Chapter 3
Fractional Positive 2D Linear Systems

3.1 Definition of (Backward) Fractional Difference of 2D Function

Definition 1.3 of (backward) fractional difference of α-order will be extended to two-dimensional (shortly 2D) discrete function x_{ij}.

Definition 3.1. The 2D discrete function

$$\Delta^{\alpha} x_{ij} = \sum_{k=0}^{i} \sum_{l=0}^{j-k} c_{\alpha}(k,l) x_{i-k,j-l}, \quad 0 < \alpha < 1, \tag{3.1a}$$

is called the (backward) fractional difference of α order of the 2D function x_{ij} where

$$c_{\alpha}(k,l) = \begin{cases} 1 & \text{for} \quad k = l = 0 \quad k,l \in \mathbb{Z}_+ \\ (-1)^{k+l} \frac{\alpha(\alpha-1)\cdots(\alpha-k-l+1)}{k!l!} & \text{for} \quad k+l > 0 \end{cases} \tag{3.1b}$$

3.2 State Equation of Fractional 2D Linear Systems

The model described by the state equation:

$$\begin{aligned} \Delta^{\alpha} x_{i+1,j+1} &= A_0 x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1} \\ &+ B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1}, \end{aligned} \tag{3.2a}$$

$$y_{ij} = C x_{ij} + D u_{ij}, \tag{3.2b}$$

is called the fractional general model of α order of 2D linear systems where $x_{ij} \in \mathbb{R}^n$, $u_{ij} \in \mathbb{R}^m$, $y_{ij} \in \mathbb{R}^p$ are state, input and output vectors and $A_k \in \mathbb{R}^{n \times n}$, $B_k \in \mathbb{R}^{n \times m}$, $k = 0,1,2$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$.

Using Definition 3.1 we may write the equations (3.2a) in the form

$$\begin{aligned} x_{i+1,j+1} &+ \sum_{k=0}^{i+1} \sum_{\substack{l=0 \\ k+l>0}}^{j-k+1} c_{\alpha}(k,l) x_{i-k+1,j-l+1} \\ &= A_0 x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1}. \end{aligned} \tag{3.3a}$$

T. Kaczorek: Selected Problems of Fractional Systems Theory, LNCIS 411, pp. 53–80.
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From (3.1b) it follows that the coefficients $c_\alpha(k,l)$ in (3.1a) strongly decrease with increasing k and l. In practice usually it is assumed that i and j are bounded by some natural numbers L_1 and L_2. In this case the equation (3.3a) takes the form

$$x_{i+1,j+1} + \sum_{k=0}^{L_1+1} \sum_{\substack{l=0 \\ k+l>0}}^{L_2-k+1} c_\alpha(k,l) x_{i-k+1,j-l+1} \\ = A_0 x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1}. \tag{3.3b}$$

Remark 3.1. From (3.3a) it follows that the fractional 2D linear system is a linear system with increasing number of delays in state vector.

Boundary conditions for (3.3a) have the form:

$$x_{i0},\ i \in \mathbb{Z}_+, \quad \text{and} \quad x_{0j},\ j \in \mathbb{Z}_+. \tag{3.4}$$

3.3 Solution of the State Equation of the Fractional 2D Linear System

Applying 2D z-transform we shall derive the solution of the state equation (3.3a) with boundary conditions (3.4).

Theorem 3.1. *The solution of the state equation (3.3a) with the boundary conditions (3.4) has the form*

$$\begin{aligned} x_{ij} &= \sum_{p=1}^{i} T_{i-p,j-1} \left(\overline{A}_1 x_{p0} + B_1 u_{p0}\right) + \sum_{q=1}^{j} T_{i-1,j-q} \left(\overline{A}_2 x_{0q} + B_2 u_{0q}\right) \\ &+ \sum_{p=1}^{i-1} T_{i-p-1,j-1} \overline{A}_0 x_{p0} + \sum_{q=1}^{j-1} T_{i-1,j-q-1} \overline{A}_0 x_{0q} + T_{i-1,j-1} \overline{A}_0 u_{00} \\ &+ \sum_{p=0}^{i-1} \sum_{q=0}^{j-1} T_{i-p-1,j-q-1} B_0 u_{pq} + \sum_{p=0}^{i} \sum_{q=0}^{j} \left(T_{i-p-1,j-q} B_1 + T_{i-p,j-q-1} B_2\right) u_{pq} \end{aligned} \tag{3.5}$$

where the matrices T_{pq} are defined as

$$T_{pq} = \begin{cases} I_n & \text{for} \quad p = q = 0 \\ \overline{A}_0 T_{p-1,q-1} + \overline{A}_1 T_{p,q-1} + \overline{A}_2 T_{p-1,q} & \\ - \sum_{k=0}^{p} \sum_{l=0}^{q} c_\alpha(p-k,q-l) T_{kl} & \text{for} \quad p+q > 0, \\ & \text{and} \quad k+l < p+q-2 \\ 0 \quad \text{(zero matrix)} & \text{for} \quad p < 0 \quad \text{and/or} \quad q < 0 \end{cases} \tag{3.6}$$

and $\overline{A}_k = A_k - \alpha I_n$ for $k = 0,1,2$.

Proof. Let $X(z_1,z_2)$ be the 2D z-transform of the discrete function x_{ij}, defined by (A.15). Applying the 2D z-transform to equation (3.3a) and using Appendix (A.3), we obtain

$$\begin{aligned} X(z_1,z_2) &= G^{-1}(z_1,z_2)\{(B_0+B_1z_1+B_2z_2)U(z_1,z_2) \\ &+ z_1z_2[X(z_1,0)+X(0,z_2)-x_{00}] \\ &+ \sum_{l=1}^{j+1} c_\alpha(0,l)z_1z_2^{-l+1}X(0,z_2)+\sum_{k=1}^{i+1} c_\alpha(k,0)z_1^{-k+1}z_2X(z_1,0) \\ &- z_1\left[\overline{A}_1 B_1\right]\begin{bmatrix} X(0,z_2) \\ U(0,z_2)\end{bmatrix} - z_2\left[\overline{A}_2 B_2\right]\begin{bmatrix} X(z_1,0) \\ U(z_1,0)\end{bmatrix}\Bigg\}, \end{aligned} \tag{3.7a}$$

where

$$G(z_1,z_2) = \left[z_1z_2I_n + \sum_{k=0}^{i+1}\sum_{\substack{l=0 \\ k+l>1}}^{j-k+1} c_\alpha(k,l)z_1^{-(k-1)}z_2^{-(l-1)}I_n - \overline{A}_0 - \overline{A}_1z_1 - \overline{A}_2z_2 \right] \tag{3.7b}$$

and $U(z_1,z_2) = \mathscr{Z}[u_{ij}]$.
Let

$$G^{-1}(z_1,z_2) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq}z_1^{-(p+1)}z_2^{-(q+1)}. \tag{3.8}$$

From the equality

$$G^{-1}(z_1,z_2)G(z_1,z_2) = G(z_1,z_2)G^{-1}(z_1,z_2) = I_n,$$

we have

$$\begin{aligned} I_n &= \left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq}z_1^{-(p+1)}z_2^{-(q+1)}\right)G(z_1,z_2) \\ &= G(z_1,z_2)\left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq}z_1^{-(p+1)}z_2^{-(q+1)}\right). \end{aligned} \tag{3.9}$$

Comparing of the coefficients at the same power of z_1 i z_2 in the equation (3.9), we obtain (3.6). Substituting of (3.8) into (3.7a), yields

$$X(z_1,z_2) = \left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq} z_1^{-(p+1)} z_2^{-(q+1)}\right)\{(B_0+B_1z_1+B_2z_2)U(z_1,z_2) \\ + z_1z_2[X(z_1,0)+X(0,z_2)-x_{00}] \\ - z_1\left[\overline{A}_1B_1\right]\begin{bmatrix}X(0,z_2)\\U(0,z_2)\end{bmatrix} - z_2\left[\overline{A}_2B_2\right]\begin{bmatrix}X(z_1,0)\\U(z_1,0)\end{bmatrix} \\ + \sum_{l=2}^{j+1} c_\alpha(0,l)z_1z_2^{-l+1}X(0,z_2) + \sum_{k=2}^{i+1} c_\alpha(k,0)z_1^{-k+1}z_2X(z_1,0)\Big\}, \tag{3.10}$$

Applying the inverse 2D z-transform and the convolution theorem we obtain the desired solution (3.5). □

3.4 Extension of the Cayley-Hamilton Theorem

From (3.7b) we have

$$G(z_1,z_2) = z_1z_2\overline{G}(z_1,z_2), \tag{3.11}$$

where

$$\overline{G}(z_1,z_2) = I_n + \sum_{k=0}^{i+1}\sum_{l=0}^{j-k+1} I_n c_\alpha(k,l) z_1^{-k}z_2^{-l} - \overline{A}_0 z^{-1}z_2^{-1} - \overline{A}_1 z_2^{-1} - \overline{A}_2 z_1^{-1}. \tag{3.12}$$

and

$$\det\left[\overline{G}(z_1,z_2)\right] = \sum_{k=0}^{N_1}\sum_{l=0}^{N_2} a_{N_1-k,N_2-l} z_1^{-k}z_2^{-l}. \tag{3.13}$$

It is assumed that i and j are bounded by some natural numbers L_1 i L_2, which determine the degrees N_1 and N_2.

From (3.11) and (3.8) it follows that

$$G^{-1}(z_1,z_2) = z_1^{-1}z_2^{-1}\overline{G}^{-1}(z_1,z_2) = z_1^{-1}z_2^{-1}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq}z_1^{-p}z_2^{-q}, \tag{3.14}$$

and

$$\overline{G}^{-1}(z_1,z_2) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq}z_1^{-p}z_2^{-q}, \tag{3.15}$$

where T_{pq} is defined by (3.6).

Theorem 3.2. *Let (3.13) be the characteristic polynomial of the system (3.2). Then the matrices T_{kl} satisfy the equation*

$$\sum_{k=0}^{N_1}\sum_{l=0}^{N_2} a_{kl}T_{kl} = 0. \tag{3.16}$$

Proof. From definition of inverse matrix, (3.13) and (3.15) we have

$$\operatorname{Adj}\left[\overline{G}(z_1,z_2)\right] = \left(\sum_{k=0}^{N_1}\sum_{l=0}^{N_2} a_{N_1-k,N_2-l} z_1^{-k} z_2^{-l}\right)\left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq} z_1^{-p} z_2^{-q}\right), \quad (3.17)$$

where $\operatorname{Adj}\left[\overline{G}(z_1,z_2)\right]$ is the adjoint matrix of $\overline{G}(z_1,z_2)$.

