, x_1, x_2)$ has to be determined as well as potential function ϕ (t, x_1, x_2, x_3).

For the free surface problem (for the function $\phi(t, x_1, x_2, x_3)$ and $\eta(t, x_1, x_2)$) governing the non-linear waves on water, we can consider two physical problems.

First is the so-called "signalling" (two-dimensional) problem, in which we have as initial conditions, when the water is initially at rest in a semi-infinite channel, the following conditions:

$$\phi(0, x_1, x_3) = 0 \text{ and } \eta(0, x_1) = 0, \text{ when } x_1 > 0, \tag{2.40a}$$

and at initial time t = 0 an "idealized wave-maker", at $x_1 = 0$, will generate a horizontal velocity disturbance, such that the initial condition is:

$$\partial\phi/\partial x_1 = W^\circ B(t/t^\circ), \text{ for } x_1 = 0 \text{ and } t > 0, \tag{2.40b}$$

where W° and t° are the characteristic velocity and time scales associated with the wave-maker idealized by the function $B(t/t^\circ)$.

A second category of the problem for water waves, in the infinite channel, is obtained by specifying an initial surface shape but zero velocity:

$$\text{for } t = 0 : \eta = a^\circ\zeta^\circ(x_1/l^\circ, x_2/m^\circ) \text{ and}$$

$$\phi(0, x_1, x_2, x_3) = 0, \tag{2.40c}$$

where l° and m° are the characteristic wavelengths (in the x_1 and x_2 directions) for the three-dimensional water wave motion.

In (2.40c) the scalar a° is a characteristic amplitude for the initial elevation of the free surface characterized by the function $\zeta^\circ(x_1/l^\circ, x_2/m^\circ)$. (Concerning the boundary conditions (kinematic and dynamic) for this free surface problem, see the next Sect. 2.4.2.)

For meteorological motions (considered in the Chap. 9), when we consider various approximate model equations – f°-plane equations, primitive equations, quasi-geostrophic equations, or Boussinesq equations – it is necessary, in fact (mainly because the filtering acoustic waves), to resolve associated unsteady adjustment problems for the formulation of consistent initial conditions for these simplified model equations.

2.4.2 Unsteady Adjustment Problems

For the study of a compressible fluid flow it is necessary to have for the determination of the solution of the corresponding evolution unsteady equations – Euler, Navier, and NS–F – a set of initial data for ρ, **u**, and T (the Cauchy problem). However, when we consider, for example, the incompressible equations ((2.13a) or (2.14a) with $\nabla.\mathbf{u} = 0$) for **u** and p, one is allowed to specify a set of initial conditions less in number than for the full compressible baroclinic equations. This is due to the fact that the "main" low-Mach-number limiting process (in fact, M tends to zero, with t and $\boldsymbol{x}$ both fixed), which leads to the approximate incompressible (model) equations, filters out some time derivatives – these corresponding to acoustic fast waves, because such waves are of no importance for low-speed aerodynamics and various atmospheric and oceanic motions – at least for steady flows.

When $\mathrm{Re}\uparrow\infty$ (large Reynolds number), from the Navier (incompressible and viscous) equations we derive the Prandtl boundary layer equations, and accordingly, for an unsteady flow the term $\partial u_3/\partial t$ disappears in the limiting momentum equation for the vertical (x_3-direction) component (u_3) of velocity! For low Reynolds number ($\mathrm{Re}\downarrow 0$) in the Stokes and Oseen limiting (steady) equations, the unsteady terms also disappear! Due to this, one encounters the problem of finding an answer to the following question: "What is the initial condition that is necessary to prescribe for **u** a solution of an incompressible equation, and in what way is this condition related to the starting initial conditions (with given data) associated with the exact, compressible equations?"

It is important to note that the exact initial conditions for the full compressible equations are not in general consistent with the estimates of basic orders of magnitude implied by the approximate model (without acoustic waves) equations. A physical process of time evolution is necessary to bring the initial set to a consistent level as far as the orders of magnitude are concerned.

