Comparison of Upwind and Centered Schemes for Low Mach Number Flows

Thu–Huyen DAO, Michael NDJINGA, and Frédéric MAGOULES

Abstract In this paper, fully implicit schemes are used for the numerical simulation of compressible flows at low Mach number. The compressible Navier–Stokes equations are discretized classically using the finite volume framework and a Roe type scheme for the convection flux. Though explicit Godunov type schemes are inaccurate for low Mach number flows on Cartesian meshes, we claim that their implicit counterpart can be more precise for that type of flow. Numerical evidence from the lid driven cavity benchmark shows that the centered implicit scheme can capture low Mach vortices, unlike the upwind scheme. We also propose a Scaling strategy based on the convection spectrum to reduce the computational cost and accelerate the convergence of both linear system and Newton scheme iterations.

Keywords Low Mach number, centered scheme, upwind scheme, compressible flows, scaling preconditioner
MSC2010: 35L65, 65F35, 76E19, 76M12

1 Introduction

Accurate numerical simulation of compressible flows at low Mach number is of great practical importance in the design and safety analysis of nuclear reactors and core thermal-hydraulic studies (see [6] and [7]). The numerical solutions of the

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corresponding two-phase flow models are based on Riemann approximate solvers which are robust and can efficiently capture shock wave solutions using an upwind strategy. However, when the flow is at low Mach number, especially on Cartesian meshes, these schemes are inaccurate, and corrections have to be made to capture the correct dynamics (see for example [8]). In [5], a detailed analysis of the behavior of Godunov type schemes applied to the compressible Euler system at low Mach number is proposed. The upwind part of the Roe scheme is identified as bringing excessive numerical diffusion and several corrections are proposed. These corrections aim at reducing the numerical diffusion of the explicit schemes, as well as maintaining their stability.

In this paper we present a more general strategy that could be easily applied to simulate various multiphase models at low Mach number. Such a strategy is inspired by single phase analysis and is first tested on the compressible Navier–Stokes equations in the present paper. In order to reduce the numerical diffusion, we consider a scheme that is order two in space such as the implicit centered scheme, already studied for example in [9].

In Sect. 2, we briefly recall the mathematical model and the considered numerical schemes. In Sect. 4 we give numerical evidence that the centered implicit scheme is much less diffusive than the upwind scheme (whether explicit or implicit) and can capture low Mach vortices. In order to reduce the computational cost involved by the resolution of many linear systems, Sect. 3 presents preconditioning strategy based on the scaling of the linear system matrix. This strategy is based on the underlying hyperbolic operator and could be applied to other set of equations.

2 Mathematical model and Numerical method

2.1 Mathematical model

The model consists of the following three balance laws for the mass, the momentum and the energy:

$$\begin{cases} \frac{\partial \rho}{\partial t} & + \nabla.\mathbf{q} & & = 0 \\ \frac{\partial \mathbf{q}}{\partial t} & + \nabla.\left(\mathbf{q} \otimes \frac{\mathbf{q}}{\rho} + p\mathbb{I}_d\right) & - \nu\Delta(\frac{\mathbf{q}}{\rho}) & = 0 \\ \frac{\partial(\rho E)}{\partial t} & + \nabla.\left[(\rho E + p)\frac{\mathbf{q}}{\rho}\right] & - \lambda\Delta T & = 0 \end{cases} \tag{1}$$

where ρ is the density, $\mathbf{v}$ the velocity, $\mathbf{q} = \rho\mathbf{v}$ the momentum, p the pressure, ρe the internal energy, $\rho E = \rho e + \frac{||\mathbf{q}||^2}{2\rho}$ the total energy, T the absolute temperature, ν the viscosity and λ the thermal conductivity. We close the system (1) by the ideal gas law $p = (\gamma - 1)\rho e$. For the sake of simplicity, we consider constant viscosity and conductivity, and neglect the contribution of viscous forces in the energy equation. By denoting $U = (\rho, \mathbf{q}, \rho E)^t$ the vector of conserved variables, the Navier–Stokes system (1) can be written as a nonlinear system of conservation laws:

$$\frac{\partial U}{\partial t} + \nabla . (\mathscr{F}^{conv}(U)) + \nabla . \left(\mathscr{F}^{diff}(U)\right) = 0, \tag{2}$$

where $\mathscr{F}^{conv}(U) = \begin{pmatrix} \mathbf{q} \\ \mathbf{q} \otimes \frac{\mathbf{q}}{\rho} + p\mathbb{I}_d \\ (\rho E + p)\,\frac{\mathbf{q}}{\rho} \end{pmatrix}$, $\mathscr{F}^{diff}(U) = \begin{pmatrix} 0 \\ -\nu \nabla(\frac{\mathbf{q}}{\rho}) \\ -\lambda \nabla T \end{pmatrix}$.

