A Unified Framework for a posteriori Error Estimation in Elliptic and Parabolic Problems with Application to Finite Volumes

Alexandre Ern and Martin Vohralík

Abstract We present a unified framework based on potential and flux reconstruction for guaranteed and efficient a posteriori error estimation. We consider as model problems the Laplace equation, the singularly perturbed convection-diffusion-reaction equation, and the heat equation. The analysis is performed for a wide class of space discretization schemes. Three simple conditions need to be verified, which we do for cell- and vertex-centered finite volumes for all model problems.

Keywords a posteriori error estimation, guaranteed upper bound, efficiency, robustness, elliptic and parabolic problems
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1 Introduction

A posteriori error estimation is an important tool in practical computations for error control and computational efficiency by adapting the discretization parameters. In the context of finite element methods, residual-based a posteriori error estimation has been initiated by Babuška and Rheinboldt [2] over three decades ago. The application to finite volume (FV) schemes is more recent; we refer, among others, to Achdou, Bernardi, and Coquel [1], Nicaise [19], and Ohlberger [20, 21].

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The purpose of this work is to present some recent results (and extensions thereof) by the authors [9, 11, 28–30] in a general framework. The salient features of this framework can be summarized as follows. Firstly, the error upper bound is formulated in terms of a *potential* and a *flux reconstruction* which must comply with some basic physical properties related to the model problem at hand. This approach allows one to achieve *guaranteed* error upper bounds, that is, upper bounds *without undetermined constants*, which is a key feature in the context of error control. Flux-based a posteriori error estimation for elliptic problems hinges on the Prager–Synge equality [22] and was first developed, among others, by Ladevèze [18] and Haslinger and Hlaváček [14].

Next, the present approach does not rely on a specific discretization scheme (in space), that is, we bound the difference between the exact solution and an arbitrary approximate solution which is only required to be piecewise smooth. Owing to this generality, the approach encompasses a wide class of schemes including FVs and many other schemes (discontinuous Galerkin, mixed finite elements, etc.) in a *unified setting*. At this stage, quite *general meshes* (e.g., with polygonal elements and so-called hanging nodes) can be considered as well. Turning next to *local efficiency*, that is, to local lower bounds on the error, we still proceed generally without resorting to any specific discretization scheme under two additional assumptions. On the one hand, we suppose that the approximate solution, the potential and flux reconstructions, and the problem data are piecewise polynomials and that the meshes possess some regularity which we formulate by introducing a matching simplicial, shape-regular submesh. On the other hand, we assume that the potential and flux reconstructions satisfy some local approximation properties which are expressed in terms of suitable local residuals of the approximate solution (plus its jumps). Local lower bounds on the error then result from the combination of these two assumptions and the fact that the local residuals provide local lower bounds on the approximation error, as previously shown, e.g., in Verfürth [24].

This paper is organized as follows. In §2, we collect some useful notation and basic ingredients for the analysis. Then, we present our results on three model problems. In §3, we consider the Laplace equation. The aim is to present in detail the key ideas in the context of a simple model problem. In §4, we turn to the convection-diffusion-reaction equation. We focus on singularly perturbed regimes resulting from dominant convection or reaction and show how the present approach can achieve *robustness* with respect to physical parameters. In §5, we consider the heat equation and the backward Euler scheme to discretize in time. The purpose is to show how the present approach handles evolution problems including time-varying meshes. In all cases, we first derive upper and lower bounds on the approximation error in an abstract framework applicable to a wide class of discretization schemes in space. Then, we show how the framework can be applied to cell- and vertex-centered FV schemes. For the sake of simplicity, we only consider model problems with homogeneous Dirichlet boundary conditions. Inhomogeneous Dirichlet and Neumann boundary conditions can be taken into account following [29]. Finally, we observe that some interesting applications of a posteriori error estimates are not covered herein; we mention, in particular, the use of such estimates as adaptive stopping criteria for linear [15] and nonlinear [7] iterative solvers.

2 Basic ingredients

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a polygonal (polyhedral) domain (open, bounded, and connected set). Let $\mathscr{T}_h$ be a partition of Ω into polygonal elements. The elements K can be *nonconvex* or *non star-shaped*. We denote by h_K the diameter of $K \in \mathscr{T}_h$ and by $\mathbf{n}_K$ its outward normal. The partition $\mathscr{T}_h$ can be *nonmatching*, that is, so-called hanging nodes are allowed. We only suppose later on (cf. Assumption 3 below) the existence of a simplicial matching and shape-regular submesh $\mathscr{S}_h$. We say that σ is a mesh side if σ has positive $(d-1)$-dimensional measure and if there are distinct $K, L \in \mathscr{T}_h$ such that $\sigma = \partial K \cap \partial L$ or if there is $K \in \mathscr{T}_h$ such that $\sigma = \partial K \cap \partial \Omega$. Mesh sides are collected in the set $\mathscr{E}_h$. We denote by h_σ the diameter of $\sigma \in \mathscr{E}_h$, we fix a unit normal to σ denoted by $\mathbf{n}_\sigma$, and define the jump across σ as the difference following the direction of $\mathbf{n}_\sigma$. Besides the usual Sobolev spaces $H^1(\Omega)$ and $H^1_0(\Omega)$, we consider the so-called broken Sobolev space $H^1(\mathscr{T}_h)$ spanned by those functions whose restriction to each element $K \in \mathscr{T}_h$ belongs to $H^1(K)$ and the so-called broken gradient operator ∇_h acting elementwise on functions in $H^1(\mathscr{T}_h)$. Additionally, we need the space $\mathbf{H}(\mathrm{div}, \Omega)$ spanned by those functions in $[L^2(\Omega)]^d$ with square-integrable weak divergence. The notation $\mathbb{P}_k(\mathscr{T}_h)$ stands for the space of piecewise polynomials of total degree $\leq k$ on $\mathscr{T}_h$, whereas, for $\mathscr{T}_h$ simplicial and matching, $\mathbf{RTN}(\mathscr{T}_h) \subset \mathbf{H}(\mathrm{div}, \Omega)$ stands for the (lowest-order) Raviart–Thomas–Nédélec finite element space [3]. For all $\mathbf{v}_h \in \mathbf{RTN}(\mathscr{T}_h)$, $\mathbf{v}_h \cdot \mathbf{n}_\sigma$ is constant on all sides $\sigma \in \mathscr{E}_h$, the univalued side fluxes $\langle \mathbf{v}_h \cdot \mathbf{n}_\sigma, 1\rangle_\sigma$ representing the degrees of freedom.

