Multi-dimensional Optimal Order Detection (MOOD) — a Very High-Order Finite Volume Scheme for Conservation Laws on Unstructured Meshes

S. Clain, S. Diot, and R. Loubère

Abstract The Multi-dimensional Optimal Order Detection (MOOD) method is an original Very High-Order Finite Volume (FV) method for conservation laws on unstructured meshes. The method is based on an *a posteriori* degree reduction of local polynomial reconstructions on cells where prescribed stability conditions are not fulfilled. Numerical experiments on advection and Euler equations problems are drawn to prove the efficiency and competitiveness of the MOOD method.

Keywords MOOD, high-order, finite volume, unstructured meshes, conservation laws
MSC2010: 65M08, 65Z05, 76M12

1 Introduction

The Multi-dimensional Optimal Order Detection has been introduced in [6] as an original High-Order Finite Volume method for conservation laws on unstructured meshes. As multi-dimensional MUSCL [2–4, 8] or ENO/WENO methods [1, 7, 10], the MOOD method is based on a high-order space discretization with local polynomial reconstructions coupled with a high-order TVD Runge–Kutta method for time discretization.

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The main difference between classical high-order methods and the MOOD one is that the limitation procedure is done *a posteriori*. Inside a time step, a first solution is computed with numerical fluxes evaluated from unlimited high-order polynomial reconstructions. Then polynomial degrees are reduced on cells where prescribed stability conditions are not fulfilled and the solution is re-evaluated. That iterative procedure provides a solution which respects the stability constraints.

The present article is devoted to an extension of the MOOD method to a sixth-order space discretization on triangular meshes. Numerical tests for the advection problem and Euler equations with gravity are given in last section.

2 Framework

We consider the scalar hyperbolic equation defined on a bounded polygonal domain $\Omega \subset \mathbb{R}^2$ written in its conservative form

$$\begin{aligned} \partial_t u + \nabla \cdot F(u) &= 0, \\ u(\cdot, 0) &= u_0, \end{aligned} \tag{1}$$

where $u = u(\mathbf{x}, t)$ is the unknown function with $t > 0$, $\mathbf{x} \in \Omega$, F is the physical flux and u_0 stands for the initial condition. We consider a triangular tessellation of Ω where K_i is a generic triangle with centroid $\mathbf{c}_i$. Moreover $\mathbf{n}_{ij}$ is the unit normal vector of edge e_{ij} from K_i to K_j and $q_{ij}^r, r = 1, 2, 3$, are the Gaussian quadrature points of e_{ij}. Finally $\underline{v}(i)$ (resp. $\overline{v}(i)$) is the index set of cells which share an edge (resp. an edge or a node with K_i). This notation is summarized in Fig. 1.
We recall the generic first-order Finite Volume discretization of (1)

$$u_i^{n+1} = u_i^n - \Delta t \sum_{j \in \underline{v}(i)} \frac{|e_{ij}|}{|K_i|} G(u_i^n, u_j^n, \mathbf{n}_{ij}), \tag{2}$$

where u_i^n is an approximation of the mean value of u on cell K_i at time t^n and $|e_{ij}|$, $|K_i|$ stand for the edge length and the cell surface respectively. We assume that the numerical flux $G(u_i^n, u_j^n, \mathbf{n}_{ij})$ satisfies the consistency and monotonicity properties such that, under an adequate CFL condition, the following Discrete Maximum Principle (DMP) is fulfilled

$$\min_{j \in \overline{v}(i)} (u_i^n, u_j^n) \leq u_i^{n+1} \leq \max_{j \in \overline{v}(i)} (u_i^n, u_j^n). \tag{3}$$

Only few modifications of (2) are needed to get the following High-Order Finite Volume scheme

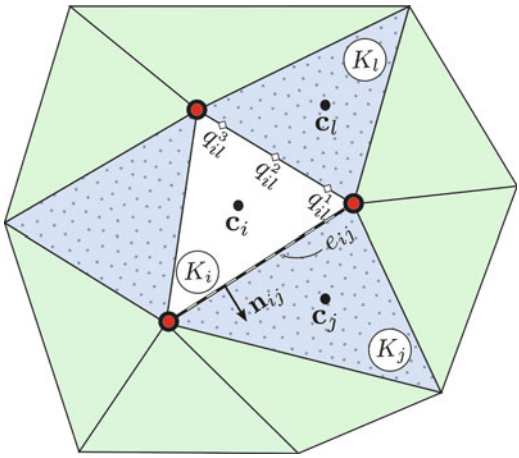

Fig. 1 Mesh notation. Index set $\underline{v}(i)$ corresponds to blue cells with dots and $\overline{v}(i)$ corresponds to every non-white cells

