Chapter 10
Stabilization of Positive and Fractional Linear Systems

10.1 Fractional Discrete-Time Linear Systems with Delays

Consider the fractional discrete-time linear system with h delays:

$$x_{i+1} = \sum_{j=1}^{i+1} (-1)^{j+1} \binom{\alpha}{j} x_{i-j+1} + \sum_{k=0}^{h} (A_k x_{i-k} + B_k u_{i-k}),\ i \in \mathbb{Z}_+, \tag{10.1a}$$

$$y_i = Cx_i + Du_i, \quad 0 < \alpha < 1, \tag{10.1b}$$

and with the state-feedback

$$u_i = Kx_i, \quad i \in \mathbb{Z}_+, \tag{10.2}$$

where $K \in \mathbb{R}^{m\times n}$ is a gain matrix.

We are looking for a gain matrix K such that the closed-loop system

$$x_{i+1} = \sum_{k=0}^{h} (A_k + B_k K + I_n c_{k+1}) x_{i-k} + \sum_{j=h+2}^{i+1} c_j x_{i-j+1}, \tag{10.3}$$

$$c_j = (-1)^{j+1} \binom{\alpha}{j} \quad \text{for} \quad j = 1,\dots,i+1;$$

is positive and asymptotically stable.

Theorem 10.1. *The fractional closed-loop system (10.3) is positive and asymptotically stable if and only if there exists a diagonal matrix*

$$\Lambda = \mathrm{diag}\left[\, \lambda_1 \ \dots \ \lambda_n \,\right], \tag{10.4}$$

with positive diagonal entries $\lambda_k > 0$, $k = 1,\dots,n$ and a matrix $D \in \mathbb{R}^{m\times n}$ such that

$$(A_k + I_n c_{k+1})\Lambda + B_k D \in \mathbb{R}_+^{n\times n}, \quad k = 0,1,\dots,h \tag{10.5}$$

T. Kaczorek: Selected Problems of Fractional Systems Theory, LNCIS 411, pp. 221–243.
springerlink.com

and

$$\sum_{k=0}^{h} (A_k \Lambda + B_k D) \mathbb{1}_n < 0, \tag{10.6}$$

where $\mathbb{1}_n = [1, \ldots, 1]^T \in \mathbb{R}^{n \times n}$.

The matrix K is given by

$$K = D\Lambda^{-1}. \tag{10.7}$$

Proof. First we shall show that the closed-loop system (10.3) is positive if and only if the condition (10.5) is satisfied. Using (10.7) and (10.3) we obtain

$$\begin{aligned} A_k + B_k K + I_n c_{k+1} &= A_k + B_k D\Lambda^{-1} + I_n c_{k+1} \\ &= [(A_k + I_n c_{k+1})\Lambda + B_k D]\Lambda^{-1} \in \mathbb{R}_+^{n \times n} \end{aligned} \tag{10.8}$$

since the condition (10.5) is met.

Taking into account that

$$K\Lambda \mathbb{1}_n = D\Lambda^{-1}\Lambda \mathbb{1}_n = D\mathbb{1}_n \quad \text{and} \quad \Lambda \mathbb{1}_n = \lambda, \tag{10.9}$$

and using (8.47) we obtain

$$\begin{aligned} \left[\sum_{k=0}^{h} (A_k + B_k K) + \sum_{j=1}^{\infty} I_n c_j - I_n\right]\lambda &= \sum_{k=0}^{h} (A_k + B_k K)\Lambda \mathbb{1}_n \\ &= \sum_{k=0}^{h} (A_k \Lambda + B_k D)\mathbb{1}_n < 0, \end{aligned} \tag{10.10}$$

when the conditions (8.47) and (10.6) are satisfied.

By Theorem 8.12 the closed-loop system (10.3) is asymptotically stable if and only if the condition (10.6) is met. □

If the conditions (10.5) and (10.6) are satisfied then the gain matrix K can be found by the use of the following procedure:

Procedure 10.1

Step 1. Choose a diagonal matrix (10.4) with $\lambda_k > 0$, $k = 1, \ldots, n$ and a matrix $D \in \mathbb{R}^{m \times n}$ satisfying the conditions (10.5) and (10.6).
Step 2. Using (10.7) find the gain matrix K.

Example 10.1. Consider the fractional system (10.1) witch $\alpha = 0.5$, $h = 2$ and

$$A_0 = \begin{bmatrix} -0.4 & 0.4 \\ 0.6 & -0.3 \end{bmatrix}, \; A_1 = \begin{bmatrix} -0.1 & 0.05 \\ 0.1 & -0.1 \end{bmatrix}, \; A_2 = \begin{bmatrix} -0.04 & 0.05 \\ 0.05 & -0.05 \end{bmatrix}, \tag{10.11}$$

$$B_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0.1 \\ 0.05 \end{bmatrix}. \tag{10.12}$$

Find a gain matrix $K \in \mathbb{R}^{1\times 2}$ such that the closed-loop system is positive and asymptotically stable.

The fractional system is positive but unstable since the matrices

$$\begin{aligned}
A_0 + c_1 I_n &= \begin{bmatrix} -0.4 & 0.4 \\ 0.6 & -0.3 \end{bmatrix} + 0.5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.4 \\ 0.6 & 0.2 \end{bmatrix}, \\
A_1 + c_2 I_n &= \begin{bmatrix} -0.1 & 0.05 \\ 0.1 & -0.1 \end{bmatrix} + 0.125 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.025 & 0.05 \\ 0.1 & 0.025 \end{bmatrix}, \\
A_2 + c_3 I_n &= \begin{bmatrix} -0.04 & 0.05 \\ 0.05 & -0.05 \end{bmatrix} + 0.0625 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.0225 & 0.05 \\ 0.05 & 0.0125 \end{bmatrix},
\end{aligned} \tag{10.13}$$

have positive entries and the characteristic polynomial of the matrix $A = A_0 + A_1 + A_2$

$$\det[Iz - A] = \begin{bmatrix} z+0.54 & -0.5 \\ -0.75 & z+0.45 \end{bmatrix} = z^2 + 0.99z - 0.132, \tag{10.14}$$

has one ($a_0 = -0.132$) negative entry. Using Procedure 10.1 we obtain the following:

Step 1. We choose:

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -0.1 & -0.2 \end{bmatrix}, \tag{10.15}$$

and we check the conditions (10.5) and (10.6)

$$\begin{aligned}
(A_0 + I_n c_1)\Lambda + B_0 D &= \begin{bmatrix} 0 & 0.2 \\ 0.5 & 0 \end{bmatrix}, \\
(A_1 + I_n c_2)\Lambda + B_1 D &= \begin{bmatrix} 0.005 & 0.01 \\ 0.09 & 0.005 \end{bmatrix}, \\
(A_2 + I_n c_3)\Lambda + B_2 D &= \begin{bmatrix} 0.0125 & 0.03 \\ 0.045 & 0.0025 \end{bmatrix},
\end{aligned}$$

and

$$\sum_{k=0}^{2} (A_k \Lambda + B_k D)\mathbb{1}_n = \begin{bmatrix} -0.43 \\ -0.045 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{10.16}$$

Thus, the conditions (10.5) and (10.6) are satisfied.

Step 2. Using (10.7), we obtain the desired gain matrix

$$K = D\Lambda^{-1} = \begin{bmatrix} -0.1 & -0.2 \end{bmatrix}. \tag{10.17}$$

The closed-loop system is positive and asymptotically stable since the matrices (10.13) have positive entries and the condition (10.16) is satisfied.

10.2 Fractional Continuous-Time Linear Systems with Delays

Consider the fractional continuous-time linear system with delays

$$\frac{d^\alpha x(t)}{dt^\alpha} = \sum_{k=0}^{q} \left[A_k x(t-d_k) + B_k u(t-d_k)\right], \tag{10.18}$$

with the state-feedback

$$u(t) = Kx(t), \tag{10.19}$$

where $K \in \mathbb{R}^{m\times n}$ is a gain matrix.