Let $D \subset \Omega$ be a polygon or polyhedron. The Poincaré inequality states that

$$\|\varphi - \varphi_D\|_D^2 \leq C_{\mathrm{P},D} h_D^2 \|\nabla \varphi\|_D^2 \qquad \forall \varphi \in H^1(D), \tag{1}$$

where φ_D is the mean of φ over D given by $\varphi_D := (\varphi, 1)_D / |D|$. When D is convex, the constant $C_{\mathrm{P},D}$ can be evaluated as $1/\pi^2$. The constant $C_{\mathrm{P},D}$ can also be evaluated for nonconvex D, cf. [12, Lemma 10.2] or [5, §2]. Let now $K \subset \Omega$ be a simplex and let σ be one of its sides. The trace inequality states that

$$\|\varphi\|_\sigma^2 \leq C_{\mathrm{t},K,\sigma} (h_K^{-1} \|\varphi\|_K^2 + \|\varphi\|_K \|\nabla \varphi\|_K) \qquad \forall \varphi \in H^1(K). \tag{2}$$

It follows from [23, Lemma 3.12] that the constant $C_{\mathrm{t},K,\sigma}$ can be evaluated as $|\sigma| h_K / |K|$, see also [5, Theorem 4.1] for $d = 2$.

3 Laplace equation

We consider the second-order elliptic problem

$$-\Delta p = f \qquad \text{in } \Omega, \tag{3a}$$

$$p = 0 \qquad \text{on } \partial\Omega, \tag{3b}$$

with $f \in L^2(\Omega)$. The weak formulation consists in finding $p \in H_0^1(\Omega)$ such that

$$(\nabla p, \nabla\varphi) = (f, \varphi) \qquad \forall \varphi \in H_0^1(\Omega). \tag{4}$$

The scalar-valued function $p \in H_0^1(\Omega)$ is called the *potential* and the vector-valued function $\mathbf{t} := -\nabla p \in \mathbf{H}(\mathrm{div}, \Omega)$ the (diffusive) *flux*.

3.1 Abstract framework

The purpose of this section is to present a unified abstract framework for a posteriori error estimation in problem (3a)–(3b). In order to proceed generally, without the specification of the numerical scheme at hand, we merely suppose that we are given a function $p_h \in H^1(\mathcal{T}_h)$ (which will represent the discrete solution later on). We define the energy (semi-)norm as $|||v||| := \|\nabla_h v\|$ for all $v \in H^1(\mathcal{T}_h)$. The a posteriori estimate for the energy error $|||p - p_h|||$ is formulated in terms of a *potential reconstruction* s_h and a *flux reconstruction* $\mathbf{t}_h$. These reconstructions must comply with the following assumption.

Assumption 1 (Potential and flux reconstruction for (3a)–(3b)) *There holds* $s_h \in H_0^1(\Omega)$, $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$, *and*

$$(\nabla{\cdot}\mathbf{t}_h, 1)_K = (f, 1)_K \qquad \forall K \in \mathcal{T}_h. \tag{5}$$

Remark 1 (Assumption 1). Assumption 1 is concerned with basic physical *constraints* and *local conservation*. For the exact solution, $p \in H_0^1(\Omega)$ and $\mathbf{t} \in \mathbf{H}(\mathrm{div}, \Omega)$ (physical constraints); moreover, $\nabla{\cdot}\mathbf{t} = f$ (conservation). The potential and flux reconstructions mimic these continuous properties.

We can now state and prove our main result concerning the error upper bound, see [27, Theorem 4.2] and [30, Theorem 4.5].

Theorem 2 (A posteriori estimate for (3a)–(3b)). *Let p be the solution of (4) and let $p_h \in H^1(\mathcal{T}_h)$ be arbitrary. Let Assumption 1 be satisfied. Then,*

$$|||p - p_h||| \le \left\{ \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{NC},K}^2 + (\eta_{\mathrm{R},K} + \eta_{\mathrm{DF},K})^2 \right\}^{1/2},$$

where, for all $K \in \mathcal{T}_h$, the diffusive flux estimator*, the* nonconformity estimator*, and the* residual estimator *are respectively given by*

$$\eta_{\mathrm{DF},K} := \|\nabla p_h + \mathbf{t}_h\|_K, \tag{6a}$$

$$\eta_{\mathrm{NC},K} := \|\nabla(p_h - s_h)\|_K, \tag{6b}$$

$$\eta_{\mathrm{R},K} := C_{\mathrm{P},K}^{1/2} h_K \|f - \nabla{\cdot}\mathbf{t}_h\|_K. \tag{6c}$$

Proof. Following [17, Lemma 4.4], we obtain using $s_h \in H_0^1(\Omega)$,

$$|||p - p_h|||^2 \le |||p_h - s_h|||^2 + \left\{ \sup_{\varphi \in H_0^1(\Omega), |||\varphi|||=1} (\nabla_h(p - p_h), \nabla\varphi) \right\}^2.$$

The first term equals the Hilbertian sum of the nonconformity estimators, and we are thus left with bounding the second term. Using (4) and $\mathbf{t}_h \in \mathbf{H}(\mathrm{div}, \Omega)$, we obtain

$$\begin{aligned}(\nabla_h(p - p_h), \nabla\varphi) &= (f, \varphi) - (\nabla_h p_h, \nabla\varphi) = (f, \varphi) - (\nabla_h p_h + \mathbf{t}_h, \nabla\varphi) + (\mathbf{t}_h, \nabla\varphi) \\ &= (f - \nabla{\cdot}\mathbf{t}_h, \varphi) - (\nabla_h p_h + \mathbf{t}_h, \nabla\varphi).\end{aligned}$$

We now bound the two above terms separately. For all $K \in \mathcal{T}_h$, let φ_K be the mean value of φ over K. Then, using (5), the Poincaré inequality (1), and the Cauchy–Schwarz inequality, we infer

$$|(f - \nabla{\cdot}\mathbf{t}_h, \varphi)_K| = |(f - \nabla{\cdot}\mathbf{t}_h, \varphi - \varphi_K)_K| \le \eta_{\mathrm{R},K} |||\varphi|||_K.$$

Moreover, bounding $|(\nabla p_h + \mathbf{t}_h, \nabla\varphi)_K| \le \eta_{\mathrm{DF},K} |||\varphi|||_K$ is immediate using the Cauchy–Schwarz inequality. The conclusion is straightforward. □

We now address local efficiency and we still proceed generally, without any notion of a particular numerical scheme. We make two more assumptions.