Comparison of the coefficients at the same power of $z_1^{-N_1} z_2^{-N_2}$ in equation (3.17), yields the equality (3.16), since the degrees of the polynomial matrix (3.17) with respect to z_1^{-1} and z_2^{-1} are less than N_1 and N_2. □

Theorem 3.2 is an extension of the classical Cayley-Hamilton theorem to fractional 2D linear system described by (3.2).

3.5 Positivity of Fractional 2D Linear Systems

Lemma 3.1. *If* $|\alpha| < 1$*, then:*

$$c_\alpha(k,l)\begin{cases} < 0 & \text{for} \quad 0 < \alpha < 1 \\ > 0 & \text{for} \quad -1 < \alpha < 0 \end{cases} \quad k,l \in \mathbb{Z}_+ \ . \quad (3.18)$$

Proof. Using (3.1b) for $0 < \alpha < 1$, we obtain

$$\begin{aligned} c_\alpha(k,l) &= (-1)^{k+l}\frac{\alpha(\alpha-1)\cdots(\alpha+1-k-l)}{k!l!} \\ &= \begin{cases} -\alpha & \text{for} \quad k+l=1 \\ -\frac{\alpha(1-\alpha)\cdots(k+l-1-\alpha)}{k!l!} & \text{for} \quad k+l>1 \end{cases} . \end{aligned}$$

since

$$\alpha(\alpha-1)\cdots(\alpha+1-k-l) = (-1)^{k+l-1}\alpha(1-\alpha)\cdots(k+l-1-\alpha) \quad \text{for} \quad k+l>1.$$

The proof of the second part is similar. □

Lemma 3.2. *If* $0 < \alpha < 1$ *and*

$$\overline{A}_k \in \mathbb{R}_+^{n\times n} \quad \textit{for} \quad k = 0,1,2, \quad (3.19)$$

then

$$T_{pq} \in \mathbb{R}_+^{n\times n} \quad \textit{for} \quad p,q \in \mathbb{Z}_+. \quad (3.20)$$

Proof. If the conditions (3.18), (3.19), are satisfied then from (3.6), we obtain (3.20). □

Theorem 3.3. *The fractional 2D linear system(3.2) for* $0 < \alpha < 1$ *is positive if and only if:*

$$\overline{A}_k \in \mathbb{R}_+^{n\times n},\ B_k \in \mathbb{R}_+^{n\times m}\ \textit{for}\ k = 0,1,2,\ C \in \mathbb{R}_+^{p\times n},\ D \in \mathbb{R}_+^{p\times m}. \quad (3.21)$$

Proof. Sufficiency. If the conditions (3.21) are satisfied then by Lemma 3.2 $T_{pq} \in \mathbb{R}_+^{n\times n}$ and from (3.5) we have $x_{ij} \in \mathbb{R}_+^n$ for $i, j \in \mathbb{Z}_+$ since $x_{i0} \in \mathbb{R}_+^n$, $x_{0j} \in \mathbb{R}_+^n$ and $u_{ij} \in \mathbb{R}_+^m$ for $i, j \in \mathbb{Z}_+$. From (3.2b) we have $y_{ij} \in \mathbb{R}_+^p$ since $C \in \mathbb{R}_+^{p\times n}$, $D \in \mathbb{R}_+^{p\times m}$ and $x_{ij} \in \mathbb{R}_+^n$, $u_{ij} \in \mathbb{R}_+^m$ for $i, j \in \mathbb{Z}_+$.

Necessity. It is assumed that the system is positive and $x_{00} = e_{ni}$, $i = 1,\dots,n$ (e_{ni} is i-th column od the identity matrix I_n), $x_{01} = x_{10} = 0$, $u_{ij} = 0$, $i, j \in \mathbb{Z}_+$. From equation (3.3a) for $i = j = 0$ and $u_{ij} = 0$, $i, j \in \mathbb{Z}_+$ we obtain $x_{11} = \overline{A}_0 e_{ni} = \overline{A}_{0i} \in \mathbb{R}_+^n$ where $\overline{A}_{0i}$ is i-th column of $\overline{A}_0$. This implies $\overline{A}_0 \in \mathbb{R}_+^{n\times n}$ since $i = 1,\dots,n$. If we assume that $x_{10} = e_{ni}$, $x_{00} = x_{01} = 0$ and $u_{ij} = 0$, $i, j \in \mathbb{Z}_+$ then from (3.3a) for $i = j = 0$ we obtain $x_{11} = \overline{A}_1 e_{ni} = \overline{A}_{1i} \in \mathbb{R}_+^n$ what implies $\overline{A}_1 \in \mathbb{R}_+^{n\times n}$. In a similar way we may show that $\overline{A}_2 \in \mathbb{R}_+^{n\times n}$. Assuming $u_{00} = e_{ni}$, $u_{ij} = 0$, $i, j \in \mathbb{Z}_+$ $i + j > 0$ and $x_{00} = x_{10} = x_{01} = 0$ from (3.3a), for $i = j = 0$, we obtain $x_{11} = B_0 e_{mi} = B_{0i} \in \mathbb{R}_+^m$ for $i = 1,\dots,m$ what implies $B_0 \in \mathbb{R}_+^{n\times m}$. The proof of $B_k \in \mathbb{R}_+^{n\times m}$ for $k = 1, 2$ and $C \in \mathbb{R}_+^{p\times n}$, $D \in \mathbb{R}_+^{p\times m}$ is similar. □

3.6 Reachability and Controllability of Positive Fractional 2D Linear Systems

Definition 3.2. The positive fractional 2D linear system (3.2) is called reachable at the point $(h,k) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ if for zero boundary conditions (3.4) and every vector $x_f \in \mathbb{R}_+^n$ there exists a sequence of inputs $u_{ij} \in \mathbb{R}_+^m$ for

$$(i, j) \in D_{hk} = \{(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : 0 \le i \le h, 0 \le j \le k, i + j \ne h + k\}, \tag{3.22}$$

which steers the state of the system from zero boundary conditions to the state x_f, i.e. $x_{hk} = x_f$.

Theorem 3.4. *The positive fractional 2D linear system (3.2) is reachable at the point (h,k) if and only if the reachability matrix*

$$R_{hk} = \begin{bmatrix} M_0 \; M_1^1 \; \dots \; M_h^1 \; M_1^2 \; \dots \; M_k^2 \; M_{11} \; \dots \; M_{1k} \; M_{21} \; \dots \; M_{h,k} \end{bmatrix} \tag{3.23}$$

contains n linearly independent monomial columns, where:

$$\begin{aligned} M_0 &= T_{h-1,k-1} B_0, \\ M_i^1 &= T_{h-i,k-1} B_1 + T_{h-i-1,k-1} B_0, \quad i = 1,\dots,h; \\ M_j^2 &= T_{h-1,k-1} B_2 + T_{h-1,k-j-1} B_0, \quad j = 1,\dots,k; \\ M_{ij} &= T_{h-i-1,k-j-1} B_0 + T_{h-i,k-j-1} B_1 + T_{h-i-1,k-1} B_2, \\ &\quad i = 1,\dots,h; \quad j = 1,\dots,k; \end{aligned} \tag{3.24}$$

Proof. Using (3.5) for $i = h$, $j = k$ and zero boundary conditions we obtain

$$x_f = R_{hk} u(h,k), \tag{3.25}$$

where

$$u(h,k) = \left[u_{00}^T \ u_{10}^T \ \dots \ u_{h0}^T \ u_{01}^T \ \dots \ u_{0k}^T \ u_{11}^T \ \dots \ u_{1k}^T \ u_{21}^T \ \dots \ u_{h,k}^T \right]^T \tag{3.26}$$

and T denotes the transpose.

For positive fractional system (3.2) from (3.23) and (3.24) we have $M_0 \in \mathbb{R}_+^{n\times m}$, $M_i^1 \in \mathbb{R}_+^{n\times m}$, $M_j^2 \in \mathbb{R}_+^{n\times m}$, $M_{ij} \in \mathbb{R}_+^{n\times m}$, $i = 1,\dots,h$; $j = 1,\dots,k$; and matrix $R_{hk} \in \mathbb{R}_+^{n\times[(h+1)(k+1)-1]m}$. From (3.25) it follows that there exists a sequence of inputs $u_{ij} \in \mathbb{R}_+^m$ for $(i,j) \in D_{hk}$ for every $x_f \in \mathbb{R}_+^n$ if and only if the matrix (3.23) contains n linearly independent monomial columns. □

The following theorem formulates only the sufficient conditions for reachability of the positive fractional system (3.2).

Theorem 3.5. *The positive fractional 2D linear system(3.2) is reachable at the point (h,k), if* $\operatorname{rank} R_{hk} = n$ *and the right inverse R_{hk}^r of the matrix (3.23) has nonnegative entries*

$$R_{hk}^r = R_{hk}^T \left[R_{hk}R_{hk}^T\right]^{-1} \in \mathbb{R}_+^{[(h+1)(k+1)-1]m\times n}. \tag{3.27}$$

Proof. If $rank R_{hk} = n$, then there exists the right inverse of R_{hk} and (3.27) holds then from equation (3.25) we obtain

$$u(h,k) = R_{hk}^r x_f \in \mathbb{R}_+^{[(h+1)(k+1)-1]m},$$

for every $x_f \in \mathbb{R}_+^n$. □

Example 3.1. Consider the positive fractional 2D linear system (3.2) with the matrices:

$$\overline{A}_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \overline{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \overline{A}_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \tag{3.28a}$$

$$B_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{3.28b}$$

To check the reachability at the point $(h,k) = (1,1)$, of the system we use Theorem 3.4. From (3.23) and (3.24) we obtain:

$$M_0 = B_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad M_1^1 = B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad M_1^2 = B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{3.29}$$

and

$$R_{11} = \left[M_0 \ M_1^1 \ M_1^2 \right] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}. \tag{3.30}$$

The first two columns of the matrix (3.30) are linearly independent monomial columns. By Theorem 3.4 the positive fractional system (3.2) with (3.28) is reachable at the point $(1,1)$. The input sequence which steers the state of the system from zero boundary conditions to any given state $x_f \in \mathbb{R}_+^2$ at the point $(1,1)$ is given by $\begin{bmatrix} u_{00} \\ u_{10} \end{bmatrix} = x_f$ and $u_{01} = 0$.

Using (3.27) and (3.30), we obtain

$$R_{hk}^r = R_{hk}^T \left[R_{hk} R_{hk}^T\right]^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}. \tag{3.31}$$

From (3.31) it follows that the condition (3.27), is not satisfied although the system is reachable at the point $(1,1)$. Note that the system is reachable at the paint $(1,1)$ for the arbitary order α, $0 < \alpha < 1$ and any matrices $\overline{A}_k$, $k = 0,1,2$.

