An Extension of the MAC Scheme to some Unstructured Meshes

Eric Chénier, Robert Eymard, and Raphaèle Herbin

Abstract We give a variational formulation of the standard MAC scheme for the approximation of the Navier-Stokes problem. This allows an extension of the MAC scheme to locally refined Cartesian meshes. A numerical example is presented, which shows an efficient computation of the solution of the Navier-Stokes problem for a general 2D or 3D domain, using locally refined meshes.

Keywords MAC scheme, incompressible Navier-Stokes, non conforming grid
MSC2010: 65N08,76D05

1 Introduction

Our aim is the approximation on an unstructured mesh, of the weak solution to the steady-state Navier-Stokes equations, defined by

$$\begin{cases} \boldsymbol{u} \in E(\Omega),\ p \in L^2(\Omega) \text{ with } \displaystyle\int_\Omega p(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0, \\ \displaystyle\int_\Omega \nabla \boldsymbol{u}(\boldsymbol{x}) : \nabla \boldsymbol{v}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} + \mathrm{R}\int_\Omega (\boldsymbol{u}(\boldsymbol{x})\cdot\nabla)\boldsymbol{u}(x)\cdot\boldsymbol{v}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \\ \qquad\qquad - \displaystyle\int_\Omega p(\boldsymbol{x})\mathrm{div}\boldsymbol{v}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}(\boldsymbol{x})\cdot\boldsymbol{v}(\boldsymbol{x})\mathrm{d}\boldsymbol{x},\ \forall \boldsymbol{v} \in H_0^1(\Omega)^d, \end{cases} \tag{1}$$

where

E. Chénier and R. Eymard
Université Paris-Est, e-mail: eric.chenier@univ-mlv.fr, robert.eymard@univ-mlv.fr

R. Herbin
Université Aix-Marseille, e-mail: Raphaele.Herbin@latp.univ-mrs.fr

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$d \in \{2, 3\}$ denotes the space dimension,
Ω is an open polygonal bounded and connected subset of $\mathbb{R}^d$,
with Lipschitz-continuous boundary $\partial\Omega$,

$$\mathrm{R} \in [0, +\infty),\ \boldsymbol{f} \in L^2(\Omega)^d,$$
$$E(\Omega) := \{\boldsymbol{v} = (v^{(i)})_{i=1,\ldots,d} \in H_0^1(\Omega)^d, \mathrm{div}\boldsymbol{v} = 0 \text{ a.e. in } \Omega\},$$

and, for all $\boldsymbol{u}, \boldsymbol{v} \in H_0^1(\Omega)^d$ and for a.e. $\boldsymbol{x} \in \Omega$, $\nabla\boldsymbol{u}(\boldsymbol{x}) : \nabla\boldsymbol{v}(\boldsymbol{x}) = \sum_{i=1}^{d} \nabla u^{(i)}(\boldsymbol{x}) \cdot \nabla v^{(i)}(\boldsymbol{x})$. The approximation of Problem (1) may be performed with several schemes among which the MAC scheme: see e.g. [7] for a presentation of its implementation and [3–6] for its mathematical analysis; the MAC scheme is very popular, in particular because it is simple and needs no stabilisation procedure. Its main drawback is that it only holds on domains which can be gridded by rectangular conforming meshes, in the sense that no hanging node is permitted. This paper is devoted to the presentation of a simple way to extend this scheme to any geometry and to possibly refined meshes, while keeping simplicity and convergence properties. In Sect. 2, we first write a discrete variational formulation of the standard MAC scheme on the Stokes problem, which is (1) with $\mathrm{R} = 0$. Thanks to this formulation, we are able in Sect. 3 to extend this scheme to more complex geometries and to the Navier-Stokes equation (1). Section 3 proposes a numerical example on a non-rectangular domain, using local refinement along the boundary of the domain.

