Some Abstract Error Estimates of a Finite Volume Scheme for the Wave Equation on General Nonconforming Multidimensional Spatial Meshes

Abdallah Bradji

Abstract A general class of nonconforming meshes has been recently studied for sationary anisotropic heterogeneous diffusion problems, see [2]. The aim of this contribution is to deal with error estimates, using this new class of meshes, for the wave equation. We present an implicit time scheme to approximate the wave equation. We prove that, when the discrete flux is calculated using a stabilized discrete gradient, the convergence order is $h_{\mathscr{D}} + k$, where $h_{\mathscr{D}}$ (resp. k) is the mesh size of the spatial (resp. time) discretization. This estimate is valid for discrete norms $\mathbb{L}^\infty(0,T;H_0^1(\Omega))$ and $\mathscr{W}^{1,\infty}(0,T;L^2(\Omega))$ under the regularity assumption $u \in \mathscr{C}^3([0,T];\mathscr{C}^2(\overline{\Omega}))$ for the exact solution u. These error estimates are useful because they allow to obtain approximations to the exact solution and its first derivatives of order $h_{\mathscr{D}} + k$.

Keywords second order hyperbolic equation, wave equation, non–conforming grid, SUSHI scheme, implicit scheme, discrete gradient
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1 Motivation and aim of this paper

We consider the wave equation, as a model for second order hyperbolic equations:

$$u_{tt}(x,t) - \Delta u(x,t) = f(x,t), \;\; (x,t) \in \Omega \times (0,T), \tag{1}$$

where Ω is an open polygonal bounded subset in $\mathbb{R}^d$, $T > 0$, and f is a given function.
An initial condition is given by: for given functions u^0 and u^1 defined on Ω

Abdallah Bradji
Department of Mathematics, university of Annaba–Algeria, e-mail: bradji@cmi.univ-mrs.fr

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$$u(x,0) = u^0(x) \text{ and } u_t(x,0) = u^1(x) \; x \in \Omega, \tag{2}$$

Homogeneous Dirichlet boundary conditions are given by

$$u(x,t) = 0, \; (x,t) \in \partial\Omega \times (0,T). \tag{3}$$

2 Definition of the scheme

The discretization of Ω is performed using the mesh $\mathscr{D} = (\mathscr{M}, \mathscr{E}, \mathscr{P})$ described in [2, Definition 2.1] which we recall here for the sake of completeness.

Definition 1. (Definition of the spatial mesh, cf. [2, Definition 2.1, Page 1012]) Let Ω be a polyhedral open bounded subset of $\mathbb{R}^d$, where $d \in \mathbb{N} \setminus \{0\}$, and $\partial\Omega = \overline{\Omega} \setminus \Omega$ its boundary. A discretisation of Ω, denoted by $\mathscr{D}$, is defined as the triplet $\mathscr{D} = (\mathscr{M}, \mathscr{E}, \mathscr{P})$, where:

1. $\mathscr{M}$ is a finite family of non empty connected open disjoint subsets of Ω (the "control volumes") such that $\overline{\Omega} = \cup_{K\in\mathscr{M}}\overline{K}$. For any $K \in \mathscr{M}$, let $\partial K = \overline{K} \setminus K$ be the boundary of K; let $\mathrm{m}(K) > 0$ denote the measure of K and h_K denote the diameter of K.
2. $\mathscr{E}$ is a finite family of disjoint subsets of $\overline{\Omega}$ (the "edges" of the mesh), such that, for all $\sigma \in \mathscr{E}$, σ is a non empty open subset of a hyperplane of $\mathbb{R}^d$, whose $(d-1)$–dimensional measure is strictly positive. We also assume that, for all $K \in \mathscr{M}$, there exists a subset $\mathscr{E}_K$ of $\mathscr{E}$ such that $\partial K = \cup_{\sigma\in\mathscr{E}_K}\overline{\sigma}$. For any $\sigma \in \mathscr{E}$, we denote by $\mathscr{M}_\sigma = \{K;\; \sigma \in \mathscr{E}_K\}$. We then assume that, for any $\sigma \in \mathscr{E}$, either $\mathscr{M}_\sigma$ has exactly one element and then $\sigma \subset \partial\Omega$ (the set of these interfaces, called boundary interfaces, denoted by $\mathscr{E}_{\mathrm{ext}}$) or $\mathscr{M}_\sigma$ has exactly two elements (the set of these interfaces, called interior interfaces, denoted by $\mathscr{E}_{\mathrm{int}}$). For all $\sigma \in \mathscr{E}$, we denote by x_σ the barycentre of σ. For all $K \in \mathscr{M}$ and $\sigma \in \mathscr{E}$, we denote by $\mathbf{n}_{K,\sigma}$ the unit vector normal to σ outward to K.
3. $\mathscr{P}$ is a family of points of Ω indexed by $\mathscr{M}$, denoted by $\mathscr{P} = (x_K)_{K\in\mathscr{M}}$, such that for all $K \in \mathscr{M}$, $x_K \in K$ and K is assumed to be x_K–star-shaped, which means that for all $x \in K$, the property $[x_K, x] \subset K$ holds. Denoting by $d_{K,\sigma}$ the Euclidean distance between x_K and the hyperplane including σ, one assumes that $d_{K,\sigma} > 0$. We then denote by $\mathscr{D}_{K,\sigma}$ the cone with vertex x_K and basis σ.

