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Selected Problems of Fractional Systems Theory

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Preface

This monograph covers some selected problems of positive fractional 1D and 2D linear systems. It is an extended and modified English version of its preceding Polish edition published by Technical University of Bialystok in 2009. This book is based on the lectures delivered by the author to the Ph.D. students of the Faculty of Electrical Engineering of Bialystok University of Technology and of Warsaw University of Technology and on invited lectures in several foreign universities in the last three years.

The monograph consists of 12 chapters. 4 appendices and a list of references.

Chapter 1 is devoted to the fractional discrete-time systems. The solution to the state equations of fractional discrete-time linear systems is derived. Necessary and sufficient conditions for the positivity, the reachability and the controllability to zero are established. The minimum energy control problem for the positive fractional systems is formulated and solved. The solution to the fractional different orders discrete-time linear systems is also derived. The fractional continuous-time linear systems are considered in Chapter 2. The definitions of fractional derivatives are recalled and the solutions to the fractional state equations of continuous-time linear systems are derived. Necessary and sufficient conditions for the internal and external positivity and the reachability of the systems without and with delays are established. The fractional positive 2D linear systems are addressed in Chapter 3. The solution to the state equations of fractional 2D linear systems is derived. The necessary and sufficient conditions for the positivity, the reachability and the controllability of the systems are established. The positive Roesser type model, the Fornasini-Marchesini type models and the general 2D model are discussed. The positive 2D linear systems with delays are also considered. The pointwise completeness and the pointwise degeneracy of standard and positive linear systems are considered in Chapter 4. First the pointwise completeness and the pointwise degeneracy of the standard discrete-time and continuous-time linear systems without and with delays are analyzed. Next the considerations are extended to the positive and fractional linear systems. The pointwise completeness and the pointwise degeneracy of electrical circuits are also investigated. The pointwise completeness and the pointwise degeneracy of linear systems with state feedbacks are considered in Chapter 5. Gain matrices of the state feedbacks are chosen so that the closed-loop systems are positive and pointwise complete. Conditions are established under which there exist such gain matrices. It is shown that the pointwise completeness and the pointwise degeneracy of the standard and fractional continuous-time linear systems are invariant under the state feedbacks. The realization problem for positive fractional discrete-time and continuous-time linear systems is addressed in Chapter 6. Sufficient conditions for the existence of positive fractional realizations of a given transfer matrix are established for SISO and MIMO linear systems. Procedures for computation of the realizations are given and illustrated by numerical examples. Chapter 7 is devoted to the cone discrete -time and continuous-time linear systems. The relationship between the cone and positive systems is discussed. Necessary and sufficient conditions for the reachability and controllability to zero of the cone fractional systems are established. Sufficient conditions for the existence and a procedure for computation of cone realizations for continuous-time linear systems with delays are also given. The stability of positive and fractional linear 1D and 2D systems is addressed in Chapter 8. Necessary and sufficient conditions for the asymptotic stability of the positive systems without and with delays are established. The relationship between the asymptotic stability of positive 1D and 2D linear systems is given. It is shown that checking of the stability of positive fractional 2D linear systems can be reduced to checking of the stability of corresponding positive 1D linear systems. The practical stability of the fractional 1D and 2D linear systems is also investigated. Chapter 9 is devoted to the stability analysis of fractional linear systems in frequency domain. It is based on M. Busowicz papers. First the stability of fractional continuous-time linear systems and of fractional systems with delays of retarded type is investigated. Next the stability of fractional discrete-time linear systems of commensurate orders in frequency domain is analyzed. The robust stability of the convex combination of two fractional polynomials is also investigated. In Chapter 10 the stabilization problem of positive and fractional 1D and 2D linear systems by state feedbacks is addressed. Gain matrices of the state feedbacks are chosen so that the closed loop-systems are positive and asymptotically stable. The LMI approaches are applied to the stabilization problem for 1D and 2D linear systems. Chapter 11 is addressed to the singular fractional continuous-time and discrete-time linear systems. Using the Weierstrass decomposition of the singular pencil of matrices the solutions to the singular fractional systems are derived. Singular fractional electrical circuits are considered. It is shown that the singular systems can be reduced to equivalent standard fractional systems. The decomposition of the singular systems into dynamical and static parts is also presented. The last Chapter 12 is devoted to the continuous-discrete (hybrid 2D) linear systems. The solution to the general model of the systems is derived and the necessary and sufficient conditions for the positivity of the model are established. The reachability of the positive continuous-discrete linear systems is analyzed. Necessary and sufficient conditions for the asymptotic stability of the positive systems are established. The robust stability of the scalar general model of the systems is investigated. The positive realization problem for the linear systems is also considered.

In Appendix A some basic definitions and properties of the Laplace transform and the z-transform are recalled. The infinite long cable with zero inductance is presented in Appendix B as an example of linear fractional continuous-time system. In Appendix C the right inverses of matrices and their application to finding the solution of the matrix linear equation are presented. Basic definitions of element operations on real matrices are recalled in Appendix D.

It is hoped that this monograph will be value to Ph.D. students and researches from the field of fractional systems.

I would like to express my gratitude to Professors J. Klamka and P. Ostalczyk, the reviewers of the Polish edition of the book, for their valuable comments, suggestions and remarks which helped to improve the monograph. My special thanks go to Professor M. Busowicz who was the scientific editor of the Polish edition of the book.

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Tadeusz Kaczorek

List of Symbols

Adj A	adjoint matrix of the matrix A
block diag A	block diagonal matrix A
$\det A$	determinant of the matrix A
rank A	rank of the matrix A
$A > 0 (x > 0)$	strictly positive matrix (vector)
$a \in A$	a is a element of the set A
$a \notin A$	a is not a element of the set A
$A \succ 0$	positive definite matrix
A^T	transpose matrix
A^{-1}	inverse matrix
x_k	discrete-time function of k
$\Delta^{\alpha,\beta} x_{ij}$	(α, β) orders fractional difference of 2D discrete function x_{ij}
${}_0\Delta_k^n x_k$	n-order backwards difference of x_k on the interval $[0, k]$
${}_0I_x^n$	n-multiple integral on the interval $(0, x)$
${}_aD_t^\alpha f(t)$	α-order derivative of the function $f(t)$ in the interval $[a, t]$
$E_\alpha(z)$	one parameter Mittag-Leffler function
$E_{\alpha,\beta}(z)$	two parameter Mittag-Leffler function
$\forall$	for all
$\Gamma(x)$	the Euler gamma function
I_n	the $n \times n$ identity matrix
$L[i + j \times b]$	addition to the i-th row of the j-th row multiplied by the number b
$L[i, j]$	the interchange of the i-th and the j-th rows
$L[i \times a]$	multiplication of the i-th row by the number $a \neq 0$
$R[i + j \times b]$	addition to the i-th column of the j-th column multiplied by the number b
$R[i, j]$	the interchange of the i-th and the j-th columns
$R[i \times a]$	multiplication of the i-th column by the number $a \neq 0$
$\mathcal{L}[f(t)] = F(s)$	Laplace transform of the function $f(t)$
$\mathcal{Z}[f(k)] = F(z)$	z-transform of the function $f(t)$
$\mathbb{C}$	the set of complex numbers

$\mathbb{N}$	the set of natural numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{R}_+$	the set of real nonnegative numbers
$\mathbb{R}_+^{n\times m}$	the set of $n \times m$ real matrices with nonnegative entries
$\mathbb{Z}$	the set of integers
$\mathbb{Z}_+$	the set of nonnegative integers
M_n	the set of $n \times n$ Metzler matrices

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Chapter 1
Fractional Discrete-Time Linear Systems

1.1 Definition of n-Order Difference

Definition 1.1. A discrete-time function defined by

$$\Delta^n x_i = \Delta^{n-1} x_i - \Delta^{n-1} x_{i-1} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x_{i-k}, \tag{1.1}$$

$$i = 1,2,3,\ldots\ , \quad n \in \mathbb{Z}, \quad x_i \in \mathbb{R},$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}\ , \tag{1.2}$$

is called the n-order (backward) difference of the function x_i.

Definition 1.2. The fractional n-order (backward) difference on the interval $[0,k]$ of the function x_i is defined as follows

$$_0\Delta_k^n x_i = \sum_{j=0}^{k} (-1)^j \binom{n}{j} x_{i-j}. \tag{1.3}$$

From (1.1) it follows that the n-order difference can be written as a linear combination of the values of discrete-time function in $n+1$ points.

The definitions are valid for n being natural numbers and integers.

Note that (1.2) is also well defined for fractional and real numbers. In general case n can be also a complex number.

Example 1.1. From (1.1) we have for:

$$\begin{aligned}
n=1: &\quad \Delta x_i = x_i - x_{i-1},\\
n=2: &\quad \Delta^2 x_i = \Delta x_i - \Delta x_{i-1} = x_i - 2x_{i-1} + x_{i-2},\\
n=3: &\quad \Delta^3 x_i = \Delta^2 x_i - \Delta^2 x_{i-1} = x_i - 3x_{i-1} + 3x_{i-2} - x_{i-3}.
\end{aligned}$$

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From (1.3) we obtain for:
$n = -1$:

$$ {}_0\Delta_k^{-1} x_k = \sum_{j=0}^{k} (-1)^j \binom{-1}{j} x_{k-j} = x_k + x_{k-1} + \cdots + x_0 = \sum_{j=0}^{k} x_{k-j}, $$

$n = -2$:

$$ {}_0\Delta_k^{-2} x_k = \sum_{j=0}^{k} (-1)^j \binom{-2}{j} x_{k-j} = x_k + \cdots + (k+1)x_0 = \sum_{j=0}^{k} (j+1) x_{k-j}. $$

Definition 1.3. The discrete-time function

$$ \Delta^\alpha x_k = \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} x_{k-j}, \tag{1.4} $$

where $0 < \alpha < 1$, $\alpha \in \mathbb{R}$ and

$$ \binom{\alpha}{k} = \begin{cases} 1 & \text{for} \quad k = 0 \\ \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} & \text{for} \quad k = 1, 2, 3, \ldots \end{cases}, \tag{1.5} $$

is called the fractional α-order difference of the function x_k.

Example 1.2. Using (1.5) for $0 < \alpha < 1$ we obtain for:

$$ \begin{aligned} k = 1 : \qquad & (-1)^1 \binom{\alpha}{1} = -\alpha < 0, \\ k = 2 : \qquad & (-1)^2 \binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{2!} < 0, \\ k = 3 : \qquad & (-1)^3 \binom{\alpha}{3} = -\frac{\alpha(\alpha-1)(\alpha-2)}{3!} < 0. \end{aligned} $$

1.2 State Equations of the Fractional Linear Systems

1.2.1 Fractional Systems without Delays

The state equations of the fractional discrete-time linear system have the form:

$$ \Delta^\alpha x_{k+1} = Ax_k + Bu_k, \quad 0 \le \alpha \le 1, \tag{1.6a} $$
$$ y_k = Cx_k + Du_k, \quad k \in \mathbb{Z}_+, \tag{1.6b} $$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^p$ are the state, input and output vectors and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Substituting the definition of fractional difference (1.4) into (1.6a), we obtain

$$x_{k+1} + \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} x_{k-j+1} = Ax_k + Bu_k, \tag{1.7a}$$

or

$$\begin{aligned} x_{k+1} &= Ax_k + \sum_{j=1}^{k+1} (-1)^{j+1} \binom{\alpha}{j} x_{k-j+1} + Bu_k \\ &= A_\alpha x_k + \sum_{j=2}^{k+1} (-1)^{j+1} \binom{\alpha}{j} x_{k-j+1} + Bu_k, \end{aligned} \tag{1.7b}$$

where

$$A_\alpha = A + \alpha I_n. \tag{1.8}$$

Remark 1.1. From (1.7b) it follows that the fractional system is equivalent to the system with increasing number of delays.

In practice it is assumed that j is bounded by natural number h. In this case the equations (1.6) take the form:

$$x_{k+1} = A_\alpha x_k + \sum_{j=1}^{h} (-1)^j \binom{\alpha}{j+1} x_{k-j} + Bu_k, \quad k \in \mathbb{Z}_+, \tag{1.9a}$$

$$y_k = Cx_k + Du_k. \tag{1.9b}$$

Remark 1.2. The equations (1.9) describe a discrete-time linear system with h delays.

1.2.2 Fractional Systems with Delays

Consider the fractional discrete-time linear system with h delays:

$$\Delta^\alpha x_{i+1} = \sum_{k=0}^{h} \left(A_k x_{i-k} + B_k u_{i-k}\right), \quad i \in \mathbb{Z}_+, \tag{1.10a}$$

$$y_i = Cx_i + Du_i, \tag{1.10b}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^p$ are the state, input and output vectors and $A_k \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times m}$, $k = 0,\dots,h$; $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Substituting the definition of fractional difference (1.4) into (1.10a) we obtain:

$$x_{i+1} = \sum_{j=1}^{i+1} (-1)^{j+1} \binom{\alpha}{j} x_{i-j+1} + \sum_{k=0}^{h} \left(A_k x_{i-k} + B_k u_{i-k}\right), \tag{1.11a}$$

$$y_i = Cx_i + Du_i, \quad i \in \mathbb{Z}_+, \tag{1.11b}$$

If i is bounded by the natural number L then from (1.11) we obtain:

$$x_{i+1} = \sum_{j=1}^{L+1} (-1)^{j+1} \binom{\alpha}{j} x_{i-j+1} + \sum_{k=0}^{h} (A_k x_{i-k} + B_k u_{i-k}), \tag{1.12a}$$

$$y_i = Cx_i + Du_i, \quad i \in \mathbb{Z}_+ . \tag{1.12b}$$

1.3 Solution of the State Equations of the Fractional Discrete-Time Linear System with Delays

1.3.1 Fractional Systems with Delays

The state equations of the fractional discrete-time linear system with q delays has the form:

$$x_{k+1} + \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} x_{k-j+1} = \sum_{r=0}^{q} (A_r x_{k-r} + B_r u_{k-r}), \quad k \in \mathbb{Z}_+, \tag{1.13a}$$

$$y_k = Cx_k + Du_k, \quad 0 \le \alpha \le 1, \tag{1.13b}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^p$ are the state, input and output vectors and $A_r \in \mathbb{R}^{n\times n}$, $B_r \in \mathbb{R}^{n\times m}$, $r = 0,1,\ldots,q$; $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, q is the number of delays.

Applying the z-transform ($\mathscr{Z}$) method we shall derive the solution of the state equation (1.13a) of the fractional system [79].

Theorem 1.1. *The solution of the equation (1.13a) has the form*

$$x_k = \Phi_k x_0 + \sum_{r=0}^{q} \sum_{i=0}^{k-r-1} \Phi_{k-r-1-i} B_r u_i + \sum_{j=1}^{k+1} \sum_{l=-1}^{-j+1} (-1)^{j+1} \binom{\alpha}{j} \Phi_{k-l-j} x_l$$
$$+ \sum_{r=0}^{q} \sum_{l=-1}^{-r} \Phi_{k-r-l-1} A_r x_l + \sum_{r=0}^{q} \sum_{l=-1}^{-r} \Phi_{k-r-l-1} B_r u_l, \tag{1.14}$$

where

$$x_k \neq 0, \quad u_k \neq 0, \quad k = 0,-1,\ldots,-q \ , \tag{1.15}$$

are initial conditions and the matrices Φ_k *are determined by the equation*

$$\Phi_{k+1} = \Phi_k (A_0 + \alpha I_n) + \sum_{i=2}^{k+1} (-1)^{i+1} \binom{\alpha}{i} \Phi_{k-i+1} + \sum_{i=1}^{k} \Phi_{k-i} A_i, \tag{1.16a}$$

$$\Phi_0 = I_n, \tag{1.16b}$$

for $k = 0,1,\ldots$.

Proof. Let $X(z)$ be the z-transform ($\mathscr{Z}$) of the discrete-time function x_i defined by (A.13). Applying the z-transform (Appendix A.3) to the equation (1.13a) we obtain

$$\mathscr{Z}[x_{k+1}]+\sum_{j=1}^{k+1}(-1)^j\binom{\alpha}{j}\mathscr{Z}[x_{k-j+1}]=\sum_{r=0}^{q}A_r\mathscr{Z}[x_{k-r}]+\sum_{r=0}^{q}B_r\mathscr{Z}[u_{k-r}]. \tag{1.17}$$

Using (A.14) to (1.17) we get

$$\begin{aligned} zX(z)-zx_0+\sum_{j=1}^{k+1}(-1)^j\binom{\alpha}{j}z^{-j+1}\left[X(z)+\sum_{l=-1}^{-j+1}x_lz^{-l}\right] \qquad (1.18)\\ =\sum_{r=0}^{q}A_rz^{-r}\left[X(z)+\sum_{l=-1}^{-r}x_lz^{-l}\right]+\sum_{r=0}^{q}B_rz^{-r}\left[U(z)+\sum_{l=-1}^{-r}u_lz^{-l}\right], \end{aligned}$$

where $U(z)=\mathscr{Z}[u_k]$.

Multiplying (1.18) by z^{-1} and solving with respect to $X(z)$ we obtain

$$\begin{aligned} X(z) &= \left[\sum_{j=0}^{k+1}(-1)^j\binom{\alpha}{j}z^{-j}I_n-\sum_{r=0}^{q}A_rz^{-r-1}\right]^{-1}\\ &\times\left\{x_0+\sum_{j=1}^{k+1}(-1)^{j+1}\binom{\alpha}{j}z^{-j}\sum_{l=-1}^{-j+1}x_lz^{-l}+\sum_{r=0}^{q}A_rz^{-r-1}\sum_{l=-1}^{-r}x_lz^{-l}\right.\\ &\left.+\sum_{r=0}^{q}B_rz^{-r-1}\left[U(z)+\sum_{l=-1}^{-r}u_lz^{-l}\right]\right\}. \end{aligned} \tag{1.19}$$

Substituting of the expansion

$$\left[\sum_{j=0}^{k+1}(-1)^j\binom{\alpha}{j}z^{-j}I_n-\sum_{r=0}^{q}A_rz^{-r-1}\right]^{-1}=\sum_{k=0}^{\infty}\Phi_kz^{-k}, \tag{1.20}$$

into (1.19) yields

$$\begin{aligned} X(z) &= \sum_{k=0}^{\infty}\Phi_kz^{-k}x_0+\sum_{k=0}^{\infty}\sum_{r=0}^{q}\Phi_kz^{-k-r-1}B_rU(z)\\ &+\sum_{k=0}^{\infty}\Phi_kz^{-k}\left[\sum_{j=1}^{k+1}\sum_{l=-1}^{-j+1}(-1)^{j+1}\binom{\alpha}{j}x_lz^{-j-l}\right.\\ &\left.+\sum_{r=0}^{q}\sum_{l=-1}^{-r}A_rx_lz^{-r-l-1}+\sum_{r=0}^{q}\sum_{l=-1}^{-r}B_ru_lz^{-r-l-1}\right]. \end{aligned} \tag{1.21}$$

Applying the inverse z-transform and the convolution theorem (Appendix A.1) to (1.21) we obtain the desired solution (1.14).

From definition of the inverse matrix we have

$$\left[\sum_{j=0}^{k+1}(-1)^j\binom{\alpha}{j}z^{-j}I_n-\sum_{r=0}^{q}A_rz^{-r-1}\right]\left[\sum_{k=0}^{\infty}\Phi_kz^{-k}\right]=I_n. \tag{1.22}$$

Comparison of the coefficients at the same powers of z^{-k}, $k = 0, 1, \ldots$; from (1.22) yields:

$$\begin{aligned}
&z^0: \quad \Phi_0 \cdot I_n = I_n, \\
&z^{-1}: -A_0 + \Phi_1 - \alpha I_n = 0 \qquad\qquad \Rightarrow \Phi_1 = A_0 + \alpha I_n, \\
&z^{-2}: \Phi_2 - \Phi_1 (A_0 + \alpha I_n) + \cdots = 0 \Rightarrow \Phi_2 = \Phi_1 (A_0 + \alpha I_n) - \Phi_0 \left(I_n \tbinom{\alpha}{2} - A_1 \right) \\
&\vdots
\end{aligned}$$

and in general case the equation (1.16). □

1.3.2 Fractional Systems without Delays

In this section we shall consider the fractional discrete-time linear system without delays. Substituting in (1.14) $q = 0$ we obtain the following theorem.

Theorem 1.2. *The solution of the equation (1.7) has the form*

$$x_k = \Phi_k x_0 + \sum_{i=0}^{k-1} \Phi_{k-i-1} B u_i, \tag{1.23}$$

where the matrices Φ_k are determined by the equation

$$\Phi_{k+1} = \Phi_k (A + \alpha I_n) + \sum_{i=2}^{k+1} (-1)^{i+1} \binom{\alpha}{i} \Phi_{k-i+1}, \quad \Phi_0 = I_n. \tag{1.24}$$

Theorem 1.3. *Let*

$$\det \left[\sum_{j=0}^{k+1} (-1)^j \binom{\alpha}{j} I_n z^{-j} - A z^{-1} \right] = \sum_{i=0}^{M} a_{M-i} z^{-i}, \tag{1.25}$$

be the characteristic polynomial of the fractional system (1.7) for $k = L$. The matrices $\Phi_1, \ldots, \Phi_M$ satisfy the equation

$$\sum_{i=0}^{M} a_i \Phi_i = 0. \tag{1.26}$$

Proof. From definition of the adjoint matrix and (1.25) we have

$$\operatorname{Adj} \left[\sum_{j=0}^{L+1} (-1)^j \binom{\alpha}{j} I_n z^{-j} - A z^{-1} \right] = \left(\sum_{i=0}^{\infty} \Phi_i z^{-i} \right) \left(\sum_{i=0}^{M} a_{M-i} z^{-i} \right), \tag{1.27}$$

where $\operatorname{Adj} F$ denotes the adjoint matrix of F.

Comparing the coefficients of the same powers of z^{-1} in (1.27), we obtain (1.26), since the degree of the matrix

$$\mathrm{Adj}\left[\sum_{j=0}^{L+1}(-1)^j\binom{\alpha}{j}I_n z^{-j} - Az^{-1}\right],$$

is less than M. □

Theorem 1.3 is an extension of the well-known Cayley-Hamilton theorem for fractional discrete-time linear systems.

Remark 1.3. The degree M of the characteristic polynomial (1.25) depends on k and it increases to infinity for $k \to \infty$. Usually it is assumed that k is bounded by natural number L. If $k = L$ then $M = N(L+1)$.

1.4 Positive Fractional Linear Systems

In this section the necessary and sufficient conditions for the positivity of the fractional discrete-time linear system:

$$x_{k+1} + \sum_{j=1}^{k+1}(-1)^j\binom{\alpha}{j}x_{k-j+1} = Ax_k + Bu_k, \quad k \in \mathbb{Z}_+, \tag{1.28a}$$

$$y_k = Cx_k + Du_k, \tag{1.28b}$$

will be established, where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^p$ are state, input and output vectors and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Let $\mathbb{R}_+^{n\times m}$ be the set of real $n\times m$ matrices with the nonnegative entries and $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$.

Definition 1.4. The system (1.28) is called (internally) positive fractional system if $x_k \in \mathbb{R}_+^n$, $y_k \in \mathbb{R}_+^p$ for every initial conditions $x_0 \in \mathbb{R}_+^n$ and all inputs $u_k \in \mathbb{R}_+^m$, $k \in \mathbb{Z}_+$.

Lemma 1.1. *If* $0 < \alpha < 1$, *then*

$$(-1)^{i+1}\binom{\alpha}{i} > 0, \quad i = 1, 2, \ldots \,. \tag{1.29}$$

Proof. The proof will be accomplished by induction. The hypothesis is true for $i = 1$ since

$$(-1)^{1+1}\binom{\alpha}{1} = \alpha > 0.$$

Assuming that $(-1)^{k+1}\binom{\alpha}{k} > 0$ for $k \geq 1$ we shall show that the hypothesis is valid for $k+1$. From (1.5) we have

$$
\begin{aligned}
(-1)^{k+2}\binom{\alpha}{k+1} &= (-1)^{k+2}\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)(\alpha-k)}{k!(k+1)} \\
&= (-1)^{k+1}\binom{\alpha}{k}\frac{k-\alpha}{k+1} > 0.
\end{aligned}
$$

Therefore, the hypothesis is true for $k+1$. This completes the proof. □

Remark 1.4. In a similar way it can be shown that for $1 < \alpha < 2$

$$
(-1)^{i+1}\binom{\alpha}{i} < 0, \quad i = 2, 3, \ldots \ .
$$

Lemma 1.2. *Let* $0 < \alpha < 1$ *and*

$$
[A + \alpha I_n] \in \mathbb{R}_+^{n\times n}, \tag{1.30}
$$

then

$$
\Phi_k \in \mathbb{R}_+^{n\times n}, \quad k = 1, 2, \ldots \ . \tag{1.31}
$$

Proof. The proof follows immediately from (1.24). □

Theorem 1.4. *The fractional system (1.28) is (internally) positive if and only if:*

$$
A_\alpha = [A + \alpha I_n] \in \mathbb{R}_+^{n\times n} \quad \text{and} \quad B \in \mathbb{R}_+^{n\times m}, \quad C \in \mathbb{R}_+^{p\times n}, \quad D \in \mathbb{R}_+^{p\times m}. \tag{1.32}
$$

Proof. Sufficiency follows from Lemma 1.2 and the equation (1.23). From (1.23) it follows that if $\Phi_k \in \mathbb{R}_+^{n\times n}$, $B \in \mathbb{R}_+^{n\times m}$, $x_0 \in \mathbb{R}_+^n$ then $x_k \in \mathbb{R}_+^n$, $k \in \mathbb{Z}_+$. Similarly from (1.28b) we have $y_k \in \mathbb{R}_+^p$ if the conditions (1.32) are satisfied.

Necessity. Let $u_k = 0$ for $k \in \mathbb{Z}_+$. For positive system from (1.28) for $k = 0$ we have $x_1 = [A + \alpha I_n]x_0 = A_\alpha x_0 = A_{\alpha 1} \in \mathbb{R}_+^n$, and $y_0 = Cx_0 \in \mathbb{R}_+^p$. Therefore $A_\alpha \in \mathbb{R}_+^{n\times n}$ and $C \in \mathbb{R}_+^{p\times n}$, since $x_0 \in \mathbb{R}_+^n$ and by Definition 1.4 it is arbitrary. Assuming $x_0 = 0$ from (1.28) for $k = 0$ we obtain $x_1 = Bu_0 \in \mathbb{R}_+^n$ and $y_0 = Du_0 \in \mathbb{R}_+^p$, and this implies $B \in \mathbb{R}_+^{n\times m}$ and $D \in \mathbb{R}_+^{p\times m}$, since $u_0 \in \mathbb{R}_+^m$ and it is arbitrary. □

Definition 1.5. The fractional discrete-time linear system (1.11) with h delays is called (internally) positive if $x_i \in \mathbb{R}_+^n$ and $y_i \in \mathbb{R}_+^p$ for any initial conditions $x_k \in \mathbb{R}_+^n$, $k = 0, -1, \ldots, -h$ and all inputs $u_i \in \mathbb{R}_+^m$, $i \in \mathbb{Z}_+$.

Theorem 1.5. *The fractional discrete-time linear system (1.11) with h delays is (internally) positive for* $0 < \alpha < 1$ *if and only if*

$$
\begin{aligned}
&A_k + c_{k+1}I_n \in \mathbb{R}_+^{n\times n}, \quad c_k = (-1)^{k+1}\binom{\alpha}{k}, \quad B_k \in \mathbb{R}_+^{n\times m}, \quad k = 1, \ldots, h; \\
&C \in \mathbb{R}_+^{p\times n}, \quad D \in \mathbb{R}_+^{p\times m}.
\end{aligned} \tag{1.33}
$$

Proof. The proof is similar to the proof of Theorem 1.4. □

1.5 Externally Positive Fractional Systems

Definition 1.6. The fractional discrete-time linear system (1.28) is called externally positive if for any inputs $u_k \in \mathbb{R}_+^m$, $k \in \mathbb{Z}_+$ and $x_0 = 0$ we have $y_k \in \mathbb{R}_+^p$, $k \in \mathbb{Z}_+$.

Definition 1.7. The output of the single-input single-output (SISO) linear system for the unit impulse

$$u_i = \begin{cases} 1 & \text{for} \quad i = 0 \\ 0 & \text{for} \quad i > 0 \end{cases},$$

and zero initial conditions is called the impulse response of the system.

In a similar way we define the matrix of impulse responses g_k of the multi-input multi-output (MIMO) linear systems.

Theorem 1.6. *The fractional discrete-time linear system (1.28) is externally positive if and only if*

$$g_k \in \mathbb{R}_+^{p\times m}, \quad k \in \mathbb{Z}_+, \tag{1.34}$$

and the matrix of impulse responses is given by

$$g_k = \begin{cases} D & \text{for} \quad k = 0 \\ C\Phi_{k-1}B & \text{for} \quad k = 1, 2, \ldots \end{cases}. \tag{1.35}$$

Proof. Sufficiency. The output of the system (1.28) with zero initial conditions and any input $u_i \in \mathbb{R}_+^m$ is given by

$$y_k = \sum_{i=0}^{k} g_{k-i} u_i, \quad k \in \mathbb{Z}_+. \tag{1.36}$$

If (1.34) holds and $u_i \in \mathbb{R}_+^m$, then from (1.36) we have $y_k \in \mathbb{R}_+^p$, $k \in \mathbb{Z}_+$.

Necessity follows immediately from Definition 1.7. □

Remark 1.5. Every (internally) positive linear system is always externally positive. This follows from Definitions 1.4 and 1.6.

Example 1.3. Consider the fractional system (1.6a) for $0 < \alpha < 1$ with the matrices:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -\alpha \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (n = 2). \tag{1.37}$$

The system is positive since

$$[A + \alpha I_n] = \begin{bmatrix} (1+\alpha) & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}_+^{2\times 2}.$$

Using (1.24) for $k = 0, 1, \dots$ we obtain:

$$\Phi_1 = (A + \alpha I_n)\Phi_0 = \begin{bmatrix} (1+\alpha) & 0 \\ 0 & 0 \end{bmatrix}, \tag{1.38a}$$

$$\Phi_2 = (A + \alpha I_n)\Phi_1 - \binom{\alpha}{2}\Phi_0 = \begin{bmatrix} \frac{\alpha^2+5\alpha+2}{2} & 0 \\ 0 & \frac{\alpha(1-\alpha)}{2} \end{bmatrix}, \tag{1.38b}$$

$$\begin{aligned} \Phi_3 &= (A + \alpha I_n)\Phi_2 - \binom{\alpha}{2}\Phi_1 + \binom{\alpha}{3}\Phi_0 \\ &= \begin{bmatrix} \frac{3(\alpha^2+5\alpha+2)(\alpha+1)-\alpha(\alpha-1)(2\alpha+5)}{6} & 0 \\ 0 & \frac{\alpha(1-\alpha)(2-\alpha)}{6} \end{bmatrix}, \end{aligned} \tag{1.38c}$$

$$\vdots$$

From (1.23) and (1.24) we have

$$x_k = \Phi_k x_0 + \sum_{i=0}^{k-1} \Phi_{k-i-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_i, \tag{1.39}$$

where Φ_k is defined by (1.38).

1.6 Reachability of Fractional Discrete-Time Linear Systems

Definition 1.8. A state $x_f \in \mathbb{R}^n$ is called reachable in (given) q steps if there exists an input sequence $u_0, u_1, \dots, u_{q-1}$, which steers the state of the system (1.28) from $x_0 = 0$ to the state x_f, i.e. $x_q = x_f$. If every given state $x_f \in \mathbb{R}^n$ is reachable in q steps then the system (1.28) is called reachable in q steps. If for every state $x_f \in \mathbb{R}^n$ there exists a number q of steps such that the system is reachable in q steps then the system is called reachable.

Theorem 1.7. *The fractional system (1.28) is reachable in q steps if and only if*

$$\operatorname{rank}\begin{bmatrix} B & \Phi_1 B & \dots & \Phi_{q-1}B \end{bmatrix} = n. \tag{1.40}$$

Proof. From (1.23) for $k = q$ and $x_0 = 0$ we have

$$x_f = \Phi_q x_0 + \sum_{i=0}^{q-1} \Phi_{q-i-1} B u_i = \begin{bmatrix} B & \Phi_1 B & \dots & \Phi_{q-1}B \end{bmatrix} \begin{bmatrix} u_{q-1} \\ u_{q-2} \\ \vdots \\ u_1 \\ u_0 \end{bmatrix}. \tag{1.41}$$

From Kronecker-Capelly theorem it follows that the equation (1.41) has a solution for every x_f if and only if the condition (1.40) is satisfied. □

Theorem 1.8. *In the condition (1.40) the matrices $\Phi_1, \ldots, \Phi_{q-1}$ can be substituted by the matrices $A_\alpha, \ldots, A_\alpha^{q-1}$, i.e.*

$$\operatorname{rank}\left[B\ \Phi_1 B\ \ldots\ \Phi_{q-1}B\right] = \operatorname{rank}\left[B\ A_\alpha B\ \ldots\ A_\alpha^{q-1}B\right] = n. \tag{1.42}$$

Proof. To simplify the notation the proof will be accomplished for $n = 4$. From (1.24) for $c_i = (-1)^{i+1}\binom{\alpha}{i}$, we have

$$\begin{aligned}
&\left[B\ \Phi_1 B\ \Phi_2 B\ \Phi_3 B\right] \\
&= \left[B\ A_\alpha B\ (A_\alpha^2 + c_2 I_n)B\ (A_\alpha^3 + 2c_2 A_\alpha + c_3 I_n)B\right] \\
&= \left[B\ A_\alpha B\ A_\alpha^2 B\ A_\alpha^3 B\right] \begin{bmatrix} I_n & 0 & c_2 I_n & c_3 I_n \\ 0 & I_n & 0 & 2c_2 I_n \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix}.
\end{aligned}$$

Hence $\operatorname{rank}\left[B\ \Phi_1 B\ \Phi_2 B\ \Phi_3 B\right] = \operatorname{rank}\left[B\ A_\alpha B\ A_\alpha^2 B\ A_\alpha^3 B\right]$, since postmultiplication of the matrix $\left[B\ A_\alpha B\ A_\alpha^2 B\ A_\alpha^3 B\right]$ by the nonsingular matrix does not change the rank of the matrix. □

Theorem 1.9. *The fractional system (1.28) is reachable if and only if one of the equivalent conditions is satisfied:*

a) The matrix $[I_n z - A_\alpha, B]$ has full rank, i.e.

$$\operatorname{rank}[I_n z - A_\alpha, B] = n, \quad \forall z \in \mathbb{C}. \tag{1.43}$$

b) The matrices $[I_n z - A_\alpha]$, B are relatively left prime or equivalently it is possible using elementary column operations (R) to reduce the matrix $[I_n z - A_\alpha, B]$ to the form $[I_n, 0]$, i.e.

$$[I_n z - A_\alpha, B] \xrightarrow{R} [I_n, 0]. \tag{1.44}$$

Proof. First we shall show that the condition (1.43) is equivalent to the condition (1.42). Let $v \in \mathbb{C}^n$ be a vector such that $v^T B = 0$ and $v^T A_\alpha = z v^T$ for $z \in \mathbb{C}$. In this case $v^T A_\alpha B = z v^T B = 0, v^T A_\alpha^2 B = z v^T A_\alpha B = 0, \ldots, v^T A_\alpha^{q-1} B = 0$ and

$$v^T\left[B\ A_\alpha B\ \ldots\ A_\alpha^{q-1}B\right] = 0. \tag{1.45}$$

From (1.45) it follows that the condition implies $v = 0$ and $v^T[I_n z - A_\alpha, B] = 0$, and this is equivalent to a (1.43). If the condition (1.42) is not satisfied then there exists a vector v satisfying (1.45) or $\operatorname{rank}[I_n z - A_\alpha, B] < n$ for $z \in \mathbb{C}$. The reduction holds if and only if the condition (1.43) is satisfied. □

Example 1.4. Using (1.42), (1.43) and (1.44) check the reachability of the system with the matrices:

$$A_\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

a) From (1.42) for $n=3$ we have

$$\operatorname{rank}\left[B\ A_\alpha B\ A_\alpha^2 B\right] = \operatorname{rank}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 7 \end{bmatrix} = 3.$$

By Theorem 1.8 the pair (A_α, B) is reachable.

b) From (1.43) we have

$$\operatorname{rank}\left[I_n z - A_\alpha, B\right] = \operatorname{rank}\left[\begin{array}{ccc|c} z & -1 & 0 & 0 \\ 0 & z & -1 & 0 \\ 1 & 2 & z+3 & 1 \end{array}\right] = 3 \quad \text{for} \quad \forall z \in \mathbb{C}.$$

Using the elementary column operations we shall show that the matrices $[I_n z - A_\alpha]$ and B are relatively left prime:

$$\left[\begin{array}{ccc|c} z & -1 & 0 & 0 \\ 0 & z & -1 & 0 \\ 1 & 2 & z+3 & 1 \end{array}\right] \xrightarrow[\substack{R[2+4\times(-2)] \\ R[1+4\times(-1)]}]{R[3+4\times(-z-3)]} \left[\begin{array}{ccc|c} z & -1 & 0 & 0 \\ 0 & z & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$
$$\xrightarrow{R[2+3\times(z)]} \left[\begin{array}{ccc|c} z & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\substack{R[2\times(-1)] \\ R[3\times(-1)]}]{R[1+2\times(z)]} \left[\,0\ I_3\,\right].$$

Therefore, by Theorem 1.9 the pair (A_α, B) is reachable.

Remark 1.6. The fractional system is reachable only if the matrix (A_α, B) has n linearly independent columns. If the matrix (A_α, B) has no n linearly independent columns then the matrix $[B, A_\alpha B, \ldots, A_\alpha^{q-1} B]$ does not have n linearly independent columns. This follows from the condition (1.43) for $z=0$.

1.7 Reachability of Positive Fractional Discrete-Time Linear Systems

Definition 1.9. A state $x_f \in \mathbb{R}_+^n$ of the positive fractional system (1.28) is called reachable in (given) q steps if there exists an input sequence $u_k \in \mathbb{R}_+^m$, for $k = 0, 1 \ldots, q-1$, which steers the state of the system from $x_0 = 0$ to the state x_f, i.e. $x_q = x_f$. If every (given) state $x_f \in \mathbb{R}_+^n$ is reachable in q steps then the positive system is called reachable in q steps. If for every state $x_f \in \mathbb{R}_+^n$ there exists a number q of steps such that the positive system (1.28) is reachable in q steps then the system is called reachable.

Definition 1.10. A square real matrix is called monomial if its every column and its every row has only one positive entry and the remaining entries are zero.

The inverse matrix of a real matrix with nonnegative entries has nonnegative entries if and only if it is a monomial matrix. The inverse matrix of monomial matrix can be found by its transposition and replacing each element of the transpose matrix by its inverse. For example the inverse matrix of the matrix

$$A = \begin{bmatrix} 0 & b & 0 \\ a & 0 & 0 \\ 0 & 0 & c \end{bmatrix}, \quad \text{has the form} \quad A^{-1} = \begin{bmatrix} 0 & \frac{1}{a} & 0 \\ \frac{1}{b} & 0 & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix},$$

where $a, b, c > 0$.

Theorem 1.10. *The positive fractional system (1.28) is reachable in q steps if and only if the matrix*

$$R_q = \begin{bmatrix} B & \Phi_1 B & \dots & \Phi_{q-1} B \end{bmatrix}, \tag{1.46}$$

contains n linearly independent monomial columns.

Proof. In a similar way as in the proof of Theorem 1.7 we obtain the equation (1.41). For given $x_f \in \mathbb{R}_+$ we can find the input sequence $u_k \in \mathbb{R}_+^m$, $k = 0, 1, \dots, q-1$ if and only if the matrix (1.46) contains n linearly independent monomial columns. □

Remark 1.7. The matrix (1.46) can not be substitute by the matrix

$$\overline{R}_q = \left[B \; A_\alpha B \; \dots \; A_\alpha^{q-1} B \right], \tag{1.47}$$

since for positive fractional systems the matrices in general case have different number of linearly independent monomial columns.

Example 1.5. Consider the fractional positive system (1.28) with the matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -\alpha & 1 \\ 1 & 0 & -\alpha \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{1.48}$$

In this case

$$[A + \alpha I_n] = \begin{bmatrix} \alpha & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \in \mathbb{R}_+^{3 \times 3}, \tag{1.49}$$

and the matrix (1.47) for $q = 3$ has the form

$$\overline{R}_3 = \left[B \; A_\alpha B \; A_\alpha^2 B \right] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

and it contains three linearly independent monomial columns but the matrix

$$R_3 = \left[B \; \Phi_1 B \; \Phi_2 B \right] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & \frac{\alpha(\alpha-1)}{2} \end{bmatrix},$$

contains only two linearly independent monomial columns.

Theorem 1.11. *The positive fractional system (1.28) is reachable only if the matrix*

$$[A+\alpha I_n, B] \tag{1.50}$$

contains n linearly independent monomial columns.

Proof. From (1.24) for $k = 0, 1, \ldots$ it can be easily shown that

$$\Phi_k = A_\alpha^k + a_{kk-1}A_\alpha^{k-1} + \cdots + a_{k1}A_\alpha + a_{k0}I_n, \tag{1.51}$$

where $a_{ki} \in \mathbb{R}$, $i = 0, 1, \ldots, k-1$.

From matrix (1.46) and the equation (1.51) it follows that the number of linearly independent monomial columns of the matrix (1.46) can not be greater than of the matrix (1.50). □

Example 1.6. Consider the fractional system (1.28) with matrices (1.37). Using (1.46) we obtain matrix

$$R_2 = \begin{bmatrix} B & \Phi_1 B \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

which has only one monomial column. By Theorem 1.10 the system with (1.37) is unreachable. However using (1.50), we obtain matrix

$$[A+\alpha I_n, B] = \left[\begin{array}{cc|c} 1+\alpha & 0 & 0 \\ 0 & 0 & 1 \end{array}\right],$$

which has two linearly independent monomial columns.

Theorem 1.12. *The positive fractional system (1.28) is reachable only if the matrix*

$$[B, (A+\alpha I_n)B], \tag{1.52}$$

contains n linearly independent monomial columns.

Proof. From (1.51) for the positive system we have

$$\Phi_k B = \sum_{i=0}^{k} a_{ki} A_\alpha^i B, \tag{1.53}$$

where $A_\alpha = A + \alpha I_n$, $a_{ki} \geq 0$, $k = 0, 1, \ldots, q-1$, $i = 0, 1, \ldots, k$.

Note that besides the matrix B, only the matrix $\Phi_1 B$ may have additional linearly independent monomial columns. The matrix (1.53) for $k = 2, 3, \ldots, q-1$ does not introduce additional linearly independent monomial columns to the matrix (1.46). □

Remark 1.8. If all m columns of the matrix B are linearly independent and monomial then the matrix (1.52) has n linearly independent monomial columns only if the matrix A_α has at least $n-m$ linearly independent monomial columns.

Example 1.7. Consider the positive fractional system (1.28) with the matrices:

$$A = \begin{bmatrix} a_{11}-\alpha & a_{12} & 1 & 0 \\ a_{21} & a_{22}-\alpha & 0 & 1 \\ a_{31} & a_{32} & -\alpha & 0 \\ a_{41} & a_{42} & 0 & -\alpha \end{bmatrix}, \quad a_{ij} \geq 0, \quad i = 1,2,3,4; \quad j = 1,2,$$

$$a) \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad b) \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Taking into account that

$$A_\alpha = \begin{bmatrix} a_{11} & a_{12} & 1 & 0 \\ a_{21} & a_{22} & 0 & 1 \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{bmatrix},$$

in the case $a)$ we obtain the matrix

$$\begin{bmatrix} B \ \Phi_1 B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

which has $n = 4$ linearly independent monomial columns. Therefore, in this case the system is reachable in $q = 2$ steps. In the case $b)$ we obtain the matrix

$$\begin{bmatrix} B \ \Phi_1 B \ \Phi_2 B \ \ldots \end{bmatrix} = \begin{bmatrix} 0 & 0 & a_{12} & \ldots \\ 0 & 1 & a_{22} & \ldots \\ 0 & 0 & a_{32} & \ldots \\ 1 & 0 & a_{42} + \frac{\alpha(1-\alpha)}{2} & \ldots \end{bmatrix},$$

which contains only two linearly independent monomial columns. By Theorem 1.10 the positive fractional system is unreachable.

It is well-knew that the observability is a dual notion. The presented considerations for the reachability of the positive fractional linear systems can be extended to the observability of this class of systems.

1.8 Controllability to Zero of the Fractional Discrete-Time Linear Systems

Definition 1.11. The fractional system (1.28) is called controllable to zero in (given) number of q steps if there exists an input sequence $u_0, u_1, \ldots, u_{q-1}$, which steers the state of the system from $x_0 \neq 0$ to the final state $x_f = 0$.

The fractional system (1.28) is called controllable to zero if there exists a natural number q such that the system is controllable to zero in q steps.

Theorem 1.13. *The fractional system (1.28) is controllable to zero in q steps if*

$$\text{rank}\begin{bmatrix} B & \Phi_1 B & \dots & \Phi_{q-1}B \end{bmatrix} = n. \tag{1.54}$$

Proof. From (1.23) for $k = q$ and $x_q = 0$ we have

$$-\Phi_q x_0 = \sum_{i=0}^{q-1} \Phi_{q-i-1} B u_i = \begin{bmatrix} B & \Phi_1 B & \dots & \Phi_{q-1}B \end{bmatrix} \begin{bmatrix} u_{q-1} \\ u_{q-2} \\ \vdots \\ u_0 \end{bmatrix}. \tag{1.55}$$

The equation (1.55) has a solution u_k, $k = 0, 1, \dots, q-1$ for arbitrary vector $\Phi_q x_0$ if the condition (1.54) is satisfied. This is a sufficient but not necessary condition for the controllability to zero since even if the condition (1.54) is not satisfied the equation (1.55) can be satisfied for arbitrary x_0, when $\Phi_q = 0$ and $u_k = 0$, $k = 0, 1, \dots, q-1$. □

Theorem 1.14. *For the controllability to zero of the fractional system (1.28) the following equality holds*

$$\text{rank}\begin{bmatrix} B & A_\alpha B & \dots & A_\alpha^{q-1} B \end{bmatrix} = \text{rank}\begin{bmatrix} B & \Phi_1 B & \dots & \Phi_{q-1}B \end{bmatrix}. \tag{1.56}$$

Proof. To simplify the notation we shall accomplished the proof for $n = 4$. From (1.24), we have

$$\begin{aligned} &\begin{bmatrix} B & \Phi_1 B & \Phi_2 B & \Phi_3 B \end{bmatrix} \\ &= \begin{bmatrix} B & A_\alpha B & (A_\alpha^2 + c_2 I_n)B & (A_\alpha^3 + 2c_2 A_\alpha + c_3 I_n)B \end{bmatrix} \\ &= \begin{bmatrix} B & A_\alpha B & A_\alpha^2 B & A_\alpha^3 B \end{bmatrix} \begin{bmatrix} I & 0 & c_2 I_n & c_3 I_n \\ 0 & I & 0 & 2c_2 I_n \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \end{aligned}$$

where $c_i = (-1)^{i+1}\binom{\alpha}{i}$.

The equality (1.55) holds since postmultiplication of the matrix

$$\begin{bmatrix} B & A_\alpha B & A_\alpha^2 B & A_\alpha^3 B \end{bmatrix}$$

by nonsingular matrix does not change its rank. □

Theorem 1.15. *The fractional system (1.28) is controllable to zero if and only if*

$$\text{rank}\begin{bmatrix} B & A_\alpha B & \dots & A_\alpha^{q-1} B & \Phi_q \end{bmatrix} = \text{rank}\begin{bmatrix} B & A_\alpha B & \dots & A_\alpha^{q-1} B \end{bmatrix}. \tag{1.57}$$

Proof. From (1.23) for $k = q$ and $x_q = 0$ we have

$$0 = x_q = \Phi_q x_0 + \sum_{i=0}^{q-1} \Phi_{q-i-1} B u_i$$
$$= \Phi_q x_0 + \begin{bmatrix} B & \Phi_1 B & \dots & \Phi_{q-1} B \end{bmatrix} \begin{bmatrix} u_{q-1} \\ u_{q-2} \\ \vdots \\ u_0 \end{bmatrix}, \tag{1.58}$$

or

$$-\Phi_q x_0 = \begin{bmatrix} B & \Phi_1 B & \dots & \Phi_{q-1} B \end{bmatrix} \begin{bmatrix} u_{q-1} \\ u_{q-2} \\ \vdots \\ u_0 \end{bmatrix}, \tag{1.59}$$

The equation (1.59) has a solution $u_i, \dots, u_{q-1}$ for arbitrary x_0 if and only if

$$\operatorname{rank} \begin{bmatrix} B & \Phi_1 B & \dots & \Phi_{q-1} B & \Phi_q \end{bmatrix} = \operatorname{rank} \begin{bmatrix} B & \Phi_1 B & \dots & \Phi_{q-1} B \end{bmatrix} = n. \tag{1.60}$$

By Theorem 1.14 the conditions (1.57) and (1.60) are equivalent. □

Remark 1.9. The condition (1.54) is only sufficient condition but not necessary for the controllability to zero of the system (1.28) since condition (1.54) implies only the condition (1.57).

Theorem 1.16. *The fractional system (1.28) is controllable to zero if and only if one of the following equivalent conditions is satisfied:*

a) *The matrix $[I_n - A_\alpha d, B]$ has full row rank, i.e.*

$$\operatorname{rank}[I_n - A_\alpha d, B] = n, \quad \forall d \in \mathbb{C}. \tag{1.61}$$

b) *The matrices $[I_n - A_\alpha d]$, B are relatively left prime or equivalent it is possible using elementary column operations (R) to reduce the matrix $[I_n - A_\alpha d, B]$ to the form $[I_n, 0]$, i.e.*

$$[I_n - A_\alpha d, B] \xrightarrow{R} [I_n, 0]. \tag{1.62}$$

Proof. The equivalence of the conditions (1.57) and (1.61) follows from Kučera theorem. The controllability to zero means that if there exists an unreachable mod then it is finite. Assume that $[I_n z - A_\alpha] = [I_n - A_\alpha d]$, where $d = z^{-1}$. Substituting $[I_n - A_\alpha d]$ instead of $[I_n z - A_\alpha]$ to the reachability condition we neglect the finite mods. Therefore, the condition (1.61) is necessary and sufficient for the controllability to zero. The reduction (1.62) can be performed if and only if the condition (1.61) is satisfied. □

Example 1.8. Check the controllability to zero of the fractional system (1.28) with the matrices:

$$A_\alpha = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & a & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{1.63}$$

a) Using (1.61), we obtain

$$\text{rank}\,[I_n - A_\alpha d, B] = \text{rank} \left[\begin{array}{ccc|c} 1 & 0 & -d & 1 \\ -d & 1 & 0 & 0 \\ 0 & -ad & 1-d & 0 \end{array}\right] = 3, \quad \forall d \in \mathbb{C}$$

By Theorem 1.16 the pair (A,B) is controllable to zero if and only if $a \neq 0$.

b) Performing the following elementary column operations we can check whether the matrices $[I_n - A_\alpha d]$ and B are relatively left prime:

$$\left[\begin{array}{ccc|c} 1 & 0 & -d & 1 \\ -d & 1 & 0 & 0 \\ 0 & -ad & 1-d & 0 \end{array}\right] \xrightarrow[R[1+2\times(d)]]{\substack{R[1+4\times(-1)] \\ R[3+4\times(d)]}} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -ad^2 & -ad & 1-d & 0 \end{array}\right]$$

$$\xrightarrow{\substack{R[1-3\times(ad+a)] \\ R[1\times(-1/a)] \\ R[2+1\times(ad)]}} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1-d & 0 \end{array}\right] \xrightarrow{\substack{R[3+1\times(d-1)] \\ R[1,4] \\ R[3,4]}} \left[\begin{array}{c|c} I_3 & 0 \end{array}\right].$$

By Theorem 1.16 the pair (A,B) is controllable to zero in $q=3$ steps.

1.9 Controllability to Zero of Positive Fractional Discrete-Time Linear Systems

Definition 1.12. The positive fractional system (1.28) is called controllable to zero in q steps if there exists an input sequence $u_i \in \mathbb{R}_+^m$, $i = 0,1,\dots,q-1$, which steers the nonzero arbitrary initial state $x_0 \in \mathbb{R}_+^n$, to the final state $x_f = 0$. The positive fractional system (1.28) is called controllable to zero if there exists a natural number $q > 0$ such that the system is controllable to zero in q steps.

Theorem 1.17. *The positive fractional system (1.28) with $B \neq 0$ is called controllable to zero in q steps if and only if*

$$\Phi_q = 0. \tag{1.64}$$

Moreover, $u_i = 0$ for $i = 0,1,\dots,q-1$.

Proof. From (1.23) for $k = q$ and $x_q = 0$ we obtain the equality (1.55). This equality for positive system can be satisfied for every x_0 if and only if the condition (1.64) is satisfied and $u_i = 0$ for $i = 0,1,\dots,q-1$. □

Lemma 1.3. *For positive fractional system (1.28) the condition (1.64) is satisfied if and only if $q = 1$ and*

$$\Phi_1 = A + \alpha I_n = 0. \tag{1.65}$$

Proof. From (1.51) with $a_{ki} \geq 0$ and Lemma 1.1 it follows that the condition (1.64) is satisfied if and only if $q = 1$ and (1.65). □

Corollary 1.1. *The positive fractional system (1.28) with $B \neq 0$ is controllable to zero if and only if $q = 1$ and the condition (1.65) is satisfied.*

1.10 Minimum Energy Control of Positive Fractional Systems

Consider the positive fractional discrete-time linear system (1.28). If the system is reachable in q steps then exist many input sequences which steer the state of the system from $x_0 = 0$ to the final state $x_f \in \mathbb{R}_+^n$. Among these sequences we are looking for a sequence $u_i \in \mathbb{R}_+^m$, $i = 0, 1, \ldots, q-1$, $i \in \mathbb{Z}_+$, which minimizes the performance index

$$I(u) = \sum_{j=0}^{q-1} u_j^T Q u_j, \tag{1.66}$$

where $Q \in \mathbb{R}^{m \times m}$ is a symmetric positive definite matrix and q is the number of steps needed to steer the system from $x_0 = 0$ to the final state $x_f \in \mathbb{R}_+^n$.