Such a process is called "unsteady adjustment" of the initial data set to the approximate structure of incompressible equations under consideration. This process of adjustment, which occurs in many fields of fluid mechanics besides Boltzmann kinetic theory (first discussed by Hilbert in 1912), is short on the time-scale of approximate simplified equations, and ultimately, in an asymptotic sense, we obtain values for the consistent set of initial conditions suitable for the simplified equations.

When we consider the set of approximate simplified model equations, usually derived heuristically, with time–space fixed, then it is first necessary, for instance, to elucidate various adjustment problems – namely, concerning Prandtl boundary layer, Stokes and Oseen steady, Navier incompressible viscous, and Boussinesq equations.

A number of adjustment problems occur in meteorology for atmospheric motions (adjustment to hydrostatic balance) and to geostrophy (as in the case, for example, cited in Sect. 9.2), and the reader can find a detailed discussion of these adjustment problems in Chap. 5 of our *Meteorological Fluid Dynamics* (1991) [19].

However, it is important to note that, depending on the physical nature of the problems, we may have two kinds of behaviour when the rescaled (short) time goes to infinity. Either one may have a tendency towards a limiting steady state, or an undamped set of oscillations (as, for example, the inertial waves in the inviscid problem of spin-up for a rotating fluid; see Greenspan (1968; §2.4) [63]). The problem, considered in Sect. 7.2, is also very particular, and requires a special approach due to the persistence of acoustic oscillations. (For the terminology of the initial layer as adapted to this kind of singular perturbation problem, see Nayfeh (1973; p. 23) [64].) Finally, we note that usually, the process of the unsteady adjustment of the aerodynamical (or meteorological) fields is a result of the generation, dispersion and damping of the fast internal waves. According to method of matching asymptotic expansions (MMAE), the initial conditions for the limiting model equations are, in fact, matching conditions between the two asymptotic representations – the main one (with t fixed), and the local one (near $t = 0$), which is a necessary companion to main one!

In conclusion, we can say that the aim of the unsteady adjustment problem can be stated as follows: Clarify just how a set of initial data associated with an exact system of unsteady equations can be related to another set of initial data associated with a simpler, approximate model system of equations which is a significant degeneracy of the original system of exact equations considered at the start, but with the less time derivatives in this approximate model system.

In order to solve such a problem it is necessary to introduce an initial layer in the vicinity of $\mathrm{t} = 0$, characterized by a short fixed time τ. Obviously such an unsteady adjustment problem is very important in meteorology for the formulation of a well-posed initial/Cauchy evolution in a time-prediction problem relative to, for example: “what the weather will be like tomorrow or for the next few days?”

Concerning the rigorous mathematical results of the singular limits in compressible fluid dynamics, see, for instance, the paper by Beiräro da Veiga (1994) [65], and also the various references in this paper. More recent papers have been published concerning the passage of compressible $\Longrightarrow$ incompressible, by Desjardins, Lions, Grenier, Masmoudi, Hagstrom, Lorenz, and Iguchi; and for references see our *Topics in Hyposonic Flow Theory* (2006) [13]. Here we do not consider these contributions, but instead discuss some of these singular-limit problems (low-Mach asymptotics) which deserve a serious, consistent, fluid dynamics investigations via a RAM Approach (as in [13]).

Concerning the low-Mach asymptotic, we observe also that in the case of a flow affected by acoustic effects in a confined gas (internal flow within a bounded domain $D(\mathrm{t})$), over a long time when the wall $\partial D(\mathrm{t})$ is started impulsively from rest, a multiple-time-scale technique is necessary, because acoustic oscillations remain undamped and the unsteady adjustment problem (with matching) does not work (see Sect. 7.2 on applications in aerodynamics).

2.4.3 Boundary Conditions for the Velocity Vector u and Temperature T

Several boundary conditions could be considered with respect to different physical situations.

If we consider, as a simple example, the motion of a fluid in a rigid container Ω (with a boundary $\partial\Omega$, independent of time), a bounded connected open subset of $\boldsymbol{R}^d$ (where $d > 1$ is the physical dimension), the different structure of the equations leads to the necessity of distinguishing between viscous (NS–F, NS, or Navier) and inviscid (Eulerian) fluids.