Definition 3.3. The positive fractional 2D linear system (3.2) is called the system with finite memory if its characteristic polynomial has the form

$$\det[G(z_1,z_2)] = c z_1^{n_1} z_2^{n_2}, \tag{3.32}$$

where c is a constant coefficient.

Lemma 3.3. *If the positive fractional 2D linear system(3.2) is a system with finite memory then*

$$\begin{aligned} x_{bc}(i,j) = &\sum_{p=1}^{i} \left(T_{i-p,j-1}\overline{A}_1 + T_{i-p-1,j-1}\overline{A}_0\right) x_{p0} \\ &+ \sum_{q=1}^{j} \left(T_{i-1,j-q}\overline{A}_2 + T_{i-1,j-q-1}\overline{A}_0\right) x_{0q} \\ &+ T_{i-1,j-1}\overline{A}_0 x_{00} = 0, \end{aligned} \tag{3.33}$$

for $i \geq n_1$, $j \geq n_2$ and any nonzero boundary conditions (3.4).

Proof. Using (3.8) and (3.32), we obtain $T_{ij} = 0$ for $i \geq n_1$, $j \geq n_2$ and the equality (3.33) holds for any nonzero boundary conditions (3.4). □

Definition 3.4. The positive fractional 2D linear system (3.2) is controllable at the point $(h.k) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ for any nonzero boundary conditions:

$$x_{i0} \in \mathbb{R}_+^n, \quad i \in \mathbb{Z}_+ \quad \text{and} \quad x_{0j} \in \mathbb{R}_+^n, \quad j \in \mathbb{Z}_+, \tag{3.34}$$

if for every vector $x_f \in \mathbb{R}_+^n$ there exists a sequence of inputs $u_{ij} \in \mathbb{R}_+^m$ for $(i,j) \in D_{hk}$ such that $x_{hk} = x_f$.

Theorem 3.6. *The positive fractional 2D linear system (3.2) is controllable at the point* (h,k) $(h \geq n_1, k \geq n_2)$ *for any nonzero boundary conditions (3.4) if and only if it is a system with finite memory and the matrix (3.23) contains n linearly independent monomial columns.*

Proof. Using (3.5) for $i = h$, $j = k$ and taking into account that $x_{hk} = x_f$, we obtain

$$x_f - x_{bc}(h,k) = R_{hk}u(h,k), \tag{3.35}$$

where R_{hk} and $x_{bc}(h,k)$ are defined by (3.23) and (3.33), respectively.

If the positive fractional system (3.2) is a system with finite memory then by Lemma 3.3 there exists a point (h,k) $(h \geq n_1, k \geq n_2)$ such that the equality (3.33) is satisfied and $x_f = R_{hk}u(h,k)$. In this case by Theorem 3.4, there exists a sequence of inputs $u_{ij} \in \mathbb{R}_+^m$ for $(i,j) \in D_{hk}$, satisfied (3.25) if and only if the matrix (3.23) contains n linearly independent monomial columns. Otherwise $x_f - x_{bc}(h,k) \notin R_{hk}u(h,k)$, since boundary conditions (3.34) are arbitrary and the vector $x_f \in \mathbb{R}_+^n$ is also arbitrary. In this case does not exist a sequence of inputs $u_{ij} \in \mathbb{R}_+^m$ for $(i,j) \in D_{hk}$, satisfying the equality (3.35). □

3.7 Controllability to Zero of Positive Fractional 2D Linear System

Definition 3.5. The positive fractional 2D linear system (3.2) is called controllable to zero at the point (h,k) $(h \geq n_1, k \geq n_2)$ if for any nonzero boundary conditions (3.34) there exists a sequence of inputs $u_{ij} \in \mathbb{R}_+^m$ for $(i,j) \in D_{hk}$, which steers the state of the system from nonzero boundary conditions to the zero state $x_{hk} = 0$.

Theorem 3.7. *The positive fractional 2D linear system (3.2) is controllable to zero at the point* (h,k) $(h \geq n_1, k \geq n_2)$ *if and only if it is a system with finite memory.*

Proof. By Lemma 3.3 for a system with finite memory the condition (3.33) is satisfied for $h \geq n_1$, $k \geq n_2$. For $x_f = 0$ from (3.35) we have

$$x_{bc}(h,k) + R_{hk}u(h,k) = 0. \tag{3.36}$$

The equation (3.36) is satisfied for $u(h,k) = 0$. If the condition (3.33), is not satisfied then does not exist $u(h,k) \in \mathbb{R}_+^{[(h+1)(k+1)-1]m}$ satisfying (3.36), since for positive system $R_{hk} \in \mathbb{R}_+^{n \times [(h+1)(k+1)-1]m}$ and $x_{bc}(h,k) \in \mathbb{R}_+^n$. □

3.8 Models of 2D Linear Systems

3.8.1 Positive 2D Linear Systems

The model described by the equations:

$$x_{i+1,j+1} = A_0 x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1}, \tag{3.37a}$$

$$y_{ij} = C x_{ij} + D u_{ij}, \quad i,j \in \mathbb{Z}_+, \tag{3.37b}$$

is called the general model of 2D linear systems, where $x_{ij} \in \mathbb{R}^n$, $u_{ij} \in \mathbb{R}^m$, $y_{ij} \in \mathbb{R}^p$ are state, input and output vectors and $A_k \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times m}$, $k = 0,1,2$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Boundary conditions for (3.37) have the form:

$$x_{i0} \in \mathbb{R}^n, \quad i \in \mathbb{Z}_+, \quad x_{0j} \in \mathbb{R}^n, \quad j \in \mathbb{Z}_+ \ . \tag{3.38}$$

Definition 3.6. The model (system) (3.37) is called (internally) positive if $x_{ij} \in \mathbb{R}^n_+$ and $y_{ij} \in \mathbb{R}^p_+$, $i \in \mathbb{Z}_+$ for all boundary conditions $x_{i0} \in \mathbb{R}^n_+$, $i \in \mathbb{Z}_+$, $x_{0j} \in \mathbb{R}^n_+$, $j \in \mathbb{Z}_+$ and all inputs $u_{ij} \in \mathbb{R}^m_+$, $i,j \in \mathbb{Z}_+$.

Theorem 3.8. *The model (system) (3.37) is positive if and only if*

$$A_k \in \mathbb{R}^{n\times n}_+,\ B_k \in \mathbb{R}^{n\times m}_+ \textit{ for } k = 0,1,2,\ C \in \mathbb{R}^{p\times n}_+,\ D \in \mathbb{R}^{p\times m}_+. \tag{3.39}$$

Proof. The proof is given in [77].

Substituting (3.37a) $B_1 = B_2 = 0$ and $B_0 = B$, we obtain the first Fornasini-Marchesini model (FF-MM) and substituting in (3.37a) $A_0 = 0$ and $B_0 = 0$, we obtain the second Fornasini-Marchesini model (SF-MM).

The Roesser model of 2D linear system has the form:

$$\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix} u_{ij}, \tag{3.40a}$$

$$y_{ij} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} + D u_{ij}, \quad i,j \in \mathbb{Z}_+, \tag{3.40b}$$

where $x^h_{ij} \in \mathbb{R}^{n_1}$ and $x^v_{ij} \in \mathbb{R}^{n_2}$ are horizontal and vertical state vectors at the point (i,j), $u_{ij} \in \mathbb{R}^m$, $y_{ij} \in \mathbb{R}^p$ are input and output vectors and $A_{kl} \in \mathbb{R}^{n_k\times n_l}$, $k,l = 1,2$; $B_{11} \in \mathbb{R}^{n_1\times m}$, $B_{22} \in \mathbb{R}^{n_2\times m}$, $C_1 \in \mathbb{R}^{p\times n_1}$, $C_2 \in \mathbb{R}^{p\times n_2}$, $D \in \mathbb{R}^{p\times m}$.

Boundary conditions for (3.40) have the form:

$$x^h_{0j} \in \mathbb{R}^{n_1}, \quad j \in \mathbb{Z}_+, \quad x^v_{i0} \in \mathbb{R}^{n_2}, \quad i \in \mathbb{Z}_+ \ . \tag{3.41}$$

Definition 3.7. The Roesser model (3.40) is called (internally) positive if $x^h_{ij} \in \mathbb{R}^{n_1}_+$, $x^v_{ij} \in \mathbb{R}^{n_2}_+$ and $y_{ij} \in \mathbb{R}^p_+$, $i,j \in \mathbb{Z}_+$ for all $x^h_{0j} \in \mathbb{R}^{n_1}$, $j \in \mathbb{Z}_+$, $x^v_{i0} \in \mathbb{R}^{n_2}$, $i \in \mathbb{Z}_+$ and all inputs $u_{ij} \in \mathbb{R}^m_+$, $i,j \in \mathbb{Z}_+$.

Theorem 3.9. *The Roesser model is positive if and only if:*

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathbb{R}_+^{n\times n}, \quad \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix} \in \mathbb{R}_+^{n\times m}, \tag{3.42a}$$

$$\begin{bmatrix} C_1 & C_2 \end{bmatrix} \in \mathbb{R}_+^{p\times n}, \ D \in \mathbb{R}_+^{p\times m}, \ n = n_1 + n_2. \tag{3.42b}$$

The proof is given in [77].

Defining:

$$x_{ij} = \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix}, \ A_1 = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}, \ A_2 = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}, \tag{3.43a}$$

$$B_1 = \begin{bmatrix} 0 \\ B_{22} \end{bmatrix}, \ B_2 = \begin{bmatrix} B_{11} \\ 0 \end{bmatrix}, \tag{3.43b}$$

we may write the Roesser model in the form of SF-MM

$$x_{i+1,j+1} = A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_1 u_{i+1,j} + B_2 u_{i,j+1}. \tag{3.44}$$

3.8.2 Positive Fractional 2D Linear Systems

Definition 3.8. The fractional horizontal difference of of α-order of the discrete function x_{ij} is defined by the relation [166]

$$\Delta_\alpha^h x_{ij} = \sum_{k=0}^{i} c_\alpha(k) x_{i-k,j}, \tag{3.45a}$$

where $\alpha \in \mathbb{R}$, $n-1 < \alpha < n \in \mathbb{N} = 1,2,\dots$ and

$$c_\alpha(k) = \begin{cases} 1 & \text{for} \quad k = 0 \\ (-1)^k \binom{\alpha}{k} = (-1)^k \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} & \text{for} \quad k > 0 \end{cases} \tag{3.45b}$$

Definition 3.9. The fractional vertical difference of β-order of the discrete function x_{ij} is defined by the relation [166]

$$\Delta_\beta^v x_{ij} = \sum_{l=0}^{j} c_\beta(l) x_{i,j-l}, \tag{3.46a}$$

where $\beta \in \mathbb{R}$, $n-1 < \beta < n \in \mathbb{N} = 1,2,\dots$ and

$$c_\beta(l) = \begin{cases} 1 & \text{for} \quad l = 0 \\ (-1)^l \binom{\beta}{l} = (-1)^l \frac{\beta(\beta-1)\cdots(\beta-l+1)}{l!} & \text{for} \quad l > 0 \end{cases} \tag{3.46b}$$