The time discretization is performed with a constant time step $k = \frac{T}{N+1}$, where $N \in \mathbb{N}^\star$, and we shall denote $t_n = nk$, for $n \in [\![0, N+1]\!]$. Throughout this paper, the letter C stands for a positive constant independent of the parameters of the space and time discretizations and its values may be different in different appearance.

We define the space $\mathscr{X}_\mathscr{D}$ as the set of all $((v_K)_{K\in\mathscr{M}}, (v_\sigma)_{\sigma\in\mathscr{E}})$, and $\mathscr{X}_{\mathscr{D},0} \subset \mathscr{X}_\mathscr{D}$ is the set of all $v \in \mathscr{X}_\mathscr{D}$ such that $v_\sigma = 0$ for all $\sigma \in \mathscr{E}_{\mathrm{ext}}$. Let $H_\mathscr{M}(\Omega) \subset \mathbb{L}^2(\Omega)$ be the space of piecewise constant functions on the control volumes of the mesh $\mathscr{M}$. For all $v \in \mathscr{X}_\mathscr{D}$, we denote by $\Pi_\mathscr{M} v \in H_\mathscr{M}(\Omega)$ the function defined by $\Pi_\mathscr{M} v(x) = v_K$, for a.e. $x \in K$, for all $K \in \mathscr{M}$.

For all $\varphi \in \mathscr{C}(\Omega)$, we define $\mathscr{P}_{\mathscr{D}}\varphi = ((\varphi(x_K))_{K\in\mathscr{M}}, (\varphi(x_\sigma))_{\sigma\in\mathscr{E}}) \in \mathscr{X}_{\mathscr{D}}$. We denote by $\mathscr{P}_{\mathscr{M}}\varphi \in H_{\mathscr{M}}(\Omega)$ the function defined by $\mathscr{P}_{\mathscr{M}}\varphi(x) = \varphi(x_K)$, for a.e. $x \in K$, for all $K \in \mathscr{M}$.
In order to analyze the convergence, we need to consider the size of the discretization $\mathscr{D}$ defined by $h_{\mathscr{D}} = \sup\{\mathrm{diam(K)}, K \in \mathscr{M}\}$ and the regularity of the mesh given by $\theta_{\mathscr{D}} = \max\left(\max_{\sigma\in\mathscr{E}_{\mathrm{int}}, K,L\in\mathscr{M}} \frac{d_{K,\sigma}}{d_{L,\sigma}}, \max_{K\in\mathscr{M},\sigma\in\mathscr{E}_K} \frac{h_K}{d_{K,\sigma}}\right)$. The scheme we want to consider in this note (A general framework will be detailed in a future paper.) is based on the use of the discrete gradient given in [2]. For $u \in \mathscr{X}_{\mathscr{D}}$, we define, for all $K \in \mathscr{M}$

$$\nabla_{\mathscr{D}} u(x) = \nabla_{K,\sigma} u, \ \text{ a. e. } x \in \mathscr{D}_{K,\sigma}, \tag{4}$$

where $\mathscr{D}_{K,\sigma}$ is the cone with vertex x_K and basis σ and

$$\nabla_{K,\sigma} u = \nabla_K u + \left(\frac{\sqrt{d}}{d_{K,\sigma}}\left(u_\sigma - u_K - \nabla_K u \cdot (x_\sigma - x_K)\right)\right)\mathbf{n}_{K,\sigma}, \tag{5}$$

where $\nabla_K u = \dfrac{1}{\mathrm{m}(K)} \sum_{\sigma\in\mathscr{E}_K} \mathrm{m}(\sigma)\,(u_\sigma - u_K)\,\mathbf{n}_{K,\sigma}$ and d is the space dimension.
We define the finite volume approximation for (1)–(3) as $\left(u^n_{\mathscr{D}}\right)_{n=0}^{N+1} \in \mathscr{X}^{N+2}_{\mathscr{D},0}$ with $u^n_{\mathscr{D}} = \left(\left(u^n_K\right)_{K\in\mathscr{M}}, \left(u^n_\sigma\right)_{\sigma\in\mathscr{E}}\right)$, for all $n \in [\![0, N+1]\!]$ and