(a) For a viscous (NS–F or Navier) fluid: $\mu > 0$ and $\mu_v \equiv \lambda + (2/3)\mu > 0$
In this case, the physical effects due to the presence of the dynamic (shear) viscosity coefficient μ yield the validity of the steady no-slip condition:

$$\mathbf{u} = 0 \text{ on } \partial\Omega, \tag{2.41a}$$

(b) For a bulk-viscous fluids: $\mu = 0, \mu_v > 0$
Since only the bulk viscosity coefficient μ_v is different from zero, in this situation the slip boundary condition

$$\mathbf{u}.\mathbf{n} = 0 \text{ on } \partial\Omega \tag{2.41b}$$

where it is assumed here, and in what follows, that $\mathbf{n} = \mathbf{n}(\mathbf{x})$ denotes the unit outward normal vector to $\partial\Omega$.

(c) For an inviscid (Eulerian) fluid: $\mu = 0, \mu_v = 0$
Also in this case, the slip boundary condition (2.41b) is assumed.

As concerns the (absolute) temperature T, the boundary condition takes different forms in the two alternative cases, $k > 0$ and $k = 0$.

(d) Conductive fluids: $k > 0$
Several boundary conditions have physical meaning. Limiting ourselves to the most common cases, we can require :

$$\mathrm{T} = \mathrm{T_w} \text{ on } \partial\Omega(\text{Dirichlet}) \tag{2.42a}$$

$$k\partial\mathrm{T}/\partial\mathrm{n} = \Xi \text{ on } \partial\Omega(\text{Neumann}) \tag{2.42b}$$

$$k\partial\mathrm{T}/\partial\mathrm{n} + h(\mathrm{T} - \mathrm{T_0}) = \Xi \text{ on } \partial\Omega(\text{third type}), \tag{2.42c}$$

where $\mathrm{T_w} > 0$ and Ξ are known functions, and $h > 0$ is a given constant.

(e) Non-conductive (adiabatic case) fluids: $k = 0$

No boundary condition have to be imposed on temperature T if (2.41a) or (2.41b) are satisfied, since in these cases the temperature is not subjected to transport phenomena through the boundary.

According to Gresho (1992, pp. 47–52) [66], if $\mathbf{u}_N{}^0(\mathbf{x})$ is the initial (t = 0) velocity field for the Navier equation (2.13a), then in the domain Ω it is necessary to impose the above incompressibility constraint (2.13b):

$$\nabla \cdot \mathbf{u}_N{}^0(\mathbf{x}) = 0, \tag{2.43a}$$

and on the boundary $\partial\Omega = \Gamma(s)$:

$$\mathbf{n} \cdot \mathbf{u}_N{}^o(s) = \mathbf{n} \cdot \mathbf{w}(s,\ 0) = \mathbf{n} \cdot \mathbf{w}^0(s), \tag{2.43b}$$

where $\mathbf{w}(s, t)$ is the specified boundary condition for the Navier velocity vector which satisfies Eq. 2.13a.

2.4.4 *Other Types of Boundary Conditions*

In many situations (inflow–outflow problems) the velocity cannot be assumed to vanish on $\partial\Omega$. This is the case, for instance, for the flow around an airfoil, where an inflow region is naturally present upstream (and an outflow region appears in the wake), or the flow near a rigid body, where the velocity can be assumed to vanish only on the boundary of the body. In these cases, several different boundary conditions may be prescribed.

Let us begin by considering the viscous case. Concerning the velocity-field, a (non-zero) Dirichlet boundary condition can be imposed everywhere, or, alternatively, only in the inflow region: that is, the subset of $\partial\Omega$ where $\mathbf{u} \cdot \mathbf{n} < 0$, whereas, on the remaining part of the boundary, the conditions

$$\mathbf{u} \cdot \mathbf{n} = U^{+} > 0 \text{ and } (\mathbf{n} \cdot \mathbf{D}) \cdot \mathbf{t} = 0, \tag{2.44a}$$

have to be prescribed. (Here, $\mathbf{t}$ is a unit tangent vector on $\partial\Omega$, and $\mathbf{D} = \mathbf{D}(\mathbf{u})$ is the rate of strain (deformation) tensor.)

