# An A Posteriori Error Estimator for a Finite Volume Discretization of the Two-phase Flow

**Daniele A. Di Pietro, Martin Vohralík, and Carole Widmer**

**Abstract** We derive a posteriori error estimates for a multi-point finite volume discretization of the two-phase Darcy problem. The proposed estimators yield a fully computable upper bound for the selected error measure. The estimate also allows to distinguish, estimate separately, and compare the linearization and algebraic errors and the time and space discretization errors. This enables, in particular, to design a discretization algorithm so that all the sources of error are properly balanced. Namely, the linear and nonlinear solvers can be stopped as soon as the algebraic and linearization errors drop to the level at which they do not affect to the overall error. This can lead to significant computational savings, since performing an excessive number of unnecessary iterations can be avoided. Similarly, the errors in space and in time can be equilibrated by time step and local mesh adaptivity.

**Keywords** Finite volumes, a posteriori error estimates, darcy model, fully computable upperbound, twophase flow.
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## 1 The two-phase flow model

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, denote a bounded connected polygonal domain and let $t_{\mathrm{F}} > 0$. Let w denote the wetting phase (e.g., water) and o the non-wetting phase (e.g., oil), and let there be given sources $f_{\mathrm{o}},\ f_{\mathrm{w}} \in L^2((0,t_{\mathrm{F}}); L^2(\Omega))$ and a (constant) porosity

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$\phi \in (0, 1]$. We consider the two-phase flow (see, e.g., [3]): Find $\mathbf{U} := \{P, S_\mathrm{o}, S_\mathrm{w}\}$, with $P$ the pressure and $S_p$, $p \in \{\mathrm{o}, \mathrm{w}\}$, the saturations, such that

$$\begin{aligned} \partial_t(\phi S_\mathrm{o}) + \nabla\cdot(\nu_\mathrm{o}(P,S_\mathrm{o})\mathbf{u}_\mathrm{o}(P,S_\mathrm{o})) &= f_\mathrm{o} \qquad \text{in } \Omega\times(0,t_\mathrm{F}),\\ \partial_t(\phi S_\mathrm{w}) + \nabla\cdot(\nu_\mathrm{w}(P,S_\mathrm{w})\mathbf{u}_\mathrm{w}(P,S_\mathrm{w})) &= f_\mathrm{w} \qquad \text{in } \Omega\times(0,t_\mathrm{F}),\\ S_\mathrm{o}+S_\mathrm{w} &= 1 \qquad \text{in } \Omega\times(0,t_\mathrm{F}). \end{aligned} \tag{1}$$

For $p \in \{\mathrm{o}, \mathrm{w}\}$, $\nu_p$ denotes here the mobility of the phase $p$ defined as the ratio of the relative permeability to the viscosity. In (1), $\mathbf{u}_\mathrm{o}$ and $\mathbf{u}_\mathrm{w}$ are such that

$$\mathbf{u}_p(P,S_p) := -K\nabla\left(P + P_{\mathrm{c},p}(S_p)\right), \quad \text{for } p\in\{\mathrm{o},\mathrm{w}\}, \text{ in } \Omega\times(0,t_\mathrm{F}), \tag{2}$$

where $P_{\mathrm{c},p}(S_p)$ is the capillary pressure and $K$ denotes a piecewise constant, uniformly elliptic tensor-valued field corresponding to the absolute permeability. To find some example of the physics laws (capillarity pressure, phase mobility) or of the absolute permeability see [7].

Problem (1) is complemented by the initial conditions:

$$S_\mathrm{o}(\cdot,0) = S_\mathrm{o}^0 \text{ and } P(\cdot,0) = P^0, \quad \text{in } \Omega, \tag{3}$$

as well as by no-flow boundary conditions:

$$\mathbf{u}_p(P,S_p)\cdot\mathbf{n}_\Omega = 0, \qquad \text{in } \partial\Omega\times(0,t_\mathrm{F}). \tag{4}$$

The purpose of this paper is to propose fully computable a posteriori error estimates for the discretization of (1)–(4) by cell-centered finite volume methods in space and the backward Euler scheme in time. In particular, we consider the multi-point finite volume method proposed in [1]. Using a dual error norm is motivated by, e.g., [8]. Developing the ideas of [4–6, 9], we in particular separate the estimate into contributions representing the *space discretization error*, *time discretization error*, *linearization error*, and *algebraic error*. Then, at each time step, the linearization algorithm and the iterative algebraic solver can be stopped as soon as the corresponding errors no longer affect the total error, and space and the time errors can be equilibrated.