2.2 Numerical method

The conservation form (2) enables to define the concept of weak solutions, which can be discontinuous ones. Discontinuous solutions such as shock waves are of great importance in transient calculations. In order to correctly capture shock waves, one needs a robust, low diffusive conservative scheme. The finite volume framework is the best appropriate setup to build such schemes as it enables to write discrete equations that express the conservation laws at each cell (see for example [1]).

We decompose the computational domain into N disjoint cells C_i with volume v_i. Two neighboring cells C_i and C_j have a common boundary ∂C_{ij} with area s_{ij}. We denote $N(i)$ the set of neighbors of a given cell C_i and $\mathbf{n}_{ij}$ the exterior unit normal vector of ∂C_{ij}. Integrating the system (2) over C_i and setting $U_i(t) = \frac{1}{v_i}\int_{C_i} U(x,t)dx$ and $U_i^n = U_i(n\Delta t)$, the discretized equations can be written:

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + \sum_{j \in N(i)} \frac{s_{ij}}{v_i}\left(\overrightarrow{\Phi}_{ij}^{conv} + \overrightarrow{\Phi}_{ij}^{diff}\right) = 0. \tag{3}$$

with: $\overrightarrow{\Phi}_{ij}^{conv} = \frac{1}{s_{ij}}\int_{\partial C_{ij}} \mathscr{F}^{conv}(U).\mathbf{n}_{ij}\,ds$, $\overrightarrow{\Phi}_{ij}^{diff} = \frac{1}{s_{ij}}\int_{\partial C_{ij}} \mathscr{F}^{diff}(U).\mathbf{n}_{ij}\,ds$.

To approximate the convection numerical flux $\overrightarrow{\Phi}_{ij}^{conv}$ we solve an approximate Riemann problem at the interface ∂C_{ij}. Using the Roe local linearisation of the fluxes [2], we obtain the following formula:

$$\overrightarrow{\Phi}_{ij}^{conv} = \frac{\mathscr{F}^{conv}(U_i) + \mathscr{F}^{conv}(U_j)}{2}.\mathbf{n}_{ij} - \mathscr{D}(U_i, U_j)\frac{U_j - U_i}{2} \tag{4}$$

$$= \mathscr{F}^{conv}(U_i)\mathbf{n}_{ij} + A^{-}(U_i, U_j)(U_j - U_i), \tag{5}$$

where $\mathscr{D}$ is an upwinding matrix, $A(U_i, U_j)$ the Roe matrix and $A^{-} = \frac{A - \mathscr{D}}{2}$. The choice $\mathscr{D} = 0$ gives the centered scheme, whereas $\mathscr{D} = |A|$ gives the upwind scheme. For the Euler equations, we can build $A(U_i, U_j)$ explicitly using the Roe averaged state (see [1]).

The diffusion numerical flux $\overrightarrow{\Phi}_{ij}^{diff}$ is approximated on structured meshes using the formula:

$$\overrightarrow{\Phi}_{ij}^{diff} = D(\frac{U_i + U_j}{2})(U_j - U_i) \tag{6}$$

with the matrix $D(U) = \begin{pmatrix} 0 & \mathbf{0} & 0 \\ \frac{\nu \mathbf{q}}{\rho^2} & \frac{-\nu}{\rho}\mathbb{I}_d & 0 \\ \frac{\lambda}{c_v}\left(\frac{c_v T}{\rho} - \frac{||\mathbf{q}||^2}{2\rho^3}\right) & \frac{\mathbf{q}^t \lambda}{\rho^2 c_v} & -\frac{\lambda}{c_v \rho} \end{pmatrix}$, where c_v is the heat capacity at constant volume.

