On the Godunov Scheme Applied to the Variable Cross-Section Linear Wave Equation

Stéphane Dellacherie and Pascal Omnes

Abstract We investigate the accuracy of the Godunov scheme applied to the variable cross-section acoustic equations. Contrarily to the constant cross-section case, the accuracy issue of this scheme in the low Mach number regime appears even in the one-dimensional case; on the other hand, we show that it is possible to construct another Godunov type scheme which is accurate in the low Mach number regime.

Keywords Variable cross-section, wave equation, low Mach, Godunov scheme
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1 Introduction

It is well-known that Godunov type schemes suffer from an accuracy problem at low Mach number. The analysis of this scheme applied to the linear wave equation has shown that this problem already occurs for such a simple submodel, except in the one-dimensional case [1]. However, it has also been proved that in higher dimensions, simplicial meshes perform much better than rectangular meshes [2]. These results are obtained by the analysis of the invariant space of the discrete wave operator: when this invariant space is rich enough to approach well the invariant space of the continuous wave operator (that is to say the incompressible fields), then the Godunov scheme is accurate at low Mach number. With the same type of analysis, we show in the present work that accuracy problems may already occur in the one-dimensional case for the variable cross-section linear wave equation, if one

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is not careful about the expression of the diffusion terms inherent to the Godunov scheme. This equation may be seen as a simple model for diphasic flows in which the volumic fraction plays the role of the variable cross-section.

2 The variable cross-section wave equation

For regular solutions, the dimensionless barotropic Euler system with variable cross-section may be written as

$$\partial_t(A\rho) + \nabla \cdot (A\rho u) = 0 \qquad \text{and} \qquad \rho(\partial_t u + u \cdot \nabla u) + \frac{\nabla p}{M^2} = 0, \tag{1}$$

where the Mach number M is supposed to be small and where $p = p(\rho)$ with $p'(\rho) > 0$. Denoting by a_* a reference sound velocity, and setting

$$\rho(t,x) := \rho_* \left[1 + \frac{M}{Aa_*} s(t,x) \right], \tag{2}$$

system (1) may be written, after some simplifications

$$\partial_t q + \mathscr{H}(q) + \frac{\mathscr{L}_{A,M}}{M}(q) = 0 \tag{3}$$

with

$$\begin{cases} q = \big(s \ , u\big)^T , & \text{(a)} \\ \mathscr{H}(q) = \big(\nabla \cdot (us) \ , (u \cdot \nabla) u\big)^T , & \text{(b)} \\ \mathscr{L}_{A,M}(q) = \left(a_* \nabla \cdot (Au) \ , \dfrac{p'[\rho_*(1 + \frac{M}{Aa_*} s)]}{a_* + \frac{M}{A} s} \nabla \left(\dfrac{s}{A} \right) \right)^T . & \text{(c)} \end{cases} \tag{4}$$

2.1 The linear wave equation with variable cross-section

When A is bounded by below and by above independently of M, we formally have that $\frac{M}{Aa_*} s(t,x) \ll 1$ in (2) and $\mathscr{O}(||\mathscr{L}_{A,M}(q)||) = 1$ in (3) when $\mathscr{O}(||q||) = 1$. In that case, (3) contains a transport contribution whose characteristic time scale is a $\mathscr{O}(1)$ and a non-linear acoustic contribution whose characteristic time scale is a $\mathscr{O}(M)$, like in the usual barotropic low Mach number Euler system. In that case, at least in a first approach, we may drop the transport contribution and study the linearized cross-section acoustic equation which reads

$$\partial_t q + \frac{L_A}{M} q = 0 \qquad \text{with} \qquad L_A q = a_* \left(\nabla \cdot (Au) \ , \ \nabla \left(\frac{s}{A} \right) \right)^T . \tag{5}$$