## 2 Discretization by the finite volume method

### *2.1 Notations*

Let $\mathscr{T} = \{T\}$ denotes a partition of $\Omega$ into simplices or rectangular parallelepipeds (the extension to general polygonal meshes is possible via the introduction of simplicial submeshes). For rectangular parallelepipeds, we further assume that $K$

is diagonal to perform $H(\mathrm{div};\Omega)$-conforming reconstructions. For every element $T \in \mathscr{T}$, we denote by $|T|$ its measure and by $h_T$ its diameter. Let $\mathscr{F} = \{\sigma\}$ be the set of faces of the mesh and, for all $T \in \mathscr{T}$, set $\mathscr{F}_T := \{\sigma \in \mathscr{F} \mid \sigma \subset \partial T\}$. The time discretization is defined by a strictly increasing sequence of discrete times $\{t^n\}_{0\le n\le N}$ such that $t^0 = 0$ and $t^N = t_{\mathrm{F}}$. For $1 \le n \le N$, we define the time interval $I_n := (t^{n-1}, t^n]$ and the time step $\tau^n := t^n - t^{n-1}$.

## 2.2 The finite volume scheme

The discrete problem reads: For all $1 \le n \le N$, all $T \in \mathscr{T}$, and all $p \in \{\mathrm{o},\mathrm{w}\}$, find $\mathbf{U}_T^n := \{P_T^n, S_{\mathrm{o},T}^n, S_{\mathrm{w},T}^n\}$ such that

$$\phi\frac{|T|}{\tau^n}\left(S_{p,T}^n - S_{p,T}^{n-1}\right) + \sum_{\sigma\in\mathscr{F}_T} \nu_p(P_{T_p^\star(\sigma)}^{n-1}, S_{p,T_p^\star(\sigma)}^{n-1})F_{p,T,\sigma}^n - f_{p,T}^n = 0, \tag{5}$$

where $f_{p,T}^n = (f_p^n, 1)_T$ and $f_p^n = \frac{1}{\tau^n}\int_{t^{n-1}}^{t^n} f_p(t)\,\mathrm{d}t$. We set $P_T^0 := (P^0, 1)_T\,/|T|$, $S_{\mathrm{o},T}^0 := (S_\mathrm{o}^0, 1)_T/|T|$, and impose $S_{\mathrm{o},T}^n + S_{\mathrm{w},T}^n = 1$ for all $0 \le n \le N$. Furthermore, $F_{p,T,\sigma}^n = F_{p,T,\sigma}(\{\mathbf{U}_{T'}^n\}_{\mathscr{S}_\sigma})$ is a multi-point approximation of the flux of the phase $p$ leaving $T \in \mathscr{T}$ through the face $\sigma \in \mathscr{F}_T$ that depends on the unknowns associated to the elements of the face stencil $\mathscr{S}_\sigma \subset \mathscr{T}$. The numerical flux is assumed to be conservative, i.e., for all internal faces $\sigma \subset \partial T_1 \cap \partial T_2$, there holds $F_{p,T_1,\sigma}^n = -F_{p,T_2,\sigma}^n$. The upwind cell $T_p^\star(\sigma)$ is equal to $T_1$ if $F_{p,T_1,\sigma}^n \ge 0$, to $T_2$ otherwise. For boundary faces $\sigma \subset \partial T \cap \partial\Omega$, $F_{p,T,\sigma}^n = 0$ to honor the no-flow boundary condition (4), and we can leave $T_p^\star(\sigma)$ undefined.