Let us, moreover, remark that the condition

$$\mathbf{u} \cdot \mathbf{n} = 0 \text{ and } (\mathbf{n} \cdot \mathbf{D}) \cdot \mathbf{t} = 0, \tag{2.44b}$$

could also be considered, on the whole, $\partial\Omega$. In this case, however, no inflow or outflow regions would be present.

More important is to analyse the boundary condition for the density ρ, since now it turns out that it is necessary to prescribe it on the inflow region. In fact, the first-order hyperbolic continuity equation (2.13a) can be solved by means of the theory of characteristics, and the boundary datum for ρ on the inflow region is indeed a (necessary) Cauchy datum for the density on a non-characteristic surface.

Let us note also, that if the heat conductivity coefficient k is vanishing and the fluid is inviscid, the same type of Dirichlet-inflow boundary condition has to be imposed on the temperature T, since in such a case Eq. 2.6 is also of the hyperbolic type for T. More complicated is the situation when the inviscid (Euler) case $(\mu = \lambda = k = 0)$ is considered.

In fact, in this case the Eulerian system is a first-order hyperbolic one, and the number of boundary conditions, in the case of an "open boundary" (or a boundary located in the interior of a body or fluid), is different depending on whether the flow is

$$\text{subsonic} : |\mathbf{u}| < \mathrm{a},$$

or

$$\text{supersonic} : |\mathbf{u}| > \mathrm{a},$$

where

$$\mathrm{a} \; = [\gamma \; \mathrm{RT} \;]^{1/2},$$

is the local sound speed for the perfect gas.

Take, for example, $d = 3$. An analysis of the sign of the eigenvalues of the associated characteristic matrix yields the following conclusion: The number of boundary conditions must be five or four on an inflow boundary, depending on whether the flow is supersonic or subsonic, and zero or one on an outflow boundary, again depending on whether the flow is supersonic or subsonic.

Obviously, in the case of an "open boundary" the normal velocity is non-zero on the boundary, except at certain points. In both cases, no obvious physical boundary conditions are known. We will not enter more deeply into this argument, and will only briefly discuss the inviscid case subjected to the slip boundary condition (2.41b), for which the boundary is a characteristic surface.

Further information on inflow–outflow boundary-value problems for compressible N–S and inviscid Euler equations can be found in two pertinent papers produced by Gustafsson and Sundström (1978)[67] and Oliger and Sundström (1978) [60]. Here, we note only that the solid-wall slip stationary boundary condition (2.41b) – the normal velocity

$$u_{\mathrm{n}} \equiv \mathbf{u} \cdot \mathbf{n} = 0,$$

should vanish at the boundary, is consistent with the number of inward characteristics (one).

The reader can find also in two papers by Viviand and Veuillot (1978)[68] and Viviand (1983)[69], a discussion of boundary conditions for steady Euler flow, considered as the limit (when time tends to infinity – the "pseudo-unsteady" method) of an unsteady flow (which does not have a precise physical meaning).

It is necessary to note that in numerical, computational, fluid dynamics, these problems of boundary conditions are very thoroughly considered by often taking into account the constraints related with the various particularity of the considered fluid flow problem and associated with numerical algorithms.

Another interesting set of boundary conditions appears when we consider the free-boundary problem; that is, a problem for which the fluid is not contained in a given domain but can move freely. In this case the vector $\mathbf{n} \cdot \boldsymbol{T}$ is prescribed on (interface) $\partial\Sigma$, where, moreover, $\mathbf{u} \cdot \mathbf{n}$ is required to be zero (stationary case) or equal to the normal velocity of the boundary itself (non-stationary case).

The value of $\mathbf{n} \cdot \boldsymbol{T}$ can be zero (free expansion of a fluid in the vacuum), or to (see, for example, the case of the well-known Bénard problem considered in Chap. 8):

$$-\,\mathrm{p}_e\mathbf{n} + 2\sigma K\mathbf{n} + \nabla_s\sigma, \text{ on interface,} \tag{2.45a}$$

where p_e is the external pressure, $\sigma = \sigma(\mathrm{T})$ is the surface tension (temperature-dependent, when the fluid is an expansible liquid), K is the mean interfacial curvature, and:

$$\nabla_s = \nabla - \mathbf{n}(\mathbf{n} \cdot \nabla), \tag{2.45b}$$

is the surface (projected) gradient at the interface, respectively.