2.3 Newton scheme

Finally, since $\sum_{j \in N(i)} \mathscr{F}^{conv}(U_i).\mathbf{n}_{ij} = 0$, using (5) and (6) the equation (3) of the numerical scheme becomes:

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + \sum_{j \in N(i)} \frac{s_{ij}}{v_i}\{(A^- + D)(U_i^{n+1}, U_j^{n+1})\}(U_j^{n+1} - U_i^{n+1}) = 0. \tag{7}$$

The system (7) is nonlinear. We use the following Newton iterative method to obtain the required solutions:

$$\begin{aligned} &\frac{\delta U_i^{k+1}}{\Delta t} + \sum_{j \in N(i)} \frac{s_{ij}}{v_i}\left[(A^- + D)(U_i^k, U_j^k)\right]\left(\delta U_j^{k+1} - \delta U_i^{k+1}\right) \\ &\qquad = -\frac{U_i^k - U_i^n}{\Delta t} - \sum_{j \in N(i)} \frac{s_{ij}}{v_i}\left[(A^- + D)(U_i^k, U_j^k)\right](U_j^k - U_i^k), \end{aligned}$$

where $\delta U_i^{k+1} = U_i^{k+1} - U_i^k$ is the variation of the k-th iterate that approximate the solution at time $n+1$. Defining the unknown vector $\mathscr{U} = (U_1, \ldots, U_N)^t$, each Newton iteration for the computation of $\mathscr{U}$ at time step $n+1$ requires the numerical solution of the following linear system:

$$\mathscr{A}(\mathscr{U}^k)\delta\mathscr{U}^{k+1} = b(\mathscr{U}^n, \mathscr{U}^k). \tag{8}$$

2.4 The low Mach problem

When the flow is smooth and the Mach number $\frac{||\mathbf{v}||}{c}$ (where $c = \sqrt{\frac{\gamma p}{\rho}}$ is the sound speed) is small, the solutions of the system (1) should behave as those of an incompressible Navier–Stokes model (see [10]). However, in general, Godunov type schemes do not preserve the asymptotic behavior and generate spurious solutions when applied to low Mach number flows (see [5]). The analysis presented in [5] suggests that the inaccuracies originate from the anisotropy of the upwind matrix $\mathscr{D}$, and various " Low Mach Schemes " are proposed in the explicit context. In order to avoid the stability issue, we propose to use implicit schemes and to consider the simpler case $\mathscr{D} = 0$ (no upwinding). The resulting centered scheme can be applied to any system of conservation law, and we present in Sect. 4 our first numerical experiments.

3 Description of the Scaling strategy

The larger the time step, the worse the condition number of the matrix $\mathscr{A}$ in (8). As a consequence, it is important to apply a preconditioner before solving the linear system. The most popular choice is the Incomplete LU factorisation (later named ILU, see [3] for more details). The error made by the approximate factorisation using an ILU preconditioner depends on the size of the off diagonal coefficients of the matrix. For a better performance of the preconditioner, it is desirable that off diagonal entries of the matrix have small magnitudes.

As we are interested in convection dominated flows, the main contributions to the matrix $\mathscr{A}$ come from the convective part discretisation of the equations through the matrix A^-. Unfortunately, the coefficients of the Roe matrix have very different magnitudes for low Mach number flows. Consequently, A^- and hence $\mathscr{A}$ have coefficients with very different magnitudes.

We are now going to detail a procedure that scales the matrix coefficients so that they have the same magnitude. The matrix A^- can be expressed using a complete eigenstructure decomposition of the Roe matrix: $A = \sum_k \lambda_k L^k \otimes R^k$. The three eigenvalues of the Roe matrix are $v_n + c$, v_n (multiplicity d), and $v_n - c$. As we are interested in flows at low Mach number, we can assume $\mathbf{v} \approx 0$ and in that case the eigenvalues of A become $\lambda^- = -c, \lambda_v = 0$, and $\lambda^+ = +c$. The right and left eigenvectors $R^\pm$ and $L^\pm$ associated to the sound waves are:

$$R^\pm = (1,\ \pm c\mathbf{n},\ \frac{c^2}{\gamma - 1})^t, \qquad L^\pm = \frac{1}{2}(0,\ \pm\frac{1}{c}\mathbf{n},\ \frac{\gamma - 1}{c^2})^t. \tag{9}$$

We have:

$$A^- = -cL^- \otimes R^- \qquad \text{for the upwind scheme,}$$

$$A^- = \frac{1}{2}(cL^+ \otimes R^+ - cL^- \otimes R^-) \text{ for the centered scheme.}$$

One sees from (9) that the disequilibrium in A^- coefficients comes from the difference in the magnitude of the components of the left and right eigenvectors of A.

