Chapter 1
Fractional Discrete-Time Linear Systems

1.1 Definition of n-Order Difference

Definition 1.1. A discrete-time function defined by

$$\Delta^n x_i = \Delta^{n-1} x_i - \Delta^{n-1} x_{i-1} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x_{i-k}, \tag{1.1}$$

$$i = 1, 2, 3, \ldots\ , \quad n \in \mathbb{Z}, \quad x_i \in \mathbb{R},$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}\ , \tag{1.2}$$

is called the n-order (backward) difference of the function x_i.

Definition 1.2. The fractional n-order (backward) difference on the interval $[0,k]$ of the function x_i is defined as follows

$$_0\Delta_k^n x_i = \sum_{j=0}^{k} (-1)^j \binom{n}{j} x_{i-j}. \tag{1.3}$$

From (1.1) it follows that the n-order difference can be written as a linear combination of the values of discrete-time function in $n+1$ points.

The definitions are valid for n being natural numbers and integers.

Note that (1.2) is also well defined for fractional and real numbers. In general case n can be also a complex number.

Example 1.1. From (1.1) we have for:

$$\begin{aligned}
n = 1: &\quad \Delta x_i = x_i - x_{i-1}, \\
n = 2: &\quad \Delta^2 x_i = \Delta x_i - \Delta x_{i-1} = x_i - 2x_{i-1} + x_{i-2}, \\
n = 3: &\quad \Delta^3 x_i = \Delta^2 x_i - \Delta^2 x_{i-1} = x_i - 3x_{i-1} + 3x_{i-2} - x_{i-3}.
\end{aligned}$$

T. Kaczorek: Selected Problems of Fractional Systems Theory, LNCIS 411, pp. 1–25.
springerlink.com

From (1.3) we obtain for:
$n = -1$:

$${}_0\Delta_k^{-1} x_k = \sum_{j=0}^{k} (-1)^j \binom{-1}{j} x_{k-j} = x_k + x_{k-1} + \cdots + x_0 = \sum_{j=0}^{k} x_{k-j},$$

$n = -2$:

$${}_0\Delta_k^{-2} x_k = \sum_{j=0}^{k} (-1)^j \binom{-2}{j} x_{k-j} = x_k + \cdots + (k+1)x_0 = \sum_{j=0}^{k} (j+1) x_{k-j}.$$

Definition 1.3. The discrete-time function

$$\Delta^\alpha x_k = \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} x_{k-j}, \tag{1.4}$$

where $0 < \alpha < 1$, $\alpha \in \mathbb{R}$ and

$$\binom{\alpha}{k} = \begin{cases} 1 & \text{for} \quad k = 0 \\ \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} & \text{for} \quad k = 1, 2, 3, \ldots \end{cases}, \tag{1.5}$$

is called the fractional α-order difference of the function x_k.

Example 1.2. Using (1.5) for $0 < \alpha < 1$ we obtain for:

$$\begin{aligned} k = 1: &\qquad (-1)^1 \binom{\alpha}{1} = -\alpha < 0, \\ k = 2: &\qquad (-1)^2 \binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{2!} < 0, \\ k = 3: &\qquad (-1)^3 \binom{\alpha}{3} = -\frac{\alpha(\alpha-1)(\alpha-2)}{3!} < 0. \end{aligned}$$

1.2 State Equations of the Fractional Linear Systems

1.2.1 Fractional Systems without Delays

The state equations of the fractional discrete-time linear system have the form:

$$\Delta^\alpha x_{k+1} = Ax_k + Bu_k, \quad 0 \le \alpha \le 1, \tag{1.6a}$$

$$y_k = Cx_k + Du_k, \quad k \in \mathbb{Z}_+, \tag{1.6b}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^p$ are the state, input and output vectors and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Substituting the definition of fractional difference (1.4) into (1.6a), we obtain

$$x_{k+1} + \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} x_{k-j+1} = Ax_k + Bu_k, \tag{1.7a}$$

or

$$\begin{aligned} x_{k+1} &= Ax_k + \sum_{j=1}^{k+1} (-1)^{j+1} \binom{\alpha}{j} x_{k-j+1} + Bu_k \\ &= A_\alpha x_k + \sum_{j=2}^{k+1} (-1)^{j+1} \binom{\alpha}{j} x_{k-j+1} + Bu_k, \end{aligned} \tag{1.7b}$$

where

$$A_\alpha = A + \alpha I_n. \tag{1.8}$$

Remark 1.1. From (1.7b) it follows that the fractional system is equivalent to the system with increasing number of delays.

In practice it is assumed that j is bounded by natural number h. In this case the equations (1.6) take the form:

$$x_{k+1} = A_\alpha x_k + \sum_{j=1}^{h} (-1)^j \binom{\alpha}{j+1} x_{k-j} + Bu_k, \quad k \in \mathbb{Z}_+, \tag{1.9a}$$

$$y_k = Cx_k + Du_k. \tag{1.9b}$$

Remark 1.2. The equations (1.9) describe a discrete-time linear system with h delays.

1.2.2 Fractional Systems with Delays

Consider the fractional discrete-time linear system with h delays:

$$\Delta^\alpha x_{i+1} = \sum_{k=0}^{h} \left(A_k x_{i-k} + B_k u_{i-k}\right), \quad i \in \mathbb{Z}_+, \tag{1.10a}$$

$$y_i = Cx_i + Du_i, \tag{1.10b}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^p$ are the state, input and output vectors and $A_k \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times m}$, $k = 0,\dots,h$; $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Substituting the definition of fractional difference (1.4) into (1.10a) we obtain:

$$x_{i+1} = \sum_{j=1}^{i+1} (-1)^{j+1} \binom{\alpha}{j} x_{i-j+1} + \sum_{k=0}^{h} \left(A_k x_{i-k} + B_k u_{i-k}\right), \tag{1.11a}$$

$$y_i = Cx_i + Du_i, \quad i \in \mathbb{Z}_+, \tag{1.11b}$$

If i is bounded by the natural number L then from (1.11) we obtain:

$$x_{i+1} = \sum_{j=1}^{L+1} (-1)^{j+1} \binom{\alpha}{j} x_{i-j+1} + \sum_{k=0}^{h} (A_k x_{i-k} + B_k u_{i-k}), \tag{1.12a}$$

$$y_i = Cx_i + Du_i, \quad i \in \mathbb{Z}_+ . \tag{1.12b}$$

1.3 Solution of the State Equations of the Fractional Discrete-Time Linear System with Delays

1.3.1 Fractional Systems with Delays

The state equations of the fractional discrete-time linear system with q delays has the form:

$$x_{k+1} + \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} x_{k-j+1} = \sum_{r=0}^{q} (A_r x_{k-r} + B_r u_{k-r}), \quad k \in \mathbb{Z}_+, \tag{1.13a}$$

$$y_k = Cx_k + Du_k, \quad 0 \le \alpha \le 1, \tag{1.13b}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^p$ are the state, input and output vectors and $A_r \in \mathbb{R}^{n\times n}$, $B_r \in \mathbb{R}^{n\times m}$, $r = 0,1,\dots,q$; $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, q is the number of delays.

Applying the z-transform ($\mathscr{Z}$) method we shall derive the solution of the state equation (1.13a) of the fractional system [79].

Theorem 1.1. *The solution of the equation (1.13a) has the form*

$$x_k = \Phi_k x_0 + \sum_{r=0}^{q} \sum_{i=0}^{k-r-1} \Phi_{k-r-1-i} B_r u_i + \sum_{j=1}^{k+1} \sum_{l=-1}^{-j+1} (-1)^{j+1} \binom{\alpha}{j} \Phi_{k-l-j} x_l$$
$$+ \sum_{r=0}^{q} \sum_{l=-1}^{-r} \Phi_{k-r-l-1} A_r x_l + \sum_{r=0}^{q} \sum_{l=-1}^{-r} \Phi_{k-r-l-1} B_r u_l, \tag{1.14}$$

where

$$x_k \neq 0, \quad u_k \neq 0, \quad k = 0,-1,\dots,-q \ , \tag{1.15}$$

are initial conditions and the matrices Φ_k are determined by the equation

$$\Phi_{k+1} = \Phi_k (A_0 + \alpha I_n) + \sum_{i=2}^{k+1} (-1)^{i+1} \binom{\alpha}{i} \Phi_{k-i+1} + \sum_{i=1}^{k} \Phi_{k-i} A_i, \tag{1.16a}$$

$$\Phi_0 = I_n, \tag{1.16b}$$

for $k = 0,1,\dots$.