The minimum energy control problem for the positive fractional system (1.28) can be stated as follows [102]: Given the matrices A, B, degree α of the system (1.28), number of steps q, the finite state $x_f \in \mathbb{R}_+^n$ and the matrix Q of (1.66). Find an input sequence $u_i \in \mathbb{R}_+^m$, $i = 0, 1, \ldots, q-1$, which steers the state of the system from $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$ and minimizes the performance index (1.66).

To solve the problem we define the matrix

$$W = W(q, Q) = R_q \overline{Q} R_q^T \in \mathbb{R}^{n \times n}, \tag{1.67}$$

where R_q is given by (1.46) and

$$\overline{Q} = \text{block diag}\left[\, Q^{-1} \;\ldots\; Q^{-1} \,\right] \in \mathbb{R}^{qm \times qm}. \tag{1.68}$$

From (1.67) it follows that the matrix W is nonsingular if and only if $\operatorname{rank} R_q = n$. If this condition is satisfied then the system is reachable in q steps. In this case we can define for given $x_f \in \mathbb{R}_+^n$ the following input sequence

$$\hat{u}_{0q} = \begin{bmatrix} \hat{u}_{q-1} \\ \hat{u}_{q-2} \\ \vdots \\ \hat{u}_0 \end{bmatrix} = \overline{Q} R_q^T W^{-1} x_f. \tag{1.69}$$

From (1.69) it follows that $\hat{u}_i \in \mathbb{R}_+^m$ for $i = 0, 1, \ldots, q-1$ if

$$\overline{Q} R_q^T W^{-1} \in \mathbb{R}_+^{m \times n}, \tag{1.70}$$

and this implies

$$Q^{-1} \in \mathbb{R}_+^{m\times m}, \quad W^{-1} \in \mathbb{R}_+^{n\times n}. \tag{1.71}$$

Theorem 1.18. *Let the fractional system (1.28) be reachable in q steps and the condition (1.71) be satisfied. Moreover let* $\overline{u}_i \in \mathbb{R}_+^m$, $i = 0,1,\dots,q-1$ *be an input sequence which steers the state of the system from* $x_0 = 0$ *to the final state* $x_f \in \mathbb{R}_+^n$. *The input sequence* $\hat{u}_i \in \mathbb{R}_+^m$, $i = 0,1,\dots,q-1$ *also steers the state of the system from* $x_0 = 0$ *to* $x_f \in \mathbb{R}_+^n$ *and minimizes the performance index (1.66), i.e.*

$$I(\hat{u}) \le I(\overline{u}). \tag{1.72}$$

The minimal value of the performance index (1.66) for (1.69) is

$$I(\hat{u}) = x_f^T W^{-1} x_f. \tag{1.73}$$

Proof. If the fractional system (1.28) is positive, reachable in q steps and the assumption (1.71) are satisfied then for $x_f \in \mathbb{R}_+^n$ we have $\hat{u}_i \in \mathbb{R}_+^m$ for $i = 0,1,\dots,q-1$. We shall show that the input sequence (1.69) steers the state of the system from $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$. Using (1.23) for $k = q$, $x_0 = 0$ and (1.67), (1.69), we obtain

$$x_q = R_q \hat{u}_{0q} = R_q \overline{Q} R_q^T W^{-1} x_f = x_f,$$

since $R_q \overline{Q} R_q^T W^{-1} = I_n$.

Both input sequences $\overline{u}_{0q}$ and $\hat{u}_{0q}$ steer the state of the system from $x_0 = 0$ to $x_f \in \mathbb{R}_+^n$ and we have $x_f = R_q \hat{u}_{0q} = R_q \overline{u}_{0q}$, i.e.

$$R_q \left[\hat{u}_{0q} - \overline{u}_{0q}\right] = 0. \tag{1.74}$$

Using (1.74) and (1.69), we shall show

$$\left[\hat{u}_{0q} - \overline{u}_{0q}\right]^T \hat{Q} \hat{u}_{0q} = 0, \tag{1.75}$$

where $\hat{Q} = \text{block diag}[Q,\dots,Q]$.

Transposing (1.74) and postmultiplying it by $W^{-1}x_f$ we obtain

$$\left[\hat{u}_{0q} - \overline{u}_{0q}\right]^T R_q^T W^{-1} x_f = 0. \tag{1.76}$$

Using (1.69) and (1.76), we obtain (1.75), since

$$\left[\hat{u}_{0q} - \overline{u}_{0q}\right]^T \hat{Q} \hat{u}_{0q} = \left[\hat{u}_{0q} - \overline{u}_{0q}\right]^T \hat{Q}\overline{Q} R_q^T W^{-1} x_f = \left[\hat{u}_{0q} - \overline{u}_{0q}\right]^T R_q^T W^{-1} x_f = 0,$$

where $\hat{Q}\overline{Q} = I_{qm}$.

Using (1.75) it is easy to show that

$$\overline{u}_{0q}^T \overline{Q} \overline{u}_{0q} = \hat{u}_{0q}^T \hat{Q} \hat{u}_{0q} + \left[\overline{u}_{0q} - \hat{u}_{0q}\right]^T \hat{Q} \left[\overline{u}_{0q} - \hat{u}_{0q}\right]. \tag{1.77}$$

From(1.77) it follows that the inequality (1.72), is satisfied since

$$\left[\overline{u}_{0q} - \hat{u}_{0q}\right]^T \hat{Q} \left[\overline{u}_{0q} - \hat{u}_{0q}\right] \geq 0.$$

To find the minimal value of the performance index we substitute (1.69) into (1.66) and using (1.67), we obtain

$$\begin{aligned} I(\hat{u}) &= \hat{u}_{0q}^T \hat{Q} \hat{u}_{0q} = \left[\overline{Q} R_q^T W^{-1} x_f\right]^T \hat{Q} \left[\overline{Q} R_q^T W^{-1} x_f\right] \\ &= x_f^T W^{-1} R_q \overline{Q} R_q^T W^{-1} x_f = x_f^T W^{-1} x_f, \end{aligned}$$

since $W^{-1} R_q \overline{Q} R_q^T = I_n$. □

If the assumption of Theorem 1.18 are satisfied then the minimal energy control problem can be solved by the use of the following procedure.

Procedure 1.1

Step 1. Knowing the matrices A, B, Q and α, q find the matrices R_q and $\overline{Q}$, using (1.46) and (1.68).
Step 2. Knowing R_q and $\overline{Q}$, and using (1.67) find the matrix W.
Step 3. Using (1.69), find the input sequence $\hat{u}_0, \hat{u}_1, \ldots, \hat{u}_{q-1}$.
Step 4. Using (1.73), find the value of $I(\hat{u})$.

Example 1.9. Consider the fractional system (1.28) for $0 < \alpha < 1$ with the matrices:

$$A = \begin{bmatrix} -\alpha & 0 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad n = 2. \tag{1.78}$$

Find the optimal input sequence which steers the state of the system from $x_0 = 0$ to the final state $x_f = [1\ 1]^T$ in $q = 2$ steps and minimizes the performance index (1.66) for $Q = [2]$.

The fractional system (1.28) with (1.78) is reachable in $q = 2$ steps. It is easy to check that the assumption of Theorem 1.18 are satisfied. Using Procedure 1.1 we obtain the following:

Step 1. In this case

$$R_2 = \begin{bmatrix} B & \Phi_1 B \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$\overline{Q} = \operatorname{diag}\begin{bmatrix} Q^{-1} & Q^{-1} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Step 2. Using (1.67), we obtain

$$W = R_2 \overline{Q} R_2^T = \overline{Q} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Step 3. Using (1.69), we obtain

$$\hat{u}_{02} = \begin{bmatrix} \hat{u}_1 \\ \hat{u}_0 \end{bmatrix} = \overline{Q} R_2^T W^{-1} x_f = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{1.79}$$

It is easy to check that the input sequence (1.79) steers the state of system in $q = 2$ steps from $x_0 = 0$ to $x_f = [1\ 1]^T$.

Step 4. In this case the minimal value of the performance index is

$$I(\hat{u}) = x_f^T W^{-1} x_f = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4. \tag{1.80}$$

Corollary 1.2. *Note that in the case of positive fractional system by suitable choice of state-feedback we may modify the reachability matrix R_q, and the minimal value of the performance index.*

1.11 Fractional Different Orders Discrete-Time Linear Systems

Consider the fractional different orders discrete-time linear system

$$\Delta^\alpha x_1(k+1) = A_{11}x_1(k) + A_{12}x_2(k) + B_1u(k), \quad k \in \mathbb{Z}_+, \tag{1.81a}$$

$$\Delta^\beta x_2(k+1) = A_{21}x_1(k) + A_{22}x_2(k) + B_2u(k), \tag{1.81b}$$

where $x_1(k) \in \mathbb{R}^{n_1}$, $x_2(k) \in \mathbb{R}^{n_2}$, $u(k) \in \mathbb{R}^m$ are state and input vectors, respectively and $A_{ij} \in \mathbb{R}^{n_i \times n_j}$, $B_i \in \mathbb{R}^{n_i \times m}$, $i, j = 1, 2$.

The fractional derivative of α order is defined by

$$\Delta^\alpha x(k) = \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} x(k-j) = \sum_{j=0}^{k} c_\alpha(j) x(k-j), \tag{1.82a}$$

$$c_\alpha(j) = (-1)^j \binom{j}{\alpha} = (-1)^j \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!}, \tag{1.82b}$$

$$c_\alpha(0) = 1, \quad j = 1, 2, \dots .$$

Using (1.82) can write the equation (1.81) in the form

$$x_1(k+1) = A_{1\alpha}x_1(k) + A_{12}x_2(k) - \sum_{j=2}^{k+1} c_\alpha(j) x_1(k-j+1) + B_1u(k), \tag{1.83a}$$

$$x_2(k+1) = A_{21}x_1(k) + A_{2\beta}x_2(k) - \sum_{j=2}^{k+1} c_\beta(j) x_2(k-j+1) + B_2u(k). \tag{1.83b}$$

where $A_{1\alpha} = A_{11} + \alpha I_{n_1}$, $A_{2\beta} = A_{22} + \beta I_{n_2}$.

Applying to (1.83) the z-transform we obtain

$$\begin{bmatrix} I_{n_1}z - A_{1\alpha} + \sum_{j=2}^{k+1} c_\alpha(j) I_{n_1} z^{-j+1} & -A_{12} \\ -A_{21} & I_{n_2}z - A_{2\beta} + \sum_{j=2}^{k+1} I_{n_2} c_\beta(j) z^{-j+1} \end{bmatrix} \times \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix} = \begin{bmatrix} zx_{10} \\ zx_{20} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(z), \tag{1.84}$$

where $X_i(z) = \mathscr{Z}[x_i(k)] = \sum_{k=0}^{\infty} x_i(k) z^{-k}$, $i = 1,2$; $U(z) = \mathscr{Z}[u(k)]$ and $x_{10} = x_1(0)$, $x_{20} = x_2(0)$.

From (1.84) we have

$$\begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix} = \begin{bmatrix} I_{n_1}z - A_{1\alpha} + \sum_{j=2}^{k+1} c_\alpha(j) I_{n_1} z^{-j+1} & -A_{12} \\ -A_{21} & I_{n_2}z - A_{2\beta} + \sum_{j=2}^{k+1} I_{n_2} c_\beta(j) z^{-j+1} \end{bmatrix}^{-1} \times \left\{ \begin{bmatrix} zx_{10} \\ zx_{20} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(z) \right\}. \tag{1.85}$$

Let

$$\begin{bmatrix} I_{n_1}z - A_{1\alpha} + \sum_{j=2}^{k+1} c_\alpha(j) I_{n_1} z^{-j+1} & -A_{12} \\ -A_{21} & I_{n_2}z - A_{2\beta} + \sum_{j=2}^{k+1} I_{n_2} c_\beta(j) z^{-j+1} \end{bmatrix}^{-1} = \sum_{j=0}^{\infty} \Phi_j z^{-(j+1)}, \tag{1.86}$$

where the matrices Φ_k are defined by

$$\Phi_i = \begin{cases} I_n \quad (n = n_1 + n_2) & \text{for} \quad i = 0 \\ A\Phi_{i-1} - D_1\Phi_{i-2} - \cdots - D_{i-1}\Phi_0 & \text{for} \quad i = 1,2,\ldots,k \\ A\Phi_{i-1} - D_1\Phi_{i-2} - \cdots - D_k\Phi_{i-k-1} & \text{for} \quad i = k+1, k+2, \ldots \end{cases}. \tag{1.87}$$

From definition of inverse matrix we have

$$\left[I_n z - A - D_1 z^{-1} - D_2 z^{-2} - \cdots - D_k z^{-k} \right] \left[\Phi_0 z^{-1} + \Phi_1 z^{-2} + \Phi_2 z^{-3} + \cdots \right] = I_n, \tag{1.88}$$

where

$$A = \begin{bmatrix} A_{1\alpha} & A_{12} \\ A_{21} & A_{2\beta} \end{bmatrix}, \quad D_k = \begin{bmatrix} c_\alpha(k+1) I_{n_1} & 0 \\ 0 & c_\beta(k+1) I_{n_2} \end{bmatrix}. \tag{1.89}$$

Comparison of the coefficient at the same power of z^{-1} we obtain

$$\Phi_0 = I_n, \quad \Phi_1 = A\Phi_0, \quad \Phi_2 = A\Phi_1 - D_1\Phi_0, \\ \Phi_3 = A\Phi_2 - D_1\Phi_1 - D_2\Phi_0, \ldots \; . \tag{1.90}$$

which can be written in the form (1.87).

Substitution of (1.86) into (1.85) yields

$$\begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix} = \sum_{j=0}^{\infty} \Phi_j z^{-j} \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} + \sum_{j=0}^{\infty} \Phi_j z^{-(j+1)} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(z). \tag{1.91}$$

Applying the inverse z-transform and the convolution theorem to (1.91) we obtain

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \Phi_k \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} + \sum_{i=0}^{k-1} \Phi_{k-i-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_i. \tag{1.92}$$

Therefore, the following theorem has been proved.

Theorem 1.19. *The solution to the fractional equation (1.81) with initial conditions $x_1(0) = x_{10}$, $x_2(0) = x_{20}$ is given by (1.92), where Φ_k is defined by (1.87).*

1.12 Positive Fractional Different Orders Discrete-Time Linear Systems

Consider the fractional different orders discrete-time linear systems described by the equation (1.81) and

$$y(k) = C \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + Du(k), \tag{1.93}$$

where $x_1(k) \in \mathbb{R}^{n_1}$, $x_2(k) \in \mathbb{R}^{n_2}$, $u(k) \in \mathbb{R}^m$, $y(k) \in \mathbb{R}^p$ are the state, input and output vectors and $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Definition 1.13. The fractional system (1.81), (1.93) is called positive if $x_1(k) \in \mathbb{R}_+^{n_1}$, $x_2(k) \in \mathbb{R}_+^{n_2}$, $y(k) \in \mathbb{R}_+^p$ for any initial conditions $x_{10} \in \mathbb{R}_+^{n_1}$, $x_{20} \in \mathbb{R}_+^{n_2}$ and all inputs $u(k) \in \mathbb{R}_+^m$ for $k \in \mathbb{Z}_+$.

Theorem 1.20. *The fractional discrete-time linear system (1.81), (1.93) with $0 < \alpha < 1$, $0 < \beta < 1$ is positive if and only if*

$$A = \begin{bmatrix} A_{1\alpha} & A_{12} \\ A_{21} & A_{2\beta} \end{bmatrix} \in \mathbb{R}_+^{n\times n}, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \mathbb{R}_+^{n\times m}, \quad C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}. \tag{1.94}$$

Proof. Necessity. Let $e_i^{n_j}$ be i-th column of the $n_j \times n_j$ identity matrix, $j = 1,2$. From (1.83) for $k = 0$, $u(0) = 0$, $x_{20} = 0$ and $x_{10} = e_i^{n_1}$ we have $x_1(1) = A_{1\alpha} e_i^{n_1} \in \mathbb{R}_+^{n_1}$ and $x_2(1) = A_{21} e_i^{n_1} \in \mathbb{R}_+^{n_1}$. This implies the nonnegativity of i-th ($i = 1,\dots,n$) columns of the matrices $A_{1\alpha}$ and A_{21}. Similarly for $k = 0$, $u(0) = 0$, $x_{10} = 0$ and $x_{20} = e_i^{n_2}$ we have $x_1(1) = A_{12} e_i^{n_2} \in \mathbb{R}_+^{n_2}$ and $x_2(1) = A_{2\beta} e_i^{n_2} \in \mathbb{R}_+^{n_2}$. To show that $B_1 \in \mathbb{R}_+^{n_1\times m}$ and $B_2 \in \mathbb{R}_+^{n_2\times m}$ we assume in (1.83) for $k = 0$, $x_1(0) = 0$, $x_2(0) = 0$ and $u(0) = e_i^m$ and we obtain $x_1(0) = B_1 e_i^m \in \mathbb{R}_+^{n_1}$ and $x_2(0) = B_2 e_i^m \in \mathbb{R}_+^{n_2}$. In a similar way we prove $C \in \mathbb{R}_+^{p\times n}$ and $D \in \mathbb{R}_+^{p\times m}$.

Sufficiency. In Lemma 1.1 was shown that if $0 < \alpha < 1$ and $0 < \beta < 1$ then $c_\alpha(j) < 0$ and $c_\beta(j) < 0$ for $j = 2, \dots, k+1$. From (1.89) if follows that $D_i \in \mathbb{R}_+^n$ for $i = 1, \dots, n$ and from(1.87) we have $\Phi_i \in \mathbb{R}_+^{n \times n}$ for $i = 0, 1 \dots$ since $A \in \mathbb{R}_+^{n \times n}$. From (1.92) we have $x_1(k) \in \mathbb{R}_+^{n_1}$, $x_2(k) \in \mathbb{R}_+^{n_2}$, $k \in \mathbb{Z}_+$ since $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \mathbb{R}_+^{n \times m}$ and $u(i) \in \mathbb{R}_+^m$, $i \in \mathbb{Z}_+$. Finaly from (1.93) we have $y(k) \in \mathbb{R}_+^p$, $k \in \mathbb{Z}_+$ since $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$, $x_1(k) \in \mathbb{R}_+^{n_1}$, $x_2(k) \in \mathbb{R}_+^{n_2}$ and $u(k) \in \mathbb{R}_+^m$, $k \in \mathbb{Z}_+$. □

These considerations can be easy extended to fractional system consisting of n subsystems of different fractional order [161].

Chapter 2
Fractional Continuous-Time Linear Systems

2.1 Definition of Euler Gamma Function and Its Properties

There exist the following two definitions of the Euler gamma function.

Definition 2.1. A function given by the integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad \mathrm{Re}(x) > 0, \tag{2.1}$$

is called the Euler gamma function.

The Euler gamma function can be also defined by

$$\Gamma(x) = \lim_{n\to\infty} \frac{n! n^x}{x(x+1)\cdots(x+n)}, \quad x \in \mathbb{C}\backslash\{0,-1,-2,\ldots\} \tag{2.2}$$

where $\mathbb{C}$ is the field of complex numbers.

We shall show that $\Gamma(x)$ satisfies the equality

$$\Gamma(x+1) = x\Gamma(x). \tag{2.3}$$

Proof. Using (2.1), we obtain

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt = \int_0^\infty t^x de^{-t} = t^x e^{-t}\Big|_0^\infty = x\int_0^\infty t^{x-1} e^{-t} dt = x\Gamma(x).$$

□

Example 2.1. From (2.3) we have for:

$$\begin{aligned}
x = 1: &\quad \Gamma(2) = 1\cdot\Gamma(1) = 1, \quad \text{since} \quad \Gamma(1) = \int_0^\infty e^{-t} dt = 1, \\
x = 2: &\quad \Gamma(3) = 2\cdot\Gamma(2) = 1\cdot 2 = 2!, \\
x = 3: &\quad \Gamma(4) = 3\cdot\Gamma(3) = 3\cdot 2\cdot\Gamma(2) = 3!.
\end{aligned}$$

T. Kaczorek: Selected Problems of Fractional Systems Theory, LNCIS 411, pp. 27–52.
springerlink.com

In general case for $x \in \mathbb{N}$ we have

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\cdots(1) = n!$$

The gamma function is also well-define for x being any real (complex) numbers. For example we have for

$$\begin{aligned} x = 1.5 \ : \qquad & \Gamma(2.5) = 1.5 \cdot \Gamma(1.5) = 1.5 \cdot 0.5\Gamma(0.5), \\ x = -0.5 : \qquad & \Gamma(0.5) = -0.5 \cdot \Gamma(-0.5) = -0.5 \cdot (-1.5)\Gamma(-1.5). \end{aligned}$$

2.2 Mittag-Leffler Function

The Mittag-Leffler function is a generalization of the exponential function $e^{s_i t}$ and it plays important role in solution of the fractional differential equations.

Definition 2.2. A function of the complex variable z defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \tag{2.4}$$

is called the one parameter Mittag-Leffler function.

Example 2.2. For $\alpha = 1$ we obtain

$$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z,$$

i.e. the classical exponential function.

An extension of the one parameter Mittag-Leffler function is the following two parameters function.

Definition 2.3. A function of the complex variable z defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \tag{2.5}$$

is called two parameters Mittag-Leffler function.

For $\beta = 1$ from (2.5) we obtain (2.4).

2.3 Definitions of Fractional Derivative-Integral

2.3.1 Riemann-Liouville Definition

It is well known that to reduce n-multiple integral to 1-tiple integral the following formula

$$
{}_0I_x^n = \int_0^x \int_0^{u_1} \cdots \int_0^{u_{n-1}} f(u_n)du_n \cdots du_2 du_1 = \frac{1}{(n-1)!}\int_0^x (x-u)^{n-1} f(u)du, \quad (2.6)
$$

can be used, where $f(u)$ is a given function. Using the equality $(n-1)! = \Gamma(n)$, the formula (2.6) can be extended for any $n \in \mathbb{R}$ and we obtain Riemann-Liouville fractional integral

$$
{}_0I_t^\alpha = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \quad (2.7)
$$

where $\alpha \in \mathbb{R}_+$ is the order of integral.

Definition 2.4. The function defined by

$$
\begin{aligned}
{}_0^{RL}D_t^\alpha f(t) &= \frac{d^\alpha}{dt^\alpha} f(t) = \frac{d^n}{dt^n}\left[{}_0I_t^{(n-\alpha)} f(t)\right] \\
&= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau, \quad (2.8)
\end{aligned}
$$

is called Riemann-Liouville fractional derivative-integral, where $n-1 \leq \alpha \leq n$, $n \in \mathbb{N}$.

Example 2.3. Consider the unit-step function

$$
f(t) = \mathbb{1}(t) = \begin{cases} 1 & \text{for} \quad t \geq 0 \\ 0 & \text{for} \quad t < 0 \end{cases} .
$$

Using (2.8), we obtain

$$
\begin{aligned}
\frac{d^\alpha}{dt^\alpha}\mathbb{1}(t) &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1} d\tau \\
&= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\left[\frac{-1}{n-\alpha}(t-\tau)^{n-\alpha}\right]_0^t = \frac{1}{\Gamma(n-\alpha)}\frac{1}{n-\alpha}\frac{d^n}{dt^n} t^{n-\alpha} \\
&= \frac{1}{\Gamma(n-\alpha)}\frac{1}{n-\alpha}(n-\alpha)(n-\alpha-1)\cdots(1-\alpha)t^{-\alpha} = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}.
\end{aligned}
$$

Therefore, the α order Riemann-Liouville derivative of unit-step function is a decreasing in time function.

Theorem 2.1. *The Riemann-Liouville derivative-integral operator is linear satisfying the relation*

$$
{}_0^{RL}D_t^\alpha[\lambda f(t) + \mu g(t)] = \lambda\, {}_0^{RL}D_t^\alpha f(t) + \mu\, {}_0^{RL}D_t^\alpha g(t), \quad \lambda, \mu \in \mathbb{R}. \quad (2.9)
$$

Proof.

$$
\begin{aligned}
{}_{0}^{RL}D_{t}^{\alpha}(\lambda f(t)+\mu g(t)) &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1}[\lambda f(\tau)+\mu g(\tau)]d\tau \\
&= \frac{\lambda}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau \\
&+ \frac{\mu}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1} g(\tau)d\tau \\
&= \lambda\, {}_{0}^{RL}D_{t}^{\alpha} f(t) + \mu\, {}_{0}^{RL}D_{t}^{\alpha} g(t).
\end{aligned}
$$

□

Theorem 2.2. *The Laplace transform of the derivative-integral (2.8) has the form*

$$
\mathscr{L}\left[{}_{0}^{RL}D_{t}^{\alpha} f(t)\right] = s^{\alpha}F(s) - \sum_{k=1}^{n} s^{k-1} f^{(\alpha-k)}(0^{+}). \tag{2.10}
$$

Proof. Using the definition given in Appendix A.2 we obtain

$$
\begin{aligned}
\mathscr{L}\left[{}_{0}^{RL}D_{t}^{\alpha} f(t)\right] &= \mathscr{L}\left[\frac{d^n}{dt^n}\left(\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau\right)\right] \\
&= \mathscr{L}\left[\frac{d^n}{dt^n}\left({}_{0}I_{t}^{n-\alpha} f(t)\right)\right] \\
&= \frac{s^n F(s)}{s^{n-\alpha}} - \sum_{k=1}^{n} s^{n-k}\frac{d^{k-1}}{dt^{k-1}}\left[{}_{0}I_{t}^{n-\alpha} f(t)\right].
\end{aligned}
$$

□

2.3.2 *Caputo Definition*

Definition 2.5. The function defined by

$$
{}_{0}^{C}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau, \tag{2.11}
$$

is called the Caputo fractional derivative-integral, where $n-1<\alpha<n$, $n\in\mathbb{N}$.

Remark 2.1. From definition 2.5 it follows that the Caputo derivative of constant is equal to zero.

Theorem 2.3. *The Caputo derivative-integral operator is linear satisfying the relation*

$$
{}_{0}^{C}D_{t}^{\alpha}[\lambda f(t)+\mu g(t)] = \lambda\, {}_{0}^{C}D_{t}^{\alpha} f(t) + \mu\, {}_{0}^{C}D_{t}^{\alpha} g(t). \tag{2.12}
$$

Proof. The proof is similar to the proof of Theorem 2.1. □

Theorem 2.4. *The Laplace transform of the derivative-integral (2.11) has the form*

$$\mathscr{L}\left[{}_0^C D_t^\alpha f(t)\right] = s^\alpha F(s) - \sum_{k=1}^{n} s^{\alpha-k} f^{(k-1)}(0^+). \tag{2.13}$$

Proof. Using the definition given in Appendix A.2, we obtain

$$\begin{aligned}
\mathscr{L}\left[{}_0^C D_t^\alpha f(t)\right] &= \mathscr{L}\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau\right] \\
&= \frac{1}{\Gamma(n-\alpha)}\mathscr{L}\left[t^{n-\alpha-1}\right]\mathscr{L}\left[f^{(n)}(t)\right] \\
&= \frac{1}{\Gamma(n-\alpha)}\frac{\Gamma(n-\alpha)}{s^{n-\alpha}}\left[s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0^+)\right] \\
&= s^\alpha F(s) - \sum_{k=1}^{n} s^{\alpha-k} f^{(k-1)}(0^+)
\end{aligned}$$

□

2.4 Solution of the Fractional State Equation of Continuous-Time Linear System

Consider the continuous-time linear system described by the equation [100]:

$$_0D_t^\alpha x(t) = \frac{d^\alpha x(t)}{dt^\alpha} = Ax(t) + Bu(t), \quad 0 < \alpha \le 1, \tag{2.14a}$$

$$y(t) = Cx(t) + Du(t), \tag{2.14b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are state, input and output vectors and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Theorem 2.5. *The solution of the equation (2.14a) has the form*

$$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau, \quad x(0) = x_0, \tag{2.15}$$

where

$$\Phi_0(t) = E_\alpha(At^\alpha) = \sum_{k=0}^{\infty}\frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \tag{2.16}$$

$$\Phi(t) = \sum_{k=0}^{\infty}\frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}. \tag{2.17}$$

$E_\alpha(At^\alpha)$ is the Mittag-Leffler function and $\Gamma(x)$ is the Euler gamma function.

Proof. Applying the Laplace transform to (2.14a) and taking in to account

$$X(s) = \mathscr{L}[x(t)] = \int_0^\infty x(t)e^{-st}dt, \tag{2.18a}$$

$$\mathscr{L}\left[D^\alpha x(t)\right] = s^\alpha X(s) - s^{\alpha-1}x_0, \tag{2.18b}$$

we obtain

$$X(s) = [I_n s^\alpha - A]^{-1}\left[s^{\alpha-1}x_0 + BU(s)\right], \quad U(s) = \mathscr{L}[u(t)]. \tag{2.19}$$

It is easy to show that

$$[I_n s^\alpha - A]^{-1} = \sum_{k=0}^{\infty} A^k s^{-(k+1)\alpha}, \tag{2.20}$$

since

$$[I_n s^\alpha - A]\left(\sum_{k=0}^{\infty} A^k s^{-(k+1)\alpha}\right) = I_n. \tag{2.21}$$

Substituting of (2.20) to (2.19), yields

$$X(s) = \sum_{k=0}^{\infty} A^k s^{-(k\alpha+1)}x_0 + \sum_{k=0}^{\infty} A^k s^{-(k+1)\alpha}BU(s). \tag{2.22}$$

Using the inverse Laplace transform and the convolution theorem (Appendix A.1) to (2.22) we obtain

$$\begin{aligned} x(t) &= \mathscr{L}^{-1}[X(s)] = \sum_{k=0}^{\infty} A^k \mathscr{L}^{-1}\left[s^{-(k\alpha+1)}\right]x_0 + \sum_{k=0}^{\infty} A^k \mathscr{L}^{-1}\left[s^{-(k+1)\alpha}BU(s)\right] \\ &= \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau, \end{aligned} \tag{2.23}$$

where

$$\begin{aligned} \Phi_0(t) &= \sum_{k=0}^{\infty} A^k \mathscr{L}^{-1}\left[s^{-(k\alpha+1)}\right] = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \\ \Phi(t) &= \mathscr{L}^{-1}\left\{[I_n s^\alpha - A]^{-1}\right\} = \sum_{k=0}^{\infty} A^k \mathscr{L}^{-1}\left[s^{-(k+1)\alpha}\right] = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}. \end{aligned}$$

□

Remark 2.2. From (2.16) and (2.17) for $\alpha = 1$ mamy

$$\Phi_0(t) = \Phi(t) = \sum_{k=0}^{\infty} \frac{At^k}{\Gamma(k+1)} = e^{At}.$$

Remark 2.3. From classical Cayley-Hamilton theorem it follows that if

$$\det\left[I_n s^\alpha - A\right] = (s^\alpha)^n + a_{n-1}(s^\alpha)^{n-1} + \cdots + a_1 s^\alpha + a_0, \tag{2.24}$$

then

$$A^n + a_{n-1}(A)^{n-1} + \cdots + a_1 A^\alpha + a_0 I_n = 0. \tag{2.25}$$

Example 2.4. Find the solution of the equation (2.14a) for $0 < \alpha \leq 1$ and:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad u(t) = \mathbb{1}(t) \tag{2.26}$$

Using (2.16) and (2.17), we obtain:

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)} = I_2 + \frac{At^\alpha}{\Gamma(\alpha+1)}, \tag{2.27a}$$

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} = I_2 \frac{t^{\alpha-1}}{\Gamma(\alpha)} + A\frac{t^{2\alpha-1}}{\Gamma(2\alpha)}, \tag{2.27b}$$

Substituting (2.27) and $u(t) = 1$ into (2.15), we obtain

$$\begin{aligned} x(t) &= x_0 + \frac{Ax_0 t^\alpha}{\Gamma(\alpha+1)} + \int_0^\infty \left(\frac{B}{\Gamma(\alpha)}(t-\tau)^{\alpha-1} + \frac{AB}{\Gamma(2\alpha)}(t-\tau)^{2\alpha-1} \right) d\tau \\ &= x_0 + \frac{Ax_0 t^\alpha}{\Gamma(\alpha+1)} + \frac{Bt^\alpha}{\Gamma(\alpha+1)} + \frac{ABt^{2\alpha}}{\Gamma(2\alpha+1)} = \begin{bmatrix} 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ 1 + \frac{t^\alpha}{\Gamma(\alpha+1)} \end{bmatrix}, \end{aligned}$$

where $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

Theorem 2.6. *The solution of the equation (2.14a) for $n-1 \leq \alpha \leq n$ and Caputo definition has the form*

$$x(t) = \sum_{l=1}^{n} \Phi_l(t) x^{(l-1)}(0^+) + \int_0^t \Phi(t-\tau) Bu(\tau) d\tau, \tag{2.28}$$

where

$$\Phi_l(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k\alpha+l)-1}}{\Gamma(k\alpha+l)}, \quad \Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}.$$

Proof. Taking into account (A.1), (2.13) from (2.14a) we obtain:

$$X(s) = \left[I_n s^\alpha - A\right]^{-1} \left[\sum_{l=1}^{n} s^{\alpha-l} x^{(l-1)}(0^+) + BU(s) \right], \quad U(s) = \mathscr{L}[u(t)]. \tag{2.29}$$

Substituting of (2.20) into (2.29), yields

$$X(s) = \sum_{k=0}^{\infty} A^k s^{-(k+1)\alpha} \left[\sum_{l=1}^{n} s^{\alpha-l} x^{(l-1)}(0^+) + BU(s) \right]$$
$$= \sum_{k=0}^{\infty} \sum_{l=1}^{n} A^k s^{-(k\alpha+l)} x^{(l-1)}(0^+) + \sum_{k=0}^{\infty} A^k s^{-(k+1)\alpha} BU(s). \tag{2.30}$$

Applying the inverse Laplace transform and the convolution theorem (Appendix A.1) to (2.30), we obtain

$$x(t) = \sum_{k=0}^{\infty} \sum_{l=1}^{n} A^k \mathscr{L}^{-1} \left[s^{-(k\alpha+l)} \right] x^{(l-1)}(0^+) + \sum_{k=0}^{\infty} A^k \mathscr{L}^{-1} \left[s^{-(k+1)\alpha} BU(s) \right]$$
$$= \sum_{l=1}^{n} \Phi_l(t) x^{(l-1)}(0^+) + \int_0^t \Phi(t-\tau) Bu(\tau) d\tau, \tag{2.31}$$

where

$$\Phi_l(t) = \sum_{k=0}^{\infty} A^k \mathscr{L}^{-1} \left[s^{-(k\alpha+l)} \right] = \sum_{k=0}^{\infty} \frac{A^k t^{(k\alpha+l)-1}}{\Gamma(k\alpha+l)},$$
$$\Phi(t) = \sum_{k=0}^{\infty} A^k \mathscr{L}^{-1} \left[s^{-(k+1)\alpha} \right] = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}.$$

□

Theorem 2.7. *The solution of the equation (2.14a) for $n-1 \le \alpha \le n$ and the Riemann-Liouville definition has form*

$$x(t) = \sum_{l=1}^{n} \Phi_l(t) x^{(\alpha-l)}(0^+) + \int_0^t \Phi(t-\tau) Bu(\tau) d\tau, \tag{2.32}$$

where

$$\Phi_l(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-l}}{\Gamma[(k+l)\alpha-l+1]}, \quad \Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}.$$

Proof. Taking into account (A.1) and (2.10), from (2.14a) we obtain:

$$X(s) = [I_n s^\alpha - A]^{-1} \left[\sum_{l=1}^{n} s^{l-1} x^{(\alpha-l)}(0^+) + BU(s) \right], \quad U(s) = \mathscr{L}[u(t)]. \tag{2.33}$$

Substituting of (2.20) to (2.33), yields

$$X(s) = \sum_{k=0}^{\infty} A^k s^{-(k+1)\alpha} \left[\sum_{l=1}^{n} s^{l-1} x^{(\alpha-l)}(0^+) + BU(s) \right]$$
$$= \sum_{k=0}^{\infty} \sum_{l=1}^{n} A^k s^{-(k+1)\alpha+l-1} x^{(\alpha-l)}(0^+) + \sum_{k=0}^{\infty} A^k s^{-(k+1)\alpha} BU(s). \tag{2.34}$$

Applying the inverse Laplace transform and the convolution theorem (Appendix A.1) to (2.34), we obtain

$$\begin{aligned} x(t) &= \sum_{k=0}^{\infty}\sum_{l=1}^{n} A^k \mathscr{L}^{-1}\left[s^{-(k+1)\alpha+l-1}\right] x^{(\alpha-l)}(0^+) \\ &+ \sum_{k=0}^{\infty} A^k \mathscr{L}^{-1}\left[s^{-(k+1)\alpha} BU(s)\right] \\ &= \sum_{l=1}^{n} \Phi_l(t) x^{(\alpha-l)}(0^+) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau, \end{aligned} \tag{2.35}$$

where

$$\begin{aligned} \Phi_l(t) &= \sum_{k=0}^{\infty} A^k \mathscr{L}^{-1}\left[s^{-(k+1)\alpha+l-1}\right] = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-l}}{\Gamma[(k+l)\alpha-l+1]}, \\ \Phi(t) &= \sum_{k=0}^{\infty} A^k \mathscr{L}^{-1}\left[s^{-(k+1)\alpha}\right] = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}. \end{aligned}$$

□

Remark 2.4. From comparison of (2.28) and (2.32) it follows that the component of the solution corresponding to $u(t)$ is the same.

2.5 Positivity of the Fractional Systems

Definition 2.6. The fractional system (2.14) is called (internally) positive if the state vector $x(t) \in \mathbb{R}_+^n$ and the output vector $y(t) \in \mathbb{R}_+^p$ for $t \geq 0$ for all initial conditions $x_0 \in \mathbb{R}_+^n$ and all inputs $u(t) \in \mathbb{R}_+^m$, $t \geq 0$.

Definition 2.7. A real square matrix $A = [a_{ij}]$ is called Metzler matrix if its off diagonal entries are nonnegative, i.e. $a_{ij} \geq 0$ for $i \neq j$.

Lemma 2.1. *Let* $A \in \mathbb{R}^{n\times n}$ *and* $0 < \alpha \leq 1$. *Then*

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha+1)} \in \mathbb{R}_+^{n\times n} \quad \text{for} \quad t \geq 0, \tag{2.36}$$

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \in \mathbb{R}_+^{n\times n} \quad \text{for} \quad t \geq 0. \tag{2.37}$$

if and only if A is a Metzler matrix.

Proof. Necessity. From:

$$\begin{aligned} \Phi_0(t) &= I_n + \frac{At^\alpha}{\Gamma(\alpha+1)} + \cdots, \\ \Phi(t) &= I_n \frac{t^{\alpha-1}}{\Gamma(\alpha)} + A\frac{t^{2\alpha-1}}{\Gamma(2\alpha)} + \cdots \end{aligned}$$

it follows that $\Phi_0(t) \in \mathbb{R}_+^{n\times n}$ i $\Phi(t) \in \mathbb{R}_+^{n\times n}$ for small value $t > 0$ only if A is a Metzler matrix.

Sufficiency. It is well-known [77] that

$$e^{At} \in \mathbb{R}_+^{n\times n} \quad \text{for} \quad t \geq 0 \tag{2.38}$$

if and only if A is a Metzler matrix.

Using (2.36), we can write

$$\Phi_0(t) - e^{At^\alpha} = \sum_{k=0}^{\infty}\left(\frac{(At^\alpha)^k}{\Gamma(k\alpha+1)} - \frac{(At^\alpha)^k}{k!}\right) = \sum_{k=0}^{\infty}\frac{k! - \Gamma(k\alpha+1)}{\Gamma(k\alpha+1)}\cdot\frac{(At^\alpha)^k}{k!}$$

for $t \geq 0$, since $k! \geq \Gamma(k\alpha+1)$ for $0 < \alpha \leq 1$. From (2.38) and (2.5) we have $\Phi_0(t) \geq e^{At^\alpha} \geq 0$ for $t \geq 0$. The proof for (2.37) is similar. □

Theorem 2.8. *The fractional continuous-time system(2.14) is (internally) positive if and only if:*

$$A \in M_n, \quad B \in \mathbb{R}_+^{n\times m}, \quad C \in \mathbb{R}_+^{p\times n}, \quad D \in \mathbb{R}_+^{p\times m}. \tag{2.39}$$

Proof. Sufficiency. By Theorem 2.5 the solution (2.14a) has the form (2.15) and $x(t) \in \mathbb{R}_+^n$, $t \geq 0$, if the condition (2.39) is satisfied since $\Phi_0 \in \mathbb{R}_+^{n\times n}$, $x_0 \in \mathbb{R}_+^n$ and $u(t) \in \mathbb{R}_+^m$ for $t \geq 0$.

Necessity. Let $u(t) = 0$, $t \geq 0$ and $x_0 = e_i$ (i-th column of the identity matrix I_n). The trajectory does not leave the orthant $\mathbb{R}_+^n$ only if the derivative of order α, $x^\alpha(0) = Ae_i \geq 0$, what implies $a_{ij} \geq 0$ for $i \neq j$. The matrix A is a Metzler matrix. From the same reason for $x_0 = 0$ we have $x^\alpha(0) = Bu(0) \geq 0$, what implies $B \in \mathbb{R}_+^{n\times m}$, since $u(0) \in \mathbb{R}_+^m$ can be arbitrary. From (2.14b) for $u(t) = 0$, $t \geq 0$ we have $y(0) = Cx_0 \geq 0$ and $C \in \mathbb{R}_+^{p\times n}$, since $x_0 \in \mathbb{R}_+^n$ can be arbitrary. In a similar way assuming $x_0 = 0$, we obtain $y(0) = Du(0) \geq 0$ and $D \in \mathbb{R}_+^{p\times m}$, since $u(0) \in \mathbb{R}_+^m$ is arbitrary. □

2.6 External Positivity of the Fractional Systems

Definition 2.8. The fractional system (2.14) is called externally positive if for all $u(t) \in \mathbb{R}_+^m$, $t \geq 0$ and zero initial conditions $x_0 = 0$ the output vector $y(t) \in \mathbb{R}_+^p$, $t \geq 0$.

Definition 2.9. Output of the fractional SISO system with zero initial conditions for Dirac impulse $u(t) = \delta(t)$ is called the impulse response of the system. In a similar way we define the matrix of impulse response of the MIMO fractional system (2.14).

Lemma 2.2. *Matrix of the impulse responses $g(t)$ of the fractional system (2.14)is given by*

$$g(t) = C\Phi(t)B + D\delta(t), \quad t \geq 0. \tag{2.40}$$

Proof. Substituting (2.15) into (2.14b) and taking into account $x_0 = 0$, $u(t) = \delta(t)$, $y(t) = g(t)$ we obtain

$$g(t) = \int_0^t C\Phi(t-\tau)B\delta(\tau)d\tau + D\delta(t) = C\Phi(t)B + D\delta(t). \tag{2.41}$$

□

Theorem 2.9. *The fractional system (2.14) is externally positive if and only if*

$$g(t) \in \mathbb{R}_+^{p\times m} \quad \text{for} \quad t \geq 0. \tag{2.42}$$

Proof. Sufficiency. The output $y(t)$ of the system (2.14) with zero initial conditions for the input $u(t)$ is given by

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau. \tag{2.43}$$

If the condition (2.42) is satisfied then from (2.43) we have $y(t) \in \mathbb{R}_+^p$, $t \geq 0$.

Necessity. The necessity follows immediately from the fact that the matrix of impulse responses in a particular case of the output of the system for $u(t) = \delta(t)$ and $\delta(t)$ is nonnegative for $t \geq 0$. □

Corollary 2.1. *The matrix of impulse responses (2.40) of internally positive system (2.14)is nonnegative for $t \geq 0$.*

Between the internal and external positivity we have the following relationship.

Corollary 2.2. *Every fractional continuous-time (internally) positive system (2.14) is always externally positive.*

2.7 Reachability of Fractional Positive Continuous-Time Linear System

Definition 2.10. A state $x_f \in \mathbb{R}_+^n$ of the fractional system (2.14) is called reachable in time t_f if there exists an input $u(t) \in \mathbb{R}_+^m$ for $t \in [0, t_f]$ which steers the state of system from zero initial condition $x_0 = 0$ to the finial state $x_f = x(t_f)$. If every state $x_f \in \mathbb{R}_+^n$ is reachable in time t_f, then the system is called reachable in time t_f. The system (2.14) is called reachable if for every $x_f \in \mathbb{R}_+^n$ there exist t_f and an input $u(t) \in \mathbb{R}_+^m$ for $t \in [0, t_f]$, which steers the state of system from $x_0 = 0$ to x_f.

Theorem 2.10. *The fractional system(2.14) is reachable in time t_f, if the matrix*

$$R(t_f) = \int_0^{t_f} \Phi(t)BB^T\Phi^T(t)dt, \tag{2.44}$$

is monomial. Moreover the input which steers the state from $x_0 = 0$ to x_f is given by

$$u(t) = B^T\Phi^T(t_f - t)R^{-1}(t_f)x_f, \quad t \in [0, t_f], \tag{2.45}$$

where T denotes transpose.

Proof. We shall show that the input (2.45) steers the state of the system (2.14) from $x_0 = 0$ to x_f.

Substituting of (2.45) into (2.15) we obtain

$$\begin{aligned} x(t_f) &= \int_0^{t_f} \Phi(t_f - \tau)BB^T\Phi^T(t_f - \tau)R^{-1}(t_f)x_f d\tau \\ &= \int_0^{t_f} \Phi(t_f - \tau)BB^T\Phi^T(t_f - \tau)d\tau R^{-1}(t_f)x_f \\ &= R(t_f)R^{-1}(t_f)x_f = x_f. \end{aligned} \tag{2.46}$$

□

Theorem 2.11. *If the matrix $A = \text{diag}\begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \in \mathbb{R}_+^{n\times n}$ and $B \in \mathbb{R}_+^{n\times m}$ for $m = n$ are monomial matrices then the system (2.14) is reachable.*

Proof. From (2.17) it follows that if A is diagonal then the matrix $\Phi(t)$ and $\Phi(t)B$ are also monomial for monomial matrix B From (2.44)written in the form

$$R(t_f) = \int_0^{t_f} \Phi(\tau)B[\Phi(\tau)B]^T d\tau, \tag{2.47}$$

it follows that the matrix (2.47) is monomial. By Theorem 2.10 the fractional system is reachable. □

Example 2.5. We shall show that the fractional system (2.14) with the matrices:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{2.48}$$

is reachable. Taking into account that

$$A^k = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^k = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{for} \quad k = 1, 2, \dots,$$

and using (2.17) we obtain

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} = \begin{bmatrix} \Phi_1(t) & 0 \\ 0 & \Phi_2(t) \end{bmatrix}, \tag{2.49}$$

where

$$\Phi_1(t) = \sum_{k=0}^{\infty} \frac{t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}, \quad \Phi_2(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)},$$

and

$$\Phi(t)B = \begin{bmatrix} 0 & \Phi_1(t) \\ \Phi_2(t) & 0 \end{bmatrix}.$$

In this case from (2.47) we have

$$R(t_f) = \int_0^{t_f} \Phi(\tau)B[\Phi(\tau)B]^T d\tau = \int_0^{t_f} \begin{bmatrix} \Phi_1^2(\tau) & 0 \\ 0 & \Phi_2^2(\tau) \end{bmatrix} dt. \tag{2.50}$$

The matrix (2.50) is monomial and by Theorem 2.9 the fractional system is reachable.

Remark 2.5. It is well-knew that the standard system

$$\dot{x} = Ax + Bu \tag{2.51}$$

with the matrices:

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & a_0 \\ 1 & 0 & \dots & 0 & a_1 \\ 0 & 1 & \dots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{2.52}$$

is reachable for all values of the coefficients a_i, $i = 0, 1, \dots, n-1$, since the reachability matrix

$$\begin{bmatrix} B \; AB \; \dots \; A^{n-1}B \end{bmatrix} = I_n. \tag{2.53}$$

The system (2.51) is also reachable as a positive system if $a_i \geq 0$, $i = 0, 1, \dots, n-2$. The fractional system (2.14) with (2.52) is reachable even for $a_i = 0$, $i = 1, \dots, n-1$ if there exist $u(t) \geq 0$, $t \in [0, t_f]$ which satisfied condition

$$x_f = \int_0^{t_f} \begin{bmatrix} \frac{(t_f-\tau)^{\alpha-1}}{\Gamma(\alpha)} \\ \frac{(t_f-\tau)^{2\alpha-1}}{\Gamma(2\alpha)} \\ \vdots \\ \frac{(t_f-\tau)^{n\alpha-1}}{\Gamma(n\alpha)} \end{bmatrix} u(\tau)d\tau. \tag{2.54}$$

This condition (2.54) follows from (2.15) for $x_0 = 0$, (2.53) and the fact that for $a_i = 0$, $i = 0, 1, \dots, n-1$, we have $A^k = 0$ for $k = n, n+1, \dots$ and

$$\Phi(t)B = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} B = \sum_{k=0}^{n-1} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} B = \begin{bmatrix} \frac{t^{\alpha-1}}{\Gamma(\alpha)} \\ \frac{t^{2\alpha-1}}{\Gamma(2\alpha)} \\ \vdots \\ \frac{t^{n\alpha-1}}{\Gamma(n\alpha)} \end{bmatrix}.$$

This example shows that the reachability conditions for the fractional system (2.14) are much stronger than for positive system (2.51) [100].

2.8 Positive Continuous-Time Linear Systems with Delays

Consider the continuous-time linear system with q delays described by the state equations

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{q} A_k x(t-d_k) + Bu(t), \tag{2.55a}$$

$$y(t) = Cx(t) + Du(t), \tag{2.55b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the state, input and output vectors and $A_k \in \mathbb{R}_+^{n\times n}$, $k = 0,1,\dots,q$; $B \in \mathbb{R}_+^{n\times m}$, $C \in \mathbb{R}_+^{p\times n}$, $D \in \mathbb{R}_+^{p\times m}$, and d_k ($d_k \geq 0$), $k = 1,2,\dots,q$ are delays.

Initial conditions for (2.55a) have the form

$$x(t) = x_0(t) \quad \text{for} \quad t \in [-d,0], \quad d = max(d_k), \tag{2.56}$$

where $x_0(t) \in \mathbb{R}^n$ is given.

Definition 2.11. The system (2.55) is called (internally) positive if $x(t) \in \mathbb{R}_+^n$, $y(t) \in \mathbb{R}_+^p$ for any $x_0(t) \in \mathbb{R}_+^n$ and all inputs $u(t) \in \mathbb{R}_+^m$, $t \geq 0$.

Theorem 2.12. *The system (2.55) is (internally) positive if and only if*

$$A_0 \in M_n, \quad A_k \in \mathbb{R}_+^{n\times n}, \quad k = 1,\dots,q; \tag{2.57a}$$

$$B \in \mathbb{R}_+^{n\times m}, \quad C \in \mathbb{R}_+^{p\times n}, \quad D \in \mathbb{R}_+^{p\times m}. \tag{2.57b}$$

Proof. Necessity. The equation (2.55a) for $x(t-d_k) = 0$, $t \in [d,0]$ and $u(t) = 0$, $t \geq 0$ takes the form

$$\dot{x}(t) = A_0 x(t), \quad t \in [0,d]. \tag{2.58}$$

It is well-known [52, 77] that $x(t) \in \mathbb{R}_+^n$ of (2.58) only if $A_0 \in M_n$. Assuming in (2.55a) $u(t) = 0$, $t \geq 0$, $x_0(-d_k) = e_i$, $i = 1,\dots,n$ (i-th column of the identity matrix I_n), $x(-d_j) = 0$, $j = 0,1,\dots,k-1,k+1,\dots,n$ for $t = 0$ we obtain $\dot{x}(0) = A_k e_i = A_{ki} \in \mathbb{R}_+^n$, where A_{ki} is i-th column of $A_k \in \mathbb{R}_+^{n\times n}$, $k = 1,\dots,q$. From (2.55a) for $t = 0$ and $x_0(t) = 0$, $t \in [-d,0]$ we have $\dot{x}(0) = Bu(t)$ and $B \in \mathbb{R}_+^{n\times m}$, since by definition $u(0) \in \mathbb{R}_+^m$ is arbitrary. The necessity of $C \in \mathbb{R}_+^{p\times n}$, $D \in \mathbb{R}_+^{p\times m}$ can be shown in a similar way as for positive systems without delays [52, 77].

Sufficiency. The solution of the equation (2.55a) for $t \in [0,d]$ has the form

$$x(t) = e^{A_0 t} + \int_0^t e^{A_0(t-\tau)} \left(\sum_{k=1}^{q} A_k x_0(\tau - d_k) + Bu(\tau) \right) d\tau \ . \tag{2.59}$$

Taking in to account that $e^{A_0 t} \in \mathbb{R}_+^{n\times n}$, $t \geq 0$, for $A_0 \in M_n$, and the condition (2.57), from (2.59) we obtain $x(t) \in \mathbb{R}_+^n$, $t \in [0,d]$, since $x_0(t) \in \mathbb{R}_+^n$, $t \in [-d,0]$ and $u(t) \in \mathbb{R}_+^m$, $t \geq 0$. From (2.55b) we have $y(t) \in \mathbb{R}_+^p$, $t \in [0,d]$, since $x(t) \in \mathbb{R}_+^n$ and $u(t) \in \mathbb{R}_+^m$. Using the step method we can extend the considerations for the intervals $[d,2d]$, $[2d,3d]$, ... □

Definition 2.12. Let to the asymptotically stable positive system (2.55) a constant input $u(t) = u \in \mathbb{R}_+^m$ be applied. The vector x_e satisfying the equation

$$0 = \sum_{k=0}^{q} A_k x_e + Bu \tag{2.60}$$

is called the equilibrium point (state) of the system (2.55) for constant input u.