Substituting (10.19) in (10.18) we obtain the closed-loop system

$$\frac{d^\alpha x(t)}{dt^\alpha} = \sum_{k=0}^{q} (A_k + B_k K)\, x(t-d_k), \quad 0 < \alpha \le 1. \tag{10.20}$$

The positive system with delays (10.20) is asymptotically stable if and only if the positive system without delays

$$\frac{d^\alpha x(t)}{dt^\alpha} = (A+BK)x(t), \quad A = \sum_{k=0}^{q} A_k, \quad B = \sum_{k=0}^{q} B_k, \tag{10.21}$$

is asymptotically stable.

We are looking for a gain matrix K such that the closed-loop system (10.20) is positive and the zeros of the characteristic polynomial

$$\det\left[I_n s^\alpha - (A+BK)\right] = (s^\alpha)^n + \overline{a}_{n-1}(s^\alpha)^{n-1} + \cdots + \overline{a}_1 s + \overline{a}_0, \tag{10.22}$$

are located in the sector $\phi = \frac{\pi}{2\alpha}$.

Theorem 10.2. *The closed-loop fractional system* (10.20) *is positive and the zeros of the polynomial* (10.22) *are located in the sector* $\phi = \frac{\pi}{2\alpha}$ *if and only if there exist a diagonal matrix*

$$\Lambda = \operatorname{diag}[\lambda_1, \ldots, \lambda_n] \quad \text{with} \quad \lambda_k > 0, \quad k = 1, \ldots, n; \tag{10.23}$$

and a real matrix $D \in \mathbb{R}^{m\times n}$ *such that the following conditions are satisfied*

$$A\Lambda + BD \in M_n, \tag{10.24}$$

$$(A\Lambda + BD)\mathbb{1}_n < 0. \tag{10.25}$$

The gain matrix K is given by the formula

$$K = D\Lambda^{-1}. \tag{10.26}$$

Proof. First we shall show that the closed-loop system (10.20) is positive if and only if (10.24) holds. Using (10.20), (10.21) and (10.26) we obtain

$$\sum_{k=0}^{q}(A_k + B_k K) = A + BK = A + BD\Lambda^{-1} = (A\Lambda + BD)\Lambda^{-1} \in M_n, \tag{10.27}$$

if and only if the condition (10.24) is satisfied.

Taking into account that

$$K\Lambda\mathbb{1}_n = D\Lambda^{-1}\Lambda\mathbb{1}_n = D\mathbb{1}_n \quad \text{and} \quad \Lambda\mathbb{1}_n = \lambda = [\lambda_1, \dots, \lambda_n]^T, \tag{10.28}$$

and using (10.25) we obtain

$$(A + BK)\lambda = (A + BK)\Lambda\mathbb{1}_n = (A\Lambda + BD)\mathbb{1}_n < 0. \tag{10.29}$$

Therefore, by Theorem 8.4 the zeros of the characteristic polynomial (10.22) are located in the sector $\phi = \frac{\pi}{2\alpha}$ if and only if the condition (10.25) is met. □

If the conditions of Theorem 10.2 are satisfied then the problem of stabilization can be solved by the use of the following procedure:

Procedure 10.2

Step 1. Choose a diagonal matrix (10.23) with $\lambda_k > 0$, $k = 1, \dots, n$ and a real matrix $D \in \mathbb{R}^{m\times n}$ satisfying the conditions (10.24) and (10.25).
Step 2. Using (10.26) find the gain matrix K.

Example 10.2. Given the fractional system (10.18) with $\alpha = 0.8$, $q = 2$ and the matrices

$$A_0 = \begin{bmatrix} 0.5 & 0.3 & -0.2 \\ 0.2 & -1 & 0 \\ 0 & -0.2 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.3 & 0.4 & -0.3 \\ 0.1 & -0.5 & 0 \\ 0 & -0.1 & 1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0.2 & 0.3 & -0.5 \\ 0.7 & -1.5 & 0 \\ 0 & -0.7 & 0.5 \end{bmatrix}, \tag{10.30}$$
$$B_0 = \begin{bmatrix} 0 & 0.1 \\ 0 & 0 \\ 0.2 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \\ 0.3 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0.4 \\ 0 & 0 \\ 0.5 & 0 \end{bmatrix}.$$

Find a gain matrix $K \in \mathbb{R}^{2\times 3}$ such that the closed-loop system is positive and the zeros of its characteristic polynomial are located in the sector $\phi = \frac{\pi}{2\alpha}$.

Note that the fractional system with (10.30) is not positive since the matrices A_0, A_1 and A_2 have negative off-diagonal entries.

In this case

$$A = \sum_{k=0}^{2} A_k = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -3 & 0 \\ 0 & -1 & 2.5 \end{bmatrix}, \quad B = \sum_{k=0}^{2} B_k = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{10.31}$$

Using Procedure and (10.31) we obtain the following

Step 1. We choose

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0.5 & 2 & -3.5 \\ -4 & 0 & 1.4 \end{bmatrix}, \tag{10.32}$$

and we check the condition (10.24)

$$A\Lambda + BD = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -3 & 0 \\ 0 & -1 & 2.5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & 2 & -3.5 \\ -4 & 0 & 1.4 \end{bmatrix}$$
$$= \begin{bmatrix} -3 & 2 & 0.4 \\ 1 & -6 & 0 \\ 0.5 & 0 & -1 \end{bmatrix} \in M_3,$$

and the condition (10.25)

$$(A\Lambda + BD)\mathbb{1}_n = \begin{bmatrix} -3 & 2 & 0.4 \\ 1 & -6 & 0 \\ 0.5 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.6 \\ -5 \\ -0.5 \end{bmatrix}.$$

Therefore, the conditions are satisfied.

Step 2. Using (10.26) we obtain the gain matrix

$$K = D\Lambda^{-1} = \begin{bmatrix} 0.5 & 2 & -3.5 \\ -4 & 0 & 1.4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.5 & 1 & -3.5 \\ -4 & 0 & 1.4 \end{bmatrix}.$$

The closed-loop system is positive since the matrix

$$A_c = A + BK = \begin{bmatrix} -3 & 1 & 0.4 \\ 1 & -3 & 0 \\ 0.5 & 0 & -1 \end{bmatrix},$$

is a Metzler matrix.

The characteristic polynomial

$$\det[I_n\lambda - A_c] = \begin{vmatrix} \lambda+3 & -1 & -0.4 \\ -1 & \lambda+3 & 0 \\ -0.5 & 0 & \lambda+1 \end{vmatrix} = \lambda^3 + 7\lambda^2 + 13.8\lambda + 7.4\,,$$

has positive coefficients. Therefore, zeros of the characteristic polynomial of the closed-loop system are located in the desired sector $\phi = \frac{5}{8}\pi$.

10.3 Application of LMI to Stabilization of Fractional Linear Systems

10.3.1 Fractional 1D Linear Systems

Definition 10.1. An inequality of the form

$$F(x) + \mathscr{F} > 0 \tag{10.33}$$

where x takes values in real vector space V, the mapping $F : V \longrightarrow S^n$ is linear, and $\mathscr{F} \in S^n$, is called the linear matrix inequality (LMI). The LMI is called feasible if there exists an $x \in V$ such that the inequality (10.33) is satisfied, otherwise the LMI is called infeasible.

A matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is called the Metzler matrix if its off-diagonal entries are nonnegative, i.e. $a_{ij} \geq 0$ for $i \neq j, i, j = 1, \ldots, n$. The matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is called Hurwitz matrix if it has all eigenvalues with negative real parts (the system $\dot{x} = Ax$ is asymptotically stable). The matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is called Schur matrix if it has all eigenvalues with module less than one (the system $x_{i+1} = Ax_i$ is asymptotically stable).

Lemma 10.1. *A Metzler matrix $A \in \mathbb{R}^{n\times n}$ is Hurwitz matrix if and only if the LMI*

$$\text{block diag}\left[-\left(A^T P + PA\right) \ , \ P \right] \succ 0 \ , \tag{10.34}$$

is feasible with respect to the diagonal matrix P.

Remark 10.1. It is well-known that $A \in \mathbb{R}^n_+$ is Schur matrix if and only if $(A - I_n)$ is Hurwitz matrix.