For all $0 \le n \le N$ and $T \in \mathscr{T}$, the unknown $S_{\mathrm{w},T}^n$ is eliminated using the local volume conservation equation $S_{\mathrm{o},T}^n + S_{\mathrm{w},T}^n = 1$. We introduce the reduced set of unknowns $\overline{\mathbf{U}}^n := \{\mathbf{P}^n, \mathbf{S}_\mathrm{o}^n\}$, where $\mathbf{P}^n = \{P_T^n\}_{T\in\mathscr{T}}$ and $\mathbf{S}_\mathrm{o}^n = \{S_{\mathrm{o},T}^n\}_{T\in\mathscr{T}}$. With a little abuse of notation, for a function $\xi(S_\mathrm{w})$, we write $\xi(S_\mathrm{o})$ to mean $\xi(1 - S_\mathrm{o})$. As a consequence, $\nu_\mathrm{w}(S_\mathrm{o})$ and $P_{\mathrm{c,w}}(S_\mathrm{o})$ are equal to $\nu_\mathrm{w}(1 - S_\mathrm{o})$, $P_{\mathrm{c,w}}(1 - S_\mathrm{o})$ and $\mathbf{u}_\mathrm{w}(P, 1 - S_\mathrm{o})$ respectively. Equation (5) becomes, for all $1 \le n \le N$, all $T \in \mathscr{T}$, and all $p \in \{\mathrm{o},\mathrm{w}\}$

$$\mathbf{D}_{p,T}^n(\overline{\mathbf{U}}^n) = 0, \text{ with,} \tag{6}$$

$$\mathbf{D}_{p,T}^n(\overline{\mathbf{U}}^n) := \phi\frac{|T|}{\tau^n}(-1)^j(S_{\mathrm{o},T}^n - S_{\mathrm{o},T}^{n-1}) + \sum_{\sigma\in\mathscr{F}_T} \nu_p(P_{T_p^\star(\sigma)}^{n-1}, S_{\mathrm{o},T_p^\star(\sigma)}^{n-1})F_{p,T,\sigma}^n - f_{p,T}^n, \tag{7}$$

where $j = 1$ if $p = \mathrm{w}$ and 0 otherwise.

### 2.3 Linearization

Problem (6) is a system of nonlinear algebraic equations that can be solved using the Newton algorithm. For a fixed $1 \le n \le N$, let $\overline{\mathbf{U}}^{n,0}$ be given (typically, $\overline{\mathbf{U}}^{n,0} = \overline{\mathbf{U}}^{n-1}$). For $1 \le k$, a new estimate $\overline{\mathbf{U}}^{n,k}$ is computed from the previous $\overline{\mathbf{U}}^{n,k-1}$ by solving the following system of linear algebraic equations: For all $T \in \mathscr{T}$ and all $p \in \{\mathrm{o}, \mathrm{w}\}$,

$$\sum_{T' \in \mathscr{T}} \frac{\partial \mathbf{D}^n_{p,T}}{\partial \overline{\mathbf{U}}_{T'}} \left( \overline{\mathbf{U}}^{n,k}_{T'} - \overline{\mathbf{U}}^{n,k-1}_{T'} \right) = -\mathbf{D}^n_{p,T}(\overline{\mathbf{U}}^{n,k-1}), \tag{8}$$

where $\overline{\mathbf{U}}^{n,k}_T = \{P^{n,k}_T, S^{n,k}_{\mathrm{o},T}\}$ denotes the approximate solutions in $T$ at the $n$-th time step and $k$-th Newton iteration. We suppose that (8) is solved using an iterative linear solver. For a fixed $1 \le n \le N$ and $k \ge 1$, let $\overline{\mathbf{U}}^{n,k,0}$ be given (typically, $\overline{\mathbf{U}}^{n,k,0} = \overline{\mathbf{U}}^{n,k-1}$). Then, at a given step $i \ge 1$, we have, for all $T \in \mathscr{T}$ and $p \in \{\mathrm{o}, \mathrm{w}\}$,

$$\sum_{T' \in \mathscr{T}} \frac{\partial \mathbf{D}^n_{p,T}}{\partial \overline{\mathbf{U}}_{T'}} \left( \overline{\mathbf{U}}^{n,k,i}_{T'} - \overline{\mathbf{U}}^{n,k-1}_{T'} \right) + \mathbf{D}^n_{p,T}(\overline{\mathbf{U}}^{n,k-1}) = \mathbf{R}^{n,k,i}_{p,T}, \tag{9}$$

where $\mathbf{R}^{n,k,i}_{p,T}$ is the algebraic residual, while $\overline{\mathbf{U}}^{n,k,i}_T = \{P^{n,k,i}_T, S^{n,k,i}_{\mathrm{o},T}\}$ denotes the approximate solution at the $n$-th time step, $k$-th Newton iteration, and $i$-th linear solver iteration.