3 Basic properties of the variable cross-section linear wave equation

3.1 General properties

In this section, we are interested in basic properties of (5) solved on a periodic torus $\mathbb{T}^{d \in \{1,2,3\}}$. For this, we define the energy space

$$(L^2_A(\mathbb{T}^d))^{1+d} := \left\{ q := \left(s \ , \ u \right)^T \text{ such that } \int_{\mathbb{T}^d} s^2 \frac{dx}{A(x)} + \int_{\mathbb{T}^d} |u|^2 A(x) dx < +\infty \right\}$$

endowed with the scalar product

$$\langle q_1, q_2 \rangle_A = \int_{\mathbb{T}^d} s_1 s_2 \frac{dx}{A(x)} + \int_{\mathbb{T}^d} u_1 \cdot u_2 \, A(x) dx. \tag{6}$$

On the other hand, we set

$$\begin{cases} \mathscr{E}_A = \left\{ q := \left(s \ , \ u \right)^T \in (L^2_A(\mathbb{T}^d))^{1+d} \text{ such that } s = aA, \ a \in \mathbb{R} \text{ and } \nabla \cdot (Au) = 0 \right\}, \\ \mathscr{E}^\perp = \left\{ q := \left(s \ , \ u \right)^T \in (L^2_A(\mathbb{T}^d))^{1+d} \right. \\ \qquad\qquad \left. \text{such that } \int_{\mathbb{T}^d} s dx = 0 \text{ and } \exists \phi \in H^1(\mathbb{T}^d), u = \nabla \phi \right\}. \end{cases}$$

We remark that $\mathscr{E}_A \perp \mathscr{E}^\perp$ for the scalar product (6). We shall admit the following extension of the Hodge decomposition $(L^2_A(\mathbb{T}^d))^{1+d} = \mathscr{E}_A \oplus \mathscr{E}^\perp$. Moreover, we have

$$\mathscr{E}_A = Ker L_A. \tag{7}$$

Finally, for all $q \in (L^2_A(\mathbb{T}^d))^{1+d}$, we define the energy $E_A := \langle q, q \rangle_A$. The following lemma is an easy extension of the energy conservation property to the variable cross-section case:

Lemma 1. *Let $q(t,x)$ be the solution of (5) on $\mathbb{T}^{d \in \{1,2,3\}}$. Then:*

$$E_A(t \geq 0) = E_A(t = 0).$$

We also have the following result:

Lemma 2. *Let $q(t,x)$ be the solution of (5) on $\mathbb{T}^{d\in\{1,2,3\}}$ with initial condition q^0. Then:*
1) $\forall q^0 \in \mathscr{E}_A : \; q(t \geq 0) \in \mathscr{E}_A$;
2) $\forall q^0 \in \mathscr{E}^\perp : \; q(t \geq 0) \in \mathscr{E}^\perp$.

Proof of Lemma 2: The first point is a direct consequence of (7). The second point is inferred from the first item, from Lemma 1, and from the following Lemma, a proof of which may be found in the appendix A of [1].

Lemma 3. *Let $\mathscr{A}$ be a linear isometry in a Hilbert space $\mathbb{H}$ and let $\mathscr{E}$ be a linear subspace of $\mathbb{H}$. Then:* $\quad \mathscr{A}\mathscr{E} = \mathscr{E} \quad \Longrightarrow \quad \mathscr{A}\mathscr{E}^\perp \subset \mathscr{E}^\perp$.

3.2 The one-dimensional case

In the particular case of the one-dimensional geometry, equation (5) is now set in $\mathbb{T}^{d=1}$ and writes

$$\partial_t q + \frac{L_A}{M} q = 0 \tag{8}$$

with

$$L_A q = a_* \left(\partial_x (Au) \;,\; \partial_x \left(\frac{s}{A} \right) \right)^T . \tag{9}$$

The subspaces $\mathscr{E}_A$ and $\mathscr{E}^\perp$ are now characterized by

$$\begin{cases} \mathscr{E}_A = \left\{ q := \left(s \;,\; u \right)^T \in (L^2_A(\mathbb{T}^1))^2 \text{ such that } s = aA \text{ and } u = \dfrac{b}{A},\; (a,b) \in \mathbb{R}^2 \right\}, \\ \mathscr{E}^\perp = \left\{ q := \left(s \;,\; u \right)^T \in (L^2_A(\mathbb{T}^1))^2 \text{ such that } \displaystyle\int_{\mathbb{T}^d} s \, dx = \int_{\mathbb{T}^d} u \, dx = 0 \right\} . \end{cases}$$

In the one-dimensional case, we remark that, as soon as $A'(x) \neq 0$, the variables s and u do not play the same role, while when $A = 1$, they do play symmetrical roles.