Lemma 3.4. *If* $0 < \alpha < 1$ *(*$0 < \beta < 1$*), then*

$$c_\alpha(k) < 0, \ \big(c_\beta(l) < 0\big) \quad \textit{for} \quad k = 1,2,\dots \ . \tag{3.47}$$

Proof. The proof will be accomplished by induction with respect to k. The hypothesis is true for $k=1$, since from (3.45b) for $k=1$ we have $c_\alpha(1)=-\alpha<0$. Assuming that the hypothesis is valid for $k\geq 1$ we shall show that it is also true for $k+1$. From (3.45b) we have

$$c_\alpha(k+1)=(-1)^{k+1}\binom{\alpha}{k+1}=-(-1)^k\binom{\alpha}{k}\frac{(\alpha-k)}{k+1}=c_\alpha(k)\frac{k-\alpha}{k+1}<0,$$

since $c_\alpha(k)<0$. □

Consider the fractional 2D linear system:

$$\begin{bmatrix}\Delta_\alpha^h x_{i+1,j}^h\\ \Delta_\beta^v x_{i,j+1}^v\end{bmatrix}=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}x_{ij}^h\\ x_{ij}^v\end{bmatrix}+\begin{bmatrix}B_1\\ B_2\end{bmatrix}u_{ij}, \tag{3.48a}$$

$$y_{ij}=\begin{bmatrix}C_1 & C_2\end{bmatrix}\begin{bmatrix}x_{ij}^h\\ x_{ij}^v\end{bmatrix}+Du_{ij},\quad i,j\in\mathbb{Z}_+, \tag{3.48b}$$

where $x_{ij}^h\in\mathbb{R}^{n_1}$ and $x_{ij}^v\in\mathbb{R}^{n_2}$ are horizontal and vertical state vectors at the point (i,j), $u_{ij}\in\mathbb{R}^m$, $y_{ij}\in\mathbb{R}^p$ are input and output vectors at the point (i,j) and $A_{kl}\in\mathbb{R}^{n_k\times n_l}$, $k,l=1,2$; $B_1\in\mathbb{R}^{n_1\times m}$, $B_2\in\mathbb{R}^{n_n\times m}$, $C_1\in\mathbb{R}^{p\times n_1}$, $C_2\in\mathbb{R}^{p\times n_2}$, $D\in\mathbb{R}^{p\times m}$.

Using Definitions 3.8 and 3.9, we may write the equation (3.48a) in the form

$$\begin{bmatrix}x_{i+1,j}^h\\ x_{i,j+1}^v\end{bmatrix}=\begin{bmatrix}\overline{A}_{11} & A_{12}\\ A_{21} & \overline{A}_{22}\end{bmatrix}\begin{bmatrix}x_{ij}^h\\ x_{ij}^v\end{bmatrix}-\begin{bmatrix}\sum_{k=2}^{i+1}c_\alpha(k)x_{i-k+1,j}^h\\ \sum_{l=2}^{j+1}c_\beta(l)x_{i,j-l+1}^v\end{bmatrix}+\begin{bmatrix}B_1\\ B_2\end{bmatrix}u_{ij}, \tag{3.49}$$

where $\overline{A}_{11}=A_{11}+\alpha I_{n_1}$ and $\overline{A}_{22}=A_{22}+\beta I_{n_2}$.

From (3.49) it follows that the fractional 2D linear system is a system with increasing number of delays in state vectors. From (3.45b) and (3.46b) it follows that the coefficients $c_\alpha(k)$ and $c_\beta(l)$ in (3.49) strongly decrease with increasing k and l. In practice usually it is assumed that k and l are bounded by some natural numbers L_1 and L_2. In this case the equation (3.49) takes the form

$$\begin{bmatrix}x_{i+1,j}^h\\ x_{i,j+1}^v\end{bmatrix}=\begin{bmatrix}\overline{A}_{11} & A_{12}\\ A_{21} & \overline{A}_{22}\end{bmatrix}\begin{bmatrix}x_{ij}^h\\ x_{ij}^v\end{bmatrix}-\begin{bmatrix}\sum_{k=2}^{L_1+1}c_\alpha(k)x_{i-k+1,j}^h\\ \sum_{l=2}^{L_2+1}c_\beta(l)x_{i,j-l+1}^v\end{bmatrix}+\begin{bmatrix}B_1\\ B_2\end{bmatrix}u_{ij}. \tag{3.50}$$

Boundary conditions for (3.48), (3.49) and (3.50) have the form:

$$x_{0j}^h\quad\text{for}\quad j\in\mathbb{Z}_+,\quad\text{and}\quad x_{i0}^v\quad\text{for}\quad i\in\mathbb{Z}_+\ . \tag{3.51}$$

Theorem 3.10. *The solution of state equation (3.49) for the boundary conditions (3.51) has the form*

$$\begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} = \sum_{p=0}^{i} T_{i-p,j} \begin{bmatrix} 0 \\ x_{p0}^v \end{bmatrix} + \sum_{q=0}^{j} T_{i,j-q} \begin{bmatrix} x_{0q}^h \\ 0 \end{bmatrix} + \sum_{p=0}^{i} \sum_{q=0}^{j} \left(T_{i-p-1,j-q} B_{10} + T_{i-p,j-q-1} B_{01} \right) u_{pq}, \tag{3.52a}$$

where

$$B_{10} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad B_{01} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \tag{3.52b}$$

and the transition matrix $T_{pq} \in \mathbb{R}^{n \times n}$ is defined as follows

$$T_{pq} = \begin{cases} I_n & \text{for} \quad p=0,\ q=0 \\ T_{10}T_{p-1,q} + T_{01}T_{p,q-1} + \Upsilon & \text{for} \quad p+q>0\ (p,q \in \mathbb{Z}_+) \\ 0 \text{ (zero matrix)} & \text{for} \quad p<0 \quad \text{and/or} \quad q<0 \end{cases} \tag{3.52c}$$

where

$$\Upsilon = -\sum_{k=2}^{p} \begin{bmatrix} c_\alpha(k) I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} T_{p-k,q} - \sum_{l=2}^{q} \begin{bmatrix} 0 & 0 \\ 0 & c_\beta(l) I_{n_2} \end{bmatrix} T_{p,q-l} \tag{3.52d}$$

$$T_{10} = \begin{bmatrix} \overline{A}_{11} & A_{12} \\ 0 & 0 \end{bmatrix}, \quad T_{01} = \begin{bmatrix} 0 & 0 \\ A_{21} & \overline{A}_{22} \end{bmatrix}. \tag{3.52e}$$

Proof. Let $X(z_1,z_2)$ be the 2D z-transform of the discrete function x_{ij}

$$X(z_1,z_2) = \mathscr{Z}[x_{ij}] = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} x_{ij} z_1^{-i} z_2^{-j}. \tag{3.53}$$

Using (3.53), we obtain

$$\mathscr{Z}\left[x_{i+1,j}^h\right] = z_1 \left[X^h(z_1,z_2) - X^h(0,z_2)\right], \tag{3.54a}$$

where $X^h(0,z_2) = \sum_{j=0}^{\infty} x_{0j}^h z_2^{-j}$,

$$\mathscr{Z}\left[x_{i,j+1}^v\right] = z_2 \left[X^v(z_1,z_2) - X^v(z_1,0)\right], \tag{3.54b}$$

where $X^v(z_1,0)=\sum_{i=0}^{\infty}x_{i0}^v z_1^{-i}$,

$$\mathscr{Z}\left[\sum_{k=2}^{i+1}c_\alpha(k)x_{i-k+1,j}^h\right]=\sum_{k=2}^{i+1}c_\alpha(k)z_1^{-k+1}X^h(z_1,z_2), \tag{3.54c}$$

since

$$\begin{aligned}\mathscr{Z}\left[x_{i-k,j}^h\right]&=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}x_{i-k,j}^h z_1^{-i}z_2^{-j}=\sum_{i=-k}^{\infty}\sum_{j=0}^{\infty}x_{ij}^h z_1^{-i-k}z_2^{-j}\\&=z_1^{-k}\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}x_{ij}^h z_1^{-i}z_2^{-j}\right]=z_1^{-k}X^h(z_1,z_2).\end{aligned} \tag{3.54d}$$

Similarly

$$\mathscr{Z}\left[\sum_{l=2}^{j+1}c_\beta(l)x_{i,j-l+1}^v\right]=\sum_{l=2}^{j+1}c_\beta(l)z_2^{-l+1}X^v(z_1,z_2), \tag{3.54e}$$

since

$$\begin{aligned}\mathscr{Z}\left[x_{i,j-l}^v\right]&=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}x_{i,j-l}^v z_1^{-i}z_2^{-j}=\sum_{i=0}^{\infty}\sum_{j=-l}^{\infty}x_{ij}^v z_1^{-i}z_2^{-j-l}\\&=z_2^{-l}\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}x_{ij}^v z_1^{-i}z_2^{-j}\right]=z_2^{-l}X^v(z_1,z_2).\end{aligned} \tag{3.54f}$$

Using (3.54), to (3.49) we obtain

$$\begin{aligned}&\begin{bmatrix}z_1X^h(z_1,z_2)-z_1X^h(0,z_2)\\ z_2X^v(z_1,z_2)-z_2X^v(z_1,0)\end{bmatrix}=\begin{bmatrix}\overline{A}_{11} & A_{12}\\ A_{21} & \overline{A}_{22}\end{bmatrix}\begin{bmatrix}X^h(z_1,z_2)\\ X^v(z_1,z_2)\end{bmatrix}\\&-\begin{bmatrix}\sum_{k=2}^{i+1}c_\alpha(k)z_1^{-k+1}X^h(z_1,z_2)\\ \sum_{l=2}^{j+1}c_\beta(l)z_2^{-l+1}X^v(z_1,z_2)\end{bmatrix}+\begin{bmatrix}B_1\\ B_2\end{bmatrix}U(z_1,z_2)\end{aligned} \tag{3.55}$$

where $U(z_1,z_2)=\mathscr{Z}(u_{ij})$.