$$u_i^{n+1} = u_i^n - \Delta t \sum_{j \in \underline{v}(i)} \frac{|e_{ij}|}{|K_i|} \sum_{r=1}^{3} \xi_r G(u_{ij,r}^n, u_{ji,r}^n, \mathbf{n}_{ij}), \tag{4}$$

namely the use of a sixth-order Gaussian quadrature rule with weights ξ_r ($r = 1, 2, 3$) and the replacement of u_i^n (resp. u_j^n) by $u_{ij,r}^n$ (resp. $u_{ji,r}^n$) which is an approximation of $u(q_{ij}^r, t^n)$ from the high-order polynomial reconstruction on K_i (resp. K_j). Notice that the high-order scheme (4) corresponds to a convex combination of the first-order one (2), that is important from a practical point of view for an easy and effective implementation.

It is well known that methods based on high-order reconstructions without limiting procedure produce spurious oscillations in the vicinity of discontinuities. In order to prevent such oscillations, the today's effective high-order methods (MUSCL, WENO...) use *a priori* limitation procedures.
The Multi-dimensional Optimal Order Detection (MOOD) method breaks away from this approach through an original effective iterative procedure based on an *a posteriori* detection of such unphysical oscillations (see Fig. 2). The details of MOOD method are recalled in next section

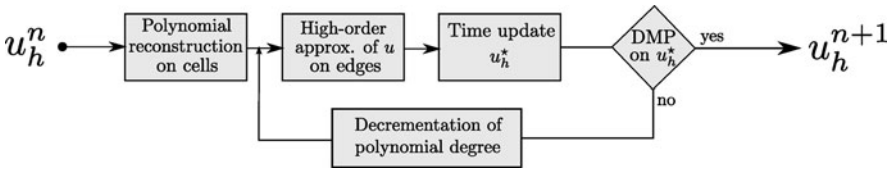

Fig. 2 A simplistic view of the Multi-dimensional Optimal Order Detection concept

3 MOOD method

For the sake of clarity, we only consider a forward Euler method and one quadrature point per edge. Consequently we denote by u_{ij} (resp. u_{ji}) the high-order approximation of u on edge e_{ij} from cell K_i (resp. K_j).

3.1 Basics

Polynomial reconstruction.

High-order approximations of the solution at quadrature points are mandatory. To this end, multi-dimensional polynomial reconstructions from mean values are carried out. There exist several techniques [1, 5] to obtain such reconstructions, but we choose to use the one from [7] where a over-determined linear system is solved using a QR decomposition. The reconstructed polynomial of arbitrary high-order $d_{max}+1$ has the form

$$\widetilde{u}(x,y) = \bar{u} + \sum_{1\leq\alpha+\beta\leq d_{max}} \mathcal{R}_{\alpha\beta}\left((x-c_x)^\alpha(y-c_y)^\beta - \frac{1}{|K|}\int_K (x-c_x)^\alpha(y-c_y)^\beta\, dxdy\right),$$

where (c_x, c_y) is the centroid of a generic cell K and $\mathcal{R}_{\alpha\beta}$ are the unknowns polynomial coefficients. In this way mean value on K is conserved and the truncation of all terms of degree $\alpha+\beta > \bar{d}$ produces a relevant approximation of u as a polynomial of degree $\bar{d} \leq d_{max}$.
At least $\mathcal{N}(d) = (d+1)(d+2)/2 - 1$ neighbors are needed to perform reconstructions. However for the sake of robustness at least $1.5\times\mathcal{N}(d)$ elements are involved. We first take the neighbors by nodes of K and then the neighbors by faces of already picked elements. Lastly, since the condition number of the generated system is dependent of spatial characteristic length, we use the technique proposed in [5] to overcome this problem.

CellPD and EdgePD.

We recall the fundamental notions introduced in [6].

- d_i is the Cell Polynomial Degree (CellPD) which represents the degree of the polynomial reconstruction on cell K_i.
- d_{ij} and d_{ji} are the Edge Polynomial Degrees (EdgePD) which correspond to the effective degrees used to respectively build u_{ij} and u_{ji} on both sides of edge e_{ij}.

We now detail the MOOD method using both notions in the case of the scalar problem (1).

3.2 *Algorithm for the scalar case.*

The MOOD method consists of the following iterative procedure which details the concept depicted in Fig. 2.

