A Low Degree Non–Conforming Approximation of the Steady Stokes Problem with an Eddy Viscosity

F. Boyer, F. Dardalhon, C. Lapuerta, and J.-C. Latché

Abstract In the context of Large Eddy Simulation, the use of a turbulence model brings the question of the implementation of the eddy–viscosity. In this communication, we propose to assess the discretization of the diffusive term based on a low–order non–conforming finite element. For this, we build a manufactured solution of the incompressible steady Stokes problem, for which the turbulent viscosity is given either by the Smagorinsky or WALE models. Numerical tests are performed for both models with the finite element approximation and the MAC scheme.

Keywords Large Eddy Simulation, WALE and Smagorinsky models, incompressible steady Stokes equations, low-order finite element approximation
MSC2010:65N30, 76D05

1 Introduction

In the context of turbulence modelling, there is an increasing interest in the Large Eddy Simulation approach (LES), resulting from the augmentation of the computer resources. LES modelling solves large turbulent structures, while small–scale effects are modelled (see [2, 13]). The approach consists in averaging the Navier–Stokes equations in space (by convolution), and then commuting this filtering operation (denoted with the overbar symbol) with space and time derivatives. This

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yields balance equations, which keep the same form as the original ones, for the resolved (large scales) velocity $\overline{\mathbf{u}}$ and pressure $\overline{p}$. Due to the presence of the nonlinear convection term, the unclosed quantity $-\mathrm{div}(\overline{\mathsf{T}})$, with $\overline{\mathsf{T}} = \overline{\mathbf{u}\mathbf{u}} - \overline{\mathbf{u}}\,\overline{\mathbf{u}}$ appears at the right–hand side, and must be modelled, *i.e.* recast as a function of the unknowns $\overline{\mathbf{u}}$ and $\overline{p}$.

The key issue in the LES approach is thus to find a suitable expression for the subgrid–scale tensor $\overline{\mathsf{T}}$. A common assumption is to suppose a proportional relation between $\overline{\mathsf{T}}$ and the deformation tensor $\overline{\mathsf{S}}$:

$$\overline{\mathsf{T}} = -2\,\nu_t\,\overline{\mathsf{S}}, \quad \text{with} \quad \overline{\mathsf{S}}_{i,j} = \frac{1}{2}(\partial_j \overline{\mathbf{u}}_i + \partial_i \overline{\mathbf{u}}_j), \quad \forall i,j \in \{1,\cdots,d\},$$

the scalar ν_t being referred to as the turbulent viscosity. We propose to study here two subgrid–scale models often encountered in the literature, namely the Smagorinsky [14] and WALE [10] models.

The Smagorinsky model is the most frequently used because of its quite simple form and reads:

$$\nu_t = (C_s\,\overline{\Delta})^2\,\sqrt{2\,\mathrm{Trace}(\overline{\mathsf{S}}\,\overline{\mathsf{S}}^T)} = (C_s\,\overline{\Delta})^2\left(2\sum_{i,j\in\{1,\cdots,d\}}\overline{\mathsf{S}}_{i,j}\,\overline{\mathsf{S}}_{i,j}\right)^{\frac{1}{2}}, \qquad (1)$$

where C_s is a constant adjusted as a function of the flow and $\overline{\Delta}$ is the cut–off length scale, usually identified to a characteristic size of the cell. However, the viscosity obtained in this way behaves like $\mathcal{O}(1)$ near a wall, contrary to the scaling $\nu_t = \mathcal{O}(y^3)$, where y stands for the distance to the wall, which may be inferred by asymptotic analysis [13]. So this model dissipates the large scales too much near a wall.