Premultiplying (3.55) by the matrix blockdiag$[I_{n_1}z_1^{-1},I_{n_2}z_2^{-1}]$, we obtain

$$\begin{bmatrix}X^h(z_1,z_2)\\ X^v(z_1,z_2)\end{bmatrix}=G^{-1}(z_1,z_2)\left\{\begin{bmatrix}z_1^{-1}B_1\\ z_2^{-1}B_2\end{bmatrix}U(z_1,z_2)+\begin{bmatrix}X^h(0,z_2)\\ X^v(z_1,0)\end{bmatrix}\right\} \tag{3.56}$$

where

$$G(z_1,z_2)=\begin{bmatrix}G_{11}(z_1,z_2) & -z_1^{-1}A_{12}\\ -z_2^{-1}A_{21} & G_{22}(z_1,z_2)\end{bmatrix} \tag{3.57}$$

and

$$G_{11}(z_1,z_2) = I_{n_1} - z_1^{-1}\overline{A}_{11} + \sum_{k=2}^{i} c_\alpha(k) z_1^{-k} I_{n_1},$$
$$G_{22}(z_1,z_2) = I_{n_2} - z_2^{-1}\overline{A}_{22} + \sum_{l=2}^{j} c_\beta(l) z_2^{-l} I_{n_2}.$$

Let

$$G^{-1}(z_1,z_2) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq} z_1^{-p} z_2^{-q} \tag{3.58}$$

and

$$T_{pq} = \begin{bmatrix} T_{pq}^{11} & T_{pq}^{12} \\ T_{pq}^{21} & T_{pq}^{22} \end{bmatrix} \tag{3.59}$$

where T_{pq}^{kl} have the same dimension as A_{kl} for $k,l = 1,2$.

From the equality

$$G^{-1}(z_1,z_2)G(z_1,z_2) = G(z_1,z_2)G^{-1}(z_1,z_2) = I_n$$

and (3.58) and (3.59) we have

$$\begin{bmatrix} I_{n_1} - z_1^{-1}\overline{A}_{11} + \sum_{k=2}^{i} c_\alpha(k) z_1^{-k} I_{n_1} & -z_1^{-1}A_{12} \\ -z_2^{-1}A_{21} & I_{n_2} - z_2^{-1}\overline{A}_{22} + \sum_{l=2}^{j} c_\beta(l) z_2^{-l} I_{n_2} \end{bmatrix}$$
$$\times \left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \begin{bmatrix} T_{pq}^{11} & T_{pq}^{12} \\ T_{pq}^{21} & T_{pq}^{22} \end{bmatrix} z_1^{-p} z_2^{-q} \right) = \begin{bmatrix} I_{n_1} & 0 \\ 0 & I_{n_2} \end{bmatrix} \tag{3.60}$$

From (3.60) it follows that

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \left(T_{pq}^{11} - \overline{A}_{11} T_{p-1,q}^{11} + \sum_{k=2}^{p} c_\alpha(k) T_{p-k,q}^{11} - A_{12} T_{p-1,q}^{21} \right) z_1^{-p} z_2^{-q} = I_{n_1} \tag{3.61a}$$
$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \left(T_{pq}^{12} - \overline{A}_{11} T_{p-1,q}^{12} + \sum_{k=2}^{p} c_\alpha(k) T_{p-k,q}^{12} - A_{12} T_{p-1,q}^{22} \right) z_1^{-p} z_2^{-q} = 0 \tag{3.61b}$$

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \left(T_{pq}^{21} - \overline{A}_{22} T_{p,q-1}^{21} + \sum_{l=2}^{q} c_\beta(l) T_{p,q-l}^{21} - A_{21} T_{p,q-1}^{11} \right) z_1^{-p} z_2^{-q} = 0 \tag{3.61c}$$
$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \left(T_{pq}^{22} - \overline{A}_{22} T_{p,q-1}^{22} + \sum_{l=2}^{q} c_\beta(l) T_{p,q-l}^{22} - A_{21} T_{p,q-1}^{12} \right) z_1^{-p} z_2^{-q} = I_{n_2} \tag{3.61d}$$

Comparing the coefficients at the same powers of z_1 i z_2 in the equation (3.61), we obtain (3.52c).

Using (3.58) and applying the inverse z-transform and the convolution theorem to (3.56), we obtain (3.52a). □

Consider the system (3.50) and

$$\overline{G}(z_1,z_2) = \begin{bmatrix} \overline{G}_{11}(z_1,z_2) & -z_1^{-1}A_{12} \\ -z_2^{-1}A_{21} & \overline{G}_{22}(z_1,z_2) \end{bmatrix} \tag{3.62}$$

where

$$\overline{G}_{11}(z_1,z_2) = I_{n_1} - z_1^{-1}\overline{A}_{11} + \sum_{k=2}^{L_1} c_\alpha(k) z_1^{-k} I_{n_1},$$
$$\overline{G}_{22}(z_1,z_2) = I_{n_2} - z_2^{-1}\overline{A}_{22} + \sum_{l=2}^{L_2} c_\beta(l) z_2^{-l} I_{n_2}.$$

Let

$$\det\overline{G}(z_1,z_2) = \sum_{p=0}^{N_1}\sum_{q=0}^{N_2} a_{N_1-p,N_2-q} z_1^{-p} z_2^{-q}, \tag{3.63}$$

where $N_1, N_2 \in \mathbb{Z}_+$ are defined by the natural numbers L_1 i L_2 in (3.50).

Theorem 3.11. *Let (3.63) be the characteristic polynomial of the system (3.50). The matrices T_{pq} satisfy the equation*

$$\sum_{p=0}^{N_1}\sum_{q=0}^{N_2} a_{pq} T_{pq} = 0. \tag{3.64}$$

The proof is similar to the proof of Theorem 3.2 [166].

Theorem 3.11 is an extension of the classical Cayley-Hamilton theorem to the fractional 2D linear systems described by the Roesser model (3.49).

Definition 3.10. The system (3.49) is called (internally) positive fractional 2D Roesser model if $x_{ij}^h \in \mathbb{R}_+^{n_1}$, $x_{ij}^v \in \mathbb{R}_+^{n_2}$ and $y_{ij} \in \mathbb{R}_+^p$, $i,j \in \mathbb{Z}_+$ for any boundary conditions $x_{0j}^h \in \mathbb{R}_+^{n_1}$, $j \in \mathbb{Z}_+$, $x_{i0}^v \in \mathbb{R}_+^{n_2}$, $i \in \mathbb{Z}_+$ and all inputs $u_{ij} \in \mathbb{R}_+^m$, $i,j \in \mathbb{Z}_+$.

Theorem 3.12. *The fractional Roesser model (3.49) for $\alpha,\beta \in \mathbb{R}$, $0 < \alpha \le 1$, $0 < \beta \le 1$ is positive if and only if*

$$\begin{bmatrix} \overline{A}_{11} & A_{12} \\ A_{21} & \overline{A}_{22} \end{bmatrix} \in \mathbb{R}_+^{n\times n}, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \mathbb{R}_+^{n\times m}, \quad \begin{bmatrix} C_1 & C_2 \end{bmatrix} \in \mathbb{R}_+^{p\times n}, \quad D \in \mathbb{R}_+^{p\times m}, \tag{3.65}$$

The proof is similar to the proof of Theorem 3.3 [166].

3.8.3 Positive 2D Linear Systems with Delays

Consider the autonomous 2D Roesser model with q delays in state vector

$$\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = \sum_{k=0}^{q} A_k \begin{bmatrix} x^h_{i-k,j} \\ x^v_{i,j-k} \end{bmatrix}, \quad i,j \in \mathbb{Z}_+, \tag{3.66}$$

where $x^h_{ij} \in \mathbb{R}^{n_1}_+$, $x^v_{ij} \in \mathbb{R}^{n_2}_+$ are horizontal and vertical state vectors at the point (i,j) and

$$A_k = \begin{bmatrix} A^k_{11} & A^k_{12} \\ A^k_{21} & A^k_{22} \end{bmatrix}, \quad k = 0,1,\ldots,q \ . \tag{3.67}$$

Defining:

$$\overline{x}^h_{ij} = \begin{bmatrix} x^h_{ij} \\ x^h_{i-1,j} \\ \vdots \\ x^h_{i-q,j} \end{bmatrix}, \quad \overline{x}^v_{ij} = \begin{bmatrix} x^v_{ij} \\ x^v_{i,j-1} \\ \vdots \\ x^v_{i,j-q} \end{bmatrix}, \tag{3.68}$$

we may write the equation (3.66) in the form

$$\begin{bmatrix} \overline{x}^h_{i+1,j} \\ \overline{x}^v_{i,j+1} \end{bmatrix} = A \begin{bmatrix} \overline{x}^h_{i,j} \\ \overline{x}^v_{i,j} \end{bmatrix}, \quad i,j \in \mathbb{Z}_+, \tag{3.69}$$

where

$$A = \begin{bmatrix} A^0_{11} & A^1_{11} & \ldots & A^{q-1}_{11} & A^q_{11} & A^0_{12} & A^1_{12} & \ldots & A^{q-1}_{12} & A^q_{12} \\ I_{n_1} & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & I_{n_1} & 0 & 0 & 0 & \ldots & 0 & 0 \\ A^0_{21} & A^1_{21} & \ldots & A^{q-1}_{21} & A^q_{21} & A^0_{22} & A^1_{22} & \ldots & A^{q-1}_{22} & A^q_{22} \\ 0 & 0 & \ldots & 0 & 0 & I_{n_2} & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & I_{n_2} & 0 \end{bmatrix} \in \mathbb{R}^{N \times N}, \tag{3.70}$$

$$N = (q+1)(n_1+n_2).$$

The Roesser model with q delays (3.66) has been reduced to Roesser model without delays but with greater dimensions.

Theorem 3.13. *The Roesser model with q delays (3.66) is positive if and only if*

$$A_k \in \mathbb{R}^{(n_1+n_1)\times(n_2+n_2)}_+ \quad \text{for} \quad k = 0,1,\ldots,q \quad \text{or equivalently} \quad A \in \mathbb{R}^{N\times N}_+. \tag{3.71}$$

The proof follows immediately from Theorem 3.9, applied to the model (3.69).

Consider the autonomous general model with q delays

$$x_{i+1,j+1} = \sum_{k=0}^{q} \left(A_k^0 x_{i-k,j-k} + A_k^1 x_{i+1-k,j-k} + A_k^2 x_{i-k,j+1-k} \right), \quad i,j \in \mathbb{Z}_+, \tag{3.72}$$

where $x_{ij} \in \mathbb{R}_+^n$ is the state vector at the point (i,j) and $A_k^t \in \mathbb{R}^{n \times n}$, $k = 0,1,\ldots,q$; $t = 0,1,2$.