Assumption 3 (Local efficiency) *We suppose that*

- *there exists a shape-regular matching simplicial submesh $\mathcal{S}_h$ of $\mathcal{T}_h$ such that, for each $K \in \mathcal{T}_h$, the number of subelements $L \subset K$, $L \in \mathcal{S}_h$, is uniformly bounded;*
- *for a fixed integer $k \ge 1$, the approximate solution p_h and the datum f are in $\mathbb{P}_k(\mathcal{T}_h)$, and the flux reconstruction $\mathbf{t}_h$ is in $[\mathbb{P}_k(\mathcal{S}_h)]^d$;*

Henceforth, we use $A \lesssim B$ when there exists a positive constant C, that can only depend on the space dimension d, the shape-regularity parameter of the mesh $\mathcal{S}_h$, and the polynomial degree k, such that $A \leq CB$. For all $K \in \mathcal{T}_h$, let $\mathfrak{T}_K$ denote all the elements in $\mathcal{T}_h$ having a nonempty intersection with K, $\mathfrak{E}_K$ all the sides in $\mathcal{E}_h$ having a nonempty intersection with K, and $\mathfrak{E}_K^{\mathrm{int}}$ the subset of $\mathfrak{E}_K$ collecting those sides lying in the interior of Ω. We introduce the *classical residual estimators* for problem (3a)–(3b) (cf. [24] for conforming methods and [1, 6] for nonconforming methods) given by

$$\eta_{\mathrm{res},K} := h_K \| f + \Delta p_h \|_{\mathfrak{T}_K} + h_K^{1/2} \| [\![\nabla_h p_h \cdot \mathbf{n}]\!] \|_{\mathfrak{E}_K^{\mathrm{int}}}, \tag{7a}$$

$$|p_h|_{\mathrm{J},K} := h_K^{-1/2} \| [\![p_h]\!] \|_{\mathfrak{E}_K}. \tag{7b}$$

Assumption 4 (Approximation property for (3a)–(3b)) *We assume that, for all $K \in \mathcal{T}_h$,*

$$\| \nabla (p_h - s_h) \|_K + \| \nabla p_h + \mathbf{t}_h \|_K \lesssim \eta_{\mathrm{res},K} + |p_h|_{\mathrm{J},K}. \tag{8}$$

We can now state and prove our main result concerning efficiency.

Theorem 5 (Efficiency of the estimate of Theorem 2). *Let p be the solution of (4) and let Assumptions 3 and 4 be satisfied. Then, for all $K \in \mathcal{T}_h$,*

$$\eta_{\mathrm{NC},K} + \eta_{\mathrm{R},K} + \eta_{\mathrm{DF},K} \lesssim |||p - p_h|||_{\mathfrak{T}_K} + |p_h|_{\mathrm{J},K}.$$

Proof. Our first step is to observe that $\eta_{\mathrm{NC},K} + \eta_{\mathrm{R},K} + \eta_{\mathrm{DF},K} \lesssim \eta_{\mathrm{res},K} + |p_h|_{\mathrm{J},K}$. This bound is immediate for $\eta_{\mathrm{NC},K}$ and $\eta_{\mathrm{DF},K}$ owing to Assumption 4, while for $\eta_{\mathrm{R},K}$, the triangle and inverse inequalities yield $\eta_{\mathrm{R},K} \lesssim h_K \| f + \Delta p_h \|_K + \| \nabla p_h + \mathbf{t}_h \|_K \lesssim \eta_{\mathrm{res},K} + |p_h|_{\mathrm{J},K}$, owing to Assumptions 3 and 4. Our second step is to observe that $\eta_{\mathrm{res},K} \lesssim |||p - p_h|||_{\mathfrak{T}_K}$, as can be derived using suitable bubble functions [24]. □

Remark 2 (Equivalence result). If p_h is in $H_0^1(\Omega)$, the jump seminorm $|p_h|_{\mathrm{J},K}$ vanishes. If the jumps of p_h have zero mean on each side, proceeding as in [1, Theorem 10] yields $|p_h|_{\mathrm{J},K} \lesssim |||p - p_h|||_{\mathfrak{T}_K}$. Finally, in the general case, an equivalence result is achieved by adding the jump seminorm $|p - p_h|_{\mathrm{J},K} = |p_h|_{\mathrm{J},K}$ to both the error measure and the nonconformity estimator.

3.2 Application to finite volumes

We apply here the framework of §3.1 to cell- and vertex-centered finite volume schemes, i.e., we specify s_h and $\mathbf{t}_h$, and we verify Assumptions 1, 3, and 4.

3.2.1 Cell-centered finite volumes

Definition 1 (Cell-centered FVs for (3a)–(3b)). A cell-centered FV scheme for discretizing (3a)–(3b), cf. [12], reads: find $\bar{p}_h \in \mathbb{P}_0(\mathcal{T}_h)$ such that

$$\sum_{\sigma \in \mathscr{E}_K} F_{K,\sigma} = (f,1)_K \qquad \forall K \in \mathscr{T}_h. \tag{9}$$

Here, $\mathscr{E}_K$ collects the sides of K and $F_{K,\sigma}$ is the diffusive flux through the side σ, which depends on $\bar{p}_h$. A simple example is the so-called "two-point" scheme. In what follows, we do not need the specific form of $F_{K,\sigma}$, but only the conservation property $F_{K,\sigma} = -F_{L,\sigma}$ for all interior sides σ shared by the elements K and L.

Let us first suppose that $\mathscr{T}_h$ is simplicial and matching. Following [13], let $\mathbf{t}_h \in \mathbf{RTN}(\mathscr{T}_h)$ be prescribed on all $K \in \mathscr{T}_h$ by the fluxes $F_{K,\sigma}$ as

$$(\mathbf{t}_h|_K \cdot \mathbf{n}_K)|_\sigma := F_{K,\sigma}/|\sigma|. \tag{10}$$

Since $\bar{p}_h$ is piecewise constant, the energy error $|||p - \bar{p}_h||| = \|\nabla p\|$ is not relevant. Instead, following [28, §3.2], we first postprocess $\bar{p}_h$ locally into $p_h \in \mathbb{P}_2(\mathscr{T}_h)$ such that, for all $K \in \mathscr{T}_h$,

$$-\nabla p_h|_K = \mathbf{t}_h|_K, \qquad (p_h, 1)_K/|K| = \bar{p}_h|_K. \tag{11}$$

The potential s_h is constructed by applying an averaging operator $\mathscr{I}_{\mathrm{av}} : \mathbb{P}_k(\mathscr{T}_h) \to \mathbb{P}_k(\mathscr{T}_h) \cap H_0^1(\Omega)$ to p_h. This operator sets the Lagrangian degrees of freedom inside Ω to the average of the values and sets 0 on $\partial\Omega$. Theorem 2 can now used to bound the error $|||p - p_h|||$ observing that (5) in Assumption 1 results from $(\nabla \cdot \mathbf{t}_h, 1)_K = \langle \mathbf{t}_h \cdot \mathbf{n}_K, 1\rangle_{\partial K} = \sum_{\sigma \in \mathscr{E}_K} F_{K,\sigma} = (f,1)_K$. Note that $\eta_{\mathrm{DF},K} = 0$ from (11), which is typical for cell-centered finite volumes. To apply Theorem 5, we verify Assumptions 3 and 4. Assumption 3 is straightforward with $\mathscr{S}_h = \mathscr{T}_h$, whereas Assumption 4 is trivial for $\mathbf{t}_h$ since $\|\nabla p_h + \mathbf{t}_h\|_K = 0$, while the bound on $\|\nabla(p_h - \mathscr{I}_{\mathrm{av}}(p_h))\|_K$ results from [1, 4, 16].