Lemma 10.2. *A nonnegative matrix $A \in \mathbb{R}^n_+$ is Schur matrix if and only if the LMI*

$$\text{block diag}\left[-\left((A - I_n)^T P + P(A - I_n) \right) \ , \ P \right] \succ 0 \ , \tag{10.35}$$

is feasible with respect to the diagonal matrix P.

Lemma 10.3. *A nonnegative matrix $A \in \mathbb{R}^n_+$ is Schur matrix if and only if the LMI*

$$\text{block diag}\left[P - A^T PA \ , \ P \right] \succ 0 \ , \tag{10.36}$$

is feasible with respect to the diagonal matrix P.

Theorem 10.3. *The positive fractional system (8.68) is practically stable if and only if one of the following equivalent conditions holds:*

a) The LMI

$$\text{block diag}\left\{\begin{bmatrix} \tilde{P}_1 - A_\alpha^T P_1 A_\alpha & -c_1 A_\alpha^T P_1 & \dots & -c_{h-1} A_\alpha^T P_1 & -c_h A_\alpha^T P_1 \\ -c_1 P_1 A_\alpha & \tilde{P}_2 - c_1^2 P_1 & \dots & -c_1 c_{h-1} P_1 & -c_1 c_h P_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_{h-1} P_1 A_\alpha & -c_1 c_{h-1} P_1 & \dots & \tilde{P}_h - c_{h-1}^2 P_1 & -c_{h-1} c_h P_1 \\ -c_h P_1 A_\alpha & -c_1 c_h P_1 & \dots & -c_{h-1} c_h P_1 & P_{h+1} - c_h^2 P_1 \end{bmatrix}, \begin{bmatrix} P_1 & 0 & \dots & 0 & 0 \\ 0 & P_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P_h & 0 \\ 0 & 0 & \dots & 0 & P_{h+1} \end{bmatrix}\right\} \succ 0 \ , \tag{10.37}$$

$$\tilde{P}_i = P_i - P_{i+1}, \quad i = 1, \dots, h;$$

is feasible with respect to the diagonal matrix $P_1, \dots, P_{h+1}$.

b) The LMI

$$\text{block diag}\left\{\begin{bmatrix} A_\alpha^T P_1 + P_1 A_\alpha - 2P_1 & P_2 + c_1 P_1 & \dots & c_{h-1} P_1 & c_h P_1 \\ P_2 + c_1 P_1 & -2P_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{h-1} P_1 & 0 & \dots & -2P_{h-1} & P_{h+1} \\ c_h P_1 & 0 & \dots & P_{h+1} & -2P_h \end{bmatrix}, \begin{bmatrix} P_1 & 0 & \dots & 0 & 0 \\ 0 & P_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P_h & 0 \\ 0 & 0 & \dots & 0 & P_{h+1} \end{bmatrix}\right\} \succ 0 \ , \tag{10.38}$$

is feasible with respect to the diagonal matrix $P_1, \dots, P_{h+1}$.

c) The LMI

$$\text{block diag}\left\{\begin{bmatrix} P_1 & 0 & \dots & 0 & -A_\alpha^T P_1 & -P_2 & \dots & 0 \\ 0 & P_2 & \dots & 0 & -c_1 P_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_{h+1} & -c_h P_1 & 0 & \dots & -P_{h+1} \\ -P_1 A_\alpha & -c_1 P_1 & \dots & -c_h P_1 & P_1 & 0 & \dots & 0 \\ -P_2 & 0 & \dots & 0 & 0 & P_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -P_{h+1} & 0 & 0 & \dots & P_{h+1} \end{bmatrix},\right.$$

$$\left.\begin{bmatrix} P_1 & 0 & \dots & 0 & 0 \\ 0 & P_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P_h & 0 \\ 0 & 0 & \dots & 0 & P_{h+1} \end{bmatrix}\right\} \succ 0 \ , \tag{10.39}$$

is feasible with respect to the diagonal matrix $P_1, \dots, P_{h+1}$.

Proof. Proof is given in [121]. □

Example 10.3. Using the LMI approaches check the practical stability of the positive fractional system

$$\Delta^\alpha x_{k+1} = 0.1 x_k, \quad k \in \mathbb{Z}_+, \tag{10.40}$$

for $\alpha = 0.5$ and $h = 2$.
In this case we have:

$$c_1 = \frac{1}{8}, \quad c_2 = \frac{1}{16}, \quad A_\alpha = 0.6 \ ,$$

and

$$\tilde{A} = \begin{bmatrix} A_\alpha & c_1 & c_2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0.6 & \frac{1}{8} & \frac{1}{16} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Applying Theorem 10.3 and using MATLAB environment together with SEDUMI solver and YALMIP parser we obtain for the LMI (10.37)

$$\text{block diag}\left\{\begin{bmatrix} P_1 - P_2 - A_\alpha^T P_1 A_\alpha & -c_1 A_\alpha^T P_1 & -c_2 A_\alpha^T P_1 \\ -c_1 P_1 A_\alpha & P_2 - P_3 - c_1^2 P_1 & -c_1 c_2 P_1 \\ -c_2 P_1 A_\alpha & -c_1 c_2 P_1 & P_3 - c_2^2 P_1 \end{bmatrix}, \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix}\right\} \succ 0$$

where

$$\text{block diag}\begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix} = \text{diag}\begin{bmatrix} 7.8921 & 3.5026 & 2.1132 \end{bmatrix}.$$

For LMI (10.38)

$$\text{block diag}\left\{\begin{bmatrix} A_\alpha^T P_1 + P_1 A_\alpha - 2P_1 & P_2 + c_1 P_1 & c_2 P_1 \\ P_2 + c_1 P_1 & -2P_1 & P_3 \\ c_2 P_1 & P_3 & -2P_2 \end{bmatrix}, \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix}\right\} \succ 0$$

where

$$\text{block diag}\begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix} = \text{diag}\begin{bmatrix} 6.9266 & 3.1156 & 2.6096 \end{bmatrix},$$

and for LMI (10.39)

$$\text{block diag}\left\{\begin{bmatrix} P_1 & 0 & 0 & -A_\alpha^T P_1 & -P_2 & 0 \\ 0 & P_2 & 0 & -c_1 P_1 & 0 & -P_3 \\ 0 & 0 & P_3 & -c_2 P_1 & 0 & 0 \\ -P_1 A_\alpha & -c_1 P_1 & -c_2 P_1 & P_1 & 0 & 0 \\ -P_2 & 0 & 0 & 0 & P_2 & 0 \\ 0 & -P_3 & 0 & 0 & 0 & P_3 \end{bmatrix}, \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix}\right\} \succ 0 \tag{10.41}$$

where

$$\text{block diag}\begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix} = \text{diag}\begin{bmatrix} 7.7203 & 3.6738 & 2.2765 \end{bmatrix}.$$

Therefore, the LMIs are feasible with respect to the matrices P_1, P_2, P_3 and the positive fractional system (10.40) is practically stable.

Example 10.4. Using the LMI approaches check the practical stability of the positive fractional system

$$\Delta^\alpha x_{k+1} = \begin{bmatrix} -0.2 & 1 \\ 0.1 & b \end{bmatrix} x_k, \quad k \in \mathbb{Z}_+, \tag{10.42}$$

for $\alpha = 0.8$ and $h = 2$ and the following two values of the coefficient b:

$$a) \quad b = -0.5, \qquad b) \quad b = 0.5.$$

In this case we have:

$$c_1 = 0.08, \quad c_2 = 0.032.$$

In Case a) we have:

$$A_{\alpha_1} = \begin{bmatrix} 0.6 & 1 \\ 0.1 & 0.3 \end{bmatrix},$$

$$\tilde{A}_1 = \begin{bmatrix} A_\alpha & c_1 I_2 & c_2 I_2 \\ I_2 & 0 & 0 \\ 0 & I_2 & 0 \end{bmatrix} = \begin{bmatrix} 0.6 & 1 & 0.08 & 0 & 0.032 & 0 \\ 0.1 & 0.3 & 0 & 0.08 & 0 & 0.032 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