Defining vector

$$\overline{x}_{ij} = \begin{bmatrix} x_{ij} \\ x_{i-1,j-1} \\ \vdots \\ x_{i-q,j-q} \end{bmatrix}, \tag{3.73}$$

and the matrices

$$\overline{A}_0 = \begin{bmatrix} A_0^0 & A_1^0 & \ldots & A_{q-1}^0 & A_q^0 \\ I_n & 0 & \ldots & 0 & 0 \\ 0 & I_n & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & I_n & 0 \end{bmatrix}, \quad \overline{A}_1 = \begin{bmatrix} A_0^1 & A_1^1 & \ldots & A_{q-1}^1 & A_q^1 \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix},$$
$$\overline{A}_2 = \begin{bmatrix} A_0^2 & A_1^2 & \ldots & A_{q-1}^2 & A_q^2 \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix}, \tag{3.74}$$

we may write (3.72) in the form

$$\overline{x}_{i+1,j+1} = \overline{A}_0 \overline{x}_{i,j} + \overline{A}_1 \overline{x}_{i+1,j} + \overline{A}_2 \overline{x}_{i,j+1}, \quad i,j \in \mathbb{Z}_+, \tag{3.75}$$

The general 2D model with q delays has been reduced to the equivalent general 2D model without delays but with greater dimensions.

Theorem 3.14. *The general 2D model with q delays (3.72) is positive if and only if* $A_k^t \in \mathbb{R}_+^{n \times n}$ *for* $k = 0,1,\ldots,q$; $t = 0,1,2$ *or equivalently if and only if* $\overline{A}_t \in \mathbb{R}_+^{\overline{N} \times \overline{N}}$, $t = 0,1,2$; $\overline{N} = (q+1)n$.

The proof follows immediately from Theorem 3.8 applied to the model (3.75).

3.9 Positive Fractional 2D Linear System of Different Orders

3.9.1 Definition of (Backward) Difference of (α,β) Order of 2D Function

Definition 3.11. The function defined by

$$\Delta^{\alpha,\beta} x_{ij} = \sum_{k=0}^{i}\sum_{l=0}^{j}(-1)^{k+l}\binom{\alpha}{k}\binom{\beta}{l}x_{i-k,j-l} = \sum_{k=0}^{i}\sum_{l=0}^{j}c_{\alpha\beta}(k,l)x_{i-k,j-l}, \quad (3.76a)$$

$$n_1-1<\alpha<n_1, \quad n_2-1<\beta<n_2, \quad n_1,n_2\in\mathbb{N}, \quad \alpha,\beta\in\mathbb{R},$$

is called the (backward) difference of (α,β)-order of the function x_{ij} where

$$\Delta^{\alpha,\beta} x_{ij} = \Delta_i^{\alpha}\Delta_j^{\beta} x_{ij},$$

and

$$c_{\alpha\beta}(k,l) = \begin{cases} 1 & \text{for } k=0 \text{ and } l=0 \\ (-1)^k \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} & \text{for } k>0 \text{ and } l=0 \\ (-1)^l \frac{\beta(\beta-1)\cdots(\beta-l+1)}{l!} & \text{for } k=0 \text{ and } l>0 \\ (-1)^{k+l} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)\beta(\beta-1)\cdots(\beta-l+1)}{k!l!} & \text{for } k>0 \text{ and } l>0 \end{cases} \quad (3.76b)$$

3.9.2 State Equations of Fractional 2D Linear System

The state equations of the general fractional 2D model of linear systems have the form:

$$\begin{aligned}\Delta^{\alpha,\beta} x_{i+1,j+1} &= A_0x_{ij}+A_1x_{i+1,j}+A_2x_{i,j+1} \\ &+ B_0u_{ij}+B_1u_{i+1,j}+B_2u_{i,j+1}, \quad (3.77a)\\ y_{ij} &= Cx_{ij}+Du_{ij}, \quad (3.77b)\end{aligned}$$

where $x_{ij}\in\mathbb{R}^n$, $u_{ij}\in\mathbb{R}^m$, $y_{ij}\in\mathbb{R}^p$ are state, input and output vectors and $A_k\in\mathbb{R}^{n\times n}$, $B_k\in\mathbb{R}^{n\times m}$, $k=0,1,2$, $C\in\mathbb{R}^{p\times n}$, $D\in\mathbb{R}^{p\times m}$.

From (3.76a) we have

$$\begin{aligned}\Delta^{\alpha,\beta} x_{i+1,j+1} &= \sum_{k=0}^{i+1}\sum_{l=0}^{j+1}c_{\alpha\beta}(k,l)x_{i-k+1,j-l+1} \\ &= x_{i+1,j+1}+\sum_{k=0}^{i+1}\sum_{\substack{l=0\\k+l>0}}^{j+1}c_{\alpha\beta}(k,l)x_{i-k+1,j-l+1}. \quad (3.78)\end{aligned}$$

Using (3.78) we may write the equation (3.77a) in the form

$$x_{i+1,j+1} = \overline{A}_0 x_{ij} + \overline{A}_1 x_{i+1,j} + \overline{A}_2 x_{i,j+1} - \sum_{\substack{k=0 \\ k,l\neq 1}}^{i+1} \sum_{\substack{l=0 \\ k+l>0}}^{j+1} c_{\alpha\beta}(k,l) x_{i-k+1,j-l+1}$$
$$+ B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1}, \tag{3.79a}$$

where:

$$\begin{aligned} \overline{A}_0 &= A_0 - c_{\alpha\beta}(1,1) = A_0 - \alpha\beta I_n, \\ \overline{A}_1 &= A_1 - c_{\alpha\beta}(0,1) = A_1 + \beta I_n, \\ \overline{A}_2 &= A_2 - c_{\alpha\beta}(1,0) = A_2 + \alpha I_n. \end{aligned} \tag{3.79b}$$

Let

$$D_{ij} = \{k,l \in \mathbb{Z}_+, \quad 0 \le k \le i, \quad 0 \le l \le j\}, \quad D = D_{i+1,j+1} \setminus D_{11},$$

then the equation (3.79a) takes the form

$$x_{i+1,j+1} = \overline{A}_0 x_{ij} + \overline{A}_1 x_{i+1,j} + \overline{A}_2 x_{i,j+1} - \sum_{i,j\in D}\sum c_{\alpha\beta}(k,l) x_{i-k+1,j-l+1}$$
$$+ B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1}. \tag{3.79c}$$

From (3.76b) it follows that the coefficients $c_{\alpha\beta}(k,l)$ strongly decrease when k and l increase. In practice usually it is assumed that i and j are bounded by some natural numbers L_1 and L_2. In this case the equation (3.79a) takes the form

$$x_{i+1,j+1} = \overline{A}_0 x_{ij} + \overline{A}_1 x_{i+1,j} + \overline{A}_2 x_{i,j+1} - \sum_{\substack{k=0 \\ k,l\neq 1}}^{L_1+1} \sum_{\substack{l=0 \\ k+l>0}}^{L_2+1} c_{\alpha\beta}(k,l) x_{i-k+1,j-l+1}$$
$$+ B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1}. \tag{3.79d}$$

Remark 3.2. From (3.79) if follows that the fractional 2D linear system is a system with increasing numbers of delays in state vector.

Boundary conditions for (3.79) have the form:

$$x_{i0}, i \in \mathbb{Z}_+, \quad \text{and} \quad x_{0j}, j \in \mathbb{Z}_+. \tag{3.80}$$

3.9.3 Solution of the State Equations of the Fractional 2D Linear Systems

Applying the 2D z-transform ($\mathscr{Z}$) we shall derive the solution of the state equation (3.79a) of the fractional 2D linear system.

Theorem 3.15. *The solution of the state equation (3.79a) with boundary conditions (3.80) has the form*

$$
\begin{aligned}
x_{ij} &= \sum_{p=0}^{i-1}\sum_{q=0}^{j-1} T_{i-p-1,j-q-1}B_0u_{pq} + \sum_{p=0}^{i}\sum_{q=0}^{j-1} T_{i-p,j-q-1}B_1u_{pq} \\
&+ \sum_{p=0}^{i-1}\sum_{q=0}^{j} T_{i-p-1,j-q}B_2u_{pq} + \sum_{p=0}^{i} T_{i-p,j}x_{p0} + \sum_{q=0}^{j} T_{i,j-q}x_{0q} - T_{ij}x_{00} \\
&- \sum_{q=0}^{j-1} T_{i,j-q-1}\left[\overline{A}_1\ B_1\right]\begin{bmatrix} x_{0q} \\ u_{0q} \end{bmatrix} - \sum_{p=0}^{i-1} T_{i-p-1,j}\left[\overline{A}_2\ B_2\right]\begin{bmatrix} x_{p0} \\ u_{p0} \end{bmatrix} \\
&+ \sum_{k=2}^{i+1}\sum_{p=0}^{i-k} c_{\alpha\beta}(k,0)T_{i-p-k,j}x_{p0} + \sum_{l=2}^{j+1}\sum_{q=0}^{j-l} c_{\alpha\beta}(0,l)T_{i,j-q-l}x_{0q}
\end{aligned} \tag{3.81}
$$

where the matrices T_{pq} are defined as follows

$$
T_{pq} = \begin{cases} I_n & \textit{for} \quad p = q = 0 \\ \overline{A}_0T_{p-1,q-1} + \overline{A}_1T_{p,q-1} + \overline{A}_2T_{p-1,q} - \Upsilon & \textit{for} \quad p+q > 0 \\ 0 \quad \text{(zero matrix)} & \textit{for} \quad p < 0 \quad \textit{and/or} \quad q < 0 \end{cases} \tag{3.82}
$$

and

$$
\Upsilon = \sum_{k=0}^{p}\sum_{l=0}^{q} c_{\alpha\beta}(p-k, q-l)T_{kl} \quad \textit{for} \quad k,l \neq p-1, q-1 \quad \textit{and} \quad k+l < p+q-1.
$$