Proof. Let $X(z)$ be the z-transform ($\mathscr{Z}$) of the discrete-time function x_i defined by (A.13). Applying the z-transform (Appendix A.3) to the equation (1.13a) we obtain

$$\mathscr{Z}[x_{k+1}]+\sum_{j=1}^{k+1}(-1)^j\binom{\alpha}{j}\mathscr{Z}[x_{k-j+1}]=\sum_{r=0}^{q}A_r\mathscr{Z}[x_{k-r}]+\sum_{r=0}^{q}B_r\mathscr{Z}[u_{k-r}]. \tag{1.17}$$

Using (A.14) to (1.17) we get

$$\begin{aligned} zX(z) &- zx_0+\sum_{j=1}^{k+1}(-1)^j\binom{\alpha}{j}z^{-j+1}\left[X(z)+\sum_{l=-1}^{-j+1}x_l z^{-l}\right] \\ &=\sum_{r=0}^{q}A_r z^{-r}\left[X(z)+\sum_{l=-1}^{-r}x_l z^{-l}\right]+\sum_{r=0}^{q}B_r z^{-r}\left[U(z)+\sum_{l=-1}^{-r}u_l z^{-l}\right], \end{aligned} \tag{1.18}$$

where $U(z)=\mathscr{Z}[u_k]$.

Multiplying (1.18) by z^{-1} and solving with respect to $X(z)$ we obtain

$$\begin{aligned} X(z) &= \left[\sum_{j=0}^{k+1}(-1)^j\binom{\alpha}{j}z^{-j}I_n-\sum_{r=0}^{q}A_r z^{-r-1}\right]^{-1} \\ &\times\left\{x_0+\sum_{j=1}^{k+1}(-1)^{j+1}\binom{\alpha}{j}z^{-j}\sum_{l=-1}^{-j+1}x_l z^{-l}+\sum_{r=0}^{q}A_r z^{-r-1}\sum_{l=-1}^{-r}x_l z^{-l}\right. \\ &\left.+\sum_{r=0}^{q}B_r z^{-r-1}\left[U(z)+\sum_{l=-1}^{-r}u_l z^{-l}\right]\right\}. \end{aligned} \tag{1.19}$$

Substituting of the expansion

$$\left[\sum_{j=0}^{k+1}(-1)^j\binom{\alpha}{j}z^{-j}I_n-\sum_{r=0}^{q}A_r z^{-r-1}\right]^{-1}=\sum_{k=0}^{\infty}\Phi_k z^{-k}, \tag{1.20}$$

into (1.19) yields

$$\begin{aligned} X(z) &= \sum_{k=0}^{\infty}\Phi_k z^{-k}x_0+\sum_{k=0}^{\infty}\sum_{r=0}^{q}\Phi_k z^{-k-r-1}B_r U(z) \\ &+\sum_{k=0}^{\infty}\Phi_k z^{-k}\left[\sum_{j=1}^{k+1}\sum_{l=-1}^{-j+1}(-1)^{j+1}\binom{\alpha}{j}x_l z^{-j-l}\right. \\ &\left.+\sum_{r=0}^{q}\sum_{l=-1}^{-r}A_r x_l z^{-r-l-1}+\sum_{r=0}^{q}\sum_{l=-1}^{-r}B_r u_l z^{-r-l-1}\right]. \end{aligned} \tag{1.21}$$

Applying the inverse z-transform and the convolution theorem (Appendix A.1) to (1.21) we obtain the desired solution (1.14).

From definition of the inverse matrix we have

$$\left[\sum_{j=0}^{k+1}(-1)^j\binom{\alpha}{j}z^{-j}I_n-\sum_{r=0}^{q}A_r z^{-r-1}\right]\left[\sum_{k=0}^{\infty}\Phi_k z^{-k}\right]=I_n. \tag{1.22}$$

Comparison of the coefficients at the same powers of z^{-k}, $k = 0, 1, \ldots$; from (1.22) yields:

$$\begin{aligned}
&z^0: \quad \Phi_0 \cdot I_n = I_n, \\
&z^{-1}: -A_0 + \Phi_1 - \alpha I_n = 0 \qquad \Rightarrow \Phi_1 = A_0 + \alpha I_n, \\
&z^{-2}: \Phi_2 - \Phi_1(A_0 + \alpha I_n) + \cdots = 0 \Rightarrow \Phi_2 = \Phi_1(A_0 + \alpha I_n) - \Phi_0\left(I_n\tbinom{\alpha}{2} - A_1\right) \\
&\vdots
\end{aligned}$$

and in general case the equation (1.16). □

1.3.2 Fractional Systems without Delays

In this section we shall consider the fractional discrete-time linear system without delays. Substituting in (1.14) $q = 0$ we obtain the following theorem.

Theorem 1.2. *The solution of the equation (1.7) has the form*

$$x_k = \Phi_k x_0 + \sum_{i=0}^{k-1} \Phi_{k-i-1} B u_i, \tag{1.23}$$

where the matrices Φ_k are determined by the equation

$$\Phi_{k+1} = \Phi_k\left(A + \alpha I_n\right) + \sum_{i=2}^{k+1} (-1)^{i+1} \binom{\alpha}{i} \Phi_{k-i+1}, \quad \Phi_0 = I_n. \tag{1.24}$$

Theorem 1.3. *Let*

$$\det\left[\sum_{j=0}^{k+1} (-1)^j \binom{\alpha}{j} I_n z^{-j} - A z^{-1}\right] = \sum_{i=0}^{M} a_{M-i} z^{-i}, \tag{1.25}$$

be the characteristic polynomial of the fractional system (1.7) for $k = L$. The matrices $\Phi_1, \ldots, \Phi_M$ satisfy the equation

$$\sum_{i=0}^{M} a_i \Phi_i = 0. \tag{1.26}$$

Proof. From definition of the adjoint matrix and (1.25) we have

$$\operatorname{Adj}\left[\sum_{j=0}^{L+1} (-1)^j \binom{\alpha}{j} I_n z^{-j} - A z^{-1}\right] = \left(\sum_{i=0}^{\infty} \Phi_i z^{-i}\right)\left(\sum_{i=0}^{M} a_{M-i} z^{-i}\right), \tag{1.27}$$

where $\operatorname{Adj} F$ denotes the adjoint matrix of F.

Comparing the coefficients of the same powers of z^{-1} in (1.27), we obtain (1.26), since the degree of the matrix

$$\mathrm{Adj}\left[\sum_{j=0}^{L+1}(-1)^j\binom{\alpha}{j}I_n z^{-j} - Az^{-1}\right],$$

is less than M. □

Theorem 1.3 is an extension of the well-known Cayley-Hamilton theorem for fractional discrete-time linear systems.

Remark 1.3. The degree M of the characteristic polynomial (1.25) depends on k and it increases to infinity for $k \to \infty$. Usually it is assumed that k is bounded by natural number L. If $k = L$ then $M = N(L+1)$.

1.4 Positive Fractional Linear Systems

In this section the necessary and sufficient conditions for the positivity of the fractional discrete-time linear system:

$$x_{k+1} + \sum_{j=1}^{k+1}(-1)^j\binom{\alpha}{j}x_{k-j+1} = Ax_k + Bu_k, \quad k \in \mathbb{Z}_+, \tag{1.28a}$$

$$y_k = Cx_k + Du_k, \tag{1.28b}$$

will be established, where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^p$ are state, input and output vectors and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Let $\mathbb{R}_+^{n\times m}$ be the set of real $n \times m$ matrices with the nonnegative entries and $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$.

Definition 1.4. The system (1.28) is called (internally) positive fractional system if $x_k \in \mathbb{R}_+^n$, $y_k \in \mathbb{R}_+^p$ for every initial conditions $x_0 \in \mathbb{R}_+^n$ and all inputs $u_k \in \mathbb{R}_+^m$, $k \in \mathbb{Z}_+$.