In Case b) we have:

$$A_{\alpha_2} = \begin{bmatrix} 0.6 & 1 \\ 0.1 & 1.3 \end{bmatrix},$$

$$\tilde{A}_2 = \begin{bmatrix} A_\alpha & c_1 I_2 & c_2 I_2 \\ I_2 & 0 & 0 \\ 0 & I_2 & 0 \end{bmatrix} = \begin{bmatrix} 0.6 & 1 & 0.08 & 0 & 0.032 & 0 \\ 0.1 & 1.3 & 0 & 0.08 & 0 & 0.032 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

In Case a) applying Theorem 10.3 and using MATLAB environment together with SEDUMI solver and YALMIP parser we obtain for the LMI (10.37)

$$\begin{aligned} &\text{block diag}\begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix} = \\ &\text{block diag}\left\{ \begin{bmatrix} 16.0915 & 0 \\ 0 & 84.368 \end{bmatrix}, \begin{bmatrix} 4.2540 & 0 \\ 0 & 16.3556 \end{bmatrix}, \begin{bmatrix} 2.5726 & 0 \\ 0 & 8.6007 \end{bmatrix} \right\}, \end{aligned}$$

for LMI (10.38)

$$\begin{aligned} &\text{block diag}\begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix} = \\ &\text{block diag}\left\{ \begin{bmatrix} 8.8848 & 0 \\ 0 & 35.5971 \end{bmatrix}, \begin{bmatrix} 2.5601 & 0 \\ 0 & 7.2962 \end{bmatrix}, \begin{bmatrix} 2.2771 & 0 \\ 0 & 5.2364 \end{bmatrix} \right\}. \end{aligned}$$

In Case b) for LMI (10.39) we obtain

$$\begin{aligned} &\text{block diag}\begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix} = \\ &\text{block diag}\left\{ \begin{bmatrix} -0.0834 & 0 \\ 0 & -0.3933 \end{bmatrix}, \begin{bmatrix} 0.4152 & 0 \\ 0 & 0.316 \end{bmatrix}, \begin{bmatrix} 0.4417 & 0 \\ 0 & 0.6885 \end{bmatrix} \right\}. \end{aligned}$$

In Case a) the positive fractional system (10.42) is practically stable. In Case b) the positive fractional system (10.42) is unstable for any h (not only for $h = 2$) since the matrix A_{α_2} has one diagonal entry grater then 1.

The characteristic polynomial of the matrix $A_{\alpha_2} - I_n$

$$p(z) = \det[I_n(z+1) - A_{\alpha_2}] = \begin{vmatrix} z - 0.4 & -1 \\ -0.1 & z - 0.3 \end{vmatrix} = z^2 - 0.7z - 0.22\,,$$

has two negative coefficients. Therefore, the system (10.42) is also unstable for any h.

10.3.2 Positive 2D Linear Systems

Theorem 10.4. *The positive Roesser model (3.40) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:*

a) LMI

$$\text{block diag}\left\{\begin{bmatrix} 2P_1 - A_{11}^T P_1 - P_1 A_{11} & -A_{21}^T P_2 - P_1 A_{12} \\ -A_{12}^T P_1 - P_2 A_{21} & 2P_2 - A_{22}^T P_2 - P_2 A_{22} \end{bmatrix}, \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\right\} \succ 0 , \tag{10.43}$$

is feasible with respect to the diagonal matrices P_1 and P_2.

b) LMI

$$\text{block diag}\left\{\begin{bmatrix} P_1 - A_{11}^T P_1 A_{11} - A_{21}^T P_2 A_{21} & -A_{11}^T P_1 A_{12} - A_{21}^T P_2 A_{22} \\ -A_{12}^T P_1 A_{11} - A_{22}^T P_2 A_{21} & P_2 - A_{12}^T P_1 A_{12} - A_{22}^T P_2 A_{22} \end{bmatrix}, \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\right\} \succ 0, \tag{10.44}$$

is feasible with respect to the diagonal matrices P_1 and P_2.

Proof. By Theorem 8.10 the positive Roesser model (3.40) is asymptotically stable if and only if the equivalent 1D system (8.42) is asymptotically stable. Using Lemma 10.2 to the system (8.42) we obtain LMI (10.43) since

$$\begin{aligned} \text{block diag} & \left\{\begin{bmatrix} I_{n_1} - A_{11}^T & -A_{21}^T \\ -A_{12}^T & I_{n_2} - A_{22}^T \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \right. \\ & \left. + \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} I_{n_1} - A_{11} & -A_{12} \\ -A_{21} & I_{n_2} - A_{22} \end{bmatrix}\right\} \\ & = \text{block diag}\left\{\begin{bmatrix} 2P_1 - A_{11}^T P_1 - P_1 A_{11} & -A_{21}^T P_2 - P_1 A_{12} \\ -A_{12}^T P_1 - P_2 A_{21} & 2P_2 - A_{22}^T P_2 - P_2 A_{22} \end{bmatrix}, \right. \\ & \left. \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\right\} \succ 0 . \end{aligned}$$

Similarly, using Lemma 10.3 to the system (8.42) we obtain LMI (10.44) since

$$\begin{aligned} \text{block diag} & \left\{\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} - \begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}, \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\right\} \\ = \text{block diag} & \left\{\begin{bmatrix} P_1 - A_{11}^T P_1 A_{11} - A_{21}^T P_2 A_{21} & -A_{11}^T P_1 A_{12} - A_{21}^T P_2 A_{22} \\ -A_{12}^T P_1 A_{11} - A_{22}^T P_2 A_{21} & P_2 - A_{12}^T P_1 A_{12} - A_{22}^T P_2 A_{22} \end{bmatrix}, \right. \\ & \left. \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\right\} \succ 0 . \end{aligned}$$

□

Theorem 10.5. *The positive (general model) system (8.10) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:*

a) LMI

$$\text{block diag}\left[2P - \sum_{k=0}^{2}\left(A_k^T P + PA_k\right)\ ,\ P\right] \succ 0\ , \tag{10.45}$$

is feasible with respect to the diagonal matrix P.

b) LMI

$$\text{block diag}\left[P - \sum_{k=0}^{2}\sum_{l=0}^{2}\left(A_k^T PA_l\right)\ ,\ P\right] \succ 0\ , \tag{10.46}$$

is feasible with respect to the diagonal matrix P.

c) LMI

$$\text{block diag}\ \left\{\begin{bmatrix} 2P_1 - \left(A_1^T + A_2^T\right)P_1 - P_1\left(A_1 + A_2\right) & -P_2 - P_1A_0 \\ -P_2 - A_0^T P_1 & 2P_2 \end{bmatrix},\right.$$
$$\left.\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\right\} \succ 0\ , \tag{10.47}$$

is feasible with respect to the diagonal matrices P_1 and P_2.

d) LMI

$$\text{block diag}\ \left\{\begin{bmatrix} P_1 - \left(A_1 + A_2\right)^T P_1\left(A_1 + A_2\right) - P_2 & -\left(A_1 + A_2\right)^T P_1A_0 \\ -A_0^T P_1\left(A_1 + A_2\right) & P_2 - A_0^T P_1 A_0 \end{bmatrix},\right.$$
$$\left.\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\right\} \succ 0\ , \tag{10.48}$$

is feasible with respect to the diagonal matrices P_1 and P_2.

Corollary 10.1. *The positive 2D SF-MM is asymptotically stable if and only if one of the following equivalent conditions is satisfied:*

a) LMI

$$\text{block diag}\left[2P - \sum_{k=1}^{2}\left(A_k^T P + PA_k\right)\ ,\ P\right] \succ 0, \tag{10.49}$$

is feasible with respect to the diagonal matrix P.

b) LMI

$$\text{block diag}\left[P - \sum_{k=1}^{2}\sum_{l=1}^{2}\left(A_k^T PA_l\right)\ ,\ P\right] \succ 0, \tag{10.50}$$

is feasible with respect to the diagonal matrix P.

