Chapter 2
Fractional Continuous-Time Linear Systems

2.1 Definition of Euler Gamma Function and Its Properties

There exist the following two definitions of the Euler gamma function.

Definition 2.1. A function given by the integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad \mathrm{Re}(x) > 0, \tag{2.1}$$

is called the Euler gamma function.

The Euler gamma function can be also defined by

$$\Gamma(x) = \lim_{n\to\infty} \frac{n! n^x}{x(x+1)\cdots(x+n)}, \quad x \in \mathbb{C}\backslash\{0,-1,-2,\ldots\} \tag{2.2}$$

where $\mathbb{C}$ is the field of complex numbers.

We shall show that $\Gamma(x)$ satisfies the equality

$$\Gamma(x+1) = x\Gamma(x). \tag{2.3}$$

Proof. Using (2.1), we obtain

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt = \int_0^\infty t^x de^{-t} = t^x e^{-t}\Big|_0^\infty = x\int_0^\infty t^{x-1} e^{-t} dt = x\Gamma(x).$$

□

Example 2.1. From (2.3) we have for:

$$\begin{aligned}
x = 1: &\quad \Gamma(2) = 1\cdot\Gamma(1) = 1, \quad \text{since} \quad \Gamma(1) = \int_0^\infty e^{-t} dt = 1,\\
x = 2: &\quad \Gamma(3) = 2\cdot\Gamma(2) = 1\cdot 2 = 2!,\\
x = 3: &\quad \Gamma(4) = 3\cdot\Gamma(3) = 3\cdot 2\cdot\Gamma(2) = 3!.
\end{aligned}$$

T. Kaczorek: Selected Problems of Fractional Systems Theory, LNCIS 411, pp. 27–52.
springerlink.com

In general case for $x \in \mathbb{N}$ we have

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\cdots(1) = n!$$

The gamma function is also well-define for x being any real (complex) numbers. For example we have for

$$\begin{aligned} x = 1.5 \ : &\qquad \Gamma(2.5) = 1.5 \cdot \Gamma(1.5) = 1.5 \cdot 0.5\Gamma(0.5), \\ x = -0.5 : &\qquad \Gamma(0.5) = -0.5 \cdot \Gamma(-0.5) = -0.5 \cdot (-1.5)\Gamma(-1.5). \end{aligned}$$

2.2 Mittag-Leffler Function

The Mittag-Leffler function is a generalization of the exponential function $e^{s_i t}$ and it plays important role in solution of the fractional differential equations.

Definition 2.2. A function of the complex variable z defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \tag{2.4}$$

is called the one parameter Mittag-Leffler function.

Example 2.2. For $\alpha = 1$ we obtain

$$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z,$$

i.e. the classical exponential function.

An extension of the one parameter Mittag-Leffler function is the following two parameters function.

Definition 2.3. A function of the complex variable z defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \tag{2.5}$$

is called two parameters Mittag-Leffler function.

For $\beta = 1$ from (2.5) we obtain (2.4).

2.3 Definitions of Fractional Derivative-Integral

2.3.1 Riemann-Liouville Definition

It is well known that to reduce n-multiple integral to 1-tiple integral the following formula

$$ {}_0I_x^n = \int_0^x \int_0^{u_1} \cdots \int_0^{u_{n-1}} f(u_n)du_n \cdots du_2 du_1 = \frac{1}{(n-1)!}\int_0^x (x-u)^{n-1} f(u)du, \quad (2.6) $$

can be used, where $f(u)$ is a given function. Using the equality $(n-1)! = \Gamma(n)$, the formula (2.6) can be extended for any $n \in \mathbb{R}$ and we obtain Riemann-Liouville fractional integral

$$ {}_0I_t^\alpha = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \quad (2.7) $$

where $\alpha \in \mathbb{R}_+$ is the order of integral.

Definition 2.4. The function defined by

$$ \begin{aligned} {}_0^{RL}D_t^\alpha f(t) &= \frac{d^\alpha}{dt^\alpha} f(t) = \frac{d^n}{dt^n}\left[{}_0I_t^{(n-\alpha)} f(t)\right] \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau, \end{aligned} \quad (2.8) $$

is called Riemann-Liouville fractional derivative-integral, where $n-1 \le \alpha \le n$, $n \in \mathbb{N}$.

Example 2.3. Consider the unit-step function

$$ f(t) = \mathbb{1}(t) = \begin{cases} 1 & \text{for} \quad t \ge 0 \\ 0 & \text{for} \quad t < 0 \end{cases}. $$

Using (2.8), we obtain

$$ \begin{aligned} \frac{d^\alpha}{dt^\alpha}\mathbb{1}(t) &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1} d\tau \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\left[\frac{-1}{n-\alpha}(t-\tau)^{n-\alpha}\right]_0^t = \frac{1}{\Gamma(n-\alpha)}\frac{1}{n-\alpha}\frac{d^n}{dt^n}t^{n-\alpha} \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{1}{n-\alpha}(n-\alpha)(n-\alpha-1)\cdots(1-\alpha)t^{-\alpha} = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}. \end{aligned} $$

Therefore, the α order Riemann-Liouville derivative of unit-step function is a decreasing in time function.

Theorem 2.1. *The Riemann-Liouville derivative-integral operator is linear satisfying the relation*

$$ {}_0^{RL}D_t^\alpha[\lambda f(t) + \mu g(t)] = \lambda\, {}_0^{RL}D_t^\alpha f(t) + \mu\, {}_0^{RL}D_t^\alpha g(t), \quad \lambda, \mu \in \mathbb{R}. \quad (2.9) $$

Proof.

$$
\begin{aligned}
{}_0^{RL}D_t^{\alpha}(\lambda f(t)+\mu g(t)) &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1}[\lambda f(\tau)+\mu g(\tau)]d\tau \\
&= \frac{\lambda}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau \\
&+ \frac{\mu}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1} g(\tau)d\tau \\
&= \lambda\, {}_0^{RL}D_t^{\alpha} f(t) + \mu\, {}_0^{RL}D_t^{\alpha} g(t).
\end{aligned}
$$

□

Theorem 2.2. *The Laplace transform of the derivative-integral (2.8) has the form*

$$\mathscr{L}\left[{}_0^{RL}D_t^{\alpha} f(t)\right] = s^{\alpha}F(s) - \sum_{k=1}^{n} s^{k-1} f^{(\alpha-k)}(0^+). \tag{2.10}$$

Proof. Using the definition given in Appendix A.2 we obtain

$$
\begin{aligned}
\mathscr{L}\left[{}_0^{RL}D_t^{\alpha} f(t)\right] &= \mathscr{L}\left[\frac{d^n}{dt^n}\left(\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau\right)\right] \\
&= \mathscr{L}\left[\frac{d^n}{dt^n}\left({}_0I_t^{n-\alpha} f(t)\right)\right] \\
&= \frac{s^n F(s)}{s^{n-\alpha}} - \sum_{k=1}^{n} s^{n-k}\frac{d^{k-1}}{dt^{k-1}}\left[{}_0I_t^{n-\alpha} f(t)\right].
\end{aligned}
$$

□

2.3.2 Caputo Definition

Definition 2.5. The function defined by

$${}_0^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau, \tag{2.11}$$

is called the Caputo fractional derivative-integral, where $n-1<\alpha<n$, $n\in\mathbb{N}$.

Remark 2.1. From definition 2.5 it follows that the Caputo derivative of constant is equal to zero.