Lemma 1.1. *If* $0 < \alpha < 1$*, then*

$$(-1)^{i+1}\binom{\alpha}{i} > 0, \quad i = 1, 2, \ldots \; . \tag{1.29}$$

Proof. The proof will be accomplished by induction. The hypothesis is true for $i = 1$ since

$$(-1)^{1+1}\binom{\alpha}{1} = \alpha > 0.$$

Assuming that $(-1)^{k+1}\binom{\alpha}{k} > 0$ for $k \geq 1$ we shall show that the hypothesis is valid for $k+1$. From (1.5) we have

$$(-1)^{k+2}\binom{\alpha}{k+1} = (-1)^{k+2}\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)(\alpha-k)}{k!(k+1)}$$
$$= (-1)^{k+1}\binom{\alpha}{k}\frac{k-\alpha}{k+1} > 0.$$

Therefore, the hypothesis is true for $k+1$. This completes the proof. □

Remark 1.4. In a similar way it can be shown that for $1 < \alpha < 2$

$$(-1)^{i+1}\binom{\alpha}{i} < 0, \quad i = 2, 3, \ldots \ .$$

Lemma 1.2. *Let* $0 < \alpha < 1$ *and*

$$[A + \alpha I_n] \in \mathbb{R}_+^{n\times n}, \tag{1.30}$$

then

$$\Phi_k \in \mathbb{R}_+^{n\times n}, \quad k = 1, 2, \ldots \ . \tag{1.31}$$

Proof. The proof follows immediately from (1.24). □

Theorem 1.4. *The fractional system (1.28) is (internally) positive if and only if:*

$$A_\alpha = [A + \alpha I_n] \in \mathbb{R}_+^{n\times n} \quad \text{and} \quad B \in \mathbb{R}_+^{n\times m}, \quad C \in \mathbb{R}_+^{p\times n}, \quad D \in \mathbb{R}_+^{p\times m}. \tag{1.32}$$

Proof. Sufficiency follows from Lemma 1.2 and the equation (1.23). From (1.23) it follows that if $\Phi_k \in \mathbb{R}_+^{n\times n}$, $B \in \mathbb{R}_+^{n\times m}$, $x_0 \in \mathbb{R}_+^n$ then $x_k \in \mathbb{R}_+^n$, $k \in \mathbb{Z}_+$. Similarly from (1.28b) we have $y_k \in \mathbb{R}_+^p$ if the conditions (1.32) are satisfied.

Necessity. Let $u_k = 0$ for $k \in \mathbb{Z}_+$. For positive system from (1.28) for $k = 0$ we have $x_1 = [A + \alpha I_n]x_0 = A_\alpha x_0 = A_{\alpha 1} \in \mathbb{R}_+^n$, and $y_0 = Cx_0 \in \mathbb{R}_+^p$. Therefore $A_\alpha \in \mathbb{R}_+^{n\times n}$ and $C \in \mathbb{R}_+^{p\times n}$, since $x_0 \in \mathbb{R}_+^n$ and by Definition 1.4 it is arbitrary. Assuming $x_0 = 0$ from (1.28) for $k = 0$ we obtain $x_1 = Bu_0 \in \mathbb{R}_+^n$ and $y_0 = Du_0 \in \mathbb{R}_+^p$, and this implies $B \in \mathbb{R}_+^{n\times m}$ and $D \in \mathbb{R}_+^{p\times m}$, since $u_0 \in \mathbb{R}_+^m$ and it is arbitrary. □

Definition 1.5. The fractional discrete-time linear system (1.11) with h delays is called (internally) positive if $x_i \in \mathbb{R}_+^n$ and $y_i \in \mathbb{R}_+^p$ for any initial conditions $x_k \in \mathbb{R}_+^n$, $k = 0, -1, \ldots, -h$ and all inputs $u_i \in \mathbb{R}_+^m$, $i \in \mathbb{Z}_+$.

Theorem 1.5. *The fractional discrete-time linear system (1.11) with h delays is (internally) positive for* $0 < \alpha < 1$ *if and only if*

$$A_k + c_{k+1}I_n \in \mathbb{R}_+^{n\times n}, \quad c_k = (-1)^{k+1}\binom{\alpha}{k}, \quad B_k \in \mathbb{R}_+^{n\times m}, \quad k = 1, \ldots, h;$$
$$C \in \mathbb{R}_+^{p\times n}, \quad D \in \mathbb{R}_+^{p\times m}. \tag{1.33}$$

Proof. The proof is similar to the proof of Theorem 1.4. □

1.5 Externally Positive Fractional Systems

Definition 1.6. The fractional discrete-time linear system (1.28) is called externally positive if for any inputs $u_k \in \mathbb{R}_+^m$, $k \in \mathbb{Z}_+$ and $x_0 = 0$ we have $y_k \in \mathbb{R}_+^p$, $k \in \mathbb{Z}_+$.

Definition 1.7. The output of the single-input single-output (SISO) linear system for the unit impulse

$$u_i = \begin{cases} 1 & \text{for} \quad i = 0 \\ 0 & \text{for} \quad i > 0 \end{cases},$$

and zero initial conditions is called the impulse response of the system.

In a similar way we define the matrix of impulse responses g_k of the multi-input multi-output (MIMO) linear systems.

Theorem 1.6. *The fractional discrete-time linear system (1.28) is externally positive if and only if*

$$g_k \in \mathbb{R}_+^{p\times m}, \quad k \in \mathbb{Z}_+, \tag{1.34}$$

and the matrix of impulse responses is given by

$$g_k = \begin{cases} D & \mathit{for} \quad k = 0 \\ C\Phi_{k-1}B & \mathit{for} \quad k = 1,2,\dots \end{cases}. \tag{1.35}$$

Proof. Sufficiency. The output of the system (1.28) with zero initial conditions and any input $u_i \in \mathbb{R}_+^m$ is given by

$$y_k = \sum_{i=0}^{k} g_{k-i}u_i, \quad k \in \mathbb{Z}_+. \tag{1.36}$$

If (1.34) holds and $u_i \in \mathbb{R}_+^m$, then from (1.36) we have $y_k \in \mathbb{R}_+^p$, $k \in \mathbb{Z}_+$.
Necessity follows immediately from Definition 1.7. □

Remark 1.5. Every (internally) positive linear system is always externally positive. This follows from Definitions 1.4 and 1.6.

Example 1.3. Consider the fractional system (1.6a) for $0 < \alpha < 1$ with the matrices:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -\alpha \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (n = 2). \tag{1.37}$$

The system is positive since

$$[A + \alpha I_n] = \begin{bmatrix} (1+\alpha) & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}_+^{2\times 2}.$$

Using (1.24) for $k = 0, 1, \ldots$ we obtain:

$$\Phi_1 = (A + \alpha I_n)\Phi_0 = \begin{bmatrix} (1+\alpha) & 0 \\ 0 & 0 \end{bmatrix}, \tag{1.38a}$$

$$\Phi_2 = (A + \alpha I_n)\Phi_1 - \binom{\alpha}{2}\Phi_0 = \begin{bmatrix} \frac{\alpha^2+5\alpha+2}{2} & 0 \\ 0 & \frac{\alpha(1-\alpha)}{2} \end{bmatrix}, \tag{1.38b}$$

$$\begin{aligned} \Phi_3 &= (A + \alpha I_n)\Phi_2 - \binom{\alpha}{2}\Phi_1 + \binom{\alpha}{3}\Phi_0 \\ &= \begin{bmatrix} \frac{3(\alpha^2+5\alpha+2)(\alpha+1)-\alpha(\alpha-1)(2\alpha+5)}{6} & 0 \\ 0 & \frac{\alpha(1-\alpha)(2-\alpha)}{6} \end{bmatrix}, \end{aligned} \tag{1.38c}$$

$$\vdots$$

From (1.23) and (1.24) we have

$$x_k = \Phi_k x_0 + \sum_{i=0}^{k-1} \Phi_{k-i-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i, \tag{1.39}$$

where Φ_k is defined by (1.38).

1.6 Reachability of Fractional Discrete-Time Linear Systems

Definition 1.8. A state $x_f \in \mathbb{R}^n$ is called reachable in (given) q steps if there exists an input sequence $u_0, u_1, \ldots, u_{q-1}$, which steers the state of the system (1.28) from $x_0 = 0$ to the state x_f, i.e. $x_q = x_f$. If every given state $x_f \in \mathbb{R}^n$ is reachable in q steps then the system (1.28) is called reachable in q steps. If for every state $x_f \in \mathbb{R}^n$ there exists a number q of steps such that the system is reachable in q steps then the system is called reachable.