10.3.3 Positive 2D Linear Systems with Delays

Consider the positive 2D Roesser model with q delays

$$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \sum_{k=0}^{q} A_k \begin{bmatrix} x_{i-k,j}^h \\ x_{i,j-k}^v \end{bmatrix}, \quad i,j \in \mathbb{Z}_+, \tag{10.51}$$

where $x_{ij}^h \in \mathbb{R}_+^{n_1}$, $x_{ij}^v \in \mathbb{R}_+^{n_2}$ are the horizontal and vertical state vectors in the point (i,j) and

$$A_k = \begin{bmatrix} A_{11}^k & A_{12}^k \\ A_{21}^k & A_{22}^k \end{bmatrix}, \quad k = 1, \ldots, q \ . \tag{10.52}$$

Theorem 10.6. *The positive Roesser model (10.51) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:*

a) LMI

$$\text{block diag}\left\{ \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \right\} \succ 0 \ , \tag{10.53}$$

where

$$P_{11} = \begin{bmatrix} 2P_1^0 - (A_{11}^0)^T P_1^0 - P_1^0 A_{11}^0 & -P_1^1 - P_1^0 A_{11}^1 & \ldots & -P_1^0 A_{11}^{q-1} & -P_1^0 A_{11}^q \\ -(A_{11}^1)^T P_1^0 - P_1^1 & 2P_1^1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -(A_{11}^{q-1})^T P_1^0 & 0 & \ldots & 2P_1^{q-1} & -P_1^q \\ -(A_{11}^q)^T P_1^0 & 0 & \ldots & -P_1^q & 2P_1^q \end{bmatrix}, \tag{10.54a}$$

$$P_{12} = P_{21}^T = -\begin{bmatrix} (A_{21}^0)^T P_2^0 + P_1^0 A_{21}^0 & P_1^0 A_{12}^1 & \ldots & P_1^0 A_{12}^{q-1} & P_1^0 A_{12}^q \\ (A_{21}^1)^T P_2^0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (A_{21}^{q-1})^T P_2^0 & 0 & \ldots & 0 & 0 \\ (A_{21}^q)^T P_2^0 & 0 & \ldots & 0 & 0 \end{bmatrix}, \tag{10.54b}$$

$$P_{22} = \begin{bmatrix} 2P_2^0 - (A_{22}^0)^T P_2^0 - P_2^0 A_{22}^0 & -P_2^1 - P_2^0 A_{22}^1 & \ldots & -P_2^0 A_{22}^{q-1} & -P_2^0 A_{22}^q \\ -(A_{22}^1)^T P_2^0 - P_2^1 & 2P_2^1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -(A_{22}^{q-1})^T P_2^0 & 0 & \ldots & 2P_2^{q-1} & -P_2^q \\ -(A_{22}^q)^T P_2^0 & 0 & \ldots & -P_2^q & 2P_2^q \end{bmatrix}, \tag{10.54c}$$

$$P_k = \text{block diag}\left[P_k^0 \ P_k^1 \ \ldots \ P_k^q \right], \quad k = 1, 2 \ , \tag{10.54d}$$

is feasible with respect to the diagonal matrices P_1 and P_2.

b) LMI

$$\text{block diag}\left\{ \begin{bmatrix} \overline{P}_{11} & \overline{P}_{12} \\ \overline{P}_{21} & \overline{P}_{22} \end{bmatrix}, \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \right\} \succ 0 \ , \tag{10.55}$$

where

$$\overline{P}_{11} = \begin{bmatrix} \tilde{P}_{11}^1 \\ \tilde{P}_{11}^2 \\ \vdots \\ \tilde{P}_{11}^{R_{11}} \end{bmatrix}, \quad \overline{P}_{12} = \overline{P}_{21}^T = -\begin{bmatrix} \tilde{P}_{12}^1 \\ \tilde{P}_{12}^2 \\ \vdots \\ \tilde{P}_{12}^{R_{12}} \end{bmatrix}, \quad \overline{P}_{22} = \begin{bmatrix} \tilde{P}_{22}^1 \\ \tilde{P}_{22}^2 \\ \vdots \\ \tilde{P}_{22}^{R_{22}} \end{bmatrix}, \tag{10.56}$$

$$
\begin{aligned}
\tilde{P}_{11}^{1} &= \big[P_1^0 - P_1^1 - (A_{11}^0)^T P_1^0 A_{11}^0,\ -(A_{11}^0)^T P_1^0 A_{11}^1, \ldots,\ -(A_{11}^0)^T P_1^0 A_{11}^q,\\
&\quad -(A_{21}^0)^T P_2^0 A_{21}^0\ ,\ -(A_{21}^0)^T P_2^0 A_{21}^1\ ,\ \ldots\ ,\ -(A_{21}^0)^T P_2^0 A_{21}^q\big],\\
\tilde{P}_{11}^{2} &= \big[-(A_{11}^1)^T P_1^0 A_{11}^0,\ P_1^1 - P_1^2 - (A_{11}^1)^T P_1^0 A_{11}^1, \ldots,\ -(A_{11}^1)^T P_1^0 A_{11}^q\ ,\\
&\quad -(A_{21}^1)^T P_2^0 A_{21}^0\ ,\ -(A_{21}^1)^T P_2^0 A_{21}^1\ ,\ \ldots\ ,\ -(A_{21}^1)^T P_2^0 A_{21}^q\big],\\
&\ \ \vdots\\
\tilde{P}_{11}^{R_{11}} &= \big[-(A_{11}^q)^T P_1^0 A_{11}^0\ ,\ -(A_{11}^q)^T P_1^0 A_{11}^1\ ,\ \ldots\ ,\ P_1^q - (A_{11}^q)^T P_1^0 A_{11}^q\ ,\\
&\quad -(A_{21}^q)^T P_2^0 A_{21}^0\ ,\ -(A_{21}^q)^T P_2^0 A_{21}^1\ ,\ \ldots\ ,\ -(A_{21}^q)^T P_2^0 A_{21}^q\big],\\
\tilde{P}_{12}^{1} &= \big[(A_{11}^0)^T P_1^0 A_{12}^0\ ,\ (A_{11}^0)^T P_1^0 A_{12}^1\ ,\ \ldots\ ,\ (A_{11}^0)^T P_1^0 A_{12}^q\ ,\\
&\quad (A_{21}^0)^T P_2^0 A_{22}^0\ ,\ (A_{21}^0)^T P_2^0 A_{22}^1\ ,\ \ldots\ ,\ (A_{21}^0)^T P_2^0 A_{22}^q\big],\\
\tilde{P}_{12}^{2} &= \big[(A_{11}^1)^T P_1^0 A_{12}^0\ ,\ (A_{11}^1)^T P_1^0 A_{12}^1\ ,\ \ldots\ ,\ (A_{11}^1)^T P_1^0 A_{12}^q\ ,\\
&\quad (A_{21}^1)^T P_2^0 A_{22}^0\ ,\ (A_{21}^1)^T P_2^0 A_{22}^1\ ,\ \ldots\ ,\ (A_{21}^1)^T P_2^0 A_{22}^q\big],\\
&\ \ \vdots\\
\tilde{P}_{12}^{R_{12}} &= \big[(A_{11}^q)^T P_1^0 A_{12}^0\ ,\ (A_{11}^q)^T P_1^0 A_{12}^1\ ,\ \ldots\ ,\ (A_{11}^q)^T P_1^0 A_{12}^q\ ,\\
&\quad (A_{21}^q)^T P_2^0 A_{22}^0\ ,\ (A_{21}^q)^T P_2^0 A_{22}^1\ ,\ \ldots\ ,\ (A_{21}^q)^T P_2^0 A_{22}^q\big],\\
\tilde{P}_{22}^{1} &= \big[P_2^0 - (A_{22}^0)^T P_2^0 A_{22}^0\ ,\ -(A_{22}^0)^T P_2^0 A_{22}^1\ ,\ \ldots\ ,\ -(A_{22}^0)^T P_2^0 A_{22}^q\ ,\\
&\quad -(A_{12}^0)^T P_1^0 A_{12}^0\ ,\ -(A_{12}^0)^T P_1^0 A_{12}^1\ ,\ \ldots\ ,\ -(A_{12}^0)^T P_1^0 A_{12}^q\big],\\
\tilde{P}_{22}^{2} &= \big[-(A_{22}^1)^T P_2^0 A_{22}^0\ ,\ P_2^1 - (A_{22}^1)^T P_2^0 A_{22}^1\ ,\ \ldots\ ,\ -(A_{22}^1)^T P_2^0 A_{22}^q\ ,\\
&\quad -(A_{12}^1)^T P_1^0 A_{12}^0\ ,\ -(A_{12}^1)^T P_1^0 A_{12}^1\ ,\ \ldots\ ,\ -(A_{12}^1)^T P_1^0 A_{12}^q\big],\\
&\ \ \vdots\\
\tilde{P}_{22}^{R_{22}} &= \big[-(A_{22}^q)^T P_2^0 A_{22}^0\ ,\ -(A_{22}^q)^T P_2^0 A_{22}^1\ ,\ \ldots\ ,\ P_2^q - (A_{22}^q)^T P_2^0 A_{22}^q\ ,\\
&\quad -(A_{12}^q)^T P_1^0 A_{12}^0\ ,\ -(A_{12}^q)^T P_1^0 A_{12}^1\ ,\ \ldots\ ,\ -(A_{12}^q)^T P_1^0 A_{12}^q\big],
\end{aligned}
$$

is feasible with respect to the diagonal matrices P_1 and P_2.