1. discretization of the initial conditions (2):

$$\langle u^0_{\mathscr{D}}, v\rangle_F = -\left(\Delta u^0, \Pi_{\mathscr{M}} v\right)_{\mathbb{L}^2(\Omega)}, \ \forall v \in \mathscr{X}_{\mathscr{D},0}, \tag{6}$$

   and

$$\langle \frac{u^1_{\mathscr{D}} - u^0_{\mathscr{D}}}{k}, v\rangle_F = -\left(\Delta u^1, \Pi_{\mathscr{M}} v\right)_{\mathbb{L}^2(\Omega)}, \ \forall v \in \mathscr{X}_{\mathscr{D},0}, \tag{7}$$

2. discretization of equation (1): for any $n \in [\![1, N]\!]$, $v \in \mathscr{X}_{\mathscr{D},0}$

$$\left(\Pi_{\mathscr{M}}\, \partial^2 u^{n+1}_{\mathscr{D}}, \Pi_{\mathscr{M}} v\right)_{\mathbb{L}^2(\Omega)} + \langle u^{n+1}_{\mathscr{D}}, v\rangle_F = \sum_{K\in\mathscr{M}} \mathrm{m}(K) f^n_K v_K, \tag{8}$$

   where

$$\langle u, v\rangle_F = \int_\Omega \nabla_{\mathscr{D}} u(x)\cdot\nabla_{\mathscr{D}} v(x)dx, \ \forall u, v \in \mathscr{X}_{\mathscr{D}}, \tag{9}$$

$$\partial^2 v^{n+1} = \frac{v^{n+1} - 2v^n + v^{n-1}}{k^2}, \ \forall n \in [\![1, N]\!], \tag{10}$$

$$f^n_K = \frac{1}{k\mathrm{m}(K)} \int_{t_n}^{t_{n+1}} \int_K f(x, t) d\, x\, dt, \tag{11}$$

   and $(\cdot,\cdot)_{\mathbb{L}^2(\Omega)}$ denotes the $\mathbb{L}^2$ inner product.

The main result of the present contribution is the following theorem.

Theorem 1. *(Error estimates for the finite volume scheme (6)–(11)) Let Ω be a polyhedral open bounded subset of $\mathbb{R}^d$, where $d \in \mathbb{N} \setminus \{0\}$, and $\partial\Omega = \overline{\Omega} \setminus \Omega$ its boundary. Assume that the solution (weak) of (1)–(3) satisfies $u \in \mathscr{C}^3([0,T];\mathscr{C}^2(\overline{\Omega}))$. Let $k = \frac{T}{N+1}$, with $N \in \mathbb{N}^\star$, and denote by $t_n = nk$, for $n \in [\![0, N+1]\!]$. Let $\mathscr{D} = (\mathscr{M}, \mathscr{E}, \mathscr{P})$ be a discretization in the sense of [2, Definition 2.1]. Assume that $\theta_{\mathscr{D}}$ satisfies $\theta \geq \theta_{\mathscr{D}}$.*
Then there exists a unique solution $\left(u_{\mathscr{D}}^n\right)_{n=0}^{N+1} \in \mathscr{X}_{\mathscr{D},\mathscr{B}}^{N+2}$ for problem (6)–(11).
For each $n \in [\![0, N+1]\!]$, let us define the error $e_{\mathscr{M}}^n \in H_{\mathscr{M}}(\Omega)$ by:

$$e_{\mathscr{M}}^n = \mathscr{P}_{\mathscr{M}}\, u(\cdot, t_n) - \Pi_{\mathscr{M}}\, u_{\mathscr{D}}^n. \tag{12}$$

Then, the following error estimates hold

- *discrete $\mathbb{L}^\infty(0,T; H_0^1(\Omega))$–estimate: for all $n \in [\![0, N+1]\!]$*

$$\| e_{\mathscr{M}}^n \|_{1,2,\mathscr{M}} \leq C(k + h_{\mathscr{D}}) \| u\|_{\mathscr{C}^3([0,T];\, \mathscr{C}^2(\overline{\Omega}))}. \tag{13}$$