But in this (viscous) case, it is necessary to also write a heat transfer condition across the interface, for an expansible and thermally conducting fluid (liquid),

$$k(T)\partial\mathrm{T}/\partial n + h_s\mathrm{T} \;=\; \text{prescribed function,} \tag{2.45c}$$

which is a Newton's cooling law, where the heat-transfer (constant) coefficient h_s is sometimes called the Biot number. We observe that rigorously, in a Bénard convection problem (with a temperature dependent tension in a free-surface) it is necessary to take into account two Biot numbers, respectively, for the conduction (motionless/no convective motion) state and convection state (see Chap. 8).

But the problem of "two Biot numbers asymptotics" is actually widely open (see, in Chap. 8, a discussion concerning this two Biot numbers problem in the framework of the Bénard problem for an expansible liquid layer on a solid horizontal flat surface and heated from below.) Since $\mathrm{d}\sigma/\mathrm{dT} \neq \mathrm{o}$, then, for the film problem it is necessary to take into account a Marangoni number proportional to the gradient:

$$(\mathrm{d}\sigma/\mathrm{dT})_{\mathrm{T}=\mathrm{T}^o},$$

where T° is a constant temperature.

In such a case we consider a thin film Bénard–Marangoni free-surface problem, which is fundamentally different from the classical Rayleigh–Bénard thermal instability problem. (See, for instance, in Velarde and Zeytounian (2002) [70], CISM Courses and Lecture (N0 428), pp. 123–90.)

Naturally, we are now imposing one more condition on the interface $\partial\Sigma$, since it is an unknown of the problem; in the non-stationary case an initial condition for the interface has to be added too (see (2.40a) or (2.40c)).

For an inviscid incompressible fluid (water) when we consider the wave on the water (in this case the problem for an irrotational flow is governed by the Laplace equation), an obvious physical simple condition is (if we assume that the surface tension is negligible), in place of (2.45a):

$$p = p_A, \textit{on interface between water and air above}, \tag{2.46}$$

where p_A denotes the air (constant, ambient) pressure on interface Σ and usually this ambient air (above the interface) is assumed passive (at temperature $T_A =$ const, pressure $p_A =$ const, with negligible viscosity and density).

In the case of a viscous liquid (thin film Marangoni problem, discussed in Chap. 8) the above condition (2.46) is replaced by a rather complicated explicit upper free surface condition (see Sect. 8.2.2).

Now, if the equation of the interface is $x_3 = \eta(t, x_1, x_2)$, in a Cartesian system of coordinates $(0, x_1, x_2, x_3)$, then from the Bernoulli incompressible integral we obtain the following dynamic condition on interface (for the wave on the water – the inviscid fluid problem) according to (2.46):

$$\partial\phi\partial t + (1/2)(\nabla\phi)^2 + g\,\eta = 0, \text{ on } x_3 = \eta(t, x_1, x_2), \tag{2.47}$$

and since the interface is a material wave surface we have also a kinematic condition:

$$\begin{aligned}\partial\phi/\partial x_3 = \partial\eta/\partial t &+ (\partial\phi/\partial x_1)\partial\eta/\partial x_1\\ &+ (\partial\phi/\partial x_2)\partial\eta/\partial x_2, \text{on } x3 = \eta(t, x_1, x_2),\end{aligned} \tag{2.48}$$

Finally, if we assume that the water rests on a horizontal and impermeable bottom of infinite extend at $x_3 = -h_{0,}$ where $h_0 =$ const is supposed finite, then we have the following simple (flat) bottom boundary condition for the Laplace equation:

$$\partial\phi/\partial x_3 = 0, \text{ on } x_3 = -h_0, \tag{2.49}$$

The Laplace equation for the potential ϕ, with (2.40a), (2.40b) or (2.40c) and (2.47)–(2.49), constitutes a well-posed problem for the investigation of the non-linear unsteady waves on the water (see, for instance, Whitham (1974) [71], and the review paper by Zeytounian (1995) [72]).