If the positive system (2.55) is asymptotically stable then the matrix

$$A = \sum_{k=0}^{q} A_k \in M_n \tag{2.61}$$

is nonsingular and from (2.60) we have

$$x_e = -A^{-1}Bu. \tag{2.62}$$

Remark 2.6. For positive asymptotically stable system (2.55)

$$-A^{-1} \in \mathbb{R}_+^{n\times n}. \tag{2.63}$$

This follows immediately from (2.62), since $x_0 \in \mathbb{R}_+^n$ and $Bu \in \mathbb{R}_+^m$ is arbitrary [52, 77].

Theorem 2.13. *The equilibrium point x_e for positive asymptotically stable system (2.55) is strictly positive, i.e. $x_e > 0$, if $Bu > 0$.*

Proof. If $A \in M_n$ and $Bu > 0$ then from (2.60) we have $x_e \in \mathbb{R}_+^n$. The hypothesis will be proved by contradiction. Assume that $x_e = 0$ then from (2.60) we have $Bu = 0$. This contradicts that $Bu > 0$. This completes the proof. □

These considerations can be extended for positive fractional continuous-time linear systems with delays.

2.9 Positive Linear Systems Consisting of n Subsystems with Different Fractional Orders

2.9.1 *Linear Differential Equations with Different Fractional Orders*

Consider a fractional linear system described by the equation

$$\begin{bmatrix} \frac{d^{\alpha_1} x_1}{dt^{\alpha_1}} \\ \vdots \\ \frac{d^{\alpha_n} x_n}{dt^{\alpha_n}} \end{bmatrix} = \begin{bmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \dots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} u, \quad \begin{matrix} p_k - 1 < \alpha_k < p_k \\ p_k \in \mathbb{N} = \{1, 2, \dots\}, \\ k = 1, \dots, n, \end{matrix} \tag{2.64}$$

where $x_k \in \mathbb{R}^{\overline{n}_k}$, $k = 1, \dots, n$ are the state vectors, $A_{kj} \in \mathbb{R}^{\overline{n}_k \times \overline{n}_j}$, $B_k \in \mathbb{R}^{\overline{n}_k \times m}$, $k, j = 1, \dots, n$ and $u \in \mathbb{R}^m$ is the input vector.

Initial conditions for (2.64) have the form

$$x_k^{(j)}(0) = x_{k0}^{(j)} \in \mathbb{R}^{\overline{n}_k}, \quad k = 1, \dots, n; \quad j = 0, 1, \dots, p_k - 1. \tag{2.65}$$

Theorem 2.14. *The solution of the equation (2.64) for $p_k - 1 < \alpha_k < p_k$, $k = 1, \dots, n$ with initial conditions (2.65) has the form*

$$\begin{aligned} x(t) &= \int_0^t [\Phi_1(t-\tau)B_{10} + \dots + \Phi_n(t-\tau)B_{n0}]\, u(\tau) d\tau \\ &+ \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1 \dots k_n} \begin{bmatrix} \sum_{j_1=1}^{p_1} \frac{t^{k_1\alpha_1 + \dots + k_n\alpha_n + j_1 - 1}}{\Gamma(k_1\alpha_1 + \dots + k_n\alpha_n + j_1)} x_{10}^{(j_1 - 1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} \frac{t^{k_1\alpha_1 + \dots + k_n\alpha_n + j_n - 1}}{\Gamma(k_1\alpha_1 + \dots + k_n\alpha_n + j_n)} x_{n0}^{(j_n - 1)} \end{bmatrix}, \end{aligned} \tag{2.66}$$

where

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \mathbb{R}^N, \quad N = \overline{n}_1 + \dots + \overline{n}_n, \quad x_0 = \begin{bmatrix} x_{10} \\ \vdots \\ x_{n0} \end{bmatrix}, \tag{2.67a}$$

$$B_{10} = \begin{bmatrix} B_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad B_{n0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ B_n \end{bmatrix}, \tag{2.67b}$$

$$\begin{aligned}
\Phi_1(t) &= \sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} T_{k_1\ldots k_n}\frac{t^{(k_1+1)\alpha_1+k_2\alpha_2+\cdots+k_n\alpha_n-1}}{\Gamma[(k_1+1)\alpha_1+k_2\alpha_2+\cdots+k_n\alpha_n]},\\
&\vdots \\
\Phi_n(t) &= \sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} T_{k_1\ldots k_n}\frac{t^{k_1\alpha_1+\cdots+k_{n-1}\alpha_{n-1}+(k_n+1)\alpha_n-1}}{\Gamma[k_1\alpha_1+\cdots+k_{n-1}\alpha_{n-1}+(k_n+1)\alpha_n]},
\end{aligned} \tag{2.67c}$$

and

$$T_{k_1\ldots k_n} = \begin{cases}
I_N & \text{for} \quad k_1=\cdots=k_n=0\\
\begin{bmatrix} A_{11} & \cdots & A_{1n}\\ 0 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 0\end{bmatrix} & \text{for} \quad \begin{matrix} k_1=1,\\ k_2=\cdots=k_n=0\end{matrix}\\
\begin{bmatrix} 0 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 0\\ A_{i1} & \cdots & A_{in}\\ 0 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 0\end{bmatrix} & \text{for} \quad \begin{matrix} k_1=\cdots=k_{i-1}=0,\\ k_1=1,\\ k_{i+1}=\cdots=k_n=0,\end{matrix}\\
\begin{bmatrix} 0 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & 0\\ A_{n1} & \cdots & A_{nn}\end{bmatrix} & \text{for} \quad \begin{matrix} k_1=\cdots=k_{n-1}=0,\\ k_i=1\end{matrix}\\
T_{10\ldots0}T_{01\ldots1}+\cdots+T_{0\ldots01}T_{1\ldots10} & \text{for} \quad k_1=\cdots=k_n=1\\
\vdots & \\
\begin{matrix} T_{10\ldots0}T_{k_1-1,k_2,\ldots,k_n}\\ +\cdots+T_{0\ldots01}T_{k_1,k_{n-1},k_n-1}\end{matrix} & \text{for} \quad k_1+\cdots+k_n>0\\
0 \qquad \text{for at last one} & k_i<0, i=1,\ldots,n
\end{cases} \tag{2.67d}$$

Proof. Using the Laplace transforms

$$X_k(s)=\mathscr{L}[x_k(t)], \quad k=1,\ldots,n; \quad U(s)=\mathscr{L}[u(t)], \tag{2.68}$$

and (A.10) we may write the equations (2.64) for $p_k - 1 < \alpha < p_k$; $p_k \in \mathbb{N}$, $k = 1\ldots,n$ in the form

$$\begin{bmatrix} I_{\bar{n}_1}s^{\alpha_1} - A_{11} & -A_{12} & \cdots & -A_{1n-1} & A_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -A_{n1} & -A_{n2} & \cdots & -A_{nn-1} & I_{\bar{n}_n}s^{\alpha_n} - A_{nn} \end{bmatrix} \begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix} = \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} U(s) + \begin{bmatrix} \sum_{j_1=1}^{p_1} s^{\alpha_1 - j_1} x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} s^{\alpha_n - j_n} x_{n0}^{(j_n-1)} \end{bmatrix}. \tag{2.69}$$

From (2.69) we have

$$\begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix} = \begin{bmatrix} I_{\bar{n}_1}s^{\alpha_1} - A_{11} & -A_{12} & \cdots & -A_{1n-1} & A_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -A_{n1} & -A_{n2} & \cdots & -A_{nn-1} & I_{\bar{n}_n}s^{\alpha_n} - A_{nn} \end{bmatrix}^{-1} \times \left\{ \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} U(s) + \begin{bmatrix} \sum_{j_1=1}^{p_1} s^{\alpha_1 - j_1} x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} s^{\alpha_n - j_n} x_{n0}^{(j_n-1)} \end{bmatrix} \right\}. \tag{2.70}$$

Comparing the coefficients at the same powers of $s^{-\alpha_k}$ it is easy to verify that

$$\begin{bmatrix} I_{\bar{n}_1} - A_{11}s^{-\alpha_1} & \cdots & -A_{1n}s^{-\alpha_1} \\ \vdots & \ddots & \vdots \\ -A_{n1}s^{-\alpha_n} & \cdots & I_{\bar{n}_n} - A_{nn}s^{-\alpha_n} \end{bmatrix} \left[\sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1\ldots k_n} s^{-(k_1\alpha_1 + \cdots + k_n\alpha_n)} \right] = I_N, \tag{2.71}$$

where matrices $T_{k_1\ldots k_n}$ are defined by (2.67d).

Using (2.71) we obtain

$$\begin{bmatrix} I_{\bar{n}_1}s^{\alpha_1} - A_{11} & -A_{12} & \cdots & -A_{1n-1} & A_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -A_{n1} & -A_{n2} & \cdots & -A_{nn-1} & I_{\bar{n}_n}s^{\alpha_n} - A_{nn} \end{bmatrix}^{-1} =$$

$$\left\{ \begin{bmatrix} I_{\bar{n}_1}s^{\alpha_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_{\bar{n}_n}s^{\alpha_n} \end{bmatrix} \begin{bmatrix} I_{\bar{n}_1} - A_{11}s^{-\alpha_1} & \cdots & -A_{1n}s^{-\alpha_1} \\ \vdots & \ddots & \vdots \\ -A_{n1}s^{-\alpha_n} & \cdots & I_{\bar{n}_n} - A_{nn}s^{-\alpha_n} \end{bmatrix} \right\}^{-1} =$$

$$\begin{bmatrix} I_{\bar{n}_1} - A_{11}s^{-\alpha_1} & \cdots & -A_{1n}s^{-\alpha_1} \\ \vdots & \ddots & \vdots \\ -A_{n1}s^{-\alpha_n} & \cdots & I_{\bar{n}_n} - A_{nn}s^{-\alpha_n} \end{bmatrix}^{-1} \left\{ \begin{bmatrix} I_{\bar{n}_1}s^{-\alpha_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_{\bar{n}_n}s^{-\alpha_n} \end{bmatrix} \right\} =$$

$$\sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1\ldots k_n} s^{-(k_1\alpha_1+\cdots+k_n\alpha_n)} \left\{ \begin{bmatrix} I_{\bar{n}_1}s^{-\alpha_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_{\bar{n}_n}s^{-\alpha_n} \end{bmatrix} \right\}. \quad (2.72)$$

substituting of (2.72) into (2.70) yields

$$\begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix} = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1\ldots k_n} s^{-(k_1\alpha_1+\cdots+k_n\alpha_n)} \begin{bmatrix} I_{\bar{n}_1}s^{-\alpha_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & I_{\bar{n}_n}s^{-\alpha_n} \end{bmatrix}$$
$$\times \left\{ \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} U(s) + \begin{bmatrix} \sum_{j_1=1}^{p_1} s^{\alpha_1 - j_1} x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} s^{\alpha_n - j_n} x_{n0}^{(j_n-1)} \end{bmatrix} \right\}$$
$$= \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1\ldots k_n} \left\{ \left[B_{10}s^{-[(k_1+1)\alpha_1+k_2\alpha_2+\cdots+k_n\alpha_n]} \right.\right.$$
$$\left. + \cdots + B_{n0}s^{-[k_1\alpha_1+\cdots+k_{n-1}\alpha_{n-1}+(k_n+1)\alpha_n]} \right] U(s)$$
$$\left. + \; s^{-(k_1\alpha_1+\cdots+k_n\alpha_n)} \begin{bmatrix} \sum_{j_1=1}^{p_1} s^{-j_1} x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} s^{-j_n} x_{n0}^{(j_n-1)} \end{bmatrix} \right\}. \quad (2.73)$$

Applying the inverse Laplace transform and the convolution theorem to (2.73) we obtain

$$\mathscr{L}^{-1} \begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix} = \mathscr{L}^{-1} \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1\ldots k_n} \left\{ \left[B_{10}s^{-[(k_1+1)\alpha_1+k_2\alpha_2+\cdots+k_n\alpha_n]} \right.\right.$$
$$\left. + \cdots + B_{n0}s^{-[k_1\alpha_1+\cdots+k_{n-1}\alpha_{n-1}+(k_n+1)\alpha_n]} \right] U(s)$$
$$\left. + \; s^{-(k_1\alpha_1+\cdots+k_n\alpha_n)} \begin{bmatrix} \sum_{j_1=1}^{p_1} s^{-j_1} x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} s^{-j_n} x_{n0}^{(j_n-1)} \end{bmatrix} \right\},$$

$$\begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \int_0^t [\Phi_1(t-\tau)B_{10} + \cdots + \Phi_n(t-\tau)B_{n0}]\, u(\tau)d\tau$$

$$+ \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1 \dots k_n} \begin{bmatrix} \sum_{j_1=1}^{p_1} \frac{t^{k_1\alpha_1+\cdots+k_n\alpha_n+j_1-1}}{\Gamma(k_1\alpha_1+\cdots+k_n\alpha_n+j_1)} x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} \frac{t^{k_1\alpha_1+\cdots+k_n\alpha_n+j_n-1}}{\Gamma(k_1\alpha_1+\cdots+k_n\alpha_n+j_n)} x_{n0}^{(j_n-1)} \end{bmatrix}, \qquad (2.74)$$

since $\mathscr{L}^{-1}\left[\frac{1}{s^{\alpha+1}}\right] = \frac{t^{\alpha}}{\Gamma(\alpha+1)}$. □

In a particular case if $0 < \alpha_k < 1$, $k = 1, \dots, n$; $(p_1 = \cdots = p_n = 1)$, then

$$\sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1 \dots k_n} \begin{bmatrix} \sum_{j_1=1}^{p_1} \frac{t^{k_1\alpha_1+\cdots+k_n\alpha_n+j_1-1}}{\Gamma(k_1\alpha_1+\cdots+k_n\alpha_n+j_1)} x_{10}^{(j_1-1)} \\ \vdots \\ \sum_{j_n=1}^{p_n} \frac{t^{k_1\alpha_1+\cdots+k_n\alpha_n+j_n-1}}{\Gamma(k_1\alpha_1+\cdots+k_n\alpha_n+j_n)} x_{n0}^{(j_n-1)} \end{bmatrix} = \Phi_0(t)x_0, \qquad (2.75)$$

where

$$\Phi_0(t) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} T_{k_1 \dots k_n} \frac{t^{k_1\alpha_1+\cdots+k_n\alpha_n}}{\Gamma(k_1\alpha_1+\cdots+k_n\alpha_n+1)}. \qquad (2.76)$$

2.9.2 Positive Fractional Systems

Definition 2.13. The fractional system (2.64) is called positive if $x_k(t) \in \mathbb{R}_+^{\overline{n}_k}$, $k = 1, \dots, n$, $t \geq 0$ for any initial conditions $x_{k0} \in \mathbb{R}_+^{\overline{n}_k}$, $k = 1, \dots, n$, and all input vectors $u \in \mathbb{R}_+^m$, $t \geq 0$.

Let M_n be the set of $n \times n$ Metzler matrices, i.e. real matrices with nonnegative off-diagonal entries.

Theorem 2.15. *The fractional system (2.64) for $p_k - 1 < \alpha < p_k$, $p_k \in \mathbb{N}$, $k = 1, \dots, n$ is positive if and only if*

$$A = \begin{bmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \dots & A_{nn} \end{bmatrix} \in M_N, \qquad (2.77a)$$

$$\begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} \in \mathbb{R}_+^{N \times m}. \qquad (2.77b)$$

Proof. To simplify the notation the proof will be given for $n = 2$. First we shall show that

$$\Phi_k(t) \in \mathbb{R}_+^{\overline{n}\times\overline{n}}, \quad (\overline{n} = \overline{n}_1 + \overline{n}_2) \quad \text{for} \quad k = 0,1,2 \quad \text{and} \quad t \geq 0, \tag{2.78}$$

only if (2.77a) holds. From the expansion (2.67c) we have

$$\begin{aligned}
\Phi_0(t) &= \begin{bmatrix} I_{\overline{n}_1} & 0 \\ 0 & I_{\overline{n}_2} \end{bmatrix} + \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} \frac{t^{\alpha_1}}{\Gamma(\alpha_1+1)} \\
&\quad + \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} \frac{t^{\alpha_2}}{\Gamma(\alpha_2+1)} + \cdots , && (2.79a) \\
\Phi_1(t) &= \begin{bmatrix} I_{\overline{n}_1} & 0 \\ 0 & I_{\overline{n}_2} \end{bmatrix} \frac{t^{\alpha_1-1}}{\Gamma(\alpha_1)} + \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} \frac{t^{2\alpha_1-1}}{\Gamma(2\alpha_1)} \\
&\quad + \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} \frac{t^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} + \cdots , && (2.79b) \\
\Phi_2(t) &= \begin{bmatrix} I_{\overline{n}_1} & 0 \\ 0 & I_{\overline{n}_2} \end{bmatrix} \frac{t^{\alpha_2-1}}{\Gamma(\alpha_2)} + \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} \frac{t^{\alpha_1+\alpha_2-1}}{\Gamma(\alpha_1+\alpha_2)} \\
&\quad + \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} \frac{t^{2\alpha_2-1}}{\Gamma(2\alpha_2)} + \cdots . && (2.79c) \\
& && (2.79d)
\end{aligned}$$

From (2.79) it follows that $\Phi_k(t) \in \mathbb{R}_+^{\overline{n}\times\overline{n}}$, $k = 0,1,2$ for small value of $t > 0$ only if the condition (2.77a) is satisfied.

In a similar way as in [100, 135] it can be shown that if (2.77) holds then

$$\Phi_0(t) \in \mathbb{R}_+^{\overline{n}\times\overline{n}}, \quad t \geq 0, \tag{2.80}$$

and

$$\Phi_1(t)B_{10} + \Phi_2(t)B_{01} \in \mathbb{R}_+^{\overline{n}\times\overline{n}}, \quad t \geq 0. \tag{2.81}$$

In this case from (2.66) we have $x(t) \in \mathbb{R}_+^{\overline{n}}$, $t \geq 0$ since by definition $x_0 \in \mathbb{R}_+^{\overline{n}}$ and $u(t) \in \mathbb{R}_+^m$, $t \geq 0$. The remaining part of the proof is similar as in [100, 135]. □

2.9.3 Fractional Linear Electrical Circuits

Consider linear electrical circuits composed of resistors, supercondensators (ultracapacitors), coils and voltage (current) sources. As the state variables (the components of the state vector x) the voltage across the supercondensators and the currents in the coils are usually chosen. It is well-known [51, 196] that the current $i(t)$ in supercondensator with its voltage $u_C(t)$ is related by formula

$$i_C(t) = C\frac{d^\alpha u_C(t)}{dt^\alpha} \quad \text{for} \quad 0 < \alpha < 1, \tag{2.82}$$

where C is the capacity of the supercondensator.

Similarly, the voltage $u_L(t)$ on the coil with its current $i_L(t)$ is related by the formula

$$u_L(t) = L\frac{d^\beta i_L(t)}{dt^\beta} \quad \text{for} \quad 0 < \beta < 1, \tag{2.83}$$

where L is the inductance of the coil.

Using the relations (2.82), (2.83) and Kirchhoff's laws we may write for the fractional linear circuits the following state equation

$$\begin{bmatrix} \frac{d^\alpha x_C}{dt^\alpha} \\ \frac{d^\beta x_L}{dt^\beta} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_C \\ x_L \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} e, \tag{2.84}$$

where the components of $x_C \in \mathbb{R}^{n_1}$ are voltages across the supercondensators, the components of $x_L \in \mathbb{R}^{n_2}$ are currents in coils and the components of $e \in \mathbb{R}^m$ are the voltages of the circuit.

Example 2.6. Consider the linear electrical circuit shown on Fig. 2.1 with known resistances R_1, R_2, R_3, capacitances C_1, C_2, inductances L_1, L_2 and sources voltages e_1, e_2.

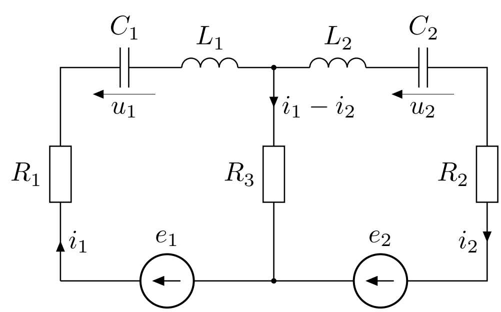

Fig. 2.1 Electrical circuit. Illustration to Example 2.6.

Using relations (2.82), (2.83) and Kirchhoff's laws we may write for the circuit the following equations:

$$i_1 = C_1\frac{d^{\alpha_1}u_1}{dt^{\alpha_1}}, \quad i_2 = C_2\frac{d^{\alpha_2}u_2}{dt^{\alpha_2}}, \tag{2.85a}$$

$$e_1 = (R_1 + R_2)i_1 + L_1\frac{d^{\beta_1}i_1}{dt^{\beta_1}} + u_1 - R_3 i_2, \tag{2.85b}$$

$$e_2 = (R_2 + R_3)i_2 + L_2\frac{d^{\beta_2}i_2}{dt^{\beta_2}} + u_2 - R_3 i_1. \tag{2.85c}$$

The equations (2.85) can be written in the form

$$\begin{bmatrix} \frac{d^{\alpha_1} u_1}{dt^{\alpha_1}} \\ \frac{d^{\alpha_2} u_2}{dt^{\alpha_2}} \\ \frac{d^{\beta_1} i_1}{dt^{\beta_1}} \\ \frac{d^{\beta_2} i_2}{dt^{\beta_2}} \end{bmatrix} = A \begin{bmatrix} u_1 \\ u_2 \\ i_1 \\ i_2 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{2.86}$$

where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{C_1} & 0 \\ 0 & 0 & 0 & \frac{1}{C_2} \\ -\frac{1}{L_1} & 0 & -\frac{R_1+R_3}{L_1} & \frac{R_3}{L_1} \\ 0 & -\frac{1}{L_2} & \frac{R_3}{L_2} & -\frac{R_2+R_3}{L_2} \end{bmatrix}, \tag{2.87a}$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}. \tag{2.87b}$$

From (2.87) it follows that the fractional electrical circuit is not positive since the matrix A has some negative off-diagonal entries.

If the fractional linear circuit is not positive but the matrix B has nonnegative entries (see for example the circuit on Fig. 2.1) then using the state-feedback

$$e = K \begin{bmatrix} x_C \\ x_L \end{bmatrix}. \tag{2.88}$$

we may usually choose the gain matrix $K \in \mathbb{R}^{m\times n}$, $(n = n_1 + n_2)$ so that the closed-loop system matrix (obtained by substituting of (2.88) into (2.84))

$$A_c = A + BK, \tag{2.89}$$

is a Metzler matrix.

Theorem 2.16. *Let A be not a Metzler matrix but $B \in \mathbb{R}_+^{n\times m}$. Then there exists a gain matrix K such that the closed-loop system matrix $A_c \in M_n$ if and only if*

$$\operatorname{rank}[B, A_c - A] = \operatorname{rank} B. \tag{2.90}$$

Proof. By Kronecker-Cappely theorem the equation

$$BK = A_c - A, \tag{2.91}$$

have a solution K for any given B and $A_c - A$ if and only if the condition (2.90) is satisfied. □

Example 2.7. (Continuation of Example 2.6). Let

$$A_c = \begin{bmatrix} 0 & 0 & \frac{1}{C_1} & 0 \\ 0 & 0 & 0 & \frac{1}{C_2} \\ \frac{a_1}{L_1} & 0 & -\frac{R_1+R_3}{L_1} & \frac{a_3}{L_1} \\ 0 & \frac{a_2}{L_2} & \frac{a_4}{L_2} & -\frac{R_2+R_3}{L_2} \end{bmatrix} \quad \text{for} \quad a_k \geq 0, \quad k = 1,2,3,4. \tag{2.92}$$

In this case the condition (2.90) is satisfied since

$$\operatorname{rank}[B, A_c - A] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{L_1} & 0 & \frac{a_1+1}{L_1} & 0 & 0 & \frac{a_3-R_3}{L_1} \\ 0 & \frac{1}{L_2} & 0 & \frac{a_2+1}{L_2} & \frac{a_4-R_3}{L_2} & 0 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix} = 2. \tag{2.93}$$

The equation (2.91) has the form

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{a_1+1}{L_1} & 0 & 0 & \frac{a_3-R_3}{L_1} \\ 0 & \frac{a_2+1}{L_2} & \frac{a_4-R_3}{L_2} & 0 \end{bmatrix}, \tag{2.94}$$

and its solution is

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix} = \begin{bmatrix} a_1+1 & 0 & 0 & a_3-R_3 \\ 0 & a_2+1 & a_4-R_3 & 0 \end{bmatrix}. \tag{2.95}$$

The matrix (2.95) has nonnegative entries if $a_k \geq 0$ for $k = 1,2,3,4$.

On the following two examples of fractional linear circuits we shall shown that it is not always possible to choose the gain matrix K so that the two conditions are satisfied:

a) the closed-loop system matrix $A_c \in M_n$,
b) the closed-loop system is asymptotically stable.

Example 2.8. Consider the fractional linear circuit shown on Fig. 2.2 with given resistances R, capacitance C, inductance L and source of voltage e.

Using (2.82), (2.83) and the second Kirchhoff's law we obtain for the circuit the state equation

$$\begin{bmatrix} \frac{d^\alpha u_C}{dt^\alpha} \\ \frac{d^\beta i}{dt^\beta} \end{bmatrix} = A \begin{bmatrix} u_C \\ i \end{bmatrix} + Be, \quad 0 < \alpha < 1; \quad 0 < \beta < 1; \tag{2.96}$$

where

$$A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}. \tag{2.97}$$

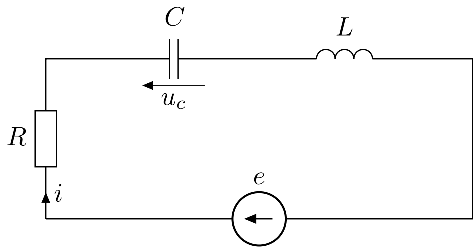

Fig. 2.2 Electrical circuit. Illustration to Example 2.8.

From (2.97) it follows that A is not a Metzler matrix but $B \in \mathbb{R}_+^2$. It is easy to see that the condition (2.90) is satisfied for

$$A_c = \begin{bmatrix} 0 & \frac{1}{C} \\ \frac{a}{L} & \frac{b-R}{L} \end{bmatrix}, \tag{2.98}$$

and

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} a+1 & b \end{bmatrix}. \tag{2.99}$$

Note that the characteristic polynomial of the matrix (2.98)

$$\det \begin{bmatrix} I_{n_1} s^\alpha - A_{11} & -A_{12} \\ -A_{21} & I_{n_2} s^\beta - A_{22} \end{bmatrix} = \begin{vmatrix} s^\alpha & -\frac{1}{C} \\ -\frac{a}{L} & s^\beta + \frac{R-b}{L} \end{vmatrix} = s^{\alpha+\beta} + \frac{R-b}{L} s^\alpha - \frac{a}{LC}, \tag{2.100}$$

has one nonnegative coefficient and closed-loop circuit is unstable for $a \geq 0$ and any b.

Example 2.9. Consider the fractional linear system shown on Fig. 2.3 with given resistances R_1, R_2, capacitance C, inductance L and source of voltage e. Using the relations (2.82), (2.83) and the second Kirchhoff's law we obtain for the circuit the state equation

$$\begin{bmatrix} \frac{d^\alpha u_C}{dt^\alpha} \\ \frac{d^\beta i}{dt^\beta} \end{bmatrix} = A \begin{bmatrix} u_C \\ i \end{bmatrix} + Be, \tag{2.101}$$

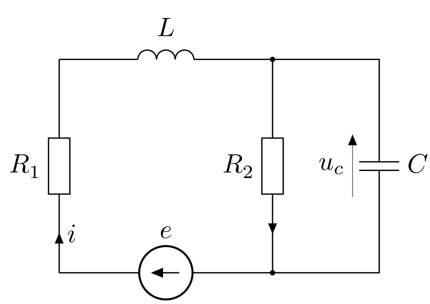

Fig. 2.3 Electrical circuit. Illustration to Example 2.9.

where

$$A = \begin{bmatrix} -\frac{1}{R_2C} & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R_1}{L} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}. \tag{2.102}$$

The matrix A is not a Metzler matrix but $B \in \mathbb{R}_+^2$. It is easy to check that the condition (2.90) is satisfied for

$$A = \begin{bmatrix} -\frac{1}{R_2C} & \frac{1}{C} \\ \frac{a}{L} & \frac{b-R_1}{L} \end{bmatrix}, \quad a,b \geq 0, \tag{2.103}$$

and from (2.91) we obtain

$$\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{a+1}{L} & \frac{b}{L} \end{bmatrix}, \tag{2.104}$$

and

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} a+1 & b \end{bmatrix}. \tag{2.105}$$

In this case the characteristic polynomial of the matrix (2.90) has the form

$$p(s) = \begin{vmatrix} s^\alpha + \frac{1}{R_2C} & -\frac{1}{C} \\ -\frac{a}{L} & s^\beta + \frac{R_1-b}{L} \end{vmatrix} = s^{\alpha+\beta} + \frac{R_1-b}{L}s^\alpha + \frac{1}{R_2C}s^\beta + \frac{R_1-aR_2-b}{R_2CL}, \tag{2.106}$$

and it is possible to choose the values of parameters a, b so that the closed-loop system is asymptotically stable [266].

Chapter 3
Fractional Positive 2D Linear Systems

3.1 Definition of (Backward) Fractional Difference of 2D Function

Definition 1.3 of (backward) fractional difference of α-order will be extended to two-dimensional (shortly 2D) discrete function x_{ij}.

Definition 3.1. The 2D discrete function

$$\Delta^{\alpha} x_{ij} = \sum_{k=0}^{i} \sum_{l=0}^{j-k} c_{\alpha}(k,l) x_{i-k,j-l}, \quad 0 < \alpha < 1, \tag{3.1a}$$

is called the (backward) fractional difference of α order of the 2D function x_{ij} where

$$c_{\alpha}(k,l) = \begin{cases} 1 & \text{for} \quad k = l = 0 \quad k,l \in \mathbb{Z}_{+} \\ (-1)^{k+l} \frac{\alpha(\alpha-1)\cdots(\alpha-k-l+1)}{k!l!} & \text{for} \quad k+l > 0 \end{cases} \tag{3.1b}$$

3.2 State Equation of Fractional 2D Linear Systems

The model described by the state equation:

$$\begin{aligned} \Delta^{\alpha} x_{i+1,j+1} &= A_0 x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1} \\ &\quad + B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1}, \end{aligned} \tag{3.2a}$$

$$y_{ij} = C x_{ij} + D u_{ij}, \tag{3.2b}$$

is called the fractional general model of α order of 2D linear systems where $x_{ij} \in \mathbb{R}^n$, $u_{ij} \in \mathbb{R}^m$, $y_{ij} \in \mathbb{R}^p$ are state, input and output vectors and $A_k \in \mathbb{R}^{n \times n}$, $B_k \in \mathbb{R}^{n \times m}$, $k = 0,1,2$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$.

Using Definition 3.1 we may write the equations (3.2a) in the form

$$\begin{aligned} x_{i+1,j+1} &+ \sum_{k=0}^{i+1} \sum_{\substack{l=0 \\ k+l>0}}^{j-k+1} c_{\alpha}(k,l) x_{i-k+1,j-l+1} \\ &= A_0 x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1}. \end{aligned} \tag{3.3a}$$

T. Kaczorek: Selected Problems of Fractional Systems Theory, LNCIS 411, pp. 53–80.
springerlink.com

From (3.1b) it follows that the coefficients $c_\alpha(k,l)$ in (3.1a) strongly decrease with increasing k and l. In practice usually it is assumed that i and j are bounded by some natural numbers L_1 and L_2. In this case the equation (3.3a) takes the form

$$\begin{aligned} x_{i+1,j+1} &+ \sum_{k=0}^{L_1+1} \sum_{\substack{l=0 \\ k+l>0}}^{L_2-k+1} c_\alpha(k,l) x_{i-k+1,j-l+1} \\ &= A_0 x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1}. \end{aligned} \tag{3.3b}$$

Remark 3.1. From (3.3a) it follows that the fractional 2D linear system is a linear system with increasing number of delays in state vector.

Boundary conditions for (3.3a) have the form:

$$x_{i0},\ i \in \mathbb{Z}_+, \quad \text{and} \quad x_{0j},\ j \in \mathbb{Z}_+. \tag{3.4}$$

3.3 Solution of the State Equation of the Fractional 2D Linear System

Applying 2D z-transform we shall derive the solution of the state equation (3.3a) with boundary conditions (3.4).

Theorem 3.1. *The solution of the state equation (3.3a) with the boundary conditions (3.4) has the form*

$$\begin{aligned} x_{ij} &= \sum_{p=1}^{i} T_{i-p,j-1} \left(\overline{A}_1 x_{p0} + B_1 u_{p0}\right) + \sum_{q=1}^{j} T_{i-1,j-q} \left(\overline{A}_2 x_{0q} + B_2 u_{0q}\right) \\ &+ \sum_{p=1}^{i-1} T_{i-p-1,j-1} \overline{A}_0 x_{p0} + \sum_{q=1}^{j-1} T_{i-1,j-q-1} \overline{A}_0 x_{0q} + T_{i-1,j-1} \overline{A}_0 u_{00} \\ &+ \sum_{p=0}^{i-1} \sum_{q=0}^{j-1} T_{i-p-1,j-q-1} B_0 u_{pq} + \sum_{p=0}^{i} \sum_{q=0}^{j} \left(T_{i-p-1,j-q} B_1 + T_{i-p,j-q-1} B_2\right) u_{pq} \end{aligned} \tag{3.5}$$

where the matrices T_{pq} are defined as

$$T_{pq} = \begin{cases} I_n & \text{for} \quad p = q = 0 \\ \overline{A}_0 T_{p-1,q-1} + \overline{A}_1 T_{p,q-1} + \overline{A}_2 T_{p-1,q} & \\ - \sum_{k=0}^{p} \sum_{l=0}^{q} c_\alpha(p-k, q-l) T_{kl} & \text{for} \quad p+q > 0, \\ & \text{and} \quad k+l < p+q-2 \\ 0 \quad \text{(zero matrix)} & \text{for} \quad p < 0 \quad \text{and/or} \quad q < 0 \end{cases} \tag{3.6}$$

and $\overline{A}_k = A_k - \alpha I_n$ for $k = 0, 1, 2$.

Proof. Let $X(z_1,z_2)$ be the 2D z-transform of the discrete function x_{ij}, defined by (A.15). Applying the 2D z-transform to equation (3.3a) and using Appendix (A.3), we obtain

$$\begin{aligned}X(z_1,z_2) &= G^{-1}(z_1,z_2)\{(B_0+B_1z_1+B_2z_2)U(z_1,z_2)\\&+ z_1z_2[X(z_1,0)+X(0,z_2)-x_{00}]\\&+\sum_{l=1}^{j+1}c_\alpha(0,l)z_1z_2^{-l+1}X(0,z_2)+\sum_{k=1}^{i+1}c_\alpha(k,0)z_1^{-k+1}z_2X(z_1,0)\\&- z_1\left[\overline{A}_1B_1\right]\begin{bmatrix}X(0,z_2)\\U(0,z_2)\end{bmatrix}-z_2\left[\overline{A}_2B_2\right]\begin{bmatrix}X(z_1,0)\\U(z_1,0)\end{bmatrix}\Bigg\},\end{aligned} \tag{3.7a}$$

where

$$G(z_1,z_2)=\left[z_1z_2I_n+\sum_{k=0}^{i+1}\sum_{\substack{l=0\\k+l>1}}^{j-k+1}c_\alpha(k,l)z_1^{-(k-1)}z_2^{-(l-1)}I_n-\overline{A}_0-\overline{A}_1z_1-\overline{A}_2z_2\right] \tag{3.7b}$$

and $U(z_1,z_2)=\mathscr{Z}[u_{ij}]$.
Let

$$G^{-1}(z_1,z_2)=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}T_{pq}z_1^{-(p+1)}z_2^{-(q+1)}. \tag{3.8}$$

From the equality

$$G^{-1}(z_1,z_2)G(z_1,z_2)=G(z_1,z_2)G^{-1}(z_1,z_2)=I_n,$$

we have

$$\begin{aligned}I_n &= \left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}T_{pq}z_1^{-(p+1)}z_2^{-(q+1)}\right)G(z_1,z_2)\\&= G(z_1,z_2)\left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}T_{pq}z_1^{-(p+1)}z_2^{-(q+1)}\right).\end{aligned} \tag{3.9}$$

Comparing of the coefficients at the same power of z_1 i z_2 in the equation (3.9), we obtain (3.6). Substituting of (3.8) into (3.7a), yields

$$
\begin{aligned}
X(z_1,z_2) = & \left(\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} T_{pq} z_1^{-(p+1)} z_2^{-(q+1)} \right) \{ (B_0 + B_1 z_1 + B_2 z_2) U(z_1,z_2) \\
& + z_1 z_2 [X(z_1,0) + X(0,z_2) - x_{00}] \\
& - z_1 \left[\overline{A}_1 B_1 \right] \begin{bmatrix} X(0,z_2) \\ U(0,z_2) \end{bmatrix} - z_2 \left[\overline{A}_2 B_2 \right] \begin{bmatrix} X(z_1,0) \\ U(z_1,0) \end{bmatrix} \\
& + \sum_{l=2}^{j+1} c_\alpha(0,l) z_1 z_2^{-l+1} X(0,z_2) + \sum_{k=2}^{i+1} c_\alpha(k,0) z_1^{-k+1} z_2 X(z_1,0) \Bigg\},
\end{aligned}
\tag{3.10}
$$

Applying the inverse 2D z-transform and the convolution theorem we obtain the desired solution (3.5). □

3.4 Extension of the Cayley-Hamilton Theorem

From (3.7b) we have

$$G(z_1,z_2) = z_1 z_2 \overline{G}(z_1,z_2), \tag{3.11}$$

where

$$\overline{G}(z_1,z_2) = I_n + \sum_{k=0}^{i+1} \sum_{l=0}^{j-k+1} I_n c_\alpha(k,l) z_1^{-k} z_2^{-l} - \overline{A}_0 z^{-1} z_2^{-1} - \overline{A}_1 z_2^{-1} - \overline{A}_2 z_1^{-1}. \tag{3.12}$$

and

$$\det\left[\overline{G}(z_1,z_2)\right] = \sum_{k=0}^{N_1} \sum_{l=0}^{N_2} a_{N_1-k,N_2-l} z_1^{-k} z_2^{-l}. \tag{3.13}$$

It is assumed that i and j are bounded by some natural numbers L_1 i L_2, which determine the degrees N_1 and N_2.

From (3.11) and (3.8) it follows that

$$G^{-1}(z_1,z_2) = z_1^{-1} z_2^{-1} \overline{G}^{-1}(z_1,z_2) = z_1^{-1} z_2^{-1} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} T_{pq} z_1^{-p} z_2^{-q}, \tag{3.14}$$

and

$$\overline{G}^{-1}(z_1,z_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} T_{pq} z_1^{-p} z_2^{-q}, \tag{3.15}$$

where T_{pq} is defined by (3.6).

Theorem 3.2. *Let (3.13) be the characteristic polynomial of the system (3.2). Then the matrices T_{kl} satisfy the equation*

$$\sum_{k=0}^{N_1} \sum_{l=0}^{N_2} a_{kl} T_{kl} = 0. \tag{3.16}$$

Proof. From definition of inverse matrix, (3.13) and (3.15) we have

$$\mathrm{Adj}\left[\overline{G}(z_1,z_2)\right] = \left(\sum_{k=0}^{N_1}\sum_{l=0}^{N_2} a_{N_1-k,N_2-l} z_1^{-k} z_2^{-l}\right)\left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq} z_1^{-p} z_2^{-q}\right), \qquad (3.17)$$

where $\mathrm{Adj}\left[\overline{G}(z_1,z_2)\right]$ is the adjoint matrix of $\overline{G}(z_1,z_2)$.

Comparison of the coefficients at the same power of $z_1^{-N_1} z_2^{-N_2}$ in equation (3.17), yields the equality (3.16), since the degrees of the polynomial matrix (3.17) with respect to z_1^{-1} and z_2^{-1} are less than N_1 and N_2. □

Theorem 3.2 is an extension of the classical Cayley-Hamilton theorem to fractional 2D linear system described by (3.2).

3.5 Positivity of Fractional 2D Linear Systems

Lemma 3.1. *If* $|\alpha| < 1$, *then:*

$$c_\alpha(k,l) \begin{cases} <0 & for \quad 0<\alpha<1 \\ >0 & for \quad -1<\alpha<0 \end{cases} \quad k,l \in \mathbb{Z}_+ \ . \qquad (3.18)$$

Proof. Using (3.1b) for $0 < \alpha < 1$, we obtain

$$\begin{aligned} c_\alpha(k,l) &= (-1)^{k+l}\frac{\alpha(\alpha-1)\cdots(\alpha+1-k-l)}{k!l!} \\ &= \begin{cases} -\alpha & \text{for} \quad k+l=1 \\ -\frac{\alpha(1-\alpha)\cdots(k+l-1-\alpha)}{k!l!} & \text{for} \quad k+l>1 \end{cases} . \end{aligned}$$

since

$$\alpha(\alpha-1)\cdots(\alpha+1-k-l) = (-1)^{k+l-1}\alpha(1-\alpha)\cdots(k+l-1-\alpha) \quad \text{for} \quad k+l>1.$$

The proof of the second part is similar. □

Lemma 3.2. *If* $0 < \alpha < 1$ *and*

$$\overline{A}_k \in \mathbb{R}_+^{n\times n} \quad for \quad k = 0,1,2, \qquad (3.19)$$

then

$$T_{pq} \in \mathbb{R}_+^{n\times n} \quad for \quad p,q \in \mathbb{Z}_+. \qquad (3.20)$$

Proof. If the conditions (3.18), (3.19), are satisfied then from (3.6), we obtain (3.20). □

Theorem 3.3. *The fractional 2D linear system(3.2) for* $0 < \alpha < 1$ *is positive if and only if:*

$$\overline{A}_k \in \mathbb{R}_+^{n\times n},\ B_k \in \mathbb{R}_+^{n\times m}\ for\ k = 0,1,2,\ C \in \mathbb{R}_+^{p\times n},\ D \in \mathbb{R}_+^{p\times m}. \qquad (3.21)$$

Proof. Sufficiency. If the conditions (3.21) are satisfied then by Lemma 3.2 $T_{pq} \in \mathbb{R}_+^{n\times n}$ and from (3.5) we have $x_{ij} \in \mathbb{R}_+^n$ for $i,j \in \mathbb{Z}_+$ since $x_{i0} \in \mathbb{R}_+^n$, $x_{0j} \in \mathbb{R}_+^n$ and $u_{ij} \in \mathbb{R}_+^m$ for $i,j \in \mathbb{Z}_+$. From (3.2b) we have $y_{ij} \in \mathbb{R}_+^p$ since $C \in \mathbb{R}_+^{p\times n}$, $D \in \mathbb{R}_+^{p\times m}$ and $x_{ij} \in \mathbb{R}_+^n$, $u_{ij} \in \mathbb{R}_+^m$ for $i,j \in \mathbb{Z}_+$.

Necessity. It is assumed that the system is positive and $x_{00} = e_{ni}$, $i = 1,\dots,n$ (e_{ni} is i-th column od the identity matrix I_n), $x_{01} = x_{10} = 0$, $u_{ij} = 0$, $i,j \in \mathbb{Z}_+$. From equation (3.3a) for $i = j = 0$ and $u_{ij} = 0$, $i,j \in \mathbb{Z}_+$ we obtain $x_{11} = \overline{A}_0 e_{ni} = \overline{A}_{0i} \in \mathbb{R}_+^n$ where $\overline{A}_{0i}$ is i-th column of $\overline{A}_0$. This implies $\overline{A}_0 \in \mathbb{R}_+^{n\times n}$ since $i = 1,\dots,n$. If we assume that $x_{10} = e_{ni}$, $x_{00} = x_{01} = 0$ and $u_{ij} = 0$, $i,j \in \mathbb{Z}_+$ then from (3.3a) for $i = j = 0$ we obtain $x_{11} = \overline{A}_1 e_{ni} = \overline{A}_{1i} \in \mathbb{R}_+^n$ what implies $\overline{A}_1 \in \mathbb{R}_+^{n\times n}$. In a similar way we may show that $\overline{A}_2 \in \mathbb{R}_+^{n\times n}$. Assuming $u_{00} = e_{ni}$, $u_{ij} = 0$, $i,j \in \mathbb{Z}_+$ $i + j > 0$ and $x_{00} = x_{10} = x_{01} = 0$ from (3.3a), for $i = j = 0$, we obtain $x_{11} = B_0 e_{mi} = B_{0i} \in \mathbb{R}_+^m$ for $i = 1,\dots,m$ what implies $B_0 \in \mathbb{R}_+^{n\times m}$. The proof of $B_k \in \mathbb{R}_+^{n\times m}$ for $k = 1,2$ and $C \in \mathbb{R}_+^{p\times n}$, $D \in \mathbb{R}_+^{p\times m}$ is similar. □

3.6 Reachability and Controllability of Positive Fractional 2D Linear Systems

Definition 3.2. The positive fractional 2D linear system (3.2) is called reachable at the point $(h,k) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ if for zero boundary conditions (3.4) and every vector $x_f \in \mathbb{R}_+^n$ there exists a sequence of inputs $u_{ij} \in \mathbb{R}_+^m$ for

$$(i,j) \in D_{hk} = \{(i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : 0 \le i \le h, 0 \le j \le k, i + j \ne h + k\}, \tag{3.22}$$

which steers the state of the system from zero boundary conditions to the state x_f, i.e. $x_{hk} = x_f$.

Theorem 3.4. *The positive fractional 2D linear system (3.2) is reachable at the point (h,k) if and only if the reachability matrix*

$$R_{hk} = \left[M_0 \; M_1^1 \; \dots \; M_h^1 \; M_1^2 \; \dots \; M_k^2 \; M_{11} \; \dots \; M_{1k} \; M_{21} \; \dots \; M_{h,k} \right] \tag{3.23}$$

contains n linearly independent monomial columns, where:

$$\begin{aligned} M_0 &= T_{h-1,k-1} B_0, \\ M_i^1 &= T_{h-i,k-1} B_1 + T_{h-i-1,k-1} B_0, \quad i = 1,\dots,h; \\ M_j^2 &= T_{h-1,k-1} B_2 + T_{h-1,k-j-1} B_0, \quad j = 1,\dots,k; \\ M_{ij} &= T_{h-i-1,k-j-1} B_0 + T_{h-i,k-j-1} B_1 + T_{h-i-1,k-1} B_2, \\ &\quad i = 1,\dots,h; \quad j = 1,\dots,k; \end{aligned} \tag{3.24}$$

Proof. Using (3.5) for $i = h$, $j = k$ and zero boundary conditions we obtain

$$x_f = R_{hk} u(h,k), \tag{3.25}$$

where

$$u(h,k) = \left[u_{00}^T \; u_{10}^T \; \dots \; u_{h0}^T \; u_{01}^T \; \dots \; u_{0k}^T \; u_{11}^T \; \dots \; u_{1k}^T \; u_{21}^T \; \dots \; u_{h,k}^T \right]^T \tag{3.26}$$

and T denotes the transpose.

For positive fractional system (3.2) from (3.23) and (3.24) we have $M_0 \in \mathbb{R}_+^{n\times m}$, $M_i^1 \in \mathbb{R}_+^{n\times m}$, $M_j^2 \in \mathbb{R}_+^{n\times m}$, $M_{ij} \in \mathbb{R}_+^{n\times m}$, $i = 1,\dots,h$; $j = 1,\dots,k$; and matrix $R_{hk} \in \mathbb{R}_+^{n\times[(h+1)(k+1)-1]m}$. From (3.25) it follows that there exists a sequence of inputs $u_{ij} \in \mathbb{R}_+^m$ for $(i,j) \in D_{hk}$ for every $x_f \in \mathbb{R}_+^n$ if and only if the matrix (3.23) contains n linearly independent monomial columns. □

The following theorem formulates only the sufficient conditions for reachability of the positive fractional system (3.2).

Theorem 3.5. *The positive fractional 2D linear system(3.2) is reachable at the point* (h,k)*, if* $\operatorname{rank} R_{hk} = n$ *and the right inverse* R_{hk}^r *of the matrix (3.23) has nonnegative entries*

$$R_{hk}^r = R_{hk}^T \left[R_{hk} R_{hk}^T \right]^{-1} \in \mathbb{R}_+^{[(h+1)(k+1)-1]m\times n}. \tag{3.27}$$

Proof. If $rank R_{hk} = n$, then there exists the right inverse of R_{hk} and (3.27) holds then from equation (3.25) we obtain

$$u(h,k) = R_{hk}^r x_f \in \mathbb{R}_+^{[(h+1)(k+1)-1]m},$$

for every $x_f \in \mathbb{R}_+^n$. □

Example 3.1. Consider the positive fractional 2D linear system (3.2) with the matrices:

$$\overline{A}_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \overline{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \overline{A}_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \tag{3.28a}$$

$$B_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{3.28b}$$

To check the reachability at the point $(h,k) = (1,1)$, of the system we use Theorem 3.4. From (3.23) and (3.24) we obtain:

$$M_0 = B_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad M_1^1 = B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad M_1^2 = B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{3.29}$$

and

$$R_{11} = \left[M_0 \; M_1^1 \; M_1^2 \right] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}. \tag{3.30}$$

The first two columns of the matrix (3.30) are linearly independent monomial columns. By Theorem 3.4 the positive fractional system (3.2) with (3.28) is reachable at the point $(1,1)$. The input sequence which steers the state of the system from zero boundary conditions to any given state $x_f \in \mathbb{R}_+^2$ at the point $(1,1)$ is given by $\begin{bmatrix} u_{00} \\ u_{10} \end{bmatrix} = x_f$ and $u_{01} = 0$.

Using (3.27) and (3.30), we obtain

$$R_{hk}^r = R_{hk}^T \left[R_{hk} R_{hk}^T\right]^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}. \tag{3.31}$$

From (3.31) it follows that the condition (3.27), is not satisfied although the system is reachable at the point $(1,1)$. Note that the system is reachable at the paint $(1,1)$ for the arbitary order α, $0 < \alpha < 1$ and any matrices $\overline{A}_k$, $k = 0,1,2$.

Definition 3.3. The positive fractional 2D linear system (3.2) is called the system with finite memory if its characteristic polynomial has the form

$$\det[G(z_1, z_2)] = c z_1^{n_1} z_2^{n_2}, \tag{3.32}$$

where c is a constant coefficient.

Lemma 3.3. *If the positive fractional 2D linear system(3.2) is a system with finite memory then*

$$\begin{aligned} x_{bc}(i,j) &= \sum_{p=1}^{i} \left(T_{i-p,j-1}\overline{A}_1 + T_{i-p-1,j-1}\overline{A}_0\right) x_{p0} \\ &+ \sum_{q=1}^{j} \left(T_{i-1,j-q}\overline{A}_2 + T_{i-1,j-q-1}\overline{A}_0\right) x_{0q} \\ &+ T_{i-1,j-1}\overline{A}_0 x_{00} = 0, \end{aligned} \tag{3.33}$$

for $i \ge n_1$, $j \ge n_2$ and any nonzero boundary conditions (3.4).

Proof. Using (3.8) and (3.32), we obtain $T_{ij} = 0$ for $i \ge n_1$, $j \ge n_2$ and the equality (3.33) holds for any nonzero boundary conditions (3.4). □

Definition 3.4. The positive fractional 2D linear system (3.2) is controllable at the point $(h.k) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ for any nonzero boundary conditions:

$$x_{i0} \in \mathbb{R}_+^n, \quad i \in \mathbb{Z}_+ \quad \text{and} \quad x_{0j} \in \mathbb{R}_+^n, \quad j \in \mathbb{Z}_+, \tag{3.34}$$

if for every vector $x_f \in \mathbb{R}_+^n$ there exists a sequence of inputs $u_{ij} \in \mathbb{R}_+^m$ for $(i,j) \in D_{hk}$ such that $x_{hk} = x_f$.

Theorem 3.6. *The positive fractional 2D linear system (3.2) is controllable at the point* (h,k) $(h \geq n_1, k \geq n_2)$ *for any nonzero boundary conditions (3.4) if and only if it is a system with finite memory and the matrix (3.23) contains n linearly independent monomial columns.*

Proof. Using (3.5) for $i = h$, $j = k$ and taking into account that $x_{hk} = x_f$, we obtain

$$x_f - x_{bc}(h,k) = R_{hk}u(h,k), \tag{3.35}$$

where R_{hk} and $x_{bc}(h,k)$ are defined by (3.23) and (3.33), respectively.

If the positive fractional system (3.2) is a system with finite memory then by Lemma 3.3 there exists a point (h,k) $(h \geq n_1, k \geq n_2)$ such that the equality (3.33) is satisfied and $x_f = R_{hk}u(h,k)$. In this case by Theorem 3.4, there exists a sequence of inputs $u_{ij} \in \mathbb{R}_+^m$ for $(i,j) \in D_{hk}$, satisfied (3.25) if and only if the matrix (3.23) contains n linearly independent monomial columns. Otherwise $x_f - x_{bc}(h,k) \notin R_{hk}u(h,k)$, since boundary conditions (3.34) are arbitrary and the vector $x_f \in \mathbb{R}_+^n$ is also arbitrary. In this case does not exist a sequence of inputs $u_{ij} \in \mathbb{R}_+^m$ for $(i,j) \in D_{hk}$, satisfying the equality (3.35). □

3.7 Controllability to Zero of Positive Fractional 2D Linear System

Definition 3.5. The positive fractional 2D linear system (3.2) is called controllable to zero at the point (h,k) $(h \geq n_1, k \geq n_2)$ if for any nonzero boundary conditions (3.34) there exists a sequence of inputs $u_{ij} \in \mathbb{R}_+^m$ for $(i,j) \in D_{hk}$, which steers the state of the system from nonzero boundary conditions to the zero state $x_{hk} = 0$.