4 Numerical approximation in the one-dimensional geometry

We now consider the numerical approximation of (8)–(9) on a mesh with N cells $[x_{i-1/2}, x_{i+1/2}]$ of constant size Δx. We denote by x_i the midpoints of the cells and by $u_i(t)$ and $s_i(t)$ the numerical approximation of u and s in the cell $[x_{i-1/2}, x_{i+1/2}]$.

4.1 A first numerical scheme

Integrating (8) over the cell $[x_{i-1/2}, x_{i+1/2}]$, we obtain

$$\begin{cases} \dfrac{d}{dt}s_i + \dfrac{a_*}{M} \cdot \dfrac{A_{i+1/2}u_{i+1/2}(t) - A_{i-1/2}u_{i-1/2}(t)}{\Delta x} = 0, \\ \\ \dfrac{d}{dt}u_i + \dfrac{a_*}{M} \cdot \dfrac{\dfrac{s_{i+1/2}(t)}{A_{i+1/2}} - \dfrac{s_{i-1/2}(t)}{A_{i-1/2}}}{\Delta x} = 0, \end{cases} \tag{10}$$

where $A_{i+1/2} := A(x_{i+1/2})$ and where the interface values $(s_{i+1/2}(t), u_{i+1/2}(t))$ are determined by the solution of a Riemann problem (R.P.) based on the equation

$$\partial_t q + \frac{a_*}{M}\left(A_{i+1/2}\partial_x u, \frac{1}{A_{i+1/2}}\partial_x s\right)^T = 0,$$

which amounts to locally neglect the variations of A in (8). The left and right initial states of the R. P. being $(s_i(t), u_i(t))$ and $(s_{i+1}(t), u_{i+1}(t))$ respectively, its solution is

$$\begin{cases} s_{i+1/2} = \frac{1}{2}(s_i + s_{i+1}) + \frac{A_{i+1/2}}{2}(u_i - u_{i+1}), \\ \\ u_{i+1/2} = \frac{1}{2A_{i+1/2}}(s_i - s_{i+1}) + \frac{1}{2}(u_i + u_{i+1}). \end{cases} \tag{11}$$

Plugging (11) into (10), we obtain the following scheme

$$\begin{cases} \dfrac{d}{dt}s_i + \dfrac{a_*}{M} \cdot \dfrac{A_{i+1/2}(u_i + u_{i+1}) - A_{i-1/2}(u_{i-1} + u_i)}{2\Delta x} = \dfrac{a_*}{2M\Delta x}\left(s_{i+1} - 2s_i + s_{i-1}\right), \\ \\ \dfrac{d}{dt}u_i + \dfrac{a_*}{M} \cdot \dfrac{\dfrac{(s_i + s_{i+1})}{A_{i+1/2}} - \dfrac{(s_{i-1} + s_i)}{A_{i-1/2}}}{2\Delta x} = \dfrac{a_*}{2M\Delta x}\left(u_{i+1} - 2u_i + u_{i-1}\right), \end{cases} \tag{12}$$

whose first-order modified equation is given by

$$\partial_t q + \frac{L_A}{M} q = \left(\nu_s \partial^2_{xx} s\ ,\ \nu_u \partial^2_{xx} u\right)^T \tag{13}$$

with $(\nu_s, \nu_u) = \frac{a_* \Delta x}{2M}(1, 1)$. This shows that for all non trivial $(\nu_s, \nu_u) \in \mathbb{R}^2$, the space $\mathscr{E}_A$ is no more invariant as soon as $A' \neq 0$. In particular, even when $\nu_u = 0$, the space $\mathscr{E}_A$ is not invariant as soon as $A' \neq 0$: this property stresses the fact that the Godunov scheme, as well as its low Mach modification obtained by simply removing the dissipative term in the right-hand side of the second equation of (12) like in [1, 2], may not be accurate at low Mach number, including in the 1D case, contrarily to the case $A' = 0$.