2 The standard MAC scheme for the Stokes equations

Let us consider in this section the standard MAC scheme for the approximation of the Stokes problem, that is (1) with $\mathrm{R} = 0$. We then consider the following case and notations, as depicted in Fig. 1. Let us consider the unit square: $\Omega =]0, 1[\times]0, 1[$, let N and M be two positive integers. With the notations of Fig.1, we denote by $\mathscr{M}$ the set of pressure grid cells:

$$\mathscr{M} = \left\{]x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}[\times]y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}[,\ 1 \le i \le N,\ 1 \le j \le M \right\},$$

and by $\mathscr{E} = \mathscr{E}^{(1)} \cup \mathscr{E}^{(2)}$ the set of the edges of the mesh, where $\mathscr{E}^{(1)}$ (resp. $\mathscr{E}^{(2)}$) is the set of vertical (resp. horizontal) edges, associated to the x (resp. y) component of the velocity. In order to define the normal velocity flux from one cell to a neighbouring one, we introduce, for any pair $\sigma, \sigma' \in \mathscr{E}^{(k)}$, $k = 1$ or 2, the transmissivity $\tau_{\sigma,\sigma'}$ between cell $K_\sigma^{(k)}$ and cell $K_{\sigma'}^{(k)}$:

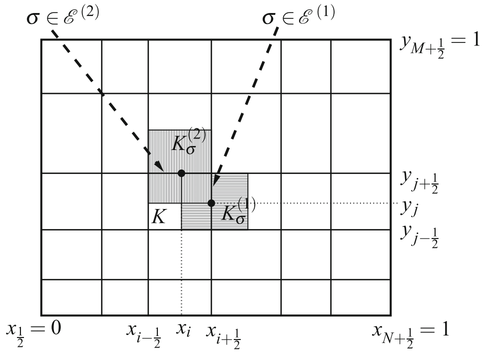

Fig. 1 Notations for the standard MAC scheme

$$\tau_{\sigma,\sigma'} = \frac{|\partial K_\sigma^{(k)} \cap \partial K_{\sigma'}^{(k)}|}{\mathrm{d}(\boldsymbol{x}_\sigma, \boldsymbol{x}_{\sigma'})}, \tag{2}$$

For instance, for a vertical edge $\sigma = \{x_{i-\frac{1}{2}}\}\times]y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}[\in \mathscr{E}^{(1)}$, one has:

$$\tau_{\sigma,\sigma'}^{(1)} = \begin{cases} \dfrac{y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}}{x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}} & \text{if } \sigma' = \{x_{i+\frac{1}{2}}\}\times]y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}[, \\ \dfrac{x_i - x_{i-1}}{y_{j+1} - y_j} & \text{if } \sigma' = \{x_{i-\frac{1}{2}}\}\times]y_{j+\frac{1}{2}}, y_{j+\frac{3}{2}}[. \end{cases} \tag{3}$$

Denoting by $(\boldsymbol{e}^{(k)})_{k=1,\dots,d}$ the canonical orthonormal basis of $\mathbb{R}^d$ and, for $K \in \mathscr{M}$, $\boldsymbol{n}_{K,\sigma}$ the unit normal vector to σ outward to K, the MAC scheme then reads:

$$\text{Find } (u_\sigma)_{\sigma\in\mathscr{E}} \subset \mathbb{R},\ (p_K)_{K\in\mathscr{M}} \subset \mathbb{R}\ ;\ \sum_{K\in\mathscr{M}} |K| p_K = 0,$$

$$\sum_{k=1}^{2} \sum_{\sigma\in\mathscr{E}_K^{(k)}} |\sigma| u_\sigma \boldsymbol{e}^{(k)} \cdot \boldsymbol{n}_{K,\sigma} = 0,\ \forall K \in \mathscr{M}, \tag{4a}$$

$$-\sum_{\sigma'\in\mathscr{E}^{(k)}} \tau_{\sigma,\sigma'}^{(k)} (u_{\sigma'} - u_\sigma) + |\sigma|(p_{L_\sigma} - p_{M_\sigma}) = \int_{K_\sigma^{(k)}} f^{(k)}(\boldsymbol{x})\mathrm{d}\boldsymbol{x},\ \forall \sigma \in \mathscr{E}^{(k)}, k = 1, 2, \tag{4b}$$

where L_σ and $M_\sigma \in \mathscr{M}$ are the two cells which share $\sigma \in \mathscr{E}^{(k)}$ as an edge, and such that $\boldsymbol{e}^{(k)}$ is oriented from L_σ and M_σ, and where the value of u_σ is set to 0 on all exterior edges.

