Finite Volumes Asymptotic Preserving Schemes for Systems of Conservation Laws with Stiff Source Terms

C. Berthon and R. Turpault

Abstract We consider here a numerical technique that allows to build asymptotic-preserving schemes for hyperbolic systems of conservation laws with eventually stiff source terms. The scheme is build in 1D and extended to unstructured 2D meshes. Its behavior is illustrated by numerical experiments on different physical applications.

Keywords systems of conservation laws, stiff, source terms, asymptotic preserving schemes, finite volumes schemes
MSC2010: 65N08, 65Z05

1 Introduction

Our objective is to develop numerical schemes adapted to the resolution of hyperbolic systems of conservation laws with source terms of the form:

$$\partial_t U + \mathrm{div}(\mathbf{F}(U)) = -\gamma R(U), \tag{1}$$

where the state vector $U \in \mathbb{R}^N$ lies in a convex set $\Omega \subset \mathbb{R}^N$. Here, $\gamma \in \mathbb{R}$, which may be a function of U, controls the stiffness of the source term. The function $R(U)$ is supposed to fulfill the compatibility properties required in [2] (see also [10]). In particular, we assume the existence of a constant $n \times N$ matrix Q with rank $n < N$ such that $QR(U) = 0$. It has been showed in [2] that when γ is large, the long-time behavior of such systems degenerates into a nonlinear parabolic system which can be written as:

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$$\partial_t u = -\mathrm{div}\left(\mathscr{M}(u)\nabla u\right), \tag{2}$$

where $u = QU$ and $\mathscr{M}(u)$ is a nonlinear diffusion matrix.

Such systems are involved in numerous physical models found for instance in radiotherapy, radiative transfer or fluid dynamics with friction. Typical applications may involve domains where the source term is neglectable (hyperbolic-dominant zones), very stiff (diffusion-dominant zones) or in-between. Therefore, it is crucial to dispose of a numerical scheme able to handle every regime. The construction of such schemes is generally very difficult. Former works (see for instance [1, 6, 8, 9, 12]) usually concentrate on modifying the HLL scheme [16] to adequately include the source term with respect to the physics of a given problem.

In this article, we will propose a generic numerical technique which extends any approximate Riemann solver into an asymptotic preserving scheme for (1). We will first introduce the construction of a finite volumes scheme adapted to the approximation of the solutions of (1) in 1D. This scheme will then be extended for 2D unstructured meshes. Finally, it will be applied on three numerical simulations that will emphasize the relevance of this numerical technique and underline a possibility to improve it.

2 Description of the Scheme

2.1 Construction in 1D

We first show the construction of the numerical technique as it was introduced in [3] and extended in [2]. It consists in a suitable modification of an approximate Riemann solver designed for the transport part of (1) (ie $\gamma = 0$).

Therefore, we start by selecting such a solver. A Riemann problem is thus approximated at each cell interface:

$$\tilde{U}_{\mathscr{R}}\left(\frac{x}{t}; U_L, U_R\right) = \begin{cases} U_L & \text{if } \dfrac{x}{t} < b^-, \\ \tilde{U}^\star & \text{if } b^- < \dfrac{x}{t} < b^+, \\ U_R & \text{if } \dfrac{x}{t} > b^+, \end{cases} \tag{3}$$

where $|b^\pm|$ are chosen to be larger than the fastest wave speed of the problem. For the sake of simplicity in the notations, we will consider in the following that $b^+ = -b^- = b > 0$. Furthermore, $\tilde{U}^\star$ represents the value of the intermediate states and hence generally depends on U_L, U_R and x/t.