When $\mathscr{T}_h$ is not simplicial or is nonmatching, the submesh $\mathscr{S}_h$ needs to be introduced. We can then proceed as in [28, §5] and [10]. The averaging operator for potential reconstruction maps into $\mathbb{P}_k(\mathscr{S}_h) \cap H_0^1(\Omega)$, while the flux is reconstructed in $\mathbf{RTN}(\mathscr{S}_h)$ either by direct prescription of its degrees of freedom or by solving local Neumann problems.

3.2.2 Vertex-centered finite volumes

We suppose here that $\mathscr{T}_h$ is simplicial and matching. Let $\mathscr{D}_h$ be a dual mesh with dual volumes D associated with the vertices of $\mathscr{T}_h$. We refer to Fig. 1, left, for an illustration. We decompose $\mathscr{D}_h$ into $\mathscr{D}_h^{\mathrm{int}}$ and $\mathscr{D}_h^{\mathrm{ext}}$, with $\mathscr{D}_h^{\mathrm{int}}$ associated with interior vertices and $\mathscr{D}_h^{\mathrm{ext}}$ with boundary ones.

Definition 2 (Vertex-centered FVs for (3a)–(3b)). A vertex-centered FV scheme for discretizing (3a)–(3b), cf. [12], reads: find $p_h \in \mathbb{P}_1(\mathscr{T}_h) \cap H_0^1(\Omega)$ such that

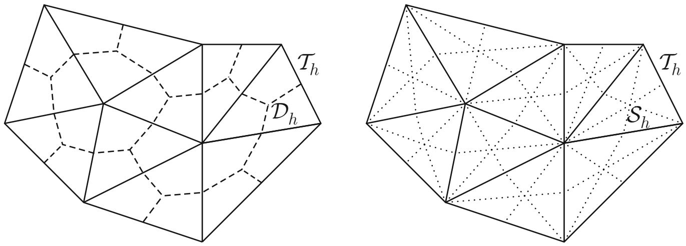

Fig. 1 Simplicial mesh $\mathscr{T}_h$ and the dual mesh $\mathscr{D}_h$ (left); simplicial submesh $\mathscr{S}_h$ (right)

$$-\langle \nabla p_h \cdot \mathbf{n}_D, 1\rangle_{\partial D} = (f, 1)_D \qquad \forall D \in \mathscr{D}_h^{\text{int}}. \tag{12}$$

To apply the framework of §3.1, we first note that, since $p_h \in H_0^1(\Omega)$, we can set $s_h = p_h$. Consequently, $\eta_{\text{NC},K} = 0$ in Theorem 2, which is typical for vertex-centered finite volumes. To construct the flux $\mathbf{t}_h$, we introduce a matching simplicial submesh $\mathscr{S}_h$, cf. Fig. 1, right. Such $\mathscr{S}_h$ is a refinement of both $\mathscr{T}_h$ and $\mathscr{D}_h$. The flux $\mathbf{t}_h$ is reconstructed in $\mathbf{RTN}(\mathscr{S}_h)$ such that, at all interior sides σ of $\mathscr{S}_h$ which lie on the boundary of some $D \in \mathscr{D}_h$, $\mathbf{t}_h \cdot \mathbf{n}_\sigma := -\nabla p_h \cdot \mathbf{n}_\sigma$. Owing to the Green theorem, $(\nabla \cdot \mathbf{t}_h, 1)_D = (f, 1)_D$ for all $D \in \mathscr{D}_h^{\text{int}}$. There are various ways of prescribing the remaining degrees of freedom of $\mathbf{t}_h$. We can merely prescribe them directly, but better computational results are obtained if a local Neumann or Neumann/Dirichlet problem is solved using mixed finite elements in each $D \in \mathscr{D}_h$ [30, §4.3]. Verifying Assumptions 1 and 3 is immediate, while Assumption 4 is proven as in [30, §5].

4 Convection-diffusion-reaction equation

We consider the convection-diffusion-reaction equation

$$-\nabla \cdot (\varepsilon \nabla p - \mathbf{w} p) + rp = f \quad \text{in } \Omega, \tag{13a}$$

$$p = 0 \quad \text{on } \partial\Omega, \tag{13b}$$

with $\varepsilon > 0$, $r \in L^\infty(\Omega)$, $\mathbf{w} \in [W^{1,\infty}(\Omega)]^d$, and $f \in L^2(\Omega)$. We assume that $\mathbf{w}$ is divergence-free with piecewise polynomial components and that r is piecewise constant taking nonnegative values. We introduce the bilinear form $\mathscr{B} := \mathscr{B}_{\text{S}} + \mathscr{B}_{\text{A}}$ on $H_0^1(\Omega) \times H_0^1(\Omega)$ such that

$$\mathscr{B}_{\text{S}}(p, \varphi) := \varepsilon(\nabla p, \nabla \varphi) + (rp, \varphi), \tag{14a}$$

$$\mathscr{B}_{\text{A}}(p, \varphi) := -(\mathbf{w}p, \nabla \varphi). \tag{14b}$$

The weak formulation consists in finding $p \in H_0^1(\Omega)$ such that

$$\mathscr{B}(p,\varphi) = (f,\varphi) \qquad \forall \varphi \in H_0^1(\Omega). \tag{15}$$

The vector-valued functions $\mathbf{t} := -\varepsilon \nabla p$ and $\mathbf{q} := \mathbf{w} p$ are in $\mathbf{H}(\mathrm{div},\Omega)$ and are, respectively, called the *diffusive* and *convective flux*.

4.1 Abstract framework

We present here a unified abstract framework for a posteriori error estimation in problem (13a)–(13b). Extending the above bilinear forms to $H^1(\mathscr{T}_h) \times H^1(\mathscr{T}_h)$ using broken gradients, we now define the energy (semi-)norm as

$$|||v||| := \mathscr{B}_{\mathrm{S}}(v,v)^{1/2} = \left(\|\varepsilon^{1/2}\nabla_h v\|^2 + \|r^{1/2}v\|^2\right)^{1/2} \qquad \forall v \in H^1(\mathscr{T}_h). \tag{16}$$

To achieve robustness of the a posteriori error estimates in the singularly perturbed regime resulting from dominant convection, we introduce, following Verfürth [26], the augmented (semi-)norm defined as

$$|||v|||_{\oplus} := |||v||| + \sup_{\varphi \in H_0^1(\Omega), |||\varphi|||=1} \mathscr{B}_{\mathrm{A}}(v,\varphi) \qquad \forall v \in H^1(\mathscr{T}_h). \tag{17}$$

The a posteriori error estimate for $|||p - p_h|||_{\oplus}$ is formulated in terms of a *potential reconstruction* s_h, a *diffusive flux reconstruction* $\mathbf{t}_h$, and a *convective flux reconstruction* $\mathbf{q}_h$. These reconstructions must comply with the following assumption.