The WALE model (Wall Adaptating Local Eddy–viscosity) aims at solving this problem and reads:

$$\nu_t = (C_w\overline{\Delta})^2\frac{\left(\sum_{i,j}\overline{\varsigma}_{i,j}\,\overline{\varsigma}_{i,j}\right)^{3/2}}{\left(\sum_{i,j}\overline{\mathsf{S}}_{i,j}\,\overline{\mathsf{S}}_{i,j}\right)^{5/2}+\left(\sum_{i,j}\overline{\varsigma}_{i,j}\,\overline{\varsigma}_{i,j}\right)^{5/4}}, \qquad (2)$$

C_w being a real constant adjusted as a function of the flow and

$$\overline{\varsigma} = \frac{1}{2}\left(\nabla\overline{\mathbf{u}}^2 + (\nabla\overline{\mathbf{u}}^2)^T\right) - \frac{1}{d}\,\mathrm{Trace}(\nabla\overline{\mathbf{u}}^2)\,I_d,$$

I_d being the $d \times d$ identity matrix. Asymptotic analysis of Eq. (2) shows that the proper behaviour $\mathcal{O}(y^3)$ of the viscosity is recovered, without any near–wall modification, which makes this model particularly attractive to deal with complex geometries.

As a first step toward the construction of a scheme for LES equations, we propose in this paper to study the discretization of the nonlinear (due to the presence of the subgrid model) diffusion term of the momentum balance equation. To this purpose, we address a simplified problem, namely the steady incompressible Stokes problem obtained by dropping the time derivative and convection terms in the original Navier-Stokes equations:

$$\begin{cases} -\mathrm{div}(2\nu \mathsf{S}(\overline{\mathbf{u}})) + \boldsymbol{\nabla}\overline{p} = \bar{\mathbf{f}} & \text{in } \Omega, \\ \mathrm{div}\,\overline{\mathbf{u}} = 0 & \text{in } \Omega, \\ \overline{\mathbf{u}} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $\mathsf{S}(\overline{\mathbf{u}}) = \frac{1}{2}(\boldsymbol{\nabla}\overline{\mathbf{u}} + \boldsymbol{\nabla}\overline{\mathbf{u}}^T)$ is the symmetric part of the gradient of $\overline{\mathbf{u}}$ and $\bar{\mathbf{f}}$ is a known forcing term. This problem is posed in Ω, an open, connected, bounded domain of $\mathbb{R}^d$ ($d = 2, 3$), supposed to be polygonal for the sake of simplicity. The effective viscosity ν is equal to the sum of the laminar and turbulent viscosities denoted by ν_l and ν_t, respectively, the latter one being given as a function of the velocity by Eqs. (1) or (2) with a coefficient $\overline{\Delta}$ supposed here to be fixed (*i.e.* independent of the mesh). Since the velocity is prescribed to zero on the whole boundary, the pressure must be supposed to be mean-valued to obtain a well-posed problem.

Two approaches are considered for the spatial discretization: low order finite element (Rannacher–Turek element) and MAC scheme. We focus the paper on the finite element version, the description of the MAC scheme used for comparison in the numerical experiments being given in [6–8]. The obtained schemes are assessed by numerical experiments, using a manufactured solution technique.

The outline of the article is as follows. After the introduction of the Rannacher–Turek finite element (Sect. 2), we describe the resulting discretization of Problem (3), *i.e.*, essentially, the proposed discrete expression for the Smagorinsky or WALE subgrid viscosity (Sect. 3). Numerical tests are reported in Sect. 4.

To alleviate the notations, we drop in the remainder of this paper the overbar symbol to denote the averaged fields.

2 Mesh and discrete spaces

Let $\mathcal{M}$ be a decomposition of the domain Ω into quadrangles, supposed to be regular in the usual sense of the finite element literature [4]. We denote by $\mathcal{E}$ the set of all faces σ of the mesh; by $\mathcal{E}_{ext}$ the set of faces included in the boundary of Ω, by $\mathcal{E}_{int}$ the set of internal faces (*i.e.* $\mathcal{E} \setminus \mathcal{E}_{\mathrm{ext}}$) and by $\mathcal{E}(K)$ the faces of a particular cell $K \in \mathcal{M}$. By $|K|$ and $|\sigma|$ we denote the measure, respectively, of the control volume K and of the face σ.