As soon as the CFL condition $b\frac{\Delta t}{\Delta x} \leq \frac{1}{2}$ holds, we are considering a juxtaposition of non-interacting approximate Riemann solvers denoted $\tilde{U}^n_{\Delta x}(x, t^n + t)$ for $t \in [0, \Delta t)$. The updated approximated solution at time t^{n+1} is then naturally defined as follows:

$$\tilde{U}_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \tilde{U}_{\Delta x}^n(x, t^n + \Delta t) dx. \tag{4}$$

This scheme can be written in the following usual conservation form:

$$\tilde{U}_i^{n+1} = U_i^n - \frac{\Delta t}{\Delta x}(\mathscr{F}_{i+1/2} - \mathscr{F}_{i-1/2}), \tag{5}$$

where $\mathscr{F}_{i+1/2}$ denotes the numerical flux at the interface $x_{i+1/2}$. Any (approximate) Riemann solver enter this framework, including for instance Godunov, HLL, HLLC and relaxation schemes. As an example, in the case of the well-known HLL scheme [16], $\tilde{U}^\star$ and $\mathscr{F}_{i+1/2}$ are given by:

$$\tilde{U}^{\star,HLL} = \frac{1}{2}(U_L + U_R) - \frac{1}{2b}(F(U_R) - F(U_L)), \tag{6}$$

$$\mathscr{F}_{i+1/2} = \frac{1}{2}(F(U_i^n) + F(U_{i+1}^n)) - \frac{b}{2}(U_{i+1}^n - U_i^n). \tag{7}$$

In order to take into account the source term, we now modify the approximate Riemann solver (3) as follows:

$$U_{\mathscr{R}}(\frac{x}{t}; U_L, U_R) = \begin{cases} U_L & \text{if } \frac{x}{t} < -b, \\ U^{\star L} & \text{if } -b < \frac{x}{t} < 0, \\ U^{\star R} & \text{if } 0 < \frac{x}{t} < b, \\ U_R & \text{if } \frac{x}{t} > b, \end{cases} \tag{8}$$

where we have set:

$$\begin{aligned} U^{\star L} &= \underline{\alpha}\tilde{U}^\star + (\mathbb{I}_\mathrm{d} - \underline{\alpha})(U_L - \bar{R}(U_L)), \\ U^{\star R} &= \underline{\alpha}\tilde{U}^\star + (\mathbb{I}_\mathrm{d} - \underline{\alpha})(U_R - \bar{R}(U_R)). \end{aligned} \tag{9}$$

Here, $\underline{\alpha}$, which denotes a $N \times N$ matrix, and $\bar{R}(U)$ are defined by:

$$\underline{\alpha} = \left(\mathbb{I}_\mathrm{d} + \frac{\gamma \Delta x}{2b}(\mathbb{I}_\mathrm{d} + \underline{\sigma})\right)^{-1}, \ \bar{R}(U) = (\mathbb{I}_\mathrm{d} + \underline{\sigma})^{-1} R(U). \tag{10}$$

The $N \times N$ matrices $\mathbb{I}_\mathrm{d}$ and $\underline{\sigma}$ respectively denote the identity matrix and a parameter matrix to be defined. The updated approximated solution at time t^{n+1} is once again naturally defined:

$$U_i^{n+1} = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} U_{\Delta x}^n(x, t^n + \Delta t)dx. \tag{11}$$

A straightforward computation leads to:

$$\begin{aligned}\frac{1}{\Delta t}(U_i^{n+1} - U_i^n) + \frac{1}{\Delta x}(\underline{\alpha}_{i+1/2}\mathscr{F}_{i+1/2} - \underline{\alpha}_{i-1/2}\mathscr{F}_{i-1/2})\\ = \frac{1}{\Delta x}(\underline{\alpha}_{i+1/2} - \underline{\alpha}_{i-1/2})F(U_i^n) - \frac{\gamma}{2}(\underline{\alpha}_{i+1/2} + \underline{\alpha}_{i-1/2})R(U_i^n).\end{aligned} \tag{12}$$

Observe that whenever $\gamma = 0$, then $\underline{\alpha} = \mathbb{I}_d$ and (12) is nothing but (5).