Theorem 2.3. *The Caputo derivative-integral operator is linear satisfying the relation*

$${}_0^{C}D_t^{\alpha}[\lambda f(t)+\mu g(t)] = \lambda\, {}_0^{C}D_t^{\alpha} f(t) + \mu\, {}_0^{C}D_t^{\alpha} g(t). \tag{2.12}$$

Proof. The proof is similar to the proof of Theorem 2.1. □

Theorem 2.4. *The Laplace transform of the derivative-integral (2.11) has the form*

$$\mathscr{L}\left[{}_0^C D_t^\alpha f(t)\right] = s^\alpha F(s) - \sum_{k=1}^{n} s^{\alpha-k} f^{(k-1)}(0^+). \tag{2.13}$$

Proof. Using the definition given in Appendix A.2, we obtain

$$\begin{aligned}
\mathscr{L}\left[{}_0^C D_t^\alpha f(t)\right] &= \mathscr{L}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau\right] \\
&= \frac{1}{\Gamma(n-\alpha)}\mathscr{L}\left[t^{n-\alpha-1}\right]\mathscr{L}\left[f^{(n)}(t)\right] \\
&= \frac{1}{\Gamma(n-\alpha)}\frac{\Gamma(n-\alpha)}{s^{n-\alpha}}\left[s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0^+)\right] \\
&= s^\alpha F(s) - \sum_{k=1}^{n} s^{\alpha-k} f^{(k-1)}(0^+)
\end{aligned}$$

□

2.4 Solution of the Fractional State Equation of Continuous-Time Linear System

Consider the continuous-time linear system described by the equation [100]:

$$ {}_0D_t^\alpha x(t) = \frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t), \quad 0 < \alpha \le 1, \tag{2.14a}$$

$$ y(t) = Cx(t) + Du(t), \tag{2.14b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are state, input and output vectors and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Theorem 2.5. *The solution of the equation (2.14a) has the form*

$$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau, \quad x(0) = x_0, \tag{2.15}$$

where

$$\Phi_0(t) = E_\alpha(At^\alpha) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \tag{2.16}$$

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}. \tag{2.17}$$

$E_\alpha(At^\alpha)$ is the Mittag-Leffler function and $\Gamma(x)$ is the Euler gamma function.

Proof. Applying the Laplace transform to (2.14a) and taking in to account

$$X(s) = \mathscr{L}[x(t)] = \int_0^\infty x(t)e^{-st}dt, \tag{2.18a}$$

$$\mathscr{L}\left[D^\alpha x(t)\right] = s^\alpha X(s) - s^{\alpha-1}x_0, \tag{2.18b}$$

we obtain

$$X(s) = [I_n s^\alpha - A]^{-1}\left[s^{\alpha-1}x_0 + BU(s)\right], \quad U(s) = \mathscr{L}[u(t)]. \tag{2.19}$$

It is easy to show that

$$[I_n s^\alpha - A]^{-1} = \sum_{k=0}^{\infty} A^k s^{-(k+1)\alpha}, \tag{2.20}$$

since

$$[I_n s^\alpha - A]\left(\sum_{k=0}^{\infty} A^k s^{-(k+1)\alpha}\right) = I_n. \tag{2.21}$$

Substituting of (2.20) to (2.19), yields

$$X(s) = \sum_{k=0}^{\infty} A^k s^{-(k\alpha+1)}x_0 + \sum_{k=0}^{\infty} A^k s^{-(k+1)\alpha}BU(s). \tag{2.22}$$

Using the inverse Laplace transform and the convolution theorem (Appendix A.1) to (2.22) we obtain

$$\begin{aligned} x(t) &= \mathscr{L}^{-1}[X(s)] = \sum_{k=0}^{\infty} A^k \mathscr{L}^{-1}\left[s^{-(k\alpha+1)}\right]x_0 + \sum_{k=0}^{\infty} A^k \mathscr{L}^{-1}\left[s^{-(k+1)\alpha}BU(s)\right] \\ &= \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau, \end{aligned} \tag{2.23}$$

where

$$\begin{aligned} \Phi_0(t) &= \sum_{k=0}^{\infty} A^k \mathscr{L}^{-1}\left[s^{-(k\alpha+1)}\right] = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \\ \Phi(t) &= \mathscr{L}^{-1}\left\{[I_n s^\alpha - A]^{-1}\right\} = \sum_{k=0}^{\infty} A^k \mathscr{L}^{-1}\left[s^{-(k+1)\alpha}\right] = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}. \end{aligned}$$

□

Remark 2.2. From (2.16) and (2.17) for $\alpha = 1$ mamy

$$\Phi_0(t) = \Phi(t) = \sum_{k=0}^{\infty} \frac{At^k}{\Gamma(k+1)} = e^{At}.$$

Remark 2.3. From classical Cayley-Hamilton theorem it follows that if

$$\det\left[I_n s^\alpha - A\right] = (s^\alpha)^n + a_{n-1}(s^\alpha)^{n-1} + \cdots + a_1 s^\alpha + a_0, \tag{2.24}$$

then

$$A^n + a_{n-1}(A)^{n-1} + \cdots + a_1 A^\alpha + a_0 I_n = 0. \tag{2.25}$$

Example 2.4. Find the solution of the equation (2.14a) for $0 < \alpha \leq 1$ and:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad u(t) = \mathbb{1}(t) \tag{2.26}$$

Using (2.16) and (2.17), we obtain:

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)} = I_2 + \frac{At^\alpha}{\Gamma(\alpha+1)}, \tag{2.27a}$$

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} = I_2 \frac{t^{\alpha-1}}{\Gamma(\alpha)} + A\frac{t^{2\alpha-1}}{\Gamma(2\alpha)}, \tag{2.27b}$$

Substituting (2.27) and $u(t) = 1$ into (2.15), we obtain

$$\begin{aligned} x(t) &= x_0 + \frac{Ax_0 t^\alpha}{\Gamma(\alpha+1)} + \int_0^\infty \left(\frac{B}{\Gamma(\alpha)}(t-\tau)^{\alpha-1} + \frac{AB}{\Gamma(2\alpha)}(t-\tau)^{2\alpha-1} \right) d\tau \\ &= x_0 + \frac{Ax_0 t^\alpha}{\Gamma(\alpha+1)} + \frac{Bt^\alpha}{\Gamma(\alpha+1)} + \frac{ABt^{2\alpha}}{\Gamma(2\alpha+1)} = \begin{bmatrix} 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} \end{bmatrix}, \end{aligned}$$

where $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

Theorem 2.6. *The solution of the equation (2.14a) for $n-1 \leq \alpha \leq n$ and Caputo definition has the form*

$$x(t) = \sum_{l=1}^{n} \Phi_l(t) x^{(l-1)}(0^+) + \int_0^t \Phi(t-\tau) Bu(\tau) d\tau, \tag{2.28}$$

where

$$\Phi_l(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k\alpha+l)-1}}{\Gamma(k\alpha+l)}, \quad \Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}.$$

Proof. Taking into account (A.1), (2.13) from (2.14a) we obtain:

$$X(s) = \left[I_n s^\alpha - A\right]^{-1} \left[\sum_{l=1}^{n} s^{\alpha-l} x^{(l-1)}(0^+) + BU(s) \right], \quad U(s) = \mathscr{L}[u(t)]. \tag{2.29}$$

Substituting of (2.20) into (2.29), yields

$$\begin{aligned} X(s) &= \sum_{k=0}^{\infty} A^k s^{-(k+1)\alpha} \left[\sum_{l=1}^{n} s^{\alpha-l} x^{(l-1)}(0^+) + BU(s) \right] \\ &= \sum_{k=0}^{\infty} \sum_{l=1}^{n} A^k s^{-(k\alpha+l)} x^{(l-1)}(0^+) + \sum_{k=0}^{\infty} A^k s^{-(k+1)\alpha} BU(s). \end{aligned} \tag{2.30}$$

Applying the inverse Laplace transform and the convolution theorem (Appendix A.1) to (2.30), we obtain

$$\begin{aligned} x(t) &= \sum_{k=0}^{\infty} \sum_{l=1}^{n} A^k \mathscr{L}^{-1} \left[s^{-(k\alpha+l)} \right] x^{(l-1)}(0^+) + \sum_{k=0}^{\infty} A^k \mathscr{L}^{-1} \left[s^{-(k+1)\alpha} BU(s) \right] \\ &= \sum_{l=1}^{n} \Phi_l(t) x^{(l-1)}(0^+) + \int_0^t \Phi(t-\tau) Bu(\tau) d\tau, \end{aligned} \tag{2.31}$$

where

$$\begin{aligned} \Phi_l(t) &= \sum_{k=0}^{\infty} A^k \mathscr{L}^{-1} \left[s^{-(k\alpha+l)} \right] = \sum_{k=0}^{\infty} \frac{A^k t^{(k\alpha+l)-1}}{\Gamma(k\alpha+l)}, \\ \Phi(t) &= \sum_{k=0}^{\infty} A^k \mathscr{L}^{-1} \left[s^{-(k+1)\alpha} \right] = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}. \end{aligned}$$