In order to extend the MAC scheme, the idea is to rewrite (4a) and (4b) under a variational formulation. We first define $H_{\mathscr{M}}(\Omega)$ as the set of piecewise functions constant in $K \in \mathscr{M}$, and $H_{\mathscr{E}}^{(k)}(\Omega)$ as the set of piecewise functions which are constant in K_σ, for $\sigma \in \mathscr{E}^{(k)}$, and which are meant to approximate the kth component of the velocity. We finally denote by $H_{\mathscr{E}}(\Omega)$ the set of all $\boldsymbol{v} = (v^{(k)})_{k=1,\ldots,d}$ with $v^{(k)} \in H_{\mathscr{E}}^{(k)}(\Omega)$. We then define the discrete divergence by:

$$\mathrm{div}_K \boldsymbol{v} = \frac{1}{|K|} \sum_{\sigma \in \mathscr{E}_K} |\sigma| v_{K,\sigma}, \ \forall K \in \mathscr{M}, \ \forall \boldsymbol{v} \in H_{\mathscr{E}}(\Omega), \tag{5}$$

where, denoting by $\mathscr{E}_{\mathrm{int}}$ (resp. $\mathscr{E}_{\mathrm{ext}}$) the set of internal (resp. boundary) edges,

$$v_{K,\sigma} = \begin{cases} v_\sigma \boldsymbol{n}_\sigma \cdot \boldsymbol{n}_{K,\sigma} & \forall \sigma \in \mathscr{E}_K \cap \mathscr{E}_{\mathrm{int}}, \\ 0 & \forall \sigma \in \mathscr{E}_K \cap \mathscr{E}_{\mathrm{ext}}, \end{cases} \quad \forall K \in \mathscr{M}, \ \forall \boldsymbol{v} \in H_{\mathscr{E}}(\Omega), \tag{6}$$

where $\boldsymbol{n}_\sigma$ denotes the basis vector $\boldsymbol{e}$ to which σ is orthogonal. Using (5), we may define the following operator:

$$\mathrm{div}_{\mathscr{D}} \boldsymbol{v}(\boldsymbol{x}) = \mathrm{div}_K \boldsymbol{v}, \ \text{ for a.e. } \boldsymbol{x} \in K, \ \forall K \in \mathscr{M}, \ \forall \boldsymbol{v} \in H_{\mathscr{E}}(\Omega), \tag{7}$$

and remark that (4a) can be written

$$\mathrm{div}_{\mathscr{D}} \boldsymbol{u}(\boldsymbol{x}) = 0, \ \text{for a.e. } \boldsymbol{x} \in \Omega. \tag{8}$$

Next, for $k = 1, \ldots, d$, we define an inner product on the space $H_{\mathscr{E}}^{(k)}$:

$$\langle u, v \rangle_k = \sum_{\{\sigma,\sigma'\} \subset \mathscr{E}^{(k)}} \tau_{\sigma,\sigma'}^{(k)} (u_\sigma - u_{\sigma'})(v_\sigma - v_{\sigma'}), \ \forall u, v \in H_{\mathscr{E}}^{(k)}(\Omega); \tag{9}$$

this allows the definition of the following inner product on $H_{\mathscr{E}}(\Omega)$ which is expected to approximate $\int_\Omega \nabla \boldsymbol{u}(\boldsymbol{x}) : \nabla \boldsymbol{v}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}$:

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\mathscr{E}} = \sum_{k=1}^{d} \langle u^{(k)}, v^{(k)} \rangle_k, \ \forall \boldsymbol{u}, \boldsymbol{v} \in H_{\mathscr{E}}(\Omega). \tag{10}$$

We then obtain, multiplying (4b) by v_σ and summing on $k = 1, 2$ and $\sigma \in \mathscr{E}^{(k)}$,

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\mathscr{E}} - \int_\Omega p(\boldsymbol{x}) \mathrm{div}_{\mathscr{D}} \boldsymbol{v}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \ \ \forall \boldsymbol{v} \in H_{\mathscr{E}}(\Omega), \tag{11}$$

A discrete variational formulation of the MAC scheme (4) is therefore:

$$\text{Find } \boldsymbol{u} \in H_{\mathscr{E}}(\Omega) \text{ and } p \in H_{\mathscr{M}}(\Omega) \text{ s. t. } \sum_{K \in \mathscr{M}} |K| p_K = 0 \text{ and (8) and (11) hold.} \tag{12}$$

3 The extended MAC scheme for the Navier-Stokes equations

We extend the standard MAC scheme to cases where all internal edges (2D) or faces (3D) whose normal is parallel to a basis vector $\boldsymbol{e}^{(k)}$, such as the pressure grid depicted in Fig. 2 (left). Because of possibly hanging nodes, we may no longer define the velocity meshes by dual rectangles, but use instead the Voronoi cells associated with the barycentres of the edges $(\boldsymbol{x}_\sigma)_{\sigma\in\mathcal{E}}$; they are defined as follows:

$$K_\sigma^{(k)} = \{\boldsymbol{x} \in \Omega, \mathrm{d}(\boldsymbol{x},\boldsymbol{x}_\sigma) < \mathrm{d}(\boldsymbol{x},\boldsymbol{x}_{\sigma'}), \sigma' \in \mathcal{E}^{(k)} \setminus \{\sigma\}\},\ \forall \sigma \in \mathcal{E}^{(k)},$$

Note that in the case of a uniform rectangular mesh, the Voronoi cells thus defined are equal to the velocity cells defined in the previous section. However, this is no longer true if a non uniform mesh is used, even in the conforming case; indeed, in this latter case, the Voronoi cells $K_\sigma^{(k)}$ are again rectangles, but they are not equal to the rectangular cells $K_\sigma^{(k)}$ defined previously. In the case of hanging nodes, they are no longer rectangular, as can be seen in Fig. 2, where we depict the pressure mesh, the horizontal and vertical velocity grids.

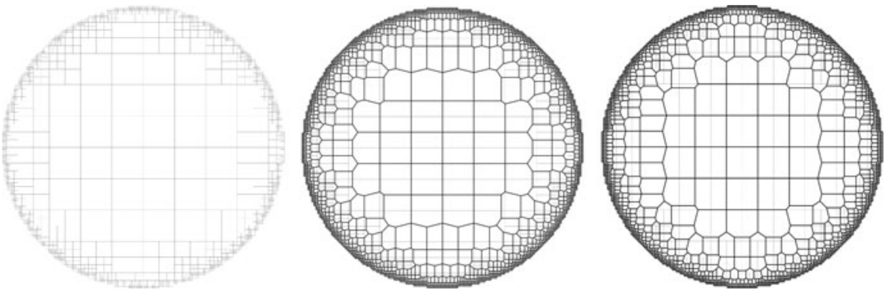

Fig. 2 The pressure and velocity grids

The diffusion term is again approximated by the discrete inner product defined by (9)-(10)-(2), but the expression of $\tau_{\sigma,\sigma'}$ given by (2) can no longer be written as in (3) for non rectangular cells. For Voronoï cells $K_\sigma^{(k)}$ and $K_{\sigma'}^{(k)}$ separated by a (dual) edge ε, such as those depicted in Fig. 3, one has