It is important to note that each physical problem has specific boundary conditions related to the intrinsic nature of the problem. For example, in gas dynamics problems the boundary conditions are different if the fluid flow is subsonic ($M < 1$), supersonic ($M > 1$), transonic ($M \sim 1$) or hypersonic ($M >> 1$). If, for instance, the undisturbed basic flow is in the $x > 0$ direction and the body in question (in x, y plane) is located on the x-axis, with its leading edge at the origin and its trailing edge at $x = 1$ (with non-dimensional variables), then we can assume that the body shape is described by:

$$y = \delta\, h(x), \tag{2.50}$$

where the non-dimensional parameter δ is the maximum value of y of the body. The 2D steady velocity potential φ(x, y) is a solution of the steady 2D Steichen dimensionless equation:

$$\left[a^2 - M^2(\partial\varphi/\partial x)^2\right]\partial^2\varphi/\partial x^2 + \left[a^2 - M^2(\partial\varphi/\partial y)^2\right]\partial^2\varphi/\partial y^2$$

$$- 2M^2(\partial\varphi/\partial x)(\partial\varphi/\partial y)\partial^2\varphi/\partial x\partial y = 0, \tag{2.51}$$

with the following relation for the local sound speed:

$$a^2 = 1 + [(\gamma + 1)/2]M^2\{1 - [(\partial\varphi/\partial x)^2 + (\partial\varphi/\partial y)^2]\}. \tag{2.52}$$

In this case the slip condition is:

$$\partial\varphi/\partial y = \delta(dh(x)/dx)\partial\varphi/\partial x,$$

$$\text{when } x \in [0, 1], \text{ on } y = \delta\, h(x). \tag{2.53}$$

Far away, upstream, from the body the flow should be undisturbed, which requires:

$$\partial\varphi/\partial x \to 1 \text{ and } \partial\varphi/\partial y \to o \text{ as } x \to -\infty. \tag{2.54}$$

In most applications, the bodies of interest are thin and streamlined, so that generally δ is a small non-dimensional parameter ($\delta << 1$). We note here only that the classical linear, subsonic and supersonic theory is invalid when respectively:

$$\left[M^2 - 1\right]/\delta^{3/2} = O(1) - \text{transonic similarity}$$

$$\delta\, M = O(1) - \text{hypersonic similarity}$$

$$\eta\delta = O(1) - \text{far field similarity},$$

where $\eta(= x + [M^2 - 1]^{1/2})\, y$ is a characteristic coordinate, such that:

$$\eta \sim \infty \text{ with } M \text{ fixed}.$$

In the case of justification of the well-known Boussinesq (1903) [8] assertion (see, for instance, Chap. 4) concerning the convection in fluids [13]: "The derivatives of the density can be neglected except when they intervene in the calculation of the force of Archimedes."

It is also necessary to consider the hyposonic ($M << 1$) case, and in such a case, for the atmospheric motions it is necessary to take into account the following constraint:

$$M/Bo = O(1) - \text{hyposonic similarity}, \tag{2.55a}$$

where

$$Bo = L^{\circ}/(RT^{\circ}/g) \tag{2.55b}$$

a ratio of two lengths, is the so-called Boussinesq number (see our (1990) [12], p. 15).

The lee waves problem (related mainly with the dynamic influence of a mountain in a baroclinic, stratified, adiabatic atmosphere) is strongly influenced by the relief slip condition and also by the upstream flow conditions. In an unbounded atmosphere the radiation (in a simple Boussinesq model case) Sommerfeld condition for the Helmholtz equation at infinity (in altitude) plays an essential role. (See, in [13], various typical examples considered by Guiraud and Zeytounian.)

In the low Rossby model for atmospheric flow, the effect of the solid (earth) surface is taken into account (by matching) through the so-called viscous Ekman layer. Indeed, the viscous coefficients are so small that we should expect the boundary conditions to be close to those valid for the corresponding inviscid system. The viscous equations do, however, require additional boundary conditions, and as an effect, viscous boundary layers may occur at the boundaries.