- *discrete $\mathscr{W}^{1,\infty}(0,T;\mathbb{L}^2(\Omega))$–estimate: for all $n \in [\![1, N+1]\!]$*

$$\| \partial^1 e_{\mathscr{M}}^n \|_{\mathbb{L}^2(\Omega)} \leq C(k + h_{\mathscr{D}}) \| u\|_{\mathscr{C}^3([0,T];\, \mathscr{C}^2(\overline{\Omega}))}, \tag{14}$$

  where $\partial^1 v^n = \dfrac{1}{k}\left(v^n - v^{n-1}\right)$.
- *error estimate in the gradient approximation: for all $n \in [\![0, N+1]\!]$*

$$\|\nabla_{\mathscr{D}}\, u_{\mathscr{D}}^n - \nabla\, u(\cdot, t_n)\|_{\mathbb{L}^2(\Omega)} \leq C(k + h_{\mathscr{D}}) \| u\|_{\mathscr{C}^3([0,T];\, \mathscr{C}^2(\overline{\Omega}))}. \tag{15}$$

The following lemma will help us to prove Theorem 1

Lemma 1. *Let Ω be a polyhedral open bounded subset of $\mathbb{R}^d$, where $d \in \mathbb{N} \setminus \{0\}$, and $\partial\Omega = \overline{\Omega} \setminus \Omega$ its boundary. Let $k = \frac{T}{N+1}$, with $N \in \mathbb{N}^\star$, and denote by $t_n = nk$, for $n \in [\![0, N+1]\!]$. Let $\mathscr{D} = (\mathscr{M}, \mathscr{E}, \mathscr{P})$ be a discretization in the sense of [2, Definition 2.1]. Assume that $\theta_{\mathscr{D}}$ satisfies $\theta \geq \theta_{\mathscr{D}}$. Assume in addition that there exists $\left(\eta_{\mathscr{D}}^n\right)_{n=0}^{N+1} \in \mathscr{X}_{\mathscr{D}}^{N+2}$ such that for any $n \in [\![1, N]\!]$, for all $v \in \mathscr{X}_{\mathscr{D}}$*

$$\left(\Pi_{\mathscr{M}}\, \partial^2\, \eta_{\mathscr{D}}^{n+1}, \Pi_{\mathscr{M}}\, v\right)_{\mathbb{L}^2(\Omega)} + \langle\, \eta_{\mathscr{D}}^{n+1}, v\rangle_F = \sum_{K\in\mathscr{M}} \mathrm{m}(K) \mathscr{S}_K^n v_K, \tag{16}$$

where $\mathscr{S}_K^n \in \mathbb{R}$, for all $n \in [\![1, N]\!]$ and for all $K \in \mathscr{M}$.
Then the following estimate holds, for all $j \in [\![1, N]\!]$.

$$\begin{aligned}&\|\Pi_{\mathscr{M}}\, \partial^1\, \eta_{\mathscr{D}}^{j+1}\|_{\mathbb{L}^2(\Omega)}^2 + C\,|\, \eta_{\mathscr{D}}^{j+1}|_{\mathscr{X}}^2\\ &\leq C\left(\|\Pi_{\mathscr{M}}\, \partial^1\, \eta_{\mathscr{D}}^1\|_{\mathbb{L}^2(\Omega)}^2 + |\eta_{\mathscr{D}}^1|_{\mathscr{X}}^2 + (\mathscr{S})^2\right),\end{aligned} \tag{17}$$

where

$$\mathscr{S} = \max\left\{\left(\sum_{K\in\mathscr{M}} \mathrm{m}(K)\left(\mathscr{S}_K^n\right)^2\right)^{\frac{1}{2}},\ n \in [\![1, N]\!]\right\}. \tag{18}$$

Proof. Taking $v = \partial^1 \eta_{\mathscr{D}}^{n+1}$ in (16) and summing the result over $n \in [\![1, j]\!]$, where $j \in [\![1, N]\!]$, we get

$$\sum_{n=1}^{j}\left(\Pi_{\mathscr{M}}\,\partial^2 \eta_{\mathscr{D}}^{n+1}, \Pi_{\mathscr{M}}\,\partial^1 \eta_{\mathscr{D}}^{n+1}\right)_{\mathbb{L}^2(\Omega)} + \sum_{n=1}^{j}\langle \eta_{\mathscr{D}}^{n+1}, \partial^1 \eta_{\mathscr{D}}^{n+1}\rangle_F$$
$$= \sum_{n=1}^{j}\sum_{K\in\mathscr{M}} \mathrm{m}(K)\mathscr{S}_K^n \partial^1 \eta_K^{n+1}. \tag{19}$$