Assumption 6 (Potential and flux reconstruction for (13a)–(13b)) *There holds* $s_h \in H_0^1(\Omega)$, $\mathbf{t}_h, \mathbf{q}_h \in \mathbf{H}(\mathrm{div},\Omega)$, *and*

$$(\nabla\cdot\mathbf{t}_h + \nabla\cdot\mathbf{q}_h + r p_h, 1)_K = (f,1)_K \qquad \forall K \in \mathscr{T}_h. \tag{18}$$

We can now state and prove our main result concerning the error upper bound. For simplicity, we assume that the mesh $\mathscr{T}_h$ is matching and simplicial so as to use the trace inequality (2). The general case can be treated by resorting to a matching simplicial submesh.

Theorem 7 (A posteriori estimate for (13a)–(13b)). *Let p be the solution of (15) and let $p_h \in H^1(\mathscr{T}_h)$ be arbitrary. Let Assumption 6 be satisfied. Assume that $\mathscr{T}_h$ is matching and simplicial. Then,*

$$|||p - p_h|||_{\oplus} \le \eta := 2\left\{\sum_{K\in\mathscr{T}_h} \eta_{\mathrm{NC},K}^2\right\}^{1/2} + \left\{\sum_{K\in\mathscr{T}_h} \widetilde{\eta}_{\mathrm{NC},K}^2\right\}^{1/2}$$
$$+ 3\left\{\sum_{K\in\mathscr{T}_h} (\eta_{\mathrm{R},K} + \eta_{\mathrm{CDF},K})^2\right\}^{1/2}.$$

For all $K \in \mathcal{T}_h$*, the* convective-diffusive flux estimator *is given by*

$$\eta_{\mathrm{CDF},K} := \min(\eta_{\mathrm{CDF},1,K}, \eta_{\mathrm{CDF},2,K}), \tag{19a}$$

$$\eta_{\mathrm{CDF},1,K} := \varepsilon^{-1/2}\|\mathbf{a}_h\|_K, \tag{19b}$$

$$\eta_{\mathrm{CDF},2,K} := m_K\|(I-\Pi_0)\nabla\cdot\mathbf{a}_h\|_K + \widetilde{m}_K^{1/2}\sum_{\sigma\in\mathcal{E}_K} C_{\mathrm{t},K,\sigma}^{1/2}\|\mathbf{a}_h\cdot\mathbf{n}_\sigma\|_\sigma, \tag{19c}$$

with $\mathbf{a}_h := \mathbf{t}_h + \mathbf{q}_h + \varepsilon\nabla_h p_h - \mathbf{w}s_h$ *and* Π_0 *the* L^2*-orthogonal projector onto* $\mathbb{P}_0(\mathcal{T}_h)$*, the* nonconformity estimators *by*

$$\eta_{\mathrm{NC},K} := |||p_h - s_h|||_K, \qquad \widetilde{\eta}_{\mathrm{NC},K} := \min(\widetilde{\eta}_{\mathrm{NC},1,K}, \widetilde{\eta}_{\mathrm{NC},2,K}), \tag{20a}$$

$$\widetilde{\eta}_{\mathrm{NC},1,K} := \varepsilon^{-1/2}\|\mathbf{b}_h\|_K, \tag{20b}$$

$$\widetilde{\eta}_{\mathrm{NC},2,K} := m_K\|(I-\Pi_0)\nabla\cdot\mathbf{b}_h\|_K + \widetilde{m}_K^{1/2}\sum_{\sigma\in\mathcal{E}_K} C_{\mathrm{t},K,\sigma}^{1/2}\|\mathbf{b}_h\cdot\mathbf{n}_\sigma\|_\sigma, \tag{20c}$$

with $\mathbf{b}_h := \mathbf{w}(p_h - s_h)$*, and the* residual estimator *by*

$$\eta_{\mathrm{R},K} := m_K\|f - \nabla\cdot\mathbf{t}_h - \nabla\cdot\mathbf{q}_h - rp_h\|_K. \tag{21}$$

Here $m_K := \min(C_{\mathrm{P},K}^{1/2}\varepsilon^{-1/2}h_K, r_K^{-1/2})$ *and* $\widetilde{m}_K := 2(1 + C_{\mathrm{P},K}^{1/2})\varepsilon^{-1/2}m_K$*.*

Proof. Following [27, Lemma 7.1] and [8, Lemma 3.1], we infer

$$|||p - p_h||| \le |||p_h - s_h||| + \sup_{\varphi\in H_0^1(\Omega),|||\varphi|||=1}\{\mathcal{B}(p - p_h, \varphi) + \mathcal{B}_{\mathrm{A}}(p_h - s_h, \varphi)\},$$

and proceeding as in [9, Lemma 4.2] yields

$$\begin{aligned} |||p - p_h|||_\oplus \le 2|||p_h - s_h||| &+ \sup_{\varphi\in H_0^1(\Omega),|||\varphi|||=1}\mathcal{B}_{\mathrm{A}}(p_h - s_h, \varphi) \\ &+ 3\sup_{\varphi\in H_0^1(\Omega),|||\varphi|||=1}\{\mathcal{B}(p - p_h, \varphi) + \mathcal{B}_{\mathrm{A}}(p_h - s_h, \varphi)\}. \end{aligned}$$

For the second term on the right-hand side, we obtain

$$\mathcal{B}_{\mathrm{A}}(p_h - s_h, \varphi) = -(\mathbf{b}_h, \nabla\varphi) \le \sum_{K\in\mathcal{T}_h}\widetilde{\eta}_{\mathrm{NC},K}|||\varphi|||_K.$$

Indeed, for all $K \in \mathcal{T}_h$, the Cauchy–Schwarz inequality on the one hand yields $-(\mathbf{b}_h, \nabla\varphi)_K \le \varepsilon^{-1/2}\|\mathbf{b}_h\|_K|||\varphi|||_K = \widetilde{\eta}_{\mathrm{NC},1,K}|||\varphi|||_K$, while integrating by parts on K leads to

$$-(\mathbf{b}_h, \nabla\varphi)_K = ((I-\Pi_0)\nabla\cdot\mathbf{b}_h, \varphi-\varphi_K)_K - \sum_{\sigma\in\mathscr{E}_K} (\mathbf{b}_h\cdot\mathbf{n}_\sigma, \varphi-\varphi_K)_\sigma \leq \widetilde{\eta}_{\mathrm{NC},2,K}|||\varphi|||_K,$$

owing to the Poincaré inequality (1) and the trace inequality (2). For the third term on the right-hand side, we observe that

$$\begin{aligned}\mathscr{B}(p-p_h,\varphi) + \mathscr{B}_{\mathrm{A}}(p_h - s_h, \varphi) &= (f - \nabla\cdot\mathbf{t}_h - \nabla\cdot\mathbf{q}_h - rp_h, \varphi) - (\mathbf{a}_h, \nabla\varphi) \\ &\leq \sum_{K\in\mathscr{T}_h} (\eta_{\mathrm{R},K} + \eta_{\mathrm{CDF},K})|||\varphi|||_K,\end{aligned}$$

using Assumption 6 for the residual term and proceeding for $\mathbf{a}_h$ as for $\mathbf{b}_h$. □