4.2 Study of a second numerical scheme

In order to propose a numerical scheme which will be accurate at low Mach number, we proceed like in [1]. That is to say:

- First, we try to modify the diffusion term in (13) such that the new equation preserves the invariance of $\mathscr{E}_A$.
- Then, we identify a numerical scheme whose modified equation corresponds to the equation with the new diffusion term.

To these two points, we add something new with respect to what is done in [1]: we shall show that it is possible to define a Godunov type scheme which corresponds to the numerical scheme proposed in the second point above. This stresses the fact that it is possible to build a particular Godunov scheme that is accurate at low Mach number for the linear wave equation with variable cross-section, if one discretizes equation (8) in a adequate set of variables. Another interest of this scheme is that it doesn't suffer from any checkerboard mode (see [3] when $A = 1$).

4.2.1 Modification of the diffusion term

Let us replace the diffusion term

$$\left(\nu_s \partial^2_{xx} s \ , \ \nu_u \partial^2_{xx} u \right)^T \tag{14}$$

in equation (13) by the diffusion term

$$\left(\nu_s \partial_x \left[A \partial_x \left(\frac{s}{A} \right) \right] \ , \ \nu_u \partial_x \left[\frac{1}{A} \partial_x \left(Au \right) \right] \right)^T \tag{15}$$

with $(\nu_s, \nu_u) \in \mathbb{R}^2$. Then, by construction, the space $\mathscr{E}_A$ is invariant for equation

$$\partial_t q + \frac{L_A}{M} q = \left(\nu_s \partial_x \left[A \partial_x \left(\frac{s}{A} \right) \right] \ , \ \nu_u \partial_x \left[\frac{1}{A} \partial_x \left(Au \right) \right] \right)^T . \tag{16}$$

Moreover, we have the following result:

Lemma 4. *Let $q(t, x)$ be solution of (16) over $\mathbb{T}^1$. Then:*

$$E_A(t \geq 0) \leq E_A(t = 0).$$

A numerical scheme associated to (16) is then likely to be stable.

Proof of Lemma 4: Let $q(t,x)$ be solution of (16). There holds

$$\frac{1}{2}\frac{d}{dt}E_A = \nu_s \int_{\mathbb{T}^d} s\partial_x \left[A\partial_x\left(\frac{s}{A}\right)\right] \frac{dx}{A(x)} + \nu_u \int_{\mathbb{T}^d} u\partial_x \left[\frac{1}{A}\partial_x\left(Au\right)\right] A(x)dx$$

$$= -\nu_s \int_{\mathbb{T}^d} \left[\partial_x\left(\frac{s}{A}\right)\right]^2 A(x)dx - \nu_u \int_{\mathbb{T}^d} \left[\partial_x\left(Au\right)\right]^2 \frac{dx}{A(x)} \leq 0.$$

This proves that $E_A(t \geq 0) \leq E_A(t=0)$.□

4.2.2 Identifying the numerical scheme

A numerical scheme whose modified equation corresponds to (16) is given by

$$\begin{cases} \dfrac{d}{dt}s_i + \dfrac{a_*}{M}\cdot\dfrac{A_{i+1}u_{i+1} - A_{i-1}u_{i-1}}{2\Delta x} = \\ \qquad \dfrac{a_*}{2M\Delta x}\left[\dfrac{A_{i+1/2}}{A_{i+1}}s_{i+1} - \left(\dfrac{A_{i+1/2}+A_{i-1/2}}{A_i}\right)s_i + \dfrac{A_{i-1/2}}{A_{i-1}}s_{i-1}\right] \quad \text{(a)} \\ \dfrac{d}{dt}u_i + \dfrac{a_*}{M}\cdot\dfrac{\dfrac{s_{i+1}}{A_{i+1}} - \dfrac{s_{i-1}}{A_{i-1}}}{2\Delta x} = \\ \qquad \dfrac{a_*}{2M\Delta x}\left[\dfrac{A_{i+1}}{A_{i+1/2}}u_{i+1} - A_i\left(\dfrac{1}{A_{i+1/2}} + \dfrac{1}{A_{i-1/2}}\right)u_i + \dfrac{A_{i-1}}{A_{i-1/2}}u_{i-1}\right] \text{(b)} \end{cases}, \tag{17}$$