1. **CellPD initialization.** Each CellPD is initialized with d_{max}.
2. **EdgePD evaluation.** Each EdgePD is set up as the minimum of the two neighboring CellPD.
3. **Quadrature points evaluation.** Each u_{ij} is evaluated with the polynomial reconstruction of degree d_{ij}.
4. **Mean values update.** The updated values $u_h^\star$ are computed using the finite volume scheme (4).
5. **DMP test.** The DMP criterion is checked on each cell K_i

$$\min_{j\in\overline{\nu}(i)}(u_i^n, u_j^n) \leq u_i^\star \leq \max_{j\in\overline{\nu}(i)}(u_i^n, u_j^n). \tag{5}$$

   If $u_i^\star$ does not satisfy (5) the CellPD is decremented, $\mathsf{d}_i := \max(0, \mathsf{d}_i - 1)$.
6. **Stopping criterion.** If all cells satisfy the DMP property, the iterative procedure stops with $u_h^{n+1} = u_h^\star$ else go to Step 2.

Since only problematic cells and their neighbors in the compact stencil $\underline{\nu}(i)$ have to be checked and re-updated during the iterative MOOD procedure, the computational cost is dramatically reduced.

3.3 *Algorithm for the Euler equations case.*

We now extend the MOOD method to the Euler system, namely

$$\partial_t \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix} + \partial_x \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(E+p) \end{pmatrix} + \partial_y \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(E+p) \end{pmatrix} = 0, \tag{6}$$

where ρ, $\mathbf{V} = (u, v)$ and p are the density, velocity and pressure respectively while the total energy per unit volume E is given by

$$E = \rho\left(\frac{1}{2}\mathbf{V}^2 + e\right), \quad \mathbf{V}^2 = u_1^2 + u_2^2, \quad e = \frac{p}{\rho(\gamma - 1)},$$

where e is the specific internal energy and γ the ratio of specific heats.

The reconstruction is classically done on the primitive variables ρ, u, v, p while $U = (\rho, \rho u, \rho v, E)$ and we use the same CellPD and EdgePD for all variables in a cell. In other words, the two notions are linked to cells and edges and not affected by the number of variables. Furthermore steps 5 and 6 of the previous MOOD algorithm are substituted with the following stages.

5. **Density DMP test.** The DMP criterion is checked on the density

$$\min_{j\in\overline{v}(i)} (\rho_i^n, \rho_j^n) \leq \rho_i^\star \leq \max_{j\in\overline{v}(i)} (\rho_i^n, \rho_j^n). \tag{7}$$

   If $\rho_i^\star$ does not satisfy (7) the CellPD is decremented, $\mathsf{d}_i := \max(0, \mathsf{d}_i - 1)$.
6. **Pressure positivity test.** The pressure positivity is checked and if $p_i^\star \leq 0$ and d_i has not been altered by step 5 then the CellPD is decremented, $\mathsf{d}_i := \max(0, \mathsf{d}_i - 1)$.
7. **Stopping criterion.** If for all $i \in \mathcal{E}_{el}$, d_i has not been altered by steps 5 and 6 then the iterative procedure stops with $U_h^{n+1} = U_h^\star$ else go to step 2.

4 Numerical results

The reader should refer to [6] for a study on the effective convergence rate and for more hydrodynamics test cases. In this paper, we restrict the presentation to two representative tests.

Scalar case

We first deal with the classical Solid Body Rotation (see [6] for details) test case for the advection problem. We plot in Fig. 3 isolines top views of the solution obtained with the MOOD method applied to different polynomial degrees and meshes. Method name, triangles number and computational times are embedded in each figure. Time is given in relative time units (r.t.u) where MOOD-P1 is taken as reference with 100 r.t.u.

First solutions obtained on the 5190 cells mesh (3 top) clearly show that the MOOD method is able to handle high-order polynomials with a great improvement of solutions while enforcing a strict DMP. Then for the sake of comparison, results with lower degrees on finer meshes are given in the bottom line of Fig. 3. Finally notice that the computational cost increase is mainly due to the reconstruction step. However since profiles are not smooth the DMP is often violated and the iterative procedure cost more than in a smooth case. For example a sixth-order unlimited version of the scheme costs 586 r.t.u., thus the iterative procedure costs about a third of the total time of the MOOD-P5 computation.