Such boundary layers may sometimes be appropriate, as in the rigid wall situation (for example, the Ekman boundary layer). However, at open boundaries they are inappropriate.

In comparison to flows in interior or exterior domains, there are two new issues when the boundary extends to infinity. First, in addition to the usual initial and boundary conditions there needs to be some prescription of fluxes or pressure drops when the flow domain has several exits to infinity (as in (2.54)). Second, the solutions of interest often have infinite energy integrals, and recently a technique of integral estimates to deal with this problem has been developed. These estimates are called Saint Venant's type, because the method was first used in the study of Saint Venant's principle in elasticity.

Concerning, more precisely, the behaviour of an incompressible fluid velocity field at infinity, we note that in Dobrokhotov and Shafarevich (1996) [73], a simple method is given which makes it possible to determine an upper bound for the decay rate at infinity of an incompressible fluid velocity field of general form; that is, to determine a lower bound for the field itself.

This method is based on the use of simple integral identities which are valid for solutions of the Navier incompressible, viscous equations, in the external region which decrease quickly enough. For the equations in entire space, some of these identities were obtained by the two authors noted above.

The property of slow decay or spreading of localized fluid flow is a consequence of incompressibility, and is not associated with viscosity alone (in contrast to the case described by Serrin (1959) [31, 74]), so that it also holds good for an inviscid Eulerian fluid flow (in this case the reasons for spreading are related with the non-uniform external flow and non-linearity).

In fact, in order to compute in a bounded region a fluid flow modelled by a problem formulated on an infinite domain, one often introduces an artificial boundary Σ and tries to write on the domain $\Omega^* \subset \Omega$, bounded by Σ, a new problem whose solution is as close as possible to the original exact problem. When the solution of this new problem in Ω^* coincides with the restriction of the original problem, the boundary Σ is said to be transparent.

Here, we note also that the reader can find valuable information concerning this approach with applications to both inviscid and viscous fluid flows in various recently published papers in the leading journals devoted to numerical fluid dynamics (see, for instance, the recent issues of *Journal of Computational Physics*).

The general slip condition in an unsteady case:

$$\mathbf{n} \cdot (\mathbf{u} - \mathbf{u}_P) = 0, \tag{2.56}$$

is satisfied, in any case, for an impermeable solid wall, where $\mathbf{u}_P$ is the velocity of the moving wall. On the other hand, from the kinetic theory of gases, when the Knudsen number, Kn is small, we obtain

$$\mathbf{n} \wedge (\mathbf{u} - \mathbf{u}_P) = 0 \text{ on a moving wall.} \tag{2.57}$$

As a consequence of (2.56) and (2.57), we again deduce the no-slip condition (but for a moving wall):

$$\mathbf{u} = \mathbf{u}_P, \text{ on the moving wall.} \tag{2.58}$$

The above condition (2.57) is the so-called weak form of the no-slip condition on the moving wall.

Concerning the boundary condition for the temperature T on the wall, from the kinetic theory of gases, again when the Knudsen number Kn is small, we obtain:

$$\mathrm{T} = \mathrm{T}_P - \beta \mathbf{q} \cdot \mathbf{n}, \tag{2.59}$$

where β is a scalar function (related with the kinetic, Knudsen, sub-layer).

An interesting case of boundary condition is related to the so-called Prandtl–Batchelor condition (see, for instance, the papers by Batchelor (1956)[75] and Wood (1957)[76]).

For a 2D incompressible, steady Eulerian fluid flow, from Eq. 2.13a, when $\nu_\circ \equiv 0$, we derive the following equation for the 2D steady stream function $\psi(\mathrm{x}, \mathrm{y})$:

$$\nabla^2\psi = F(\psi), \tag{2.60}$$

where the function $F(\psi)$ is arbitrary! But, if the domain Ω where the flow is considered is a bounded connected open subset of $\mathbf{R}^2$, then we do not have the possibility of utilizing the behaviour condition at infinity for the determination of this function $F(\psi)$.