It was proved in [2] that the scheme (12) is consistant with (1) and preserves Ω as soon as the approximate Riemann solver for the transport part does so. These properties hold for any relevant choice of the parameter matrices $\underline{\sigma}$. These matrices may therefore be chosen to enforce the scheme (12) to be consistant with (2) in the asymptotic regimes. Indeed, an asymptotic analysis of the scheme shows that it is asymptotic preserving if $\underline{\sigma}_{i+1/2}$ is chosen so that the following relation holds:

$$Q(\mathbb{I}_d + \underline{\sigma}_{i+1/2})^{-1} = \frac{1}{b^2}\mathscr{M}_{i+1/2}Q, \tag{13}$$

where $\mathscr{M}_{i+1/2}$ is a discretization of the diffusion matrix $\mathscr{M}(u)$ at the interface $x_{i+1/2}$. One of the edges of this scheme is that it allows to consider applications where γ is a nonlinear function of x and U (see examples in [3] and [4]).

2.2 Extension for 2D unstructured grids

In the case of unstructured grids, the 1D scheme (12) can be extended into the following scheme:

$$\begin{aligned}U_K^{n+1} = U_K^n - \frac{\Delta t}{|C_K|}\sum_{e\in\partial K}|e|\underline{\alpha_e}\Big[\mathscr{F}_e.n_e - F(U_K^n)n_x - G(U_K^n)n_y\Big]\\ + \frac{c\Delta t}{|C_K|}\sum_{e\in\partial K}|e|\beta_e b_e(\mathbb{I}_d - \underline{\alpha_e})\bar{R}(U_K^n),\end{aligned} \tag{14}$$

where $|K|$ is the measure of the cell K and $|e|$ is the measure of the interface e.

Furthermore, $\underline{\alpha}_e$ is chosen as:

$$\underline{\alpha}_e = |e|\left(|e|\mathbb{I}_d + \frac{\gamma|K|}{2b}(\mathbb{I}_d + \underline{\sigma})\right)^{-1},$$

Finally, β is set to $1/2$. It is to note that the choices of $\underline{\alpha}$ and β are the simplest admissible ones. However, they are not unique and other expressions may even improve the accuracy of the scheme.

This scheme has successfully been used in the case of cartesian grids in 2D (see for example [5]). In the case of unstructured grids however, in order to enforce the asymptotic preserveness of (14), the choice of $\underline{\sigma}_e$ implies the knowledge of a relevant scheme for the diffusion equation (2). Due to the nonlinear nature of the anisotropy of the diffusion matrix $\mathscr{M}(u)$, the classical two-point scheme (aka FV4, see [15]) lacks of consistance. Therefore, efficient compact schemes have to be considered in order to discretize the diffusion operator. In this framework, we are considering Discrete-Duality Finite Volumes schemes (see for instance [7, 11, 13, 17]). The rich structure of the DDFV schemes can obviously also be used to improve the hyperbolic solvers.

3 Numerical Results

In this section, numerical examples illustrate the behavior of the scheme (12) on three different test-cases. For the sake of simplicity, we used the HLL solver for the transport part.

TC1: Euler with friction

We first consider the 1D isentropic Euler equations with friction. The system reads:

$$\partial_t \rho + \partial_x q = 0,$$
$$\partial_t q + \partial_x \Big(\frac{q^2}{\rho} + p(\rho) \Big) = -\kappa q,$$

where $\rho > 0$ denotes the density and $q \in \mathbb{R}$ is the fluid momentum. The pressure function $p : \mathbb{R}_+ \to \mathbb{R}_+$ is assumed to be regular enough and to satisfy $p'(\rho) > 0$ in order to ensure the first-order homogeneous associated system to be hyperbolic.

The associated diffusive regime is governed by:

$$\partial_t \rho = \partial_x (p^{'}(\rho) \partial_x \rho). \tag{15}$$

Figure 1 shows the density computed at time $t = 20$. The reference solution is a grid-converged result with a scheme that approximates the diffusion equation (15).

The results of the scheme (12) are in very good agreement with the reference solution even on a coarse grid ($\Delta x = 0.02$). The results of the scheme with $\underline{\sigma} = 0$ are also plotted on Fig. 1. They are representative of what happens with a scheme which is not asymptotic-preserving (although consistant). Indeed, an asymptotic analysis of this scheme shows that it is consistant with a diffusion equation with the wrong diffusion coefficient (see [3]).