Theorem 1.7. *The fractional system (1.28) is reachable in q steps if and only if*

$$\operatorname{rank}\begin{bmatrix} B & \Phi_1 B & \ldots & \Phi_{q-1} B \end{bmatrix} = n. \tag{1.40}$$

Proof. From (1.23) for $k = q$ and $x_0 = 0$ we have

$$x_f = \Phi_q x_0 + \sum_{i=0}^{q-1} \Phi_{q-i-1} B u_i = \begin{bmatrix} B & \Phi_1 B & \ldots & \Phi_{q-1} B \end{bmatrix} \begin{bmatrix} u_{q-1} \\ u_{q-2} \\ \vdots \\ u_1 \\ u_0 \end{bmatrix}. \tag{1.41}$$

From Kronecker-Capelly theorem it follows that the equation (1.41) has a solution for every x_f if and only if the condition (1.40) is satisfied. □

Theorem 1.8. *In the condition (1.40) the matrices* $\Phi_1,\dots,\Phi_{q-1}$ *can be substituted by the matrices* $A_\alpha,\dots,A_\alpha^{q-1}$*, i.e.*

$$\operatorname{rank}\left[B\ \Phi_1 B\ \dots\ \Phi_{q-1}B\right] = \operatorname{rank}\left[B\ A_\alpha B\ \dots\ A_\alpha^{q-1}B\right] = n. \tag{1.42}$$

Proof. To simplify the notation the proof will be accomplished for $n = 4$. From (1.24) for $c_i = (-1)^{i+1}\binom{\alpha}{i}$, we have

$$\begin{aligned} &\left[B\ \Phi_1 B\ \Phi_2 B\ \Phi_3 B\right] \\ = &\left[B\ A_\alpha B\ (A_\alpha^2 + c_2 I_n)B\ (A_\alpha^3 + 2c_2 A_\alpha + c_3 I_n)B\right] \\ = &\left[B\ A_\alpha B\ A_\alpha^2 B\ A_\alpha^3 B\right] \begin{bmatrix} I_n & 0 & c_2 I_n & c_3 I_n \\ 0 & I_n & 0 & 2c_2 I_n \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix}. \end{aligned}$$

Hence $\operatorname{rank}\left[B\ \Phi_1 B\ \Phi_2 B\ \Phi_3 B\right] = \operatorname{rank}\left[B\ A_\alpha B\ A_\alpha^2 B\ A_\alpha^3 B\right]$, since postmultiplication of the matrix $\left[B\ A_\alpha B\ A_\alpha^2 B\ A_\alpha^3 B\right]$ by the nonsingular matrix does not change the rank of the matrix. □

Theorem 1.9. *The fractional system (1.28) is reachable if and only if one of the equivalent conditions is satisfied:*

a) *The matrix* $[I_n z - A_\alpha, B]$ *has full rank, i.e.*

$$\operatorname{rank}[I_n z - A_\alpha, B] = n, \quad \forall z \in \mathbb{C}. \tag{1.43}$$

b) *The matrices* $[I_n z - A_\alpha]$, B *are relatively left prime or equivalently it is possible using elementary column operations (R) to reduce the matrix* $[I_n z - A_\alpha, B]$ *to the form* $[I_n, 0]$*, i.e.*

$$[I_n z - A_\alpha, B] \xrightarrow{R} [I_n, 0]. \tag{1.44}$$

Proof. First we shall show that the condition (1.43) is equivalent to the condition (1.42). Let $v \in \mathbb{C}^n$ be a vector such that $v^T B = 0$ and $v^T A_\alpha = z v^T$ for $z \in \mathbb{C}$. In this case $v^T A_\alpha B = z v^T B = 0, v^T A_\alpha^2 B = z v^T A_\alpha B = 0, \dots, v^T A_\alpha^{q-1} B = 0$ and

$$v^T\left[B\ A_\alpha B\ \dots\ A_\alpha^{q-1}B\right] = 0. \tag{1.45}$$

From (1.45) it follows that the condition implies $v = 0$ and $v^T[I_n z - A_\alpha, B] = 0$, and this is equivalent to a (1.43). If the condition (1.42) is not satisfied then there exists a vector v satisfying (1.45) or $\operatorname{rank}[I_n z - A_\alpha, B] < n$ for $z \in \mathbb{C}$. The reduction holds if and only if the condition (1.43) is satisfied. □

Example 1.4. Using (1.42), (1.43) and (1.44) check the reachability of the system with the matrices:

$$A_\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

a) From (1.42) for $n = 3$ we have

$$\text{rank}\left[B\ A_\alpha B\ A_\alpha^2 B\right] = \text{rank}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 7 \end{bmatrix} = 3.$$

By Theorem 1.8 the pair (A_α, B) is reachable.

b) From (1.43) we have

$$\text{rank}\,[I_n z - A_\alpha, B] = \text{rank}\left[\begin{array}{ccc|c} z & -1 & 0 & 0 \\ 0 & z & -1 & 0 \\ 1 & 2 & z+3 & 1 \end{array}\right] = 3 \quad \text{for} \quad \forall z \in \mathbb{C}.$$

Using the elementary column operations we shall show that the matrices $[I_n z - A_\alpha]$ and B are relatively left prime:

$$\left[\begin{array}{ccc|c} z & -1 & 0 & 0 \\ 0 & z & -1 & 0 \\ 1 & 2 & z+3 & 1 \end{array}\right] \xrightarrow[R[1+4\times(-1)]]{\substack{R[3+4\times(-z-3)] \\ R[2+4\times(-2)]}} \left[\begin{array}{ccc|c} z & -1 & 0 & 0 \\ 0 & z & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R[2+3\times(z)]} \left[\begin{array}{ccc|c} z & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R[3\times(-1)]]{\substack{R[1+2\times(z)] \\ R[2\times(-1)]}} \left[0\ I_3\right].$$

Therefore, by Theorem 1.9 the pair (A_α, B) is reachable.

Remark 1.6. The fractional system is reachable only if the matrix (A_α, B) has n linearly independent columns. If the matrix (A_α, B) has no n linearly independent columns then the matrix $[B, A_\alpha B, \ldots, A_\alpha^{q-1} B]$ does not have n linearly independent columns. This follows from the condition (1.43) for $z = 0$.

1.7 Reachability of Positive Fractional Discrete-Time Linear Systems

Definition 1.9. A state $x_f \in \mathbb{R}_+^n$ of the positive fractional system (1.28) is called reachable in (given) q steps if there exists an input sequence $u_k \in \mathbb{R}_+^m$, for $k = 0, 1 \ldots, q-1$, which steers the state of the system from $x_0 = 0$ to the state x_f, i.e. $x_q = x_f$. If every (given) state $x_f \in \mathbb{R}_+^n$ is reachable in q steps then the positive system is called reachable in q steps. If for every state $x_f \in \mathbb{R}_+^n$ there exists a number q of steps such that the positive system (1.28) is reachable in q steps then the system is called reachable.

Definition 1.10. A square real matrix is called monomial if its every column and its every row has only one positive entry and the remaining entries are zero.

The inverse matrix of a real matrix with nonnegative entries has nonnegative entries if and only if it is a monomial matrix. The inverse matrix of monomial matrix can be found by its transposition and replacing each element of the transpose matrix by its inverse. For example the inverse matrix of the matrix

$$A = \begin{bmatrix} 0 & b & 0 \\ a & 0 & 0 \\ 0 & 0 & c \end{bmatrix}, \quad \text{has the form} \quad A^{-1} = \begin{bmatrix} 0 & \frac{1}{a} & 0 \\ \frac{1}{b} & 0 & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix},$$

where $a, b, c > 0$.

Theorem 1.10. *The positive fractional system (1.28) is reachable in q steps if and only if the matrix*

$$R_q = \begin{bmatrix} B & \Phi_1 B & \dots & \Phi_{q-1} B \end{bmatrix}, \tag{1.46}$$

contains n linearly independent monomial columns.

Proof. In a similar way as in the proof of Theorem 1.7 we obtain the equation (1.41). For given $x_f \in \mathbb{R}_+$ we can find the input sequence $u_k \in \mathbb{R}_+^m$, $k = 0, 1, \dots, q-1$ if and only if the matrix (1.46) contains n linearly independent monomial columns. □

Remark 1.7. The matrix (1.46) can not be substitute by the matrix

$$\overline{R}_q = \begin{bmatrix} B & A_\alpha B & \dots & A_\alpha^{q-1} B \end{bmatrix}, \tag{1.47}$$

since for positive fractional systems the matrices in general case have different number of linearly independent monomial columns.