Proof. The positive Roesser model (10.51) is asymptotically stable if and only if the reduced model (3.69) is asymptotically stable. Applying to the reduced model (3.69) Theorem 10.4 and using (3.70) and (10.43) we obtain (10.54). Similarly, using (3.70) and (10.44) we obtain (10.56). □

Example 10.5. Using LMI check the asymptotic stability of the positive Roesser model (10.51) for $q = 1$ with the matrices:

$$A_0 = \begin{bmatrix} 0.1 & 0.2 & | & 0 \\ 0 & 0.1 & | & 0.3 \\ -- & -- & -|- & -- \\ 0 & 0 & | & 0.2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.2 & 0.1 & | & 0.2 \\ 0 & 0.1 & | & 0.2 \\ -- & -- & -|- & -- \\ 0 & 0 & | & 0.2 \end{bmatrix}. \tag{10.57}$$

In this case the matrix (3.70) of the reduced positive Roesser model has the form

$$A = \begin{bmatrix} A_{11}^0 & A_{11}^1 & A_{12}^0 & A_{12}^1 \\ I_2 & 0 & 0 & 0 \\ A_{21}^0 & A_{21}^1 & A_{22}^0 & A_{22}^1 \\ 0 & 0 & I_1 & 0 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 & 0.2 & 0.1 & 0 & 0.2 \\ 0 & 0.1 & 0 & 0.1 & 0.3 & 0.2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{10.58}$$

Using Theorem 10.6 and using MATLAB environment together with SEDUMI solver and YALMIP parser we obtain LMI (10.53).

$$\text{block diag}\left[P_1 \ P_2\right] = \text{diag}\left[0.7799 \ 0.7883 \ 0.5625 \ 0.5615 \ 0.9452 \ 0.5931\right],$$

and for LMI (10.55)

$$\text{block diag}\left[P_1 \ P_2\right] = \text{diag}\left[1.5526 \ 1.5897 \ 0.8374 \ 0.8074 \ 1.8736 \ 0.9290\right].$$

Moreover, LMI is feasible with respect to the diagonal matrices P_1 and P_2 and the positive Roesser model is asymptotically stable.

The above considerations can be easily extended for the positive Roesser model with delays of the form

$$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \sum_{k=0}^{q_1} \sum_{l=0}^{q_2} A_{kl} \begin{bmatrix} x_{i-k,j}^h \\ x_{i,j-l}^v \end{bmatrix}, \quad i,j \in \mathbb{Z}_+,$$

where $x_{ij}^h \in \mathbb{R}_+^{n_1}$, $x_{ij}^v \in \mathbb{R}_+^{n_2}$ are the horizontal and vertical state vectors in the point (i,j) and $A_{kl} \in \mathbb{R}_+^{(n_1+n_2)\times(n_1+n_2)}$.

Theorem 10.7. *The positive 2D (general model) system with q delays (3.72) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:*

a) LMI

$$\text{block diag}\left[2P - \left(\hat{P}_0 + \hat{P}_1 + \hat{P}_2\right), \ P\right] \succ 0 , \tag{10.59}$$

where

$$P = \text{block diag}\left[\, P_0 \; P_1 \; \dots \; P_q \,\right],$$

$$\hat{P}_0 = \begin{bmatrix} (A_0^0)^T P_0 + P_0 A_0^0 & P_1 + P_0 A_1^0 & \dots & P_0 A_{q-1}^0 & P_0 A_q^0 \\ (A_1^0)^T P_0 + P_1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (A_{q-1}^0)^T P_0 & 0 & \dots & 0 & P_q \\ (A_q^0)^T P_0 & 0 & \dots & P_q & 0 \end{bmatrix}, \qquad (10.60a)$$

$$\hat{P}_k = \begin{bmatrix} (A_0^k)^T P_0 + P_0 A_0^k & P_0 A_1^k & \dots & P_0 A_{q-1}^k & P_0 A_q^k \\ (A_1^k)^T P_0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (A_{q-1}^k)^T P_0 & 0 & \dots & 0 & 0 \\ (A_q^k)^T P_0 & 0 & \dots & 0 & 0 \end{bmatrix}, \qquad (10.60b)$$

is feasible with respect to the diagonal matrix P.

b) *LMI*

$$\text{block diag}\left[\, P - \hat{P} \;,\; P \,\right] \succ 0 \;. \qquad (10.61)$$

where

$$P = \text{block diag}\left[\, P_0 \; P_1 \; \dots \; P_q \,\right],$$

$$\hat{P} = \begin{bmatrix} (A_0)^T P_0 A_0 + P_1 & (A_0)^T P_0 A_1 & \dots & (A_0)^T P_0 A_{q-1} & (A_0)^T P_0 A_q \\ (A_1)^T P_0 A_0 & (A_1)^T P_0 A_1 + P_2 & \dots & (A_1)^T P_0 A_{q-1} & (A_1)^T P_0 A_q \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (A_{q-1})^T P_0 A_0 & (A_{q-1})^T P_0 A_1 & \dots & (A_{q-1})^T P_0 A_{q-1} + P_q & (A_{q-1})^T P_0 A_q \\ (A_q)^T P_0 A_0 & (A_q)^T P_0 A_1 & \dots & (A_q)^T P_0 A_{q-1} & (A_q)^T P_0 A_q \end{bmatrix}$$

$$A_k = A_k^0 + A_k^1 + A_k^2, \quad k = 0, 1, \dots, q \;, \qquad (10.62)$$

is feasible with respect to the diagonal matrix P.

Proof. The positive 2D system (3.72) is asymptotically stable if and only if the reduced system (3.75) is asymptotically stable. Applying to the reduced system (3.75) LMI (10.45), we obtain LMI (10.60). Similarly, applying to the reduced system (3.75) LMI (10.46), we obtain LMI (10.62). □

Remark 10.2. In a similar way using LMI (10.47) and (10.48) to the reduced system (3.75), we obtain LMI for the positive system (3.72).

Remark 10.3. Substituting $A_k^0 = 0$, $k = 0, 1, \dots, q$ in Theorem 10.7, we obtain the corresponding LMI conditions for the positive 2D SF-MM.

Example 10.6. Using LMI check the asymptotic stability of the positive 2D system (3.72) for $q = 1$ with the matrices:

$$A_0^0 = \begin{bmatrix} 0.6 & 0.1 \\ 0 & 0.2 \end{bmatrix}, \quad A_0^1 = \begin{bmatrix} 0.1 & 0.2 \\ 0 & 0.21 \end{bmatrix}, \quad A_0^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$A_1^1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1^1 = \begin{bmatrix} 0.2 & 0.1 \\ 0 & 0.1 \end{bmatrix}, \quad A_1^2 = \begin{bmatrix} 0 & 0.2 \\ 0 & 0.4 \end{bmatrix}.$$

The matrices (3.74) of the reduced system (3.75) have the form:

$$\overline{A}_0 = \begin{bmatrix} 0.6 & 0.1 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \overline{A}_1 = \begin{bmatrix} 0.1 & 0.2 & 0.2 & 0.1 \\ 0 & 0.21 & 0 & 0.1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \overline{A}_2 = \begin{bmatrix} 0 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Using Theorem 10.7 and using MATLAB environment together with SEDUMI solver and YALMIP parser we obtain LMI (10.59)

$$P = \text{diag}\left[0.1382 \ 2.0346 \ 0.0414 \ 1.0731\right],$$

and for LMI (10.61)

$$P = \text{diag}\left[0.2681 \ 3.5981 \ 0.0647 \ 1.8976\right].$$

Moreover, LMI is feasible with respect to the diagonal matrix P and the positive 2D system is asymptotically stable.