The space discretization relies on the Rannacher–Turek mixed finite element. The degrees of freedom for the velocity are located at the mass center of the faces of the mesh, and we use the version of the element where they represent the average of the velocity over the face. The set of degrees of freedom thus reads:

$$\{\mathbf{u}_{\sigma,i},\ \sigma \in \mathcal{E},\ i = 1, \cdots, d\}.$$

The discrete functional space over a cell K is obtained through the usual Q_1 mapping from the space $span\left\{1, (x_i)_{i=1,\ldots,d}, (x_i^2 - x_{i+1}^2)_{i=1,\ldots,d-1}\right\}$ over the reference element. The approximation for the velocity is non–conforming: the space X_h is composed of discrete functions which are discontinuous through an edge, but the jump of their integral is imposed to be zero. We denote by $\varphi_\sigma^{(i)}$ the vector shape function associated to $\mathbf{u}_{\sigma,i}$, which, by definition, reads $\varphi_\sigma^{(i)} = \varphi_\sigma\, \mathbf{e}^{(i)}$, where φ_σ is the Rannacher–Turek scalar shape function and $\mathbf{e}^{(i)}$ is the i^{th} vector of the canonical basis of $\mathbb{R}^d$, and we define $\mathbf{u}_\sigma$ by $\mathbf{u}_\sigma = \sum_i \mathbf{u}_{\sigma,i}\, \mathbf{e}^{(i)}$. With these definitions, we have:

$$\mathbf{u} = \sum_{\sigma\in\mathcal{E}} \sum_{i=1,\cdots,d} \mathbf{u}_{\sigma,i}\ \varphi_\sigma^{(i)}(\mathbf{x}) = \sum_{\sigma\in\mathcal{E}} \mathbf{u}_\sigma\ \varphi_\sigma(\mathbf{x}), \qquad \text{for a.e. } \mathbf{x} \in \Omega.$$

Dirichlet boundary conditions are built in the definition of the discrete velocity space X_h by fixing $\mathbf{u}_{\sigma,i} = 0$ for all faces in $\mathcal{E}_{\text{ext}}$ and any component i.

The pressure is piecewise constant, and its degrees of freedom are denoted by p_K for any cell $K \in \mathcal{M}$. We denote by M_h the discrete pressure space.

3 The scheme

In this section, we begin with describing the approximation of the turbulent viscosity, which is chosen piecewise constant by cell, and we then present the discretization of Problem (3).

Expression of the cell viscosity ν_K for the Smagorinsky model – We propose to study two discretizations of the term S in Eq. (1) of the turbulent viscosity. The first one consists in approximating the expression Trace($\mathsf{S}\,\mathsf{S}^T$) by its mean value over a cell K:

$$\overline{\mathsf{S}^2}^K = \frac{1}{|K|}\int_K \mathsf{S}(\mathbf{u}) : \mathsf{S}(\mathbf{u})\, \mathrm{d}x.$$

The second approach is to compute the mean value of the velocity gradient over K and then to use it in the definition of S:

$$\overline{\mathsf{S}_{ij}}^K = \frac{1}{2}\left(\overline{\partial_j \mathbf{u}_i}^K + \overline{\partial_i \mathbf{u}_j}^K\right) = \frac{1}{2}\left(\frac{1}{|K|}\int_K \partial_j \mathbf{u}_i\, \mathrm{d}x + \frac{1}{|K|}\int_K \partial_i \mathbf{u}_j\, \mathrm{d}x\right). \quad (4)$$

Finally, the expression of the effective viscosity for both approximations is:

– for the method 1:

$$\nu_K = \nu_l + (C_s\ \overline{\Delta})^2 \left(2\,\overline{\mathsf{S}^2}^K\right)^{\frac{1}{2}}, \tag{5}$$