□

Theorem 2.7. *The solution of the equation (2.14a) for* $n-1 \le \alpha \le n$ *and the Riemann-Liouville definition has form*

$$x(t) = \sum_{l=1}^{n} \Phi_l(t) x^{(\alpha-l)}(0^+) + \int_0^t \Phi(t-\tau) Bu(\tau) d\tau, \tag{2.32}$$

where

$$\Phi_l(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-l}}{\Gamma[(k+l)\alpha-l+1]}, \quad \Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}.$$

Proof. Taking into account (A.1) and (2.10), from (2.14a) we obtain:

$$X(s) = [I_n s^{\alpha} - A]^{-1} \left[\sum_{l=1}^{n} s^{l-1} x^{(\alpha-l)}(0^+) + BU(s) \right], \quad U(s) = \mathscr{L}[u(t)]. \tag{2.33}$$

Substituting of (2.20) to (2.33), yields

$$\begin{aligned} X(s) &= \sum_{k=0}^{\infty} A^k s^{-(k+1)\alpha} \left[\sum_{l=1}^{n} s^{l-1} x^{(\alpha-l)}(0^+) + BU(s) \right] \\ &= \sum_{k=0}^{\infty} \sum_{l=1}^{n} A^k s^{-(k+1)\alpha+l-1} x^{(\alpha-l)}(0^+) + \sum_{k=0}^{\infty} A^k s^{-(k+1)\alpha} BU(s). \end{aligned} \tag{2.34}$$

Applying the inverse Laplace transform and the convolution theorem (Appendix A.1) to (2.34), we obtain

$$\begin{aligned}x(t) &= \sum_{k=0}^{\infty}\sum_{l=1}^{n} A^k \mathscr{L}^{-1}\left[s^{-(k+1)\alpha+l-1}\right] x^{(\alpha-l)}(0^+)\\ &+ \sum_{k=0}^{\infty} A^k \mathscr{L}^{-1}\left[s^{-(k+1)\alpha} BU(s)\right]\\ &= \sum_{l=1}^{n} \Phi_l(t) x^{(\alpha-l)}(0^+) + \int_0^t \Phi(t-\tau) Bu(\tau) d\tau, \end{aligned} \tag{2.35}$$

where

$$\begin{aligned}\Phi_l(t) &= \sum_{k=0}^{\infty} A^k \mathscr{L}^{-1}\left[s^{-(k+1)\alpha+l-1}\right] = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-l}}{\Gamma[(k+l)\alpha-l+1]},\\ \Phi(t) &= \sum_{k=0}^{\infty} A^k \mathscr{L}^{-1}\left[s^{-(k+1)\alpha}\right] = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}.\end{aligned}$$

□

Remark 2.4. From comparison of (2.28) and (2.32) it follows that the component of the solution corresponding to $u(t)$ is the same.

2.5 Positivity of the Fractional Systems

Definition 2.6. The fractional system (2.14) is called (internally) positive if the state vector $x(t) \in \mathbb{R}_+^n$ and the output vector $y(t) \in \mathbb{R}_+^p$ for $t \geq 0$ for all initial conditions $x_0 \in \mathbb{R}_+^n$ and all inputs $u(t) \in \mathbb{R}_+^m$, $t \geq 0$.

Definition 2.7. A real square matrix $A = [a_{ij}]$ is called Metzler matrix if its off diagonal entries are nonnegative, i.e. $a_{ij} \geq 0$ for $i \neq j$.

Lemma 2.1. *Let $A \in \mathbb{R}^{n\times n}$ and $0 < \alpha \leq 1$. Then*

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)} \in \mathbb{R}_+^{n\times n} \quad \text{for} \quad t \geq 0, \tag{2.36}$$

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \in \mathbb{R}_+^{n\times n} \quad \text{for} \quad t \geq 0. \tag{2.37}$$

if and only if A is a Metzler matrix.

Proof. Necessity. From:

$$\begin{aligned}\Phi_0(t) &= I_n + \frac{At^{\alpha}}{\Gamma(\alpha+1)} + \cdots,\\ \Phi(t) &= I_n \frac{t^{\alpha-1}}{\Gamma(\alpha)} + A\frac{t^{2\alpha-1}}{\Gamma(2\alpha)} + \cdots\end{aligned}$$

it follows that $\Phi_0(t) \in \mathbb{R}_+^{n\times n}$ i $\Phi(t) \in \mathbb{R}_+^{n\times n}$ for small value $t > 0$ only if A is a Metzler matrix.

Sufficiency. It is well-known [77] that

$$e^{At} \in \mathbb{R}_+^{n\times n} \quad \text{for} \quad t \geq 0 \tag{2.38}$$

if and only if A is a Metzler matrix.

Using (2.36), we can write

$$\Phi_0(t) - e^{At^\alpha} = \sum_{k=0}^{\infty}\left(\frac{(At^\alpha)^k}{\Gamma(k\alpha+1)} - \frac{(At^\alpha)^k}{k!}\right) = \sum_{k=0}^{\infty}\frac{k! - \Gamma(k\alpha+1)}{\Gamma(k\alpha+1)}\cdot\frac{(At^\alpha)^k}{k!}$$

for $t \geq 0$, since $k! \geq \Gamma(k\alpha+1)$ for $0 < \alpha \leq 1$. From (2.38) and (2.5) we have $\Phi_0(t) \geq e^{At^\alpha} \geq 0$ for $t \geq 0$. The proof for (2.37) is similar. □

Theorem 2.8. *The fractional continuous-time system(2.14) is (internally) positive if and only if:*

$$A \in M_n, \quad B \in \mathbb{R}_+^{n\times m}, \quad C \in \mathbb{R}_+^{p\times n}, \quad D \in \mathbb{R}_+^{p\times m}. \tag{2.39}$$

Proof. Sufficiency. By Theorem 2.5 the solution (2.14a) has the form (2.15) and $x(t) \in \mathbb{R}_+^n$, $t \geq 0$, if the condition (2.39) is satisfied since $\Phi_0 \in \mathbb{R}_+^{n\times n}$, $x_0 \in \mathbb{R}_+^n$ and $u(t) \in \mathbb{R}_+^m$ for $t \geq 0$.

Necessity. Let $u(t) = 0$, $t \geq 0$ and $x_0 = e_i$ (i-th column of the identity matrix I_n). The trajectory does not leave the orthant $\mathbb{R}_+^n$ only if the derivative of order α, $x^\alpha(0) = Ae_i \geq 0$, what implies $a_{ij} \geq 0$ for $i \neq j$. The matrix A is a Metzler matrix. From the same reason for $x_0 = 0$ we have $x^\alpha(0) = Bu(0) \geq 0$, what implies $B \in \mathbb{R}_+^{n\times m}$, since $u(0) \in \mathbb{R}_+^m$ can be arbitrary. From (2.14b) for $u(t) = 0$, $t \geq 0$ we have $y(0) = Cx_0 \geq 0$ and $C \in \mathbb{R}_+^{p\times n}$, since $x_0 \in \mathbb{R}_+^n$ can be arbitrary. In a similar way assuming $x_0 = 0$, we obtain $y(0) = Du(0) \geq 0$ and $D \in \mathbb{R}_+^{p\times m}$, since $u(0) \in \mathbb{R}_+^m$ is arbitrary. □

2.6 External Positivity of the Fractional Systems

Definition 2.8. The fractional system (2.14) is called externally positive if for all $u(t) \in \mathbb{R}_+^m$, $t \geq 0$ and zero initial conditions $x_0 = 0$ the output vector $y(t) \in \mathbb{R}_+^p$, $t \geq 0$.

Definition 2.9. Output of the fractional SISO system with zero initial conditions for Dirac impulse $u(t) = \delta(t)$ is called the impulse response of the system. In a similar way we define the matrix of impulse response of the MIMO fractional system (2.14).