The consideration can be easily extended for the positive 2D system of the form

$$x_{i+1,j+1} = \sum_{k=0}^{q_1} \sum_{l=0}^{q_2} \left(A_{kl}^0 x_{i-k,j-l} + A_{kl}^1 x_{i+1-k,j-l} + A_{kl}^2 x_{i-k,j+1-l}\right), \quad i,j \in \mathbb{Z}_+, \tag{10.63}$$

where $x_{ij} \in \mathbb{R}_+^n$ is the state vector in the point (i,j) and $A_{kl}^t \in \mathbb{R}^{n\times n}$, $k = 0,1,\ldots,q_1$; $l = 0,1,\ldots,q_2$; $t = 0,1,2$.

10.3.4 Fractional 2D Roesser Model

Consider the positive fractional Roesser model (3.49) with the state-feedback

$$u_{ij} = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix}, \tag{10.64}$$

where $K = [K_1, K_2] \in \mathbb{R}^{m\times n}$, $K_j \in \mathbb{R}^{m\times n}$, $j = 1,2$ is a gain matrix.

We are looking for a gain matrix K such that the closed-loop system

$$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} \overline{A}_{11}+B_1K_1 & A_{12}+B_1K_2 \\ A_{21}+B_2K_1 & \overline{A}_{22}+B_2K_2 \end{bmatrix} \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} - \begin{bmatrix} \sum_{k=2}^{i+1} c_\alpha(k) x_{i-k+1,j}^h \\ \sum_{l=2}^{j+1} c_\beta(l) x_{i,j-l+1}^v \end{bmatrix}, \tag{10.65}$$

is positive and asymptotically stable [127].

Theorem 10.8. *The positive fractional closed-loop system (10.65) is positive and asymptotically stable if and only if there exist a block diagonal matrix*

$$\Lambda = \text{block diag}\left[\Lambda_1 \ \Lambda_2\right], \quad \Lambda_k = \text{diag}\left[\lambda_{k1},\ldots,\lambda_{kn_k}\right], \quad \lambda_{kj} > 0, \tag{10.66}$$

$$k = 1,2; \quad j = 1,\ldots,n_k;$$

and a real matrix

$$D = \left[D_1 \ D_2\right], \quad D_k \in \mathbb{R}^{m\times n_k}, \quad k = 1,2; \tag{10.67}$$

satisfying conditions

$$\begin{bmatrix} \overline{A}_{11}\Lambda_1 + B_1D_1 & A_{12}\Lambda_2 + B_1D_2 \\ A_{21}\Lambda_1 + B_2D_1 & \overline{A}_{22}\Lambda_2 + B_2D_2 \end{bmatrix} \in \mathbb{R}_+^{n\times n} \tag{10.68}$$

and

$$\begin{bmatrix} A_{11}\Lambda_1 + B_1D_1 & A_{12}\Lambda_2 + B_1D_2 \\ A_{21}\Lambda_1 + B_2D_1 & A_{22}\Lambda_2 + B_2D_2 \end{bmatrix} \begin{bmatrix} \mathbb{1}_{n_1} \\ \mathbb{1}_{n_2} \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{10.69}$$

where $\mathbb{1}_{n_k} = [1,\ldots,1]^T \in \mathbb{R}_+^{n_k}$, $k = 1,2$. *The gain matrix is given by the formula*

$$K = \left[K_1 \ K_2\right] = \left[D_1 \ D_2\right]\Lambda^{-1} = \left[D_1\Lambda_1^{-1} \ D_2\Lambda_2^{-1}\right] \tag{10.70}$$

Proof. Proof of this Theorem is similar to the proof of Theorem 10.1 [166]. □

It is well-known that the positive closed-loop system (10.65) is asymptotically stable if and only if the positive 1D system with the matrix

$$\begin{bmatrix} \overline{A}_{11}+B_1K_1 & A_{12}+B_1K_2 \\ A_{21}+B_2K_1 & \overline{A}_{22}+B_2K_2 \end{bmatrix} - \sum_{k=2}^{\infty} \begin{bmatrix} I_{n_1}c_\alpha(k) & 0 \\ 0 & I_{n_2}c_\beta(k) \end{bmatrix}, \tag{10.71}$$

is asymptotically stable.

Taking into account that

$$\sum_{k=2}^{\infty} c_\alpha(k) = \alpha - 1, \quad \sum_{l=2}^{\infty} c_\beta(l) = \beta - 1, \tag{10.72}$$

and $\overline{A}_{11} = A_{11} + I_{n_1}\alpha, \overline{A}_{22} = A_{22} + I_{n_2}\beta$, we may write the matrix (10.71) in the form

$$\begin{bmatrix} \hat{A}_{11} + B_1K_1 & A_{12} + B_1K_2 \\ A_{21} + B_2K_1 & \hat{A}_{22} + B_2K_2 \end{bmatrix} = A + BK, \tag{10.73}$$

where $\hat{A}_{11} = A_{11} + I_{n_1}, \hat{A}_{22} = A_{22} + I_{n_2}$ and

$$A = \begin{bmatrix} \hat{A}_{11} & A_{12} \\ A_{21} & \hat{A}_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \tag{10.74}$$

Theorem 10.9. *The fractional closed-loop system (10.65) is positive and asymptotically stable if and only if there exist a positive definite block diagonal matrix (10.66) and a real matrix (10.67)such that the condition (10.68) is satisfied and the LMI*

$$\begin{bmatrix} -\Lambda & A\Lambda + BD \\ (A\Lambda + BD)^T & -\Lambda \end{bmatrix} \prec 0 \ , \tag{10.75}$$

is feasible with respect to the positive definite diagonal matrix Λ.

Proof. The closed-loop system (10.65) is positive if and only if the condition (10.68) is satisfied since the condition

$$\begin{aligned} \begin{bmatrix} \overline{A}_{11} + B_1K_1 & A_{12} + B_1K_2 \\ A_{21} + B_2K_1 & \overline{A}_{22} + B_2K_2 \end{bmatrix} &= \begin{bmatrix} \overline{A}_{11} + B_1D_1\Lambda_1^{-1} & A_{12} + B_1D_2\Lambda_2^{-1} \\ A_{21} + B_2D_1\Lambda_1^{-1} & \overline{A}_{22} + B_2D_2\Lambda_2^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \overline{A}_{11}\Lambda_1 + B_1D_1 & A_{12}\Lambda_2 + B_1D_2 \\ A_{21}\Lambda_1 + B_2D_1 & \overline{A}_{22}\Lambda_2 + B_2D_2 \end{bmatrix} \\ &\times \begin{bmatrix} \Lambda_1^{-1} & 0 \\ 0 & \Lambda_2^{-1} \end{bmatrix}, \end{aligned}$$

is equivalent to (10.68).

The closed-loop system (10.65) is asymptotically stable if and only if the LMI

$$P - (A + BK)^T P (A + BK) \succ 0 \ , \tag{10.76}$$

is feasible with respect to a positive definite diagonal matrix P.