Theorem 3.7. *The positive fractional 2D linear system (3.2) is controllable to zero at the point* (h,k) $(h \geq n_1, k \geq n_2)$ *if and only if it is a system with finite memory.*

Proof. By Lemma 3.3 for a system with finite memory the condition (3.33) is satisfied for $h \geq n_1$, $k \geq n_2$. For $x_f = 0$ from (3.35) we have

$$x_{bc}(h,k) + R_{hk}u(h,k) = 0. \tag{3.36}$$

The equation (3.36) is satisfied for $u(h,k) = 0$. If the condition (3.33), is not satisfied then does not exist $u(h,k) \in \mathbb{R}_+^{[(h+1)(k+1)-1]m}$ satisfying (3.36), since for positive system $R_{hk} \in \mathbb{R}_+^{n \times [(h+1)(k+1)-1]m}$ and $x_{bc}(h,k) \in \mathbb{R}_+^n$. □

3.8 Models of 2D Linear Systems

3.8.1 Positive 2D Linear Systems

The model described by the equations:

$$\begin{aligned} x_{i+1,j+1} &= A_0 x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1} \\ &\quad + B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1}, \end{aligned} \tag{3.37a}$$

$$y_{ij} = C x_{ij} + D u_{ij}, \quad i, j \in \mathbb{Z}_+, \tag{3.37b}$$

is called the general model of 2D linear systems, where $x_{ij} \in \mathbb{R}^n$, $u_{ij} \in \mathbb{R}^m$, $y_{ij} \in \mathbb{R}^p$ are state, input and output vectors and $A_k \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times m}$, $k = 0,1,2$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Boundary conditions for (3.37) have the form:

$$x_{i0} \in \mathbb{R}^n, \quad i \in \mathbb{Z}_+, \quad x_{0j} \in \mathbb{R}^n, \quad j \in \mathbb{Z}_+ \ . \tag{3.38}$$

Definition 3.6. The model (system) (3.37) is called (internally) positive if $x_{ij} \in \mathbb{R}^n_+$ and $y_{ij} \in \mathbb{R}^p_+$, $i \in \mathbb{Z}_+$ for all boundary conditions $x_{i0} \in \mathbb{R}^n_+$, $i \in \mathbb{Z}_+$, $x_{0j} \in \mathbb{R}^n_+$, $j \in \mathbb{Z}_+$ and all inputs $u_{ij} \in \mathbb{R}^m_+$, $i, j \in \mathbb{Z}_+$.

Theorem 3.8. *The model (system) (3.37) is positive if and only if*

$$A_k \in \mathbb{R}^{n\times n}_+,\ B_k \in \mathbb{R}^{n\times m}_+ \textit{ for } k = 0,1,2,\ C \in \mathbb{R}^{p\times n}_+,\ D \in \mathbb{R}^{p\times m}_+. \tag{3.39}$$

Proof. The proof is given in [77].

Substituting (3.37a) $B_1 = B_2 = 0$ and $B_0 = B$, we obtain the first Fornasini-Marchesini model (FF-MM) and substituting in (3.37a) $A_0 = 0$ and $B_0 = 0$, we obtain the second Fornasini-Marchesini model (SF-MM).

The Roesser model of 2D linear system has the form:

$$\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix} u_{ij}, \tag{3.40a}$$

$$y_{ij} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} + D u_{ij}, \quad i, j \in \mathbb{Z}_+, \tag{3.40b}$$

where $x^h_{ij} \in \mathbb{R}^{n_1}$ and $x^v_{ij} \in \mathbb{R}^{n_2}$ are horizontal and vertical state vectors at the point (i, j), $u_{ij} \in \mathbb{R}^m$, $y_{ij} \in \mathbb{R}^p$ are input and output vectors and $A_{kl} \in \mathbb{R}^{n_k\times n_l}$, $k,l = 1,2$; $B_{11} \in \mathbb{R}^{n_1\times m}$, $B_{22} \in \mathbb{R}^{n_2\times m}$, $C_1 \in \mathbb{R}^{p\times n_1}$, $C_2 \in \mathbb{R}^{p\times n_2}$, $D \in \mathbb{R}^{p\times m}$.

Boundary conditions for (3.40) have the form:

$$x^h_{0j} \in \mathbb{R}^{n_1}, \quad j \in \mathbb{Z}_+, \quad x^v_{i0} \in \mathbb{R}^{n_2}, \quad i \in \mathbb{Z}_+ \ . \tag{3.41}$$

Definition 3.7. The Roesser model (3.40) is called (internally) positive if $x^h_{ij} \in \mathbb{R}^{n_1}_+$, $x^v_{ij} \in \mathbb{R}^{n_2}_+$ and $y_{ij} \in \mathbb{R}^p_+$, $i, j \in \mathbb{Z}_+$ for all $x^h_{0j} \in \mathbb{R}^{n_1}$, $j \in \mathbb{Z}_+$, $x^v_{i0} \in \mathbb{R}^{n_2}$, $i \in \mathbb{Z}_+$ and all inputs $u_{ij} \in \mathbb{R}^m_+$, $i, j \in \mathbb{Z}_+$.

Theorem 3.9. *The Roesser model is positive if and only if:*

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathbb{R}_+^{n\times n}, \quad \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix} \in \mathbb{R}_+^{n\times m}, \tag{3.42a}$$

$$\begin{bmatrix} C_1 & C_2 \end{bmatrix} \in \mathbb{R}_+^{p\times n}, \; D \in \mathbb{R}_+^{p\times m}, \; n = n_1 + n_2. \tag{3.42b}$$

The proof is given in [77].

Defining:

$$x_{ij} = \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix}, \; A_1 = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}, \; A_2 = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}, \tag{3.43a}$$

$$B_1 = \begin{bmatrix} 0 \\ B_{22} \end{bmatrix}, \; B_2 = \begin{bmatrix} B_{11} \\ 0 \end{bmatrix}, \tag{3.43b}$$

we may write the Roesser model in the form of SF-MM

$$x_{i+1,j+1} = A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_1 u_{i+1,j} + B_2 u_{i,j+1}. \tag{3.44}$$

3.8.2 Positive Fractional 2D Linear Systems

Definition 3.8. The fractional horizontal difference of of α-order of the discrete function x_{ij} is defined by the relation [166]

$$\Delta_\alpha^h x_{ij} = \sum_{k=0}^{i} c_\alpha(k) x_{i-k,j}, \tag{3.45a}$$

where $\alpha \in \mathbb{R}$, $n-1 < \alpha < n \in \mathbb{N} = 1,2,\ldots$ and

$$c_\alpha(k) = \begin{cases} 1 & \text{for} \quad k = 0 \\ (-1)^k \binom{\alpha}{k} = (-1)^k \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} & \text{for} \quad k > 0 \end{cases} \tag{3.45b}$$

Definition 3.9. The fractional vertical difference of β-order of the discrete function x_{ij} is defined by the relation [166]

$$\Delta_\beta^v x_{ij} = \sum_{l=0}^{j} c_\beta(l) x_{i,j-l}, \tag{3.46a}$$

where $\beta \in \mathbb{R}$, $n-1 < \beta < n \in \mathbb{N} = 1,2,\ldots$ and

$$c_\beta(l) = \begin{cases} 1 & \text{for} \quad l = 0 \\ (-1)^l \binom{\beta}{l} = (-1)^l \frac{\beta(\beta-1)\cdots(\beta-l+1)}{l!} & \text{for} \quad l > 0 \end{cases} \tag{3.46b}$$

Lemma 3.4. *If* $0 < \alpha < 1$ *(*$0 < \beta < 1$*), then*

$$c_\alpha(k) < 0, \; \big(c_\beta(l) < 0\big) \quad \textit{for} \quad k = 1,2,\ldots \; . \tag{3.47}$$

Proof. The proof will be accomplished by induction with respect to k. The hypothesis is true for $k = 1$, since from (3.45b) for $k = 1$ we have $c_\alpha(1) = -\alpha < 0$. Assuming that the hypothesis is valid for $k \geq 1$ we shall show that it is also true for $k+1$. From (3.45b) we have

$$c_\alpha(k+1) = (-1)^{k+1}\binom{\alpha}{k+1} = -(-1)^k\binom{\alpha}{k}\frac{(\alpha-k)}{k+1} = c_\alpha(k)\frac{k-\alpha}{k+1} < 0,$$

since $c_\alpha(k) < 0$. □

Consider the fractional 2D linear system:

$$\begin{bmatrix} \Delta_\alpha^h x_{i+1,j}^h \\ \Delta_\beta^v x_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{ij}, \tag{3.48a}$$

$$y_{ij} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}\begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} + Du_{ij}, \quad i,j \in \mathbb{Z}_+, \tag{3.48b}$$

where $x_{ij}^h \in \mathbb{R}^{n_1}$ and $x_{ij}^v \in \mathbb{R}^{n_2}$ are horizontal and vertical state vectors at the point (i,j), $u_{ij} \in \mathbb{R}^m$, $y_{ij} \in \mathbb{R}^p$ are input and output vectors at the point (i,j) and $A_{kl} \in \mathbb{R}^{n_k \times n_l}$, $k,l = 1,2$; $B_1 \in \mathbb{R}^{n_1 \times m}$, $B_2 \in \mathbb{R}^{n_n \times m}$, $C_1 \in \mathbb{R}^{p \times n_1}$, $C_2 \in \mathbb{R}^{p \times n_2}$, $D \in \mathbb{R}^{p \times m}$.

Using Definitions 3.8 and 3.9, we may write the equation (3.48a) in the form

$$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} \overline{A}_{11} & A_{12} \\ A_{21} & \overline{A}_{22} \end{bmatrix}\begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} - \begin{bmatrix} \sum_{k=2}^{i+1} c_\alpha(k) x_{i-k+1,j}^h \\ \sum_{l=2}^{j+1} c_\beta(l) x_{i,j-l+1}^v \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{ij}, \tag{3.49}$$

where $\overline{A}_{11} = A_{11} + \alpha I_{n_1}$ and $\overline{A}_{22} = A_{22} + \beta I_{n_2}$.

From (3.49) it follows that the fractional 2D linear system is a system with increasing number of delays in state vectors. From (3.45b) and (3.46b) it follows that the coefficients $c_\alpha(k)$ and $c_\beta(l)$ in (3.49) strongly decrease with increasing k and l. In practice usually it is assumed that k and l are bounded by some natural numbers L_1 and L_2. In this case the equation (3.49) takes the form

$$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \begin{bmatrix} \overline{A}_{11} & A_{12} \\ A_{21} & \overline{A}_{22} \end{bmatrix}\begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} - \begin{bmatrix} \sum_{k=2}^{L_1+1} c_\alpha(k) x_{i-k+1,j}^h \\ \sum_{l=2}^{L_2+1} c_\beta(l) x_{i,j-l+1}^v \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{ij}. \tag{3.50}$$

Boundary conditions for (3.48), (3.49) and (3.50) have the form:

$$x_{0j}^h \quad \text{for} \quad j \in \mathbb{Z}_+, \quad \text{and} \quad x_{i0}^v \quad \text{for} \quad i \in \mathbb{Z}_+ \ . \tag{3.51}$$

Theorem 3.10. *The solution of state equation (3.49) for the boundary conditions (3.51) has the form*

$$\begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} = \sum_{p=0}^{i} T_{i-p,j} \begin{bmatrix} 0 \\ x_{p0}^v \end{bmatrix} + \sum_{q=0}^{j} T_{i,j-q} \begin{bmatrix} x_{0q}^h \\ 0 \end{bmatrix} + \sum_{p=0}^{i} \sum_{q=0}^{j} \left(T_{i-p-1,j-q} B_{10} + T_{i-p,j-q-1} B_{01} \right) u_{pq}, \tag{3.52a}$$

where

$$B_{10} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad B_{01} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \tag{3.52b}$$

and the transition matrix $T_{pq} \in \mathbb{R}^{n \times n}$ is defined as follows

$$T_{pq} = \begin{cases} I_n & \text{for} \quad p = 0,\ q = 0 \\ T_{10} T_{p-1,q} + T_{01} T_{p,q-1} + \Upsilon & \text{for} \quad p + q > 0\ (p, q \in \mathbb{Z}_+) \\ 0 \ (\text{zero matrix}) & \text{for} \quad p < 0 \quad \text{and/or} \quad q < 0 \end{cases} \tag{3.52c}$$

where

$$\Upsilon = -\sum_{k=2}^{p} \begin{bmatrix} c_\alpha(k) I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} T_{p-k,q} - \sum_{l=2}^{q} \begin{bmatrix} 0 & 0 \\ 0 & c_\beta(l) I_{n_2} \end{bmatrix} T_{p,q-l} \tag{3.52d}$$

$$T_{10} = \begin{bmatrix} \overline{A}_{11} & A_{12} \\ 0 & 0 \end{bmatrix}, \quad T_{01} = \begin{bmatrix} 0 & 0 \\ A_{21} & \overline{A}_{22} \end{bmatrix}. \tag{3.52e}$$

Proof. Let $X(z_1, z_2)$ be the 2D z-transform of the discrete function x_{ij}

$$X(z_1, z_2) = \mathscr{Z}[x_{ij}] = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} x_{ij} z_1^{-i} z_2^{-j}. \tag{3.53}$$

Using (3.53), we obtain

$$\mathscr{Z}\left[x_{i+1,j}^h\right] = z_1 \left[X^h(z_1, z_2) - X^h(0, z_2)\right], \tag{3.54a}$$

where $X^h(0, z_2) = \sum_{j=0}^{\infty} x_{0j}^h z_2^{-j}$,

$$\mathscr{Z}\left[x_{i,j+1}^v\right] = z_2 \left[X^v(z_1, z_2) - X^v(z_1, 0)\right], \tag{3.54b}$$

where $X^v(z_1,0) = \sum_{i=0}^{\infty} x_{i0}^v z_1^{-i}$,

$$\mathscr{Z}\left[\sum_{k=2}^{i+1} c_\alpha(k) x_{i-k+1,j}^h\right] = \sum_{k=2}^{i+1} c_\alpha(k) z_1^{-k+1} X^h(z_1,z_2), \tag{3.54c}$$

since

$$\mathscr{Z}\left[x_{i-k,j}^h\right] = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{i-k,j}^h z_1^{-i} z_2^{-j} = \sum_{i=-k}^{\infty}\sum_{j=0}^{\infty} x_{ij}^h z_1^{-i-k} z_2^{-j}$$
$$= z_1^{-k}\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{ij}^h z_1^{-i} z_2^{-j}\right] = z_1^{-k} X^h(z_1,z_2). \tag{3.54d}$$

Similarly

$$\mathscr{Z}\left[\sum_{l=2}^{j+1} c_\beta(l) x_{i,j-l+1}^v\right] = \sum_{l=2}^{j+1} c_\beta(l) z_2^{-l+1} X^v(z_1,z_2), \tag{3.54e}$$

since

$$\mathscr{Z}\left[x_{i,j-l}^v\right] = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{i,j-l}^v z_1^{-i} z_2^{-j} = \sum_{i=0}^{\infty}\sum_{j=-l}^{\infty} x_{ij}^v z_1^{-i} z_2^{-j-l}$$
$$= z_2^{-l}\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{ij}^v z_1^{-i} z_2^{-j}\right] = z_2^{-l} X^v(z_1,z_2). \tag{3.54f}$$

Using (3.54), to (3.49) we obtain

$$\begin{bmatrix} z_1 X^h(z_1,z_2) - z_1 X^h(0,z_2) \\ z_2 X^v(z_1,z_2) - z_2 X^v(z_1,0) \end{bmatrix} = \begin{bmatrix} \overline{A}_{11} & A_{12} \\ A_{21} & \overline{A}_{22} \end{bmatrix} \begin{bmatrix} X^h(z_1,z_2) \\ X^v(z_1,z_2) \end{bmatrix}$$
$$- \begin{bmatrix} \sum_{k=2}^{i+1} c_\alpha(k) z_1^{-k+1} X^h(z_1,z_2) \\ \sum_{l=2}^{j+1} c_\beta(l) z_2^{-l+1} X^v(z_1,z_2) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(z_1,z_2) \tag{3.55}$$

where $U(z_1,z_2) = \mathscr{Z}(u_{ij})$.

Premultiplying (3.55) by the matrix blockdiag$[I_{n_1} z_1^{-1}, I_{n_2} z_2^{-1}]$, we obtain

$$\begin{bmatrix} X^h(z_1,z_2) \\ X^v(z_1,z_2) \end{bmatrix} = G^{-1}(z_1,z_2)\left\{\begin{bmatrix} z_1^{-1} B_1 \\ z_2^{-1} B_2 \end{bmatrix} U(z_1,z_2) + \begin{bmatrix} X^h(0,z_2) \\ X^v(z_1,0) \end{bmatrix}\right\} \tag{3.56}$$

where

$$G(z_1,z_2) = \begin{bmatrix} G_{11}(z_1,z_2) & -z_1^{-1} A_{12} \\ -z_2^{-1} A_{21} & G_{22}(z_1,z_2) \end{bmatrix} \tag{3.57}$$

and

$$G_{11}(z_1,z_2) = I_{n_1} - z_1^{-1}\overline{A}_{11} + \sum_{k=2}^{i} c_\alpha(k) z_1^{-k} I_{n_1},$$

$$G_{22}(z_1,z_2) = I_{n_2} - z_2^{-1}\overline{A}_{22} + \sum_{l=2}^{j} c_\beta(l) z_2^{-l} I_{n_2}.$$

Let

$$G^{-1}(z_1,z_2) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq} z_1^{-p} z_2^{-q} \tag{3.58}$$

and

$$T_{pq} = \begin{bmatrix} T_{pq}^{11} & T_{pq}^{12} \\ T_{pq}^{21} & T_{pq}^{22} \end{bmatrix} \tag{3.59}$$

where T_{pq}^{kl} have the same dimension as A_{kl} for $k,l = 1,2$.

From the equality

$$G^{-1}(z_1,z_2)G(z_1,z_2) = G(z_1,z_2)G^{-1}(z_1,z_2) = I_n$$

and (3.58) and (3.59) we have

$$\begin{bmatrix} I_{n_1} - z_1^{-1}\overline{A}_{11} + \sum_{k=2}^{i} c_\alpha(k) z_1^{-k} I_{n_1} & -z_1^{-1}A_{12} \\ -z_2^{-1}A_{21} & I_{n_2} - z_2^{-1}\overline{A}_{22} + \sum_{l=2}^{j} c_\beta(l) z_2^{-l} I_{n_2} \end{bmatrix}$$
$$\times \left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \begin{bmatrix} T_{pq}^{11} & T_{pq}^{12} \\ T_{pq}^{21} & T_{pq}^{22} \end{bmatrix} z_1^{-p} z_2^{-q} \right) = \begin{bmatrix} I_{n_1} & 0 \\ 0 & I_{n_2} \end{bmatrix} \tag{3.60}$$

From (3.60) it follows that

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \left(T_{pq}^{11} - \overline{A}_{11} T_{p-1,q}^{11} + \sum_{k=2}^{p} c_\alpha(k) T_{p-k,q}^{11} - A_{12} T_{p-1,q}^{21} \right) z_1^{-p} z_2^{-q} = I_{n_1} \tag{3.61a}$$

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \left(T_{pq}^{12} - \overline{A}_{11} T_{p-1,q}^{12} + \sum_{k=2}^{p} c_\alpha(k) T_{p-k,q}^{12} - A_{12} T_{p-1,q}^{22} \right) z_1^{-p} z_2^{-q} = 0 \tag{3.61b}$$

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \left(T_{pq}^{21} - \overline{A}_{22} T_{p,q-1}^{21} + \sum_{l=2}^{q} c_\beta(l) T_{p,q-l}^{21} - A_{21} T_{p,q-1}^{11} \right) z_1^{-p} z_2^{-q} = 0 \tag{3.61c}$$

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \left(T_{pq}^{22} - \overline{A}_{22} T_{p,q-1}^{22} + \sum_{l=2}^{q} c_\beta(l) T_{p,q-l}^{22} - A_{21} T_{p,q-1}^{12} \right) z_1^{-p} z_2^{-q} = I_{n_2} \tag{3.61d}$$

Comparing the coefficients at the same powers of z_1 i z_2 in the equation (3.61), we obtain (3.52c).

Using (3.58) and applying the inverse z-transform and the convolution theorem to (3.56), we obtain (3.52a). □

Consider the system (3.50) and

$$\overline{G}(z_1,z_2) = \begin{bmatrix} \overline{G}_{11}(z_1,z_2) & -z_1^{-1}A_{12} \\ -z_2^{-1}A_{21} & \overline{G}_{22}(z_1,z_2) \end{bmatrix} \tag{3.62}$$

where

$$\overline{G}_{11}(z_1,z_2) = I_{n_1} - z_1^{-1}\overline{A}_{11} + \sum_{k=2}^{L_1} c_\alpha(k) z_1^{-k} I_{n_1},$$
$$\overline{G}_{22}(z_1,z_2) = I_{n_2} - z_2^{-1}\overline{A}_{22} + \sum_{l=2}^{L_2} c_\beta(l) z_2^{-l} I_{n_2}.$$

Let

$$\det \overline{G}(z_1,z_2) = \sum_{p=0}^{N_1}\sum_{q=0}^{N_2} a_{N_1-p,N_2-q} z_1^{-p} z_2^{-q}, \tag{3.63}$$

where $N_1, N_2 \in \mathbb{Z}_+$ are defined by the natural numbers L_1 i L_2 in (3.50).

Theorem 3.11. *Let (3.63) be the characteristic polynomial of the system (3.50). The matrices T_{pq} satisfy the equation*

$$\sum_{p=0}^{N_1}\sum_{q=0}^{N_2} a_{pq} T_{pq} = 0. \tag{3.64}$$

The proof is similar to the proof of Theorem 3.2 [166].

Theorem 3.11 is an extension of the classical Cayley-Hamilton theorem to the fractional 2D linear systems described by the Roesser model (3.49).

Definition 3.10. The system (3.49) is called (internally) positive fractional 2D Roesser model if $x_{ij}^h \in \mathbb{R}_+^{n_1}$, $x_{ij}^v \in \mathbb{R}_+^{n_2}$ and $y_{ij} \in \mathbb{R}_+^p$, $i,j \in \mathbb{Z}_+$ for any boundary conditions $x_{0j}^h \in \mathbb{R}_+^{n_1}$, $j \in \mathbb{Z}_+$, $x_{i0}^v \in \mathbb{R}_+^{n_2}$, $i \in \mathbb{Z}_+$ and all inputs $u_{ij} \in \mathbb{R}_+^m$, $i,j \in \mathbb{Z}_+$.

Theorem 3.12. *The fractional Roesser model (3.49) for $\alpha,\beta \in \mathbb{R}$, $0 < \alpha \le 1$, $0 < \beta \le 1$ is positive if and only if*

$$\begin{bmatrix} \overline{A}_{11} & A_{12} \\ A_{21} & \overline{A}_{22} \end{bmatrix} \in \mathbb{R}_+^{n\times n}, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \in \mathbb{R}_+^{n\times m}, \quad \begin{bmatrix} C_1 & C_2 \end{bmatrix} \in \mathbb{R}_+^{p\times n}, \quad D \in \mathbb{R}_+^{p\times m}, \tag{3.65}$$

The proof is similar to the proof of Theorem 3.3 [166].

3.8.3 Positive 2D Linear Systems with Delays

Consider the autonomous 2D Roesser model with q delays in state vector

$$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \sum_{k=0}^{q} A_k \begin{bmatrix} x_{i-k,j}^h \\ x_{i,j-k}^v \end{bmatrix}, \quad i,j \in \mathbb{Z}_+, \tag{3.66}$$

where $x_{ij}^h \in \mathbb{R}_+^{n_1}$, $x_{ij}^v \in \mathbb{R}_+^{n_2}$ are horizontal and vertical state vectors at the point (i,j) and

$$A_k = \begin{bmatrix} A_{11}^k & A_{12}^k \\ A_{21}^k & A_{22}^k \end{bmatrix}, \quad k = 0,1,\dots,q \ . \tag{3.67}$$

Defining:

$$\overline{x}_{ij}^h = \begin{bmatrix} x_{ij}^h \\ x_{i-1,j}^h \\ \vdots \\ x_{i-q,j}^h \end{bmatrix}, \quad \overline{x}_{ij}^v = \begin{bmatrix} x_{ij}^v \\ x_{i,j-1}^v \\ \vdots \\ x_{i,j-q}^v \end{bmatrix}, \tag{3.68}$$

we may write the equation (3.66) in the form

$$\begin{bmatrix} \overline{x}_{i+1,j}^h \\ \overline{x}_{i,j+1}^v \end{bmatrix} = A \begin{bmatrix} \overline{x}_{i,j}^h \\ \overline{x}_{i,j}^v \end{bmatrix}, \quad i,j \in \mathbb{Z}_+, \tag{3.69}$$

where

$$A = \begin{bmatrix} A_{11}^0 & A_{11}^1 & \dots & A_{11}^{q-1} & A_{11}^q & A_{12}^0 & A_{12}^1 & \dots & A_{12}^{q-1} & A_{12}^q \\ I_{n_1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I_{n_1} & 0 & 0 & 0 & \dots & 0 & 0 \\ A_{21}^0 & A_{21}^1 & \dots & A_{21}^{q-1} & A_{21}^q & A_{22}^0 & A_{22}^1 & \dots & A_{22}^{q-1} & A_{22}^q \\ 0 & 0 & \dots & 0 & 0 & I_{n_2} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & I_{n_2} & 0 \end{bmatrix} \in \mathbb{R}^{N \times N}, \tag{3.70}$$

$$N = (q+1)(n_1+n_2).$$

The Roesser model with q delays (3.66) has been reduced to Roesser model without delays but with greater dimensions.

Theorem 3.13. *The Roesser model with q delays (3.66) is positive if and only if*

$$A_k \in \mathbb{R}_+^{(n_1+n_1)\times(n_2+n_2)} \quad \textit{for} \quad k = 0,1,\dots,q \quad \textit{or equivalently} \quad A \in \mathbb{R}_+^{N\times N}. \tag{3.71}$$

The proof follows immediately from Theorem 3.9, applied to the model (3.69).

Consider the autonomous general model with q delays

$$x_{i+1,j+1} = \sum_{k=0}^{q} \left(A_k^0 x_{i-k,j-k} + A_k^1 x_{i+1-k,j-k} + A_k^2 x_{i-k,j+1-k} \right), \quad i,j \in \mathbb{Z}_+, \tag{3.72}$$

where $x_{ij} \in \mathbb{R}_+^n$ is the state vector at the point (i,j) and $A_k^t \in \mathbb{R}^{n\times n}$, $k = 0,1,\ldots,q$; $t = 0,1,2$.

Defining vector

$$\bar{x}_{ij} = \begin{bmatrix} x_{ij} \\ x_{i-1,j-1} \\ \vdots \\ x_{i-q,j-q} \end{bmatrix}, \tag{3.73}$$

and the matrices

$$\overline{A}_0 = \begin{bmatrix} A_0^0 & A_1^0 & \ldots & A_{q-1}^0 & A_q^0 \\ I_n & 0 & \ldots & 0 & 0 \\ 0 & I_n & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & I_n & 0 \end{bmatrix}, \quad \overline{A}_1 = \begin{bmatrix} A_0^1 & A_1^1 & \ldots & A_{q-1}^1 & A_q^1 \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix},$$

$$\overline{A}_2 = \begin{bmatrix} A_0^2 & A_1^2 & \ldots & A_{q-1}^2 & A_q^2 \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \end{bmatrix}, \tag{3.74}$$

we may write (3.72) in the form

$$\bar{x}_{i+1,j+1} = \overline{A}_0 \bar{x}_{i,j} + \overline{A}_1 \bar{x}_{i+1,j} + \overline{A}_2 \bar{x}_{i,j+1}, \quad i,j \in \mathbb{Z}_+, \tag{3.75}$$

The general 2D model with q delays has been reduced to the equivalent general 2D model without delays but with greater dimensions.

Theorem 3.14. *The general 2D model with q delays (3.72) is positive if and only if* $A_k^t \in \mathbb{R}_+^{n\times n}$ *for* $k = 0,1,\ldots,q$*;* $t = 0,1,2$ *or equivalently if and only if* $\overline{A}_t \in \mathbb{R}_+^{\overline{N}\times\overline{N}}$*,* $t = 0,1,2$*;* $\overline{N} = (q+1)n$*.*

The proof follows immediately from Theorem 3.8 applied to the model (3.75).

3.9 Positive Fractional 2D Linear System of Different Orders

3.9.1 Definition of (Backward) Difference of (α,β) Order of 2D Function

Definition 3.11. The function defined by

$$\Delta^{\alpha,\beta}x_{ij}=\sum_{k=0}^{i}\sum_{l=0}^{j}(-1)^{k+l}\binom{\alpha}{k}\binom{\beta}{l}x_{i-k,j-l}=\sum_{k=0}^{i}\sum_{l=0}^{j}c_{\alpha\beta}(k,l)x_{i-k,j-l}, \quad (3.76a)$$

$$n_1-1<\alpha<n_1,\quad n_2-1<\beta<n_2,\quad n_1,n_2\in\mathbb{N},\quad \alpha,\beta\in\mathbb{R},$$

is called the (backward) difference of (α,β)-order of the function x_{ij} where

$$\Delta^{\alpha,\beta}x_{ij}=\Delta_i^{\alpha}\Delta_j^{\beta}x_{ij},$$

and

$$c_{\alpha\beta}(k,l)=\begin{cases}1 & \text{for } k=0 \text{ and } l=0\\ (-1)^k\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} & \text{for } k>0 \text{ and } l=0\\ (-1)^l\frac{\beta(\beta-1)\cdots(\beta-l+1)}{l!} & \text{for } k=0 \text{ and } l>0\\ (-1)^{k+l}\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)\beta(\beta-1)\cdots(\beta-l+1)}{k!l!} & \text{for } k>0 \text{ and } l>0\end{cases} \quad (3.76b)$$

3.9.2 State Equations of Fractional 2D Linear System

The state equations of the general fractional 2D model of linear systems have the form:

$$\begin{aligned}\Delta^{\alpha,\beta}x_{i+1,j+1} &= A_0x_{ij}+A_1x_{i+1,j}+A_2x_{i,j+1}\\ &+ B_0u_{ij}+B_1u_{i+1,j}+B_2u_{i,j+1}, \quad (3.77a)\\ y_{ij} &= Cx_{ij}+Du_{ij}, \quad (3.77b)\end{aligned}$$

where $x_{ij}\in\mathbb{R}^n$, $u_{ij}\in\mathbb{R}^m$, $y_{ij}\in\mathbb{R}^p$ are state, input and output vectors and $A_k\in\mathbb{R}^{n\times n}$, $B_k\in\mathbb{R}^{n\times m}$, $k=0,1,2$, $C\in\mathbb{R}^{p\times n}$, $D\in\mathbb{R}^{p\times m}$.

From (3.76a) we have

$$\begin{aligned}\Delta^{\alpha,\beta}x_{i+1,j+1} &= \sum_{k=0}^{i+1}\sum_{l=0}^{j+1}c_{\alpha\beta}(k,l)x_{i-k+1,j-l+1}\\ &= x_{i+1,j+1}+\sum_{k=0}^{i+1}\sum_{\substack{l=0\\k+l>0}}^{j+1}c_{\alpha\beta}(k,l)x_{i-k+1,j-l+1}. \quad (3.78)\end{aligned}$$

Using (3.78) we may write the equation (3.77a) in the form

$$x_{i+1,j+1} = \overline{A}_0 x_{ij} + \overline{A}_1 x_{i+1,j} + \overline{A}_2 x_{i,j+1} - \sum_{\substack{k=0 \\ k,l\neq 1}}^{i+1} \sum_{\substack{l=0 \\ k+l>0}}^{j+1} c_{\alpha\beta}(k,l) x_{i-k+1,j-l+1} + B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1}, \tag{3.79a}$$

where:

$$\begin{aligned} \overline{A}_0 &= A_0 - c_{\alpha\beta}(1,1) = A_0 - \alpha\beta I_n, \\ \overline{A}_1 &= A_1 - c_{\alpha\beta}(0,1) = A_1 + \beta I_n, \\ \overline{A}_2 &= A_2 - c_{\alpha\beta}(1,0) = A_2 + \alpha I_n. \end{aligned} \tag{3.79b}$$

Let

$$D_{ij} = \{k,l \in \mathbb{Z}_+, \quad 0 \le k \le i, \quad 0 \le l \le j\}, \quad D = D_{i+1,j+1} \setminus D_{11},$$

then the equation (3.79a) takes the form

$$x_{i+1,j+1} = \overline{A}_0 x_{ij} + \overline{A}_1 x_{i+1,j} + \overline{A}_2 x_{i,j+1} - \sum_{i,j\in D}\sum c_{\alpha\beta}(k,l) x_{i-k+1,j-l+1} + B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1}. \tag{3.79c}$$

From (3.76b) it follows that the coefficients $c_{\alpha\beta}(k,l)$ strongly decrease when k and l increase. In practice usually it is assumed that i and j are bounded by some natural numbers L_1 and L_2. In this case the equation (3.79a) takes the form

$$x_{i+1,j+1} = \overline{A}_0 x_{ij} + \overline{A}_1 x_{i+1,j} + \overline{A}_2 x_{i,j+1} - \sum_{\substack{k=0 \\ k,l\neq 1}}^{L_1+1} \sum_{\substack{l=0 \\ k+l>0}}^{L_2+1} c_{\alpha\beta}(k,l) x_{i-k+1,j-l+1} + B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1}. \tag{3.79d}$$

Remark 3.2. From (3.79) if follows that the fractional 2D linear system is a system with increasing numbers of delays in state vector.

Boundary conditions for (3.79) have the form:

$$x_{i0}, i \in \mathbb{Z}_+, \quad \text{and} \quad x_{0j}, j \in \mathbb{Z}_+. \tag{3.80}$$

3.9.3 Solution of the State Equations of the Fractional 2D Linear Systems

Applying the 2D z-transform ($\mathscr{Z}$) we shall derive the solution of the state equation (3.79a) of the fractional 2D linear system.

Theorem 3.15. *The solution of the state equation (3.79a) with boundary conditions (3.80) has the form*

$$\begin{aligned}
x_{ij} = &\sum_{p=0}^{i-1}\sum_{q=0}^{j-1} T_{i-p-1,j-q-1}B_0 u_{pq} + \sum_{p=0}^{i}\sum_{q=0}^{j-1} T_{i-p,j-q-1}B_1 u_{pq} \\
&+ \sum_{p=0}^{i-1}\sum_{q=0}^{j} T_{i-p-1,j-q}B_2 u_{pq} + \sum_{p=0}^{i} T_{i-p,j}x_{p0} + \sum_{q=0}^{j} T_{i,j-q}x_{0q} - T_{ij}x_{00} \\
&- \sum_{q=0}^{j-1} T_{i,j-q-1}\left[\overline{A}_1 \; B_1\right]\begin{bmatrix} x_{0q} \\ u_{0q}\end{bmatrix} - \sum_{p=0}^{i-1} T_{i-p-1,j}\left[\overline{A}_2 \; B_2\right]\begin{bmatrix} x_{p0} \\ u_{p0}\end{bmatrix} \\
&+ \sum_{k=2}^{i+1}\sum_{p=0}^{i-k} c_{\alpha\beta}(k,0)T_{i-p-k,j}x_{p0} + \sum_{l=2}^{j+1}\sum_{q=0}^{j-l} c_{\alpha\beta}(0,l)T_{i,j-q-l}x_{0q}
\end{aligned} \tag{3.81}$$

where the matrices T_{pq} are defined as follows

$$T_{pq} = \begin{cases} I_n & \text{for} \quad p=q=0 \\ \overline{A}_0 T_{p-1,q-1} + \overline{A}_1 T_{p,q-1} + \overline{A}_2 T_{p-1,q} - \Upsilon & \text{for} \quad p+q>0 \\ 0 \quad \text{(zero matrix)} & \text{for} \quad p<0 \quad \text{and/or} \quad q<0 \end{cases} \tag{3.82}$$

and

$$\Upsilon = \sum_{k=0}^{p}\sum_{l=0}^{q} c_{\alpha\beta}(p-k,q-l)T_{kl} \quad \text{for} \quad k,l \neq p-1,q-1 \quad \text{and} \quad k+l<p+q-1.$$

Proof. Let $X(z_1,z_2)$ be the 2D z-transform of the discrete function x_{ij}, defined by (A.15). Taking in to account

$$
\begin{aligned}
&\mathscr{Z}\left[\sum_{\substack{k=0\\k,l\neq 1}}^{i+1}\sum_{\substack{l=0\\k+l>0}}^{j+1} c_{\alpha\beta}(k,l)x_{i-k+1,j-l+1}\right] = \mathscr{Z}\left[\sum_{\substack{k=1\\k,l\neq 1}}^{i+1}\sum_{l=1}^{j+1} c_{\alpha\beta}(k,l)x_{i-k+1,j-l+1}\right.\\
&\left. + \sum_{k=2}^{i+1} c_{\alpha\beta}(k,0)x_{i-k+1,j+1} + \sum_{l=2}^{j+1} c_{\alpha\beta}(0,l)x_{i+1,j-l+1}\right]\\
&= \sum_{\substack{k=1\\k,l\neq 1}}^{i+1}\sum_{l=1}^{j+1} c_{\alpha\beta}(k,l)z_1^{-k+1}z_2^{-l+1}X(z_1,z_2)\\
&+ \sum_{k=2}^{i+1} c_{\alpha\beta}(k,0)z_1^{-k+1}z_2[X(z_1,z_2)-X(z_1,0)]\\
&+ \sum_{l=2}^{j+1} c_{\alpha\beta}(0,l)z_1z_2^{-l+1}[X(z_1,z_2)-X(0,z_2)]\\
&= \sum_{\substack{k=0\\k,l\neq 1}}^{i+1}\sum_{\substack{l=0\\k+l>1}}^{j+1} c_{\alpha\beta}(k,l)z_1^{-k+1}z_2^{-l+1}X(z_1,z_2)\\
&- \sum_{k=2}^{i+1} c_{\alpha\beta}(k,0)z_1^{-k+1}z_2X(z_1,0) - \sum_{l=2}^{j+1} c_{\alpha\beta}(0,l)z_1z_2^{-l+1}X(0,z_2),
\end{aligned}
$$

and applying the 2D z-transform to (3.79a) and using Appendix A.3, we obtain

$$
\begin{aligned}
X(z_1,z_2) &= G^{-1}(z_1,z_2)\{(B_0+B_1z_1+B_2z_2)U(z_1,z_2)\\
&+ z_1z_2[X(z_1,0)+X(0,z_2)-x_{00}]\\
&+ \sum_{l=2}^{j+1} c_{\alpha\beta}(0,l)z_1z_2^{-l+1}X(0,z_2) + \sum_{k=2}^{i+1} c_{\alpha\beta}(k,0)z_1^{-k+1}z_2X(z_1,0)\\
&- z_1\left[\overline{A}_1 B_1\right]\begin{bmatrix} X(0,z_2)\\ U(0,z_2)\end{bmatrix} - z_2\left[\overline{A}_2 B_2\right]\begin{bmatrix} X(z_1,0)\\ U(z_1,0)\end{bmatrix}\Bigg\},
\end{aligned}
\tag{3.83a}
$$

where

$$
G(z_1,z_2) = \left[z_1z_2I_n + \sum_{\substack{k=0\\k,l\neq 1}}^{i+1}\sum_{\substack{l=0\\k+l>1}}^{j+1} c_{\alpha\beta}(k,l)z_1^{-(k-1)}z_2^{-(l-1)} - \overline{A}_0 - \overline{A}_1z_1 - \overline{A}_2z_2\right]
\tag{3.83b}
$$

and $U(z_1,z_2) = \mathscr{Z}[u_{ij}]$.

Let

$$
G^{-1}(z_1,z_2) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq}z_1^{-(p+1)}z_2^{-(q+1)}.
\tag{3.84}
$$

From the equality

$$G^{-1}(z_1,z_2)G(z_1,z_2) = G(z_1,z_2)G^{-1}(z_1,z_2) = I_n,$$

we have

$$\begin{aligned} I_n &= \left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq} z_1^{-(p+1)} z_2^{-(q+1)}\right) G(z_1,z_2) \\ &= G(z_1,z_2)\left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq} z_1^{-(p+1)} z_2^{-(q+1)}\right). \end{aligned} \tag{3.85}$$

Substituting of (3.83b) into (3.85) yields

$$\begin{aligned} I_n &= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq} z_1^{-p} z_2^{-q} + \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \sum_{\substack{k=0 \\ k,l\neq 1}}^{i+1} \sum_{\substack{l=0 \\ k+l>1}}^{j+1} c_{\alpha\beta}(k,l) z_1^{-(p+k)} z_2^{-(q+l)} \\ &- \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \overline{A}_0 T_{pq} z_1^{-(p+1)} z_2^{-(q+1)} - \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \overline{A}_1 T_{pq} z_1^{-p} z_2^{-(q+1)} \\ &- \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \overline{A}_2 T_{pq} z_1^{-(p+1)} z_2^{-q}, \end{aligned}$$

and

$$\begin{aligned} I_n = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \Bigg[& T_{pq} + \sum_{\substack{k=0 \\ k,l\neq 1}}^{i+1} \sum_{\substack{l=0 \\ k+l>1}}^{j+1} c_{\alpha\beta}(k,l) T_{p-k,q-l} - \overline{A}_0 T_{p-1,q-1} \\ & - \overline{A}_1 T_{p,q-1} - \overline{A}_2 T_{p-1,q} \Bigg] z_1^{-p} z_2^{-q}. \end{aligned}$$

Comparing the coefficients at the same powers of z_1 and z_2 in equation (3.85) we obtain (3.82). Substituting (3.84) into (3.83a) we obtain

$$\begin{aligned} X(z_1,z_2) = & \left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq} z_1^{-(p+1)} z_2^{-(q+1)}\right) \{(B_0 + B_1 z_1 + B_2 z_2)U(z_1,z_2) \\ & + z_1 z_2 [X(z_1,0) + X(0,z_2) - x_{00}] \\ & - z_1 \left[\overline{A}_1 B_1\right] \begin{bmatrix} X(0,z_2) \\ U(0,z_2) \end{bmatrix} - z_2 \left[\overline{A}_2 B_2\right] \begin{bmatrix} X(z_1,0) \\ U(z_1,0) \end{bmatrix} \\ & + \sum_{l=2}^{j+1} c_{\alpha\beta}(0,l) z_1 z_2^{-l+1} X(0,z_2) + \sum_{k=2}^{i+1} c_{\alpha\beta}(k,0) z_1^{-k+1} z_2 X(z_1,0) \Bigg\}, \end{aligned} \tag{3.86}$$

Applying the inverse 2D z-transform to the equation (3.86) and taking into account $T_{pq} = 0$ for $p < 0$ and $q < 0$, we obtain the desired solution (3.81). □

Using (3.82) it can be easily shown that for $i, j \in \mathbb{Z}_+$ we have

$$
\begin{aligned}
x_{ij} = &\sum_{p=0}^{i}\sum_{q=0}^{j} T_{i-p-1,j-q-1}B_0u_{pq} + \sum_{p=1}^{i}\sum_{q=1}^{j}\left(T_{i-p,j-q-1}B_1 + T_{i-p-1,j-q}B_2\right)u_{pq} \\
&+ \sum_{p=1}^{i} T_{i-p,j-1}B_1u_{p0} + \sum_{q=1}^{j} T_{i-1,j-q}B_2u_{0q} + \sum_{k=2}^{i}\sum_{p=0}^{i-k} c_{\alpha\beta}(k,0)T_{i-p-k,j}x_{p0} \\
&+ \sum_{p=1}^{i}\left(T_{i-p-1,j-1}\overline{A}_0 + T_{i-p,j-1}\overline{A}_1 - \sum_{\substack{k=0\\D_1}}^{i-p}\sum_{\substack{l=0\\D_2}}^{j} c_{\alpha\beta}(i-p-k,j-l)T_{k,l}\right)x_{p0} \\
&+ \sum_{q=1}^{j}\left(T_{i-1,j-q-1}\overline{A}_0 + T_{i-1,j-q}\overline{A}_2 - \sum_{\substack{k=0\\D_3}}^{i}\sum_{\substack{l=0\\D_4}}^{j-q} c_{\alpha\beta}(i-k,j-q-l)T_{kl}\right)x_{0q} \\
&+ \sum_{l=2}^{j}\sum_{q=0}^{j-l} c_{\alpha\beta}(0,l)T_{i,j-q-l}x_{0q} \\
&+ \left(T_{i-1,j-1}\overline{A}_0 - \sum_{\substack{k=0\\D_5}}^{i}\sum_{\substack{l=0\\D_6}}^{j} c_{\alpha\beta}(i-k,j-l)T_{kl}\right)x_{00},
\end{aligned}
$$

where:

$$
\begin{aligned}
&D_1 = k+l < i+j-p-1, \quad D_2 = k,l \neq i-p-1, j-1, \\
&D_3 = k+l < i+j-q-1, \quad D_4 = k,l \neq i-1, j-q-1, \\
&D_5 = k+l < i+j-1, \qquad D_6 = k,l \neq i-1, j-1.
\end{aligned}
\tag{3.87}
$$

After some manipulations the solution can be rewritten in the form

$$
\begin{aligned}
x_{ij} &= \sum_{p=0}^{i}\sum_{q=0}^{j} T_{i-p-1,j-q-1}B_0u_{pq} + \sum_{p=1}^{i}\sum_{q=1}^{j}\left(T_{i-p,j-q-1}B_1 + T_{i-p-1,j-q}B_2\right)u_{pq} \\
&+ \sum_{p=1}^{i} T_{i-p,j-1}B_1u_{p0} + \sum_{q=1}^{j} T_{i-1,j-q}B_2u_{0q} \\
&+ \sum_{p=1}^{i}\left(T_{i-p-1,j-1}\overline{A}_0 + T_{i-p,j-1}\overline{A}_1 - \sum_{\substack{k=0\\k+l>1}}^{i-p}\sum_{\substack{l=1\\k,l\neq 1}}^{j} c_{\alpha\beta}(k,l)T_{i-p-k,j-l}\right)x_{p0} \\
&+ \sum_{q=1}^{j}\left(T_{i-1,j-q-1}\overline{A}_0 + T_{i-1,j-q}\overline{A}_2 - \sum_{\substack{k=1\\k+l>1}}^{i}\sum_{\substack{l=0\\k,l\neq 1}}^{j-q} c_{\alpha\beta}(k,l)T_{i-k,j-q-l}\right)x_{0q} \\
&+ \left(T_{i-1,j-1}\overline{A}_0 - \sum_{\substack{k=1\\k,l\neq 1}}^{i}\sum_{l=1}^{j} c_{\alpha\beta}(k,l)T_{i-k,j-l}\right)x_{00},
\end{aligned}
$$

or

$$
\begin{aligned}
x_{ij} &= \sum_{p=0}^{i}\sum_{q=0}^{j} T_{i-p-1,j-q-1}B_0u_{pq} + \sum_{p=1}^{i}\sum_{q=1}^{j}\left(T_{i-p,j-q-1}B_1 + T_{i-p-1,j-q}B_2\right)u_{pq} \\
&+ \sum_{p=1}^{i} T_{i-p,j-1}B_1u_{p0} + \sum_{q=1}^{j} T_{i-1,j-q}B_2u_{0q} \\
&+ \sum_{p=1}^{i}\left(T_{i-p-1,j-1}\overline{A}_0 + T_{i-p,j-1}\overline{A}_1 - \sum_{\substack{k=0\\D_1}}^{i-p}\sum_{\substack{l=0\\D_2}}^{j-1} c_{\alpha\beta}(i-p-k,j-l)T_{k,l}\right)x_{p0} \\
&+ \sum_{q=1}^{j}\left(T_{i-1,j-q-1}\overline{A}_0 + T_{i-1,j-q}\overline{A}_2 - \sum_{\substack{k=0\\D_3}}^{i-1}\sum_{\substack{l=0\\D_4}}^{j-q} c_{\alpha\beta}(i-k,j-q-l)T_{k,l}\right)x_{0q} \\
&+ \left(T_{i-1,j-1}\overline{A}_0 - \sum_{\substack{k=0\\D_7}}^{i-1}\sum_{l=0}^{j-1} c_{\alpha\beta}(i-k,j-l)T_{k,l}\right)x_{00}.
\end{aligned}
$$

where D_i, $i = 1,2,3,4$; are given by (3.87) and $D_7 = k+l < i+j-2$.

3.9.4 Extension of the Cayley-Hamilton Theorem

From (3.83b) we have

$$
G(z_1,z_2) = z_1z_2\overline{G}(z_1,z_2), \tag{3.88}
$$

where

$$\overline{G}(z_1,z_2) = I_n + \sum_{k=0}^{L_1+1}\sum_{l=0}^{L_2+1} I_n c_{\alpha\beta}(k,l) z_1^{-k} z_2^{-l} - \overline{A}_0 z^{-1} z_2^{-1} - \overline{A}_q z_2^{-1} - \overline{A}_2 z_1^{-1}. \quad (3.89)$$

and

$$\det\left[\overline{G}(z_1,z_2)\right] = \sum_{k=0}^{N_1}\sum_{l=0}^{N_2} a_{N_1-k,N_2-l} z_1^{-k} z_2^{-l}. \quad (3.90)$$

It is assumed that i and j are bounded by some natural numbers L_1 i L_2, which determine the degrees N_1 and N_2.

From (3.88) and (3.84) it follows that

$$G^{-1}(z_1,z_2) = z_1^{-1} z_2^{-1} \overline{G}^{-1}(z_1,z_2) = z_1^{-1} z_2^{-1} \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq} z_1^{-p} z_2^{-q}, \quad (3.91)$$

and

$$\overline{G}^{-1}(z_1,z_2) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq} z_1^{-p} z_2^{-q}, \quad (3.92)$$

where T_{pq} are defined by (3.82).

Theorem 3.16. *Let(3.90) be the characteristic polynomial of (3.77). The matrices T_{kl} satisfy the equation*

$$\sum_{k=0}^{N_1}\sum_{l=0}^{N_2} a_{kl} T_{kl} = 0. \quad (3.93)$$

Proof. From definition of the inverse matrix and (3.90), (3.92) we have

$$\operatorname{Adj}\left[\overline{G}(z_1,z_2)\right] = \left(\sum_{k=0}^{N_1}\sum_{l=0}^{N_2} a_{N_1-k,N_2-l} z_1^{-k} z_2^{-l}\right)\left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} T_{pq} z_1^{-p} z_2^{-q}\right), \quad (3.94)$$

where Adj $\left[\overline{G}(z_1,z_2)\right]$ is adjoint matrix of $\overline{G}(z_1,z_2)$.

Comparing the coefficients at the same power of $z_1^{-N_1} z_2^{-N_2}$ in equation (3.94), we obtain (3.93), since the degree of the polynomial matrix (3.94) is less than N_1 i N_2. □

Theorem 3.16 is an extension of the classical Cayley-Hamilton theorem to the fractional 2D linear systems described by (3.77).

3.9.5 Positivity of the Fractional 2D Linear Systems

Lemma 3.5. *If*

a) $0 < \alpha < 1$ *and* $1 < \beta < 2$ *then*

$$c_{\alpha\beta}(k,l) < 0 \quad \textit{for} \quad k = 1,2,\ldots; \quad l = 2,3,\ldots; \quad (3.95a)$$

b) $1 < \alpha < 2$ *and* $0 < \beta < 1$ *then*

$$c_{\alpha\beta}(k,l) < 0 \quad for \quad k = 2,3,\ldots; \quad l = 1,2,\ldots; \tag{3.95b}$$

c) $0 < \alpha < 1$ *and* $1 < \beta < 2$ *then*

$$c_{\alpha\beta}(k,1) > 0 \quad for \quad k = 2,3,\ldots; \tag{3.95c}$$

and

$$c_{\alpha\beta}(0,l) > 0 \quad for \quad l = 2,3,\ldots; \tag{3.95d}$$

Proof. The proof will be accomplished by induction. The hypothesis (3.95a) is true for $k = 1$ and $l = 2$ since

$$c_{\alpha\beta}(1,2) = (-1)^3 \frac{\alpha\beta(\beta-1)}{2} < 0.$$

Assuming that the hypothesis is true for the pair (k,l), $k+l \geq 3$, we shall show that it is also valid for the pairs $(k+1,l)$, $(k,l+1)$ and $(k+1,l+1)$.

From (3.76b) we have

$$c_{\alpha\beta}(k+1,l) = c_{\alpha\beta}(k,l)\frac{k-\alpha}{k+1} < 0,$$

since $c_{\alpha\beta}(k,l) < 0$ for $k = 1,2,\ldots$; $l = 2,3,\ldots$.

Similarly

$$c_{\alpha\beta}(k,l+1) = c_{\alpha\beta}(k,l)\frac{l-\beta}{l+1} < 0,$$

since $c_{\alpha\beta}(k,l) < 0$ for $k = 1,2,\ldots$; $l = 2,3,\ldots$; and

$$c_{\alpha\beta}(k+1,l+1) = c_{\alpha\beta}(k,l)\frac{(k-\alpha)(l-\beta)}{(k+1)(l+1)} < 0,$$

since $c_{\alpha\beta}(k,l) < 0$ for $k = 1,2,\ldots$; $l = 2,3,\ldots$. Proofs for (3.95b), (3.95c) and (3.95d) are similar. □

Remark 3.3. Taking in to account (3.95c) and (3.95d) we shall assume that for $0 < \alpha < 1$ and $1 < \beta < 2$

$$\sum_{k=2}^{i+1} c_{\alpha\beta}(k,1)x_{i-k+1,j} = 0, \quad \sum_{l=2}^{j+1} c_{\alpha\beta}(0,l)x_{i+1,j-l+1} = 0. \tag{3.96}$$

Lemma 3.6. *If the conditions (3.95) are satisfied and*

$$\overline{A}_k \in \mathbb{R}_+^{n\times n} \quad for \quad k = 0,1,2, \tag{3.97}$$

then

$$T_{pq} \in \mathbb{R}_+^{n\times n} \quad \textit{for} \quad p,q \in \mathbb{Z}_+. \tag{3.98}$$

Proof. If the conditions (3.95), (3.96), (3.97) are satisfied then from (3.82) we obtain (3.98). □

Definition 3.12. The system (3.77) is called (internally) positive fractional 2D linear system if $x_{ij} \in \mathbb{R}_+^n$ and $y_{ij} \in \mathbb{R}_+^p$, $i,j \in \mathbb{Z}_+$ for any boundary conditions $x_{i0} \in \mathbb{R}_+^n$, $i \in \mathbb{Z}_+$, $x_{0j} \in \mathbb{R}_+^n$, $i \in \mathbb{Z}_+$ and all inputs $u_{ij} \in \mathbb{R}_+^p$, $i,j \in \mathbb{Z}_+$.