Lemma 2.2. *Matrix of the impulse responses $g(t)$ of the fractional system (2.14)is given by*

$$g(t) = C\Phi(t)B + D\delta(t), \quad t \geq 0. \tag{2.40}$$

Proof. Substituting (2.15) into (2.14b) and taking into account $x_0 = 0$, $u(t) = \delta(t)$, $y(t) = g(t)$ we obtain

$$g(t) = \int_0^t C\Phi(t-\tau)B\delta(\tau)d\tau + D\delta(t) = C\Phi(t)B + D\delta(t). \tag{2.41}$$

□

Theorem 2.9. *The fractional system (2.14) is externally positive if and only if*

$$g(t) \in \mathbb{R}_+^{p\times m} \quad \text{for} \quad t \geq 0. \tag{2.42}$$

Proof. Sufficiency. The output $y(t)$ of the system (2.14) with zero initial conditions for the input $u(t)$ is given by

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau. \tag{2.43}$$

If the condition (2.42) is satisfied then from (2.43) we have $y(t) \in \mathbb{R}_+^p$, $t \geq 0$.

Necessity. The necessity follows immediately from the fact that the matrix of impulse responses in a particular case of the output of the system for $u(t) = \delta(t)$ and $\delta(t)$ is nonnegative for $t \geq 0$. □

Corollary 2.1. *The matrix of impulse responses (2.40) of internally positive system (2.14)is nonnegative for $t \geq 0$.*

Between the internal and external positivity we have the following relationship.

Corollary 2.2. *Every fractional continuous-time (internally) positive system (2.14) is always externally positive.*

2.7 Reachability of Fractional Positive Continuous-Time Linear System

Definition 2.10. A state $x_f \in \mathbb{R}_+^n$ of the fractional system (2.14) is called reachable in time t_f if there exists an input $u(t) \in \mathbb{R}_+^m$ for $t \in [0, t_f]$ which steers the state of system from zero initial condition $x_0 = 0$ to the finial state $x_f = x(t_f)$. If every state $x_f \in \mathbb{R}_+^n$ is reachable in time t_f, then the system is called reachable in time t_f. The system (2.14) is called reachable if for every $x_f \in \mathbb{R}_+^n$ there exist t_f and an input $u(t) \in \mathbb{R}_+^m$ for $t \in [0, t_f]$, which steers the state of system from $x_0 = 0$ to x_f.

Theorem 2.10. *The fractional system(2.14) is reachable in time t_f, if the matrix*

$$R(t_f) = \int_0^{t_f} \Phi(t)BB^T\Phi^T(t)dt, \tag{2.44}$$

is monomial. Moreover the input which steers the state from $x_0 = 0$ to x_f is given by

$$u(t) = B^T\Phi^T(t_f - t)R^{-1}(t_f)x_f, \quad t \in [0, t_f], \tag{2.45}$$

where T denotes transpose.

Proof. We shall show that the input (2.45) steers the state of the system (2.14) from $x_0 = 0$ to x_f.

Substituting of (2.45) into (2.15) we obtain

$$\begin{aligned} x(t_f) &= \int_0^{t_f} \Phi(t_f - \tau)BB^T\Phi^T(t_f - \tau)R^{-1}(t_f)x_f d\tau \\ &= \int_0^{t_f} \Phi(t_f - \tau)BB^T\Phi^T(t_f - \tau)d\tau R^{-1}(t_f)x_f \\ &= R(t_f)R^{-1}(t_f)x_f = x_f. \end{aligned} \tag{2.46}$$

□

Theorem 2.11. *If the matrix* $A = \text{diag}\begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \in \mathbb{R}_+^{n\times n}$ *and* $B \in \mathbb{R}_+^{n\times m}$ *for* $m = n$ *are monomial matrices then the system (2.14) is reachable.*

Proof. From (2.17) it follows that if A is diagonal then the matrix $\Phi(t)$ and $\Phi(t)B$ are also monomial for monomial matrix B From (2.44)written in the form

$$R(t_f) = \int_0^{t_f} \Phi(\tau)B[\Phi(\tau)B]^T d\tau, \tag{2.47}$$

it follows that the matrix (2.47) is monomial. By Theorem 2.10 the fractional system is reachable. □

Example 2.5. We shall show that the fractional system (2.14) with the matrices:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{2.48}$$

is reachable. Taking into account that

$$A^k = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^k = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{for} \quad k = 1, 2, \dots,$$

and using (2.17) we obtain

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} = \begin{bmatrix} \Phi_1(t) & 0 \\ 0 & \Phi_2(t) \end{bmatrix}, \tag{2.49}$$

where

$$\Phi_1(t) = \sum_{k=0}^{\infty} \frac{t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}, \quad \Phi_2(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)},$$

and

$$\Phi(t)B = \begin{bmatrix} 0 & \Phi_1(t) \\ \Phi_2(t) & 0 \end{bmatrix}.$$

In this case from (2.47) we have

$$R(t_f) = \int_0^{t_f} \Phi(\tau)B[\Phi(\tau)B]^T d\tau = \int_0^{t_f} \begin{bmatrix} \Phi_1^2(\tau) & 0 \\ 0 & \Phi_2^2(\tau) \end{bmatrix} dt. \tag{2.50}$$

The matrix (2.50) is monomial and by Theorem 2.9 the fractional system is reachable.

Remark 2.5. It is well-knew that the standard system

$$\dot{x} = Ax + Bu \tag{2.51}$$

with the matrices:

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & a_0 \\ 1 & 0 & \dots & 0 & a_1 \\ 0 & 1 & \dots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{2.52}$$

is reachable for all values of the coefficients a_i, $i = 0, 1, \dots, n-1$, since the reachability matrix

$$\begin{bmatrix} B \; AB \; \dots \; A^{n-1}B \end{bmatrix} = I_n. \tag{2.53}$$

The system (2.51) is also reachable as a positive system if $a_i \geq 0$, $i = 0, 1, \dots, n-2$. The fractional system (2.14) with (2.52) is reachable even for $a_i = 0$, $i = 1, \dots, n-1$ if there exist $u(t) \geq 0$, $t \in [0, t_f]$ which satisfied condition

$$x_f = \int_0^{t_f} \begin{bmatrix} \frac{(t_f-\tau)^{\alpha-1}}{\Gamma(\alpha)} \\ \frac{(t_f-\tau)^{2\alpha-1}}{\Gamma(2\alpha)} \\ \vdots \\ \frac{(t_f-\tau)^{n\alpha-1}}{\Gamma(n\alpha)} \end{bmatrix} u(\tau)d\tau. \tag{2.54}$$

This condition (2.54) follows from (2.15) for $x_0 = 0$, (2.53) and the fact that for $a_i = 0$, $i = 0, 1, \dots, n-1$, we have $A^k = 0$ for $k = n, n+1, \dots$ and

$$\Phi(t)B = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} B = \sum_{k=0}^{n-1} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} B = \begin{bmatrix} \frac{t^{\alpha-1}}{\Gamma(\alpha)} \\ \frac{t^{2\alpha-1}}{\Gamma(2\alpha)} \\ \vdots \\ \frac{t^{n\alpha-1}}{\Gamma(n\alpha)} \end{bmatrix}.$$

This example shows that the reachability conditions for the fractional system (2.14) are much stronger than for positive system (2.51) [100].

2.8 Positive Continuous-Time Linear Systems with Delays

Consider the continuous-time linear system with q delays described by the state equations

$$\dot{x}(t) = A_0x(t) + \sum_{k=1}^{q} A_k x(t-d_k) + Bu(t), \tag{2.55a}$$

$$y(t) = Cx(t) + Du(t), \tag{2.55b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the state, input and output vectors and $A_k \in \mathbb{R}_+^{n\times n}$, $k = 0,1,\ldots,q$; $B \in \mathbb{R}_+^{n\times m}$, $C \in \mathbb{R}_+^{p\times n}$, $D \in \mathbb{R}_+^{p\times m}$, and d_k ($d_k \geq 0$), $k = 1,2,\ldots,q$ are delays.

Initial conditions for (2.55a) have the form

$$x(t) = x_0(t) \quad \text{for} \quad t \in [-d,0], \quad d = max(d_k), \tag{2.56}$$

where $x_0(t) \in \mathbb{R}^n$ is given.