– for the method 2:

$$\nu_K = \nu_l + (C_s\ \overline{\Delta})^2 \left(2\sum_{i,j} \overline{\mathsf{S}_{ij}}^K\ \overline{\mathsf{S}_{ij}}^K\right)^{\frac{1}{2}}. \tag{6}$$

These discretizations are different since the discrete velocity field is not piecewise affine. However, as reported hereafter in Sect. 4, they give similar results, so only Method 2 is retained for the discretization of the WALE model, to avoid the computation of integrals needing high order quadrature formulas.

Expression of the cell viscosity ν_K for the WALE model – The discretization of the tensor ς in a cell $K \in \mathcal{M}$ is:

$$\begin{aligned}\overline{\varsigma_{ij}}^K = \frac{1}{2} \sum_{\ell\in\{1,\cdots,d\}} &\left(\overline{\partial_\ell \mathbf{u}_i}^K \overline{\partial_j \mathbf{u}_\ell}^K + \overline{\partial_i \mathbf{u}_\ell}^K \overline{\partial_\ell \mathbf{u}_j}^K\right) \\ &- \left(\frac{1}{d} \sum_{m,n\in\{1,\cdots,d\}} \overline{\partial_m \mathbf{u}_n}^K \overline{\partial_n \mathbf{u}_m}^K\right) \delta_{i,j}, \quad \forall i,j \in \{1,\cdots,d\},\end{aligned} \tag{7}$$

where δ is the Kronecker symbol. Using the approximations of S and ς given respectively by (4) and (7), the effective viscosity on K reads:

$$\nu_K = \nu_l + (C_w\ \overline{\Delta})^2 \frac{\left(\sum_{i,j} \overline{\varsigma_{ij}}^K\ \overline{\varsigma_{ij}}^K\right)^{3/2}}{\left(\sum_{i,j} \overline{\mathsf{S}_{ij}}^K \overline{\mathsf{S}_{ij}}^K\right)^{5/2} + \left(\sum_{i,j} \overline{\varsigma_{ij}}^K\ \overline{\varsigma_{ij}}^K\right)^{5/4}}.$$

Discretization of Problem (3) – The scheme for the solution of Problem (3) consists in searching for $\mathbf{u} \in X_h$ and $p \in M_h$ such that the mean value of p over Ω is zero and:

for $1 \leq i \leq d$ and for any $\sigma \in \mathcal{E}_{\text{int}}$,

$$\sum_{K\in\mathcal{M}} 2\nu_K \int_K \mathsf{S}(\mathbf{u}) : \mathsf{S}(\varphi_\sigma^{(i)})\,\mathrm{d}x - \sum_{K\in\mathcal{M}} \int_K p\,\mathrm{div}(\varphi_\sigma^{(i)})\,\mathrm{d}x = \int_\Omega \mathbf{f}\cdot\varphi_\sigma^{(i)}\mathrm{d}x,$$

$$\forall K \in \mathcal{M}, \quad \int_K \mathrm{div}(\mathbf{u})\,\mathrm{d}x = 0. \tag{8}$$

4 Numerical tests

In this section, we build a manufactured solution to Problem (3) in 2D, and compare the results obtained with the considered discretizations to the analytical solution and to the discrete solutions obtained with the MAC scheme. The simulations are performed with the ISIS software based on the platform PELICANS, both developed at IRSN [9, 11].

Description of the numerical test – The computational domain Ω is the unit square $(0, 1)^2$ and we calculate the forcing term $\mathbf{f}$ such that the exact velocity and pressure fields, $\mathbf{u}_{exact}$ and p_{exact}, are given by:

$$\mathbf{u}_{exact} = \mathbf{curl}(\sin(\pi x)\ \sin(\pi y)), \quad p_{exact} = \cos(\pi x)\ \sin(\pi y).$$

Note that $\mathbf{u}_{exact}$ indeed satisfies homogeneous Dirichlet boundary conditions on $\partial\Omega$, and the mean value over Ω of p_{exact} is zero.