Proof. Let $X(z_1, z_2)$ be the 2D z-transform of the discrete function x_{ij}, defined by (A.15). Taking in to account

$$
\begin{aligned}
&\mathscr{Z}\left[\sum_{\substack{k=0\\k,l\neq 1}}^{i+1}\sum_{\substack{l=0\\k+l>0}}^{j+1} c_{\alpha\beta}(k,l)x_{i-k+1,j-l+1}\right] = \mathscr{Z}\left[\sum_{\substack{k=1\\k,l\neq 1}}^{i+1}\sum_{l=1}^{j+1} c_{\alpha\beta}(k,l)x_{i-k+1,j-l+1}\right.\\
&+\left.\sum_{k=2}^{i+1} c_{\alpha\beta}(k,0)x_{i-k+1,j+1} + \sum_{l=2}^{j+1} c_{\alpha\beta}(0,l)x_{i+1,j-l+1}\right]\\
&= \sum_{\substack{k=1\\k,l\neq 1}}^{i+1}\sum_{l=1}^{j+1} c_{\alpha\beta}(k,l)z_1^{-k+1}z_2^{-l+1}X(z_1,z_2)\\
&+ \sum_{k=2}^{i+1} c_{\alpha\beta}(k,0)z_1^{-k+1}z_2[X(z_1,z_2)-X(z_1,0)]\\
&+ \sum_{l=2}^{j+1} c_{\alpha\beta}(0,l)z_1z_2^{-l+1}[X(z_1,z_2)-X(0,z_2)]\\
&= \sum_{\substack{k=0\\k,l\neq 1}}^{i+1}\sum_{\substack{l=0\\k+l>1}}^{j+1} c_{\alpha\beta}(k,l)z_1^{-k+1}z_2^{-l+1}X(z_1,z_2)\\
&- \sum_{k=2}^{i+1} c_{\alpha\beta}(k,0)z_1^{-k+1}z_2X(z_1,0) - \sum_{l=2}^{j+1} c_{\alpha\beta}(0,l)z_1z_2^{-l+1}X(0,z_2),
\end{aligned}
$$

and applying the 2D z-transform to (3.79a) and using Appendix A.3, we obtain

$$
\begin{aligned}
X(z_1,z_2) &= G^{-1}(z_1,z_2)\{(B_0+B_1z_1+B_2z_2)U(z_1,z_2)\\
&+ z_1z_2[X(z_1,0)+X(0,z_2)-x_{00}]\\
&+ \sum_{l=2}^{j+1} c_{\alpha\beta}(0,l)z_1z_2^{-l+1}X(0,z_2) + \sum_{k=2}^{i+1} c_{\alpha\beta}(k,0)z_1^{-k+1}z_2X(z_1,0)\\
&- z_1\left[\overline{A}_1 B_1\right]\begin{bmatrix} X(0,z_2)\\ U(0,z_2)\end{bmatrix} - z_2\left[\overline{A}_2 B_2\right]\begin{bmatrix} X(z_1,0)\\ U(z_1,0)\end{bmatrix}\Bigg\},
\end{aligned} \tag{3.83a}
$$

where

$$
G(z_1,z_2) = \left[z_1z_2I_n + \sum_{\substack{k=0\\k,l\neq 1}}^{i+1}\sum_{\substack{l=0\\k+l>1}}^{j+1} c_{\alpha\beta}(k,l)z_1^{-(k-1)}z_2^{-(l-1)} - \overline{A}_0 - \overline{A}_1z_1 - \overline{A}_2z_2\right] \tag{3.83b}
$$

and $U(z_1,z_2) = \mathscr{Z}[u_{ij}]$.

Let

$$
G^{-1}(z_1,z_2) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq}z_1^{-(p+1)}z_2^{-(q+1)}. \tag{3.84}
$$

From the equality

$$G^{-1}(z_1,z_2)G(z_1,z_2) = G(z_1,z_2)G^{-1}(z_1,z_2) = I_n,$$

we have

$$\begin{aligned} I_n &= \left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq} z_1^{-(p+1)} z_2^{-(q+1)}\right) G(z_1,z_2) \\ &= G(z_1,z_2)\left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq} z_1^{-(p+1)} z_2^{-(q+1)}\right). \end{aligned} \tag{3.85}$$

Substituting of (3.83b) into (3.85) yields

$$\begin{aligned} I_n = &\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq} z_1^{-p} z_2^{-q} + \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \sum_{\substack{k=0 \\ k,l\neq 1}}^{i+1} \sum_{\substack{l=0 \\ k+l>1}}^{j+1} c_{\alpha\beta}(k,l) z_1^{-(p+k)} z_2^{-(q+l)} \\ &- \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \overline{A}_0 T_{pq} z_1^{-(p+1)} z_2^{-(q+1)} - \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \overline{A}_1 T_{pq} z_1^{-p} z_2^{-(q+1)} \\ &- \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \overline{A}_2 T_{pq} z_1^{-(p+1)} z_2^{-q}, \end{aligned}$$

and

$$\begin{aligned} I_n = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} &\Bigg[T_{pq} + \sum_{\substack{k=0 \\ k,l\neq 1}}^{i+1} \sum_{\substack{l=0 \\ k+l>1}}^{j+1} c_{\alpha\beta}(k,l) T_{p-k,q-l} - \overline{A}_0 T_{p-1,q-1} \\ &- \overline{A}_1 T_{p,q-1} - \overline{A}_2 T_{p-1,q}\Bigg] z_1^{-p} z_2^{-q}. \end{aligned}$$

Comparing the coefficients at the same powers of z_1 and z_2 in equation (3.85) we obtain (3.82). Substituting (3.84) into (3.83a) we obtain

$$\begin{aligned} X(z_1,z_2) = &\left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq} z_1^{-(p+1)} z_2^{-(q+1)}\right) \{(B_0 + B_1 z_1 + B_2 z_2) U(z_1,z_2) \\ &+ z_1 z_2 [X(z_1,0) + X(0,z_2) - x_{00}] \\ &- z_1 \left[\overline{A}_1 B_1\right] \begin{bmatrix} X(0,z_2) \\ U(0,z_2) \end{bmatrix} - z_2 \left[\overline{A}_2 B_2\right] \begin{bmatrix} X(z_1,0) \\ U(z_1,0) \end{bmatrix} \\ &+ \sum_{l=2}^{j+1} c_{\alpha\beta}(0,l) z_1 z_2^{-l+1} X(0,z_2) + \sum_{k=2}^{i+1} c_{\alpha\beta}(k,0) z_1^{-k+1} z_2 X(z_1,0) \Bigg\}, \end{aligned} \tag{3.86}$$

Applying the inverse 2D z-transform to the equation (3.86) and taking into account $T_{pq} = 0$ for $p < 0$ and $q < 0$, we obtain the desired solution (3.81). □

Using (3.82) it can be easily shown that for $i, j \in \mathbb{Z}_+$ we have

$$
\begin{aligned}
x_{ij} = {} & \sum_{p=0}^{i}\sum_{q=0}^{j} T_{i-p-1,j-q-1}B_0 u_{pq} + \sum_{p=1}^{i}\sum_{q=1}^{j}\left(T_{i-p,j-q-1}B_1 + T_{i-p-1,j-q}B_2\right)u_{pq} \\
& + \sum_{p=1}^{i} T_{i-p,j-1}B_1 u_{p0} + \sum_{q=1}^{j} T_{i-1,j-q}B_2 u_{0q} + \sum_{k=2}^{i}\sum_{p=0}^{i-k} c_{\alpha\beta}(k,0)T_{i-p-k,j}x_{p0} \\
& + \sum_{p=1}^{i}\left(T_{i-p-1,j-1}\overline{A}_0 + T_{i-p,j-1}\overline{A}_1 - \sum_{\substack{k=0 \\ D_1}}^{i-p}\sum_{\substack{l=0 \\ D_2}}^{j} c_{\alpha\beta}(i-p-k,j-l)T_{k,l}\right)x_{p0} \\
& + \sum_{q=1}^{j}\left(T_{i-1,j-q-1}\overline{A}_0 + T_{i-1,j-q}\overline{A}_2 - \sum_{\substack{k=0 \\ D_3}}^{i}\sum_{\substack{l=0 \\ D_4}}^{j-q} c_{\alpha\beta}(i-k,j-q-l)T_{kl}\right)x_{0q} \\
& + \sum_{l=2}^{j}\sum_{q=0}^{j-l} c_{\alpha\beta}(0,l)T_{i,j-q-l}x_{0q} \\
& + \left(T_{i-1,j-1}\overline{A}_0 - \sum_{\substack{k=0 \\ D_5}}^{i}\sum_{\substack{l=0 \\ D_6}}^{j} c_{\alpha\beta}(i-k,j-l)T_{kl}\right)x_{00},
\end{aligned}
$$

where:

$$
\begin{aligned}
& D_1 = k+l < i+j-p-1, \quad D_2 = k,l \neq i-p-1, j-1, \\
& D_3 = k+l < i+j-q-1, \quad D_4 = k,l \neq i-1, j-q-1, \\
& D_5 = k+l < i+j-1, \qquad D_6 = k,l \neq i-1, j-1.
\end{aligned}
\tag{3.87}
$$

After some manipulations the solution can be rewritten in the form

$$
\begin{aligned}
x_{ij} &= \sum_{p=0}^{i}\sum_{q=0}^{j} T_{i-p-1,j-q-1}B_0u_{pq} + \sum_{p=1}^{i}\sum_{q=1}^{j}\left(T_{i-p,j-q-1}B_1 + T_{i-p-1,j-q}B_2\right)u_{pq} \\
&+ \sum_{p=1}^{i} T_{i-p,j-1}B_1u_{p0} + \sum_{q=1}^{j} T_{i-1,j-q}B_2u_{0q} \\
&+ \sum_{p=1}^{i}\left(T_{i-p-1,j-1}\overline{A}_0 + T_{i-p,j-1}\overline{A}_1 - \sum_{\substack{k=0\\k+l>1}}^{i-p}\ \sum_{\substack{l=1\\k,l\neq 1}}^{j} c_{\alpha\beta}(k,l)T_{i-p-k,j-l}\right)x_{p0} \\
&+ \sum_{q=1}^{j}\left(T_{i-1,j-q-1}\overline{A}_0 + T_{i-1,j-q}\overline{A}_2 - \sum_{\substack{k=1\\k+l>1}}^{i}\ \sum_{\substack{l=0\\k,l\neq 1}}^{j-q} c_{\alpha\beta}(k,l)T_{i-k,j-q-l}\right)x_{0q} \\
&+ \left(T_{i-1,j-1}\overline{A}_0 - \sum_{\substack{k=1\\k,l\neq 1}}^{i}\ \sum_{l=1}^{j} c_{\alpha\beta}(k,l)T_{i-k,j-l}\right)x_{00},
\end{aligned}
$$

or

$$
\begin{aligned}
x_{ij} &= \sum_{p=0}^{i}\sum_{q=0}^{j} T_{i-p-1,j-q-1}B_0u_{pq} + \sum_{p=1}^{i}\sum_{q=1}^{j}\left(T_{i-p,j-q-1}B_1 + T_{i-p-1,j-q}B_2\right)u_{pq} \\
&+ \sum_{p=1}^{i} T_{i-p,j-1}B_1u_{p0} + \sum_{q=1}^{j} T_{i-1,j-q}B_2u_{0q} \\
&+ \sum_{p=1}^{i}\left(T_{i-p-1,j-1}\overline{A}_0 + T_{i-p,j-1}\overline{A}_1 - \sum_{\substack{k=0\\D_1}}^{i-p}\ \sum_{\substack{l=0\\D_2}}^{j-1} c_{\alpha\beta}(i-p-k,j-l)T_{k,l}\right)x_{p0} \\
&+ \sum_{q=1}^{j}\left(T_{i-1,j-q-1}\overline{A}_0 + T_{i-1,j-q}\overline{A}_2 - \sum_{\substack{k=0\\D_3}}^{i-1}\ \sum_{\substack{l=0\\D_4}}^{j-q} c_{\alpha\beta}(i-k,j-q-l)T_{k,l}\right)x_{0q} \\
&+ \left(T_{i-1,j-1}\overline{A}_0 - \sum_{\substack{k=0\\D_7}}^{i-1}\ \sum_{l=0}^{j-1} c_{\alpha\beta}(i-k,j-l)T_{k,l}\right)x_{00}.
\end{aligned}
$$

where D_i, $i = 1,2,3,4$; are given by (3.87) and $D_7 = k+l < i+j-2$.