We now address the efficiency of the estimate of Theorem 7. In what follows, $\lesssim$ can include factors depending on the maximal ratio m_K/m_L for K, L having a nonempty intersection. We introduce the *classical residual estimators* for problem (13a)–(13b) given by

$$\eta_{\mathrm{res},K} := m_K\|f + \nabla\cdot(\varepsilon\nabla p_h - \mathbf{w}p_h) - rp_h\|_{\mathfrak{T}_K} + m_K^{1/2}\varepsilon^{-1/4}\|[\![\varepsilon\nabla_h p_h]\!]\cdot\mathbf{n}\|_{\mathfrak{E}_K^{\mathrm{int}}}, \tag{22a}$$

$$|p_h|_{\mathrm{J},K} := (\varepsilon^{1/2}h_K^{-1/2} + m_K^{1/2}\varepsilon^{-1/4}\|\mathbf{w}\|_{[L^\infty(K)]^d} + r_K^{1/2}h_K^{1/2})\|[\![p_h]\!]\|_{\mathfrak{E}_K}. \tag{22b}$$

We also set $|v|_{\mathrm{J}} := \left\{\sum_{K\in\mathscr{T}_h}|v|_{\mathrm{J},K}^2\right\}^{1/2}$ for all $v \in H^1(\mathscr{T}_h)$.

Assumption 8 (Approximation property for (13a)–(13b)) *We assume that, for all $K \in \mathscr{T}_h$, with $\mathbf{c}_h = \mathbf{a}_h$ or $\mathbf{b}_h$,*

$$m_K\|(I-\Pi_0)\nabla\cdot\mathbf{c}_h\|_K + m_K^{1/2}\varepsilon^{-1/4}\sum_{\sigma\in\mathscr{E}_K}\|\mathbf{c}_h\cdot\mathbf{n}_\sigma\|_\sigma \lesssim \eta_{\mathrm{res},K} + |p_h|_{\mathrm{J},K}.$$

Proceeding as in [9, Theorems 3.2 and 3.4] leads to the following lower bound, which is global in space owing to the use of a dual norm.

Theorem 9 (Efficiency of the estimate of Theorem 7). *Let p be the solution of (15) and let Assumption 8, and the second item of Assumption 3, be satisfied. Then,*

$$\eta \lesssim |||p - p_h|||_\oplus + |p - p_h|_{\mathrm{J}}. \tag{23}$$

Remark 3 (Fully robust equivalence result). Adding the jump seminorm $|\cdot|_{\mathrm{J}}$ to the error measure, a fully robust equivalence result is finally achieved in the form

$$|||p - p_h|||_\oplus + |p - p_h|_{\mathrm{J}} \leq \eta + |p_h|_{\mathrm{J}} \lesssim |||p - p_h|||_\oplus + |p - p_h|_{\mathrm{J}}. \tag{24}$$

4.2 Application to finite volumes

We apply here the framework of §4.1 to cell- and vertex-centered finite volume schemes, i.e., we specify s_h, $\mathbf{t}_h$, and $\mathbf{q}_h$, and we verify Assumption 6, and, at least in some cases, Assumption 8.

4.2.1 Cell-centered finite volumes

Definition 3 (Cell-centered FVs for (13a)–(13b)). A cell-centered FV scheme for discretizing (13a)–(13b), cf. [12], reads: find $\bar{p}_h \in \mathbb{P}_0(\mathscr{T}_h)$ such that

$$\sum_{\sigma \in \mathscr{E}_K} F_{K,\sigma} + \sum_{\sigma \in \mathscr{E}_K} W_{K,\sigma} + r_K \bar{p}_h|_K = (f,1)_K \qquad \forall K \in \mathscr{T}_h. \tag{25}$$

In addition to the diffusive fluxes $F_{K,\sigma}$, $W_{K,\sigma}$ are the convective fluxes, also depending on $\bar{p}_h$. We do not need the precise form of the fluxes, but only $F_{K,\sigma} = -F_{L,\sigma}$ and $W_{K,\sigma} = -W_{L,\sigma}$ for all interior sides σ shared by the elements K and L.

Following the ideas exposed in §3.2.1, we first define $\mathbf{t}_h, \mathbf{q}_h \in \mathbf{RTN}(\mathscr{T}_h)$ by

$$(\mathbf{t}_h|_K \cdot \mathbf{n}_K)|_\sigma := F_{K,\sigma}/|\sigma|, \qquad (\mathbf{q}_h|_K \cdot \mathbf{n}_K)|_\sigma := W_{K,\sigma}/|\sigma|. \tag{26}$$

Define p_h similarly to (11). It is immediate to see using the Green theorem that (26) and (25) yield (18). A reasonable condition on $W_{K,\sigma}$ in the context of upwind or centered convective fluxes is that

$$\| W_{K,\sigma}/|\sigma| - \mathbf{w} \cdot \mathbf{n}_K p_h|_K \|_\sigma \lesssim \|\mathbf{w}\|_{[L^\infty(K)]^d} \| [\![\bar{p}_h]\!] \|_\sigma. \tag{27}$$

Then, Assumption 8 holds, up to the oscillation terms $m_K \|(I - \Pi_0)\nabla\cdot(\mathbf{w} p_h)\|_K$, when additionally including $|\bar{p}_h|_{\mathrm{J},K}$ on the right-hand side, and the efficiency result (23) holds when additionally including $|p - \bar{p}_h|_{\mathrm{J}}$ on the right-hand side.

4.2.2 Vertex-centered finite volumes

Definition 4 (Vertex-centered FVs for (13a)–(13b)). A vertex-centered FV scheme for discretizing (13a)–(13b), cf. [12], reads: find $p_h \in \mathbb{P}_1(\mathscr{T}_h) \cap H_0^1(\Omega)$ such that

$$-\langle \varepsilon \nabla p_h \cdot \mathbf{n}_D, 1 \rangle_{\partial D} + \langle \mathbf{w} \cdot \mathbf{n}_D p_h, 1 \rangle_{\partial D} + (r p_h, 1)_D = (f,1)_D \qquad \forall D \in \mathscr{D}_h^{\mathrm{int}}. \tag{28}$$

Note that we only consider a centered convective flux.