Example 1.5. Consider the fractional positive system (1.28) with the matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -\alpha & 1 \\ 1 & 0 & -\alpha \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{1.48}$$

In this case

$$[A + \alpha I_n] = \begin{bmatrix} \alpha & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \in \mathbb{R}_+^{3\times 3}, \tag{1.49}$$

and the matrix (1.47) for $q = 3$ has the form

$$\overline{R}_3 = \begin{bmatrix} B & A_\alpha B & A_\alpha^2 B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

and it contains three linearly independent monomial columns but the matrix

$$R_3 = \begin{bmatrix} B & \Phi_1 B & \Phi_2 B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & \frac{\alpha(\alpha-1)}{2} \end{bmatrix},$$

contains only two linearly independent monomial columns.

Theorem 1.11. *The positive fractional system (1.28) is reachable only if the matrix*

$$[A+\alpha I_n, B] \tag{1.50}$$

contains n linearly independent monomial columns.

Proof. From (1.24) for $k = 0,1,\dots$ it can be easily shown that

$$\Phi_k = A_\alpha^k + a_{kk-1}A_\alpha^{k-1} + \cdots + a_{k1}A_\alpha + a_{k0}I_n, \tag{1.51}$$

where $a_{ki} \in \mathbb{R}$, $i = 0,1,\dots,k-1$.

From matrix (1.46) and the equation (1.51) it follows that the number of linearly independent monomial columns of the matrix (1.46) can not be greater than of the matrix (1.50). □

Example 1.6. Consider the fractional system (1.28) with matrices (1.37). Using (1.46) we obtain matrix

$$R_2 = \begin{bmatrix} B & \Phi_1 B \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

which has only one monomial column. By Theorem 1.10 the system with (1.37) is unreachable. However using (1.50), we obtain matrix

$$[A+\alpha I_n, B] = \left[\begin{array}{cc|c} 1+\alpha & 0 & 0 \\ 0 & 0 & 1 \end{array}\right],$$

which has two linearly independent monomial columns.

Theorem 1.12. *The positive fractional system (1.28) is reachable only if the matrix*

$$[B, (A+\alpha I_n)B], \tag{1.52}$$

contains n linearly independent monomial columns.

Proof. From (1.51) for the positive system we have

$$\Phi_k B = \sum_{i=0}^{k} a_{ki} A_\alpha^i B, \tag{1.53}$$

where $A_\alpha = A + \alpha I_n$, $a_{ki} \geq 0$, $k = 0,1,\dots,q-1$, $i = 0,1,\dots,k$.

Note that besides the matrix B, only the matrix $\Phi_1 B$ may have additional linearly independent monomial columns. The matrix (1.53) for $k = 2,3,\dots,q-1$ does not introduce additional linearly independent monomial columns to the matrix (1.46). □

Remark 1.8. If all m columns of the matrix B are linearly independent and monomial then the matrix (1.52) has n linearly independent monomial columns only if the matrix A_α has at least $n-m$ linearly independent monomial columns.

Example 1.7. Consider the positive fractional system (1.28) with the matrices:

$$A = \begin{bmatrix} a_{11}-\alpha & a_{12} & 1 & 0 \\ a_{21} & a_{22}-\alpha & 0 & 1 \\ a_{31} & a_{32} & -\alpha & 0 \\ a_{41} & a_{42} & 0 & -\alpha \end{bmatrix}, \quad a_{ij} \geq 0, \quad i=1,2,3,4; \quad j=1,2,$$

$$a) \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad b) \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Taking into account that

$$A_\alpha = \begin{bmatrix} a_{11} & a_{12} & 1 & 0 \\ a_{21} & a_{22} & 0 & 1 \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{bmatrix},$$

in the case $a)$ we obtain the matrix

$$\begin{bmatrix} B & \Phi_1 B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

which has $n=4$ linearly independent monomial columns. Therefore, in this case the system is reachable in $q=2$ steps. In the case $b)$ we obtain the matrix

$$\begin{bmatrix} B & \Phi_1 B & \Phi_2 B & \ldots \end{bmatrix} = \begin{bmatrix} 0 & 0 & a_{12} & \ldots \\ 0 & 1 & a_{22} & \ldots \\ 0 & 0 & a_{32} & \ldots \\ 1 & 0 & a_{42}+\frac{\alpha(1-\alpha)}{2} & \ldots \end{bmatrix},$$

which contains only two linearly independent monomial columns. By Theorem 1.10 the positive fractional system is unreachable.

It is well-knew that the observability is a dual notion. The presented considerations for the reachability of the positive fractional linear systems can be extended to the observability of this class of systems.

1.8 Controllability to Zero of the Fractional Discrete-Time Linear Systems

Definition 1.11. The fractional system (1.28) is called controllable to zero in (given) number of q steps if there exists an input sequence $u_0, u_1, \ldots, u_{q-1}$, which steers the state of the system from $x_0 \neq 0$ to the final state $x_f = 0$.

The fractional system (1.28) is called controllable to zero if there exists a natural number q such that the system is controllable to zero in q steps.

Theorem 1.13. *The fractional system (1.28) is controllable to zero in q steps if*

$$\operatorname{rank}\left[B\ \Phi_1 B\ \ldots\ \Phi_{q-1}B\right] = n. \tag{1.54}$$

Proof. From (1.23) for $k = q$ and $x_q = 0$ we have

$$-\Phi_q x_0 = \sum_{i=0}^{q-1} \Phi_{q-i-1} B u_i = \left[B\ \Phi_1 B\ \ldots\ \Phi_{q-1}B\right] \begin{bmatrix} u_{q-1} \\ u_{q-2} \\ \vdots \\ u_0 \end{bmatrix}. \tag{1.55}$$

The equation (1.55) has a solution u_k, $k = 0,1,\ldots,q-1$ for arbitrary vector $\Phi_q x_0$ if the condition (1.54) is satisfied. This is a sufficient but not necessary condition for the controllability to zero since even if the condition (1.54) is not satisfied the equation (1.55) can be satisfied for arbitrary x_0, when $\Phi_q = 0$ and $u_k = 0$, $k = 0,1,\ldots,q-1$. □

Theorem 1.14. *For the controllability to zero of the fractional system (1.28) the following equality holds*

$$\operatorname{rank}\left[B\ A_\alpha B\ \ldots\ A_\alpha^{q-1}B\right] = \operatorname{rank}\left[B\ \Phi_1 B\ \ldots\ \Phi_{q-1}B\right]. \tag{1.56}$$

Proof. To simplify the notation we shall accomplished the proof for $n = 4$. From (1.24), we have

$$\begin{aligned} &\left[B\ \Phi_1 B\ \Phi_2 B\ \Phi_3 B\right] \\ =\ &\left[B\ A_\alpha B\ (A_\alpha^2 + c_2 I_n)B\ (A_\alpha^3 + 2c_2 A_\alpha + c_3 I_n)B\right] \\ =\ &\left[B\ A_\alpha B\ A_\alpha^2 B\ A_\alpha^3 B\right] \begin{bmatrix} I & 0 & c_2 I_n & c_3 I_n \\ 0 & I & 0 & 2c_2 I_n \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}. \end{aligned}$$

where $c_i = (-1)^{i+1}\binom{\alpha}{i}$.