## 3 A posteriori error estimate

### 3.1 Space-time approximate solutions

Let, for $0 \le n \le N$ and $p \in \{\mathrm{o}, \mathrm{w}\}$, $S^n_{p,h}$ be the piecewise constant function such that $S_{p,h|T} = S_{p,T}$ for all $T \in \mathscr{T}$. We introduce the space-time function $S_{p,h\tau}$ continuous and piecewise affine in time, and such that $S_{p,h\tau}(t^n) = S^n_{p,h}$ for $0 \le n \le N$. In order to give a meaning to the gradient operator appearing in (2), we need to postprocess the approximate cell pressures $\{P^n_T\}_{T \in \mathscr{T}}$ and capillary pressures $\{P^n_{\mathrm{c},p,T}\}_{T \in \mathscr{T}}$, $P^n_{\mathrm{c},p,T} := P_{\mathrm{c},p}(S^n_{p,T})$, $p \in \{\mathrm{o}, \mathrm{w}\}$. As in [5, 6, 9], we introduce an elementwise postprocessing of $\{P^n_T\}_{T \in \mathscr{T}}$ and $\{P^n_{\mathrm{c},p,T}\}_{T \in \mathscr{T}}$, $1 \le n \le N$, yielding piecewise quadratic functions $\tilde{P}^n_h$ and $\tilde{P}^n_{\mathrm{c},p,h}$ ($\tilde{P}^0_h$ is given by a projection of the initial pressure $P^0$). As for the saturations, $\tilde{P}_{h\tau}$ and $\tilde{P}_{\mathrm{c},p,h\tau}$ are the space-time functions, continuous and piecewise affine in time, and such that $\tilde{P}_{p,h\tau}(t^n) := \tilde{P}^n_h$ and $\tilde{P}_{\mathrm{c},p,h\tau}(t^n) := \tilde{P}^n_{\mathrm{c},p,h}$, respectively.

### 3.2 Error measure

Set $X := L^2((0,t_{\rm F});H^1(\Omega))$. For $\varphi \in X$, let $\|\varphi\|_X^2 := \int_0^{t_{\rm F}} \|\nabla\varphi\|^2 {\rm d}t$ and $\|\cdot\|$ denotes the $L^2$-norm on $\Omega$. We suppose that the solution $(P, S_{\rm o}, S_{\rm w})$ of the problem (1)–(4) has the necessary regularity to permit the following weak formulation characterization: For all $\varphi \in X$, and all $p \in \{{\rm o},{\rm w}\}$,

$$\int_0^{t_{\rm F}} \left\{\langle \partial_t(\phi S_p), \varphi\rangle - \big(\nu_p(P,S_p)\mathbf{u}_p(P,S_p), \nabla\varphi\big)_\Omega - (f_p,\varphi)_\Omega\right\} {\rm d}t = 0. \quad (10)$$

The aim of the following measure is to evaluate the residual of the approximate solution and the nonconformity of the approximate pressure (i.e., the facts that $(\tilde{P}_{h\tau}, S_{{\rm o},h\tau}, S_{{\rm w},h\tau})$ do not satisfy (10) and that $\tilde{P}_{h\tau} \notin X$ in general). Note that if $S_{p,h\tau}$ coincide with $S_p$, $p \in \{{\rm o},{\rm w}\}$, and $\tilde{P}_{h\tau}$ with $P$, the error measure equals zero:

$$\begin{aligned} &|||(S_p - S_{p,h\tau}, P - \tilde{P}_{h\tau})||| \\ := &\sup_{\varphi\in X,\, \|\varphi\|_X=1} \int_0^{t_{\rm F}} \Bigg\{ \langle \partial_t(\phi S_p) - \partial_t(\phi S_{p,h\tau}), \varphi\rangle \\ &- \big(\nu_p(P,S_p)\mathbf{u}_p(P,S_p) - \nu_p(\tilde{P}_{h\tau}, S_{p,h\tau})\mathbf{u}_p(\tilde{P}_{h\tau}, S_{p,h\tau}), \nabla\varphi\big) \Bigg\} {\rm d}t \\ &+ \inf_{\delta\in X} \left\{ \int_0^{t_{\rm F}} \left\| \nu_p(\tilde{P}_{h\tau}, S_{p,h\tau})\mathbf{u}_p(\tilde{P}_{h\tau}, S_{p,h\tau}) - \nu_p(\delta, S_{p,h\tau})\mathbf{u}_p(\delta, S_{p,h\tau})\right\|^2 {\rm d}t \right\}^{\frac{1}{2}}. \end{aligned} \quad (11)$$