Definition 2.11. The system (2.55) is called (internally) positive if $x(t) \in \mathbb{R}_+^n$, $y(t) \in \mathbb{R}_+^p$ for any $x_0(t) \in \mathbb{R}_+^n$ and all inputs $u(t) \in \mathbb{R}_+^m$, $t \geq 0$.

Theorem 2.12. *The system (2.55) is (internally) positive if and only if*

$$A_0 \in M_n, \quad A_k \in \mathbb{R}_+^{n\times n}, \quad k = 1,\ldots,q; \tag{2.57a}$$

$$B \in \mathbb{R}_+^{n\times m}, \quad C \in \mathbb{R}_+^{p\times n}, \quad D \in \mathbb{R}_+^{p\times m}. \tag{2.57b}$$

Proof. Necessity. The equation (2.55a) for $x(t-d_k) = 0$, $t \in [d,0]$ and $u(t) = 0$, $t \geq 0$ takes the form

$$\dot{x}(t) = A_0x(t), \quad t \in [0,d]. \tag{2.58}$$

It is well-known [52, 77] that $x(t) \in \mathbb{R}_+^n$ of (2.58) only if $A_0 \in M_n$. Assuming in (2.55a) $u(t) = 0$, $t \geq 0$, $x_0(-d_k) = e_i$, $i = 1,\ldots,n$ (i-th column of the identity matrix I_n), $x(-d_j) = 0$, $j = 0,1,\ldots,k-1,k+1,\ldots,n$ for $t = 0$ we obtain $\dot{x}(0) = A_k e_i = A_{ki} \in \mathbb{R}_+^n$, where A_{ki} is i-th column of $A_k \in \mathbb{R}_+^{n\times n}$, $k = 1,\ldots,q$. From (2.55a) for $t = 0$ and $x_0(t) = 0$, $t \in [-d,0]$ we have $\dot{x}(0) = Bu(t)$ and $B \in \mathbb{R}_+^{n\times m}$, since by definition $u(0) \in \mathbb{R}_+^m$ is arbitrary. The necessity of $C \in \mathbb{R}_+^{p\times n}$, $D \in \mathbb{R}_+^{p\times m}$ can be shown in a similar way as for positive systems without delays [52, 77].

Sufficiency. The solution of the equation (2.55a) for $t \in [0,d]$ has the form

$$x(t) = e^{A_0t} + \int_0^t e^{A_0(t-\tau)} \left(\sum_{k=1}^{q} A_k x_0(\tau - d_k) + Bu(\tau) \right) d\tau \ . \tag{2.59}$$

Taking in to account that $e^{A_0 t} \in \mathbb{R}_+^{n\times n}$, $t \geq 0$, for $A_0 \in M_n$, and the condition (2.57), from (2.59) we obtain $x(t) \in \mathbb{R}_+^n$, $t \in [0,d]$, since $x_0(t) \in \mathbb{R}_+^n$, $t \in [-d,0]$ and $u(t) \in \mathbb{R}_+^m$, $t \geq 0$. From (2.55b) we have $y(t) \in \mathbb{R}_+^p$, $t \in [0,d]$, since $x(t) \in \mathbb{R}_+^n$ and $u(t) \in \mathbb{R}_+^m$. Using the step method we can extend the considerations for the intervals $[d,2d]$, $[2d,3d], \ldots$. □

Definition 2.12. Let to the asymptotically stable positive system (2.55) a constant input $u(t) = u \in \mathbb{R}_+^m$ be applied. The vector x_e satisfying the equation

$$0 = \sum_{k=0}^{q} A_k x_e + Bu \tag{2.60}$$

is called the equilibrium point (state) of the system (2.55) for constant input u.

If the positive system (2.55) is asymptotically stable then the matrix

$$A = \sum_{k=0}^{q} A_k \in M_n \tag{2.61}$$

is nonsingular and from (2.60) we have

$$x_e = -A^{-1}Bu. \tag{2.62}$$

Remark 2.6. For positive asymptotically stable system (2.55)

$$-A^{-1} \in \mathbb{R}_+^{n\times n}. \tag{2.63}$$

This follows immediately from (2.62), since $x_0 \in \mathbb{R}_+^n$ and $Bu \in \mathbb{R}_+^m$ is arbitrary [52, 77].

Theorem 2.13. *The equilibrium point x_e for positive asymptotically stable system (2.55) is strictly positive, i.e. $x_e > 0$, if $Bu > 0$.*

Proof. If $A \in M_n$ and $Bu > 0$ then from (2.60) we have $x_e \in \mathbb{R}_+^n$. The hypothesis will be proved by contradiction. Assume that $x_e = 0$ then from (2.60) we have $Bu = 0$. This contradicts that $Bu > 0$. This completes the proof. □

These considerations can be extended for positive fractional continuous-time linear systems with delays.

2.9 Positive Linear Systems Consisting of n Subsystems with Different Fractional Orders

2.9.1 Linear Differential Equations with Different Fractional Orders

Consider a fractional linear system described by the equation

$$\begin{bmatrix} \frac{d^{\alpha_1}x_1}{dt^{\alpha_1}} \\ \vdots \\ \frac{d^{\alpha_n}x_n}{dt^{\alpha_n}} \end{bmatrix} = \begin{bmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \dots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} u, \quad \begin{matrix} p_k - 1 < \alpha_k < p_k \\ p_k \in \mathbb{N} = \{1,2,\dots\}, \\ k = 1,\dots,n, \end{matrix} \tag{2.64}$$

where $x_k \in \mathbb{R}^{\overline{n}_k}$, $k = 1,\dots,n$ are the state vectors, $A_{kj} \in \mathbb{R}^{\overline{n}_k \times \overline{n}_j}$, $B_k \in \mathbb{R}^{\overline{n}_k \times m}$, $k, j = 1,\dots,n$ and $u \in \mathbb{R}^m$ is the input vector.

Initial conditions for (2.64) have the form

$$x_k^{(j)}(0) = x_{k0}^{(j)} \in \mathbb{R}^{\overline{n}_k}, \quad k = 1,\dots,n; \quad j = 0,1,\dots,p_k - 1. \tag{2.65}$$

Theorem 2.14. *The solution of the equation (2.64) for $p_k - 1 < \alpha_k < p_k$, $k = 1,\dots,n$ with initial conditions (2.65) has the form*

$$\begin{aligned} x(t) &= \int_0^t [\Phi_1(t-\tau)B_{10} + \dots + \Phi_n(t-\tau)B_{n0}]\, u(\tau)d\tau \\ &+ \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1\dots k_n} \begin{bmatrix} \sum_{j_1=1}^{p_1} \frac{t^{k_1\alpha_1+\dots+k_n\alpha_n+j_1-1}}{\Gamma(k_1\alpha_1+\dots+k_n\alpha_n+j_1)} x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} \frac{t^{k_1\alpha_1+\dots+k_n\alpha_n+j_n-1}}{\Gamma(k_1\alpha_1+\dots+k_n\alpha_n+j_n)} x_{n0}^{(j_n-1)} \end{bmatrix}, \end{aligned} \tag{2.66}$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \mathbb{R}^N, \quad N = \overline{n}_1 + \dots + \overline{n}_n, \quad x_0 = \begin{bmatrix} x_{10} \\ \vdots \\ x_{n0} \end{bmatrix}, \tag{2.67a}$$

$$B_{10} = \begin{bmatrix} B_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad B_{n0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ B_n \end{bmatrix}, \tag{2.67b}$$

$$\begin{aligned}
\Phi_1(t) &= \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1\dots k_n} \frac{t^{(k_1+1)\alpha_1+k_2\alpha_2+\cdots+k_n\alpha_n-1}}{\Gamma[(k_1+1)\alpha_1+k_2\alpha_2+\cdots+k_n\alpha_n]}, \\
&\vdots \\
\Phi_n(t) &= \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1\dots k_n} \frac{t^{k_1\alpha_1+\cdots+k_{n-1}\alpha_{n-1}+(k_n+1)\alpha_n-1}}{\Gamma[k_1\alpha_1+\cdots+k_{n-1}\alpha_{n-1}+(k_n+1)\alpha_n]},
\end{aligned} \tag{2.67c}$$