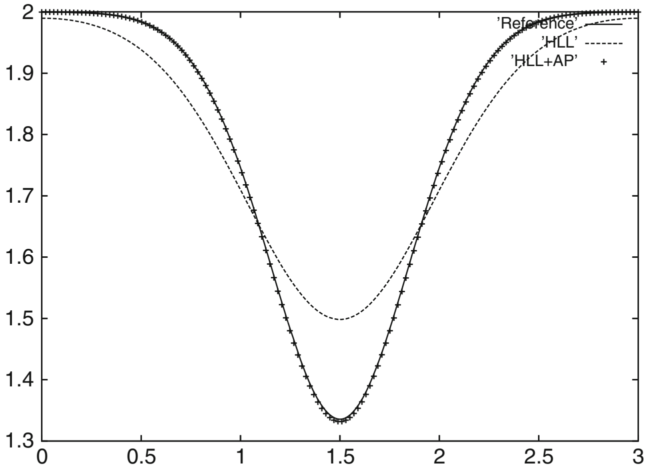

Fig. 1 TC1: computed values of ρ at time $t = 20$. Reference solution (full line) and HLL scheme with (+) or without (dashed line) AP correction

TC2: M1 model for radiative transfer

Now we are interested in the 2D $M1$ model for radiative transfer:

$$\begin{aligned} \partial_t E + \partial_x \mathbf{F}_x + \partial_y \mathbf{F}_y &= c\sigma(aT^4 - E), \\ \partial_t \mathbf{F}_x + c^2\partial_x \mathbf{P}_{xx} + c^2\partial_y \mathbf{P}_{xy} &= -c\sigma \mathbf{F}_x, \\ \partial_t \mathbf{F}_y + c^2\partial_y \mathbf{P}_{xy} + c^2\partial_y \mathbf{P}_{yy} &= -c\sigma \mathbf{F}_y, \\ \rho C_v \partial_t T &= c\sigma(E - aT^4), \end{aligned}$$

where E, $\mathbf{F}$ and $\mathbf{P}$ respectively denote the radiative energy, the radiative flux vector and the radiative pressure tensor. Moreover, T is the material temperature, σ is the opacity, a and c are physical parameters. Finally $\mathbf{P} = \mathbf{P}(\frac{\|\mathbf{F}\|}{cE})$ is a prescribed function (see [14]).

The associated asymptotic regime is described by the so-called equilibrium diffusion equation:

$$\partial_t(\rho C_v T + aT^4) + \mathrm{div}\Big(\frac{4acT^3}{3\sigma}\nabla T\Big) = 0.$$

In order to obtain a scheme which is consistant with the diffusion operator, unknowns on the triangular mesh were considered at the orthocenter and the classical FV4 scheme (see [15]) was used. Of course, this trick is not valid in general so that other approaches have to be considered as was mentionned in Sect. 2.

Figure 2 shows the results of the scheme (14) on a left-entering Marshak wave inside a square 1m-wide domain with an obstacle. The parameters are $E(t = 0) =$