3.9.4 Extension of the Cayley-Hamilton Theorem

From (3.83b) we have

$$G(z_1,z_2) = z_1z_2\overline{G}(z_1,z_2), \tag{3.88}$$

where

$$\overline{G}(z_1,z_2) = I_n + \sum_{k=0}^{L_1+1}\sum_{l=0}^{L_2+1} I_n c_{\alpha\beta}(k,l) z_1^{-k} z_2^{-l} - \overline{A}_0 z^{-1} z_2^{-1} - \overline{A}_q z_2^{-1} - \overline{A}_2 z_1^{-1}. \quad (3.89)$$

and

$$\det\left[\overline{G}(z_1,z_2)\right] = \sum_{k=0}^{N_1}\sum_{l=0}^{N_2} a_{N_1-k,N_2-l} z_1^{-k} z_2^{-l}. \quad (3.90)$$

It is assumed that i and j are bounded by some natural numbers L_1 i L_2, which determine the degrees N_1 and N_2.

From (3.88) and (3.84) it follows that

$$G^{-1}(z_1,z_2) = z_1^{-1} z_2^{-1} \overline{G}^{-1}(z_1,z_2) = z_1^{-1} z_2^{-1} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq} z_1^{-p} z_2^{-q}, \quad (3.91)$$

and

$$\overline{G}^{-1}(z_1,z_2) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq} z_1^{-p} z_2^{-q}, \quad (3.92)$$

where T_{pq} are defined by (3.82).

Theorem 3.16. *Let(3.90) be the characteristic polynomial of (3.77). The matrices T_{kl} satisfy the equation*

$$\sum_{k=0}^{N_1}\sum_{l=0}^{N_2} a_{kl} T_{kl} = 0. \quad (3.93)$$

Proof. From definition of the inverse matrix and (3.90), (3.92) we have

$$\mathrm{Adj}\left[\overline{G}(z_1,z_2)\right] = \left(\sum_{k=0}^{N_1}\sum_{l=0}^{N_2} a_{N_1-k,N_2-l} z_1^{-k} z_2^{-l}\right)\left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq} z_1^{-p} z_2^{-q}\right), \quad (3.94)$$

where Adj $\left[\overline{G}(z_1,z_2)\right]$ is adjoint matrix of $\overline{G}(z_1,z_2)$.

Comparing the coefficients at the same power of $z_1^{-N_1} z_2^{-N_2}$ in equation (3.94), we obtain (3.93), since the degree of the polynomial matrix (3.94) is less than N_1 i N_2. □

Theorem 3.16 is an extension of the classical Cayley-Hamilton theorem to the fractional 2D linear systems described by (3.77).

3.9.5 Positivity of the Fractional 2D Linear Systems

Lemma 3.5. *If*

a) $0 < \alpha < 1$ *and* $1 < \beta < 2$ *then*

$$c_{\alpha\beta}(k,l) < 0 \quad \textit{for} \quad k = 1,2,\ldots; \quad l = 2,3,\ldots; \quad (3.95a)$$

b) $1 < \alpha < 2$ *and* $0 < \beta < 1$ *then*

$$c_{\alpha\beta}(k,l) < 0 \quad for \quad k = 2,3,\ldots; \quad l = 1,2,\ldots; \tag{3.95b}$$

c) $0 < \alpha < 1$ *and* $1 < \beta < 2$ *then*

$$c_{\alpha\beta}(k,1) > 0 \quad for \quad k = 2,3,\ldots; \tag{3.95c}$$

and

$$c_{\alpha\beta}(0,l) > 0 \quad for \quad l = 2,3,\ldots; \tag{3.95d}$$

Proof. The proof will be accomplished by induction. The hypothesis (3.95a) is true for $k = 1$ and $l = 2$ since

$$c_{\alpha\beta}(1,2) = (-1)^3 \frac{\alpha\beta(\beta-1)}{2} < 0.$$

Assuming that the hypothesis is true for the pair (k,l), $k+l \geq 3$, we shall show that it is also valid for the pairs $(k+1,l)$, $(k,l+1)$ and $(k+1,l+1)$.

From (3.76b) we have

$$c_{\alpha\beta}(k+1,l) = c_{\alpha\beta}(k,l)\frac{k-\alpha}{k+1} < 0,$$

since $c_{\alpha\beta}(k,l) < 0$ for $k = 1,2,\ldots$; $l = 2,3,\ldots$.

Similarly

$$c_{\alpha\beta}(k,l+1) = c_{\alpha\beta}(k,l)\frac{l-\beta}{l+1} < 0,$$

since $c_{\alpha\beta}(k,l) < 0$ for $k = 1,2,\ldots$; $l = 2,3,\ldots$; and

$$c_{\alpha\beta}(k+1,l+1) = c_{\alpha\beta}(k,l)\frac{(k-\alpha)(l-\beta)}{(k+1)(l+1)} < 0,$$

since $c_{\alpha\beta}(k,l) < 0$ for $k = 1,2,\ldots$; $l = 2,3,\ldots$. Proofs for (3.95b), (3.95c) and (3.95d) are similar. □

Remark 3.3. Taking in to account (3.95c) and (3.95d) we shall assume that for $0 < \alpha < 1$ and $1 < \beta < 2$

$$\sum_{k=2}^{i+1} c_{\alpha\beta}(k,1)x_{i-k+1,j} = 0, \quad \sum_{l=2}^{j+1} c_{\alpha\beta}(0,l)x_{i+1,j-l+1} = 0. \tag{3.96}$$

Lemma 3.6. *If the conditions (3.95) are satisfied and*

$$\overline{A}_k \in \mathbb{R}_+^{n\times n} \quad for \quad k = 0,1,2, \tag{3.97}$$

then

$$T_{pq} \in \mathbb{R}_+^{n\times n} \quad \textit{for} \quad p,q \in \mathbb{Z}_+. \tag{3.98}$$

Proof. If the conditions (3.95), (3.96), (3.97) are satisfied then from (3.82) we obtain (3.98). □

Definition 3.12. The system (3.77) is called (internally) positive fractional 2D linear system if $x_{ij} \in \mathbb{R}_+^n$ and $y_{ij} \in \mathbb{R}_+^p$, $i, j \in \mathbb{Z}_+$ for any boundary conditions $x_{i0} \in \mathbb{R}_+^n$, $i \in \mathbb{Z}_+$, $x_{0j} \in \mathbb{R}_+^n$, $i \in \mathbb{Z}_+$ and all inputs $u_{ij} \in \mathbb{R}_+^p$, $i, j \in \mathbb{Z}_+$.

Theorem 3.17. *Let the assumptions (3.96) be satisfied. The fractional 2D linear system (3.77) for* $0 < \alpha < 1$ *and* $1 < \beta < 2$ *(or* $1 < \alpha < 2$ *and* $0 < \beta < 1$*) is positive if and only if:*

$$\overline{A}_k \in \mathbb{R}_+^{n\times n},\ B_k \in \mathbb{R}_+^{n\times m}\ \textit{for}\ k = 0,1,2,\ C \in \mathbb{R}_+^{p\times n},\ D \in \mathbb{R}_+^{p\times m}. \tag{3.99}$$

Proof. Sufficiency. If the conditions (3.99) are satisfied then by Lemma 3.6 $T_{pq} \in \mathbb{R}_+^{n\times n}$ and from (3.81) we have $x_{ij} \in \mathbb{R}_+^n$ for $i, j \in \mathbb{Z}_+$, since $x_{i0} \in \mathbb{R}_+^n$, $x_{0j} \in \mathbb{R}_+^n$ and $u_{ij} \in \mathbb{R}_+^m$ for $i, j \in \mathbb{Z}_+$. From (3.77b) we have $y_{ij} \in \mathbb{R}_+^p$ since $C \in \mathbb{R}_+^{p\times n}$, $D \in \mathbb{R}_+^{p\times m}$ and $x_{ij} \in \mathbb{R}_+^n$, $u_{ij} \in \mathbb{R}_+^m$ for $i, j \in \mathbb{Z}_+$.

Necessity. Let the system be positive and $x_{00} = e_{ni}$, $i = 1,\dots,n$ (e_{ni} is i-th column of the identity matrix I_n), $x_{01} = x_{10} = 0$, $u_{ij} = 0$, $i, j \in \mathbb{Z}_+$. From (3.79a) for $i = j = 0$ and $u_{ij} = 0$, $i, j \in \mathbb{Z}+$ we obtain $x_{11} = \overline{A}_0 e_{ni} = \overline{A}_{0i} \in \mathbb{R}_+^n$, where $\overline{A}_{0i}$ is i-th column of $\overline{A}_0$. This implies $\overline{A}_0 \in \mathbb{R}_+^{n\times n}$, since $i = 1,\dots,n$. If we assume that $x_{10} = e_{ni}$, $x_{00} = x_{01} = 0$ and $u_{ij} = 0$, $i, j \in \mathbb{Z}_+$, then from (3.79a) for $i = j = 0$ we have $x_{11} = \overline{A}_1 e_{ni} = \overline{A}_{1i} \in \mathbb{R}_+^n$, what implies $\overline{A}_1 \in \mathbb{R}_+^{n\times n}$. In a similar way we may show that $\overline{A}_2 \in \mathbb{R}_+^{n\times n}$. Assuming $u_{00} = e_{ni}$, $u_{ij} = 0$, $i, j \in \mathbb{Z}_+$ $i + j > 0$ and $x_{00} = x_{10} = x_{01} = 0$ from (3.79a), for $i = j = 0$, we obtain $x_{11} = B_0 e_{mi} = B_{0i} \in \mathbb{R}_+^m$ for $i = 1,\dots,m$, what implies $B_0 \in \mathbb{R}_+^{n\times m}$. In a similar way we may show that $B_k \in \mathbb{R}_+^{n\times m}$ for $k = 1,2$ and $C \in \mathbb{R}_+^{p\times n}$, $D \in \mathbb{R}_+^{p\times m}$. □

Remark 3.4. From (3.76b) and (3.79a) it follows that if $\alpha = \beta$, $0 < \alpha < 1$, then $c_{\alpha\beta}(k,l) > 0$ for $k, l = 1,2,\dots$ and the fractional 2D linear system (3.77) is not positive.

The considerations presented for Roesser model can be easily extended to the model (3.77) [103].