Theorem 3.17. *Let the assumptions (3.96) be satisfied. The fractional 2D linear system (3.77) for* $0 < \alpha < 1$ *and* $1 < \beta < 2$ *(or* $1 < \alpha < 2$ *and* $0 < \beta < 1$*) is positive if and only if:*

$$\overline{A}_k \in \mathbb{R}_+^{n\times n},\ B_k \in \mathbb{R}_+^{n\times m}\ \textit{for}\ k = 0,1,2,\ C \in \mathbb{R}_+^{p\times n},\ D \in \mathbb{R}_+^{p\times m}. \tag{3.99}$$

Proof. Sufficiency. If the conditions (3.99) are satisfied then by Lemma 3.6 $T_{pq} \in \mathbb{R}_+^{n\times n}$ and from (3.81) we have $x_{ij} \in \mathbb{R}_+^n$ for $i,j \in \mathbb{Z}_+$, since $x_{i0} \in \mathbb{R}_+^n$, $x_{0j} \in \mathbb{R}_+^n$ and $u_{ij} \in \mathbb{R}_+^m$ for $i,j \in \mathbb{Z}_+$. From (3.77b) we have $y_{ij} \in \mathbb{R}_+^p$ since $C \in \mathbb{R}_+^{p\times n}$, $D \in \mathbb{R}_+^{p\times m}$ and $x_{ij} \in \mathbb{R}_+^n$, $u_{ij} \in \mathbb{R}_+^m$ for $i,j \in \mathbb{Z}_+$.

Necessity. Let the system be positive and $x_{00} = e_{ni}$, $i = 1,\dots,n$ (e_{ni} is i-th column of the identity matrix I_n), $x_{01} = x_{10} = 0$, $u_{ij} = 0$, $i,j \in \mathbb{Z}_+$. From (3.79a) for $i = j = 0$ and $u_{ij} = 0$, $i,j \in \mathbb{Z}+$ we obtain $x_{11} = \overline{A}_0 e_{ni} = \overline{A}_{0i} \in \mathbb{R}_+^n$, where $\overline{A}_{0i}$ is i-th column of $\overline{A}_0$. This implies $\overline{A}_0 \in \mathbb{R}_+^{n\times n}$, since $i = 1,\dots,n$. If we assume that $x_{10} = e_{ni}$, $x_{00} = x_{01} = 0$ and $u_{ij} = 0$, $i,j \in \mathbb{Z}_+$, then from (3.79a) for $i = j = 0$ we have $x_{11} = \overline{A}_1 e_{ni} = \overline{A}_{1i} \in \mathbb{R}_+^n$, what implies $\overline{A}_1 \in \mathbb{R}_+^{n\times n}$. In a similar way we may show that $\overline{A}_2 \in \mathbb{R}_+^{n\times n}$. Assuming $u_{00} = e_{ni}$, $u_{ij} = 0$, $i,j \in \mathbb{Z}_+$ $i+j > 0$ and $x_{00} = x_{10} = x_{01} = 0$ from (3.79a), for $i = j = 0$, we obtain $x_{11} = B_0 e_{mi} = B_{0i} \in \mathbb{R}_+^m$ for $i = 1,\dots,m$, what implies $B_0 \in \mathbb{R}_+^{n\times m}$. In a similar way we may show that $B_k \in \mathbb{R}_+^{n\times m}$ for $k = 1,2$ and $C \in \mathbb{R}_+^{p\times n}$, $D \in \mathbb{R}_+^{p\times m}$. □

Remark 3.4. From (3.76b) and (3.79a) it follows that if $\alpha = \beta$, $0 < \alpha < 1$, then $c_{\alpha\beta}(k,l) > 0$ for $k,l = 1,2,\dots$ and the fractional 2D linear system (3.77) is not positive.

The considerations presented for Roesser model can be easily extended to the model (3.77) [103].

Chapter 4
Pointwise Completeness and Pointwise Degeneracy of Linear Systems

4.1 Standard Discrete-Time Linear Systems

Consider the discrete-time linear system described by the equation

$$x_{i+1} = Ax_i, \tag{4.1}$$

where $x_i \in \mathbb{R}^n$ is the state vector and $A \in \mathbb{R}^{n\times n}$.

Theorem 4.1. *Solution of the equation (4.1) has the form*

$$x_i = A^i x_0, \quad i \in \mathbb{Z}_+ = \{0, 1, \dots\}. \tag{4.2}$$

Definition 4.1. The standard discrete-time linear system (4.1) is called pointwise complete for $i = q$ if for every final state $x_f \in \mathbb{R}^n$ there exists an initial condition x_0 such that $x_q = x_f$.

Theorem 4.2. *The standard discrete-time linear system (4.1) is pointwise complete if and only if the matrix A is nonsingular.*

Proof. For $i = q$ from (4.2) we have $x_f = x_q = A^q x_0$. From this equation it is possible to find x_0 for any given vector x_f if and only if $\det A^q \neq 0$. Note that $\det A^q = (\det A)^q$. Therefore, $x_0 = A^{-q} x_f$ if and only if $\det A \neq 0$. □

Definition 4.2. The standard discrete-time linear system (4.1) is called pointwise degenerated for $i = q$ if there exist nonzero vector $v \in \mathbb{R}^n$ such that for all initial conditions $x_0 \in \mathbb{R}^n$ the solution (4.2) for $i = q$ satisfies the condition $v^T x_q = 0$.

Theorem 4.3. *The standard discrete-time linear system (4.1) is pointwise degenerated for $i = q$ if and only if the matrix A is singular. The vector v can be found from* $v^T A^q = 0$.

Proof. There exists a vector v such that $v^T A^q = 0$, if and only if the matrix A is singular. In this case premultiplying the equation $x_f = A^q x_0$ by v^T we obtain $v^T x_f = v^T A^q = 0$. □

T. Kaczorek: Selected Problems of Fractional Systems Theory, LNCIS 411, pp. 81–101.
springerlink.com

Example 4.1. Check the pointwise completeness and pointwise degeneracy of the system (4.1) with the matrix

$$A = \begin{bmatrix} 0 & a \\ 2 & 1 \end{bmatrix}.$$

Note that $\det A = -2a$ and the system is pointwise complete for $a \neq 0$ and it is pointwise degenerated for $a = 0$. In this case

$$v = \begin{bmatrix} 1 & 0 \end{bmatrix}^T.$$

4.2 Standard Continuous-Time Linear Systems

Consider the continuous-time linear system described by the equation

$$\dot{x}(t) = Ax(t), \tag{4.3}$$

where $x \in \mathbb{R}^n$ is the state vector and $A \in \mathbb{R}^{n \times n}$.

Theorem 4.4. *Solution of the equation (4.3) has the form*

$$x(t) = e^{At} x_0. \tag{4.4}$$

From expansion of e^{At} it follows that $\det e^{At} \neq 0$ for every matrix A and time t.

Definition 4.3. The standard continuous-time linear system (4.3) is called pointwise complete for $t = t_f$ if for every final state $x_f \in \mathbb{R}^n$ there exists an initial condition $x(0) = x_0$ such that $x(t_f) = x_f$.

Theorem 4.5. *The standard continuous-time linear system (4.3) is pointwise complete for $t = t_f$ for every (nonsingular or singular) matrix A.*

Definition 4.4. The standard continuous-time linear system (4.3) is called pointwise degenerated in the direction v and time $t = t_f$, if there exists a nonzero vector $v \in \mathbb{R}^n$ such that for every initial state $x_0 \in \mathbb{R}^n$ the solution (4.4) for $t = t_f$ satisfies the condition $v^T x_f = 0$.

Taking into account that every continuous-time linear system is pointwise complete we obtain the following theorem.

Theorem 4.6. *The standard continuous-time linear system (4.3) is not pointwise degenerated for every matrix A.*

Proof. The proof follows immediately from the fact that $\det e^{At} \neq 0$ for every A and time t. □

Corollary 4.1. *From Theorems 4.5 and 4.6 it follows that in linear circuit composed of resistances R, inductances L and capacitances C by suitable choice of initial conditions (voltages $u_C(0)$ on condensators and currents $i_L(0)$ in coils) it is possible to obtain in a given time t_f the desired values $u_C(t_f)$ and $i_L(t_f)$.*

4.3 Standard Discrete-Time Linear Systems with Delays

Consider the discrete-time linear systems with h delays

$$x_{i+1} = \sum_{j=0}^{h} A_j x_{i-j} = A_0 x_i + A_1 x_{i-1} + \cdots + A_h x_{i-h}, \tag{4.5}$$

where $x_i \in \mathbb{R}^n$ is the state vector a $A_j \in \mathbb{R}^{n\times n}$, $j = 0, 1, \ldots, h$.

In general case the initial conditions $x_0, x_{-1}, \ldots, x_{-n}$ are nonzero.

Theorem 4.7. *Solution of the equation (4.5) has the form*

$$x_i = \Phi_i x_0 + \sum_{j=0}^{h} \sum_{l=-1}^{-j} \Phi_{i-j-l-1} A_j x_l, \tag{4.6}$$

where

$$\Phi_{i+1} = \sum_{k=0}^{h} \Phi_{i-k} A_k, \quad \Phi_0 = I_n, \quad \textit{and} \quad \Phi_i = 0, \quad \textit{for} \quad i < 0.$$

Definition 4.5. The discrete-time linear system with h delays (4.5) is called pointwise complete for $i = q \geq h$ if for any given final state $x_f \in \mathbb{R}^n$ there exist initial conditions $x_0, x_{-1}, \ldots, x_{-h}$ such that the solution for $i = q$ is equal to x_f, i.e. $x_f = x_q$.

Definition 4.6. The discrete-time linear system with h delays (4.5) is called pointwise degenerated in the direction v for $i = q \geq h$ if there exists a nonzero vector $v \in \mathbb{R}^n$ such that for any initial conditions $x_0, x_{-1}, \ldots, x_{-h}$ the solution (4.6) for $i = q$ satisfies the condition $v^T x_q = 0$.

Theorem 4.8. *The discrete-time linear system with h delays (4.5) is pointwise complete for $i = q$ if and only if*

$$\operatorname{rank} H_q = n, \tag{4.7}$$

where

$$H_q = \begin{bmatrix} H_0(q) \; H_1(q) \; \ldots \; H_h(q) \end{bmatrix} \tag{4.8}$$

$$H_0(q) = \Phi_q, \quad H_j(q) = \sum_{k=1}^{h-j+1} \Phi_{q-k} A_{k+j-1}, \quad j = 1, \ldots, h. \tag{4.9}$$

Proof. From (4.6) for $i = q$ we have

$$x_q = H_q \tilde{x}_0, \tag{4.10}$$

where

$$\tilde{x}_0 = \begin{bmatrix} x_0 \\ x_{-1} \\ \vdots \\ x_{-h} \end{bmatrix} \in \mathbb{R}^{(h+1)n}. \tag{4.11}$$

The equation (4.10) has a solution $\tilde{x}_0$ for any $x_q = x_f$ if and only if the condition (4.7) is satisfied. □

Theorem 4.9. *The discrete-time linear system with h delays (4.5) is pointwise degenerated in the direction v for $i = q \geq h$ if and only if*

$$\text{rank}\, H_q < n \quad . \tag{4.12}$$

Proof. By Definition 4.6 the system (4.5) is pointwise degenerated in the direction v for $i = q$ if for any vector $\tilde{x}_0$ there exists a vector $x_q = x_f$ such that

$$v^T x_f = v^T H_q \tilde{x}_0 = 0. \tag{4.13}$$

There exists a vector v satisfying (4.13) for any vector $\tilde{x}_0$ if and only if the condition (4.12) is satisfied. □

4.4 Positive Discrete-Time Linear Systems

Consider the discrete-time linear system described by the equation (4.1).

Definition 4.7. The discrete-time linear system (4.1) is called positive if $x_i \in \mathbb{R}_+^n$, $i \in \mathbb{Z}_+$ for any initial conditions $x_0 \in \mathbb{R}_+^n$.

Theorem 4.10. *The discrete-time linear system (4.1) is positive if and only if $A \in \mathbb{R}_+^{n \times n}$.*

Proof. Necessity. Assuming that $x_i \in \mathbb{R}_+^n$, and $x_0 = e_i = [0,...,0,1,0,...,0]^T$, we obtain $x_1 = Ax_0 = A_i \in \mathbb{R}_+^n$ for $i = 1,\ldots,n$.
Sufficiency. For $x_0 \in \mathbb{R}_+^n$ and $A \in \mathbb{R}_+^{n \times n}$, we have $A^i \in \mathbb{R}_+^{n \times n}$ and from (4.2) $x_i \in \mathbb{R}_+^n$, $i \in \mathbb{Z}_+$. □

Definition 4.8. The positive discrete-time system (4.1) is called pointwise complete for $i = q$ if for every final state $x_f \in \mathbb{R}_+^n$ there exists $x_0 \in \mathbb{R}_+^n$ such that $x_q = x_f$.

Theorem 4.11. *Positive discrete-time system (4.1) is pointwise complete for $i = q$ if and only if the matrix A is monomial (see Definition 1.10).*

Proof. It is easy to show that the matrix A^q for $q = 1,2,\ldots$ is monomial if and only if the matrix A is monomial. It is well-known that $A^{-q} \in \mathbb{R}_+^{n \times n}$ if and only if the matrix A is monomial. In this case from (4.2) we have $x_0 \in \mathbb{R}_+^n$ for any $x_f \in \mathbb{R}_+^n$. □

Definition 4.9. The positive discrete-time system (4.1) is called pointwise degenerated for $i = q$ if there exists at least one final state $x_f \in \mathbb{R}_+^n$, which is unreachable in q steps from any initial state $x_0 \in \mathbb{R}_+^n$, or equivalently the equality $x_q = x_f$ is not satisfied for any $x_0 \in \mathbb{R}_+^n$.

Theorem 4.12. *The positive discrete-time system (4.1) is pointwise degenerated for $i = q$ if and only if the matrix A is not monomial.*

Proof. The equation $x_f = x_q = A^q x_0$ has a solution $x_0 \in \mathbb{R}_+^n$ for any $x_f \in \mathbb{R}_+^n$ if and only if A is a monomial matrix. □

4.5 Positive Continuous-Time Linear Systems

Consider the continuous-time linear system described by the equation (4.3).

Definition 4.10. The continuous-time system (4.3) is called positive if $x(t) \in \mathbb{R}_+^n$, $t \geq 0$ for any initial conditions $x_0 \in \mathbb{R}_+^n$.

Theorem 4.13. *The continuous-time system (4.3) is positive if and only if A is a Metzler matrix (see Definition 2.7).*

Definition 4.11. The positive continuous-time system (4.3) is called pointwise complete for $t = t_f$ if for every final state $x_f \in \mathbb{R}_+^n$ there exists $x_0 \in \mathbb{R}_+^n$ such that $x(t_f) = x_f$.

Theorem 4.14. *The positive continuous-time system (4.3) is pointwise complete for $t = t_f$ if and only if the matrix A is diagonal.*

Proof. From (4.4) it follows that for any $x_f \in \mathbb{R}_+^n$ there exists $x_0 \in \mathbb{R}_+^n$ if and only if the matrix e^{-At_f} is monomial. From the expansion

$$e^{-At_f} = \sum_{k=0}^{\infty} \frac{(-At_f)^k}{k!} = I_n - \frac{At_f}{1!} + \frac{(At_f)^2}{2!} - \cdots, \tag{4.14}$$

it follows that the matrix e^{-At_f} is monomial if and only if A is diagonal. □

Definition 4.12. The positive continuous-time system (4.3) is called pointwise degenerated for $t = t_f$ if there exists at least one final state $x_f \in \mathbb{R}_+^n$, which is reachable for $t = t_f$ from any initial state $x_0 \in \mathbb{R}_+^n$, or equivalently the equation $x(t_f) = x_f$ is not satisfied for any $x_0 \in \mathbb{R}_+^n$.

Theorem 4.15. *The positive continuous-time system (4.3) is pointwise degenerated for $t = t_f$ if and only if the matrix A is not diagonal.*

Proof. For any $x_f \in \mathbb{R}_+^n$ there exists $x_0 \in \mathbb{R}_+^n$ satisfying (4.4) if and only if the matrix e^{-At_f} is monomial. From (4.14) it follows that e^{-At_f} is monomial if and only if A is diagonal. □

4.6 Positive Discrete-Time Linear Systems with Delays

Definition 4.13. The discrete-time linear system with delays (4.5) is called positive if $x_i \in \mathbb{R}_+^n$, $i \in \mathbb{Z}_+$ for any initial conditions $x_{-j} \in \mathbb{R}_+^n$, $j = 0, \ldots, h$.

Theorem 4.16. *The discrete-time linear system with delays (4.5) is positive if and only if $A_j \in \mathbb{R}_+^{n \times n}$ for $j = 0, \ldots, h$.*

Definition 4.14. The positive discrete-time system (4.5) is called pointwise complete for $i = q \geq h$ if for every $x_f \in \mathbb{R}_+^n$ there exist initial conditions $x_{-j} \in \mathbb{R}_+^n$, $j = 0, \ldots, h$ such that $x_q = x_f$.

The solution (4.6) of the equation (4.5) with nonzero initial conditions can be rewritten in the form

$$x_q = H_q \tilde{x}_0 \tag{4.15}$$

where

$$\begin{aligned} \tilde{x}_0 &= \left[x_0^T, x_{-1}^T, \ldots, x_{-h}^T \right]^T \in \mathbb{R}_+^{(h+1)n}, \\ H_q &= \left[H_0(q) \; H_1(q) \; \ldots \; H_h(q) \right] \in \mathbb{R}_+^{n \times (h+1)n}, \end{aligned} \tag{4.16}$$

and

$$H_j(q) = \sum_{k=1}^{h-j+1} \Phi_{q-k} A_{k+j-1}, \quad H_0(q) = \Phi_q.$$

From Definition 4.14 and (4.15) the following necessary condition for pointwise completeness follows [32].

Lemma 4.1. *The positive discrete-time system with delays (4.5) is pointwise complete for $i = q$ only if the matrix H_q has full row rank.*

Theorem 4.17. *Positive discrete-time system with delays (4.5) is pointwise complete for $i = q$ if and only if one of the following equivalent conditions is satisfied:*

a) $\mathrm{Im}_+ H_q = \mathbb{R}_+^n$, *where* $\mathrm{Im}_+ H_q = \left\{ x_q \in \mathbb{R}_+^n : x_q = H_q \tilde{x}_0, \; \tilde{x}_0 \in \mathbb{R}_+^{(h+1)n} \right\}$ *is positive image of the matrix H_q defined by (4.16);*

b) The matrix H_q contains n linearly independent monomial columns;

Proof. From Definition 4.14 and (4.15) it follows that the positive system (4.5) is pointwise complete for $i = q$ if and only if for every $x_q \in \mathbb{R}_+^n$ there exists $\tilde{x}_0 \in \mathbb{R}_+^{(h+1)n}$ such that the equation (4.15) is satisfied but this is equivalent to the condition a). Note that the matrix H_q contains n linearly independent monomial columns if and only if the condition a) is satisfied. Therefore, the conditions a) and b) are equivalent.

If one of the conditions is satisfied then $\tilde{x}_0 \in \mathbb{R}_+^{(h+1)n}$ can be found from $x_0 = (\overline{H}_q)^{-1} x_f$, where $\overline{H}_q$ is monomial matrix composed of the monomial columns and $x_q = x_f \in \mathbb{R}_+^n$. □

Theorem 4.18. *The positive discrete-time system with delays (4.5) is pointwise complete for $i = q$ if the matrix H_q has full row rank and*

$$H_p = H_q^T \left[H_q H_q^T\right]^{-1} \in \mathbb{R}_+^{n(h+1)\times n}, \tag{4.17}$$

Moreover

$$\tilde{x}_0 = H_q^T \left[H_q H_q^T\right]^{-1} x_f. \tag{4.18}$$

Proof. If $\operatorname{rank} H_q = n$, then $\det[H_q H_q^T] \neq 0$ and the matrix (4.17) is well-defined. If the condition (4.17) is satisfied and $x_f \in \mathbb{R}_+^n$, then $\tilde{x}_0 \in \mathbb{R}_+^{(h+1)n}$ and $\tilde{x}_0$ is given by (4.18). □

Definition 4.15. The positive discrete-time system with delays (4.5) is pointwise degenerated for $i = q$, if there exists at least one state $x_f \in \mathbb{R}_+^n$, which is unreachable from any initial condition $x_0 \in \mathbb{R}_+^n$.

Corollary 4.2. *The positive discrete-time system with delays (4.5) can be pointwise degenerated although* $\operatorname{rank} H_q = n$ *if there exists $x_q \in \mathbb{R}_+^n$ for which it is impossible to find $\tilde{x}_0 \in \mathbb{R}_+^{(h+1)n}$ satisfying the equation (4.15).*

Theorem 4.19. *The positive discrete-time system with delays (4.5) is pointwise degenerated for $i = q$ if and only if the matrix H_q does not contain n linearly independent monomial columns.*

Proof. The proof follows immediately from Definition 4.15 and Theorem 4.17. □

Example 4.2. Check the pointwise completeness of the positive system (4.5) for $h = 1$ and the matrices:

$$A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

In this case the matrix $H_1 = [H_0(1), H_1(1)] = [A_0, A_1]$ has 3 linearly independent monomial columns. Therefore, by Theorem 4.17 the system is pointwise complete. It is easy to check for this system the condition (4.17) is not satisfied and we are not able to find the initial condition for the equation (4.18). This initial condition $\tilde{x}_0$ corresponding to $x_f = [x_{f1}, x_{f2}, x_{f3}]^T$ can be found as follows. From matrix $[A_0, A_1]$ we choose the second column of the matrix A_0 and the first two columns of the matrix A_1. The matrix composed from these monomial columns has the form

$$\overline{H}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Assuming $\tilde{x}_0 = [0, x_1, 0, x_2, x_3, 0]^T$, from (4.15) we obtain

$$x_f = \overline{H}_1 \tilde{x}_0' \quad \text{for} \quad \tilde{x}_0' = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T.$$

Hence

$$\tilde{x}_0' = \overline{H}_1^{-1} x_f = \left[\, x_{f1} \; x_{f2} \; \tfrac{1}{2} x_{f3} \,\right]^T, \tag{4.19}$$

and $\tilde{x}_0 = [0, x_{f1}, 0, x_{f2}, \frac{1}{2}x_{f3}, 0]$.

Theorem 4.20. *The positive discrete-time system with delays (4.5) is pointwise degenerated for $i = q$ if the matrix $[A_0, A_1, \dots, A_h]$ does not contain n linearly independent monomial columns.*

Proof. The matrix (4.16) contains n linearly independent monomial columns only if the matrix $[A_0, A_1, \dots, A_h]$ contains n linearly independent monomial columns. □

4.7 Fractional Discrete-Time Linear Systems

Consider the fractional autonomous discrete-time linear system described by the equation

$$\Delta^\alpha x_{i+1} = A x_i, \quad i \in \mathbb{Z}_+, \quad 0 < \alpha < 1, \tag{4.20}$$

where $x_i \in \mathbb{R}^n$ is the state vector and $A \in \mathbb{R}^{n \times n}$.

The (backward) difference of α-order has the form

$$\Delta^\alpha x_i = x_i + \sum_{j=1}^{i} (-1)^j \binom{\alpha}{j} x_{i-j}, \quad 0 < \alpha < 1. \tag{4.21}$$

Substituting of (4.21) into (4.20) yields:

$$x_{i+1} = A_\alpha x_i + \sum_{j=1}^{i} c_j(\alpha) x_{i-j}, \tag{4.22}$$

where

$$A_\alpha = A + I_n \alpha, \quad c_j(\alpha) = (-1)^j \binom{\alpha}{j+1} > 0. \tag{4.23}$$

Theorem 4.21. *The solution of the equation (4.22) has the form*

$$x_i = \Phi_i x_0, \quad i \in \mathbb{Z}_+, \tag{4.24}$$

where

$$\Phi_0 = I_n, \quad \Phi_{i+1} = A_\alpha \Phi_i + \sum_{j=1}^{i} c_j(\alpha) \Phi_{i-j}. \tag{4.25}$$

Definition 4.16. The fractional discrete-time linear system (4.22) is called pointwise complete for $i = q \geq 1$ if for every $x_f \in \mathbb{R}^n$ there exists an initial condition $x_0 \in \mathbb{R}^n$ such that $x_q = x_f$.

Theorem 4.22. *The fractional discrete-time linear system (4.22) is pointwise complete for $i = q$ if and only if* $\operatorname{rank} \Phi_q = n$ *or equivalently* $\det \Phi_q \neq 0$.

Proof. From (4.24) for $i = q$ we have $x_f = \Phi_q x_0$ and we can find x_0 for given x_f if and only if rank $\Phi_q = n$, $\det \Phi_q \neq 0$. □

Definition 4.17. The fractional discrete-time linear system (4.22) is pointwise degenerated in the direction v if there exists a nonzero vector $v \in \mathbb{R}^n$ such that for all initial conditions $x_0 \in \mathbb{R}^n$ the solution of (4.22) satisfies the condition $v^T x_q = 0$.

Theorem 4.23. *The fractional discrete-time linear system (4.22) is pointwise degenerated for $i = q$ if and only if*

$$\operatorname{rank} \Phi_q < n, \quad \textit{or equivalently} \quad \det \Phi_q = 0. \tag{4.26}$$

The direction of degeneracy v can be found from the equation $v^T \Phi_q = 0$.

Proof. If the condition (4.26) is satisfied, then there exists a vector $v \in \mathbb{R}^n$ such that $v^T x_q = v^T x_f = v^T \Phi_q x_0$ for every $x_0 \in \mathbb{R}^n$. □

4.8 Fractional Continuous-Time Linear Systems

Consider the fractional autonomous continuous-time linear system described by the equation

$$\frac{d^\alpha}{dt^\alpha} x(t) = Ax(t) \tag{4.27}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $0 < \alpha < 1$ and $\frac{d^\alpha}{dt^\alpha}$ is defined by (2.11).

Theorem 4.24. *The solution of the equation (4.27) has the form:*

$$x(t) = \Phi_0(t) x_0, \tag{4.28}$$

where

$$\Phi_0(t) = \Phi_0(A,t) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha + 1)} = I_n + \frac{At^\alpha}{\Gamma(\alpha + 1)} + \cdots \tag{4.29}$$

Lemma 4.2. *The matrix $\Phi_0(t)$ defined by (4.29) is nonsingular for any matrix $A \in \mathbb{R}^{n \times n}$ and time $t \geq 0$.*

Proof. Consider the function

$$\Phi_0(z,t) = \sum_{k=0}^{\infty} \frac{z^k t^{k\alpha}}{\Gamma(k\alpha + 1)} = I_n + \frac{zt^\alpha}{\Gamma(\alpha + 1)} + \cdots \tag{4.30}$$

We shall show that $\Phi_0(A,t) \neq 0$ for any matrix $A \in \mathbb{R}^{n \times n}$. The function (4.30) is well-defined on the spectrum of the matrix A. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues (real or complex) of the matrix A. From (4.30) it follows that $\Phi_0(\lambda_i, t) \neq 0$ for any real λ_i, $i = 1, \ldots, n$ and $\Phi_0(\lambda_i, t)\Phi_0(\lambda_{i+1}, t) \neq 0$ for any complex conjugate pair

$(\lambda_i, \lambda_{i+1})$, $i = 1, 2, \ldots, n-1$. It is well-known that the eigenvalues of the matrix $\Phi_0(A,t)$ are equal to $\Phi_0(\lambda_1,t), \Phi_0(\lambda_2,t), \ldots, \Phi_0(\lambda_n,t)$ and

$$\det \Phi_0(A,t) = \Phi_0(\lambda_1,t)\Phi_0(\lambda_2,t)\ldots\Phi_0(\lambda_n,t) \neq 0.$$

□

Lemma 4.3. *If the matrix A has distinct eigenvalues then the matrix (4.29) is given by*

$$\Phi_0(A,t) = \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{A - \lambda_j I_n}{\lambda_i - \lambda_j} \Phi_0(\lambda_i,t). \tag{4.31}$$

Example 4.3. Using the Sylvester formula (4.31) find the matrix $\Phi_0(A,t)$ for the system (4.27) with the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.32}$$

This matrix has two eigenvalues $\lambda_1 = 1$, $\lambda_2 = 0$. Using (4.31) we obtain

$$\Phi_0(A,t) = A\Phi_0(\lambda_1,t) + (I_2 - A)\Phi_0(\lambda_2,t) = \begin{bmatrix} \varphi(t) & 0 \\ 0 & 1 \end{bmatrix}, \tag{4.33}$$

where

$$\varphi(t) = \sum_{k=0}^{\infty} \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}.$$

The matrix (4.29) can be rewritten in the form

$$\Phi_0(t) = I_n + \sum_{k=1}^{\infty} A^k \varphi_k(t), \tag{4.34}$$

where

$$\varphi_k(t) = \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}. \tag{4.35}$$

From (4.34) we have the following lemma.

Lemma 4.4. *If A is a nilpotent matrix with index μ, i.e.*

$$A^k = 0 \quad for \quad k = \mu, \mu+1, \ldots \quad and \quad A^{\mu-1} \neq 0,$$

then

$$\Phi_0(t) = I_n + \sum_{k=1}^{\mu-1} A^k \varphi_k(t). \tag{4.36}$$

Definition 4.18. The fractional continuous-time linear system (4.27) is called pointwise complete for $t = t_f$ if for every final state $x_f \in \mathbb{R}^n$ there exists a vector of initial conditions $x_0 \in \mathbb{R}^n$ such that $x(t_f) = x_f$.

Theorem 4.25. *The fractional continuous-time linear system (4.27) is pointwise complete for any $t = t_f$ and every x_f.*

Proof. From (4.28) for $t = t_f$ we have

$$x(t_f) = x_f = \Phi_0(t_f)x_0,$$

and

$$x_0 = \left[\Phi_0(t_f)\right]^{-1} x_f, \tag{4.37}$$

since by Lemma 4.2 $\det \Phi_0(t_f) \neq 0$ for any matrix A and time $t \geq 0$. □

Definition 4.19. The fractional continuous-time system (4.27) is called pointwise degenerated in the direction v for $t = t_f$ if there exists a vector $v \in \mathbb{R}^n$ such that for all initial conditions $x_0 \in \mathbb{R}^n$ the solution of (4.27) for $t = t_f$ satisfies the condition $v^T x_f = 0$.

Remark 4.1. Every strictly upper (down) triangular matrix is a nilpotent matrix with the index not greater than the dimensions of the matrix.

Example 4.4. Check the pointwise completeness of the system with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{4.38}$$

The nilpotency index of the matrix (4.38) is equal to 2. From (4.29) we have

$$\Phi_0(t) = I_n + \frac{At^\alpha}{\Gamma(\alpha+1)} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \frac{t^\alpha}{\alpha} = \begin{bmatrix} 1 & \frac{t^\alpha}{\alpha} \\ 0 & 1 \end{bmatrix}.$$

Assuming $t_f = 1$, $x_f = [1,1]^T$ we obtain

$$x_0 = \begin{bmatrix} 1 & \frac{1^\alpha}{\alpha} \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{\alpha} \\ 1 \end{bmatrix}.$$

From Lemma 4.4 for $\mu = 2$ we have

$$\Phi_0(t) = I_2 + A\varphi_1(t) = \begin{bmatrix} 1 & \varphi_1(t) \\ 0 & 1 \end{bmatrix},$$

where $\varphi_1(t)$ is given by (4.35) for $k = 1$. By Theorem 4.25 the system is pointwise complete. For any given find state $x_f = [x_{f1}, x_{f2}]^T \in \mathbb{R}^2$ we may find the desired initial conditions using (4.37), i.e.

$$x_0 = \left[\Phi_0(t_f)\right]^{-1} x_f = \begin{bmatrix} 1 & -\varphi_1(t_f) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{f1} \\ x_{f2} \end{bmatrix} = \begin{bmatrix} x_{f1} - x_{f2}\varphi_1(t_f) \\ x_{f2} \end{bmatrix}.$$

Assuming $x_{f1} = 0$ and $x_{f2} > 0$, we obtain

$$x_0 = \begin{bmatrix} -x_{f2}\varphi_1(t_f) \\ x_{f2} \end{bmatrix}. \tag{4.39}$$

4.9 Positive Fractional Discrete-Time Linear System

Consider the fractional discrete-time linear system described by the equation (4.22).

Definition 4.20. The fractional discrete-time linear system (4.22) is called positive if $x_i \in \mathbb{R}_+^n$, $i \in \mathbb{Z}_+$ for all initial conditions $x_0 \in \mathbb{R}_+^n$.

Theorem 4.26. *The fractional discrete-time linear system (4.22) is positive if and only if $A + \alpha I_n = A_\alpha \in \mathbb{R}_+^{n\times n}$.*

Definition 4.21. The positive fractional discrete-time system (4.22) is pointwise complete for $i = q \geq 1$ if for any final state $x_f \in \mathbb{R}_+^n$ there exists an initial condition $x_0 \in \mathbb{R}_+^n$, such that $x_q = x_f$.

Theorem 4.27. *The positive fractional discrete-time system (4.22) is pointwise complete for $i = q \geq 1$ if and only if the matrix A_α is nonsingular and diagonal. If the matrix A_α is singular and diagonal then the system (4.22) is pointwise complete for $i = q \geq 2$.*

Proof. From (4.24) for $i = q \geq 1$ we have $x_0 = [\Phi_q]^{-1} x_q$. Hence $x_0 \in \mathbb{R}_+^n$ for $x_f \in \mathbb{R}_+^n$, if and only if $[\Phi_q]^{-1} \in \mathbb{R}_+^{n\times n}$, and this is equivalent to the condition that Φ_q is a monomial matrix. From (4.25) and that $c_j > 0$ for every $j \geq 1$ and $A_\alpha \in \mathbb{R}_+^{n\times n}$ it follows that Φ_i is monomial matrix for any $i \geq 1$, if and only if A_α is monomial matrix. From the structure of matrix A_α it follows that it is monomial if and only if it is nonsingular and diagonal. If A_α is a singular and diagonal then $\Phi_1 = A_\alpha$ is not monomial and from (4.24) it follows that Φ_i is nonsingular and diagonal for any $i \geq 2$. In this case the system (4.22) is pointwise complete for $i = q \geq 2$. □

Definition 4.22. The positive fractional discrete-time system (4.22) is called pointwise degenerated if there exists at least one final state $x_f \in \mathbb{R}_+^n$, such that is unreachable from any initial condition $x_0 \in \mathbb{R}_+^n$.

Theorem 4.28. *The positive fractional discrete-time system (4.22) is pointwise degenerated for $i = q \geq 1$ if and only if the matrix Φ_q is not monomial.*

Example 4.5. Check the pointwise completeness of the fractional system (4.22) for $0 < \alpha < 1$ with the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -\alpha \end{bmatrix}.$$

Using (4.23) and (4.25) we obtain:

$$c_1 = (-1)\frac{\alpha(\alpha-1)}{2}, \quad c_2 = \frac{\alpha(\alpha-1)(\alpha-2)}{6}, \quad \dots$$

$$\Phi_1 = A_\alpha = A + I_n\alpha = \begin{bmatrix} 1+\alpha & 0 \\ 0 & 0 \end{bmatrix},$$
$$\Phi_2 = A_\alpha\Phi_1 + c_1\Phi_0 = \begin{bmatrix} (1+\alpha)^2 + c_1 & 0 \\ 0 & c_1 \end{bmatrix},$$
$$\Phi_3 = A_\alpha\Phi_1 + c_1\Phi_1 + c_2\Phi_0 = \begin{bmatrix} [(1+\alpha)^2 + c_1](1+\alpha) + c_1(1+\alpha) + c_2 & 0 \\ 0 & c_2 \end{bmatrix}$$
$$\vdots$$

The system is positive since $A_\alpha \in \mathbb{R}_+^{2\times 2}$ and the matrices Φ_i are diagonal with nonnegative entries for $i \geq 2$. Therefore, by Theorems 4.27 and 4.28 we have the following:

a) The positive fractional system is not pointwise complete for $i = q = 1$, since the matrix Φ_1 is singular. The system is pointwise degenerated in the direction $v = [0,1]^T$;
b) The positive fractional system is pointwise complete for $i = q \geq 2$, since the matrix Φ_i is diagonal and nonsingular for every $i = q \geq 2$.

In this case it is possible to find $x_0 = [\Phi_q]^{-1}x_q$ for any given $x_q = x_f$ only for $q \geq 2$. Assuming $\alpha = 0.5$ we obtain $c_1 = 0.125$ and:

$$\Phi_1 = \begin{bmatrix} 1.5 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} 2.375 & \\ 0 & 0.125 \end{bmatrix}.$$

Therefore, any desired final state $x_f = [x_{f1}, x_{f2}]^T \in \mathbb{R}_+^2$ is reachable for $i = q = 2$ from the initial state

$$x_0 = [\Phi_2]^{-1}x_f = \begin{bmatrix} 0.4211 & 0 \\ 0 & 8.0 \end{bmatrix}\begin{bmatrix} x_{f1} \\ x_{f2} \end{bmatrix}.$$

4.10 Positive Fractional Continuous-Time Linear Systems

Consider the fractional continuous-time linear system described by the equation (4.27).

Definition 4.23. The fractional continuous-time linear system (4.27) is called positive if $x(t) \in \mathbb{R}_+^n$, $t \geq 0$ for all initial conditions $x_0 \in \mathbb{R}_+^n$.

Theorem 4.29. *The fractional continuous-time system (4.27) is positive if and only if A is a Metzler matrix.*

Definition 4.24. The positive fractional continuous-time system (4.27) is called pointwise complete for $t = t_f$ if for any final state $x_f \in \mathbb{R}_+^n$ there exists an initial condition $x_0 \in \mathbb{R}_+^n$ such that $x(t_f) = x_f$.

Theorem 4.30. *The positive fractional continuous-time system (4.27) is pointwise complete for $t = t_f$ if and only if the matrix A is diagonal.*

Proof. From (4.37) it follows that for any $x_f \in \mathbb{R}_+^n$ there exists $x_0 \in \mathbb{R}_+^n$ if and only if $\left[\Phi_0(t_f)\right]^{-1} \in \mathbb{R}_+^n$ and the matrix $\Phi_0(t_f)$ is monomial. By (4.34) the matrix $\Phi_0(t_f)$ is monomial if and only if the matrix A is diagonal. □

Definition 4.25. The positive fractional continuous-time system (4.27) is called pointwise degenerated if there exists at least one finial state $x_f \in \mathbb{R}_+^n$, which is unreachable from any initial state $x_0 \in \mathbb{R}_+^n$, in other words does not exist $t = t_f$ and $x_0 \in \mathbb{R}_+^n$ such that $x(t_f) = x_f$.

Theorem 4.31. *The positive fractional continuous-time system (4.27) is pointwise degenerated if and only if the matrix A is not diagonal.*

Proof. The proof follows immediately from Definition 4.25 and Theorem 4.30. □

Example 4.6. In Example 4.4 it was shown that the fractional system (4.27) with the Metzler matrix (4.38) is pointwise complete. From (4.39) it follows that if $x_{f1} = 0$ and $x_{f2} > 0$, then $x_0 \notin \mathbb{R}_+^2$. This means that the positive fractional system with (4.38) is pointwise degenerated. The same result follows from Theorem 4.30.

Example 4.7. Consider the positive fractional system (4.27) with the matrix (4.32). Using Theorem 4.30 and (4.37) we may find $x_0 \in \mathbb{R}_+^2$ for any final state $x_f \in \mathbb{R}_+^2$. If $x_f = [x_{f1}, x_{f2}]^T \in \mathbb{R}_+^2$ then from (4.37) we have

$$x_0 = \begin{bmatrix} \frac{1}{\varphi(t_f)} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{f1} \\ x_{f2} \end{bmatrix} = \begin{bmatrix} \frac{x_{f1}}{\varphi(t_f)} \\ x_{f2} \end{bmatrix} \in \mathbb{R}_+^2.$$

4.11 Pointwise Completeness and Pointwise Degeneracy of Electrical Circuits

Example 4.8. Consider the electrical circuit shown on Fig. 4.1 with given resistances R_1, R_2, R_3, capacitances C_1, C_2 and voltage source $e = e(t)$. The voltages on the condensators $u_1 = u_1(t)$ and $u_2 = u_2(t)$ are the state variables $x_1 = u_1$ and $x_2 = u_2$ and the voltage source is the input $u = e$.

Using the Kirchhoff's laws we may write the equations:

$$R_1C_1\dot{u}_1 + u_1 + R_3(C_1\dot{u}_1 + C_2\dot{u}_2) = e, \tag{4.40a}$$

$$R_2C_2\dot{u}_2 + u_2 + R_3(C_1\dot{u}_1 + C_2\dot{u}_2) = e, \tag{4.40b}$$

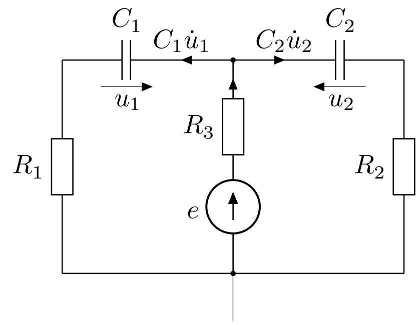

Fig. 4.1 R, C, e type electrical circuit. Illustration to Example 4.8.

The equations (4.40) can be rewritten in the form

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = A \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + Be, \tag{4.41}$$

where

$$A = \begin{bmatrix} -\frac{R_2+R_3}{C_1[R_1(R_2+R_3)+R_2R_3]} & \frac{R_3}{C_1[R_1(R_2+R_3)+R_2R_3]} \\ \frac{R_3}{C_2[R_1(R_2+R_3)+R_2R_3]} & -\frac{R_1+R_3}{C_2[R_1(R_2+R_3)+R_2R_3]} \end{bmatrix}, \tag{4.42a}$$

$$B = \begin{bmatrix} \frac{R_2}{C_1[R_1(R_2+R_3)+R_2R_3]} \\ \frac{R_1}{C_2[R_1(R_2+R_3)+R_2R_3]} \end{bmatrix}. \tag{4.42b}$$

The matrix A is a Metzler matrix and the matrix B has positive entries. This electrical circuit is an example of linear positive continuous-time system. This electrical circuit will be called shortly R, C, e type.

By Theorem 4.14 an electrical circuit is pointwise complete for $t = t_f$ if and only if its A matrix is diagonal what implies $R_3 = 0$.

By Theorem 4.15 an electrical circuit is pointwise degenerated for every $t = t_f$ if and only if the matrix A is not diagonal, what implies $R_3 > 0$.

Example 4.9. Consider the electrical circuit shown on Fig. 4.2 with given resistances R_1, R_2, R_3, inductances L_1, L_2 and voltage sources $e_1 = e_1(t)$ and $e_2 = e_2(t)$. The currents in coils $i_1 = i_1(t)$ and $i_2 = i_2(t)$ are the state variables $x_1 = i_1$ and $x_2 = i_2$ and the voltages sources $u_1 = e_1$, $u_2 = e_2$ are inputs.

Using the Kirchhoff's laws we may write the equations:

$$L_1 \frac{di_1}{dt} + R_1 i_1 + R_3(i_1 - i_2) = e_1, \tag{4.43a}$$

$$L_2 \frac{di_2}{dt} + R_2 i_2 + R_3(i_2 - i_1) = e_2, \tag{4.43b}$$

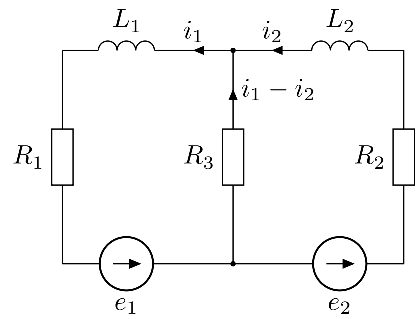

Fig. 4.2 *R*, *L*, *e* type electrical circuit. Illustration to Example 4.9.

The equations (4.43) can be rewritten in the form

$$\frac{d}{dt}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{4.44}$$

where

$$A = \begin{bmatrix} -\frac{R_1+R_3}{L_1} & \frac{R_3}{L_1} \\ \frac{R_3}{L_2} & -\frac{R_2+R_3}{L_2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}. \tag{4.45}$$

The matrix A is a Metzler matrix and the matrix B has nonnegative entries. The electrical circuit is an example of positive continuous-time linear system. This electrical circuit will be called shortly *R*, *L*, *e* type.

The presented considerations can be extended to any electrical circuits of *R*, *C*, *e* and *R*, *L*, *e* types.

Theorem 4.32. *The positive electrical circuits of R, C, e and R, L, e types are pointwise complete for every* $t = t_f$ *if and only if the matrix A is diagonal. The electrical circuits are pointwise degenerated if and only if A is not diagonal Metzler matrix.*

In general case an electrical circuits *R*, *C*, *L*, *e* types is not a positive system. If we neglect the assumption $x_0 \in \mathbb{R}^n_+$, $x(t) \in \mathbb{R}^n_+$, $u(t) \in \mathbb{R}^m_+$, $t \geq 0$ then from Theorem 4.5 we obtain the following theorem.

Theorem 4.33. *An electrical circuit of R, C, L, e type is pointwise complete for every* $t = t_f$ *(as standard linear system).*

4.12 Standard Continuous-Discrete Linear System Described by the General Model

Consider the autonomous general model

$$\dot{x}(t,i+1) = A_0x(t,i) + A_1\dot{x}(t,i) + A_2x(t,i+1), \tag{4.46}$$

$$t \in \mathbb{R}_+ = [0, +\infty], \quad i \in \mathbb{Z}_+ = \{0, 1, \dots\},$$

where $\dot{x}(t,i) = \frac{\partial x(t,i)}{\partial t}$, $x(t,i) \in \mathbb{R}^n$, $u(t,i) \in \mathbb{R}^m$, $y(t,i) \in \mathbb{R}^p$ are the state, input and output vectors.

Boundary conditions for (4.46) are given by

$$x(0,i) = x_i, \quad i \in \mathbb{Z}_+, \quad \text{and} \quad x(t,0) = x_{t0}, \quad \dot{x}(t,0) = x_{t1}, \quad t \in \mathbb{R}_+. \tag{4.47}$$

Definition 4.26. The general model (4.46) is called pointwise complete at the point (t_f, q) if for every final state $x_f \in \mathbb{R}^n$ there exist boundary conditions (4.47) such that $x(t_f, q) = x_f$.

Theorem 4.34. *The general model (4.46) is always pointwise complete at the point (t_f, q) for any $t_f > 0$ and $q = 1$.*

Proof. From (4.46) for $i = 0$ we have

$$\dot{x}(t,1) = A_2 x(t,1) + F(t,0), \tag{4.48}$$

where

$$F(t,0) = A_0 x(t,0) + A_1 \dot{x}(t,0) = A_0 x_{t0} + A_1 x_{t1}. \tag{4.49}$$

Assuming $x_{t0} = 0$, $x_{t1} = 0$ we obtain $F(t,0) = 0$ and from (4.48)

$$x(t,1) = e^{A_2 t} x(0,1). \tag{4.50}$$

Substituting $t = t_f$ and $q = 1$ we obtain

$$x_f = e^{A_2 t_f} x(0,1) \tag{4.51}$$

and

$$x(0,1) = e^{-A_2 t_f} x_f. \tag{4.52}$$

Therefore, for any final state x_f there exist boundary conditions $x_{t0} = 0$, $x_{t1} = 0$ and $x_1 = e^{-A_2 t_f} x_f$ such that $x(t_f, 1) = x_f$ since the matrix $e^{-A_2 t_f} x_f$ exists for any matrix A_2 and any $t_f > 0$. □

From Theorem 4.34 we have the following corollaries.

Corollary 4.3. *Any general model (4.46) is pointwise complete at the point $(t_f, 1)$ for arbitrary $t_f > 0$.*

Corollary 4.4. *The pointwise completeness of the general model at the point $(t_f, 1)$ is independent of the matrices A_0 and A_1 of the model.*

Definition 4.27. The general model (4.46) is called pointwise degenerated at the point (t_f, q) in the direction v if there exist a nonzero vector $v \in \mathbb{R}^n$ such that for all boundary conditions (4.47) the solution of the model for $t = t_f$, $i = q$ satisfies the condition $v^T x(t_f, q) = 0$.

Theorem 4.35. *The general model (4.46) is not pointwise degenerated at the point* $(t_f, 1)$ *for any* $t_f > 0$.

Proof. Using the solution of (4.48)

$$x(t,1) = e^{A_2 t}x(0,1) + \int_0^t e^{A_2(t-\tau)}F(\tau,0)d\tau, \tag{4.53a}$$

we obtain

$$v^T x(t,1) = v^T e^{A_2 t}x(0,1) + \int_0^t v^T e^{A_2(t-\tau)}F(\tau,0)d\tau, \tag{4.53b}$$

where $F(t,0)$ is defined by (4.49). From (4.53b) it follows that does not exist a nonzero vector $v \in \mathbb{R}^n$ such that for all boundary conditions (4.47) $v^T(t_f,1) = 0$ since the matrix $e^{A_2 t_f}$ is nonsingular for every matrix A_2 and $t_f > 0$. □

Example 4.10. Consider the general model (4.46) with the matrices

$$A_0 = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}. \tag{4.54}$$

Find the boundary conditions (4.47) at the point $(t_f, q) = (1,1)$ for $x_f = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Taking into account that the eigenvalues of A_2 are $\lambda_1 = -1$, $\lambda_2 = -2$ and using the Sylvester formula we obtain

$$e^{-A_2 t} = \frac{A_2 - \lambda_2 I_n}{\lambda_1 - \lambda_2}e^{-\lambda_1 t} + \frac{A_2 - \lambda_1 I_n}{\lambda_2 - \lambda_1}e^{-\lambda_2 t} = \begin{bmatrix} e^t & 0 \\ e^t - e^{2t} & e^{2t} \end{bmatrix}. \tag{4.55}$$

From (4.52) we have the desired boundary conditions

$$x(0,1) = e^{-A_2 t_f}x_f = \begin{bmatrix} e^{t_f} & 0 \\ e^{t_f} - e^{2t_f} & e^{2t_f} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}\Bigg|_{t_f=1} = \begin{bmatrix} 2e \\ 2e + e^2 \end{bmatrix}, \tag{4.56}$$

and $x(t,0) = 0$, $\dot{x}(t,0) = 0$, $t \geq 0$.

The above conditions can be extended as follows.

From (4.46) for $i = 1$ we have

$$\dot{x}(t,2) = A_2 x(t,2) + F(t,1), \tag{4.57}$$

where

$$F(t,1) = A_0 x(t,1) + A_1 \dot{x}(t,1). \tag{4.58}$$

Substituting of (4.48) and (4.50) for $F(t,0) = 0$ into (4.58) yields

$$F(t,1) = (A_0 + A_1 A_2)x(t,1) = (A_0 + A_1 A_2)e^{A_2 t}x(0,1). \tag{4.59}$$

Assuming $x(0,1) = 0$ we obtain $F(t,1) = 0$ and from (4.57)

$$x(t,2) = e^{A_2 t} x(0,2). \tag{4.60}$$

Continuing the procedure for $i = 2,\dots,q-1$ we obtain the following theorem, which is an extension of Theorem 4.34.

Theorem 4.36. *The general model (4.46) is always pointwise complete at the point* (t_f, q), $t_f > 0$, $q \in \mathbb{N} = \{1,2,\dots\}$ *for any matrices* A_k, $k = 0,1,2$.

Theorem 4.35 can be also extended for any point (t_f, q).

4.13 Positive Continuous-Discrete Linear System Described by the General Model

Definition 4.28. The model (4.46) is called positive if $x(t,i) \in \mathbb{R}_+^n$, $t \in \mathbb{R}_+$, $i \in \mathbb{Z}_+$ for any boundary conditions

$$x_{t0} \in \mathbb{R}_+^n, \quad x_{t1} \in \mathbb{R}_+^n, \quad t \in \mathbb{R}_+, \quad x_i \in \mathbb{R}_+^n, \quad i \in \mathbb{Z}_+. \tag{4.61}$$

Theorem 4.37. *The general model (4.46) is positive if and only if*

$$A_2 \in M_n, \tag{4.62a}$$

$$A_0, A_1 \in \mathbb{R}_+^{n\times n}, \quad A = A_0 + A_1 A_2 \in \mathbb{R}_+^{n\times n}, \tag{4.62b}$$

where M_n *is the set of* $n \times n$ *Metzler matrices (with nonnegative off-diagonal entries).*

Proof. Necessity of $A_0 \in \mathbb{R}_+^{n\times n}$ and $A_1 \in \mathbb{R}_+^{n\times n}$ follows immediately from (4.49) since $F(t,0) \in \mathbb{R}_+^n$, $t \in \mathbb{R}_+$ and x_{t0}, x_{t1} are arbitrary. From (4.50) it follows that $A_2 \in M_n$ since $e^{A_2 t} \in \mathbb{R}_+^{n\times n}$ only if A_2 is a Metzler matrix, $x(t,1) \in \mathbb{R}_+^n$, $t \in \mathbb{R}_+$ and $x(0,1)$ is arbitrary. From (4.57) it follows that $F(t,1) \in \mathbb{R}_+^n$, $t \in \mathbb{R}_+$ for any $x(0,1) \in \mathbb{R}_+^n$ only if $A = A_0 + A_1 A_2 \in \mathbb{R}_+^{n\times n}$. The proof of sufficiency is similar to the one given in [77]. □

Definition 4.29. The positive general model (4.46) is called pointwise complete at the point (t_f, q) if for every final state $x_f \in \mathbb{R}_+^n$ there exist boundary conditions (4.61) such that

$$x(t_f, q) = x_f, \quad t_f > 0, \quad q \in \mathbb{N} = \{1,2,\dots\}. \tag{4.63}$$

Theorem 4.38. *The positive general model (4.46) is pointwise complete at the point* $(t_f, 1)$ *if and only if the matrix* A_2 *is diagonal.*

Proof. In a similar way as in proof of Theorem 4.34 we may obtain the equation (4.52). It is well-known [77] that $e^{A_2 t} \in \mathbb{R}_+^{n\times n}$, $t \in \mathbb{R}_+$ if and only if A_2 is a Metzler matrix. Hence $e^{-A_2 t} \in \mathbb{R}_+^{n\times n}$ if and only if A_2 is a diagonal matrix. In this case for arbitrary $x_f \in \mathbb{R}_+^n$ if and only if $x(0,1) \in \mathbb{R}_+^n$. □

In a similar way as for standard general model we can prove the following theorem.