and

$$T_{k_1\dots k_n} = \begin{cases}
I_N & \textit{for} \quad k_1=\cdots=k_n=0 \\
\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} & \textit{for} \quad \begin{matrix} k_1=1, \\ k_2=\cdots=k_n=0 \end{matrix} \\
\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ A_{i1} & \cdots & A_{in} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} & \textit{for} \quad \begin{matrix} k_1=\cdots=k_{i-1}=0, \\ k_1=1, \\ k_{i+1}=\cdots=k_n=0, \end{matrix} \\
\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ A_{n1} & \cdots & A_{nn} \end{bmatrix} & \textit{for} \quad \begin{matrix} k_1=\cdots=k_{n-1}=0, \\ k_i=1 \end{matrix} \\
T_{10\dots0}T_{01\dots1}+\cdots+T_{0\dots01}T_{1\dots10} & \textit{for} \quad k_1=\cdots=k_n=1 \\
\vdots & \\
\begin{matrix} T_{10\dots0}T_{k_1-1,k_2,\dots,k_n} \\ +\cdots+T_{0\dots01}T_{k_1,k_{n-1},k_n-1} \end{matrix} & \textit{for} \quad k_1+\cdots+k_n>0 \\
0 \qquad \textit{for at last one} & k_i<0, i=1,\dots,n
\end{cases} \tag{2.67d}$$

Proof. Using the Laplace transforms

$$X_k(s) = \mathscr{L}[x_k(t)], \quad k=1,\dots,n; \quad U(s) = \mathscr{L}[u(t)], \tag{2.68}$$

and (A.10) we may write the equations (2.64) for $p_k - 1 < \alpha < p_k;\ p_k \in \mathbb{N}$, $k = 1\dots,n$ in the form

$$\begin{bmatrix} I_{\overline{n}_1}s^{\alpha_1} - A_{11} & -A_{12} & \cdots & -A_{1n-1} & A_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -A_{n1} & -A_{n2} & \cdots & -A_{nn-1} & I_{\overline{n}_n}s^{\alpha_n} - A_{nn} \end{bmatrix} \begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix}$$
$$= \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} U(s) + \begin{bmatrix} \sum_{j_1=1}^{p_1} s^{\alpha_1 - j_1} x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} s^{\alpha_n - j_n} x_{n0}^{(j_n-1)} \end{bmatrix}. \tag{2.69}$$

From (2.69) we have

$$\begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix} = \begin{bmatrix} I_{\overline{n}_1}s^{\alpha_1} - A_{11} & -A_{12} & \cdots & -A_{1n-1} & A_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -A_{n1} & -A_{n2} & \cdots & -A_{nn-1} & I_{\overline{n}_n}s^{\alpha_n} - A_{nn} \end{bmatrix}^{-1}$$
$$\times \left\{ \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} U(s) + \begin{bmatrix} \sum_{j_1=1}^{p_1} s^{\alpha_1 - j_1} x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} s^{\alpha_n - j_n} x_{n0}^{(j_n-1)} \end{bmatrix} \right\}. \tag{2.70}$$

Comparing the coefficients at the same powers of $s^{-\alpha_k}$ it is easy to verify that

$$\begin{bmatrix} I_{\overline{n}_1} - A_{11}s^{-\alpha_1} & \cdots & -A_{1n}s^{-\alpha_1} \\ \vdots & \ddots & \vdots \\ -A_{n1}s^{-\alpha_n} & \cdots & I_{\overline{n}_n} - A_{nn}s^{-\alpha_n} \end{bmatrix} \left[\sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1\dots k_n} s^{-(k_1\alpha_1 + \cdots + k_n\alpha_n)} \right] = I_N, \tag{2.71}$$

where matrices $T_{k_1\dots k_n}$ are defined by (2.67d).

Using (2.71) we obtain

$$\begin{bmatrix} I_{\overline{n}_1}s^{\alpha_1} - A_{11} & -A_{12} & \cdots & -A_{1n-1} & A_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -A_{n1} & -A_{n2} & \cdots & -A_{nn-1} & I_{\overline{n}_n}s^{\alpha_n} - A_{nn} \end{bmatrix}^{-1} =$$
$$\left\{ \begin{bmatrix} I_{\overline{n}_1}s^{\alpha_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_{\overline{n}_n}s^{\alpha_n} \end{bmatrix} \begin{bmatrix} I_{\overline{n}_1} - A_{11}s^{-\alpha_1} & \cdots & -A_{1n}s^{-\alpha_1} \\ \vdots & \ddots & \vdots \\ -A_{n1}s^{-\alpha_n} & \cdots & I_{\overline{n}_n} - A_{nn}s^{-\alpha_n} \end{bmatrix} \right\}^{-1} =$$

$$\begin{bmatrix} I_{\overline{n}_1} - A_{11}s^{-\alpha_1} & \cdots & -A_{1n}s^{-\alpha_1} \\ \vdots & \ddots & \vdots \\ -A_{n1}s^{-\alpha_n} & \cdots & I_{\overline{n}_n} - A_{nn}s^{-\alpha_n} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} I_{\overline{n}_1}s^{-\alpha_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_{\overline{n}_n}s^{-\alpha_n} \end{bmatrix} \right\} =$$

$$\sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1 \dots k_n} s^{-(k_1\alpha_1+\cdots+k_n\alpha_n)} \left\{ \begin{bmatrix} I_{\overline{n}_1}s^{-\alpha_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_{\overline{n}_n}s^{-\alpha_n} \end{bmatrix} \right\}. \tag{2.72}$$

substituting of (2.72) into (2.70) yields

$$\begin{aligned} \begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix} &= \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1 \dots k_n} s^{-(k_1\alpha_1+\cdots+k_n\alpha_n)} \begin{bmatrix} I_{\overline{n}_1}s^{-\alpha_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_{\overline{n}_n}s^{-\alpha_n} \end{bmatrix} \\ &\times \left\{ \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} U(s) + \begin{bmatrix} \sum_{j_1=1}^{p_1} s^{\alpha_1 - j_1} x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} s^{\alpha_n - j_n} x_{n0}^{(j_n-1)} \end{bmatrix} \right\} \\ &= \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1 \dots k_n} \left\{ \left[B_{10} s^{-[(k_1+1)\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n]} \right. \right. \\ &+ \left. \cdots + B_{n0} s^{-[k_1\alpha_1+\cdots+k_{n-1}\alpha_{n-1}+(k_n+1)\alpha_n]} \right] U(s) \\ &+ \left. s^{-(k_1\alpha_1+\cdots+k_n\alpha_n)} \begin{bmatrix} \sum_{j_1=1}^{p_1} s^{-j_1} x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} s^{-j_n} x_{n0}^{(j_n-1)} \end{bmatrix} \right\}. \end{aligned} \tag{2.73}$$

Applying the inverse Laplace transform and the convolution theorem to (2.73) we obtain

$$\begin{aligned} \mathscr{L}^{-1} \begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix} &= \mathscr{L}^{-1} \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1 \dots k_n} \left\{ \left[B_{10} s^{-[(k_1+1)\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n]} \right. \right. \\ &+ \left. \cdots + B_{n0} s^{-[k_1\alpha_1+\cdots+k_{n-1}\alpha_{n-1}+(k_n+1)\alpha_n]} \right] U(s) \\ &+ \left. s^{-(k_1\alpha_1+\cdots+k_n\alpha_n)} \begin{bmatrix} \sum_{j_1=1}^{p_1} s^{-j_1} x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} s^{-j_n} x_{n0}^{(j_n-1)} \end{bmatrix} \right\}, \end{aligned}$$

$$\begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \int_0^t [\Phi_1(t-\tau)B_{10} + \cdots + \Phi_n(t-\tau)B_{n0}]\,u(\tau)d\tau$$

$$+ \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1 \dots k_n} \begin{bmatrix} \sum_{j_1=1}^{p_1} \frac{t^{k_1\alpha_1+\cdots+k_n\alpha_n+j_1-1}}{\Gamma(k_1\alpha_1+\cdots+k_n\alpha_n+j_1)} x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} \frac{t^{k_1\alpha_1+\cdots+k_n\alpha_n+j_n-1}}{\Gamma(k_1\alpha_1+\cdots+k_n\alpha_n+j_n)} x_{n0}^{(j_n-1)} \end{bmatrix}, \tag{2.74}$$

since $\mathscr{L}^{-1}\left[\frac{1}{s^{\alpha+1}}\right] = \frac{t^\alpha}{\Gamma(\alpha+1)}$. □