### 3.3 A posteriori error estimate

We let $\mathbf{RTN}(T) := [\mathbb{P}_0(T)]^d + \mathbb{P}_0(T)\mathbf{x}$ and $\mathbf{RTN}(T) := [\mathbb{P}_0(T)]^d + [\mathbb{P}_0(T)]^d\mathbf{x}$, on simplices and on rectangular parallelepipeds respectively, and we introduce the Raviart–Thomas–Nédélec space

$$\mathbf{RTN}(\mathscr{T}) := \{\mathbf{v}_h \in \mathbf{H}({\rm div},\Omega) \mid \mathbf{v}_{h|T} \in \mathbf{RTN}(T),\ \forall T \in \mathscr{T}\}.$$

Following [2, 4, 5, 9], in order to obtain an estimate on (11), we introduce for $1 \le n \le N$ and $p \in \{{\rm o},{\rm w}\}$ the flux reconstructions $\boldsymbol{\theta}^n_{p,h} \in \mathbf{RTN}(\mathscr{T})$ such that for $1 \le n \le N$, $T \in \mathscr{T}$, $T' \in \mathscr{T}_T$, $(T \cap T' = \sigma_{T,T'})$, and $p \in \{{\rm o},{\rm w}\}$,

$$\langle \boldsymbol{\theta}^n_{p,h} \cdot \mathbf{n}_T\ |_{\sigma_{T,T'}}, 1\rangle_{\sigma_{T,T'}} := \nu_p(P^{n-1}_{T^\star_p(\sigma)}, S^{n-1}_{{\rm o},T^\star_p(\sigma)}) F^n_{p,T,\sigma}. \quad (12)$$

The following local conservation property is obtained by the Green theorem from (6) and (12):

$$(f^n_p - \partial_t(\phi S_{p,h\tau}) - \nabla\cdot\boldsymbol{\theta}^n_{p,h}, 1)_T = 0. \quad (13)$$

Let us now define the *residual estimators* $\eta^n_{\mathrm{R},T,p}$, the *diffusive flux estimators* $\eta^n_{\mathrm{DF},T,p}$, and the *nonconformity estimators* $\eta^n_{\mathrm{NC},T,p}$ as

$$\begin{aligned}
\eta^n_{\mathrm{R},T,p} &:= \frac{h_T}{\pi}\|f_p - \partial_t(\phi S_{p,h\tau}) - \nabla{\cdot}\boldsymbol{\theta}^n_{p,h}\|_T,\\
\eta^n_{\mathrm{DF},T,p}(t) &:= \left\|\boldsymbol{\theta}^n_{p,h} - \nu_p(\tilde{P}_{h\tau}, S_{p,h\tau})\mathbf{u}_p(\tilde{P}_{h\tau}, S_{p,h\tau})(t)\right\|_T, \qquad (14)\\
\eta^n_{\mathrm{NC},T,p}(t) &:= \left\|\nu_p(\tilde{P}_{h\tau}, S_{p,h\tau})\mathbf{u}_p(\tilde{P}_{h\tau}, S_{p,h\tau})(t) - \nu_p(\delta_{h\tau}, S_{p,h\tau})\mathbf{u}_p(\delta_{h\tau}, S_{p,h\tau})(t)\right\|_T.
\end{aligned}$$

Here $\delta_{h\tau} \in X$ is continuous and piecewise affine in time and such that $\delta_{h\tau}(t^n) = \delta^n_h$, with $\delta^n_h := \mathscr{I}_{av}(\tilde{P}^n_h)$ for all $0 \le n \le N$; $\mathscr{I}_{av}$ is an averaging operator as in [5, 6, 9].