As in §3.2.2, we set $s_h = p_h$ in Assumption 6. Consequently, $\eta_{\mathrm{NC},K} = \widetilde{\eta}_{\mathrm{NC},K} = 0$ in Theorem 7. For the convective flux reconstruction, we simply set $\mathbf{q}_h := \mathbf{w} p_h$. For the diffusive flux reconstruction, we introduce the mesh $\mathscr{S}_h$ (cf. Fig. 1, right) and we define $\mathbf{t}_h \in \mathbf{RTN}(\mathscr{S}_h)$ such that $\mathbf{t}_h{\cdot}\mathbf{n}_\sigma := -\varepsilon \nabla p_h{\cdot}\mathbf{n}_\sigma$ at all interior sides σ of $\mathscr{S}_h$ which lie on the boundary of some $D \in \mathscr{D}_h$. As in §3.2.2, local problems can be solved to fulfill Assumption 6. Assumption 8 can be verified as in §3.2.2 for the diffusive part, while the convective part is trivial owing to the choice of $\mathbf{q}_h$.

5 Heat equation

We consider the heat equation

$$\partial_t p - \Delta p = f \quad \text{in } \Omega \times (0,T), \tag{29a}$$

$$p = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{29b}$$

$$p(\cdot,0) = p_0 \quad \text{in } \Omega, \tag{29c}$$

with $f \in L^2(\Omega \times (0,T))$, initial condition $p_0 \in L^2(\Omega)$, and final time $T > 0$. The exact solution is in the space $Y := \{y \in X; \partial_t y \in X'\}$, with $X := L^2(0,T;H_0^1(\Omega))$ and $X' = L^2(0,T;H^{-1}(\Omega))$, satisfies (29c) in $L^2(\Omega)$, and is such that, for a.e. $t \in (0,T)$,

$$\langle \partial_t p, \varphi\rangle(t) + (\nabla p, \nabla\varphi)(t) = (f,\varphi)(t) \qquad \forall \varphi \in H_0^1(\Omega). \tag{30}$$

The space-time energy norm is given by $\|y\|_X := \left\{\int_0^T \|\nabla y\|^2(t)\,\mathrm{d}t\right\}^{1/2}$ for all $y \in X$. Following Verfürth [25], we augment the energy norm by a dual norm of the time derivative as $\|y\|_Y := \|y\|_X + \|\partial_t y\|_{X'}$ with $\|\partial_t y\|_{X'} := \left\{\int_0^T \|\partial_t y\|_{H^{-1}}^2(t)\,\mathrm{d}t\right\}^{1/2}$.

5.1 *Abstract framework*

We consider an increasing sequence of discrete times $\{t^n\}_{0 \le n \le N}$ such that $t^0 = 0$ and $t^N = T$ and introduce the time intervals $I_n := (t^{n-1}, t^n]$ and the time steps $\tau^n := t^n - t^{n-1}$ for all $1 \le n \le N$. The meshes are allowed to vary in time; we denote by $\mathscr{T}_h^n$ the mesh used to march in time from t^{n-1} to t^n, for all $1 \le n \le N$, and by $\mathscr{T}_h^0$ the initial mesh. We suppose that the approximate solution on t^n, denoted by $p_{h\tau}^n$, is in $H^1(\mathscr{T}_h^n)$, and we let $p_{h\tau}$ be the space-time approximate solution, given by $p_{h\tau}^n$ at each discrete time t^n and piecewise affine and continuous in time. We denote the space of such functions by $P_\tau^1(H^1(\mathscr{T}_h))$. We also denote by $P_\tau^1(H_0^1(\Omega))$ the space of functions that are piecewise affine and continuous in time and $H_0^1(\Omega)$ in space and by $P_\tau^0(\mathbf{H}(\mathrm{div},\Omega))$ the space of functions that are piecewise constant in time and $\mathbf{H}(\mathrm{div}, \Omega)$ in space. For all $1 \le n \le N$, we set $\widetilde{f}^n := \frac{1}{\tau^n} \int_{I_n} f(\cdot, t)\, \mathrm{d}t$, and, for $\varphi_{h\tau} \in P^1_\tau(H^1(\mathcal{T}_h))$, $\partial_t p^n_{h\tau} := \frac{1}{\tau^n}(\varphi^n_{h\tau} - \varphi^{n-1}_{h\tau})$.

We aim at measuring the error $(p - p_{h\tau})$ in the $\|\cdot\|_Y$-norm using the broken gradient operator in the energy norm. The a posteriori error estimate is formulated in terms of a *space-time potential reconstruction* $s_{h\tau}$ and a *space-time flux reconstruction* $\mathbf{t}_{h\tau}$. These reconstructions must comply with the following assumption.

Assumption 10 (Potential and flux reconstruction for (29a)–(29c)) *There holds* $s_{h\tau} \in P^1_\tau(H^1_0(\Omega))$, $\mathbf{t}_{h\tau} \in P^0_\tau(\mathbf{H}(\mathrm{div}, \Omega))$, *and, for all* $1 \le n \le N$ *and for all* $K \in \mathcal{T}_h^n$,

$$(\partial_t s^n_{h\tau}, 1)_K = (\partial_t p^n_{h\tau}, 1)_K, \tag{31a}$$

$$(\widetilde{f}^n - \partial_t p^n_{h\tau} - \nabla{\cdot}\mathbf{t}^n_{h\tau}, 1)_K = 0. \tag{31b}$$

We can now state our main result concerning the error upper bound, see [11, Theorem 3.6] and also [11, Theorem 3.2] for a slightly sharper bound.

Theorem 11 (A posteriori estimate for (29a)–(29c)). *Let p be the solution of (30) and let $p_{h\tau} \in P^1_\tau(H^1(\mathcal{T}_h))$ be arbitrary. Let Assumption 10 be satisfied. Then,*

$$\|p - p_{h\tau}\|_Y \le \left\{\sum_{n=1}^N (\eta^n_{\mathrm{sp}})^2\right\}^{1/2} + \left\{\sum_{n=1}^N (\eta^n_{\mathrm{tm}})^2\right\}^{1/2} + \eta_{\mathrm{IC}} + 3\|f - \widetilde{f}\|_{X'}, \tag{32}$$

with, for all $1 \le n \le N$, *the* space *and* time error estimators *given by*

$$(\eta^n_{\mathrm{sp}})^2 := \sum_{K \in \mathcal{T}_h^n} 3 \left\{ \tau^n (9(\eta^n_{\mathrm{R},K} + \eta^n_{\mathrm{DF},K})^2 + (\eta^n_{\mathrm{NC},2,K})^2) + \int_{I_n} (\eta^n_{\mathrm{NC},1,K})^2(t)\, \mathrm{d}t \right\}, \tag{33a}$$

$$(\eta^n_{\mathrm{tm}})^2 := \sum_{K \in \mathcal{T}_h^n} 3\tau^n \|\nabla(s^n_{h\tau} - s^{n-1}_{h\tau})\|^2_K. \tag{33b}$$