In a particular case if $0 < \alpha_k < 1$, $k = 1,\dots,n$; $(p_1 = \cdots = p_n = 1)$, then

$$\sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1 \dots k_n} \begin{bmatrix} \sum_{j_1=1}^{p_1} \frac{t^{k_1\alpha_1+\cdots+k_n\alpha_n+j_1-1}}{\Gamma(k_1\alpha_1+\cdots+k_n\alpha_n+j_1)} x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} \frac{t^{k_1\alpha_1+\cdots+k_n\alpha_n+j_n-1}}{\Gamma(k_1\alpha_1+\cdots+k_n\alpha_n+j_n)} x_{n0}^{(j_n-1)} \end{bmatrix} = \Phi_0(t)x_0, \tag{2.75}$$

where

$$\Phi_0(t) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1 \dots k_n} \frac{t^{k_1\alpha_1+\cdots+k_n\alpha_n}}{\Gamma(k_1\alpha_1+\cdots+k_n\alpha_n+1)}. \tag{2.76}$$

2.9.2 Positive Fractional Systems

Definition 2.13. The fractional system (2.64) is called positive if $x_k(t) \in \mathbb{R}_+^{\overline{n}_k}$, $k = 1,\dots,n$, $t \geq 0$ for any initial conditions $x_{k0} \in \mathbb{R}_+^{\overline{n}_k}$, $k = 1,\dots,n$, and all input vectors $u \in \mathbb{R}_+^m$, $t \geq 0$.

Let M_n be the set of $n \times n$ Metzler matrices, i.e. real matrices with nonnegative off-diagonal entries.

Theorem 2.15. *The fractional system (2.64) for $p_k - 1 < \alpha < p_k$, $p_k \in \mathbb{N}$, $k = 1,\dots,n$ is positive if and only if*

$$A = \begin{bmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \dots & A_{nn} \end{bmatrix} \in M_N, \tag{2.77a}$$

$$\begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} \in \mathbb{R}_+^{N \times m}. \tag{2.77b}$$

Proof. To simplify the notation the proof will be given for $n = 2$. First we shall show that

$$\Phi_k(t) \in \mathbb{R}_+^{\overline{n}\times\overline{n}}, \quad (\overline{n} = \overline{n}_1 + \overline{n}_2) \quad \text{for} \quad k = 0,1,2 \quad \text{and} \quad t \geq 0, \tag{2.78}$$

only if (2.77a) holds. From the expansion (2.67c) we have

$$\Phi_0(t) = \begin{bmatrix} I_{\overline{n}_1} & 0 \\ 0 & I_{\overline{n}_2} \end{bmatrix} + \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} \frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)} + \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} \frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)} + \cdots, \tag{2.79a}$$

$$\Phi_1(t) = \begin{bmatrix} I_{\overline{n}_1} & 0 \\ 0 & I_{\overline{n}_2} \end{bmatrix} \frac{t^{\alpha_1-1}}{\Gamma(\alpha_1)} + \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} \frac{t^{2\alpha_1-1}}{\Gamma(2\alpha_1)} + \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} \frac{t^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} + \cdots, \tag{2.79b}$$

$$\Phi_2(t) = \begin{bmatrix} I_{\overline{n}_1} & 0 \\ 0 & I_{\overline{n}_2} \end{bmatrix} \frac{t^{\alpha_2-1}}{\Gamma(\alpha_2)} + \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} \frac{t^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} + \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} \frac{t^{2\alpha_2-1}}{\Gamma(2\alpha_2)} + \cdots. \tag{2.79c}$$

(2.79d)

From (2.79) it follows that $\Phi_k(t) \in \mathbb{R}_+^{\overline{n}\times\overline{n}}$, $k = 0,1,2$ for small value of $t > 0$ only if the condition (2.77a) is satisfied.

In a similar way as in [100, 135] it can be shown that if (2.77) holds then

$$\Phi_0(t) \in \mathbb{R}_+^{\overline{n}\times\overline{n}}, \quad t \geq 0, \tag{2.80}$$

and

$$\Phi_1(t)B_{10} + \Phi_2(t)B_{01} \in \mathbb{R}_+^{\overline{n}\times\overline{n}}, \quad t \geq 0. \tag{2.81}$$

In this case from (2.66) we have $x(t) \in \mathbb{R}_+^{\overline{n}}$, $t \geq 0$ since by definition $x_0 \in \mathbb{R}_+^{\overline{n}}$ and $u(t) \in \mathbb{R}_+^m$, $t \geq 0$. The remaining part of the proof is similar as in [100, 135]. □

2.9.3 Fractional Linear Electrical Circuits

Consider linear electrical circuits composed of resistors, supercondensators (ultracapacitors), coils and voltage (current) sources. As the state variables (the components of the state vector x) the voltage across the supercondensators and the currents in the coils are usually chosen. It is well-known [51, 196] that the current $i(t)$ in supercondensator with its voltage $u_C(t)$ is related by formula

$$i_C(t) = C\frac{d^\alpha u_C(t)}{dt^\alpha} \quad \text{for} \quad 0 < \alpha < 1, \tag{2.82}$$

where C is the capacity of the supercondensator.

Similarly, the voltage $u_L(t)$ on the coil with its current $i_L(t)$ is related by the formula

$$u_L(t) = L\frac{d^\beta i_L(t)}{dt^\beta} \quad \text{for} \quad 0 < \beta < 1, \tag{2.83}$$

where L is the inductance of the coil.

Using the relations (2.82), (2.83) and Kirchhoff's laws we may write for the fractional linear circuits the following state equation

$$\begin{bmatrix} \frac{d^\alpha x_C}{dt^\alpha} \\ \frac{d^\beta x_L}{dt^\beta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_C \\ x_L \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} e, \tag{2.84}$$

where the components of $x_C \in \mathbb{R}^{n_1}$ are voltages across the supercondensators, the components of $x_L \in \mathbb{R}^{n_2}$ are currents in coils and the components of $e \in \mathbb{R}^m$ are the voltages of the circuit.

Example 2.6. Consider the linear electrical circuit shown on Fig. 2.1 with known resistances R_1, R_2, R_3, capacitances C_1, C_2, inductances L_1, L_2 and sources voltages e_1, e_2.

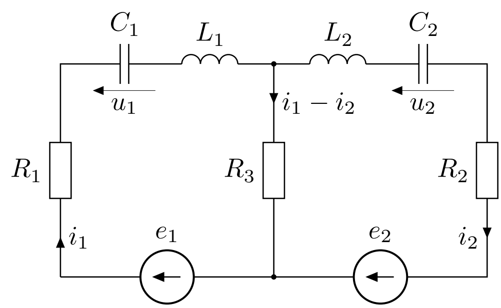

Fig. 2.1 Electrical circuit. Illustration to Example 2.6.

Using relations (2.82), (2.83) and Kirchhoff's laws we may write for the circuit the following equations:

$$i_1 = C_1\frac{d^{\alpha_1} u_1}{dt^{\alpha_1}}, \quad i_2 = C_2\frac{d^{\alpha_2} u_2}{dt^{\alpha_2}}, \tag{2.85a}$$

$$e_1 = (R_1 + R_2)i_1 + L_1\frac{d^{\beta_1} i_1}{dt^{\beta_1}} + u_1 - R_3 i_2, \tag{2.85b}$$

$$e_2 = (R_2 + R_3)i_2 + L_2\frac{d^{\beta_2} i_2}{dt^{\beta_2}} + u_2 - R_3 i_1. \tag{2.85c}$$

The equations (2.85) can be written in the form

$$\begin{bmatrix} \frac{d^{\alpha_1} u_1}{dt^{\alpha_1}} \\ \frac{d^{\alpha_2} u_2}{dt^{\alpha_2}} \\ \frac{d^{\beta_1} i_1}{dt^{\beta_1}} \\ \frac{d^{\beta_2} i_2}{dt^{\beta_2}} \end{bmatrix} = A \begin{bmatrix} u_1 \\ u_2 \\ i_1 \\ i_2 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{2.86}$$

where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{C_1} & 0 \\ 0 & 0 & 0 & \frac{1}{C_2} \\ -\frac{1}{L_1} & 0 & -\frac{R_1+R_3}{L_1} & \frac{R_3}{L_1} \\ 0 & -\frac{1}{L_2} & \frac{R_3}{L_2} & -\frac{R_2+R_3}{L_2} \end{bmatrix}, \tag{2.87a}$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}. \tag{2.87b}$$

From (2.87) it follows that the fractional electrical circuit is not positive since the matrix A has some negative off-diagonal entries.