Using the Schur complement we can write the condition (10.76) in the form

$$\begin{bmatrix} -P & P(A + BK) \\ (A + BK)^T P & -P \end{bmatrix} \prec 0 \ . \tag{10.77}$$

Substituting of (10.70) and $P = \Lambda^{-1}$ into (10.77) yields

$$\begin{aligned} &\begin{bmatrix} -\Lambda^{-1} & \Lambda^{-1}\left(A + BD\Lambda^{-1}\right) \\ \left(A + BD\Lambda^{-1}\right)^T \Lambda^{-1} & -\Lambda^{-1} \end{bmatrix} = \begin{bmatrix} -\Lambda^{-1} & 0 \\ 0 & -\Lambda^{-1} \end{bmatrix} \\ &\times \begin{bmatrix} -\Lambda & A\Lambda + BD \\ (A\Lambda + BD)^T & -\Lambda \end{bmatrix} \begin{bmatrix} -\Lambda^{-1} & 0 \\ 0 & -\Lambda^{-1} \end{bmatrix} \prec 0 \ . \end{aligned} \tag{10.78}$$

Applying the congruent transformation with the matrix block diag$[\Lambda,\Lambda]$ we obtain the condition (10.75). □

Example 10.7. Given the fractional 2D Roesser model with $\alpha = 0.4$ $\beta = 0.5$ and

$$A_{11} = \begin{bmatrix} -0.5 & -0.1 \\ 0.1 & 0.01 \end{bmatrix}, A_{12} = \begin{bmatrix} -0.1 & -0.1 \\ 0.2 & 0.1 \end{bmatrix}, A_{21} = \begin{bmatrix} -0.3 & -0.1 \\ 0.2 & 0.1 \end{bmatrix}, \tag{10.79a}$$

$$A_{22} = \begin{bmatrix} -1 & -0.1 \\ 0.4 & 0.1 \end{bmatrix}, B_1 = \begin{bmatrix} -0.2 \\ 0.1 \end{bmatrix}, B_2 = \begin{bmatrix} -0.3 \\ 0.2 \end{bmatrix}. \tag{10.79b}$$

Find a gain matrix $K = [K_1, K_2]$, $K_i \in \mathbb{R}^{1\times 2}$, $i = 1,2$ such that the closed-loop system is positive and asymptotically stable.

The fractional 2D Roesser model with (10.79a) is not positive since the matrices

$$\begin{bmatrix} \overline{A}_{11} & A_{12} \\ A_{21} & \overline{A}_{22} \end{bmatrix} = \begin{bmatrix} -0.1 & -0.1 & -0.1 & -0.1 \\ 0.1 & 0.41 & 0.2 & 0.1 \\ -0.3 & -0.1 & -0.5 & -0.1 \\ 0.2 & 0.1 & 0.4 & 0.6 \end{bmatrix}$$

have negative entries. The model is also unstable since the matrix

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} -0.5 & -0.1 & -0.1 & -0.1 \\ 0.1 & 0.01 & 0.2 & 0.1 \\ -0.3 & -0.1 & -1 & -0.1 \\ 0.2 & 0.1 & 0.4 & 0.1 \end{bmatrix} \tag{10.80}$$

has positive diagonal entries.

We choose:

$$D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}, \quad D_1 = D_2 = \begin{bmatrix} -0.4 & -0.2 \end{bmatrix}. \tag{10.81}$$

Applying Theorem 10.9 and using MATLAB environment together with SEDUMI solver and YALMIP parser for the LMI (10.75) we obtain:

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}, \quad \Lambda_1 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix}, \quad \Lambda_2 = \begin{bmatrix} 0.2258 & 0 \\ 0 & 0.2413 \end{bmatrix}. \tag{10.82}$$

Therefore, the LMI is feasible with respect to the diagonal matrix Λ.

Using (10.70) we obtain the gain matrix

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix} = \begin{bmatrix} D_1\Lambda_1^{-1} & D_2\Lambda_2^{-1} \end{bmatrix} = \begin{bmatrix} -1 & -0.5 & -1.7712 & -0.8289 \end{bmatrix}. \tag{10.83}$$

The closed-loop system is positive since matrices:

$$\overline{A}_{11} + B_1K_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.36 \end{bmatrix}, \quad A_{12} + B_1K_2 = \begin{bmatrix} 0.2542 & 0.0658 \\ 0.0229 & 0.0171 \end{bmatrix},$$
$$A_{21} + B_2K_1 = \begin{bmatrix} 0 & 0.05 \\ 0 & 0 \end{bmatrix}, \quad \overline{A}_{22} + B_2K_2 = \begin{bmatrix} 0.0313 & 0.1487 \\ 0.0458 & 0.4342 \end{bmatrix},$$

have all nonnegative entries.

The closed-loop system is asymptotically stable since its characteristic polynomial

$$\det \begin{bmatrix} I_{n_1}z - (A_{11} + B_1K_1) & -(A_{12} + B_1K_2) \\ -(A_{21} + B_2K_1) & I_{n_2}z - (A_{22} + B_2K_2) \end{bmatrix}$$
$$= z^4 + 0.8744z^3 + 0.2166z^2 + 0.0141z + 0.0003,$$

has positive coefficients.

Example 10.8. Given the positive fractional 2D Roesser model with $\alpha = 0.4$, $\beta = 0.9$ and

$$A_{11} = \begin{bmatrix} -0.4 & 0.01 \\ 0.03 & 0.001 \end{bmatrix}, A_{12} = \begin{bmatrix} 0.01 & 0.01 \\ 0.01 & 0.2 \end{bmatrix}, A_{21} = \begin{bmatrix} 0.01 & 0.2 \\ 0 & 0.01 \end{bmatrix}, \tag{10.84a}$$
$$A_{22} = \begin{bmatrix} -0.9 & 0.01 \\ 0.01 & -0.8 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0.001 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0.002 \end{bmatrix}. \tag{10.84b}$$

Find a gain matrix $K = [K_1, K_2]$, $K_i \in \mathbb{R}^{1\times 2}$, $i = 1, 2$ such that the closed-loop system is positive and asymptotically stable.

The fractional 2D Roesser model with (10.84) is unstable since the matrix

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} -0.4 & 0.01 & 0.01 & 0.01 \\ 0.03 & 0.001 & 0.01 & 0.2 \\ 0.01 & 0.2 & -0.9 & 0.01 \\ 0 & 0.01 & 0.01 & -0.8 \end{bmatrix} \tag{10.85}$$

has positive diagonal entries.

We choose:

$$D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.13 & -0.37 \end{bmatrix}, \quad D_2 = \begin{bmatrix} -3.19 & -0.11 \end{bmatrix}. \tag{10.86}$$

Applying Theorem 10.9 and using MATLAB environment together with SEDUMI solver and YALMIP parser for the LMI (10.75) we obtain:

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}, \Lambda_1 = \begin{bmatrix} 0.0554 & 0 \\ 0 & 0.0755 \end{bmatrix}, \Lambda_2 = \begin{bmatrix} 0.8659 & 0 \\ 0 & 0.0032 \end{bmatrix}. \tag{10.87}$$

Therefore, the LMI is feasible with respect to the diagonal matrix Λ.

Using (10.70) we obtain the gain matrix

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix} = \begin{bmatrix} D_1\Lambda_1^{-1} & D_2\Lambda_2^{-1} \end{bmatrix} = \begin{bmatrix} 2.3460 & -4.9035 & -3.6840 & -34.1058 \end{bmatrix}. \tag{10.88}$$

The closed-loop system is positive since matrices:

$$\overline{A}_{11} + B_1K_1 = \begin{bmatrix} 0 & 0.01 \\ 0.0323 & 0.3961 \end{bmatrix}, \quad A_{12} + B_1K_2 = \begin{bmatrix} 0.01 & 0.01 \\ 0.0063 & 0.1659 \end{bmatrix},$$
$$A_{21} + B_2K_1 = \begin{bmatrix} 0.01 & 0.2 \\ 0.0047 & 0.0002 \end{bmatrix}, \quad \overline{A}_{22} + B_2K_2 = \begin{bmatrix} 0 & 0.01 \\ 0.0026 & 0.0318 \end{bmatrix},$$

have all nonnegative entries.

The closed-loop system is asymptotically stable since its characteristic polynomial

$$\det \begin{bmatrix} I_{n_1}z - (A_{11} + B_1K_1) & -(A_{12} + B_1K_2) \\ -(A_{21} + B_2K_1) & I_{n_2}z - (A_{22} + B_2K_2) \end{bmatrix} = z^4 + 2.1721z^3 + 1.4953z^2 + 0.3159z + 0.0004,$$

has positive coefficients.

These considerations can be extended to the closed-loop systems with poles located in desired sectors of the left half complex plane [187].