If we multiply A^- to the left (respectively to the right) by a diagonal matrix with the coefficients $d_{sca} = diag(1, c\mathbf{n}, \frac{c^2}{\gamma-1})$, respectively $d_{sca}^{-1} = diag(1, \frac{1}{c}\mathbf{n}, \frac{\gamma-1}{c^2})$ ($\mathbf{n}$ is the unit normal vector), we obtain vectors and matrices with better balanced coefficients:

$$d_{sca}^{-1} R^{\pm} = (1, \pm\mathbf{n}, 1)^t, \qquad d_{sca} L^{\pm} = (0, \pm\mathbf{n}, 1)^t,$$

$$L^{\pm} \otimes R^{\pm} = \frac{1}{2}\begin{pmatrix} 0 & \mathbf{0} & 0 \\ \pm\frac{1}{c}\mathbf{n} & \mathbf{n}\otimes\mathbf{n} & \pm\frac{c}{\gamma-1}\mathbf{n} \\ \frac{\gamma-1}{c^2} & \pm\frac{\gamma-1}{c}\mathbf{n}^t & 1 \end{pmatrix}, \; d_{sca} L^{\pm} \otimes R^{\pm} d_{sca}^{-1} = \frac{1}{2}\begin{pmatrix} 0 & \mathbf{0} & 0 \\ \pm\mathbf{n} & \mathbf{n}\otimes\mathbf{n} & \pm\mathbf{n} \\ 1 & \pm\mathbf{n}^t & 1 \end{pmatrix}$$

Any mesh can be associated with two diagonal matrices D_{sca} and D_{sca}^{-1} having the size of the mesh and containing the successive coefficients of the local matrices d_{sca} and d_{sca}^{-1}. Instead of solving system (8), one can rather solve:

$$\tilde{\mathscr{A}}\mathscr{V} = \tilde{b}, \tag{10}$$

where $\tilde{\mathscr{A}} = D_{sca}\mathscr{A}D_{sca}^{-1}$, $\mathscr{V} = D_{sca}\mathscr{U}$ and $\tilde{b} = D_{sca}b$. System (10) can be resolved more easily using an ILU preconditioner. Once the solution $\mathscr{V}$ is obtained we compute $D_{sca}^{-1}\mathscr{V}$ to obtain the original unknown vector $\mathscr{U}$.

4 Numerical results

4.1 Upwind scheme vs Centered one

Figures 1 and 2 present the streamlines of the steady state result obtained using either the upwind or the centered schemes to discretize the convective part of the Navier–Stokes equations with the fully implicit scheme presented in Sect. 2.2. Our test case is a lid driven cavity flow at Reynolds number 400 solved on a cartesian 50×50 cell mesh. This case is described in [4], with the correct solution given by an incompressible solver. The lid speed is $1\,m/s$, the maximum Mach number of the flow is 0.008. The Roe approximate Riemann solver [2] employed for the convection fluxes is known to have problem solving such low Mach number flows when the scheme is explicit, especially on multidimensional cartesian meshes (see [5]). It can be seen on Fig. 1 that the upwind scheme does not capture the correct streamlines. However, on Fig. 2, it can be seen that the implicit centered scheme is much less diffusive and captures the correct solution with its expected three vortices.