where $A_i := A(x_i)$. A discrete analogue of Lemma 4 may be proved through "discrete integration by parts" and shows that the scheme is stable and that the discrete invariant space is the set

$$\mathcal{E}_A^h = \left\{q := \left(s\ ,\ u\right)^T \in (\mathbb{R}^N)^2 \text{ such that } s_i = aA_i \text{ and } u_i = \frac{b}{A_i}\ ,\ (a,b) \in \mathbb{R}^2\right\},$$

which admits the orthogonal set

$$(\mathcal{E}^h)^\perp = \left\{q := \left(s\ ,\ u\right)^T \in (\mathbb{R}^N)^2 \text{ such that } \sum_{i=1}^N s_i = \sum_{i=1}^N u_i = 0\right\}$$

for the discrete scalar product $\langle q_1, q_2\rangle_A^h := \sum_{i=1}^N \Delta x \left(\frac{(s_1)_i(s_2)_i}{A_i} + (u_1)_i(u_2)_iA_i\right)$.

4.2.3 The associated Godunov scheme

It is possible to obtain scheme (17) from (10) by the following process: we set

$$\widetilde{q} := \left(r\ ,\ j\right)^T\ ,\ r := \frac{s}{A}\ ,\ j := Au$$

and solve the Riemann Problem based on the equation

$$\partial_t \widetilde{q} + \frac{a_*}{M}\left(\partial_x\left(\frac{j}{A_{i+1/2}}\right)\ ,\ \partial_x(A_{i+1/2}r)\right)^T = 0$$

with initial left and right states given by $\left(\frac{s_i}{A_i}, A_i u_i\right)^T$ and $\left(\frac{s_{i+1}}{A_{i+1}}, A_{i+1}u_{i+1}\right)^T$ respectively. This provides an expression for $\left(r_{i+1/2}, j_{i+1/2}\right)^T$ which is plugged into (10) for the evaluation of $\frac{s_{i+1/2}}{A_{i+1/2}}$ and $A_{i+1/2}u_{i+1/2}$, and we obtain (17).

5 Numerical results in 1D

In this section, we chose $A(x) = \frac{1}{4}\sin(2\pi x) + \frac{1}{2}$. As an initial condition, we choose $s^0(x) = A(x)$ and $u^0(x) = 1/A(x)$. At the discrete level, we choose $s_i^0 = A(x_i)$ and $u_i^0 = 1/A(x_i)$, so that the initial condition belongs to $\mathscr{E}_A^h$. Then, with (17), this initial condition is left unchanged for all times, as is the case with the continuous solution. On the other hand, with (12), the solution $(s_i(t), u_i(t))_{i\in[1,N]}^T$ has a non zero component in the space $(\mathscr{E}^h)^\perp$ as soon as $t > 0$, which may be computed by an orthogonal projection. Figure 1 shows the discrete weighted L^2

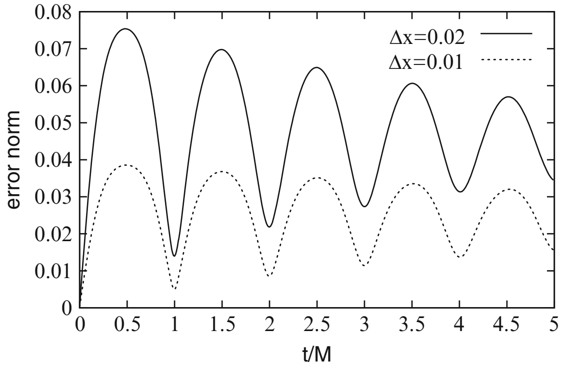

Fig. 1 norm of the spurious component for $M = 10^{-4}$ as a function of t/M

norm of this spurious component in $(\mathscr{E}^h)^\perp$ as a function of time scaled by M, with $M = 10^{-4}$ and for two different values of Δx. The size of the spurious wave grows up from 0 at $t = 0$ to $\mathscr{O}(\Delta x)$ at $t = \mathscr{O}(M)$, which is much greater than $\mathscr{O}(M)$, since $M \ll \Delta x$.

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The paper is in final form and no similar paper has been or is being submitted elsewhere.