We need the following two rules

$$\left(\Pi_{\mathscr{M}}\,\partial^2 \eta_{\mathscr{D}}^{n+1}, \Pi_{\mathscr{M}}\,\partial^1 \eta_{\mathscr{D}}^{n+1}\right)_{\mathbb{L}^2(\Omega)} = \frac{1}{2k}\| \alpha_{\mathscr{D}}^{n+1} - \alpha_{\mathscr{D}}^{n}\|^2_{\mathbb{L}^2(\Omega)}$$
$$+\frac{1}{2k}\left(\|\alpha_{\mathscr{D}}^{n+1}\|^2_{\mathbb{L}^2(\Omega)} - \|\alpha_{\mathscr{D}}^{n}\|^2_{\mathbb{L}^2(\Omega)}\right), \tag{20}$$

where $\alpha_{\mathscr{D}}^n = \Pi_{\mathscr{M}}\,\partial^1 \eta_{\mathscr{D}}^n$ and

$$\langle \eta_{\mathscr{D}}^{n+1}, \partial^1 \eta_{\mathscr{D}}^{n+1}\rangle_F = \frac{1}{2k}\langle \eta_{\mathscr{D}}^{n+1} - \eta_{\mathscr{D}}^{n}, \eta_{\mathscr{D}}^{n+1} - \eta_{\mathscr{D}}^{n}\rangle_F$$
$$+\frac{1}{2k}\left\{\langle \eta_{\mathscr{D}}^{n+1}, \eta_{\mathscr{D}}^{n+1}\rangle_F - \langle \eta_{\mathscr{D}}^{n}, \eta_{\mathscr{D}}^{n}\rangle_F\right\}. \tag{21}$$

Identities (20)–(21) yield

$$\sum_{n=1}^{j}\left(\Pi_{\mathscr{M}}\,\partial^2 \eta_{\mathscr{D}}^{n+1}, \Pi_{\mathscr{M}}\,\partial^1 \eta_{\mathscr{D}}^{n+1}\right)_{\mathbb{L}^2(\Omega)} + \sum_{n=1}^{j}\langle \eta_{\mathscr{D}}^{n+1}, \partial^1 \eta_{\mathscr{D}}^{n+1}\rangle_F$$
$$\geq \frac{1}{2k}\left(\| \alpha_{\mathscr{D}}^{j+1}\|^2_{\mathbb{L}^2(\Omega)} + \langle \eta_{\mathscr{D}}^{j+1}, \eta_{\mathscr{D}}^{j+1}\rangle_F\right) - \frac{1}{2k}\left(\| \alpha_{\mathscr{D}}^{1}\|^2_{\mathbb{L}^2(\Omega)} + \langle \eta_{\mathscr{D}}^{1}, \eta_{\mathscr{D}}^{1}\rangle_F\right).$$

This with (19) and [2, Lemma 4.2] implies

$$\frac{1}{2k}\left(\|\alpha^{j+1}\|^2_{\mathbb{L}^2(\Omega)} + C|\,\eta_{\mathscr{D}}^{j+1}|^2_{\mathscr{X}}\right) \leq \sum_{n=1}^{j}\sum_{K\in\mathscr{M}} \mathrm{m}(K)\mathscr{S}_K^n \partial^1 \eta_K^{n+1}$$
$$+\frac{1}{2k}\left(\|\alpha_{\mathscr{D}}^{1}\|^2_{\mathbb{L}^2(\Omega)} + C|\eta_{\mathscr{D}}^{1}|^2_{\mathscr{X}}\right). \tag{22}$$

Multiplying both sides of the previous inequality by $2k$ and using the Cauchy Schwarz inequality, we get

$$\|\varPi_{\mathscr{M}}\, \partial^1\, \eta_{\mathscr{D}}^{j+1}\|_{\mathbb{L}^2(\varOmega)}^2 + C\,|\,\eta_{\mathscr{D}}^{j+1}|_{\mathscr{X}}^2 \leq 2k\mathscr{S}\sum_{n=1}^{j}\|\varPi_{\mathscr{M}}\,\partial^1\,\eta_{\mathscr{D}}^{n+1}\|_{\mathbb{L}^2(\varOmega)} + \|\varPi_{\mathscr{M}}\,\partial^1\,\eta_{\mathscr{D}}^1\|_{\mathbb{L}^2(\varOmega)}^2 + C|\eta_{\mathscr{D}}^1|_{\mathscr{X}}^2, \quad (23)$$

where $\mathscr{S}$ is given by (18).
This with the inequality $ab \leq \frac{T}{k}a^2 + \frac{k}{T}b^2$, (23) implies, for all $j \in [\![1, N]\!]$