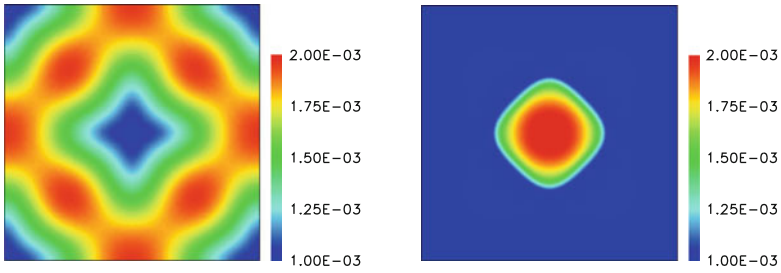
We take $\nu_l = 10^{-3}$ and the coefficient $C_w\overline{\Delta}$ in the expression of ν_t (Eqs. (1) and (2)) is set to $C_s\overline{\Delta} = 0.007$ for the Smagorinsky model and $C_w\overline{\Delta} = 0.009$ for the WALE model, which yields a turbulent and a laminar viscosity of the same range. The Smagorinsky and WALE viscosities obtained for $\mathbf{u}_{exact}$ are plotted on Fig. 1. The profiles are quite different, and one remarks that, as expected, the turbulent viscosity vanishes near the wall with the WALE model while it does not decrease with the Smagorinsky model.

The nonlinear problem (8) is solved using an iterative process analog to a time marching algorithm of pressure correction type [5], computing at each step the value of the turbulent viscosity from the beginning-of-step velocity. The steady state is supposed to be reached when velocity and pressure increments are small enough.

The discrete L^2–norm defined by

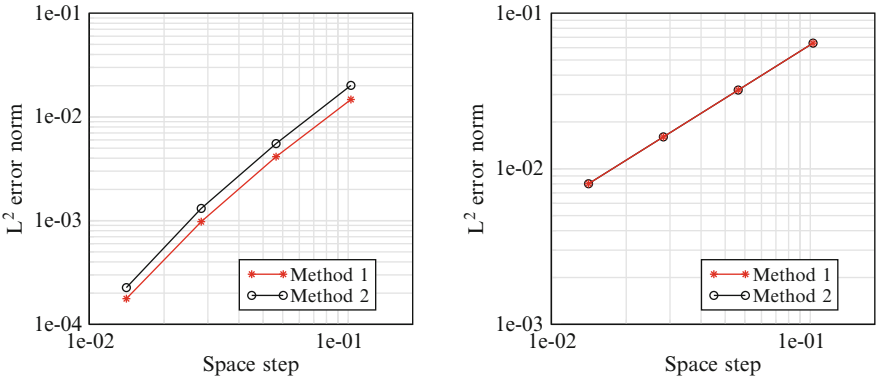
$$\|\mathbf{u}\|_0^2 = \sum_K \frac{|K|}{4} \sum_\sigma |\mathbf{u}_\sigma|^2$$

is used to measure the spatial error for $n\times n$ structured uniform meshes, with $n = 10$, 20, 40 and 80.



(a) Smagorinsky model (b) WALE model

Fig. 1 Repartition of the effective viscosity



(a) Velocity (b) Pressure

Fig. 2 L^2 error norm for the velocity and the pressure as a function of the space step for both discretizations of the Smagorinsky viscosity: Method 1 corresponds to Eq. (5), Method 2 to Eq. (6)

Comparison of both implementations for the Smagorinsky model – On Fig. 2, the spatial error in L^2–norm is plotted for both methods for the computation of the Smagorinsky viscosity. Both implementations give about the same accuracy. Consequently, Method 2 is chosen for further numerical experiments, because its implementation is simpler.