Theorem 4.39. *The positive general model (4.46) is pointwise complete at the point* (t_f, q), $t_f > 0$, $q \in \mathbb{N} = \{1, 2, \ldots\}$ *if and only if the matrix* A_2 *is diagonal.*

From Theorem 4.39 we have the following corollary.

Corollary 4.5. *The pointwise completeness of the positive general model (4.46) is independent of the matrices* A_0 *and* A_1 *of the model.*

Definition 4.30. The positive general model (4.46) is called pointwise degenerated at the point (t_f, q) if there exists at least one final state $x_f \in \mathbb{R}_+^n$ such that $x(t_f, q) \neq x_f$ for all $x(0,i) \in \mathbb{R}_+^n$ and $x(t,0) = 0$, $\dot{x}(t,0) = 0$, $t \in \mathbb{R}_+$.

Theorem 4.40. *The positive general model (4.46) is pointwise degenerated at the point* (t_f, q) *if the matrix* $A_2 \in M_n$ *is not diagonal.*

Proof. In a similar way as in proof of Theorem 4.34 we may obtain the equality (4.52) which can be satisfied for $x_f \in \mathbb{R}_+^n$ and $x(0,i) \in \mathbb{R}_+^n$ if and only if the Metzler matrix A_2 is diagonal. The proof for $q > 1$ is similar. □

These considerations can be easily extended for $x(0,i) \in \mathbb{R}_+^n$, $x(t,0) \in \mathbb{R}_+^n$ and $\dot{x}(t,0) \in \mathbb{R}_+^n$, $t \in \mathbb{R}_+$.

Example 4.11. Consider the general model (4.46) with the matrices

$$A_0 = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}. \tag{4.64}$$

The model is positive since the matrices A_0 and A_1 have nonnegative entries and

$$A = A_0 + A_1 A_2 = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in \mathbb{R}_+^{2\times 2}. \tag{4.65}$$

The matrix A_2 is diagonal and the positive model with (4.64) by Theorem 4.38 is pointwise complete at the point $(t_f, 1)$, $t_f > 0$. Using (4.52) we obtain

$$x(0,1) = e^{-A_2 t_f} x_f = \begin{bmatrix} e^{t_f} & 0 \\ 0 & e^{2t_f} \end{bmatrix} x_f \in \mathbb{R}_+^2, \tag{4.66}$$

for any $x_f \in \mathbb{R}_+^2$ and $t_f \in \mathbb{R}_+$.

Example 4.12. Consider the general model (4.46) with the matrices (4.54). The model is positive since A_0 and A_1 have nonnegative entries, $A_2 \in M_n$ and

$$A = A_0 + A_1 A_2 = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \in \mathbb{R}_+^{2\times 2}. \tag{4.67}$$

Let $x_f = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Using (4.55) and (4.52) we obtain the vector

$$x(0,1) = e^{-A_2 t_f} x_f = \begin{bmatrix} e^{t_f} & 0 \\ e^{t_f} - e^{2t_f} & e^{2t_f} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{t_f} \\ e^{t_f} - e^{2t_f} \end{bmatrix}, \tag{4.68}$$

with negative second component for $t_f > 0$. Therefore, the model is pointwise degenerated at the point $(t_f, 1)$. The same result follows from Theorem 4.40 since the matrix A_2 is not diagonal. Note that the vector $x(0,1)$ given by (4.56) for $x_f = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ has positive components.

Chapter 5
Pointwise Completeness and Pointwise Degeneracy of Linear Systems with State-Feedbacks

5.1 Standard Discrete-Time Linear Systems

Consider the discrete-time system

$$x_{i+1} = Ax_i + Bu_i, \tag{5.1}$$

with the state-feedback

$$u_i = Kx_i, \tag{5.2}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ are state and input vectors, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $K \in \mathbb{R}^{m\times n}$ is a gain matrix. Substituting (5.2) into (5.1) we obtain

$$x_{i+1} = A_c x_i, \tag{5.3}$$

where $A_c = A + BK$.

The solution of the equation (5.3) has the form

$$x_i = A_c^i x_0. \tag{5.4}$$

By Theorem 4.2 the closed-loop system (5.3) is pointwise complete if and only if the matrix A_c is nonsingular. Let the system (5.1) be pointwise degenerated. We are looking for a gain matrix $K \in \mathbb{R}^{m\times n}$ such that the closed-loop matrix $A_c = A + BK$ is nonsingular.

Theorem 5.1. *If the pair (A,B) is reachable then there exists a gain matrix $K \in \mathbb{R}^{m\times n}$ such that the closed-loop system is pointwise complete.*

Proof. It is well-known that

$$\det A = \lambda_1 \lambda_2 \dots \lambda_n. \tag{5.5}$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of the matrix A.

T. Kaczorek: Selected Problems of Fractional Systems Theory, LNCIS 411, pp. 103–116.
springerlink.com

If the system (5.1) is pointwise degenerated then at least one of the eigenvalues is zero. The closed-loop system is pointwise complete if and only if all eigenvalues $s_1, s_2, \dots, s_n$ of the matrix A_c

$$\det[A+BK] = s_1 s_2 \dots s_n \neq 0. \tag{5.6}$$

are nonzero. To obtain (5.6) we have to replace all zero eigenvalues λ_i, $i = 1, \dots, k$ of A with nonzero eigenvalues s_i, $i = 1, \dots, k$ of A_c. The problem of finding a gain matrix K such that (5.6) has been reduced to the classical eigenvalues assignment problem. It is well-known that we can arbitrary assign the eigenvalues of the closed-loop system if and only if the pair (A, B) is reachable. This complete the proof. □

Remark 5.1. Note that Theorem 5.1 formulates sufficient but not necessary conditions for the existence of a gain matrix.

Example 5.1. The following three cases will be considered.

Case 1. The pair:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{5.7}$$

is unreachable since

$$\operatorname{rank}\left[B\ AB\ A^2B\right] = \operatorname{rank}\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} = 2 < n = 3.$$

The closed-loop matrix

$$A_c = A + BK = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ k_1 & k_2 - 2 & k_3 + 1 \end{bmatrix},$$

is nonsingular for $k_1 \neq 0$ and arbitrary k_2, k_3. Therefore, although that the pair (5.7) is unreachable there exists a gain matrix $K = [k_1, k_2, k_3]$, such that the closed-loop system is pointwise complete.

Case 2. The pair:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{5.8}$$

is unreachable since

$$\operatorname{rank}\left[B\ AB\ A^2B\right] = \operatorname{rank}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & -1 \end{bmatrix} = 2 < n = 3.$$

From the form of the closed-loop matrix

$$A_c = A + BK = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ k_1 & k_2 - 2 & k_3 + 1 \end{bmatrix},$$

it follows that for any gain matrix $K \in \mathbb{R}^{m \times n}$ we have $\det A_c = 0$. In this case does not exist a gain matrix $K = [k_1, k_2, k_3]$, such that the closed-loop system is pointwise complete.

Case 3. The pair:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{5.9}$$

is reachable since

$$\operatorname{rank} \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = 3 = n.$$

The closed-loop matrix

$$A + BK = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ k_1 & k_2 - 2 & k_3 + 1 \end{bmatrix},$$

is nonsingular for $k_1 \neq 0$ and arbitrary k_2, k_3 and the closed-loop system is pointwise complete.

Theorem 5.2. *There exists a gain matrix $K \in \mathbb{R}^{m \times n}$ such that the closed-loop system (5.3) is pointwise complete if and only if*

$$\operatorname{rank}[A, B] = n. \tag{5.10}$$

Proof. Necessity. From

$$A + BK = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} I_n \\ K \end{bmatrix}, \tag{5.11}$$

it follows that $\det[A + BK] \neq 0$ implies the condition (5.10), since

$$\operatorname{rank}[A + BK] \leq min \left\{ \operatorname{rank}[AB], \operatorname{rank} \begin{bmatrix} I_n \\ K \end{bmatrix} \right\}.$$

Sufficiency. Without loss of generality we may assume that the matrices A and B have the form

$$A = \begin{bmatrix} A_1 \\ 0 \end{bmatrix}, \quad A_1 \in \mathbb{R}^{r \times n}, \quad r = \operatorname{rank} A, \tag{5.12a}$$

$$B = \begin{bmatrix} 0 & B_2 \\ I_p & \end{bmatrix}, \quad B_2 \in \mathbb{R}^{n \times (m-p)}, \quad p \leq m. \tag{5.12b}$$

If the pair (A, B) has not the form (5.12), then premultiplying (5.11) by the matrix L of elementary row operations (Appendix D), we may always obtain on the matrix $[A, B]$ in the form (5.12).

Let

$$\det \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \neq 0, \quad A_2 \in \mathbb{R}^{p \times n}, \quad p = n - r, \tag{5.13}$$

and

$$K = \begin{bmatrix} A_2 \\ 0 \end{bmatrix} \in \mathbb{R}^{m \times n}, \tag{5.14}$$

then

$$A + BK = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ I_p & \end{bmatrix} \begin{bmatrix} A_2 \\ 0 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}. \tag{5.15}$$

This completes the proof. □

Example 5.2. Consider the pair

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & 1 \end{bmatrix}. \tag{5.16}$$

In this case $n = 3$, $m = 2$ and $\det A = 0$.

Applying the following elementary row operations L we obtain

$$\begin{bmatrix} A & B \end{bmatrix} = \left[\begin{array}{ccc|cc} 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 2 & 0 & -1 & 1 \end{array}\right] \xrightarrow{L[3+1\times 2]} \left[\begin{array}{ccc|cc} 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$
$$\xrightarrow{L[1+3\times(-1)]} \left[\begin{array}{ccc|cc} 0 & -1 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \tag{5.17}$$

and

$$A_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \quad r = 2, \quad p = 1. \tag{5.18}$$

For $A_2 = [k_1, k_2, k_3]$ we have

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ k_1 & k_2 & k_3 \end{bmatrix} \quad \text{and} \quad \det \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ k_1 & k_2 & k_3 \end{vmatrix} = k_3 - k_1 \neq 0 \tag{5.19}$$

for $k_1 \neq k_3$ and arbitrary k_2.

The same result we obtain by computation of the determinant

$$\begin{aligned} \det[A+BK] &= \det \left\{ \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \\ 0 & 0 & 0 \end{bmatrix} \right\} \\ &= \begin{vmatrix} k_1 & k_2-1 & k_3 \\ 1 & 0 & 1 \\ -k_1 & 2-k_2 & -k_3 \end{vmatrix} = \begin{vmatrix} k_1 & k_2-1 & k_3 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix} \\ &= k_1 - k_3. \end{aligned} \tag{5.20}$$

Remark 5.2. Note that the reachability of the pair (A, B) implies (5.10). Substituting $z = 0$ in the reachability test of the pair (A, B)

$$\operatorname{rank}[I_n z - A, B] = n, \quad \forall z \in \mathbb{C}, \tag{5.21}$$

we obtain

$$\operatorname{rank}[-A, B] = \operatorname{rank}[A, B] = n.$$

Example 5.3. Consider the pair:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{5.22}$$

In this case

$$\operatorname{rank}[I_n z - A, B] = \operatorname{rank} \left[\begin{array}{ccc|c} z & 0 & -1 & 0 \\ 0 & z-1 & 0 & 0 \\ 0 & 2 & z-1 & 1 \end{array}\right].$$

For $z = 0$

$$\operatorname{rank} \left[\begin{array}{ccc|c} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 1 \end{array}\right] = 3,$$

and for $z = 1$

$$\operatorname{rank} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right] = 2 \leq n = 3.$$

The pair (5.22) is unreachable but it satisfies the condition (5.10). The closed-loop matrix

$$A + BK = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ k_1 & k_2 - 2 & k_3 + 1 \end{bmatrix},$$

is nonsingular for $k_1 \neq 0$ and arbitrary k_2 and k_3. Note that although the pair (5.22) is unreachable there exists $K \in \mathbb{R}^{m \times n}$ such that closed-loop system is pointwise complete.

5.2 Standard Continuous-Time Linear Systems

Consider the standard continuous-time linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{5.23}$$

with the state-feedback

$$u(t) = Kx(t), \tag{5.24}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ are state and input vectors, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $K \in \mathbb{R}^{m \times n}$ is a gain matrix.

Substitution of (5.24) into (5.23) yields

$$\dot{x}(t) = A_c x(t), \tag{5.25}$$

where $A_c = A + BK$.

The solution of the equation (5.25) has the form

$$x(t) = e^{A_c t} x_0. \tag{5.26}$$

Theorem 5.3. *The pointwise completeness and the pointwise degeneracy of the standard continuous-time linear system are invariant under the state-feedback.*

Proof. From (5.26) for $t = t_f$ we have $x_f = e^{A_c t_f} x_0$ and $x_0 = e^{-A_c t_f} x_f$, since $\det[e^{A_c t_f}] \neq 0$ for any matrix A_c and time t_f. □

Remark 5.3. Theorem 5.3 is also valid for output feedbacks, $u(t) = Fy(t) = FCx(t)$, since the closed-loop system matrix has the form $A_c = A + BFC$.

5.3 Positive Standard Discrete-Time Linear Systems

Consider the discrete-time linear system (5.1) with the state-feedback (5.2).

Definition 5.1. The discrete-time system (5.1) is called positive if the state vector $x_i \in \mathbb{R}_+^n$, $i \in \mathbb{Z}_+$ for any initial conditions $x_0 \in \mathbb{R}_+^n$ and all input sequences $u_i \in \mathbb{R}_+^m$, $i \in \mathbb{Z}_+$.

Theorem 5.4. *The discrete-time system (5.1) is positive if and only if* $A \in \mathbb{R}_+^{n\times n}$, $B \in \mathbb{R}_+^{n\times m}$.

Proof. The proof is similar to the proof of Theorem 4.10. □

By Theorem 4.10 the closed-loop system is positive if and only if $A_c \in \mathbb{R}_+^{n\times n}$ and by Theorem 4.11 it is pointwise complete if and only if the matrix A_c is monomial.

Let the matrix $B \in \mathbb{R}^{n\times m}$ have $\overline{m} \leq m$ linearly independent monomial rows $b_{i_1},...,b_{i_{\overline{m}}}$ and its remaining rows be zero. The matrix $\overline{B} \in \mathbb{R}^{\overline{m}\times n}$ is obtained from B by adding $n-m$ zero in each its $b_{i_1},\dots,b_{i_{\overline{m}}}$ rows and the matrix $\overline{A} \in \mathbb{R}^{(n-\overline{m})\times n}$ is obtained from A by removing the rows $a_{i_1},\dots,a_{i_{\overline{m}}}$ (corresponding to the rows $b_{i_1},\dots,b_{i_{\overline{m}}}$). The $k \geq \overline{m}$ rows of the matrices A and B are called linearly independent monomial rows if and only if the matrix $\begin{bmatrix} \overline{A} \\ \overline{B} \end{bmatrix}$ contains k linearly independent monomial rows.

For example if

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

then

$$\overline{A} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrices A and B contain three linearly independent monomial rows since the matrix

$$\begin{bmatrix} \overline{A} \\ \overline{B} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

is monomial.

Theorem 5.5. *Let the positive system (5.1) be pointwise degenerated. There exists a gain matrix* $K \in \mathbb{R}^{m\times n}$ *such that the closed-loop system (5.3) is positive and pointwise complete if the matrices* A *and* B *contain* n *linearly independent monomial rows.*

Proof. Without loss of generality we may assume that:

1) the matrix B has the first $\overline{m} \leq m$ linearly independent monomial rows $b_1,\dots,b_{\overline{m}}$ and remaining zero rows, i.e.

$$B = \begin{bmatrix} b_1 \\ \vdots \\ b_{\overline{m}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{5.27}$$

2) and that the rows $a_{\overline{m}+1},\dots,a_n$ of the matrix A are also linearly independent monomial rows.

We are looking for a gain matrix of the form

$$K = \begin{bmatrix} k_1 \\ \vdots \\ k_{\overline{m}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{5.28}$$

The closed-loop matrix has the form

$$A_c = A + BK = \begin{bmatrix} a_1 + b_1 k_1 \\ \vdots \\ a_{\overline{m}} + b_{\overline{m}} k_{\overline{m}} \\ a_{\overline{m}+1} \\ \vdots \\ a_n \end{bmatrix}. \tag{5.29}$$

Note that $k_1,\dots,k_{\overline{m}}$ of (5.28) can be always chosen so that the matrix (5.29) has n linearly independent monomial rows. □

Example 5.4. Consider the positive discrete-time system (5.1) with the matrices:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{5.30}$$

In this case the matrix B has only one monomial row and the matrix A two linearly independent monomial rows. We are looking for a gain matrix $K = [k_1, k_2, k_3]$ such that the closed-loop system matrix

$$A_c = A + BK = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ k_1+1 & k_2+2 & k_3+3 \end{bmatrix}, \tag{5.31}$$

is monomial. It is easy to see that for $k_1 = -1$, $k_2 = -2$, $k_3 > -3$ the matrix (5.31) is monomial and the closed-loop system is positive and pointwise complete.

Example 5.5. Consider the positive discrete-time system (5.1) with the matrices

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{5.32}$$

In this case the condition of Theorem 5.5 is not satisfied since only one row of the matrix A corresponding to zero rows of B is monomial.

For a gain of the form $K = [k_1, k_2, k_3]$ the closed-loop matrix

$$A + BK = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ k_1 & k_2 & k_3 + 2 \end{bmatrix},$$

has at most two linearly independent rows for any entries k_1, k_2, k_3. The closed-loop matrix $A_c = A + BK$ is not monomial for any choice of the gain matrix K. Therefore, for the pair (5.32) does not exist a gain matrix K such that the closed-loop system is pointwise complete.

5.4 Positive Standard Continuous-Time Linear Systems

Consider the continuous-time linear system (5.23) with the state-feedback (5.24).

Definition 5.2. The continuous-time linear system (5.23) is called positive if $x(t) \in \mathbb{R}_+^n$, $t \geq 0$ for any initial conditions $x_0 \in \mathbb{R}_+^n$ and all inputs $u(t) \in \mathbb{R}_+^m$, $t \geq 0$.

Theorem 5.6. *The continuous-time linear system (5.23) is positive if and only if $A \in M_n$ and $B \in \mathbb{R}_+^{n\times m}$.*

Proof. The proof is similar to the proof of Theorem 2.8. □

By Theorem 4.13 the closed-loop system (5.25) is positive if and only if $A_c \in M_n$. By Theorem 4.14 the positive continuous-time linear system is pointwise complete if and only if the matrix A is diagonal.

Remark 5.4. Note that the matrix e^{At} is monomial if and only if the matrix A is diagonal. The matrix e^{-At} is monomial if and only if the matrix A is diagonal.

Theorem 5.7. *Let the positive system (5.23) be pointwise degenerated. There exists a gain matrix $K \in \mathbb{R}^{m\times n}$ such that the closed-loop system (5.25) is positive and pointwise complete if the pair (A, B) of the system satisfies the condition:*

The matrix B has $l \leq m$ linearly independent monomial rows $b_{i_1}, b_{i_2}, \ldots, b_{i_l}$ and all its remaining rows are zero. Every i-th row of the matrix A has the form $a_i = c_i e_i$, $i = l+1, \ldots, n$, where c_i is a real nonzero or zero coefficients and e_i is i-th row of the identity matrix I_n.

Proof. Without loss of generality we may assume $i_1 = 1, i_2 = 2, \ldots, i_l = l$ and the matrices A and B have the form

$$A = \begin{bmatrix} a_1 \\ \vdots \\ a_l \\ c_{l+1}e_{l+1} \\ \vdots \\ c_n e_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ \vdots \\ b_l \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad a_i = \begin{bmatrix} a_{i1} & \ldots & a_{in} \end{bmatrix}. \tag{5.33}$$

The closed-loop matrix has the form

$$A_c = A + BK = \begin{bmatrix} a_1 + b_1 k_1 \\ \vdots \\ a_l + b_l k_l \\ c_{l+1} e_{l+1} \\ \vdots \\ c_n e_n \end{bmatrix}. \tag{5.34}$$

From (5.34) it follows that it is always possible to choose the gain matrix $K \in \mathbb{R}^{m \times n}$ so that the matrix A_c is diagonal if the condition of Theorem 5.7 is satisfied. □

Example 5.6. Consider the positive continuous-time system (5.23) with the matrices:

$$A = \begin{bmatrix} -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.35}$$

We are looking for a gain matrix of the form

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix},$$

such the closed-loop matrix A_c is diagonal.

The pair (5.35) satisfy the condition of Theorem 5.7 and the closed-loop matrix A_c has the form

$$\begin{aligned} A_c = A + BK &= \begin{bmatrix} -1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix} \\ &= \begin{bmatrix} k_{21} - 1 & k_{22} + 2 & k_{23} & k_{24} + 1 \\ 0 & 0 & 0 & 0 \\ k_{11} + 2 & k_{12} + 1 & k_{13} & k_{14} + 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}. \end{aligned}$$

For k_{13} and k_{21} arbitrary and $k_{22} = -2$, $k_{23} = 0$, $k_{24} = -1$, $k_{11} = -2$, $k_{12} = -1$, $k_{14} = -2$ we obtain a diagonal closed-loop matrix A_c. The desired gain matrix is:

$$K = \begin{bmatrix} -2 & -1 & k_{13} & -2 \\ k_{21} & -2 & 0 & -1 \end{bmatrix}.$$

Theorem 5.8. *Let the positive system (5.23) be pointwise degenerated. There exists a gain matrix $K \in \mathbb{R}^{m \times n}$ such that the closed-loop system (5.25) is positive and pointwise complete if and only if for a given diagonal matrix $\overline{A}_c \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}_+^{n \times m}$ the following condition is satisfied*

$$\operatorname{rank} B = \operatorname{rank}\left[\overline{A}_c - A \ , \ B\right]. \tag{5.36}$$

Proof. By Kronecker-Capelly theorem the equation

$$BK = \overline{A}_c - A, \tag{5.37}$$

has a solution K for a given $\overline{A}_c - A$ and $B \in \mathbb{R}_+^{n\times m}$ if and only if the condition (5.36) is satisfied. □

Example 5.7. Consider the positive system (5.23) with the matrices (5.35). We are looking for a gain matrix

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix},$$

such the closed-loop matrix A_c is diagonal

$$\overline{A}_c = \operatorname{diag}\left[\, a_1 \ a_2 \ a_3 \ a_4 \,\right]. \tag{5.38}$$

The condition (5.36) for (5.35) and (5.38) has the form

$$\operatorname{rank}\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} a_1+1 & -2 & 0 & -1 & 0 & 1 \\ 0 & a_2 & 0 & 0 & 0 & 0 \\ -2 & -1 & a_3 & -2 & 1 & 0 \\ 0 & 0 & 0 & a_4-2 & 0 & 0 \end{bmatrix} = 2,$$

and it is satisfied for $a_2 = 0$, $a_4 = 2$.

In this case the equation (5.37) has the form

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} K = \begin{bmatrix} a_1+1 & -2 & 0 & -1 \\ 0 & a_2 & 0 & 0 \\ -2 & -1 & a_3 & -2 \\ 0 & 0 & 0 & a_4-2 \end{bmatrix}, \tag{5.39}$$

and its solution is

$$K = \begin{bmatrix} -2 & -1 & a_3 & -2 \\ a_1+1 & -2 & 0 & -1 \end{bmatrix}, \tag{5.40}$$

where a_1, a_3 are arbitrary.

For

$$\overline{A}_c = \operatorname{diag}\left[\, -1 \ 0 \ 0 \ 2 \,\right],$$

the gain matrix K has the form

$$K = \begin{bmatrix} -2 & -1 & 0 & -2 \\ 0 & -2 & 0 & -1 \end{bmatrix}.$$

This result is consistent with the result obtain in Example 5.6.

5.5 Fractional Discrete-Time Linear Systems

Consider the fractional discrete-time linear system

$$x_{i+1} = A_\alpha x_i + \sum_{j=1}^{i} c_j(\alpha) x_{k-j} + Bu_i, \tag{5.41}$$

with the state-feedback

$$u_i = Kx_i, \tag{5.42}$$

where $c_j(\alpha)$ is defined by (4.23), $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ are the state and input vectors $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $K \in \mathbb{R}^{m\times n}$ is a gain matrix. Substitution of (5.42) into (5.41) yields

$$x_{i+1} = [A_\alpha + BK]x_i + \sum_{j=1}^{i} c_j(\alpha) x_{k-j}. \tag{5.43}$$

To investigate the pointwise completeness and pointwise degeneracy we may use Theorems 4.8 and 4.9 for $A_0 = A_\alpha + BK$, $A_k = c_k(\alpha)I_n$, $k = 1,\dots,q$.

Theorem 5.9. *Every fractional discrete-time linear system is pointwise complete for $i = q = 1$ and every gain matrix K.*

Proof. Using (4.8) for $i = q = 1$ we obtain

$$H_1 = \begin{bmatrix} H_0(1) & H_1(1) \end{bmatrix} = \begin{bmatrix} \Phi_1 & c_1(\alpha)I_n \end{bmatrix}.$$

By Theorem 4.8 this system is pointwise complete for every gain matrix K for $i = q = 1$, since

$$\operatorname{rank} H_1 = \operatorname{rank} \begin{bmatrix} \Phi_1 & c_1(\alpha)I_n \end{bmatrix} = n. \tag{5.44}$$

□

5.6 Fractional Continuous-Time Linear Systems

Consider the fractional continuous-time linear systems

$$\frac{d^\alpha}{dt^\alpha} x(t) = Ax(t) + Bu(t), \tag{5.45}$$

with the state-feedback

$$u(t) = Kx(t), \tag{5.46}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ are state and input vectors, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $K \in \mathbb{R}^{m\times n}$ is a gain matrix. Substitution of (5.46) into (5.45) yields

$$\frac{d^\alpha}{dt^\alpha} x(t) = A_c x(t), \quad A_c = A + BK. \tag{5.47}$$

Theorem 5.10. *The pointwise completeness and pointwise degeneracy of fractional continuous-time linear systems are invariant under the state-feedbacks and the output-feedbacks.*

Proof. The proof follows immediately from the fact that the fractional continuous-time system is pointwise complete for every matrix A. □

5.7 Positive Fractional Discrete-Time Linear System

Consider the positive fractional discrete-time linear system (5.41) with the state-feedback (5.42), where $x_i \in \mathbb{R}_+^n$, $u_i \in \mathbb{R}_+^m$ are state and input vectors, $A \in \mathbb{R}_+^{n\times n}$, $B \in \mathbb{R}_+^{n\times m}$ and $K \in \mathbb{R}^{m\times n}$.

Theorem 5.11. *Let the positive fractional discrete-time system (5.41) be pointwise degenerated. There exists a gain matrix $K \in \mathbb{R}^{m\times n}$ such that the closed-loop system is positive and pointwise complete if the pair (A,B) satisfies the condition:*

The matrix B has $l \le m$ linearly independent monomial rows and the remaining its rows are zero. Every row a_i of the matrix A corresponding to the zero row of B has the form $a_i = c_i e_i$ where c_i is nonzero coefficient and e_i is i-th row if I_n.

Proof. Proof is similar to the proof of Theorem 5.7. □

Example 5.8. Consider the positive fractional discrete-time system (5.41) with the matrices:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 3 & 0 & 0 \\ 2 & 1 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.48}$$

We are looking for a gain matrix of the form

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix},$$

such that the closed-loop matrix A_c is diagonal with positive diagonal entries.
The pair (5.48) satisfies the condition of Theorem 5.11. The closed-loop matrix has the form

$$\begin{aligned} A_c = A + BK &= \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 3 & 0 & 0 \\ 2 & 1 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix} \\ &= \begin{bmatrix} k_{21}+1 & k_{22}+2 & k_{23} & k_{24}+1 \\ 0 & 3 & 0 & 0 \\ k_{11}+2 & k_{12}+1 & k_{13}+1 & k_{14}+2 \\ 0 & 0 & 0 & 2 \end{bmatrix}. \end{aligned}$$

Assuming $k_{13} > -1, k_{21} > -1$ and $k_{22} = -2, k_{23} = 0, k_{24} = -1, k_{11} = -2, k_{12} = -1$, $k_{14} = -2$ we obtain diagonal matrix A_c with positive diagonal entries. The desired gain matrix has the form:

$$K = \begin{bmatrix} -2 & -1 & k_{13} & -2 \\ k_{21} & -2 & 0 & -1 \end{bmatrix}.$$

5.8 Positive Fractional Continuous-Time Linear Systems

Consider the positive fractional continuous-time linear system (5.45) with the state-feedback (5.46), where $x(t) \in \mathbb{R}^n_+$, $u(t) \in \mathbb{R}^m_+$ are state and input vectors and $A \in M_n$, $B \in \mathbb{R}^{n\times m}_+$.

Theorem 5.12. *Let the positive fractional continuous-time system (5.45) be pointwise degenerated. There exists a gain matrix $K \in \mathbb{R}^{m\times n}$ such that the closed-loop system is positive and pointwise complete if the pair (A,B)satisfies the condition:*

The matrix B has $l \leq m$ linearly independent monomial rows and the remaining its rows are zero. Every row a_i of the matrix A corresponding to zero row of B has the form $a_i = c_i e_i$, where c_i is nonzero or zero coefficient and e_i is the i-th row of I_n.

Proof. Proof is similar to the proof of Theorem 5.7. □

Example 5.9. Consider the positive fractional continuous-time system (5.45) with the matrices

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 2 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{5.49}$$

We are looking for a gain matrix of the form

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \end{bmatrix},$$

such that the closed-loop matrix

$$A_c = A + BK = \begin{bmatrix} k_{21} & k_{22}+1 & k_{23}+2 \\ 0 & 0 & 0 \\ k_{11}+2 & k_{12}+1 & k_{13}-1 \end{bmatrix},$$

is diagonal.

Assuming $k_{21} \geq 0$, $k_{13} \geq -1$ and $k_{22} = -1$, $k_{23} = -2$, $k_{11} = -2$, $k_{12} = -1$ we obtain the desired diagonal matrix A_c.

Chapter 6
Realization Problem for Positive Fractional and Continuous-Discrete 2D Linear Systems

6.1 Fractional Discrete-Time Linear Systems

Consider the fractional discrete-time linear system:

$$x_{k+1} + \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} x_{k-j+1} = Ax_k + Bu_k, \quad k \in \mathbb{Z}_+, \tag{6.1a}$$

$$y_k = Cx_k + Du_k, \tag{6.1b}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^p$ are state, input and output vectors and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Lemma 6.1. *The transfer matrix of the fractional system (6.1) has the form*

$$T(z) = C\left[I_n(z-c_\alpha) - A\right]^{-1} B + D = \frac{N(z)}{d(z)}, \tag{6.2}$$

where

$$c_\alpha = c_\alpha(k,z) = \sum_{j=1}^{k} (-1)^{j-1} \binom{\alpha}{j} z^{1-j}, \tag{6.3a}$$

$$N(z) = C\,\mathrm{Adj}\left[I_n(z-c_\alpha) - A\right]^{-1} B + Dd(z) \tag{6.3b}$$

$$= N_n(z-c_\alpha)^n + \cdots + N_1(z-c_\alpha) + N_0 = \begin{bmatrix} N_{11}(z) & \dots & N_{1m}(z) \\ \vdots & \ddots & \vdots \\ N_{p1}(z) & \dots & N_{pm}(z) \end{bmatrix},$$

$$N_i \in \mathbb{R}^{p\times m}, \quad i = 0,1,\dots,n, \tag{6.3c}$$

T. Kaczorek: Selected Problems of Fractional Systems Theory, LNCIS 411, pp. 117–140.
springerlink.com

$$N_{ij} = \sum_{l=-n}^{(k-1)n} N_{ij}^{l} z^{-l}, \quad i = 1,\dots,p; \quad j = 1,\dots,m; \tag{6.3d}$$

$$\begin{aligned} d(z) &= \det[I_n(z-c_\alpha)-A] \\ &= (z-c_\alpha)^n + a_{n-1}(z-c_\alpha)^{n-1} + \dots + a_1(z-c_\alpha) + a_0 = \sum_{l=-n}^{kn} \overline{a}_i z^{-i} \\ &\quad a_0 = \det A, \dots, a_{n-1} = \mathrm{trance} A. \end{aligned} \tag{6.3e}$$

Proof. Using $\mathscr{Z}$-transform to (6.1) with zero initial conditions and taking into account (A.14b), we obtain:

$$X(z) = [I_n(z-c_\alpha)-A]^{-1} BU(z), \tag{6.4a}$$

$$Y(z) = CX(z) + DU(z), \tag{6.4b}$$

where $X(z) = \mathscr{Z}[x_k]$, $U(z) = \mathscr{Z}[u_k]$, $Y(z) = \mathscr{Z}[y_k]$.

Substitution of (6.4b) into (6.4a) yields $Y(z) = T(z)U(z)$, where $T(z)$ is given by (6.2). □

Definition 6.1. The transfer matrix $T(z)$ is called proper if

$$\lim_{z\to\infty} T(z) = K \in \mathbb{R}^{p\times m},$$

and it is called strictly proper if $K = 0$.

From (6.2) we have

$$\lim_{z\to\infty} T(z) = D, \tag{6.5}$$

since

$$\lim_{z\to\infty} [I_n(z-c_\alpha)-A]^{-1} = 0.$$

Definition 6.2. Matrices A, B, C, D satisfying

$$A + I_n\alpha \in \mathbb{R}_+^{n\times n}, \quad B \in \mathbb{R}_+^{n\times m}, \quad C \in \mathbb{R}_+^{p\times n}, \quad D \in \mathbb{R}_+^{p\times m},$$

are called a positive realization of a given transfer matrix $T(z)$, if they satisfy the equality

$$T(z) = C[I_n(z-c_\alpha)-A]^{-1}B + D.$$

Realization is called minimal if the dimension of A is minimal among all realizations of $T(z)$.

The positive fractional realization problem can be stated as follows. Given a proper transfer matrix $T(z)$, find its positive realization.

Sufficient conditions for the existence of the positive fractional realizations of a given proper transfer matrix $T(z)$ will be established and procedures for computation of the positive fractional realizations will be proposed [110].

6.1.1 SISO Systems

From (6.2) it follows that the transfer matrix of fractional discrete-time linear system is proper rational matrix in the variable $z - c_\alpha$. From (1.5) and (6.3a) it follows that $c_\alpha = c_\alpha(k,z)$ strongly decreases when k increases. In practical problems it is assumed that k is bounded by some natural number h. In this case the fractional system (6.1) is a system with h delays in state vector.

Firstly the following two cases for single-input single-output systems (SISO) will be addressed:

Case 1). Given α, c_α and the transfer function

$$T(z) = \frac{b_n(z-c_\alpha)^n + \cdots + b_1(z-c_\alpha) + b_0}{(z-c_\alpha)^n + a_{n-1}(z-c_\alpha)^{n-1} + \cdots + a_1(z-c_\alpha) + a_0}, \tag{6.6}$$

find its positive fractional realization A, B, C, D.

Case 2). Given a transfer function of the form

$$T(z) = \frac{b_n z^n + \cdots + b_1 z + b_0 + b_{-1}z^{-1} + \cdots + b_{-qn}z^{-qn}}{z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 + a_{-1}z^{-1} + \cdots + a_{-qn}z^{-qn}}, \tag{6.7}$$

find a fractional order α and a positive fractional realization A, B, C, D.

In Case 1) the problem can be solved using the well-known realization theory of positive systems [77].

Theorem 6.1. *There exist positive fractional realizations of the forms:*

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C^T = \begin{bmatrix} \overline{b}_0 \\ \overline{b}_1 \\ \vdots \\ \overline{b}_{n-1} \end{bmatrix}, \tag{6.8a}$$

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -a_{n-1} \end{bmatrix}, \; B = \begin{bmatrix} \overline{b}_0 \\ \overline{b}_1 \\ \vdots \\ \overline{b}_{n-1} \end{bmatrix}, \; C^T = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \tag{6.8b}$$

$$A = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \dots & -a_1 & -a_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; C^T = \begin{bmatrix} \overline{b}_{n-1} \\ \overline{b}_{n-2} \\ \vdots \\ \overline{b}_0 \end{bmatrix}, \tag{6.8c}$$

$$A = \begin{bmatrix} -a_{n-1} & 1 & 0 & \dots & 0 \\ -a_{n-2} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_1 & 0 & 0 & \dots & 1 \\ -a_0 & 0 & 0 & \dots & 0 \end{bmatrix}, \; B = \begin{bmatrix} \overline{b}_{n-1} \\ \overline{b}_{n-2} \\ \vdots \\ \overline{b}_0 \end{bmatrix}, \; C^T = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{6.8d}$$

$$\overline{b}_k = b_k - a_k b_n, \quad k = 0, 1, \dots, n-1; \quad D = b_n. \tag{6.8e}$$

of the transfer function(6.6), if

a) $b_k \geq 0$ *for* $k = 0, 1, \dots, n$;
b) $a_k \leq 0$ *for* $k = 0, 1, \dots, n-2$ *and* $a_{n-1} \leq \alpha$;

Proof. The detail of the proof will be given only for (6.8a). The proofs for (6.8b), (6.8c) and (6.8d) are similar.

From (6.6) we have

$$D = \lim_{z \to \infty} T(z) = b_n, \tag{6.9}$$

and the strictly proper transfer function has the form

$$T_{sp}(z) = T(z) - D = \frac{\overline{b}_{n-1}(z - c_\alpha)^{n-1} + \dots + \overline{b}_1(z - c_\alpha) + \overline{b}_0}{(z - c_\alpha)^n + a_{n-1}(z - c_\alpha)^{n-1} + \dots + a_1(z - c_\alpha) + a_0}. \tag{6.10}$$

Taking into account that for (6.8a)

$$\det[I_n(z - c_\alpha) - A] = (z - c_\alpha)^n + a_{n-1}(z - c_\alpha)^{n-1} + \dots + a_1(z - c_\alpha) + a_0,$$

and

$$\mathrm{Adj}[I_n(z - c_\alpha) - A]B = \begin{bmatrix} 1 & (z - c_\alpha) & \dots & (z - c_\alpha)^{n-1} \end{bmatrix}^T.$$

it is easy to verify that

$$\begin{aligned} C[I_n(z - c_\alpha) - A]^{-1}B &= \frac{C\,\mathrm{Adj}[I_n(z - c_\alpha) - A]B}{\det[I_n(z - c_\alpha) - A]} \\ &= \frac{\overline{b}_{n-1}(z - c_\alpha)^{n-1} + \dots + \overline{b}_1(z - c_\alpha) + \overline{b}_0}{(z - c_\alpha)^n + a_{n-1}(z - c_\alpha)^{n-1} + \dots + a_1(z - c_\alpha) + a_0} \end{aligned}$$

The matrix $A_\alpha \in \mathbb{R}_+^{n\times n}$, if and only if the condition $b)$ of Theorem 6.1 is satisfied and the matrices C and D are nonnegative if the condition $a)$ is met. □

If the conditions of Theorem 6.1 are satisfied then the positive fractional realizations (6.8) of the transfer function (6.6) can be found by use of the following procedure:

Procedure 6.1

Step 1. Knowing transfer function (6.6) and using (6.9) find D and the strictly proper transfer function (6.10).

Step 2. Using (6.8) find the desired realizations.

Remark 6.1. The positive realizations (6.8) are minimal if and only if the transfer function (6.6) is irreducible.

Example 6.1. Find the positive minimal fractional realizations (6.8) of the irreducible transfer function

$$T(z) = \frac{2(z-c_\alpha)^2 + 5(z-c_\alpha) + 2}{(z-c_\alpha)^2 - (z-c_\alpha) - 2}. \tag{6.11}$$

Using Procedure 6.1 we obtain the following:

Step 1. From (6.9) and (6.11) we have $D = 2$ and

$$T_{sp}(z) = T(z) - D = \frac{7(z-c_\alpha)+6}{(z-c_\alpha)^2 - (z-c_\alpha) - 2}.$$

Step 2. In this case $\overline{b}_0 = 6$, $\overline{b}_1 = 7$. Using (6.8), we obtain the desired positive fractiona realizations:

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 6 & 7 \end{bmatrix}, \quad D = 2,$$
$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 6 \\ 7 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = 2,$$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 7 & 6 \end{bmatrix}, \quad D = 2,$$
$$A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 7 \\ 6 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 2.$$

Positive realization problem in the Case $2)$ will be presented on a very simple example of SISO fractional system with the strictly proper transfer function

$$T_{sp}(z) = \frac{b_1 z + b_0 + b_{-1}z^{-1} + b_{-2}z^{-2}}{z^2 + a_1 z + a_0 + a_{-1}z^{-1} + a_{-2}z^{-2} + a_{-3}z^{-3} + a_{-4}z^{-4}}. \tag{6.12}$$

In this case $n = 2$ and $q = 2$.

Without loss of generality we may assume the matrix A in the canonical Frobenius form

$$A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}. \tag{6.13}$$

Taking into account (1.5), (6.12) and (6.13), we may write

$$\begin{aligned} \det[I_n(z-c_\alpha)-A] &= \begin{bmatrix} z-c_0-c_1z^{-1}-c_2z^{-2} & -1 \\ a_0 & z+a_1-c_0-c_1z^{-1}-c_2z^{-2} \end{bmatrix} \\ &= z^2+(a_1-2c_0)z+c_0^2-a_1c_0-2c_1 \\ &+ (2c_0c_1-a_1c_1-2c_2)z^{-1}+(c_1^2+2c_0c_2-a_1c_2)z^{-2} \\ &+ 2c_1c_2z^{-3}+c_2^2z^{-4}+a_0, \end{aligned} \tag{6.14}$$

where

$$c_k = (-1)^k \frac{\alpha(\alpha-1)\cdots(\alpha-k)}{(k+1)!}, \quad k = 0,1,\dots\ . \tag{6.15}$$

From comparison of the denominator (6.12) and (6.14) we have $c_2^2 = a_{-4}$. From (6.15) for $k = 2$ we have $\alpha(\alpha-1)(\alpha-2) = 6\sqrt{a_{-4}}$, and

$$\alpha^3 - 3\alpha^2 + 2\alpha - 6\sqrt{a_{-4}} = 0. \tag{6.16}$$

Solving the equation (6.16) we may find the desired real fractional order α. Knowing α and using (6.15), we may find the coefficients c_0, c_1 and $c_\alpha = c_0 + c_1z^{-1} + c_2z^{-2}$. Then the denominator of the transfer function (6.12) can be written in the form

$$z^2+a_1z+a_0+a_{-1}z^{-1}+a_{-2}z^{-2}+a_{-3}z^{-3}+a_{-4}z^{-4} = (z-c_\alpha)^2+\overline{a}_1(z-c_\alpha)+\overline{a}_0.$$

In a similar way we proceed with the numerator of (6.12) and we may write the transfer function (6.12) in the form

$$T(z) = \frac{\overline{b}_1(z-c_\alpha)+\overline{b}_0}{(z-c_\alpha)^2+\overline{a}_1(z-c_\alpha)+\overline{a}_0}. \tag{6.17}$$

To find a positive fractional realization of (6.17) we may use Procedure 6.1.

Procedure 6.2. In general case of SISO fractional system with transfer function (6.7) we proceed as follows:

Step 1. Using (6.9), find D and strictly proper transfer function

$$T_{sp}(z) = \frac{\overline{b}_{n-1}z^{n-1}+\cdots+\overline{b}_1z+\overline{b}_0+\overline{b}_{-1}z^{-1}+\cdots+\overline{b}_{-(n-1)q}z^{-(n-1)q}}{z^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0+a_{-1}z^{-1}+\cdots+a_{-nq}z^{-nq}}. \tag{6.18}$$

Step 2. Knowing the coefficient a_{-nq} of the denominator of (6.18) and solving the equation

$$a_{-nq} = \frac{\alpha(\alpha-1)\cdots(\alpha-q)}{(q+1)!}, \tag{6.19}$$

find the desired fractional order α.

Step 3. Knowing α and using (6.15), compute the coefficients $c_0, c_1, \dots, c_{q-1}$ and $c_\alpha = c_0 + c_1 z^{-1} + \cdots + c_q z^{-q}$ and write (6.18) in the form

$$T_{sp}(z) = \frac{\hat{b}_{n-1}(z-c_\alpha)^{n-1} + \cdots + \hat{b}_1(z-c_\alpha) + \hat{b}_0}{(z-c_\alpha)^n + \overline{a}_{n-1}(z-c_\alpha)^{n-1} + \cdots + \overline{a}_1(z-c_\alpha) + \overline{a}_0}. \tag{6.20}$$

Step 4. Using Procedure 6.1 find the desired positive fractional realization (6.8) of the transfer function (6.20) (and (6.7)).

Remark 6.2. The method can be easily extended for MIMO fractional system.

6.1.2 MIMO Systems

Consider a multi-input multi-output (MIMO) positive fractional system with proper transfer matrix $T(z)$. Using (6.5), we may find the matrix D and next the strictly proper transfer matrix which can be written in the form

$$T_{sp}(z) = T(z) - D = \begin{bmatrix} \frac{N_{11}(z)}{D_1(z)} & \cdots & \frac{N_{1m}(z)}{D_m(z)} \\ \vdots & \ddots & \vdots \\ \frac{N_{p1}(z)}{D_1(z)} & \cdots & \frac{N_{pm}(z)}{D_m(z)} \end{bmatrix} = N(z)D^{-1}(z), \tag{6.21}$$

where

$$N(z) = \begin{bmatrix} N_{11}(z) & \dots & N_{1m}(z) \\ \vdots & \ddots & \vdots \\ N_{p1}(z) & \dots & N_{pm}(z) \end{bmatrix}, \quad D(z) = \operatorname{diag}\left[D_1(z) \; \dots \; D_m(z) \right],$$

$$N_{ij}(z) = c_{ij}^{d_j-1}(z-c_\alpha)^{d_j-1} + \cdots + c_{ij}^1(z-c_\alpha) + c_{ij}^0, \tag{6.22a}$$

$$D_j(z) = (z-c_\alpha)^{d_j} + a_{jd_j-1}(z-c_\alpha)^{d_j-1} + \cdots + a_{j1}(z-c_\alpha) + a_{j0}, \tag{6.22b}$$

$$i = 1, \dots, p; \quad j = 1, \dots, m;$$

Theorem 6.2. *There exists the positive fractional realization*

$$A = \text{block diag}\left[A_1 \ \dots \ A_m\right] \in \mathbb{R}_+^{n\times n}, \quad j=1,\dots,m; \quad n = d_1+\dots+d_m,$$

$$A_j = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_{j0} & -a_{j1} & -a_{j2} & \dots & -a_{jd_j-1} \end{bmatrix} \in \mathbb{R}_+^{d_j\times d_j}, \tag{6.23}$$

$$B = \text{block diag}\left[B_1 \ \dots \ B_m\right] \in \mathbb{R}_+^{n\times m}, \quad B_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}_+^{d_j}, \quad j=1,\dots,m;$$

$$C = \begin{bmatrix} c_{11}^0 & \dots & c_{11}^{d_1-1} & \dots & c_{1m}^0 & \dots & c_{1m}^{d_m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ c_{p1}^0 & \dots & c_{p1}^{d_1-1} & \dots & c_{pm}^0 & \dots & c_{pm}^{d_m-1} \end{bmatrix} \in \mathbb{R}_+^{p\times n}, \quad D = T(\infty) \in \mathbb{R}_+^{p\times m},$$

of the transfer matrix $T(z)$, if the following conditions are satisfied:

a) $T(\infty) \in \mathbb{R}_+^{p\times m}$
b) $a_{ij} \le 0$ *for* $i=1,\dots,m$; $j=0,1,\dots,d_i-2$ *and* $a_{id_{i-1}} \le \alpha$ *for* $i=1,\dots,m$;
c) $c_{ij}^l \ge 0$ *for* $i=1,\dots,p$; $j=1,\dots,m$; $l=0,1,\dots,d_{m-1}$;

Proof. First we shall show that the matrices (6.23) are a realization of the strictly proper transfer matrix (6.21). Using (6.22) and (6.23), it is easy to verify that

$$B_jD_j(z) = \left[I_{d_j}(z-c_\alpha) - A_j\right] \begin{bmatrix} 1 \\ z-c_\alpha \\ \vdots \\ (z-c_\alpha)^{d_j-1} \end{bmatrix}, \quad \text{for} \quad j=1,\dots,m;$$

and

$$BD(z) = [I_n(z-c_\alpha) - A]S, \tag{6.24}$$

where

$$S = \text{block diag}\left[S_1 \ \dots \ S_m\right], \quad S_j = \begin{bmatrix} 1 \\ (z-c_\alpha) \\ \vdots \\ (z-c_\alpha)^{d_j-1} \end{bmatrix}, \quad j=1,\dots,m;$$

Premultiplying (6.24) by $C[I_n(z-c_\alpha)-A]^{-1}$ and postmultiplying by $D(z)^{-1}$, we obtain

$$C\left[I_n(z-c_\alpha)-A\right]^{-1}B = CSD^{-1}(z) = N(z)D^{-1}(z) = T_{sp}(z)$$

since

$$N(z) = CS = \begin{bmatrix} c_{11}^0 & \dots & c_{11}^{d_1-1} & \dots & c_{1m}^0 & \dots & c_{1m}^{d_m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ c_{p1}^0 & \dots & c_{p1}^{d_1-1} & \dots & c_{pm}^0 & \dots & c_{pm}^{d_m-1} \end{bmatrix}$$
$$\times \begin{bmatrix} 1 & 0 \dots & 0 \\ z-c_\alpha & 0 \dots & 0 \\ \vdots & \vdots \ddots & \vdots \\ (z-c_\alpha)^{d_1-1} & 0 \dots & 0 \\ \vdots & \vdots \ddots & \vdots \\ 0 & 0 \dots & 1 \\ 0 & 0 \dots & z-c_\alpha \\ \vdots & \vdots \ddots & \vdots \\ 0 & 0 \dots & (z-c_\alpha)^{d_m-1} \end{bmatrix}. \tag{6.25}$$

If the condition $a)$ is met then from (6.5) we have $D \in \mathbb{R}_+^{p\times m}$. If the conditions $b)$ are satisfied then $A_\alpha \in \mathbb{R}_+^{n\times n}$. The matrix C has nonnegative entries if the conditions $c)$ are satisfied. Therefore, the matrices (6.23) are a positive fractional realization of $T(z)$. □

If the conditions of Theorem 6.2 are satisfied then the positive fractional realization (6.23) of the transfer matrix $T(z)$ can be found by use of the following procedure:

Procedure 6.3

Step 1. Knowing the proper transfer matrix $T(z)$ and using (6.5), find the matrix D and the strictly proper matrix $T_{sp}(z)$.

Step 2. Find the minimal degrees $d_1,\dots,d_m$ of the denominators $D_1(z),\dots,D_m(z)$ and write the matrix $T_{sp}(z)$ in the form (6.21).

Step 3. Using the equality

$$D(z) = \text{diag}\left[(z-c_\alpha)^{d_1} \;\dots\; (z-c_\alpha)^{d_m}\right] + \text{diag}\left[a_1 \;\dots\; a_m\right]S, \tag{6.26}$$

find

$$a_j = \left[a_{j0}\; a_{j1}\; \dots\; a_{jd_j-1}\right] \quad \text{for} \quad j=1,\dots,m; \tag{6.27}$$

and the matrix A.

Step 4. Knowing the matrix $N(z)$ and using (6.25), find the matrix C.