$$\|\varPi_{\mathscr{M}}\, \partial^1\, \eta_{\mathscr{D}}^{j+1}\|_{\mathbb{L}^2(\varOmega)}^2 + C\,|\,\eta_{\mathscr{D}}^{j+1}|_{\mathscr{X}}^2 \leq \frac{2k}{T}\sum_{n=2}^{j}\left(\|\varPi_{\mathscr{M}}\,\partial^1\,\eta_{\mathscr{D}}^{n}\|_{\mathbb{L}^2(\varOmega)}^2 + C\,|\,\eta_{\mathscr{D}}^{n}|_{\mathscr{X}}^2\right)$$
$$+2\|\varPi_{\mathscr{M}}\,\partial^1\,\eta_{\mathscr{D}}^1\|_{\mathbb{L}^2(\varOmega)}^2 + C|\eta_{\mathscr{D}}^1|_{\mathscr{X}}^2 + 8T^2\,(\mathscr{S})^2\,. \quad (24)$$

Using the discrete version of the Gronwall's Lemma, (24) implies estimate (17).

Sketch of the proof of Theorem 1: The uniqueness of $\left(u_{\mathscr{D}}^n\right)_{n\in[\![0,N+1]\!]}$ satisfying (6)–(11) can be deduced from the [2, Lemma 4.2]. As usual, we can use this uniqueness to prove the existence. To prove (13)–(15), we compare the solution $\left(u_{\mathscr{D}}^n\right)_{n\in[\![0,N+1]\!]}$ satisfying (6)–(11) with the solution (it exists and it is unique thanks to [2, Lemma 4.2]): for any $n \in [\![0, N+1]\!]$, find $\bar{u}_{\mathscr{D}}^n \in \mathscr{X}_{\mathscr{D},0}$ such that, see (9)

$$\langle\, \bar{u}_{\mathscr{D}}^n, v\rangle_F = -\sum_{K\in\mathscr{M}} v_K \int_K \varDelta\, u(x,t_n)dx, \;\; \forall\, v \in \mathscr{X}_{\mathscr{D},0}. \quad (25)$$

Taking $n = 0$ in (25), using the fact that $u(\cdot,0) = u^0(\cdot)$, and comparing this with (6), we get the following property which will be used below

$$\bar{u}_{\mathscr{D}}^0 = u_{\mathscr{D}}^0. \quad (26)$$

One remarks that the solution of (25) is the same one of [1, (12)], one can use error estimates [1, (13), (15), and (16)] as error estimates for the solution of (25).
Writing (25) in the step $n+1$ and substracting the result from (8) to get

$$\left(\varPi_{\mathscr{M}}\,\partial^2\,\eta_{\mathscr{D}}^{n+1}, \varPi_{\mathscr{M}}\,v\right)_{\mathbb{L}^2(\varOmega)} + \langle\,\eta_{\mathscr{D}}^{n+1}, v\rangle_F = \sum_{K\in\mathscr{M}} \mathrm{m}(K)\mathscr{S}_K^n v_K, \quad (27)$$

where $\eta_{\mathscr{D}}^n = u_{\mathscr{D}}^n - \bar{u}_{\mathscr{D}}^n$, for all $n \in [\![0, N+1]\!]$ and

$$\mathscr{S}_K^n = \frac{1}{k\mathrm{m}(K)}\int_{t_n}^{t_{n+1}}\int_K f(x,t)d\,x\,dt + \frac{1}{\mathrm{m}(K)}\int_K \varDelta\,u(x,t_{n+1})dx - \partial^2\bar{u}_K^{n+1}. \quad (28)$$

Equation (27) with Lemma 1 implies that, for all $n \in [\![1, N]\!]$

$$
\begin{aligned}
&\|\Pi_{\mathscr{M}}\, \partial^1\, \eta_{\mathscr{D}}^{n+1}\|_{\mathbb{L}^2(\Omega)}^2 + C\,|\,\eta_{\mathscr{D}}^{n+1}|_{\mathscr{X}}^2 \\
&\leq\; C\left(\|\Pi_{\mathscr{M}}\, \partial^1\, \eta_{\mathscr{D}}^{1}\|_{\mathbb{L}^2(\Omega)}^2 + |\eta_{\mathscr{D}}^{1}|_{\mathscr{X}}^2 + (\mathscr{S})^2\right). \qquad (29)
\end{aligned}
$$

To estimate the terms on the right hand side of the previous inequality, we consider

$$
\xi_{\mathscr{D}}^n = \bar{u}_{\mathscr{D}}^n - \mathscr{P}_{\mathscr{D}}\, u(\cdot, t_n),\;\; \forall\, n \in [\![0, N+1]\!]. \qquad (30)
$$