The equality (1.55) holds since postmultiplication of the matrix

$$\left[B\ A_\alpha B\ A_\alpha^2 B\ A_\alpha^3 B\right]$$

by nonsingular matrix does not change its rank. □

Theorem 1.15. *The fractional system (1.28) is controllable to zero if and only if*

$$\operatorname{rank}\left[B\ A_\alpha B\ \ldots\ A_\alpha^{q-1}B\ \Phi_q\right] = \operatorname{rank}\left[B\ A_\alpha B\ \ldots\ A_\alpha^{q-1}B\right]. \tag{1.57}$$

Proof. From (1.23) for $k = q$ and $x_q = 0$ we have

$$0 = x_q = \Phi_q x_0 + \sum_{i=0}^{q-1} \Phi_{q-i-1} B u_i$$

$$= \Phi_q x_0 + \begin{bmatrix} B & \Phi_1 B & \dots & \Phi_{q-1} B \end{bmatrix} \begin{bmatrix} u_{q-1} \\ u_{q-2} \\ \vdots \\ u_0 \end{bmatrix}, \tag{1.58}$$

or

$$-\Phi_q x_0 = \begin{bmatrix} B & \Phi_1 B & \dots & \Phi_{q-1} B \end{bmatrix} \begin{bmatrix} u_{q-1} \\ u_{q-2} \\ \vdots \\ u_0 \end{bmatrix}, \tag{1.59}$$

The equation (1.59) has a solution $u_i, \dots, u_{q-1}$ for arbitrary x_0 if and only if

$$\operatorname{rank} \begin{bmatrix} B & \Phi_1 B & \dots & \Phi_{q-1} B & \Phi_q \end{bmatrix} = \operatorname{rank} \begin{bmatrix} B & \Phi_1 B & \dots & \Phi_{q-1} B \end{bmatrix} = n. \tag{1.60}$$

By Theorem 1.14 the conditions (1.57) and (1.60) are equivalent. □

Remark 1.9. The condition (1.54) is only sufficient condition but not necessary for the controllability to zero of the system (1.28) since condition (1.54) implies only the condition (1.57).

Theorem 1.16. *The fractional system (1.28) is controllable to zero if and only if one of the following equivalent conditions is satisfied:*

a) The matrix $[I_n - A_\alpha d, B]$ has full row rank, i.e.

$$\operatorname{rank}[I_n - A_\alpha d, B] = n, \quad \forall d \in \mathbb{C}. \tag{1.61}$$

b) The matrices $[I_n - A_\alpha d]$, B are relatively left prime or equivalent it is possible using elementary column operations (R) to reduce the matrix $[I_n - A_\alpha d, B]$ to the form $[I_n, 0]$, i.e.

$$[I_n - A_\alpha d, B] \xrightarrow{R} [I_n, 0]. \tag{1.62}$$

Proof. The equivalence of the conditions (1.57) and (1.61) follows from Kučera theorem. The controllability to zero means that if there exists an unreachable mod then it is finite. Assume that $[I_n z - A_\alpha] = [I_n - A_\alpha d]$, where $d = z^{-1}$. Substituting $[I_n - A_\alpha d]$ instead of $[I_n z - A_\alpha]$ to the reachability condition we neglect the finite mods. Therefore, the condition (1.61) is necessary and sufficient for the controllability to zero. The reduction (1.62) can be performed if and only if the condition (1.61) is satisfied. □

Example 1.8. Check the controllability to zero of the fractional system (1.28) with the matrices:

$$A_\alpha = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & a & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{1.63}$$

a) Using (1.61), we obtain

$$\operatorname{rank}[I_n - A_\alpha d, B] = \operatorname{rank} \left[\begin{array}{ccc|c} 1 & 0 & -d & 1 \\ -d & 1 & 0 & 0 \\ 0 & -ad & 1-d & 0 \end{array}\right] = 3, \quad \forall d \in \mathbb{C}$$

By Theorem 1.16 the pair (A, B) is controllable to zero if and only if $a \neq 0$.

b) Performing the following elementary column operations we can check whether the matrices $[I_n - A_\alpha d]$ and B are relatively left prime:

$$\left[\begin{array}{ccc|c} 1 & 0 & -d & 1 \\ -d & 1 & 0 & 0 \\ 0 & -ad & 1-d & 0 \end{array}\right] \xrightarrow[R[1+2\times(d)]]{\substack{R[1+4\times(-1)] \\ R[3+4\times(d)]}} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -ad^2 & -ad & 1-d & 0 \end{array}\right]$$

$$\xrightarrow[R[2+1\times(ad)]]{\substack{R[1-3\times(ad+a)] \\ R[1\times(-1/a)]}} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1-d & 0 \end{array}\right] \xrightarrow[R[3,4]]{\substack{R[3+1\times(d-1)] \\ R[1,4]}} \left[I_3 \mid 0 \right].$$

By Theorem 1.16 the pair (A, B) is controllable to zero in $q = 3$ steps.

1.9 Controllability to Zero of Positive Fractional Discrete-Time Linear Systems

Definition 1.12. The positive fractional system (1.28) is called controllable to zero in q steps if there exists an input sequence $u_i \in \mathbb{R}_+^m$, $i = 0, 1, \ldots, q-1$, which steers the nonzero arbitrary initial state $x_0 \in \mathbb{R}_+^n$, to the final state $x_f = 0$. The positive fractional system (1.28) is called controllable to zero if there exists a natural number $q > 0$ such that the system is controllable to zero in q steps.

Theorem 1.17. *The positive fractional system (1.28) with $B \neq 0$ is called controllable to zero in q steps if and only if*

$$\Phi_q = 0. \tag{1.64}$$

Moreover, $u_i = 0$ for $i = 0, 1, \ldots, q-1$.

Proof. From (1.23) for $k = q$ and $x_q = 0$ we obtain the equality (1.55). This equality for positive system can be satisfied for every x_0 if and only if the condition (1.64) is satisfied and $u_i = 0$ for $i = 0, 1, \ldots, q-1$. □

Lemma 1.3. *For positive fractional system (1.28) the condition (1.64) is satisfied if and only if $q = 1$ and*

$$\Phi_1 = A + \alpha I_n = 0. \tag{1.65}$$

Proof. From (1.51) with $a_{ki} \geq 0$ and Lemma 1.1 it follows that the condition (1.64) is satisfied if and only if $q = 1$ and (1.65). □

Corollary 1.1. *The positive fractional system (1.28) with $B \neq 0$ is controllable to zero if and only if $q = 1$ and the condition (1.65) is satisfied.*

1.10 Minimum Energy Control of Positive Fractional Systems

Consider the positive fractional discrete-time linear system (1.28). If the system is reachable in q steps then exist many input sequences which steer the state of the system from $x_0 = 0$ to the final state $x_f \in \mathbb{R}_+^n$. Among these sequences we are looking for a sequence $u_i \in \mathbb{R}_+^m$, $i = 0, 1, \ldots, q-1$, $i \in \mathbb{Z}_+$, which minimizes the performance index

$$I(u) = \sum_{j=0}^{q-1} u_j^T Q u_j, \tag{1.66}$$

where $Q \in \mathbb{R}^{m \times m}$ is a symmetric positive definite matrix and q is the number of steps needed to steer the system from $x_0 = 0$ to the final state $x_f \in \mathbb{R}_+^n$.

The minimum energy control problem for the positive fractional system (1.28) can be stated as follows [102]: Given the matrices A, B, degree α of the system (1.28), number of steps q, the finite state $x_f \in \mathbb{R}_+^n$ and the matrix Q of (1.66). Find an input sequence $u_i \in \mathbb{R}_+^m$, $i = 0, 1, \ldots, q-1$, which steers the state of the system from $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$ and minimizes the performance index (1.66).

To solve the problem we define the matrix

$$W = W(q, Q) = R_q \overline{Q} R_q^T \in \mathbb{R}^{n \times n}, \tag{1.67}$$

where R_q is given by (1.46) and

$$\overline{Q} = \text{block diag}\left[\, Q^{-1} \ \ldots \ Q^{-1} \,\right] \in \mathbb{R}^{qm \times qm}. \tag{1.68}$$

From (1.67) it follows that the matrix W is nonsingular if and only if $\operatorname{rank} R_q = n$. If this condition is satisfied then the system is reachable in q steps. In this case we can define for given $x_f \in \mathbb{R}_+^n$ the following input sequence

$$\hat{u}_{0q} = \begin{bmatrix} \hat{u}_{q-1} \\ \hat{u}_{q-2} \\ \vdots \\ \hat{u}_0 \end{bmatrix} = \overline{Q} R_q^T W^{-1} x_f. \tag{1.69}$$

From (1.69) it follows that $\hat{u}_i \in \mathbb{R}_+^m$ for $i = 0, 1, \ldots, q-1$ if

$$\overline{Q} R_q^T W^{-1} \in \mathbb{R}_+^{m \times n}, \tag{1.70}$$

and this implies

$$Q^{-1} \in \mathbb{R}_+^{m\times m}, \quad W^{-1} \in \mathbb{R}_+^{n\times n}. \tag{1.71}$$

Theorem 1.18. *Let the fractional system (1.28) be reachable in q steps and the condition (1.71) be satisfied. Moreover let $\overline{u}_i \in \mathbb{R}_+^m$, $i = 0, 1, \dots, q-1$ be an input sequence which steers the state of the system from $x_0 = 0$ to the final state $x_f \in \mathbb{R}_+^n$. The input sequence $\hat{u}_i \in \mathbb{R}_+^m$, $i = 0, 1, \dots, q-1$ also steers the state of the system from $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$ and minimizes the performance index (1.66), i.e.*

$$I(\hat{u}) \le I(\overline{u}). \tag{1.72}$$

The minimal value of the performance index (1.66) for (1.69) is

$$I(\hat{u}) = x_f^T W^{-1} x_f. \tag{1.73}$$

Proof. If the fractional system (1.28) is positive, reachable in q steps and the assumption (1.71) are satisfied then for $x_f \in \mathbb{R}_+^n$ we have $\hat{u}_i \in \mathbb{R}_+^m$ for $i = 0, 1, \dots, q-1$. We shall show that the input sequence (1.69) steers the state of the system from $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$. Using (1.23) for $k = q$, $x_0 = 0$ and (1.67), (1.69), we obtain

$$x_q = R_q \hat{u}_{0q} = R_q \overline{Q} R_q^T W^{-1} x_f = x_f,$$

since $R_q \overline{Q} R_q^T W^{-1} = I_n$.