Example 6.2. Find the positive fractional realization (6.23) of the transfer matrix

$$T(z) = \begin{bmatrix} \frac{2(z-c_\alpha)^2+3(z-c_\alpha)+1}{(z-c_\alpha)^2+(z-c_\alpha)} & \frac{(z-c_\alpha)^2+3(z-c_\alpha)+2}{(z-c_\alpha)^2-2(z-c_\alpha)-3} \\ \frac{z-c_\alpha+3}{z-c_\alpha-1} & \frac{2(z-c_\alpha)+1}{(z-c_\alpha)^2-2(z-c_\alpha)-3} \end{bmatrix}. \tag{6.28}$$

Using Procedure 6.3 we obtain the following:

Step 1. From (6.5) and (6.28) we have

$$D = \lim_{z\to\infty} T(z) = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \tag{6.29}$$

and

$$T_{sp}(z) = T(z) - D = \begin{bmatrix} \frac{z-c_\alpha+1}{(z-c_\alpha)^2+(z-c_\alpha)} & \frac{5(z-c_\alpha)+5}{(z-c_\alpha)^2-2(z-c_\alpha)-3} \\ \frac{4}{z-c_\alpha-1} & \frac{2(z-c_\alpha)+1}{(z-c_\alpha)^2-2(z-c_\alpha)-3} \end{bmatrix}. \tag{6.30}$$

Step 2. In this case $d_1 = d_2 = 2$ and $D_1(z) = (z-c_\alpha)^2 - (z-c_\alpha)$, $D_2(z) = (z-c_\alpha)^2 - 2(z-c_\alpha) - 3$. The matrix (6.30) takes the form

$$T_{sp}(z) = \begin{bmatrix} \frac{z-c_\alpha+1}{(z-c_\alpha)^2-(z-c_\alpha)} & \frac{5(z-c_\alpha)+5}{(z-c_\alpha)^2-2(z-c_\alpha)-3} \\ \frac{4(z-c_\alpha)}{(z-c_\alpha)^2-(z-c_\alpha)} & \frac{2(z-c_\alpha)+1}{(z-c_\alpha)^2-2(z-c_\alpha)-3} \end{bmatrix}. \tag{6.31}$$

Step 3. Using (6.26) we obtain

$$\begin{bmatrix} (z-c_\alpha)^2-(z-c_\alpha) & 0 \\ 0 & (z-c_\alpha)^2-2(z-c_\alpha)-3 \end{bmatrix} =$$

$$\begin{bmatrix} (z-c_\alpha)^2 & 0 \\ 0 & (z-c_\alpha)^2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ z-c_\alpha & 0 \\ 0 & 1 \\ 0 & z-c_\alpha \end{bmatrix},$$

and

$$a_1 = \begin{bmatrix} a_{10} & a_{11} \end{bmatrix} = \begin{bmatrix} 0 & -1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} a_{20} & a_{21} \end{bmatrix} = \begin{bmatrix} -3 & -2 \end{bmatrix}.$$

Therefore, the matrix A has the form

$$A = \text{block diag}\begin{bmatrix} A_1 & A_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & 2 \end{bmatrix}. \tag{6.32}$$

Step 4. Using (6.25) and (6.31), we obtain

$$\begin{bmatrix} z-c_\alpha+1 & 5(z-c_\alpha)+5 \\ 4(z-c_\alpha) & 2(z-c_\alpha)+1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 5 & 5 \\ 0 & 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ z-c_\alpha & 0 \\ 0 & 1 \\ 0 & z-c_\alpha \end{bmatrix},$$

and

$$C = \begin{bmatrix} 1 & 1 & 5 & 5 \\ 0 & 4 & 1 & 2 \end{bmatrix}. \tag{6.33}$$

The matrix B has the form

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{6.34}$$

The desired positive fractional realization (6.23) of (6.28) is given by (6.29), (6.32), (6.33) and (6.34).

The strictly proper transfer matrix $T_{sp}(z)$ can be also written in the form

$$T_{sp}(z) = \begin{bmatrix} \frac{\overline{N}_{11}(z)}{\overline{D}_1(z)} & \cdots & \frac{\overline{N}_{1m}(z)}{\overline{D}_1(z)} \\ \vdots & \ddots & \vdots \\ \frac{\overline{N}_{p1}(z)}{\overline{D}_p(z)} & \cdots & \frac{\overline{N}_{pm}(z)}{\overline{D}_p(z)} \end{bmatrix} = \overline{D}^{-1}(z)\overline{N}(z), \tag{6.35}$$

where

$$\overline{N}(z) = \begin{bmatrix} \overline{N}_{11}(z) & \dots & \overline{N}_{1m}(z) \\ \vdots & \ddots & \vdots \\ \overline{N}_{p1}(z) & \dots & \overline{N}_{pm}(z) \end{bmatrix}, \quad \overline{D}(z) = \text{diag}\left[\overline{D}_1(z) \ \dots \ \overline{D}_p(z)\right],$$

$$\overline{N}_{ij}(z) = \overline{b}_{ij}^{\overline{d}_i-1}(z-c_\alpha)^{\overline{d}_i-1} + \dots + \overline{b}_{ij}^1(z-c_\alpha) + \overline{b}_{ij}^0, \tag{6.36a}$$

$$\overline{D}_i(z) = (z-c_\alpha)^{\overline{d}_i} + \overline{a}_{i\overline{d}_i-1}(z-c_\alpha)^{\overline{d}_i-1} + \dots + \overline{a}_{i1}(z-c_\alpha) + \overline{a}_{i0}, \tag{6.36b}$$

$$i = 1,\dots,p; \quad j = 1,\dots,m;$$

Theorem 6.3. *There exists the positive fractional realization*

$$\overline{A} = \text{block diag}\left[\overline{A}_1 \ \dots \ \overline{A}_p\right] \in \mathbb{R}_+^{\overline{n}\times\overline{n}}, \tag{6.37a}$$

$$j = 1,\dots,p; \ \overline{n} = \overline{d}_1 + \dots + \overline{d}_p,$$

$$\overline{A}_j = \begin{bmatrix} 0 & 0 & \dots & 0 & -\overline{a}_{j0} \\ 1 & 0 & \dots & 0 & -\overline{a}_{j1} \\ 0 & 1 & \dots & 0 & -\overline{a}_{j2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -\overline{a}_{j\overline{d}_j-1} \end{bmatrix} \in \mathbb{R}_+^{\overline{d}_j\times\overline{d}_j}, \tag{6.37b}$$

$$\overline{B} = \begin{bmatrix} \overline{b}_{11}^{0} & \overline{b}_{12}^{0} & \dots & \overline{b}_{1m}^{0} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{b}_{11}^{\overline{d}_1-1} & \overline{b}_{12}^{\overline{d}_1-1} & \dots & \overline{b}_{1m}^{\overline{d}_1-1} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{b}_{p1}^{0} & \overline{b}_{p2}^{0} & \dots & \overline{b}_{pm}^{0} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{b}_{p1}^{\overline{d}_p-1} & \overline{b}_{p2}^{\overline{d}_p-1} & \dots & \overline{b}_{pm}^{\overline{d}_p-1} \end{bmatrix} \in \mathbb{R}_+^{\overline{n}\times m}, \tag{6.37c}$$

$$\overline{C} = \text{block diag}\left[\overline{C}_1 \dots \overline{C}_p\right] \in \mathbb{R}_+^{p\times\overline{n}},\ \overline{C}_j = \left[0\ 0\ \dots\ 1\right] \in \mathbb{R}_+^{1\times\overline{d}_j}, \tag{6.37d}$$

$$\overline{D} = T(\infty) \in \mathbb{R}_+^{p\times m}, \tag{6.37e}$$

of the transfer matrix $T(z)$, if the following conditions are satisfied:

a) $T(\infty) \in \mathbb{R}_+^{p\times m}$
b) $\overline{a}_{ij} \leq 0$ *for* $i = 1,\dots,p;\ j = 0,1,\dots,\overline{d}_i - 2$ *and* $a_{i\overline{d}_{i-1}} \leq \alpha$ *for* $i = 1,\dots,p;$
c) $\overline{b}_{jk}^{i} \geq 0$ *for* $j = 1,\dots,p;\ k = 1,\dots,m;\ i = 0,1,\dots,\overline{d}_j - 1;$

Proof. First we shall show that the matrices (6.37) are a realization of the strictly proper transfer matrix (6.35). Using (6.36) and (6.37) it is easy to verify that

$$\overline{D}_j(z)\overline{C}_j = \left[1\ z-c_\alpha\ \dots\ (z-c_\alpha)^{\overline{d}_j-1}\right]\left[I_{\overline{d}_j}(z-c_\alpha) - \overline{A}_j\right],$$

for $j = 1,\dots,p$; and

$$\overline{D}(z)\overline{C} = \overline{S}\left[I_{\overline{n}}(z-c_\alpha) - \overline{A}\right], \tag{6.38}$$

where

$$\overline{S} = \text{block diag}\left[\overline{S}_1\ \dots\ \overline{S}_p\right],\ \overline{S}_j = \left[1\ z-c_\alpha\ \dots\ (z-c_\alpha)^{\overline{d}_j-1}\right],\ j = 1,\dots,p;$$

Premultiplying (6.38) by $\overline{D}^{-1}(z)$, and postmultiplying by $[I_{\overline{n}}(z - c_\alpha) - \overline{A}]^{-1}\overline{B}$ we obtain

$$\overline{C}[I_{\overline{n}}(z-c_\alpha) - \overline{A}]^{-1}\overline{B} = \overline{D}^{-1}(z)\overline{SB} = \overline{D}^{-1}(z)\overline{N}(z) = T_{sp}(z),$$

since $\overline{SB} = \overline{N}(z)$.

If the condition $a)$ is met then from (6.5) we have $D \in \mathbb{R}_+^{p\times m}$. If the conditions $b)$ are satisfied then the matrix $A_\alpha \in \mathbb{R}_+^{\overline{n}\times\overline{n}}$. The matrix $\overline{B}$ has nonnegative entries if the conditions $c)$ are satisfied. Therefore, the matrices (6.37) are a positive fractional realization of $T(z)$. □

If the conditions of Theorem 6.3 are satisfied then the positive fractional realization (6.37) of the transfer matrix $T(z)$ can be computed by the use of the following procedure:

Procedure 6.4

Step 1. Knowing the proper transfer matrix $T(z)$ and using (6.5) find the matrix D and the strictly proper matrix $T_{sp}(z)$.
Step 2. Find the minimal degrees $\overline{d}_1,\dots,\overline{d}_p$ of the denominators $\overline{D}_1(z),\dots,\overline{D}_p(z)$ and write the matrix $T_{sp}(z)$ in the form (6.35).
Step 3. Using the equality

$$\overline{D}(z) = \operatorname{diag}\left[(z-c_\alpha)^{\overline{d}_1} \dots (z-c_\alpha)^{\overline{d}_p}\right] + \overline{S}\operatorname{diag}\left[\overline{a}_1 \dots \overline{a}_p\right], \tag{6.39}$$

find

$$\overline{a}_j = \left[\overline{a}_{j0}\ \overline{a}_{j1} \dots \overline{a}_{j\overline{d}_j-1}\right]^T \quad \text{for} \quad j = 1,\dots,p; \tag{6.40}$$

and the matrix $\overline{A}$.
Step 4. Knowing the matrix $\overline{N}(z)$ and using the equality $\overline{S}\overline{B} = \overline{N}(z)$, find the matrix $\overline{B}$.

Example 6.3. Find the positive fractional realization (6.37) of the strictly proper transfer matrix

$$T_{sp}(z) = \begin{bmatrix} \frac{(z-c_\alpha)^2+2(z-c_\alpha)+3}{(z-c_\alpha)^3-2(z-c_\alpha)^2-3(z-c_\alpha)-1} & \frac{4(z-c_\alpha)+2}{(z-c_\alpha)^3-2(z-c_\alpha)^2-3(z-c_\alpha)-1} \\ \frac{z-c_\alpha+2}{(z-c_\alpha)^2-2(z-c_\alpha)-3} & \frac{2(z-c_\alpha)+3}{(z-c_\alpha)^2-2(z-c_\alpha)-3} \end{bmatrix}. \tag{6.41}$$

Using Procedure 6.4, we obtain the following:

Step 1. From (6.5) and (6.41) we have $D = 0$.
Step 2. In this case $\overline{D}_1(z) = (z-c_\alpha)^3 - 2(z-c_\alpha)^2 - 3(z-c_\alpha) - 1$, $\overline{D}_2(z) = (z-c_\alpha)^2 - 2(z-c_\alpha) - 3$ and $\overline{d}_1 = 3$, $\overline{d}_2 = 2$. The matrix (6.41) has already the desired form (6.35).
Step 3. Using (6.39) and (6.41) we obtain

$$\begin{bmatrix} (z-c_\alpha)^3 - 2(z-c_\alpha)^2 - 3(z-c_\alpha) - 1 & 0 \\ 0 & (z-c_\alpha)^2 - 2(z-c_\alpha) - 3 \end{bmatrix} =$$

$$\begin{bmatrix} (z-c_\alpha)^3 & 0 \\ 0 & (z-c_\alpha)^2 \end{bmatrix} - \begin{bmatrix} 1 & z-c_\alpha & (z-c_\alpha)^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & z-c_\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 0 \\ 2 & 0 \\ 0 & 3 \\ 0 & 2 \end{bmatrix},$$

and

$$\overline{a}_1 = -\left[1\ 3\ 2\right]^T, \quad \overline{a}_2 = -\left[3\ 2\right]^T.$$

Therefore, the matrix $\overline{A}$ has the form

$$\overline{A} = \text{block diag}\left[\overline{A}_1\ \overline{A}_2\right] = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}. \tag{6.42}$$

Step 4. Using the equality $\overline{SB} = \overline{N}(z)$ and (6.41), we obtain

$$\begin{bmatrix} (z-c_\alpha)^2 + 2(z-c_\alpha) + 3 & 4(z-c_\alpha) + 2 \\ z - c_\alpha + 2 & 2(z-c_\alpha) + 3 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & z-c_\alpha & (z-c_\alpha)^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & z-c_\alpha \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 4 \\ 1 & 0 \\ 2 & 3 \\ 1 & 2 \end{bmatrix},$$

and

$$\overline{B} = \begin{bmatrix} 3 & 2 \\ 2 & 4 \\ 1 & 0 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}. \tag{6.43}$$

The matrix C in this case has the form

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{6.44}$$

The desired positive fractional realization (6.37) of (6.41) is given by (6.42), (6.43), (6.44) and $D = 0$.

6.2 Fractional Continuous-Time Linear Systems

Consider the fractional continuous-time linear system (2.14). Using the Laplace transform to (2.14) with zero initial conditions, it is easy to show that the transfer matrix of the system is given by the formula [108]

$$T(s) = C\left[I_n s^\alpha - A\right]^{-1} B + D. \tag{6.45}$$

The transfer matrix is called proper if

$$\lim_{s\to\infty} T(s) = K \in \mathbb{R}^{p\times m},$$

and it is called strictly proper if and only if $K = 0$.

From (6.45) we have

$$\lim_{s\to\infty} T(s) = D, \tag{6.46}$$

since

$$\lim_{s\to\infty} [I_n s^\alpha - A]^{-1} = 0.$$

Definition 6.3. Matrices

$$A \in M_n, \quad B \in \mathbb{R}_+^{n\times m}, \quad C \in \mathbb{R}_+^{p\times n}, \quad D \in \mathbb{R}_+^{p\times m}, \tag{6.47}$$

are called a positive fractional realization of a given transfer matrix $T(s)$, if they satisfy the equality (6.45). A realization is called minimal if the dimension of A is minimal among all realizations of $T(s)$.

The positive fractional realization problem can be stated as follows. Given a proper transfer matrix $T(s)$, find its realizations (6.47).

6.2.1 SISO Systems

First the realization problem will be solved for a single-input single-output (SISO) linear fractional systems with the proper transfer function

$$T(s) = \frac{b_n(s^\alpha)^n + b_{n-1}(s^\alpha)^{n-1} + \cdots + b_1 s^\alpha + b_0}{(s^\alpha)^n - a_{n-1}(s^\alpha)^{n-1} - \cdots - a_1 s^\alpha - a_0}. \tag{6.48}$$

Using (6.46), we obtain

$$D = \lim_{s\to\infty} T(s) = b_n, \tag{6.49}$$

and the strictly proper transfer function has the form

$$T_{sp}(s) = T(s) - D = \frac{\overline{b}_{n-1}(s^\alpha)^{n-1} + \cdots + \overline{b}_1 s^\alpha + \overline{b}_0}{(s^\alpha)^n - a_{n-1}(s^\alpha)^{n-1} - \cdots - a_1 s^\alpha - a_0}, \tag{6.50}$$

where

$$\overline{b}_k = b_k + a_k b_n, \quad k = 0, 1, \ldots, n-1. \tag{6.51}$$

From (6.51) it follows that if $a_k \geq 0$ and $b_k \geq 0$ for $k = 0, 1, \ldots, n$ then also $\overline{b}_k \geq 0$ for $k = 0, 1, \ldots, n-1$.

Theorem 6.4. *There exist positive fractional realizations of the forms:*

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_0 & a_1 & a_2 & \dots & a_{n-1} \end{bmatrix},\ B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},\ C^T = \begin{bmatrix} \overline{b}_0 \\ \overline{b}_1 \\ \vdots \\ \overline{b}_{n-1} \end{bmatrix},\ D = b_n, \tag{6.52a}$$

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & a_0 \\ 1 & 0 & \dots & 0 & a_1 \\ 0 & 1 & \dots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_{n-1} \end{bmatrix},\ B = \begin{bmatrix} \overline{b}_0 \\ \overline{b}_1 \\ \vdots \\ \overline{b}_{n-1} \end{bmatrix},\ C^T = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},\ D = b_n, \tag{6.52b}$$

$$A = \begin{bmatrix} a_{n-1} & a_{n-2} & \dots & a_1 & a_0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ C^T = \begin{bmatrix} \overline{b}_{n-1} \\ \overline{b}_{n-2} \\ \vdots \\ \overline{b}_0 \end{bmatrix},\ D = b_n, \tag{6.52c}$$

$$A = \begin{bmatrix} a_{n-1} & 1 & 0 & \dots & 0 \\ a_{n-2} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & 0 & 0 & \dots & 1 \\ a_0 & 0 & 0 & \dots & 0 \end{bmatrix},\ B = \begin{bmatrix} \overline{b}_{n-1} \\ \overline{b}_{n-2} \\ \vdots \\ \overline{b}_0 \end{bmatrix},\ C^T = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},\ D = b_n, \tag{6.52d}$$

of the transfer function(6.48) if:

a) $b_k \geq 0$ *for* $k = 0, 1, \dots, n$;
b) $a_k \geq 0$ *for* $k = 0, 1, \dots, n-2$; *and* $b_{n-1} + a_{n-1}b_n \geq 0$;

Proof. Taking into account that for (6.52a)

$$\det[I_n s^\alpha - A] = (s^\alpha)^n - a_{n-1}(s^\alpha)^{n-1} - \dots - a_1 s^\alpha - a_0,$$

and

$$\operatorname{Adj}[I_n s^\alpha - A]B = \begin{bmatrix} 1 & s^\alpha & \dots & (s^\alpha)^{n-1} \end{bmatrix}^T,$$

it is easy to verify that

$$\begin{aligned} C[I_n s^\alpha - A]^{-1}B &= \frac{C \operatorname{Adj}[I_n s^\alpha - A]B}{\det[I_n s^\alpha - A]} \\ &= \frac{\overline{b}_{n-1}(s^\alpha)^{n-1} + \dots + \overline{b}_1 s^\alpha + \overline{b}_0}{(s^\alpha)^n - a_{n-1}(s^\alpha)^{n-1} - \dots - a_1 s^\alpha - a_0}. \end{aligned}$$

The matrix A is Metzler matrix if and only if $a_k \geq 0$ for $k = 0, 1, \ldots, n-2$ and arbitrary a_{n-1}. Note that the coefficients of matrices C and D are nonnegative if the condition $a)$ is met and $\overline{b}_{n-1} = b_{n-1} + a_{n-1}b_n \geq 0$. The proofs for (6.52b-6.52d) are similar (dual). □

The matrices (6.52) are minimal realizations if and only if the transfer function (6.48) is irreducible. If the conditions of Theorem 6.4, are satisfied then the positive minimal realizations (6.52) of the transfer function (6.48) can be computed by use of the following procedure:

Procedure 6.5

Step 1. Knowing $T(s)$ and using (6.49) find D and the strictly proper transfer function (6.50).

Step 2. Using (6.52) find the desired realizations.

Example 6.4. Find the positive minimal fractional realizations (6.52) of the irreducible transfer function

$$T(s) = \frac{2(s^\alpha)^2 + 5s^\alpha + 1}{(s^\alpha)^2 + 2s^\alpha - 3}. \tag{6.53}$$

Using Procedure 6.5 and (6.53) we obtain the following:

Step 1. From (6.49) and (6.53) we have $D = 2$ and

$$T_{sp} = T(z) - D = \frac{s^\alpha + 7}{(s^\alpha)^2 + 2s^\alpha - 3}.$$

Step 2. Taking into account that in this case $\overline{b}_0 = 7$, $\overline{b}_1 = 1$ and using (6.52) we obtain the desired positive minimal fractional realizations:

$$A = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 7 & 1 \end{bmatrix}, \quad D = 2,$$
$$A = \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 7 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = 2,$$

$$A = \begin{bmatrix} -2 & 3 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 7 \end{bmatrix}, \quad D = 2,$$
$$A = \begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 7 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = 2.$$

6.2.2 MIMO Systems

Consider a multi-input multi-output (MIMO) positive fractional system (2.14) with a proper transfer matrix $T(s)$. Using the formula (6.46) we can find the matrix D and the strictly proper transfer matrix which can be written in the form

$$T_{sp}(s) = T(s) - D = \begin{bmatrix} \frac{N_{11}(z)}{D_1(z)} & \cdots & \frac{N_{1m}(z)}{D_m(z)} \\ \vdots & \ddots & \vdots \\ \frac{N_{p1}(z)}{D_1(z)} & \cdots & \frac{N_{pm}(z)}{D_m(z)} \end{bmatrix} = N(s)D^{-1}(s), \tag{6.54}$$

where

$$N(s) = \begin{bmatrix} N_{11}(s) & \dots & N_{1m}(s) \\ \vdots & \ddots & \vdots \\ N_{p1}(s) & \dots & N_{pm}(s) \end{bmatrix}, \quad D(s) = \text{diag}\left[D_1(s) \ \dots \ D_m(s) \right],$$

$$N_{ik}(s) = c_{ik}^{d_k-1}(s^\alpha)^{d_k-1} + \cdots + c_{ik}^1 s^\alpha + c_{ik}^0, \tag{6.55a}$$

$$D_k(s) = (s^\alpha)^{d_k} - a_{kd_k-1}(s^\alpha)^{d_k-1} - \cdots - a_{k1}s^\alpha - a_{k0}, \tag{6.55b}$$

$$i = 1, \dots, p; \quad k = 1, \dots, m;$$

Theorem 6.5. *There exists the positive fractional realization*

$$A = \text{block diag}\left[A_1 \ \dots \ A_m \right] \in \mathbb{R}^{n \times n},$$

$$A_k = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_{k0} & a_{k1} & a_{k2} & \dots & a_{kd_k-1} \end{bmatrix} \in \mathbb{R}_+^{d_k \times d_k}, \quad \begin{matrix} k = 1, \dots, m; \\ n = d_1 + \cdots + d_m; \end{matrix} \tag{6.56a}$$

$$B = \text{block diag}\left[B_1 \ \dots \ B_m \right] \in \mathbb{R}_+^{n \times m}, \quad B_k = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}_+^{d_k}, \tag{6.56b}$$

$$C = \begin{bmatrix} c_{11}^0 & \dots & c_{11}^{d_1-1} & \dots & c_{1m}^0 & \dots & c_{1m}^{d_m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ c_{p1}^0 & \dots & c_{p1}^{d_1-1} & \dots & c_{pm}^0 & \dots & c_{pm}^{d_m-1} \end{bmatrix} \in \mathbb{R}_+^{p \times n}, \tag{6.56c}$$

$$D = T(\infty) \in \mathbb{R}_+^{p \times m}, \tag{6.56d}$$

of the transfer matrix $T(s)$, if the following conditions are satisfied:

a) $T(\infty) \in \mathbb{R}_+^{p \times m}$
b) $a_{kl} \geq 0$ *for* $k = 1, \dots, m$; $l = 0, 1, \dots, d_k - 2$; *and* a_{kd_k-1} *can be arbitrary*
c) $c_{ik}^j \geq 0$ *for* $i = 1, \dots, p$; $l = 1, \dots, m$; $k = 0, 1, \dots, d_k - 1$

Proof. First we shall show that the matrices (6.56) are a realization of strictly proper matrix (6.54). Using (6.55) and (6.56) it is easy to verify that

$$B_k D_k(s) = \left[I_{d_k} s^\alpha - A_k\right] \begin{bmatrix} 1 \\ s^\alpha \\ \vdots \\ (s^\alpha)^{d_k-1} \end{bmatrix}, \quad \text{for} \quad k = 1, \dots, m;$$

and

$$BD(s) = \left[I_{d_k} s^\alpha - A\right] S, \tag{6.57}$$

where

$$S = \text{block diag}\left[S_1 \ \dots \ S_m\right], \quad S_k = \begin{bmatrix} 1 \\ s^\alpha \\ \vdots \\ (s^\alpha)^{d_k-1} \end{bmatrix}, \quad k = 1, \dots, m;$$

Premultiplying (6.57) by $C[I_n s^\alpha - A]^{-1}$ and postmultiplying by $D(s)^{-1}$ we obtain

$$C\left[I_n s^\alpha - A\right]^{-1} B = CSD^{-1}(s) = N(s)D^{-1}(s) = T_{sp}(s)$$

since

$$N(s) = \begin{bmatrix} c_{11}^0 & \dots & c_{11}^{d_1-1} & \dots & c_{1m}^0 & \dots & c_{1m}^{d_m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ c_{p1}^0 & \dots & c_{p1}^{d_1-1} & \dots & c_{pm}^0 & \dots & c_{pm}^{d_m-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \dots & 0 \\ s^\alpha & 0 \dots & 0 \\ \vdots & \vdots \ddots & \vdots \\ (s^\alpha)^{d_1-1} & 0 \dots & 0 \\ \vdots & \vdots \ddots & \vdots \\ 0 & 0 \dots & 1 \\ 0 & 0 \dots & s^\alpha \\ \vdots & \vdots \ddots & \vdots \\ 0 & 0 \dots & (s^\alpha)^{d_m-1} \end{bmatrix}. \tag{6.58}$$

If the condition a), is met then from (6.46) we have $D \in \mathbb{R}_+^{p \times m}$. If the conditions b) are satisfied then the matrix A is a Metzler matrix and the matrix C has nonnegative entries if the conditions c) are met. □

If the conditions of Theorem 6.5 are satisfied then the positive fractional realization (6.56) of the transfer matrix $T(s)$ can be computed by use of the following procedure:

Procedure 6.6

Step 1. Knowing the proper transfer matrix $T(s)$ and using (6.46) compute matrix D and the strictly proper matrix $T_{sp}(s)$.

Step 2. Find the minimal degrees $d_1, \dots, d_m$ of the denominators $D_1(s), \dots, D_m(s)$ and write the matrix $T_{sp}(s)$ in the form (6.54).

Step 3. Using the equality

$$D(s) = \operatorname{diag}\left[\, (s^\alpha)^{d_1} \;\dots\; (s^\alpha)^{d_m} \,\right] - \operatorname{diag}\left[\, a_1 \;\dots\; a_m \,\right] S, \tag{6.59}$$

find

$$a_k = \left[\, a_{k0} \; a_{k1} \;\dots\; a_{kd_k-1} \,\right] \quad \text{for} \quad k = 1, \dots, m; \tag{6.60}$$

and the matrix A.

Step 4. Knowing the matrix $N(s)$ and using (6.58) find the matrix C.

Example 6.5. Find the positive fractional realization (6.56) of the transfer matrix

$$T(s) = \begin{bmatrix} \frac{2s^\alpha+1}{s^\alpha} & \frac{(s^\alpha)^2+3s^\alpha+2}{(s^\alpha)^2+2s^\alpha-3} \\ \frac{s^\alpha+3}{s^\alpha+1} & \frac{2s^\alpha+1}{(s^\alpha)^2+2s^\alpha-3} \end{bmatrix}. \tag{6.61}$$

Using the Procedure 6.6 we obtain the following:

Step 1. From (6.46), (6.54) and (6.61) we have

$$D = \lim_{s \to \infty} T(s) = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \tag{6.62}$$

and

$$T_{sp}(s) = T(s) - D = \begin{bmatrix} \frac{1}{s^\alpha} & \frac{s^\alpha+5}{(s^\alpha)^2+2s^\alpha-3} \\ \frac{2}{s^\alpha+1} & \frac{2s^\alpha+1}{(s^\alpha)^2+2s^\alpha-3} \end{bmatrix}. \tag{6.63}$$

Step 2. In this case $D_1(s) = (s^\alpha)^2 + s^\alpha$, $D_2(s) = (s^\alpha)^2 + 2s^\alpha - 3$, $d_1 = d_2 = 2$ and the matrix (6.63) takes the form

$$T_{sp}(s) = \begin{bmatrix} \frac{s^\alpha+1}{(s^\alpha)^2+s^\alpha} & \frac{s^\alpha+5}{(s^\alpha)^2+2s^\alpha-3} \\ \frac{2s^\alpha}{(s^\alpha)^2+s^\alpha} & \frac{2s^\alpha+1}{(s^\alpha)^2+2s^\alpha-3} \end{bmatrix}. \tag{6.64}$$

Step 3. Using (6.59) we obtain

$$\begin{bmatrix} (s^\alpha)^2 + s^\alpha & 0 \\ 0 & (s^\alpha)^2 + 2(s^\alpha) - 3 \end{bmatrix} =$$

$$\begin{bmatrix} (s^\alpha)^2 & 0 \\ 0 & (s^\alpha)^2 \end{bmatrix} - \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s^\alpha & 0 \\ 0 & 1 \\ 0 & s^\alpha \end{bmatrix},$$

and

$$a_1 = \left[\, a_{10} \; a_{11} \,\right] = \left[\, 0 \; -1 \,\right], \quad a_2 = \left[\, a_{20} \; a_{21} \,\right] = \left[\, 3 \; -2 \,\right].$$

Therefore, the matrix A has the form

$$A = \text{block diag}\left[A_1\ A_2\right] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & -2 \end{bmatrix}. \tag{6.65}$$

Step 4. Using (6.58) and (6.64) we obtain

$$\begin{bmatrix} s^\alpha + 1 & s^\alpha + 5 \\ 2s^\alpha & 2s^\alpha + 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 5 & 1 \\ 0 & 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ s^\alpha & 0 \\ 0 & 1 \\ 0 & s^\alpha \end{bmatrix},$$

and

$$C = \begin{bmatrix} 1 & 1 & 5 & 1 \\ 0 & 2 & 1 & 2 \end{bmatrix}. \tag{6.66}$$

In this case the matrix B has the form

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{6.67}$$

The desired positive fractional realization (6.56) of (6.61) is given by (6.62), (6.65), (6.66) and (6.67).

The strictly proper transfer matrix $T_{sp}(s)$ can be also written in the form

$$T_{sp}(s) = \begin{bmatrix} \frac{\overline{N}_{11}(s)}{\overline{D}_1(s)} & \cdots & \frac{\overline{N}_{1m}(s)}{\overline{D}_1(s)} \\ \vdots & \ddots & \vdots \\ \frac{\overline{N}_{p1}(s)}{\overline{D}_p(s)} & \cdots & \frac{\overline{N}_{pm}(s)}{\overline{D}_p(s)} \end{bmatrix} = \overline{D}^{-1}(s)\overline{N}(s), \tag{6.68}$$

where

$$\overline{N}(s) = \begin{bmatrix} \overline{N}_{11}(s) & \dots & \overline{N}_{1m}(s) \\ \vdots & \ddots & \vdots \\ \overline{N}_{p1}(s) & \dots & \overline{N}_{pm}(s) \end{bmatrix}, \quad \overline{D}(s) = \text{diag}\left[\overline{D}_1(s) \dots \overline{D}_p(s)\right],$$

$$\overline{N}_{jk}(s) = \overline{b}_{jk}^{\overline{d}_j-1}(s^\alpha)^{\overline{d}_j-1} + \cdots + \overline{b}_{jk}^{1} s^\alpha + \overline{b}_{jk}^{0}, \tag{6.69a}$$

$$\overline{D}_k(s) = (s^\alpha)^{\overline{d}_j} - \overline{a}_{k\overline{d}_j-1}(s^\alpha)^{\overline{d}_j-1} - \cdots - \overline{a}_{k1}s^\alpha + \overline{a}_{k0}, \tag{6.69b}$$

$$k = 1, \dots, m; \quad j = 1, \dots, p;$$

Theorem 6.6. *There exists the positive fractional realization*

$$\overline{A} = \text{block diag}\left[\overline{A}_1 \ \dots \ \overline{A}_p\right] \in \mathbb{R}^{\overline{n}\times\overline{n}}, \quad \begin{matrix} k = 1,\dots,p; \\ \overline{n} = \overline{d}_1 + \dots + \overline{d}_p; \end{matrix} \tag{6.70}$$

$$\overline{A}_k = \begin{bmatrix} 0 & 0 & \dots & 0 & \overline{a}_{k0} \\ 1 & 0 & \dots & 0 & \overline{a}_{k1} \\ 0 & 1 & \dots & 0 & \overline{a}_{k2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \overline{a}_{k\overline{d}_k-1} \end{bmatrix} \in \mathbb{R}^{\overline{d}_k\times\overline{d}_k},$$

$$\overline{B} = \begin{bmatrix} \overline{b}_{11}^0 & \overline{b}_{12}^0 & \dots & \overline{b}_{1m}^0 \\ \vdots & \vdots & \ddots & \vdots \\ \overline{b}_{11}^{\overline{d}_1-1} & \overline{b}_{12}^{\overline{d}_1-1} & \dots & \overline{b}_{1m}^{\overline{d}_1-1} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{b}_{p1}^0 & \overline{b}_{p2}^0 & \dots & \overline{b}_{pm}^0 \\ \vdots & \vdots & \ddots & \vdots \\ \overline{b}_{p1}^{\overline{d}_p-1} & \overline{b}_{p2}^{\overline{d}_p-1} & \dots & \overline{b}_{pm}^{\overline{d}_p-1} \end{bmatrix} \in \mathbb{R}_+^{\overline{n}\times m},$$

$$\overline{C} = \text{block diag}\left[\overline{C}_1 \ \dots \ \overline{C}_p\right] \in \mathbb{R}_+^{p\times\overline{n}}, \quad \overline{C}_k = \left[0 \ 0 \ \dots \ 1\right] \in \mathbb{R}_+^{1\times\overline{d}_k},$$
$$\overline{D} = T(\infty) \in \mathbb{R}_+^{p\times m},$$

of the transfer matrix $T(s)$ if the following conditions are satisfied:

a) $T(\infty) \in \mathbb{R}_+^{p\times m}$
b) $a_{kl} \geq 0$ *for* $k = 1,\dots,p;\ l = 0,1,\dots,\overline{d}_k - 2;$ *and* $\overline{a}_{k\overline{d}_k-1}$ *can be arbitrary*
c) $\overline{b}_{jk}^i \geq 0$ *for* $j = 1,\dots,p;\ k = 1,\dots,m;\ i = 0,1,\dots,\overline{d}_j - 1;$

Proof. First we shall show that the matrices (6.70) are a realization of the strictly proper matrix (6.68). Using (6.69) and (6.70) it is easy to verify that

$$\overline{D}_k(s)\overline{C}_k = \left[1 \ s^\alpha \ \dots \ (s^\alpha)^{\overline{d}_k-1}\right]\left[I_{\overline{d}_k}s^\alpha - \overline{A}_k\right], \quad k = 1,\dots,p;$$

and

$$\overline{D}(s)\overline{C} = \overline{S}\left[I_{\overline{n}}s^\alpha - \overline{A}\right], \tag{6.71}$$

where

$$\overline{S} = \text{block diag}\left[\overline{S}_1 \ \dots \ \overline{S}_p\right], \quad \overline{S}_k = \left[1 \ s^\alpha \ \dots \ (s^\alpha)^{\overline{d}_k-1}\right], \quad k = 1,\dots,p;$$

Premultiplying (6.71) by $\overline{D}^{-1}(s)$ and postmultiplying by $[I_{\overline{n}}s^\alpha - \overline{A}]^{-1}\overline{B}$ we obtain

$$\overline{C}[I_{\overline{n}}s^\alpha - \overline{A}]^{-1}\overline{B} = \overline{D}^{-1}(s)\overline{S}\overline{B} = \overline{D}^{-1}(s)\overline{N}(s) = T_{sp}(s), \quad \text{since} \quad \overline{S}\overline{B} = \overline{N}(s).$$

If the condition $a)$ is met then from (6.46) we have $D \in \mathbb{R}_+^{p\times m}$. If the conditions $b)$ are satisfied then the matrix A is a Metzler matrix and the matrix $\overline{B}$ has nonnegative entries if the conditions $c)$ are met. □

If the conditions of Theorem 6.6 are satisfied then the positive fractional realization (6.70) of the transfer matrix $T(s)$ can be computed by use of the following procedure:

Procedure 6.7

Step 1. Knowing the proper transfer matrix $T(s)$ and using (6.46) compute the matrix D and the strictly proper matrix $T_{sp}(s)$.
Step 2. Find the minimal degrees $\overline{d}_1,\dots,\overline{d}_p$ of the denominators $\overline{D}_1(s),\dots,\overline{D}_p(s)$ and write the matrix $T_{sp}(s)$ in the form (6.68).
Step 3. Using the equality

$$\overline{D}(s) = \mathrm{diag}\left[(s^\alpha)^{\overline{d}_1} \;\dots\; (s^\alpha)^{\overline{d}_p} \right] - \overline{S}\,\mathrm{diag}\left[\overline{a}_1 \;\dots\; \overline{a}_p \right], \tag{6.72}$$

find

$$\overline{a}_k = \left[\overline{a}_{k0} \; \overline{a}_{k1} \;\dots\; \overline{a}_{k\overline{d}_k-1} \right]^T \quad \text{for} \quad k = 1,\dots,p; \tag{6.73}$$

and the matrix $\overline{A}$.
Step 4. Knowing the matrix $\overline{N}(s)$ and using the equality $\overline{SB} = \overline{N}(s)$ find the matrix $\overline{B}$.

Example 6.6. Find the positive fractional realization (6.70) of the strictly proper transfer matrix

$$T_{sp}(z) = \begin{bmatrix} \frac{(s^\alpha)^2+2s^\alpha+3}{(s^\alpha)^3+2(s^\alpha)^2-3(s^\alpha)-1} & \frac{4(s^\alpha)+2}{(s^\alpha)^3+2(s^\alpha)^2-3s^\alpha-1} \\ \frac{s^\alpha+2}{(s^\alpha)^2+2s^\alpha-3} & \frac{2s^\alpha+3}{(s^\alpha)^2+2s^\alpha-3} \end{bmatrix}. \tag{6.74}$$

Using Procedure 6.7 we obtain the following:

Step 1. From (6.46) and (6.74) we have $D = 0$.
Step 2. In this case $\overline{d}_1 = 3$, $\overline{d}_2 = 2$, $\overline{D}_1(s) = (s^\alpha)^3 + 2(s^\alpha)^2 - 3s^\alpha - 1$, $\overline{D}_2(s) = (s^\alpha)^2 + 2s^\alpha - 3$. The matrix (6.74) has already the desired form (6.68).
Step 3. Using (6.72) we obtain

$$\begin{bmatrix} (s^\alpha)^3 + 2(s^\alpha)^2 - 3s^\alpha - 1 & 0 \\ 0 & (s^\alpha)^2 + 2s^\alpha - 3 \end{bmatrix} =$$

$$\begin{bmatrix} (s^\alpha)^3 & 0 \\ 0 & (s^\alpha)^2 \end{bmatrix} - \begin{bmatrix} 1 & s^\alpha & (s^\alpha)^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & s^\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 0 \\ -2 & 0 \\ 0 & 3 \\ 0 & -2 \end{bmatrix},$$

and

$$\overline{a}_1 = \begin{bmatrix} 1 & 3 & -2 \end{bmatrix}^T, \quad \overline{a}_2 = \begin{bmatrix} 3 & -2 \end{bmatrix}^T.$$

Therefore, the matrix $\overline{A}$ has the form

$$\overline{A} = \text{block diag}\begin{bmatrix} \overline{A}_1 & \overline{A}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}. \tag{6.75}$$

Step 4. Using the equality $\overline{S}\overline{B} = \overline{N}(s)$ and (6.74) we obtain

$$\begin{bmatrix} (s^\alpha)^2 + 2s^\alpha + 3 & 4s^\alpha + 2 \\ s^\alpha + 2 & 2s^\alpha + 3 \end{bmatrix} = \begin{bmatrix} 1 & s^\alpha & (s^\alpha)^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & s^\alpha \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 4 \\ 1 & 0 \\ 2 & 3 \\ 1 & 2 \end{bmatrix},$$

and

$$\overline{B} = \begin{bmatrix} 3 & 2 & 1 & 2 & 1 \\ 2 & 4 & 0 & 3 & 2 \end{bmatrix}^T, \quad C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{6.76}$$

The desired positive fractional realization (6.70) of (6.74) is given by (6.75), (6.76), and $D = 0$.

Chapter 7
Cone Discrete-Time and Continuous-Time Linear Systems

7.1 Basic Definitions

Definition 7.1. Let $P = [p_1^T, \ldots, p_n^T]^T \in \mathbb{R}^{n\times n}$ be nonsingular and p_k be the k-th ($k = 1,\ldots,n$) its row. The set

$$\mathscr{P} = \left\{ x_i \in \mathbb{R}^n : \bigcap_{k=1}^{n} p_k x_i \geq 0 \right\} \tag{7.1}$$

is called a linear cone of the state variable x_i generated by the matrix P.

In a similar way we may define for inputs u_i the linear cone of the inputs

$$\mathscr{Q} = \left\{ u_i \in \mathbb{R}^m : \bigcap_{k=1}^{m} q_k u_i \geq 0 \right\}, \tag{7.2}$$

generated by the nonsingular matrix $Q = [q_1^T, \ldots, q_m^T]^T \in \mathbb{R}^{m\times m}$ and for outputs y_i, the linear cone of the outputs

$$\mathscr{V} = \left\{ y_i \in \mathbb{R}^p : \bigcap_{k=1}^{p} v_k y_i \geq 0 \right\}, \tag{7.3}$$

generated by the nonsingular matrix $V = [v_1^T, \ldots, v_p^T]^T \in \mathbb{R}^{p\times p}$.

7.1.1 Cone Discrete-Time Systems

Consider the discrete-time linear systems

$$x_{i+1} = Ax_i + Bu_i, \tag{7.4a}$$
$$y_i = Cx_i + Du_i, \tag{7.4b}$$

T. Kaczorek: Selected Problems of Fractional Systems Theory, LNCIS 411, pp. 141–159.
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where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^p$ are the state, input and output vectors and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Definition 7.2. The linear system (7.4) is called $(\mathscr{P}, \mathscr{Q}, \mathscr{V})$ cone system if $x_i \in \mathscr{P}$ and $y_i \in \mathscr{V}$, $i \in \mathbb{Z}_+$ for every $x_0 \in \mathscr{P}$ and all $u_i \in \mathscr{Q}$, $i \in \mathbb{Z}_+$.

If $\mathscr{P} = \mathbb{R}^n_+$, $\mathscr{Q} = \mathbb{R}^m_+$, $\mathscr{V} = \mathbb{R}^p_+$ then $(\mathbb{R}^n_+, \mathbb{R}^m_+, \mathbb{R}^p_+)$ cone system is equivalent to the positive system.

Theorem 7.1. *The linear system (7.4) is $(\mathscr{P}, \mathscr{Q}, \mathscr{V})$ cone system if and only if*

$$\overline{A} = PAP^{-1} \in \mathbb{R}^{n\times n}_+, \quad \overline{B} = PBQ^{-1} \in \mathbb{R}^{n\times m}_+, \tag{7.5a}$$

$$\overline{C} = VCP^{-1} \in \mathbb{R}^{p\times n}_+, \quad \overline{D} = VDQ^{-1} \in \mathbb{R}^{p\times m}_+. \tag{7.5b}$$

Proof. Let

$$\overline{x}_k = Px_k, \quad \overline{u}_k = Qu_k, \quad \overline{y}_k = Vy_k, \quad k \in \mathbb{Z}_+. \tag{7.6}$$

From Definition 7.2 it follows that if $x_k \in \mathscr{P}$ then $\overline{x}_k \in \mathbb{R}^n_+$, if $u_k \in \mathscr{Q}$ then $\overline{u}_k \in \mathbb{R}^m_+$ and if $y_k \in \mathscr{V}$ then $\overline{y}_k \in \mathbb{R}^p_+$. From (7.4) and (7.6) we have

$$\overline{x}_{k+1} = Px_{k+1} = PAx_k + PBu_k = PAP^{-1}\overline{x}_k + PBQ^{-1}\overline{u}_k = \overline{A}\overline{x}_k + \overline{B}\overline{u}_k, \tag{7.7a}$$

and

$$\overline{y}_k = Vy_k = VCx_k + VDu_k = VCP^{-1}\overline{x}_k + VDQ^{-1}\overline{u}_k = \overline{C}\overline{x}_k + \overline{D}\overline{u}_k, \quad k \in \mathbb{Z}_+. \tag{7.7b}$$

It is well-known that the system (7.7) is positive if and only if the conditions (7.5) are satisfied. □

7.1.2 Cone Continuous-Time Systems with Delays

Consider the continuous-time linear system

$$\dot{x}(t) = \sum_{i=0}^{h} A_i x(t - id) + \sum_{j=0}^{q} B_j u(t - jd), \tag{7.8a}$$

$$y(t) = Cx(t) + Du(t), \tag{7.8b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the state, input and output vectors and $A_i \in \mathbb{R}^{n\times n}$, $i = 0, 1, \ldots, h$; $B_j \in \mathbb{R}^{n\times m}$, $j = 0, 1, \ldots, q$; $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, and $d > 0$ is a delay.

Initial conditions for (7.8) are given by

$$x_0(t) \quad \text{for} \quad t \in [-hd, 0] \quad \text{and} \quad u_0(t) \quad \text{for} \quad t \in [-qd, 0) \tag{7.9}$$

Definition 7.3. The continuous-time system (7.8) is called the $(\mathscr{P},\mathscr{Q},\mathscr{V})$ cone system if $x(t)\in\mathscr{P}$ and $y(t)\in\mathscr{V}$, $t\geq 0$ for every $x_0\in\mathscr{P}$, $t\in[-hd,0]$, $u_0\in\mathscr{Q}$, $t\in[-qd,0)$ and $u(t)\in\mathscr{Q}$, $t\geq 0$.

If $\mathscr{P}=\mathbb{R}_+^n$, $\mathscr{Q}=\mathbb{R}_+^m$, $\mathscr{V}=\mathbb{R}_+^p$, then $(\mathbb{R}_+^n,\mathbb{R}_+^m,\mathbb{R}_+^p)$ cone system is equivalent to the positive system.

Theorem 7.2. *The continuous-time system (7.8) is the $(\mathscr{P},\mathscr{Q},\mathscr{V})$ cone system if and only if*

$$\begin{aligned}
\overline{A}_0 &= PA_0P^{-1}\in M_n, \quad \overline{A}_i=PA_iP^{-1}\in\mathbb{R}_+^{n\times n}, \quad i=1,\dots,h;\\
\overline{B}_j &= PB_jQ^{-1}\in\mathbb{R}_+^{n\times m}, \quad j=0,1,\dots,q;\\
\overline{C} &= VCP^{-1}\in\mathbb{R}_+^{p\times n}, \quad \overline{D}=VDQ^{-1}\in\mathbb{R}_+^{p\times m}.
\end{aligned}\tag{7.10}$$

Proof. Let

$$\overline{x}(t)=Px(t), \quad \overline{u}(t)=Qu(t), \quad \overline{y}(t)=Vy(t). \tag{7.11}$$

From Definition 7.3 it follows that if $x(t)\in\mathscr{P}$ then $\overline{x}(t)\in\mathbb{R}_+^n$, if $u(t)\in\mathscr{Q}$ then $\overline{u}(t)\in\mathbb{R}_+^m$ and if $y(t)\in\mathscr{V}$, then $\overline{y}(t)\in\mathbb{R}_+^p$.

From (7.8) and (7.11) we have

$$\begin{aligned}
\dot{\overline{x}}(t)=P\dot{x}(t) &= \sum_{i=0}^{h}PA_ix(t-id)+\sum_{j=0}^{q}PB_ju(t-jd)\\
&= \sum_{i=0}^{h}PA_iP^{-1}\overline{x}(t-id)+\sum_{j=0}^{q}PB_jQ^{-1}\overline{u}(t-jd)\\
&= \sum_{i=0}^{h}\overline{A}_i\overline{x}(t-id)+\sum_{j=0}^{q}\overline{B}_j\overline{u}(t-jd),
\end{aligned}\tag{7.12a}$$

and

$$\begin{aligned}
\overline{y}(t) &= Vy(t)=VCx(t)+VDu(t)=VCP^{-1}\overline{x}(t)+VDQ^{-1}\overline{u}(t)\\
&= \overline{C}\overline{x}(t)+\overline{D}\overline{u}(t).
\end{aligned}\tag{7.12b}$$

It is well-known that the continuous-time system (7.12) is positive if and only if the conditions (7.10) are satisfied. □

The considerations can be easily extended to linear systems with arbitrary delays by substitution in (7.8a) $id=d_i$, $i=0,1,\dots,h$ and $jd=d_j$, $j=0,1,\dots,q$.

7.1.3 Cone Fractional Discrete-Time Systems

Consider the fractional discrete-time linear systems

$$x_{k+1} = Ax_k + \sum_{j=1}^{k+1}(-1)^{j+1}\binom{\alpha}{j}x_{k-j+1} + Bu_k$$
$$= A_\alpha x_k + \sum_{j=1}^{k}(-1)^{j}\binom{\alpha}{j+1}x_{k-j} + Bu_k, \tag{7.13a}$$
$$y_k = Cx_k + Du_k, \quad k \in \mathbb{Z}_+ \tag{7.13b}$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^p$ are the state, input and output vectors nad $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Definition 7.4. The fractional system (7.13) is called $(\mathscr{P},\mathscr{Q},\mathscr{V})$ cone fractional discrete-time linear systems if $x_k \in \mathscr{P}$ and $y_k \in \mathscr{V}$, $k \in \mathbb{Z}_+$ for every $x_0 \in \mathscr{P}$ and $u_k \in \mathscr{Q}$, $k \in \mathbb{Z}_+$.

The $(\mathscr{P},\mathscr{Q},\mathscr{V})$ cone fractional discrete-time linear systems (7.13) will be shortly called the cone fractional system. If $\mathscr{P} = \mathbb{R}_+^n$, $\mathscr{Q} = \mathbb{R}_+^m$, $\mathscr{V} = \mathbb{R}_+^p$, then $(\mathbb{R}_+^n,\mathbb{R}_+^m,\mathbb{R}_+^p)$ cone system is equivalent to the positive fractional system.

Theorem 7.3. *The fractional system (7.13) is the $(\mathscr{P},\mathscr{Q},\mathscr{V})$ cone fractional system if and only if*

$$\overline{A}_\alpha = PA_\alpha P^{-1} \in \mathbb{R}_+^{n\times n}, \ \ \overline{B} = PBQ^{-1} \in \mathbb{R}_+^{n\times m}, \tag{7.14a}$$
$$\overline{C} = VCP^{-1} \in \mathbb{R}_+^{p\times n}, \ \ \overline{D} = VDQ^{-1} \in \mathbb{R}_+^{p\times m}, \tag{7.14b}$$

Proof. Let

$$\overline{x}_k = Px_k, \quad \overline{u}_k = Qu_k, \quad \overline{y}_k = Vy_k, \quad k \in \mathbb{Z}_+. \tag{7.15}$$

From Definition 7.4 it follows that if $x_k \in \mathscr{P}$ then $\overline{x}_k \in \mathbb{R}_+^n$, if $u_k \in \mathscr{Q}$ then $\overline{u}_k \in \mathbb{R}_+^m$ and if $y_k \in \mathscr{V}$, then $\overline{y}_k \in \mathbb{R}_+^p$. From (7.13) and (7.15) we have

$$\overline{x}_{k+1} = PA_\alpha x_k + \sum_{j=1}^{k}(-1)^{j}\binom{\alpha}{j+1}Px_{k-j} + PBu_k \tag{7.16a}$$
$$= \overline{A}_\alpha \overline{x}_k + \sum_{j=1}^{k}(-1)^{j}\binom{\alpha}{j+1}\overline{x}_{k-j} + \overline{B}\overline{u}_k, \tag{7.16b}$$

and

$$\overline{y}_k = Vy_k = VCx_k + VDu_k = VCP^{-1}\overline{x}_k + VDQ^{-1}\overline{u}_k = \overline{C}\overline{x}_k + \overline{D}\overline{u}_k, \quad k \in \mathbb{Z}_+. \tag{7.16c}$$

It is well-known that the fractional system (7.16) is positive if and only if the conditions (7.14) are satisfied. □

7.1.4 Cone Fractional Continuous-Time System

Consider the fractional continuous-time system

$$\frac{d^\alpha}{dt^\alpha}x(t) = Ax(t) + Bu(t), \quad 0 < \alpha < 1, \tag{7.17a}$$

$$y(t) = Cx(t) + Du(t), \tag{7.17b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the state, input and output vectors and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Definition 7.5. The fractional system (7.17) is called $(\mathscr{P}, \mathscr{Q}, \mathscr{V})$ cone fractional continuous-time system if $x(t) \in \mathscr{P}$ and $y(t) \in \mathscr{V}$, $t \geq 0$ for every $x_0 \in \mathscr{P}$ and $u(t) \in \mathscr{Q}, t \geq 0$.

The $(\mathscr{P}, \mathscr{Q}, \mathscr{V})$ cone fractional continuous-time system (7.17) we shall shortly called the cone fractional system. If $\mathscr{P} = \mathbb{R}_+^n$, $\mathscr{Q} = \mathbb{R}_+^m$, $\mathscr{V} = \mathbb{R}_+^p$ then the cone $(\mathbb{R}_+^n, \mathbb{R}_+^m, \mathbb{R}_+^p)$ system is equivalent to the positive fractional system.