It is useful to remark that (recall that $\eta_{\mathscr{D}}^n = u_{\mathscr{D}}^n - \bar{u}_{\mathscr{D}}^n$)

$$
u_{\mathscr{D}}^n - \mathscr{P}_{\mathscr{D}}\, u(\cdot, t_n) = \eta_{\mathscr{D}}^n + \xi_{\mathscr{D}}^n. \qquad (31)
$$

1. *Estimate of* $\|\Pi_{\mathscr{M}}\, \partial^1\, \eta_{\mathscr{D}}^{1}\|_{\mathbb{L}^2(\Omega)}$: using (31), we get (recall that $u_t(\cdot, 0) = u^1(\cdot)$)

$$
\|\Pi_{\mathscr{M}}\, \partial^1\, \eta_{\mathscr{D}}^{1}\|_{\mathbb{L}^2(\Omega)} \leq \sum_{i=1}^{4} \mathbb{T}_i, \qquad (32)
$$

where

$$
\mathbb{T}_1 = \|\Pi_{\mathscr{M}} \partial^1 \xi_{\mathscr{D}}^1\|_{\mathbb{L}^2(\Omega)},\;\; \mathbb{T}_2 = \|\Pi_{\mathscr{M}}\, \partial^1\, u_{\mathscr{D}}^1 - u^1\|_{\mathbb{L}^2(\Omega)},
$$

$\mathbb{T}_3 = \|\, u_t(\cdot, 0) - \partial^1\, u(\cdot, t_1)\|_{\mathbb{L}^2(\Omega)}$, and $\mathbb{T}_4 = \|\, \partial^1\, u(\cdot, t_1) - \mathscr{P}_{\mathscr{M}} \partial^1\, u(\cdot, t_1)\|_{\mathbb{L}^2(\Omega)}$.
Estimate [1, (15)], when $j = 1$, with (30) leads to

$$
\mathbb{T}_1 \leq\; C\, h_{\mathscr{D}} \|\, u\|_{\mathscr{C}^1([0,T];\, \mathscr{C}^2(\overline{\Omega}))}. \qquad (33)
$$

Equation (7) can be written as

$$
\langle\, \partial^1\, u_{\mathscr{D}}^1, v\rangle_F = -\left(\,\Delta\, u^1, \Pi_{\mathscr{M}}\, v\right)_{\mathbb{L}^2(\Omega)},\;\; \forall\, v \in \mathscr{X}_{\mathscr{D},0}. \qquad (34)
$$

This with [2, (4.25)] and the triangle inequality implies that

$$
\mathbb{T}_i \;\leq\; C\, (k + h_{\mathscr{D}})\|\, u\|_{\mathscr{C}^1([0,T];\, \mathscr{C}^2(\overline{\Omega}))},\;\; \forall\, i \in [\![2, 4]\!]. \qquad (35)
$$

Thanks to (32), (33), and (35), we have

$$
\|\Pi_{\mathscr{M}}\, \partial^1\, \eta_{\mathscr{D}}^{1}\|_{\mathbb{L}^2(\Omega)} \leq\; C\, (k + h_{\mathscr{D}})\|\, u\|_{\mathscr{C}^1([0,T];\, \mathscr{C}^2(\overline{\Omega}))}. \qquad (36)
$$

2. *Estimate of* $|\eta_{\mathscr{D}}^{1}|_{\mathscr{X}}$: let us first remark that thanks to (6) and (7), we have

$$
\langle\, u_{\mathscr{D}}^1, v\rangle_F = -\left(\,\Delta\,(u^0 + ku^1), \Pi_{\mathscr{M}}\, v\right)_{\mathbb{L}^2(\Omega)},\;\; \forall\, v \in \mathscr{X}_{\mathscr{D},0}. \qquad (37)
$$

In order to bound $|\eta^1_{\mathscr{D}}|_{\mathscr{X}} = |u^1_{\mathscr{D}} - \bar{u}^1_{\mathscr{D}}|_{\mathscr{X}}$, we use the triangle inequality to get

$$|\eta^1_{\mathscr{D}}|_{\mathscr{X}} \leq |u^1_{\mathscr{D}} - \mathscr{P}_{\mathscr{D}}(u^0 + ku^1)|_{\mathscr{X}} + |\mathscr{P}_{\mathscr{D}}(u^0 + ku^1) - \mathscr{P}_{\mathscr{D}} u(\cdot, t_1)|_{\mathscr{X}} + |\mathscr{P}_{\mathscr{D}} u(\cdot, t_1) - \bar{u}^1_{\mathscr{D}}|_{\mathscr{X}}. \quad (38)$$