**Theorem 1 (Guaranteed a posteriori error estimate).** *Let* $p \in \{\mathrm{o}, \mathrm{w}\}$. *Then*

$$\begin{aligned}
|||(S_p - S_{p,h\tau}, P - \tilde{P}_{h\tau})||| \le &\left\{\sum_{n=1}^N \int_{I_n} \sum_{T\in\mathscr{T}} (\eta^n_{\mathrm{R},T,p} + \eta^n_{\mathrm{DF},T,p}(t))^2\,\mathrm{d}t\right\}^{\frac{1}{2}}\\
&+ \left\{\sum_{n=1}^N \int_{I_n} \sum_{T\in\mathscr{T}} (\eta^n_{\mathrm{NC},T,p}(t))^2\,\mathrm{d}t\right\}^{\frac{1}{2}}.
\end{aligned} \qquad (15)$$

*Proof.* The proof is straightforward using the definition of the error measure (11) and following the techniques of [5]. The second term in (15) clearly issues from the second term in the right hand-side of (11). We thus only have to prove that the first term is an upper bound on the first term in the right hand-side of (11). Let $\varphi \in X$, $\|\varphi\|_X = 1$, and $p \in \{\mathrm{o}, \mathrm{w}\}$. Set $\mathbf{w}_p := \nu_p(P, S_p)\mathbf{u}_p(P, S_p)$ and $\mathbf{w}_{p,h\tau} := \nu(\tilde{P}_{h\tau}, S_{p,h\tau})\mathbf{u}_p(\tilde{P}_{h\tau}, S_{p,h\tau})$. Then using the characterization of the weak solution (10),

$$\begin{aligned}
&\int_0^{t_\mathrm{F}} \{\langle\partial_t(\phi S_p) - \partial_t(\phi S_{p,h\tau}), \varphi\rangle - (\mathbf{w}_p - \mathbf{w}_{p,h\tau}, \nabla\varphi)\}\mathrm{d}t\\
&= \int_0^{t_\mathrm{F}} \{(f_p - \partial_t(\phi S_{p,h\tau}), \varphi) + (\mathbf{w}_{p,h\tau}, \nabla\varphi)\}\mathrm{d}t.
\end{aligned}$$

Let now $1 \le n \le N$ be given. Adding and subtracting $(\boldsymbol{\theta}^n_{p,h}, \nabla\varphi)$, using the Green theorem, the local conservativity property (13), the Poincaré inequality, and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&(f_p, \varphi) - (\partial_t(\phi S_{p,h\tau}), \varphi) + (\mathbf{w}_{p,h\tau}, \nabla\varphi)\\
&= (f_p - \partial_t(\phi S_{p,h\tau}) - \nabla{\cdot}\boldsymbol{\theta}^n_{p,h}, \varphi) + (\mathbf{w}_{p,h\tau} - \boldsymbol{\theta}^n_{p,h}, \nabla\varphi)\\
&= (f_p - \partial_t(\phi S_{p,h\tau}) - \nabla{\cdot}\boldsymbol{\theta}^n_{p,h}, \varphi - \Pi_0\varphi) + (\mathbf{w}_{p,h\tau} - \boldsymbol{\theta}^n_{p,h}, \nabla\varphi)\\
&\le \sum_{T\in\mathscr{T}} (\eta^n_{\mathrm{R},T,p} + \eta^n_{\mathrm{DF},T,p}(t))\|\nabla\varphi\|_T,
\end{aligned}$$

where $\Pi_0$ denotes the $L^2$-orthogonal projection onto piecewise constants on $\mathscr{T}$. The assertion follows by the Cauchy–Schwarz inequality and by $\|\varphi\|_X = 1$. □

## 3.4 Identification of different components of the error

Let $1 \leq n \leq N$, $T \in \mathscr{T}$, and $p \in \{\mathrm{o}, \mathrm{w}\}$. In Section 2.2, we define the nonlinear system (6) and we solve it in Section 2.3 using an iterative solver for the Newton algorithm. Let assume we are at the $n$-th time step, $k$-th Newton step and $i$-th linearization step. We introduce the following notations:

$$A^{n,k,i}_{p,T} := \phi \frac{|T|}{\tau^n} \left[ (S^{n,k,i}_{p,T} - S^{n,k-1}_{p,T}) - S^{n-1}_{p,T} \right], \quad B^{n,k,i}_{p,T,\sigma} := \nu_p(P^{n,k-1}_{T^\star_p(\sigma)}, S^{n,k-1}_{p,T^\star_p(\sigma)}) F^{n,k,i}_{p,T,\sigma}.$$