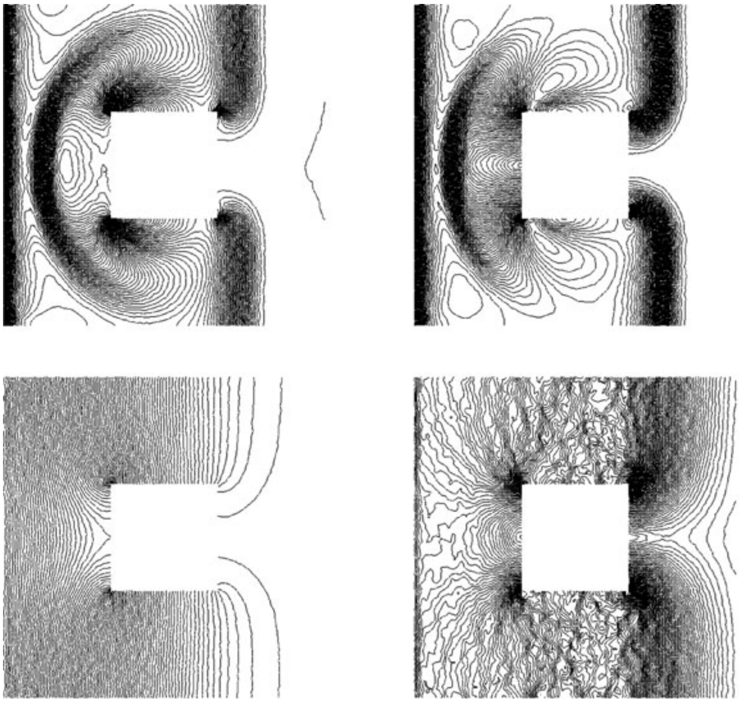

Fig. 2 TC2: Radiative energy (l) and normalized flux (r). Top: $t = 1.e - 8$ and $\sigma = 0$. Bottom: $t = 1.e - 5$ and $\sigma = 10$. Same contours for the energy, same number of contours for the flux (max$\simeq 0.8$ (T) and 0.1 (B)). Triangular mesh with $h \simeq 6.5e - 3$

$a1000^4$, $F(t = 0) = (0, 0)$, $T(t = 0) = 1000$, $E_L = a2000^4$, $F_L = (0, 0)$ and $T_L = 2000$. Two computations were carried on with $\sigma = 0$ and $\sigma = 10$.

TC3: toy model

For this last application, we consider an interesting toy model that is one of the simplest nontrivial example where the asymptotic regime is described by a system of two equations. It writes:

$$\partial_t \rho + \partial_x q = 0,$$

$$\partial_t q + \partial_x \Big(\frac{q^2}{\rho} + p(\rho) \Big) = -\kappa q + \sigma f,$$

$$\partial_t e + \partial_x f = 0,$$

$$\partial_t f + \partial_x \chi \left(\frac{f}{e} \right) e = -\sigma f,$$

where $\chi(\xi) = \frac{3+4\xi^2}{5+2\sqrt{4-3\xi^2}}$.

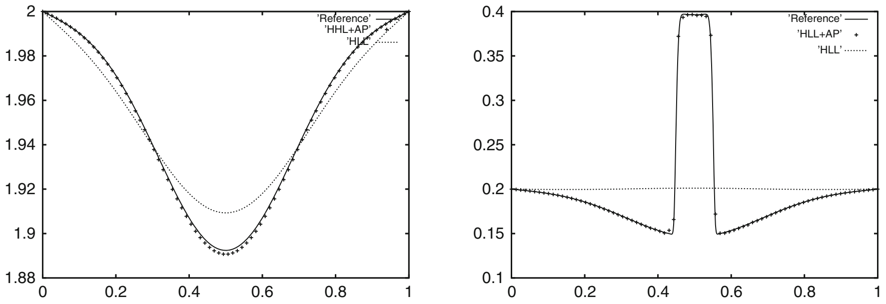

Fig. 3 TC3: computed values of e (l) and ρ (r) at time $t = 50$. Reference solution (full line) and HLL scheme with $(+)$ or without (dashed line) AP correction

The asymptotic regime of this system is given by:

$$\begin{aligned} \partial_t \rho - \frac{1}{\kappa}\partial_x^2 p(\rho) - \frac{1}{3\kappa}\partial_x^2 e = 0, \\ \partial_t e - \frac{1}{3\sigma}\partial_x^2 e = 0. \end{aligned} \tag{16}$$

Figure 3 shows the results of scheme (12) at time $t = 50$ compared with a reference solution for the following test-case: the initial values are $\rho(t = 0) = 0.2$, $q(t = 0) = f(t = 0) = 0$ and $e(t = 0) = 2 - 0.5 \times 1_{[0.45;0.55]}$. The other parameters are $\kappa = 2000$, $\sigma = 1000$, $p(\rho) = 10^{-3}\rho^2$.

With these parameters, the solution is governed by the asymptotic system (16). The results given by the AP preserving scheme (12) are in excellent agreement with the reference solution, even on a very coarse grid (only 80 points where used). It is to note that this test-case is very challenging and that a scheme which does not preserve the asymptotics gives poor results here. As a illustration, the results given by the choice $\underline{\sigma} = 0$ are also showed on Fig. 3.

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The paper is in final form and no similar paper has been or is being submitted elsewhere.