This with the proof of [2, (4.29)] and suitable Taylor expansions, we get

$$|\eta^1_{\mathscr{D}}|_{\mathscr{X}} \leq C\,(k + h_{\mathscr{D}})\|u\|_{\mathscr{C}^2([0,T];\,\mathscr{C}^2(\overline{\Omega}))}. \quad (39)$$

3. *Estimate of* $\mathscr{S}$: substituting f by $u_{tt} - \Delta u$, see (1), in the expansion of $\mathscr{S}^n_K$, we get

$$\mathscr{S}^n_K = \frac{1}{k\mathrm{m}(K)} \int_{t_n}^{t_{n+1}} \int_K u_{tt}(x,t)d\,x\,dt - \frac{1}{k\mathrm{m}(K)} \int_{t_n}^{t_{n+1}} \int_K \Delta(x,t)d\,x\,dt + \frac{1}{\mathrm{m}(K)} \int_K \Delta u(x, t_{n+1})dx - \partial^2 \bar{u}^{n+1}_K. \quad (40)$$

Thanks to the Taylor expansion and [1, (15)], when $j = 2$, we have

$$\mathscr{S} \leq C\,(k + h_{\mathscr{D}})\|u\|_{\mathscr{C}^3([0,T];\,\mathscr{C}^2(\overline{\Omega}))}. \quad (41)$$

Gathering now (29), (36), (39), and (41) yields, for all $n \in [\![2, N+1]\!]$

$$\|\Pi_{\mathscr{M}}\,\partial^1\,\eta^n_{\mathscr{D}}\|_{\mathbb{L}^2(\Omega)} \leq C\,(k + h_{\mathscr{D}})\|u\|_{\mathscr{C}^3([0,T];\,\mathscr{C}^2(\overline{\Omega}))}, \quad (42)$$

and

$$|\eta^n_{\mathscr{D}}|_{\mathscr{X}} \leq C\,(k + h_{\mathscr{D}})\|u\|_{\mathscr{C}^3([0,T];\,\mathscr{C}^2(\overline{\Omega}))}. \quad (43)$$

We now combine (42)–(43) with [1, (13), (15), and (16)] to prove the required estimates (13)–(15).

- *Proof of estimate* (13): estimate (43) with [2, (4.6)] implies

$$\|\Pi_{\mathscr{M}}\,\eta^n_{\mathscr{D}}\|_{1,2,\mathscr{M}} \leq C\,(k + h_{\mathscr{D}})\|u\|_{\mathscr{C}^3([0,T];\,\mathscr{C}^2(\overline{\Omega}))}, \quad \forall n \in [\![2, N+1]\!]. \quad (44)$$

  This with (31), the fact that $\Pi_{\mathscr{M}}\,\xi^n_{\mathscr{D}} = \Pi_{\mathscr{M}}\bar{u}^n_{\mathscr{D}} - \mathscr{P}_{\mathscr{M}}\,u(\cdot, t_n)$, estimate [1, (13)], and the triangle inequality implies estimate (13) for all $n \in [\![2, N+1]\!]$. The case when $n = 1$ in (13) can be proved by gathering (39), [2, (4.6)], and the case $n = 1$ of [1, (13)]. Property (26) with the case $n = 0$ of [1, (13)] yields the case $n = 0$ of (13).
- *Proof of estimate* (14): the case when $n \in [\![2, N+1]\!]$ of (14) can be proved by gathering (42), the case when $j = 1$ in [1, (15)], and the triangle inequality. The case $n = 1$ of (14) can be proved by gathering (36), the case when $n = 1$ and $j = 1$ in [1, (15)], and the triangle inequality.

– *Proof of estimate* (15): gathering (39) and (43), and [2, Lemma 4.2] leads to

$$\| \nabla_{\mathscr{D}} \eta_{\mathscr{D}}^{n} \|_{\mathbb{L}^2(\Omega)} \leq C\,(k + h_{\mathscr{D}}) \| u \|_{\mathscr{C}^3([0,T];\, \mathscr{C}^2(\overline{\Omega}))}, \quad \forall\, n \in [\![1, N+1]\!]. \tag{45}$$

Combining (45), [1, (16)], and the triangle inequality yields (15) for all $n \in [\![1, N+1]\!]$. The case $n = 0$ of (15) can be deduced directly from the case $n = 0$ of [1, (16)] by using (26). □

References

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The paper is in final form and no similar paper has been or is being submitted elsewhere.