$$\tau_{\sigma,\sigma'} = \frac{|\varepsilon|}{d_\varepsilon} \tag{13}$$

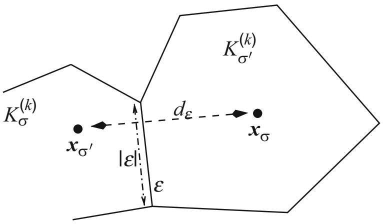

Fig. 3 Notations for a velocity cell

where $|\varepsilon|$ denotes the length of the edge ε shared by $K_\sigma^{(k)}$ and $K_{\sigma'}^{(k)}$, and $d_\varepsilon = d(\boldsymbol{x}_\sigma, \boldsymbol{x}_{\sigma'})$ the distance between the two cell centres $\boldsymbol{x}_\sigma$ and $\boldsymbol{x}_{\sigma'}$, which are also the barycentres of the edges σ and σ'. We can again define $H_{\mathcal{M}}(\Omega)$

as the set of piecewise functions constant on the pressure cells $K \in \mathscr{M}$, the set $H_{\mathscr{E}}^{(k)}(\Omega)$ of piecewise constant functions on the grid cells K_σ, for $\sigma \in \mathscr{E}_{\text{int}}^{(k)} \cup \mathscr{E}_{\text{ext}}$ which vanish on any grid cell K_σ for a boundary edge $\sigma \subset \partial\Omega$; this discrete set is the space of functions meant to approximate the k-th component of the velocity. We finally denote by $H_{\mathscr{E}}(\Omega)$ the set of all $\boldsymbol{v} = (v^{(k)})_{k=1,\ldots,d}$ with $v^{(k)} \in H_{\mathscr{E}}^{k}(\Omega)$. The extended MAC scheme for the Stokes equations (R = 0) is again (5)-(12), with the new definition (13) for $\tau_{\sigma,\sigma'}$.

In order to write this generalized scheme for the Navier-Stokes equations, we need to add the discretization of the nonlinear term $\int_\Omega (\boldsymbol{u}(\boldsymbol{x}) \cdot \nabla)\boldsymbol{u}(x) \cdot \boldsymbol{v}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$. For $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in H_{\mathscr{E}}(\Omega)$, we define the discrete nonlinear convection term

$$b_{\mathscr{E}}(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \sum_{K \in \mathscr{M}} \sum_{\substack{\sigma \in \mathscr{E}_K \\ \mathscr{M}_\sigma = \{K,L\}}} |\sigma| u_{K,\sigma} \frac{\boldsymbol{\Pi}_K \boldsymbol{v} + \boldsymbol{\Pi}_L \boldsymbol{v}}{2} \cdot \boldsymbol{\Pi}_K \boldsymbol{w},$$

where $u_{K,\sigma}$ is defined by (6) $\boldsymbol{\Pi}_K \boldsymbol{v}$ is a reconstruction of the full velocity on each pressure cell K defined by its components $(\boldsymbol{\Pi}_K \boldsymbol{v})_k, k = 1, \ldots, d$:

$$(\boldsymbol{\Pi}_K \boldsymbol{v})_k = \frac{1}{\sum_{\sigma \in \mathscr{E}_K \cap \mathscr{E}^{(k)}} |K_\sigma^{(k)}|} \sum_{\sigma \in \mathscr{E}_K \cap \mathscr{E}^{(k)}} |K_\sigma^{(k)}| v_\sigma.$$

The extended MAC scheme for the Navier-Stokes equation then reads:

$$\begin{cases} \text{Find } \boldsymbol{u} \in H_{\mathscr{E}}(\Omega) \text{ and } p \in H_{\mathscr{M}}(\Omega) \text{ s. t.} \displaystyle\sum_{K \in \mathscr{M}} |K| p_K = 0, \\ \operatorname{div}_{\mathscr{D}} \boldsymbol{u}(\boldsymbol{x}) = 0, \text{ for a.e. } \boldsymbol{x} \in \Omega. \\ \langle \boldsymbol{u}, \boldsymbol{v} \rangle_{\mathscr{E}} - \displaystyle\int_\Omega p(\boldsymbol{x}) \operatorname{div}_{\mathscr{D}} \boldsymbol{v}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} + \mathrm{R}\, b_{\mathscr{E}}(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = \displaystyle\int_\Omega \boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \forall \boldsymbol{v} \in H_{\mathscr{E}}(\Omega). \end{cases}$$