The linear system (9) is then equivalent to the following sum of diagonal terms and face fluxes:

$$\frac{\partial A^{n,k,i}_{p,T}}{\partial \overline{\mathbf{U}}_T} + \sum_{\sigma \in \mathscr{F}_T} \sum_{T' \in \mathscr{S}_\sigma} \frac{\partial B^{n,k,i}_{p,T,\sigma}}{\partial \overline{\mathbf{U}}_{T'}} + \mathbf{D}^n_{p,T}(\overline{\mathbf{U}}^{n,k-1}) = \mathbf{R}^{n,k,i}_{p,T}. \tag{16}$$

Let us now define a linearization flux $\overline{\boldsymbol{\theta}}^{n,k,i}_{p,h} \in \mathbf{RTN}(\mathscr{T})$ and algebraic solver flux $\mathbf{r}^{n,k,i}_{p,h} \in \mathbf{RTN}(\mathscr{T})$ such that $\boldsymbol{\theta}^{n,k,i}_{p,h} := \overline{\boldsymbol{\theta}}^{n,k,i}_{p,h} + \mathbf{r}^{n,k,i}_{p,h}$ and such that

$$\langle \overline{\boldsymbol{\theta}}^{n,k,i}_{p,h} \cdot \mathbf{n}_T \,|_{\sigma_{T,T'}}, 1 \rangle_{\sigma_{T,T'}} := \sum_{T' \in \mathscr{T}_T} \frac{\partial B^{n,k,i}_{p,T,\sigma}}{\partial \overline{\mathbf{U}}_{T'}} \text{ and } (\nabla \cdot \mathbf{r}^{n,k,i}_{p,h}, 1)_T = -\mathbf{R}^{n,k,i}_{p,T}. \tag{17}$$

Note that $\overline{\boldsymbol{\theta}}^{n,k,i}_{p,h}$ is fully specified; $\mathbf{r}^{n,k,i}_{p,h}$ can be constructed as in [6]. This gives

$$(f^n_p - \partial_t(\phi S^{k,i}_{p,h\tau}) - \nabla \cdot \overline{\boldsymbol{\theta}}^{n,k,i}_{p,h}, 1)_T = (\nabla \cdot \mathbf{r}^{n,k,i}_{p,h}, 1)_T, \qquad p \in \{\mathrm{o}, \mathrm{w}\}. \tag{18}$$

We can now define the same estimators as in (14) and we have:

$$\eta^{n,k,i}_{\mathrm{R},T,p} + \eta^{n,k,i}_{\mathrm{DF},T,p}(t) + \eta^{n,k,i}_{\mathrm{NC},T,p}(t) \leq \eta^{n,k,i}_{\mathrm{tm},T,p}(t) + \eta^{n,k,i}_{\mathrm{sp},T,p}(t) + \eta^{n,k,i}_{\mathrm{lin},T,p}(t) + \eta^{n,k,i}_{\mathrm{alg},T,p},$$

with

$$\begin{aligned}
\eta^{n,k,i}_{\mathrm{tm},T,p}(t) &:= \left\| \nu_p(\tilde{P}^{k,i}_{h\tau}, S^{k,i}_{p,h\tau}) \mathbf{u}_p(\tilde{P}^{k,i}_{h\tau}, S^{k,i}_{p,h\tau})(t) - \nu_p(\tilde{P}^{n,k,i}_h, S^{n,k,i}_{p,h}) \mathbf{u}_p(\tilde{P}^{n,k,i}_h, S^{n,k,i}_{p,h}) \right\|_T, \\
\eta^{n,k,i}_{\mathrm{sp},T,p}(t) &:= \eta^{n,k,i}_{\mathrm{R},T,p} + \eta^{n,k,i}_{\mathrm{NC},T,p}(t), \\
\eta^{n,k,i}_{\mathrm{lin},T,p}(t) &:= \left\| \nu_p(\tilde{P}^{n,k,i}_h, S^{n,k,i}_{p,h}) \mathbf{u}_p(\tilde{P}^{n,k,i}_h, S^{n,k,i}_{p,h}) - \overline{\boldsymbol{\theta}}^{n,k,i}_{p,h} \right\|_T, \\
\eta^{n,k,i}_{\mathrm{alg},T,p} &:= \|\mathbf{r}^{n,k,i}_p\|_T.
\end{aligned} \tag{19}$$

## 3.5 Adaptive algorithm

To solve the nonlinear system (6), let us introduce the following algorithm, for $1 \leq n \leq N$.