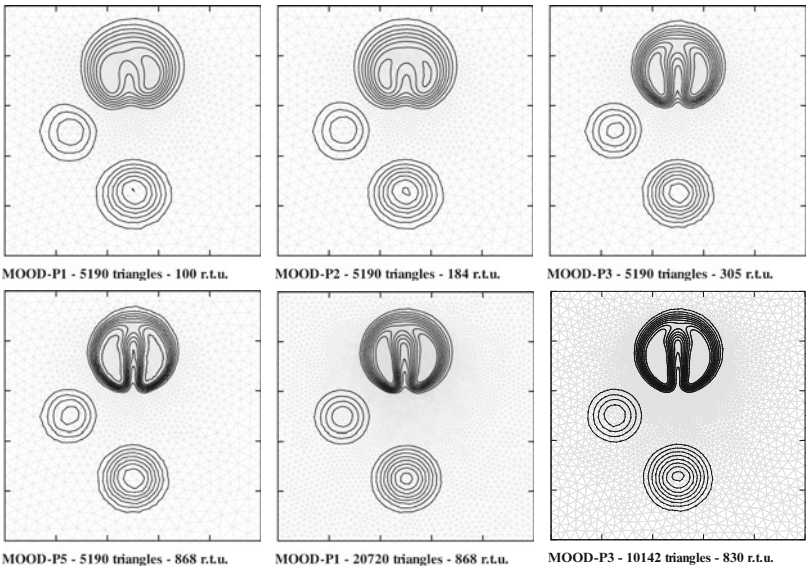

Fig. 3 Solid Body Rotation. 10 isolines (0 to 1). Time in relative time units (r.t.u.)

Euler equations case

For the system case, a Rayleigh–Taylor Instability for the Euler equations with gravity is considered. The reader should refer to [9] for complete description of the test case. A zoom on the pattern of the unstructured symmetric triangular mesh of 28800 cells and the density solutions for MOOD-P1, MOOD-P3 and MOOD-P5 are plotted in Fig. 4.

As for the scalar case the MOOD method is plainly able to improve the solution through the use of high-order polynomial reconstructions. From a computational cost point of view, computational times given in Fig. 4 prove that the MOOD iterative procedure is effective since the time raise from a degree to a bigger one is mainly due to the reconstruction cost itself.

Decrementation procedure

In Table 1, we give the mean percentage over all the calculation of polynomial degrees actually used to compute the solution, *i.e.* the CellPD at the end of the iterative procedure. Three test cases are taken as examples (see [6] for details), the Solid Body Rotation of Fig. 3 with MOOD-P3, the classical Double Mach Reflection on a 57600 cells uniform mesh with MOOD-P2 and the Mach 3 Wind Tunnel on a 4978 cells Delaunay mesh with MOOD-P3. Results show that only few cells are affected by the *a posteriori* limitation.

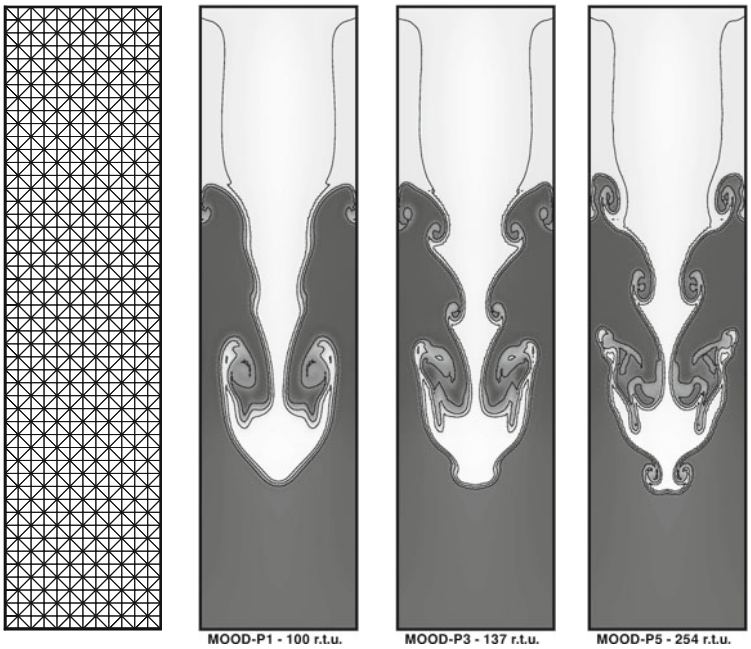

Fig. 4 Rayleigh–Taylor Instability. Density. 5 isolines from 0.8 (dark) to 2.3 (light)

Test case	P0	P1	P2	P3
Solid Body Rotation	7.16%	0.78%	0.64%	91.42%
Double Mach Reflection	5.69%	0.72%	93.69%	—
Mach 3 Wind Tunnel	3.02%	0.36%	0.16%	96.46%

Fig. 5 Mean percentage of polynomial degrees actually used with MOOD method

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The paper is in final form and no similar paper has been or is being submitted elsewhere.