With this scheme, a control over the discrete kinetic energy can be obtained, which allows to prove some discrete H^1 estimates on the velocity. Then an L^2 estimate is proved for the discrete pressure, using the standard Necas lifting, which is particularly easy thanks to the staggered grids. The proof of convergence is then completed, considering the interpolation of regular test functions. Details may be found in [2].

4 Numerical example

We consider a problem where the continuous solution of the Navier–Stokes equations (1) with R = 1 is given by:

$$
\begin{aligned}
\bar{u}_1(x_1, x_2) &= 2\pi \sin^2(\pi x_1)\cos(\pi x_2)\sin(\pi x_2) \\
\bar{u}_2(x_1, x_2) &= -2\pi \cos(\pi x_1)\sin(\pi x_1)\sin^2(\pi x_2) \\
\bar{p}(x_1, x_2) &= \sin^2(\pi x_1)\sin^2(\pi x_2)
\end{aligned}
$$

in a circle with centre $(0, 0)$ and radius 0.45. We consider four meshes for the mass conservation $\mathscr{M}_j$, $j = 0, \ldots, 3$, defined in the following way:

1. a structured square 10×10 is given on the square $[0, 1] \times [0, 1]$,
2. for $i = 0, \ldots, 3$, let us split in 4 control volumes each grid block whose centre (x_1, x_2) satisfies

$$\sqrt{(x_1 - 0.5)^2 + (x_2 - 0.5)^2} \geq 0.45 - 0.25/2^i,$$

3. for $i = 0, \ldots, j$, let us split in 4 control volumes each grid block K,
4. get rid of all the control volumes K with centre (x_1, x_2) such that

$$\sqrt{(x_1 - 0.5)^2 + (x_2 - 0.5)^2} > 0.45.$$

Let us denote card$(\mathscr{M}_j)$ the number of control volumes of the mesh $\mathscr{M}_j$. We get that card$(\mathscr{M}_0) = 1604$, card$(\mathscr{M}_1) = 6416$, card$(\mathscr{M}_2) = 25592$ and card$(\mathscr{M}_3) = 102324$. The L^2 errors of unknowns u_1, u_2, p, respectively denoted by $e_2(u_1), e_2(u_2), e_2(p)$, are respectively computed in the Voronoi grids associated to the velocity components and in $\mathscr{M}_j$.

Left part of Fig. (4) shows the errors $\log 10(e_2(u_1))$ and $\log 10(e_2(p))$ with respect to $\log 10(1/\sqrt{\text{card}(\mathscr{M}_j)})$ for $j = 0, \ldots, 3$. On right part of Fig. (4) are

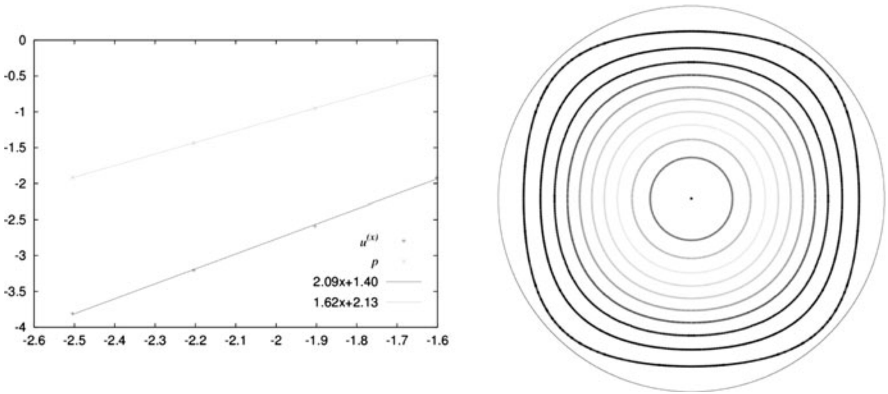

Fig. 4 Left: The L^2 error with respect to the number of control volumes. Right: Stream lines

plotted the stream lines for the finest mesh. The velocity components and the pressure are respectively shown in Figs. (5), (6) and (7). Although the velocity