1. Choose initial saturations $\mathbf{S}_{\mathrm{o}}^{n,0}$ and pressures $\mathbf{P}^{n,0}$ according to (3). Typically, we put $\mathbf{S}_{\mathrm{o}}^{n,0} = \mathbf{S}_{\mathrm{o}}^{n-1}$ and $\mathbf{P}^{n,0} = P^{n-1}$. Set $k = 1$.
2. Set up the linear system (8).
   a. Choose some initial saturation $\mathbf{S}_{\mathrm{o}}^{n,k,0}$ and pressure $\mathbf{P}^{n,k,0}$. Typically, we let $\mathbf{S}_{\mathrm{o}}^{n,k,0} = \mathbf{S}_{\mathrm{o}}^{n,k-1}$ and $\mathbf{P}^{n,k,0} = \mathbf{P}^{n,k-1}$. Set $i = 1$.
   b. Perform a step of a chosen iterative method for the solution of (8), starting from $\mathbf{S}_{\mathrm{o}}^{n,k,i-1}$ and $\mathbf{P}^{n,k,i-1}$. This gives approximations $\mathbf{S}_{\mathrm{o}}^{n,k,i}$ and $\mathbf{P}^{n,k,i}$.
   c. Postprocess locally the pressures $\mathbf{P}^{n,k,i}$.
   d. Construct the fluxes $\overline{\boldsymbol{\theta}}_{p,h}^{n,k,i} \in \mathbf{RTN}(\mathscr{T})$, $p \in \{\mathrm{o}, \mathrm{w}\}$, according to Section 3.4.
   e. For $p \in \{\mathrm{o}, \mathrm{w}\}$, from the algebraic residual vectors $\mathbf{R}_p^{n,k,i}$ construct the fluxes $\mathbf{r}_{p,h}^{n,k,i} \in \mathbf{RTN}(\mathscr{T})$, as described in Section 3.4.
   f. We evaluate all the indicators (19)and define their global versions by their Hilbertian sums. The convergence criterion for the linear solver is:

$$\eta_{\mathrm{alg},p}^{n,k,i} \leq \gamma_{\mathrm{alg}}(\eta_{\mathrm{sp},p}^{n,k,i} + \eta_{\mathrm{tm},p}^{n,k,i} + \eta_{\mathrm{lin},p}^{n,k,i}), \qquad p \in \{\mathrm{o}, \mathrm{w}\}. \tag{20}$$

   Here, $0 < \gamma_{\mathrm{alg}} \leq 1$ is a user-given weight, typically close to 1. Criterion (20) expresses that there is no need to continue with the algebraic solver iterations if the overall error is dominated by the other components. If (20) is reached, set $\mathbf{S}_{\mathrm{o}}^{n,k} := \mathbf{S}_{\mathrm{o}}^{n,k,i}$ and $\mathbf{P}^{n,k} := \mathbf{P}^{n,k,i}$. If not, $i := i + 1$ and go back to step 2(b).

3. The convergence criterion for the nonlinear solver is:

$$\eta_{\mathrm{lin},p}^{n,k,i} \leq \gamma_{\mathrm{lin}}(\eta_{\mathrm{sp},p}^{n,k,i} + \eta_{\mathrm{tm},p}^{n,k,i}), \qquad p \in \{\mathrm{o}, \mathrm{w}\}. \tag{21}$$

   Here $0 < \gamma_{\mathrm{lin}} \leq 1$ is a user-given weight, typically close to 1. Criterion (21) expresses that there is no need to continue with the linearization iterations if the overall error is dominated by the other components. If criterion (21) is reached, finish. If not, $k := k + 1$ and go back to step 1.

Additionally, for all $1 \leq n \leq N$, the space and time estimators $\eta_{\mathrm{sp},p}^n$ and $\eta_{\mathrm{tm},p}^n$ should be made of similar size.

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The paper is in final form and no similar paper has been or is being submitted elsewhere.