Comparison of the finite element approach and the MAC scheme for both models – On Fig. 3 and Fig. 4, 'FE' and 'FV' represent the discretization chosen, namely the Rannacher–Turek Finite Element and the MAC scheme (Finite Volume) respectively. Both discretizations seem to lead to the same order of convergence in space, that is 2 for the velocity and 1 for the pressure, for the Smagorinsky model. The FE discretization is more accurate than the MAC scheme but, for a given mesh, the number of degrees of freedom for the velocity for the FE discretization is twice (for $d = 2$) greater than for the MAC approximation (the number of degrees of freedom for the pressure being the same in both cases). For the WALE model, results look similar, with a more irregular convergence for the velocity.

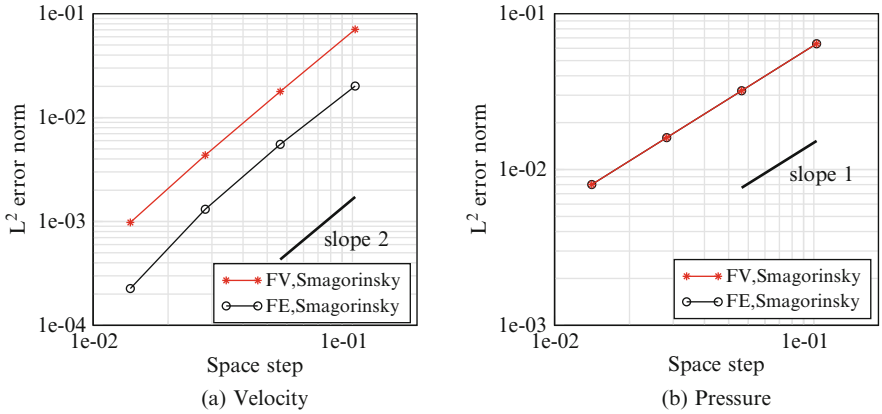


Fig. 3 L^2 error norm for the velocity and the pressure as a function of the space step for the Smagorinsky model

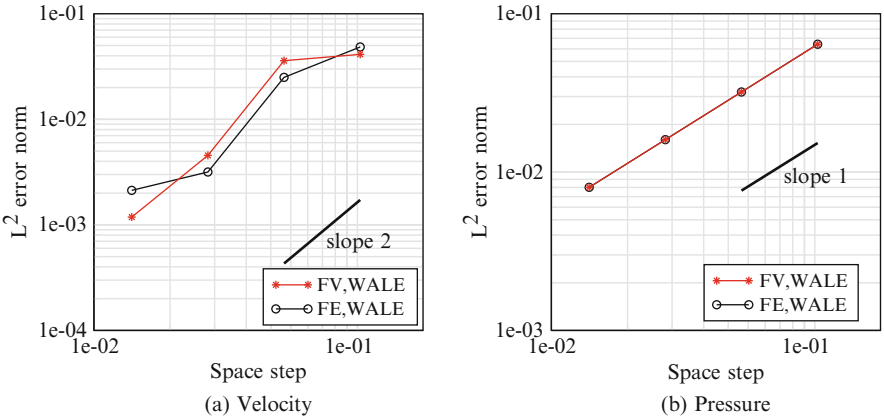


Fig. 4 L^2 error norm for the velocity and the pressure as a function of the space step for the WALE model

5 Conclusion

As a conclusion, the space discretizations retained for the Smagorinsky model and the WALE model give satisfactory results for the considered steady nonlinear Stokes problem, both for the finite element method and for the MAC scheme. Next steps will be to extend the scheme to the complete Navier–Stokes equations with the same subgrid models (see [1, 3] for a kinetic energy preserving discretization of

the convection term) and assess it on the academic test of the plane channel, before turning to more complex industrial applications.

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The paper is in final form and no similar paper has been or is being submitted elsewhere.