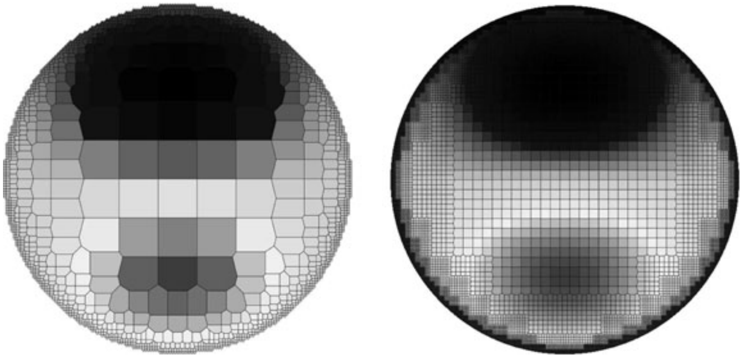

Fig. 5 Horizontal component of the velocity for $j = 0$ and $j = 2$

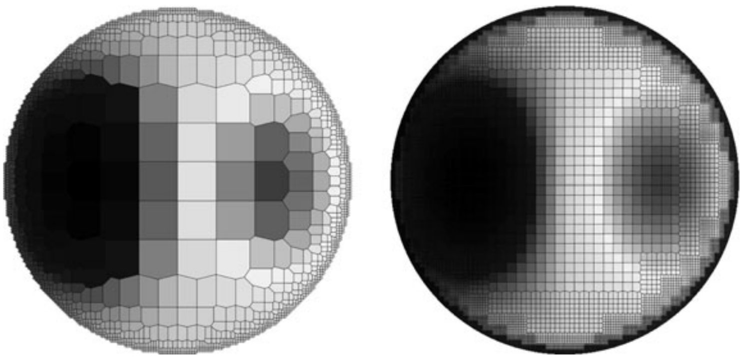

Fig. 6 Vertical component of the velocity for $j = 0$ for $j = 2$

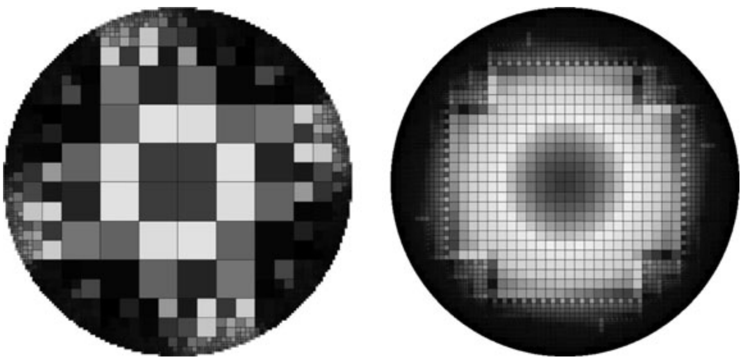

Fig. 7 Pressure for $j = 0$ and $j = 2$

fields are accurately computed on the coarsest mesh, the pressure fields show oscillations where neighbouring control volumes have contrasted sizes. However, these oscillations disappear while refining the mesh.

5 Conclusion

The generalised MAC scheme seems very efficient on meshes which are parallel to the axes. In particular, the scheme keeps a five-point stencil on all non-refined regions. It can also be extended to more general non-structured grids. However for these latter grids, the stencil may become large, which can be a problem when solving the linear systems in the Newton iteration.

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The paper is in final form and no similar paper has been or is being submitted elsewhere.