Both input sequences $\overline{u}_{0q}$ and $\hat{u}_{0q}$ steer the state of the system from $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$ and we have $x_f = R_q \hat{u}_{0q} = R_q \overline{u}_{0q}$, i.e.

$$R_q \left[\hat{u}_{0q} - \overline{u}_{0q}\right] = 0. \tag{1.74}$$

Using (1.74) and (1.69), we shall show

$$\left[\hat{u}_{0q} - \overline{u}_{0q}\right]^T \hat{Q} \hat{u}_{0q} = 0, \tag{1.75}$$

where $\hat{Q} = \text{block diag}[Q, \dots, Q]$.

Transposing (1.74) and postmultiplying it by $W^{-1}x_f$ we obtain

$$\left[\hat{u}_{0q} - \overline{u}_{0q}\right]^T R_q^T W^{-1} x_f = 0. \tag{1.76}$$

Using (1.69) and (1.76), we obtain (1.75), since

$$\left[\hat{u}_{0q} - \overline{u}_{0q}\right]^T \hat{Q} \hat{u}_{0q} = \left[\hat{u}_{0q} - \overline{u}_{0q}\right]^T \hat{Q} \overline{Q} R_q^T W^{-1} x_f = \left[\hat{u}_{0q} - \overline{u}_{0q}\right]^T R_q^T W^{-1} x_f = 0,$$

where $\hat{Q}\overline{Q} = I_{qm}$.

Using (1.75) it is easy to show that

$$\overline{u}_{0q}^T \overline{Q} \overline{u}_{0q} = \hat{u}_{0q}^T \hat{Q} \hat{u}_{0q} + \left[\overline{u}_{0q} - \hat{u}_{0q}\right]^T \hat{Q} \left[\overline{u}_{0q} - \hat{u}_{0q}\right]. \tag{1.77}$$

From(1.77) it follows that the inequality (1.72), is satisfied since

$$\left[\overline{u}_{0q} - \hat{u}_{0q}\right]^T \hat{Q}\left[\overline{u}_{0q} - \hat{u}_{0q}\right] \geq 0.$$

To find the minimal value of the performance index we substitute (1.69) into (1.66) and using (1.67), we obtain

$$\begin{aligned} I(\hat{u}) &= \hat{u}_{0q}^T \hat{Q} \hat{u}_{0q} = \left[\overline{Q} R_q^T W^{-1} x_f\right]^T \hat{Q}\left[\overline{Q} R_q^T W^{-1} x_f\right] \\ &= x_f^T W^{-1} R_q \overline{Q} R_q^T W^{-1} x_f = x_f^T W^{-1} x_f, \end{aligned}$$

since $W^{-1} R_q \overline{Q} R_q^T = I_n$. □

If the assumption of Theorem 1.18 are satisfied then the minimal energy control problem can be solved by the use of the following procedure.

Procedure 1.1

Step 1. Knowing the matrices A, B, Q and α, q find the matrices R_q and $\overline{Q}$, using (1.46) and (1.68).
Step 2. Knowing R_q and $\overline{Q}$, and using (1.67) find the matrix W.
Step 3. Using (1.69), find the input sequence $\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{q-1}$.
Step 4. Using (1.73), find the value of $I(\hat{u})$.

Example 1.9. Consider the fractional system (1.28) for $0 < \alpha < 1$ with the matrices:

$$A = \begin{bmatrix} -\alpha & 0 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad n = 2. \tag{1.78}$$

Find the optimal input sequence which steers the state of the system from $x_0 = 0$ to the final state $x_f = [1\ 1]^T$ in $q = 2$ steps and minimizes the performance index (1.66) for $Q = [2]$.

The fractional system (1.28) with (1.78) is reachable in $q = 2$ steps. It is easy to check that the assumption of Theorem 1.18 are satisfied. Using Procedure 1.1 we obtain the following:

Step 1. In this case

$$R_2 = \left[B\ \Phi_1 B\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$\overline{Q} = \mathrm{diag}\left[Q^{-1}\ Q^{-1}\right] = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Step 2. Using (1.67), we obtain

$$W = R_2 \overline{Q} R_2^T = \overline{Q} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Step 3. Using (1.69), we obtain

$$\hat{u}_{02} = \begin{bmatrix} \hat{u}_1 \\ \hat{u}_0 \end{bmatrix} = \overline{Q} R_2^T W^{-1} x_f = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{1.79}$$

It is easy to check that the input sequence (1.79) steers the state of system in $q = 2$ steps from $x_0 = 0$ to $x_f = [1\ 1]^T$.

Step 4. In this case the minimal value of the performance index is

$$I(\hat{u}) = x_f^T W^{-1} x_f = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4. \tag{1.80}$$

Corollary 1.2. *Note that in the case of positive fractional system by suitable choice of state-feedback we may modify the reachability matrix R_q, and the minimal value of the performance index.*

1.11 Fractional Different Orders Discrete-Time Linear Systems

Consider the fractional different orders discrete-time linear system

$$\Delta^\alpha x_1(k+1) = A_{11}x_1(k) + A_{12}x_2(k) + B_1 u(k), \quad k \in \mathbb{Z}_+, \tag{1.81a}$$

$$\Delta^\beta x_2(k+1) = A_{21}x_1(k) + A_{22}x_2(k) + B_2 u(k), \tag{1.81b}$$

where $x_1(k) \in \mathbb{R}^{n_1}$, $x_2(k) \in \mathbb{R}^{n_2}$, $u(k) \in \mathbb{R}^m$ are state and input vectors, respectively and $A_{ij} \in \mathbb{R}^{n_i \times n_j}$, $B_i \in \mathbb{R}^{n_i \times m}$, $i, j = 1, 2$.

The fractional derivative of α order is defined by

$$\Delta^\alpha x(k) = \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} x(k-j) = \sum_{j=0}^{k} c_\alpha(j) x(k-j), \tag{1.82a}$$

$$c_\alpha(j) = (-1)^j \binom{j}{\alpha} = (-1)^j \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!}, \tag{1.82b}$$

$$c_\alpha(0) = 1, \quad j = 1, 2, \ldots .$$

Using (1.82) can write the equation (1.81) in the form

$$x_1(k+1) = A_{1\alpha}x_1(k) + A_{12}x_2(k) - \sum_{j=2}^{k+1} c_\alpha(j) x_1(k-j+1) + B_1 u(k), \tag{1.83a}$$

$$x_2(k+1) = A_{21}x_1(k) + A_{2\beta}x_2(k) - \sum_{j=2}^{k+1} c_\beta(j) x_2(k-j+1) + B_2 u(k). \tag{1.83b}$$

where $A_{1\alpha} = A_{11} + \alpha I_{n_1}$, $A_{2\beta} = A_{22} + \beta I_{n_2}$.

Applying to (1.83) the z-transform we obtain

$$\begin{bmatrix} I_{n_1}z - A_{1\alpha} + \sum_{j=2}^{k+1} c_\alpha(j) I_{n_1} z^{-j+1} & -A_{12} \\ -A_{21} & I_{n_2}z - A_{2\beta} + \sum_{j=2}^{k+1} I_{n_2} c_\beta(j) z^{-j+1} \end{bmatrix} \times \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix} = \begin{bmatrix} zx_{10} \\ zx_{20} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(z), \tag{1.84}$$

where $X_i(z) = \mathscr{Z}[x_i(k)] = \sum_{k=0}^{\infty} x_i(k) z^{-k}$, $i = 1,2$; $U(z) = \mathscr{Z}[u(k)]$ and $x_{10} = x_1(0)$, $x_{20} = x_2(0)$.