The key of this indeterminacy is strongly related with the vanishing viscosity problem. In fact, with the limiting process Re $\uparrow \infty$(or, $\nu_\circ \downarrow 0$) in the steady form of the Navier Eq. 2.13a. Namely, if we assume that the limit streamlines are closed in Ω, then according to Batchelor [75] we derive for the limit Euler stream line $\gamma_\circ$ (which is the one of the stream lines $\psi_\circ =$ constant) the following Prandtl–Batchelor condition:

$$dF(\psi_\circ)/d\psi_\circ = 0 \text{ and } F(\psi_\circ) = F_{\circ\circ} \equiv \text{const.} \tag{2.61}$$

As a consequence, the Eulerian vorticity, $\omega_\circ = -(1/2)F(\psi_\circ)$, for a steady incompressible 2D fluid flow, is constant in any region where the streamlines are closed.

From the matching performed by Wood [76] – with the corresponding Prandtl boundary-layer in the vicinity of $\partial\Omega$ – the value of this above constant, $F_{\circ\circ}$, is well determined.

On the other hand, in Guiraud and Zeytounian's short paper (1984) [77] a process for setting in motion a viscous incompressible liquid inside a 2D cavity is considered, and it is shown that the basic process occurs for a time of the order of $t = O(\text{Re})$. Then a flow, *à la* Prandtl–Batchelor, with constant Euler vorticity is established after a time $t >> \text{Re}$.

In this same paper [77], a G–Z functional equation is derived which governs the distribution of the vorticity in the main stage of interest, and for the simple case of a cylindrical cavity it is shown that the vorticity tends towards its own steady-state value exponentially.

Finally, concerning the case of overspecified and underspecified boundary conditions, it is important to note that when for a given problem the number of boundary conditions is overspecified, the difference approximation (for a numerical calculation) may well be stable. However, the effective boundary conditions which influence the solution are, in general, difficult to determine, especially for problems in several space dimensions.

They may well be a complicated function of the conditions given and bearing little resemblance to them. An additional complication induced by over-specification is that the underlying solution being approximated is not generally continuous. In order to avoid the problems associated with the proper selection of boundary conditions, the order and type of the differential equations is often raised to obtain a problem that is easier to analyze and approximate.

For example, the Eulerian equations are usually modified by adding dissipative terms so that the number of boundary conditions is appropriate. Unfortunately, this idea seldom works. If a spurious boundary layer of appreciable size results, the effects are not unlike those for discontinuities (for a system of equations, the errors can propagate away from the discontinuity through other components of the solution), and unless the dissipative terms are very large, the error introduced at the boundary will again propagate into the interior.

Now, if the boundary conditions are underspecified there are no a priori estimates for the differential equations. In order for an approximation to be computable these must be a sufficient number of boundary conditions specified for the approximation. This cannot be fewer than the number required for the differential equation.

The well-posedness of the initial boundary value fluid flow problems follows, to some extent, from properly formulated initial and boundary conditions, and is strongly linked with the various facets (through the existence and uniqueness results) of the solvability of these fluid flow problems.

We recognize that in large part what might be called "mathematical topics in fluid dynamics" has remained closed to the mainstream of theoretical fluid dynamics and mathematical physicists, due in large part – as judiciously observed in the book by Doering and Gibbon (1995) [78] devoted to applied analysis of the Navier (Navier–Stokes incompressible) equations – to the technical nature of rigorous investigations, often phrased in the unfamiliar language of abstract (non-linear) functional analysis.[3]

The above summary of Chap. 2 presents the main theoretical concepts and principles, and also equations and associated initial and boundary conditions, of classical/Newtonian fluid dynamics. Various theoretical concepts can be found in our books devoted, respectively, to non-viscous (2002) [37] and viscous (2004) [47] fluid flows. In our survey paper on the well-posedness of problems in fluid dynamics (a fluid-dynamical point of view) (1999) [79] the problem is carefully considered, and an historical survey of some mathematical aspects of Newtonian fluid flows can be found in our (2001) [29] surveys.

[3] The curious reader can find in "Handbook of Mathematical Fluid Dynamics, vol. 1 to 4", numerous papers related with rigorous mathematical results, of existence, unicity, regularity, well-posedness and limiting processes for solution of fluid flow problems, mainly by compactnesse-a very abstract functional approach!