4.2 Assessment of the Scaling strategy

We now study the performance of our numerical methods on the same lid driven cavity test case presented in Sect. 4.1. In this section, we vary the time step

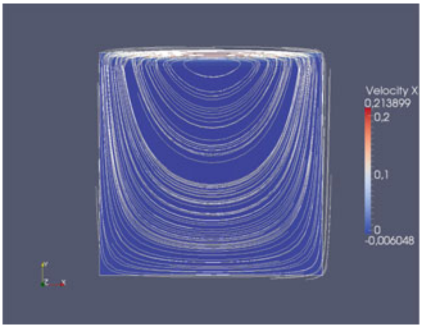

Fig. 1 Steady state, upwind scheme

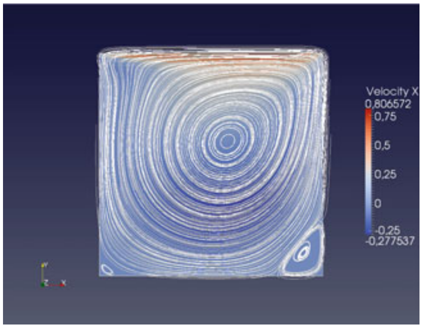

Fig. 2 Steady state, centered scheme

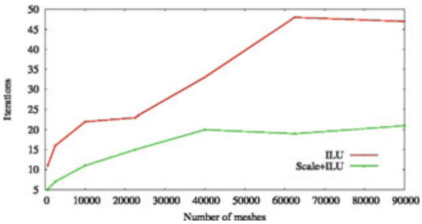

Fig. 3 Number of GMRES iterations for the upwind scheme, CFL 1000

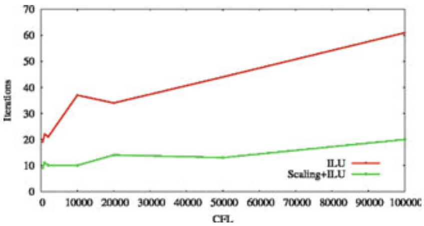

Fig. 4 Number of GMRES iterations for the upwind scheme, mesh 100 × 100

(CFL number) and the mesh size. We also compare the direct solver with the iterative one and the effect of different preconditioners on the resolution of the linear systems.

Considering first the upwind scheme, we remark that the ILU preconditioner with no level of fill-in performs well. Figs. 3 and 4, show the average number of GMRES iterations at each Newton iteration. We observe that the use of our Scaling strategy presented in Sect. 3 reduces more than twice the iteration number.

When we use the centered scheme, the system matrix has a poor diagonal, and ILU preconditioner with no fill-in is not efficient in preconditioning the linear system. One needs to use an incomplete factorisation with two levels of fill-in to solve linear system up to the CFL 100, and the Scaling strategy enables to save a considerable number of iterations (Fig. 5). Beyond that value, only a direct solver is able to solve the system. However, one can remark that the Scaling strategy enables a reduction of the number of Newton iterations using a direct solver (Fig. 6). We also stress that the steady state solution obtained with very large CFL numbers is still accurate and displays the expected vortices.

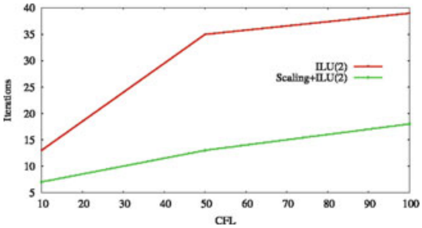

Fig. 5 Number of GMRES iterations for the centered scheme, mesh 50 × 50

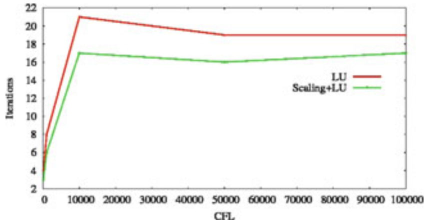

Fig. 6 Number of Newton iterations for the centered scheme, mesh 50 × 50s

5 Conclusion and Perspectives

In this paper, two simple and general fully implicit schemes have been presented for the simulation of compressible Navier–Stokes equations at low Mach number. We have shown that the centered scheme is able to capture low Mach vortices unlike the upwind scheme. However, ILU preconditioning performs better with the upwind scheme than with the centered scheme. Thanks to the particular features of Roe matrix for compressible Navier–Stokes equations, we have proposed a preconditioning strategy Scaling+ILU that considerably reduces the computation time. The centered scheme and the scaling strategy can be applied to other sets of equations than Navier–Stokes. Study of these techniques applied to two-phase flow models will follow.

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The paper is in final form and no similar paper has been or is being submitted elsewhere.