If the fractional linear circuit is not positive but the matrix B has nonnegative entries (see for example the circuit on Fig. 2.1) then using the state-feedback

$$e = K \begin{bmatrix} x_C \\ x_L \end{bmatrix}. \tag{2.88}$$

we may usually choose the gain matrix $K \in \mathbb{R}^{m\times n}$, $(n = n_1 + n_2)$ so that the closed-loop system matrix (obtained by substituting of (2.88) into (2.84))

$$A_c = A + BK, \tag{2.89}$$

is a Metzler matrix.

Theorem 2.16. *Let A be not a Metzler matrix but $B \in \mathbb{R}_+^{n\times m}$. Then there exists a gain matrix K such that the closed-loop system matrix $A_c \in M_n$ if and only if*

$$\operatorname{rank}[B, A_c - A] = \operatorname{rank} B. \tag{2.90}$$

Proof. By Kronecker-Cappely theorem the equation

$$BK = A_c - A, \tag{2.91}$$

have a solution K for any given B and $A_c - A$ if and only if the condition (2.90) is satisfied. □

Example 2.7. (Continuation of Example 2.6). Let

$$A_c = \begin{bmatrix} 0 & 0 & \frac{1}{C_1} & 0 \\ 0 & 0 & 0 & \frac{1}{C_2} \\ \frac{a_1}{L_1} & 0 & -\frac{R_1+R_3}{L_1} & \frac{a_3}{L_1} \\ 0 & \frac{a_2}{L_2} & \frac{a_4}{L_2} & -\frac{R_2+R_3}{L_2} \end{bmatrix} \quad \text{for} \quad a_k \geq 0, \quad k = 1,2,3,4. \tag{2.92}$$

In this case the condition (2.90) is satisfied since

$$\text{rank}\,[B, A_c - A] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{L_1} & 0 & \frac{a_1+1}{L_1} & 0 & 0 & \frac{a_3-R_3}{L_1} \\ 0 & \frac{1}{L_2} & 0 & \frac{a_2+1}{L_2} & \frac{a_4-R_3}{L_2} & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix} = 2. \tag{2.93}$$

The equation (2.91) has the form

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{a_1+1}{L_1} & 0 & 0 & \frac{a_3-R_3}{L_1} \\ 0 & \frac{a_2+1}{L_2} & \frac{a_4-R_3}{L_2} & 0 \end{bmatrix}, \tag{2.94}$$

and its solution is

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix} = \begin{bmatrix} a_1+1 & 0 & 0 & a_3-R_3 \\ 0 & a_2+1 & a_4-R_3 & 0 \end{bmatrix}. \tag{2.95}$$

The matrix (2.95) has nonnegative entries if $a_k \geq 0$ for $k = 1,2,3,4$.

On the following two examples of fractional linear circuits we shall shown that it is not always possible to choose the gain matrix K so that the two conditions are satisfied:

a) the closed-loop system matrix $A_c \in M_n$,
b) the closed-loop system is asymptotically stable.

Example 2.8. Consider the fractional linear circuit shown on Fig. 2.2 with given resistances R, capacitance C, inductance L and source of voltage e.

Using (2.82), (2.83) and the second Kirchhoff's law we obtain for the circuit the state equation

$$\begin{bmatrix} \frac{d^\alpha u_C}{dt^\alpha} \\ \frac{d^\beta i}{dt^\beta} \end{bmatrix} = A \begin{bmatrix} u_C \\ i \end{bmatrix} + Be, \quad 0 < \alpha < 1; \quad 0 < \beta < 1; \tag{2.96}$$

where

$$A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}. \tag{2.97}$$

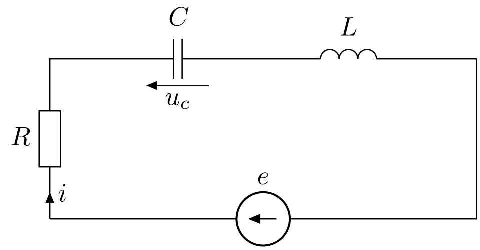

Fig. 2.2 Electrical circuit. Illustration to Example 2.8.

From (2.97) it follows that A is not a Metzler matrix but $B \in \mathbb{R}_+^2$. It is easy to see that the condition (2.90) is satisfied for

$$A_c = \begin{bmatrix} 0 & \frac{1}{C} \\ \frac{a}{L} & \frac{b-R}{L} \end{bmatrix}, \tag{2.98}$$

and

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} a+1 & b \end{bmatrix}. \tag{2.99}$$

Note that the characteristic polynomial of the matrix (2.98)

$$\det \begin{bmatrix} I_{n_1}s^\alpha - A_{11} & -A_{12} \\ -A_{21} & I_{n_2}s^\beta - A_{22} \end{bmatrix} = \begin{vmatrix} s^\alpha & -\frac{1}{C} \\ -\frac{a}{L} & s^\beta + \frac{R-b}{L} \end{vmatrix} = s^{\alpha+\beta} + \frac{R-b}{L}s^\alpha - \frac{a}{LC}, \tag{2.100}$$

has one nonnegative coefficient and closed-loop circuit is unstable for $a \geq 0$ and any b.

Example 2.9. Consider the fractional linear system shown on Fig. 2.3 with given resistances R_1, R_2, capacitance C, inductance L and source of voltage e. Using the relations (2.82), (2.83) and the second Kirchhoff's law we obtain for the circuit the state equation

$$\begin{bmatrix} \frac{d^\alpha u_C}{dt^\alpha} \\ \frac{d^\beta i}{dt^\beta} \end{bmatrix} = A \begin{bmatrix} u_C \\ i \end{bmatrix} + Be, \tag{2.101}$$

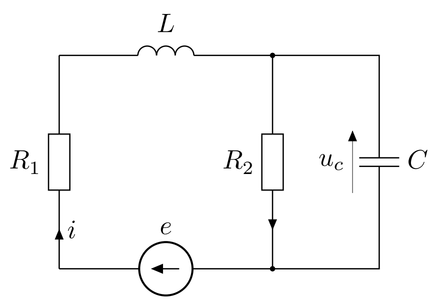

Fig. 2.3 Electrical circuit. Illustration to Example 2.9.

where

$$A = \begin{bmatrix} -\frac{1}{R_2C} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R_1}{L} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}. \tag{2.102}$$

The matrix A is not a Metzler matrix but $B \in \mathbb{R}_+^2$. It is easy to check that the condition (2.90) is satisfied for

$$A = \begin{bmatrix} -\frac{1}{R_2C} & \frac{1}{C} \\ \frac{a}{L} & \frac{b-R_1}{L} \end{bmatrix}, \quad a, b \geq 0, \tag{2.103}$$

and from (2.91) we obtain

$$\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{a+1}{L} & \frac{b}{L} \end{bmatrix}, \tag{2.104}$$

and

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} a+1 & b \end{bmatrix}. \tag{2.105}$$

In this case the characteristic polynomial of the matrix (2.90) has the form

$$p(s) = \begin{vmatrix} s^\alpha + \frac{1}{R_2C} & -\frac{1}{C} \\ -\frac{a}{L} & s^\beta + \frac{R_1-b}{L} \end{vmatrix} = s^{\alpha+\beta} + \frac{R_1-b}{L}s^\alpha + \frac{1}{R_2C}s^\beta + \frac{R_1-aR_2-b}{R_2CL}, \tag{2.106}$$

and it is possible to choose the values of parameters a, b so that the closed-loop system is asymptotically stable [266].