For all $K \in \mathcal{T}_h^n$, *the* residual estimator, *the* diffusive flux estimator, *and the* nonconformity estimators *are given by*

$$\eta^n_{\mathrm{R},K} := C^{1/2}_{\mathrm{P},K} h_K \|\widetilde{f}^n - \partial_t s^n_{h\tau} - \nabla{\cdot}\mathbf{t}^n_{h\tau}\|_K, \tag{34a}$$

$$\eta^n_{\mathrm{DF},K} := \|\nabla s^n_{h\tau} + \mathbf{t}^n_{h\tau}\|_K, \tag{34b}$$

$$\eta^n_{\mathrm{NC},1,K}(t) := \|\nabla^n_h (s_{h\tau} - p_{h\tau})(t)\|_K, \quad \forall t \in I_n, \tag{34c}$$

$$\eta^n_{\mathrm{NC},2,K} := C^{1/2}_{\mathrm{P},K} h_K \|\partial_t (s_{h\tau} - p_{h\tau})^n\|_K. \tag{34d}$$

Finally, the initial condition estimator *is given by* $\eta_{\mathrm{IC}} := 2^{1/2}\|s^0_{h\tau} - p_0\|$.

We next turn to the efficiency of the estimate of Theorem 11. We introduce the *classical residual estimators* for problem (29a)–(29c) given by

$$\eta^n_{\mathrm{res},K} := h_K \|\widetilde{f}^n - \partial_t p^n_{h\tau} + \Delta p^n_{h\tau}\|_{\mathfrak{T}_K} + h_K^{1/2} \|[\![\nabla^n_h p^n_{h\tau} \cdot \mathbf{n}]\!]\|_{\mathfrak{E}_K^{\mathrm{int}}}, \tag{35a}$$

$$|p^n_{h\tau}|_{\mathrm{J},K} := h_K^{-1/2} \|[\![p^n_{h\tau}]\!]\|_{\mathfrak{E}_K}. \tag{35b}$$

Assumption 12 (Approximation property for (29a)–(29c)) *We assume that for all $1 \le n \le N$ and for all $K \in \mathcal{T}_h^n$,*

$$\|\nabla^n_h (p^n_{h\tau} - s^n_{h\tau})\|_K + \|\nabla^n_h p^n_{h\tau} + \mathbf{t}^n_{h\tau}\|_K \lesssim \eta^n_{\mathrm{res},K} + |p^n_{h\tau}|_{\mathrm{J},K}. \tag{36}$$

We can now state our efficiency result, see [11, Theorem 3.9]. As in [25], the lower bound is local in time, but global in space.

Theorem 13 (Efficiency of the estimate of Theorem 11). *Let Assumption 12 hold, let Assumption 3 hold at all discrete times, let both the refinement and coarsening in time be not too abrupt, and let, for all $1 \le n \le N$, $(h^n)^2 \lesssim \tau^n$. Then, for all $1 \le n \le N$,*

$$\eta^n_{\mathrm{sp}} + \eta^n_{\mathrm{tm}} \lesssim \|p - p_{h\tau}\|_{Y(I_n)} + \mathscr{J}^n(p_{h\tau}) + \|f - \widetilde{f}\|_{X'(I_n)}, \tag{37}$$

where $\mathscr{J}^n(p_{h\tau}) := \left\{\tau^n \sum_{K \in \mathcal{T}_h^{n-1}} |p^{n-1}_{h\tau}|^2_{\mathrm{J},K} + \tau^n \sum_{K \in \mathcal{T}_h^n} |p^n_{h\tau}|^2_{\mathrm{J},K}\right\}^{1/2}$.

Remark 4 (Equivalence result). We refer to [11, Remark 3.10] for bounding the jumps $\mathscr{J}^n(p_{h\tau})$, see also Remark 2.

5.2 *Application to finite volumes*

We apply here the framework of §5.1 to cell- and vertex-centered finite volume schemes, i.e., we specify $s_{h\tau}$ and $\mathbf{t}_{h\tau}$, and we verify Assumptions 10 and 12. For simplicity, we only discuss matching simplicial meshes.

5.2.1 Cell-centered finite volumes

Definition 5 (Cell-centered FVs for (29a)–(29c)). A cell-centered FV scheme for (29a)–(29c), cf. [12], reads: for all $1 \le n \le N$, find $\bar{p}^n_{h\tau} \in \mathbb{P}_0(\mathcal{T}_h^n)$ s. t.

$$\frac{1}{\tau^n}(\bar{p}^n_{h\tau} - p^{n-1}_{h\tau}, 1)_K + \sum_{\sigma \in \mathscr{E}_K} F^n_{K,\sigma} = (\widetilde{f}^n, 1)_K \qquad \forall K \in \mathcal{T}_h^n. \tag{38}$$

As in §3.2.1, the fluxes $\mathbf{t}_{h\tau}^n$ are constructed from the side fluxes $F_{K,\sigma}^n$ by an equivalent of (10). An elementwise postprocessing as (11) is applied to obtain $p_{h\tau}^n$ from $\bar{p}_{h\tau}^n$. The potential is reconstructed at each discrete time from a modification of the averaging operator of §3.1 where local bubble functions are used to satisfy (31a) (cf. [11]). Then, owing to the construction of $\mathbf{t}_{h\tau}^n$, (31b) is also satisfied, whence Assumption 10 follows. Finally, we set $\mathcal{S}_h^n = \mathcal{T}_h^n$; Assumption 12 is trivial for $\mathbf{t}_{h\tau}$ since $\|\nabla_h^n p_{h\tau}^n + \mathbf{t}_{h\tau}^n\|_K = 0$ and is proven for $s_{h\tau}^n$ in [11].

5.2.2 Vertex-centered finite volumes

Definition 6 (Vertex-centered FVs for (29a)–(29c)). A vertex-centered FV scheme for (29a)–(29c), cf. [12], reads: for all $1 \leq n \leq N$, find $p_{h\tau}^n \in \mathbb{P}_1(\mathcal{T}_h^n) \cap H_0^1(\Omega)$ s. t.

$$(\partial_t p_{h\tau}^n, 1)_D - \langle \nabla p_{h\tau}^n \cdot \mathbf{n}_D, 1\rangle_{\partial D} = (\widetilde{f}^n, 1)_D \qquad \forall D \in \mathcal{D}_h^{\mathrm{int},n}. \tag{39}$$

As in §3.2.2, $p_{h\tau}^n \in H_0^1(\Omega)$ for all $1 \leq n \leq N$, so that we set $s_{h\tau}^n = p_{h\tau}^n$. Consequently, $\eta_{\mathrm{NC},1,K}^n = \eta_{\mathrm{NC},2,K}^n = 0$ in Theorem 11. The fluxes $\mathbf{t}_{h\tau}$ are constructed as in §3.2.2, using the simplicial submeshes $\mathcal{S}_h^n$. Assumptions 10 and 12 are then verified by proceeding as in §3.2.2.

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The paper is in final form and has not been or is not being submitted elsewhere.