From (1.84) we have

$$\begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix} = \begin{bmatrix} I_{n_1}z - A_{1\alpha} + \sum_{j=2}^{k+1} c_\alpha(j) I_{n_1} z^{-j+1} & -A_{12} \\ -A_{21} & I_{n_2}z - A_{2\beta} + \sum_{j=2}^{k+1} I_{n_2} c_\beta(j) z^{-j+1} \end{bmatrix}^{-1} \times \left\{ \begin{bmatrix} zx_{10} \\ zx_{20} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(z) \right\}. \tag{1.85}$$

Let

$$\begin{bmatrix} I_{n_1}z - A_{1\alpha} + \sum_{j=2}^{k+1} c_\alpha(j) I_{n_1} z^{-j+1} & -A_{12} \\ -A_{21} & I_{n_2}z - A_{2\beta} + \sum_{j=2}^{k+1} I_{n_2} c_\beta(j) z^{-j+1} \end{bmatrix}^{-1} = \sum_{j=0}^{\infty} \Phi_j z^{-(j+1)}, \tag{1.86}$$

where the matrices Φ_k are defined by

$$\Phi_i = \begin{cases} I_n \quad (n = n_1 + n_2) & \text{for} \quad i = 0 \\ A\Phi_{i-1} - D_1\Phi_{i-2} - \cdots - D_{i-1}\Phi_0 & \text{for} \quad i = 1,2,\ldots,k \\ A\Phi_{i-1} - D_1\Phi_{i-2} - \cdots - D_k\Phi_{i-k-1} & \text{for} \quad i = k+1, k+2, \ldots \end{cases}. \tag{1.87}$$

From definition of inverse matrix we have

$$\left[I_n z - A - D_1 z^{-1} - D_2 z^{-2} - \cdots - D_k z^{-k} \right] \left[\Phi_0 z^{-1} + \Phi_1 z^{-2} + \Phi_2 z^{-3} + \cdots \right] = I_n, \tag{1.88}$$

where

$$A = \begin{bmatrix} A_{1\alpha} & A_{12} \\ A_{21} & A_{2\beta} \end{bmatrix}, \quad D_k = \begin{bmatrix} c_\alpha(k+1) I_{n_1} & 0 \\ 0 & c_\beta(k+1) I_{n_2} \end{bmatrix}. \tag{1.89}$$

Comparison of the coefficient at the same power of z^{-1} we obtain

$$\begin{aligned} &\Phi_0 = I_n, \quad \Phi_1 = A\Phi_0, \quad \Phi_2 = A\Phi_1 - D_1\Phi_0, \\ &\Phi_3 = A\Phi_2 - D_1\Phi_1 - D_2\Phi_0, \ldots \quad . \end{aligned} \tag{1.90}$$

which can be written in the form (1.87).

Substitution of (1.86) into (1.85) yields

$$\begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix} = \sum_{j=0}^{\infty} \Phi_j z^{-j} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} + \sum_{j=0}^{\infty} \Phi_j z^{-(j+1)} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(z). \tag{1.91}$$

Applying the inverse z-transform and the convolution theorem to (1.91) we obtain

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \Phi_k \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} + \sum_{i=0}^{k-1} \Phi_{k-i-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_i. \tag{1.92}$$

Therefore, the following theorem has been proved.

Theorem 1.19. *The solution to the fractional equation (1.81) with initial conditions* $x_1(0) = x_{10}$*,* $x_2(0) = x_{20}$ *is given by (1.92), where* Φ_k *is defined by (1.87).*

1.12 Positive Fractional Different Orders Discrete-Time Linear Systems

Consider the fractional different orders discrete-time linear systems described by the equation (1.81) and

$$y(k) = C \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + Du(k), \tag{1.93}$$

where $x_1(k) \in \mathbb{R}^{n_1}$, $x_2(k) \in \mathbb{R}^{n_2}$, $u(k) \in \mathbb{R}^m$, $y(k) \in \mathbb{R}^p$ are the state, input and output vectors and $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$.

Definition 1.13. The fractional system (1.81), (1.93) is called positive if $x_1(k) \in \mathbb{R}_+^{n_1}$, $x_2(k) \in \mathbb{R}_+^{n_2}$, $y(k) \in \mathbb{R}_+^p$ for any initial conditions $x_{10} \in \mathbb{R}_+^{n_1}$, $x_{20} \in \mathbb{R}_+^{n_2}$ and all inputs $u(k) \in \mathbb{R}_+^m$ for $k \in \mathbb{Z}_+$.

Theorem 1.20. *The fractional discrete-time linear system (1.81), (1.93) with* $0 < \alpha < 1$*,* $0 < \beta < 1$ *is positive if and only if*

$$A = \begin{bmatrix} A_{1\alpha} & A_{12} \\ A_{21} & A_{2\beta} \end{bmatrix} \in \mathbb{R}_+^{n \times n}, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \mathbb{R}_+^{n \times m}, \quad C \in \mathbb{R}^{p \times n}, \quad D \in \mathbb{R}^{p \times m}. \tag{1.94}$$

Proof. Necessity. Let $e_i^{n_j}$ be i-th column of the $n_j \times n_j$ identity matrix, $j = 1, 2$. From (1.83) for $k = 0$, $u(0) = 0$, $x_{20} = 0$ and $x_{10} = e_i^{n_1}$ we have $x_1(1) = A_{1\alpha} e_i^{n_1} \in \mathbb{R}_+^{n_1}$ and $x_2(1) = A_{21} e_i^{n_1} \in \mathbb{R}_+^{n_1}$. This implies the nonnegativity of i-th ($i = 1, \ldots, n$) columns of the matrices $A_{1\alpha}$ and A_{21}. Similarly for $k = 0$, $u(0) = 0$, $x_{10} = 0$ and $x_{20} = e_i^{n_2}$ we have $x_1(1) = A_{12} e_i^{n_2} \in \mathbb{R}_+^{n_2}$ and $x_2(1) = A_{2\beta} e_i^{n_2} \in \mathbb{R}_+^{n_2}$. To show that $B_1 \in \mathbb{R}_+^{n_1 \times m}$ and $B_2 \in \mathbb{R}_+^{n_2 \times m}$ we assume in (1.83) for $k = 0$, $x_1(0) = 0$, $x_2(0) = 0$ and $u(0) = e_i^m$ and we obtain $x_1(0) = B_1 e_i^m \in \mathbb{R}_+^{n_1}$ and $x_2(0) = B_2 e_i^m \in \mathbb{R}_+^{n_2}$. In a similar way we prove $C \in \mathbb{R}_+^{p \times n}$ and $D \in \mathbb{R}_+^{p \times m}$.

Sufficiency. In Lemma 1.1 was shown that if $0 < \alpha < 1$ and $0 < \beta < 1$ then $c_\alpha(j) < 0$ and $c_\beta(j) < 0$ for $j = 2, \dots, k+1$. From (1.89) if follows that $D_i \in \mathbb{R}_+^n$ for $i = 1, \dots, n$ and from(1.87) we have $\Phi_i \in \mathbb{R}_+^{n\times n}$ for $i = 0, 1\dots$ since $A \in \mathbb{R}_+^{n\times n}$. From (1.92) we have $x_1(k) \in \mathbb{R}_+^{n_1}$, $x_2(k) \in \mathbb{R}_+^{n_2}$, $k \in \mathbb{Z}_+$ since $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \mathbb{R}_+^{n\times m}$ and $u(i) \in \mathbb{R}_+^m$, $i \in \mathbb{Z}_+$. Finaly from (1.93) we have $y(k) \in \mathbb{R}_+^p$, $k \in \mathbb{Z}_+$ since $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, $x_1(k) \in \mathbb{R}_+^{n_1}$, $x_2(k) \in \mathbb{R}_+^{n_2}$ and $u(k) \in \mathbb{R}_+^m$, $k \in \mathbb{Z}_+$. □

These considerations can be easy extended to fractional system consisting of n subsystems of different fractional order [161].