Theorem 7.4. *The fractional system (7.17) is the $(\mathscr{P}, \mathscr{Q}, \mathscr{V})$ cone fractional system if and only if*

$$\overline{A} = PAP^{-1} \in M_n, \;\; \overline{B} = PBQ^{-1} \in \mathbb{R}_+^{n\times m}, \tag{7.18a}$$

$$\overline{C} = VCP^{-1} \in \mathbb{R}_+^{p\times n}, \;\; \overline{D} = VDQ^{-1} \in \mathbb{R}_+^{p\times m}, \tag{7.18b}$$

Proof. Let

$$\overline{x}(t) = Px(t), \quad \overline{u}(t) = Qu(t), \quad \overline{y}(t) = Vy(t), \quad t \geq 0. \tag{7.19}$$

From Definition 7.5 it follows that if $x(t) \in \mathscr{P}$, then $\overline{x}(t) \in \mathbb{R}_+^n$, if $u(t) \in \mathscr{Q}$, then $\overline{u}(t) \in \mathbb{R}_+^m$ and if $y(t) \in \mathscr{V}$, then $\overline{y}(t) \in \mathbb{R}_+^p$. From (7.17) and (7.19) we have

$$\begin{aligned} D^\alpha \overline{x}(t) &= D^\alpha Px(t) = PAx(t) + PBu(t) = PAP^{-1}\overline{x}(t) + PBQ^{-1}\overline{u}(t) \\ &= \overline{A}\overline{x}(t) + \overline{B}\overline{u}(t), \quad t \geq 0, \end{aligned} \tag{7.20a}$$

and

$$\begin{aligned} \overline{y}(t) &= Vy(t) = VCx(t) + VDu(t) = VCP^{-1}\overline{x}(t) + VDQ^{-1}\overline{u}(t) \\ &= \overline{C}\overline{x}(t) + \overline{D}\overline{u}(t), \quad t \geq 0. \end{aligned} \tag{7.20b}$$

It is well-known the the fractional system (7.20) is positive if and only if the conditions (7.18) are satisfied. □

7.2 Reachability of Cone Fractional Systems

7.2.1 Cone Fractional Discrete-Time Systems

Definition 7.6. A state $x_f \in \mathscr{P}$ of the cone fractional system (7.13) is called reachable in q steps if there exists an inputs sequence $u_k \in \mathscr{Q}$, $k = 0, 1, \dots, q-1$, which steers the state of the system from zero initial state $x_0 = 0$ to the desired state x_f, i.e. $x_q = x_f$. If every state $x_f \in \mathscr{P}$ is reachable in q steps then the cone fractional system is called reachable in q steps. If for every state $x_f \in \mathscr{P}$ there exists a natural number q such that the state is reachable in q steps then the cone fractional system is called reachable.

Theorem 7.5. *The cone fractional system (7.13) is reachable in q steps if and only if the matrix*

$$\overline{R}_q = \left[PBQ^{-1} \; P\Phi_1 BQ^{-1} \; \dots \; P\Phi_{q-1} BQ^{-1} \right], \tag{7.21}$$

contains n linearly independent monomial columns where Φ_i, $i = 1, \dots, q-1$ are defined by (1.24).

Proof. From (7.15) it follows that if $x_k \in \mathscr{P}$, then $\overline{x}_k = Px_k \in \mathbb{R}_+^n$ and if $u_k \in \mathscr{Q}$, then $\overline{u}_k = Qu_k \in \mathbb{R}_+^m$ for $k \in \mathbb{Z}_+$. From Definitions 7.4 and 1.9 it follows that the cone fractional system (7.13) is reachable in q steps if and only if the positive fractional system (7.16) is reachable in q steps. Using (1.24) and (7.14), it is easy to show that the matrices $\overline{\Phi}_k$ of the system (7.16) are related with the matrices Φ_k of the system (7.13) as follows

$$\overline{\Phi}_k = P\Phi_k P^{-1} \quad \text{for} \quad k = 0, 1, \dots \;. \tag{7.22}$$

Taking into account that

$$\overline{\Phi}_k \overline{B} = P\Phi_k P^{-1} PBQ^{-1} = P\Phi_k BQ^{-1}, \quad k = 1, 2, \dots, q-1, \tag{7.23}$$

we obtain

$$\overline{R}_q = \left[\overline{B} \; \overline{\Phi}_1 \overline{B} \dots \overline{\Phi}_{q-1} \overline{B} \right] = \left[PBQ^{-1} \; P\Phi_1 BQ^{-1} \; \dots \; P\Phi_{q-1} BQ^{-1} \right]. \tag{7.24}$$

From Theorem 1.10 it follows that the positive fractional system (7.16) is reachable in q steps if and only if the matrix (7.24) contains n linearly independent monomial columns. □

Example 7.1. Consider the cone fractional system (7.13) with the matrices

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad Q = [1], \quad A = \begin{bmatrix} -\alpha & a \\ 1 & a - \alpha + 1 \end{bmatrix}, \quad a > 0, \quad 0 < \alpha < 1, \tag{7.25}$$

and for the following two forms of the matrix B:

$$B_1 = \begin{bmatrix} b \\ b \end{bmatrix}, \quad B_2 = \begin{bmatrix} -b \\ b \end{bmatrix}, \quad b > 0.$$

In case 1 we shall show that the cone fractional system is unreachable. Using (1.32), (7.14) and (7.25), we obtain:

$$\begin{aligned}\overline{A}_\alpha = P(A+\alpha I_n)P^{-1} &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & a \\ 1 & a+1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} a+1 & a \\ 1 & 0 \end{bmatrix}, \quad a>0, \qquad (7.26) \\ \overline{B}_1 = PB_1Q^{-1} &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} b \\ b \end{bmatrix} = \begin{bmatrix} 2b \\ 0 \end{bmatrix}, \quad b>0.\end{aligned}$$

The system with (7.26) is a positive fractional system. Using (7.21) for $q=2$, (7.25) and taking into account that $\Phi_1 = A_\alpha$, we obtain

$$\begin{aligned}\overline{R}_2 &= \begin{bmatrix} PB_1Q^{-1} & P\Phi_1B_1Q^{-1} \end{bmatrix} = P\begin{bmatrix} B_1 & A_\alpha B_1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} b & ab \\ b & (a+2)b \end{bmatrix} = \begin{bmatrix} 2b & 2(a+1)b \\ 0 & 2b \end{bmatrix}. \qquad (7.27)\end{aligned}$$

The matrix(7.27) contains only one monomial column. From Theorem 7.5 it follows that the cone fractional system is unreachable. Note that the necessary condition of reachability of Theorem 1.11 is not satisfied since the matrix

$$\begin{bmatrix} \overline{A}_\alpha & \overline{B}_1 \end{bmatrix} = \begin{bmatrix} a+1 & a & 2b \\ 1 & 0 & 0 \end{bmatrix}$$

contains only one linearly independent monomial column.

In case 2 we have

$$\overline{B}_2 = PB_2Q^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -b \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 2b \end{bmatrix}, \quad b>0. \qquad (7.28)$$

The system with (7.26) and (7.28) is also a positive fractional system. Using (7.21) for $q=2$, (7.25), we obtain the matrix

$$\overline{R}_2 = P\begin{bmatrix} B_2 & A_\alpha B_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -b & ab \\ b & ab \end{bmatrix} = \begin{bmatrix} 0 & 2ab \\ 2b & 0 \end{bmatrix}, \quad a>0, \quad b>0, \quad (7.29)$$

which contains two linearly independent monomial columns. Therefore, by Theorem 7.5 the cone fractional system is reachable. In this case the necessary condition of the reachability is satisfied since the matrix

$$\begin{bmatrix} \overline{A}_\alpha & \overline{B}_2 \end{bmatrix} = \begin{bmatrix} a+1 & a & 0 \\ 1 & 0 & 2b \end{bmatrix}$$

contains two linearly independent monomial columns.

7.2.2 Cone Fractional Continuous-Time System

Definition 7.7. A state $x_f \in \mathscr{P}$ of the cone fractional system (7.17) is called reachable in time t_f, if there exists an input $u(t) \in \mathscr{Q}$, $t \in [0, t_f]$, which steers the state of the system from zero initial state $x_0 = 0$ to the finial state $x(t_f) = x_f$. If every state $x_f \in \mathscr{P}$ is reachable in time t_f, then the cone fractional system is reachable in time t_f. If for every state $x_f \in \mathscr{P}$ there exists time t_f such that the state is reachable in time t_f, then the cone fractional system is called reachable.

Theorem 7.6. *The positive cone fractional system (7.17) is reachable in time t_f, if the matrix*

$$\overline{R}(t_f) = P\int_0^{t_f} \Phi(\tau)BQ^{-1}\left(Q^{-1}\right)^T B^T\Phi^T(\tau)d\tau P^T, \tag{7.30}$$

is monomial, where $\Phi(t)$ is defined by (2.17).

Proof. From (7.19) it follows that if $x(t) \in \mathscr{P}$, then $\overline{x}(t) = Px(t) \in \mathbb{R}_+^n$, $t \geq 0$, if $u(t) \in \mathscr{Q}$, then $\overline{u}(t) = Qu(t) \in \mathbb{R}_+^m$, $t \geq 0$. By Definitions 2.10 and 7.7 the cone fractional system (7.17) is reachable in time t_f, if the positive fractional system (7.20) is reachable in time t_f.

From (7.18) and (2.17) we have

$$\overline{\Phi}(t) = \sum_{k=0}^{\infty}\frac{\overline{A}^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} = \sum_{k=0}^{\infty}\frac{\left(PAP^{-1}\right)^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} = P\Phi(t)P^{-1}, \tag{7.31}$$

since $\overline{A}^k = (PAP^{-1})^k = PA^kP^{-1}$ for $k = 1, 2, \ldots$ and

$$\overline{\Phi}(t)\overline{B} = P\Phi(t)P^{-1}PBQ^{-1} = P\Phi(t)BQ^{-1}. \tag{7.32}$$

Using (2.44) and (7.30), we obtain

$$\begin{aligned}\overline{R}(t_f) &= \int_0^{t_f}\overline{\Phi}(\tau)\overline{B}\overline{B}^T\overline{\Phi}^T(\tau)d\tau = \int_0^{t_f}\left(P\Phi(\tau)BQ^{-1}\right)\left(P\Phi(\tau)BQ^{-1}\right)^T d\tau \\ &= P\int_0^{t_f}\Phi(\tau)BQ^{-1}\left(Q^{-1}\right)^T B^T\Phi^T(\tau)d\tau P^T \end{aligned}\tag{7.33}$$

By Theorem 2.10 the positive cone fractional system (7.17) is reachable in time t_f, if the matrix (7.30) is monomial. □

Corollary 7.1. *If $Q = I_m$, then $\overline{R}(t_f) = PR(t_f)P^T$ and the positive fractional system (7.17) is reachable in time t_f, if the positive fractional system is reachable and the matrix P is monomial.*

Example 7.2. Consider the cone fractional system (7.17) with the matrices:

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{7.34}$$

From (7.34) and (2.49) it follows that

$$\Phi(t)B = \begin{bmatrix} 0 & \Phi_1(t) \\ \Phi_2(t) & 0 \end{bmatrix}. \tag{7.35}$$

From (2.44) we have

$$R(t_f) = \int_0^{t_f} \Phi(\tau)BB^T\Phi^T(\tau)dt = \int_0^{t_f} \begin{bmatrix} \Phi_1^2(\tau) & 0 \\ 0 & \Phi_2^2(\tau) \end{bmatrix} d\tau, \tag{7.36}$$

where

$$\Phi_1(t) = \sum_{k=0}^{\infty} \frac{t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}, \quad \Phi_2(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad 0 < \alpha < 1. \tag{7.37}$$

The matrix (7.36) is monomial and by Theorem 7.6 the positive fractional system is reachable in time t_f.

In this case $Q = I_2$ and the matrix

$$\begin{aligned} \overline{R}(t_f) = PR(t_f)P^T &= \int_0^{t_f} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \Phi_1^2(\tau) & 0 \\ 0 & \Phi_2^2(\tau) \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} d\tau \\ &= \int_0^{t_f} \begin{bmatrix} \Phi_1^2(\tau)+\Phi_2^2(\tau) & \Phi_2^2(\tau)-\Phi_1^2(\tau) \\ \Phi_2^2(\tau)-\Phi_1^2(\tau) & \Phi_1^2(\tau)+\Phi_2^2(\tau) \end{bmatrix} d\tau, \end{aligned} \tag{7.38}$$

is not monomial since $\Phi_1^2(\tau) \neq \Phi_2^2(\tau)$. Therefore, the sufficient condition of the reachability of Theorem 7.6 of the cone system is not satisfied.

Corollary 7.2. *From this example and from comparison of (2.44) and (7.30) it follows that the sufficient conditions for the reachability of the cone fractional systems are much stronger than for the positive fractional systems.*

7.3 Controllability to Zero of Cone Fractional Discrete-Time Systems

Definition 7.8. The positive fractional system (7.13) is called controllable to zero in q steps if for any nonzero initial condition $x_0 \in \mathbb{R}_+^n$ there exists a sequence of inputs $u_k \in \mathbb{R}_+^m$, $k = 0, 1, \ldots, q-1$, which steers the state of the system from x_0 to the finial state $x_f = 0$.

Theorem 7.7. *Positive fractional system (7.13) is controllable to zero if and only if* $q = 1$ *and*

$$\Phi_1 = A_\alpha = A + \alpha I_n = 0. \tag{7.39}$$

Moreover $u_0 = 0$.

Proof. The proof is similar to the proof of Lemma 1.3. □

Definition 7.9. The cone fractional system (7.13) is called controllable to zero in q steps if for any initial condition $x_0 \in \mathscr{P}$ there exists a sequence of inputs $u_k \in \mathscr{Q}$, $k = 0, 1, \ldots, q-1$, which steers the state of the system from x_0 to the final state $x_f = 0$.

Theorem 7.8. *The cone fractional system (7.13) is controllable to zero if and only if $q = 1$ and the condition (7.39) is satisfied.*

Proof. From (7.15) it follows that if $x_k \in \mathscr{P}$, then $\overline{x}_k = Px_k \in \mathbb{R}_+^n$ and if $u_k \in \mathscr{Q}$, then $\overline{u}_k = Qu_k \in \mathbb{R}_+^m$ for $k \in \mathbb{Z}_+$. From Definitions 7.8 and 7.9 it follows that cone fractional system (7.13) is controllable to zero in q steps if and only if the positive fractional system (7.16) is controllable to zero in q steps if and only if $q = 1$ and

$$\overline{\Phi}_1 = P\Phi_1 P^{-1} = P\left(A + \alpha I_n\right)P^{-1} = 0. \tag{7.40}$$

This is equivalent to the condition (7.39) since $\det P \neq 0$. □

7.4 Cone Realization Problem for Linear Systems

7.4.1 Discrete-Time Linear Systems

Lemma 7.1. *The transfer matrix*

$$T(z) = C\left[I_n z - A\right]^{-1} B + D, \tag{7.41}$$

of the $(\mathscr{P}, \mathscr{Q}, \mathscr{V})$ *cone system (7.4) and the transfer matrix*

$$\overline{T}(z) = \overline{C}\left[I_n z - \overline{A}\right]^{-1} \overline{B} + \overline{D}, \tag{7.42}$$

of the positive system (7.4) are related by

$$\overline{T}(z) = VT(z)Q^{-1}. \tag{7.43}$$

Proof. Using (7.42), (7.41) and (7.5), we obtain

$$\begin{aligned}\overline{T}(z) &= \overline{C}\left[I_n z - \overline{A}\right]^{-1}\overline{B} + \overline{D} = VCP^{-1}\left[I_n z - PAP^{-1}\right]^{-1} PBQ^{-1} + VDQ^{-1} \\ &= VCP^{-1}\left[P(I_n z - A)P^{-1}\right]^{-1} PBQ^{-1} + VDQ^{-1} \\ &= VC\left[I_n z - A\right]^{-1} BQ^{-1} + VDQ^{-1} = VT(z)Q^{-1}.\end{aligned}$$

□

Definition 7.10. The matrices:

$$A \in \mathbb{R}^{n \times n} \quad \text{and} \quad B \in \mathbb{R}^{n \times m}, \quad C \in \mathbb{R}^{p \times n}, \quad D \in \mathbb{R}^{p \times m}, \tag{7.44}$$

are called a $(\mathscr{P}, \mathscr{Q}, \mathscr{V})$ cone realization of the transfer matrix $T(z) \in \mathbb{R}^{p \times m}(z)$ if they satisfy (7.41) and the conditions:

$$PAP^{-1} \in \mathbb{R}_+^{n\times n}, \quad PBQ^{-1} \in \mathbb{R}_+^{n\times m}, \tag{7.45a}$$

$$VCP^{-1} \in \mathbb{R}_+^{p\times n}, \quad VDQ^{-1} \in \mathbb{R}_+^{p\times m}, \tag{7.45b}$$

where P, Q, V are nonsingular matrices generating the cones $\mathscr{P}, \mathscr{Q}, \mathscr{V}$.

The $(\mathscr{P}, \mathscr{Q}, \mathscr{V})$ cone realization problem can be stated as follows. Given a proper transfer matrix $T(z) \in \mathbb{R}^{p\times m}(z)$ and nonsingular matrices P, Q, V generating the cones $\mathscr{P}, \mathscr{Q}, \mathscr{V}$. Find a $(\mathscr{P}, \mathscr{Q}, \mathscr{V})$ cone realization of the transfer matrix $T(z)$.

The $(\mathscr{P}, \mathscr{Q}, \mathscr{V})$ cone realization of the transfer matrix $T(z) \in \mathbb{R}^{p\times m}(z)$ for given nonsingular matrices P, Q, V can be computed by the use of the following procedure:

Procedure 7.1

Step 1. Knowing $T(z)$ and the matrices Q, V and using (7.43), find the transfer matrix $\overline{T}(z)$.

Step 2. Using the known procedures finding a positive realization $\overline{A}$, $\overline{B}$, $\overline{C}$, $\overline{D}$ of the form (6.8) of $\overline{T}(z)$.

Step 3. Using:

$$A = P^{-1}\overline{A}P, \quad B = P^{-1}\overline{B}Q, \tag{7.46a}$$

$$C = V^{-1}\overline{C}P, \quad D = V^{-1}\overline{D}Q, \tag{7.46b}$$

find the desired realization.

Theorem 7.9. *There exists a $(\mathscr{P}, \mathscr{Q}, \mathscr{V})$ cone realization of $T(z)$ if and only if there exists a positive realization of $\overline{T}(z)$.*

Proof. The proof follows from Procedure 7.1 and Lemma 7.1. □

From Theorem 7.9 for SISO ($m = p = 1$) systems we have the following corollary.

Corollary 7.3. *There exist a $(\mathscr{P}, \mathscr{Q}, \mathscr{V})$ cone realization A, B, C, D if and only if there exists a positive realization $\overline{A}$, $\overline{B}$, $\overline{C}$, $\overline{D}$ of $\overline{T}(z)$. The realizations are related by:*

$$A = P^{-1}\overline{A}P, \quad B = P^{-1}\overline{B}Q, \tag{7.47}$$

$$C = V^{-1}\overline{C}P, \quad D = k\overline{D}, \tag{7.48}$$

where $k = QV^{-1}$ is a scalar and the transfer matrices $\overline{T}(z)$ and $T(z)$ are related by $\overline{T}(z) = kT(z)$.

Example 7.3. Given the transfer function

$$T(z) = \frac{2z+1}{z^2 - 2z - 3}, \tag{7.49}$$

and

$$P = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}, \quad Q = V = 1. \tag{7.50}$$

Find its a $(\mathscr{P}, \mathscr{Q}, \mathscr{V})$ cone realization. Using Procedure 7.1 we obtain the following:

Step 1. In this case $\overline{T}(z) = T(z)$, since $Q = V = 1$.
Step 2. A positive realization of (7.49) has the form:

$$\overline{A} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad \overline{D} = 0. \tag{7.51}$$

Step 3. Using (7.46), (7.50) and (7.51), we obtain the desired cone realization:

$$A = P^{-1}\overline{A}P = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 5 & -1 \end{bmatrix}, \tag{7.52a}$$

$$B = P^{-1}\overline{B}Q = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1] = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \tag{7.52b}$$

$$C = V^{-1}\overline{C}P = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \end{bmatrix}, \quad D = V^{-1}\overline{D}Q = 0. \tag{7.52c}$$

Example 7.4. Given the transfer matrix

$$\begin{aligned} T(z) &= \frac{1}{2(z-1)(z-2)(z-3)} \\ &\times \begin{bmatrix} 3z^2 - 8z + 5 & z - 3 & 2z^2 - 11z + 15 \\ -z^2 + 8z - 11 & -2z^2 + 5z - 1 & 6z^2 - 19z + 11 \end{bmatrix}, \end{aligned} \tag{7.53}$$

and the matrices:

$$P = \begin{bmatrix} 1 & -2 & 0 & 1 & -1 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 2 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 2 & 0 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}. \tag{7.54}$$

Find a $(\mathscr{P}, \mathscr{Q}, \mathscr{V})$ cone realization of (7.53). In this case $m = 3$ and $p = 2$. Using Procedure 7.1, we obtain the following:

Step 1. From (7.43) and (7.53) we have

$$\overline{T}(z) = VT(z)Q^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \frac{1}{2(z-1)(z-2)(z-3)}$$
$$\times \begin{bmatrix} 3z^2-8z+5 & z-3 & 2z^2-11z+15 \\ -z^2+8z-11 & -2z^2+5z-1 & 6z^2-19z+11 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} \frac{2z-4}{(z-1)(z-3)} & 0 & \frac{3z-7}{(z-2)(z-3)} \\ \frac{3}{z-3} & \frac{2z-3}{(z-1)(z-2)} & \frac{2}{z-3} \end{bmatrix}. \tag{7.55}$$

Step 2. A positive realization of (7.55) has the form:

$$\overline{A} = \mathrm{diag}\begin{bmatrix} 1 & 1 & 2 & 2 & 3 & 3 \end{bmatrix}, \quad \overline{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \\ 3 & 0 & 2 \end{bmatrix} \tag{7.56a}$$

$$\overline{C} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad \overline{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{7.56b}$$

Step 3. Using (7.46), (7.54) and (7.56), we obtain the desired realization:

$$A = \frac{1}{6}\begin{bmatrix} 12 & 11 & 6 & -3 & 6 & 1 \\ 6 & 7 & 12 & 9 & 0 & 5 \\ -6 & 15 & -18 & -21 & 0 & -9 \\ 18 & -29 & 60 & 57 & 0 & 23 \\ 12 & -20 & 42 & 30 & 12 & 14 \\ -12 & 16 & -36 & -24 & 0 & 2 \end{bmatrix}, \quad B = \frac{1}{6}\begin{bmatrix} 1 & 2 & -4 \\ -1 & 4 & -8 \\ 3 & -6 & 12 \\ -7 & 10 & -14 \\ -10 & 10 & -14 \\ 14 & -8 & 10 \end{bmatrix},$$

$$C = \frac{1}{2}\begin{bmatrix} 1 & 0 & 5 & 2 & 4 & 4 \\ -3 & -2 & 3 & 2 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

7.4.2 *Cone Realization Problem for Continuous-Time Systems with Delays*

Lemma 7.2. *The transfer matrix*

$$T(s,w) = C\left[I_n s - A_0 - A_1 w - \ldots - A_h w^h\right]^{-1}[B_0 + B_1 w + \ldots + B_q w^q] + D,$$
$$w = e^{-ds}, \tag{7.57}$$

of the $(\mathscr{P}, \mathscr{Q}, \mathscr{V})$ *cone system* (7.8) *and the transfer matrix*

$$\overline{T}(s,w) = \overline{C}\left[I_n s - \overline{A}_0 - \overline{A}_1 w - \cdots - \overline{A}_h w^h\right]^{-1}\left[\overline{B}_0 + \overline{B}_1 w + \cdots + \overline{B}_q w^q\right] + \overline{D}, \tag{7.58}$$

of the positive system (7.12) are related by

$$\overline{T}(s,w) = VT(s,w)Q^{-1} \tag{7.59}$$

Proof. Using (7.10), (7.57) and (7.58), we obtain

$$\begin{aligned}
\overline{T}(s,w) &= \overline{C}\left[I_n s - \overline{A}_0 - \overline{A}_1 w - \cdots - \overline{A}_h w^h\right]^{-1} \\
&\times \left[\overline{B}_0 + \overline{B}_1 w + \cdots + \overline{B}_q w^q\right] + \overline{D} \\
&= VCP^{-1}\left[P\left(I_n s - A_0 - A_1 w - \cdots - A_h w^h\right)P^{-1}\right]^{-1} \\
&\times P\left[B_0 + B_1 w + \cdots + B_q w^q\right]Q^{-1} + VDQ^{-1} \\
&= V\left(C\left[I_n s - A_0 - A_1 w - \cdots - A_h w^h\right]^{-1}\right. \\
&\times \left.\left[B_0 + B_1 w + \cdots + B_q w^q\right] + D\right)Q^{-1} \\
&= VT(s,w)Q^{-1}.
\end{aligned}$$

□

Definition 7.11. The matrices:

$$A_i \in \mathbb{R}^{n\times n}, \quad i = 0,1,\ldots,h; \quad B_j \in \mathbb{R}^{n\times m}, \quad j = 0,1,\ldots,q;$$

$$C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m},$$

are called $(\mathscr{P},\mathscr{Q},\mathscr{V})$ cone realization of the transfer matrix $T(s,w)$ if they satisfy (7.10) and the conditions (7.57).

The $(\mathscr{P},\mathscr{Q},\mathscr{V})$ cone realization problem can be stated as follows. Given a proper transfer matrix $T(s,w) \in \mathbb{R}^{p\times m}(s,w)$ and the nonsingular matrices P,Q,V generating the cones $\mathscr{P},\mathscr{Q},\mathscr{V}$. Find a $(\mathscr{P},\mathscr{Q},\mathscr{V})$ cone realization of $T(s,w)$.

From (7.58) we have

$$\overline{D} = \lim_{s\to\infty}\overline{T}(s,w), \tag{7.60}$$

since

$$\lim_{s\to\infty}\left[I_n s - \overline{A}_0 - \overline{A}_1 w - \cdots - \overline{A}_h w^h\right]^{-1} = 0.$$

The strictly proper transfer matrix is

$$\overline{T}_{sp}(s,w) = \overline{T}(s,w) - \overline{D}. \tag{7.61}$$

The positive realization problem of (7.61) has been reduced to finding the matrices:

$$\overline{A}_0 \in M_n, \quad \overline{A}_i \in \mathbb{R}_+^{n\times n}, \quad i = 1,\ldots,h;$$

$$\overline{B}_j \in \mathbb{R}_+^{n\times m}, \quad j = 0,1,\ldots,q; \quad \overline{C} \in \mathbb{R}_+^{p\times n}.$$

If for the given $\overline{T}_{sp}(s,w)$

$$\overline{A}_0 = \begin{bmatrix} 0 & 0 & \ldots & 0 & a_{00} \\ 1 & 0 & \ldots & 0 & a_{01} \\ 0 & 1 & \ldots & 0 & a_{02} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & a_{0n-1} \end{bmatrix}, \quad \overline{A}_i = \begin{bmatrix} 0 & \ldots & 0 & a_{i0} \\ 0 & \ldots & 0 & a_{i1} \\ \vdots & \ddots & \vdots & \\ 0 & \ldots & 0 & a_{in-1} \end{bmatrix}, \quad i = 1,\ldots,h, \tag{7.62}$$

then

$$\begin{aligned} d(s,w) &= \det\left[I_n s - \overline{A}_0 - \overline{A}_1 w - \cdots - \overline{A}_h w^h\right] \\ &= s^n - d_{n-1}s^{n-1} - \cdots - d_1 s - d_0, \qquad (7.63) \\ d_j = d_j(w) &= a_{h,j}w^h + a_{h-1,j}w^{h-1} + \cdots + a_{1j}w + a_{0j}, \quad j = 0,1,\ldots,n-1; \end{aligned}$$

and the n-th row of the adjoint matrix

$$\text{Adj}\left[I_n s - \overline{A}_0 - \overline{A}_1 w - \cdots - \overline{A}_h w^h\right], \tag{7.64}$$

has the form

$$R_n(s) = \begin{bmatrix} 1 & s & \ldots & s^{n-1} \end{bmatrix}. \tag{7.65}$$

The strictly proper transfer matrix $\overline{T}_{sp}(s,w)$ can be written in the form

$$\overline{T}_{sp}(s,w) = \begin{bmatrix} \frac{N_1(s,w)}{d_1(s,w)} \\ \vdots \\ \frac{N_p(s,w)}{d_p(s,w)} \end{bmatrix}, \tag{7.66}$$

where

$$\begin{aligned} d_k(s,w) &= s^{n_k} - d^k_{n_k-1}s^{n_k-1} - \cdots - d^k_1 s - d^k_0, \quad i = 0,1,\ldots,n_k-1, \\ d^k_i = d^k_i(w) &= a^k_{h,i}w^h + a^k_{h-1,i}w^{h-1} + \cdots + a^k_{1i}w + a^k_{0i}, \quad k = 1,\ldots,p; \qquad (7.67) \end{aligned}$$

is the least common denominator of the i-th row of the matrix $\overline{T}_{sp}(s,w)$ and

$$N_k(s,w) = \begin{bmatrix} n_{k1}(s,w) & \ldots & n_{km}(s,w) \end{bmatrix}, \quad k = 1,\ldots,p; \tag{7.68a}$$

$$n_{kj}(s,w) = n_{kj}^{n_k-1}s^{n_k-1} + \cdots + n_{kj}^{1}s + n_{kj}^{0}, \quad j = 0,1,\ldots,m; \tag{7.68b}$$
$$n_{kj}^{i} = n_{kj}^{iq}w^{q} + \cdots + n_{kj}^{i1}w + n_{kj}^{i0}, \quad i = 0,1,\ldots,n_k-1.$$

With the polynomials (7.67) are associated the matrices

$$\overline{A}_{k0} = \begin{bmatrix} 0 & 0 & \ldots & 0 & a_{00}^{k} \\ 1 & 0 & \ldots & 0 & a_{01}^{k} \\ 0 & 1 & \ldots & 0 & a_{02}^{k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & a_{0n_k-1}^{k} \end{bmatrix}, \overline{A}_{ki} = \begin{bmatrix} 0 & \ldots & 0 & a_{i0}^{k} \\ 0 & \ldots & 0 & a_{i1}^{k} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & a_{in_k-1}^{k} \end{bmatrix}, \quad \begin{matrix} k = 1,\ldots,p; \\ i = 1,\ldots,h_k; \end{matrix} \tag{7.69}$$

satisfying the condition

$$d_k(s,w) = \det\left[I_{n_k}s - \overline{A}_{k0} - \overline{A}_{k1}w - \cdots - \overline{A}_{kh_k}w^{h_k}\right], \quad k = 1,\ldots,p. \tag{7.70}$$

Let

$$\overline{A}_0 = \text{block diag}\left[\overline{A}_{10} \ \ldots \ \overline{A}_{p0}\right] \in \mathbb{R}^{n\times n}, \quad n = n_1 + \cdots + n_p;$$
$$\overline{A}_i = \text{block diag}\left[\overline{A}_{1i} \ \ldots \ \overline{A}_{pi}\right] \in \mathbb{R}^{n\times n}, \quad i = 1,\ldots,p; \tag{7.71a}$$
$$\overline{B}_k = \begin{bmatrix} b_{11}^{k} & \ldots & b_{1m}^{k} \\ \vdots & \ddots & \vdots \\ b_{p1}^{k} & \ldots & b_{pm}^{k} \end{bmatrix}, \quad b_{ij}^{k} = \begin{bmatrix} b_{ij}^{k1} \\ \vdots \\ b_{ij}^{kn_i} \end{bmatrix}, \quad \begin{matrix} k = 0,1,\ldots,q; \\ j = 1,\ldots,m; \end{matrix} \tag{7.71b}$$
$$\overline{C} = \text{block diag}\left[c_1 \ \ldots \ c_p\right], \quad c_i = \left[0 \ \ldots \ 0 \ 1\right] \in \mathbb{R}^{1\times n_i}. \tag{7.71c}$$

It is assumed that the number of delays q in the input vector is equal to the polynomial degree of the matrix $N(s,w)$ with respect to w.

The entries of the matrices $\overline{B}_k$, $k = 0,1,\ldots,q$, are given by:

$$\begin{aligned} & b_{1j}^{01} = n_{1j}^{00}, \ b_{1j}^{11} = n_{1j}^{01}, \ldots, b_{1j}^{0q} = n_{1j}^{q-1,0}, \ldots, b_{1j}^{0n_1} = n_{1j}^{n_1-1,0}, \\ & \qquad b_{1j}^{1n_1} = n_{1j}^{n_1-1,1}, \ldots, b_{1j}^{qn_1} = n_{1j}^{n_1-1,q} \\ & \vdots \\ & b_{pj}^{01} = n_{pj}^{00}, \ b_{pj}^{11} = n_{pj}^{01}, \ldots, b_{pj}^{0q} = n_{pj}^{q-1,0}, \ldots, b_{pj}^{0n_p} = n_{pj}^{n_p-1,0}, \\ & \qquad b_{pj}^{1n_p} = n_{pj}^{n_p-1,1}, \ldots, b_{pj}^{qn_p} = n_{pj}^{n_p-1,q} \end{aligned} \tag{7.72}$$

for $j = 1,\ldots,m$.

Theorem 7.10. *There exists a positive realization of the transfer matrix $\overline{T}(s,w)$, if:*

a)

$$\overline{T}(\infty) = \lim_{s\to\infty}\overline{T}(s,w) \in \mathbb{R}_{+}^{p\times m} \tag{7.73}$$

b) *the coefficients of polynomials* $d_k(s,w)$, $k = 1,\dots,p$ *are nonnegative except* $a^k_{0n_{k-1}}$, $k = 1,\dots,p$, *which can be arbitrary*

$$a^k_{ij} \geq 0 \quad \textit{for} \quad i = 1,\dots,h_k; \quad j = 0,1,\dots,n_k - 2; \quad k = 0,1,\dots,p; \tag{7.74}$$

c) *the coefficients of polynomial* $N_j(s,w)$, $j = 1,\dots,p$, *are nonnegative*

$$n^{il}_{kj} \geq 0 \quad \textit{for} \quad \begin{array}{l} i = 0,1,\dots,n_k - 1;\ j = 1,\dots,m; \\ k = 1,\dots,p;\ l = 0,1,\dots,q; \end{array} \tag{7.75}$$

If the conditions of Theorem 7.10 are satisfied then the minimal positive realization of $\overline{T}(s,w)$ can be found by the use of the following procedure:

Procedure 7.2

Step 1. Using (7.60) and (7.61) find the matrix $\overline{D}$ and the strictly proper transfer matrix $\overline{T}_{sp}(s,w)$.

Step 2. Knowing the coefficients of polynomial $d_k(s,w)$, $k = 1,\dots,p$, find the matrices (7.69) and (7.71a).

Step 3. Knowing the coefficients of polynomial $N_j(s,w)$, $j = 1,\dots,p$; and using (7.72), find the matrices $\overline{B}_k$, $k = 0,1,\dots,q$, and the matrix $\overline{C}$.

For given matrices P,Q,V the $(\mathscr{P},\mathscr{Q},\mathscr{V})$ cone realization of $T(s,w) \in \mathbb{R}^{p\times m}$ can be found by the use of the following procedure:

Procedure 7.3

Step 1. Knowing the transfer matrix $T(s,w)$ and the matrices P,Q,V and using (7.59) find the transfer matrix $\overline{T}(s,w)$.

Step 2. Using Procedure 7.2 find the positive realization $\overline{A}_i$, $i = 0,1,\dots,h$; $\overline{B}_j$, $j = 0,1,\dots,q$; $\overline{C}$, $\overline{D}$ of $\overline{T}(s,w)$.

Step 3. Using (7.10), find the desired realization.

Theorem 7.11. *There exists a* $(\mathscr{P},\mathscr{Q},\mathscr{V})$ *cone realization of the transfer matrix* $T(s,w)$ *if and only if the there exists a positive realization of* $\overline{T}(s,w)$.

Proof. The Proof follows from Procedure 7.3 and Lemma 7.2. □

Example 7.5. Given the transfer function

$$T(s,w) = \frac{2s^2 + 6s - (2w+1)}{s^2 + (2-w)s - (2w+1)}\ , \tag{7.76}$$

and

$$P = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}, \quad Q = V = 1. \tag{7.77}$$

Find a $(\mathscr{P},\mathscr{Q},\mathscr{V})$ cone realization of (7.76).

Using Procedure 7.3, we obtain the following:

Step 1. In this case $\overline{T}(s,w) = T(s,w)$, since $Q = V = 1$.
Step 2. Using Procedure 7.2, we obtain $\overline{D} = 2$ and a positive realization of (7.76) in the form:

$$\overline{A}_0 = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}, \overline{A}_1 = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}, \overline{B}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \overline{B}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \overline{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Step 3. Using (7.10) we obtain the desired cone realization:

$$A_0 = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ -1 & -7 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, B_1 = \frac{1}{3}\begin{bmatrix} 4 \\ 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

and $D = \overline{D} = 2$

Example 7.6. Given the transfer matrix

$$T(s,w) = \frac{1}{2}\begin{bmatrix} \frac{(-w^2+w-1)s+w^2-w+1}{s^2+(-w^2+2)s-(2w^2+w+1)} + \frac{-2s+3w^2-1}{s-2w^2-w+1} & \frac{3s^2+(-w^2+w+8)s-5w^2-w-2}{s^2+(-w^2+2)s-(2w^2+w+1)} + \frac{4s-3w^2+5}{s-2w^2-w+1} \\ \frac{(w^2-w+1)s-w^2+w-1}{s^2+(-w^2+2)s-(2w^2+w+1)} + \frac{-2s+3w^2-1}{s-2w^2-w+1} & \frac{-3s^2+(w^2-w-8)s+5w^2+w+2}{s^2+(-w^2+2)s-(2w^2+w+1)} + \frac{4s-3w^2+5}{s-2w^2-w+1} \end{bmatrix}. \tag{7.78}$$

and the matrices

$$P = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}, \quad V = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}. \tag{7.79}$$

Find a $(\mathscr{P}, \mathscr{Q}, \mathscr{V})$ cone realization of (7.78).

In this case $m = p = 2$. Using Procedure 7.3 we obtain the following:

Step 1. From (7.59), (7.78) and (7.79) we have

$$\overline{T}(s,w) = \begin{bmatrix} \frac{s^2+(-w^2+w+2)s-w^2-w}{s^2+(-w^2+2)s-(2w^2+w+1)} & \frac{s^2+3s-(2w^2+1)}{s^2+(-w^2+2)s-(2w^2+w+1)} \\ \frac{w^2+1}{s-2w^2-w+1} & \frac{2s-2w^2+2}{s-2w^2-w+1} \end{bmatrix}. \tag{7.80}$$

Step 2. Using Procedure 7.2 we obtain

$$\overline{D} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \tag{7.81}$$

and

$$\overline{T}_{sp}(s,w) = \begin{bmatrix} \frac{ws+w^2+1}{s^2+(-w^2+2)s-(2w^2+w+1)} & \frac{(w^2+1)s+w}{s^2+(-w^2+2)s-(2w^2+w+1)} \\ \frac{w^2+1}{s-2w^2-w+1} & \frac{2w^2+2w}{s-2w^2-w+1} \end{bmatrix}. \tag{7.82}$$

Taking into account that

$$d_1(s,w) = s^2 + (-w^2+2)s - (2w^2+w+1), \quad d_2(s,w) = s - 2w^2 - w + 1,$$

and

$$n_{11}(s,w) = ws + (w^2+1), \;\; n_{21}(s,w) = w^2+1,$$
$$n_{12}(s,w) = (w^3+1)s + w, \;\; n_{22}(s,w) = 2(w^2+w),$$

we obtain a positive realization of (7.80) in the form:

$$\overline{A}_0 = \begin{bmatrix} \overline{A}_{10} & 0 \\ 0 & \overline{A}_{20} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \overline{A}_1 = \begin{bmatrix} \overline{A}_{11} & 0 \\ 0 & \overline{A}_{21} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\overline{A}_2 = \begin{bmatrix} \overline{A}_{21} & 0 \\ 0 & \overline{A}_{22} \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$
$$\overline{B}_0 = \begin{bmatrix} b_{11}^{01} & b_{12}^{01} \\ b_{11}^{02} & b_{12}^{02} \\ b_{21}^{01} & b_{22}^{01} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \overline{B}_1 = \begin{bmatrix} b_{11}^{11} & b_{12}^{11} \\ b_{11}^{12} & b_{12}^{12} \\ b_{21}^{11} & b_{22}^{11} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix},$$
$$\overline{B}_2 = \begin{bmatrix} b_{11}^{21} & b_{12}^{21} \\ b_{11}^{22} & b_{12}^{22} \\ b_{21}^{21} & b_{22}^{21} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} \overline{C}_1 & 0 \\ 0 & \overline{C}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Step 3. Using (7.10) we obtain the desired cone realization:

$$A_0 = \begin{bmatrix} -1 & 1 & -2 \\ 0 & 0 & -2 \\ 1 & -1.5 & -2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 3 & -3 & -2 \\ 2 & -2 & -2 \\ 0.5 & -0.5 & 0 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 6 & -6 & -4 \\ 4 & -4 & -4 \\ 0.5 & -0.5 & 0 \end{bmatrix},$$
$$B_0 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -3 & 6 \\ -1 & 2 \\ -0.5 & 2.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -3 & 9 \\ -2 & 4 \\ -1 & 3.5 \end{bmatrix},$$
$$C = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 3.5 \\ -1 & 0.5 \end{bmatrix}.$$

Chapter 8
Stability of Positive Fractional 1D and 2D Linear Systems

8.1 Asymptotic Stability of Discrete-Time Linear Systems

8.1.1 Positive Discrete-Time Systems

Consider the positive discrete-time linear system:

$$x_{i+1} = Ax_i + Bu_i, \tag{8.1a}$$
$$y_i = Cx_i + Du_i, \tag{8.1b}$$

where $x_i \in \mathbb{R}_+^n$, $u_i \in \mathbb{R}_+^m$, $y_i \in \mathbb{R}_+^p$ are the state, input and output vectors and $A \in \mathbb{R}_+^{n\times n}$, $B \in \mathbb{R}_+^{n\times m}$, $C \in \mathbb{R}_+^{p\times n}$, $D \in \mathbb{R}_+^{p\times m}$.

The positive system (8.1) is called asymptotically stable if the solution

$$x_i = A^i x_0, \tag{8.2}$$

of the equation

$$x_{i+1} = Ax_i, \quad A \in \mathbb{R}_+^{n\times n}, \quad i \in \mathbb{Z}_+, \tag{8.3}$$

satisfies the condition

$$\lim_{i\to\infty} x_i = 0,$$

for every $x_0 \in \mathbb{R}_+^n$.

Theorem 8.1. *The positive system (8.3) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:*

a) All eigenvalues $z_1,\dots,z_n$ of the matrix A satisfy the condition $|z_k| < 1$ for $k = 1,\dots,n$;
b) $\det[I_n z - A] \neq 0$ for $|z| \geq 1$;
c) $\rho(A) < 1$, where $\rho(A)$ is the spectral radius of the matrix A defined by

$$\rho(A) = \max_{1\leq k\leq n} |z_k|.$$

T. Kaczorek: Selected Problems of Fractional Systems Theory, LNCIS 411, pp. 161–188.
springerlink.com

d) All coefficients $\hat{a}_i$, $i = 0, 1, \ldots, n-1$ of the characteristic polynomial

$$p_A(z) = \det[I_n(z+1) - A] = z^n + \hat{a}_{n-1}z^{n-1} + \cdots + \hat{a}_1 z + \hat{a}_0, \tag{8.4}$$

are positive.

e) All principal minors of the matrix

$$\overline{A} = I_n - A = \begin{bmatrix} \overline{a}_{11} & \ldots & \overline{a}_{1n} \\ \vdots & \ddots & \vdots \\ \overline{a}_{n1} & \ldots & \overline{a}_{nn} \end{bmatrix}, \tag{8.5}$$

are positive, i.e.

$$|\overline{a}_{11}| > 0, \quad \begin{vmatrix} \overline{a}_{11} & \overline{a}_{12} \\ \overline{a}_{21} & \overline{a}_{22} \end{vmatrix} > 0, \quad \ldots, \quad \det\overline{A} > 0. \tag{8.6}$$

f) There exists a strictly positive vector $\overline{x} > 0$ (all components are positive) such that

$$[A - I_n]\overline{x} < 0. \tag{8.7}$$

g) All diagonal entries of the matrices $A_{n-k}^{(k)}$ for $k = 1, \ldots, n-1$ are negative where the matrices $A_{n-k}^{(k)}$ are defined as follows

$$A_n^{(0)} = A - I_n = \begin{bmatrix} a_{11}^{(0)} & \ldots & a_{1n}^{(0)} \\ \vdots & \ddots & \vdots \\ a_{n1}^{(0)} & \ldots & a_{nn}^{(0)} \end{bmatrix} = \begin{bmatrix} A_{n-1}^{(0)} & b_{n-1}^{(0)} \\ c_{n-1}^{(0)} & a_{nn}^{(0)} \end{bmatrix},$$

$$A_{n-1}^{(0)} = \begin{bmatrix} a_{11}^{(0)} & \ldots & a_{1,n-1}^{(0)} \\ \vdots & \ddots & \vdots \\ a_{n-1,1}^{(0)} & \ldots & a_{n-1,n-1}^{(0)} \end{bmatrix}, \tag{8.8a}$$

$$\tag{8.8b}$$

$$b_{n-1}^{(0)} = \begin{bmatrix} a_{1n}^{(0)} \\ \vdots \\ a_{n-1,n}^{(0)} \end{bmatrix}, \quad c_{n-1}^{(0)} = \begin{bmatrix} a_{n1}^{(0)} & \ldots & a_{n,n-1}^{(0)} \end{bmatrix},$$

and

$$A_{n-k}^{(k)} = A_{n-k}^{(k-1)} - \frac{b_{n-k}^{(k-1)} c_{n-k}^{(k-1)}}{a_{n-k+1,n-k+1}^{(k-1)}} = \begin{bmatrix} a_{11}^{(k)} & \dots & a_{1,n-k}^{(k)} \\ \vdots & \ddots & \vdots \\ a_{n-k,1}^{(k)} & \dots & a_{n-k,n-k}^{(k)} \end{bmatrix}$$

$$= \begin{bmatrix} A_{n-k-1}^{(k)} & b_{n-k-1}^{(k)} \\ c_{n-k-1}^{(k)} & a_{n-k,n-k}^{(k)} \end{bmatrix},$$

$$b_{n-k-1}^{(k)} = \begin{bmatrix} a_{1,n-k}^{(k)} \\ \vdots \\ a_{n-k-1,n-k}^{(k)} \end{bmatrix}, \quad c_{n-1}^{(k)} = \begin{bmatrix} a_{n-k,1}^{(k)} & \dots & a_{n-k,n-k-1}^{(k)} \end{bmatrix},$$

Proof. The proof of the conditions a)-f) is given in [77, 96]. The proof of the condition g) follows from application of Theorem 8.1 to the positive discrete-time linear system (8.3). □

Theorem 8.2. *The positive system (8.3) is unstable if at least one diagonal entry of the matrix A is greater than* 1.

Proof. The proof is given in [77, 96]. □

Example 8.1. Using the conditions of Theorem 8.1 check the asymptotic stability of the positive system (8.3) with matrix

$$A = \begin{bmatrix} 0.1 & 0.2 & 1 \\ 0 & 0.3 & 0.5 \\ 0 & 0 & 0.4 \end{bmatrix}. \tag{8.9}$$

The matrix (8.9) has the eigenvalues $z_1 = 0.1$, $z_2 = 0.3$, $z_3 = 0.4$.
The condition a) is satisfied and the system is asymptotically stable.
The condition b) is also satisfied since

$$\det[I_n z - A] = \begin{vmatrix} z-0.1 & -0.2 & -1 \\ 0 & z-0.3 & -0.5 \\ 0 & 0 & z-0.4 \end{vmatrix} \neq 0 \quad \text{for} \quad |z| \geq 1.$$

The spectral radius of the matrix is equal to

$$\rho(A) = \max_{1 \leq k \leq 3} |z_k| = 0.4 < 1,$$

and the condition c) is satisfied.

In this case the characteristic polynomial (8.4) has the form

$$p_A(z) = \det[I_n(z+1) - A] = \begin{vmatrix} z+0.9 & -0.2 & -1 \\ 0 & z+0.7 & -0.5 \\ 0 & 0 & z+0.6 \end{vmatrix}$$
$$= z^3 + 2.2z^2 + 1.59z + 0.378,$$

and all its coefficients are positive. Therefore, the condition d) is satisfied.

The condition e) is also satisfied since all principal minors of the matrix

$$\overline{A} = I_n - A = \begin{bmatrix} 0.9 & -0.2 & -1 \\ 0 & 0.7 & -0.5 \\ 0 & 0 & 0.6 \end{bmatrix},$$

are positive

$$M_1 = 0.9, \quad M_2 = \begin{vmatrix} 0.9 & -0.2 \\ 0 & 0.7 \end{vmatrix} = 0.63, \quad \det\overline{A} = 0.378.$$

As the strictly positive vector $\overline{x}$ in (8.7) we choose the equilibrium point of the system (8.1) for $Bu = \mathbb{1}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, i.e.

$$\overline{x} = [I_n - A]^{-1}\mathbb{1}_n = \begin{bmatrix} 0.9 & -0.2 & -1 \\ 0 & 0.7 & -0.5 \\ 0 & 0 & 0.6 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
$$= \frac{1}{0.378}\begin{bmatrix} 1.344 \\ 0.99 \\ 0.63 \end{bmatrix}.$$

This vector satisfies the condition (8.7) since

$$[A - I_n]\overline{x} = [A - I_n][I_n - A]^{-1}\mathbb{1}_n = -\mathbb{1}_n.$$

Therefore, the condition f) is also satisfied.

In this case using (8.8) we obtain the following matrices

$$A_3^{(0)} = \begin{bmatrix} -0.9 & 0.2 & 1 \\ 0 & -0.7 & 0.5 \\ 0 & 0 & -0.6 \end{bmatrix},$$

$$A_2^{(1)} = \begin{bmatrix} -0.9 & 0.2 \\ 0 & -0.7 \end{bmatrix} + \frac{\begin{bmatrix} 1 \\ 0.5 \end{bmatrix}\begin{bmatrix} 0 & 0 \end{bmatrix}}{0.6} = \begin{bmatrix} -0.9 & 0.2 \\ 0 & -0.7 \end{bmatrix},$$

and

$$A_1^{(2)} = \left[-0.9\right] + \frac{\left[0.2\right]\left[0\right]}{0.7} = \left[-0.9\right].$$

All these matrices have negative diagonal entries.

Therefore, the condition *g*) is also satisfied and the positive system is asymptotically stable.

8.1.2 Positive 2D Linear Systems

Consider the positive 2D linear system described by the equations

$$\begin{aligned} x_{i+1,j+1} &= A_0 x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1} \\ &\quad + B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1}, \end{aligned} \tag{8.10a}$$

$$y_{ij} = C x_{ij} + D u_{ij}, \quad i,j \in \mathbb{Z}_+, \tag{8.10b}$$

where $x_{ij} \in \mathbb{R}_+^n$, $u_{ij} \in \mathbb{R}_+^m$, $y_{ij} \in \mathbb{R}_+^p$ are the state, input and output vectors and $A_k \in \mathbb{R}_+^{n\times n}$, $B_k \in \mathbb{R}_+^{n\times m}$, $k = 0,1,2$, $C \in \mathbb{R}_+^{p\times n}$, $D \in \mathbb{R}_+^{p\times m}$.

Definition 8.1. The positive 2D system (8.10) is called asymptotically stable if for any bounded boundary conditions $x_{i0} \in \mathbb{R}_+^n$, $i \in \mathbb{Z}_+$, $x_{0j} \in \mathbb{R}_+^n$, $j \in \mathbb{Z}_+$ and zero inputs $u_{ij} = 0$, $i,j \in \mathbb{Z}_+$ the condition

$$\lim_{i,j\to\infty} x_{ij} = 0 \quad \text{for all} \quad x_{i0} \in \mathbb{R}_+^n,\ x_{0j} \in \mathbb{R}_+^n,\ i,j \in \mathbb{Z}_+, \tag{8.11}$$

is satisfied.

It is well-known [77, 96] that the positive system (model) (8.10a) is asymptotically stable if and only if

$$\det\left[I_n - A_0 z_1 z_2 - A_1 z_1 - A_2 z_2\right] \neq 0\,, \tag{8.12a}$$

for

$$\forall (z_1,z_2) \in \left\{(z_1,z_2) : |z_1| \le 1, |z_2| \le 1\right\}. \tag{8.12b}$$

To the positive asymptotically stable 2D system

$$x_{i+1,j+1} = A_0 x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + B u_{ij}\,, \tag{8.13}$$

we apply the strictly positive input $u_{ij} = \overline{u} > 0$, $\overline{u} \in \mathbb{R}_+^m$. The vector $x_e \in \mathbb{R}_+^n$ is called the equilibrium point (state) of the system (8.13) for $u_{ij} = \overline{u}$, if the condition

$$x_e = [A_0 + A_1 + A_2]\,x_e + B\overline{u}. \tag{8.14}$$

is satisfied.

If $B\overline{u} = \mathbb{1}_n = [1\ 1\ \dots\ 1]^T \in \mathbb{R}_+^n$ then from (8.14) we obtain

$$x_e = [I_n - A_0 - A_1 - A_2]^{-1}\,\mathbb{1}_n > \mathbb{1}_n, \tag{8.15}$$

since $x_e = [A_0 + A_1 + A_2]\,x_e + \mathbb{1}_n$.

Theorem 8.3. *Positive system (8.10) is asymptotically stable if and only if there exists a strictly positive vector $\lambda \in \mathbb{R}_+^n$ such that*

$$[A_0 + A_1 + A_2 - I_n]\,\lambda < 0. \tag{8.16}$$

Proof. If the positive system (8.10) is asymptotically stable then there exists a strictly positive vector λ which satisfies the condition (8.16) since substituting

$$\lambda = [I_n - A_0 - A_1 - A_2]^{-1}\,\mathbb{1}_n$$

into (8.16) we obtain

$$\begin{aligned}[A_0 + A_1 + A_2 - I_n]\,\lambda &= [A_0 + A_1 + A_2 - I_n]\,[I_n - A_0 - A_1 - A_2]^{-1}\,\mathbb{1}_n \\ &= -\mathbb{1}_n < 0. \end{aligned} \tag{8.17}$$

If there exists a strictly positive vector $\lambda > 0$ satisfying the condition (8.16) then

$$[A_0 + A_1 + A_2]\,\lambda < \lambda,$$

and this implies the asymptotic stability of the system (8.10). □

Remark 8.1. The positive system (8.10) is unstable if

$$\det[I_n - A_0 - A_1 - A_2] \le 0, \tag{8.18}$$

since the coefficient

$$a_0 = \det[I_n - A_0 - A_1 - A_2] \tag{8.19}$$

is not positive.

Theorem 8.4. *Let*

$$\det[I_n - A_0 - A_1 - A_2] > 0. \tag{8.20}$$

The positive system (8.10) is asymptotically stable if and only if the sum of entries of every row (column) of the adjoint matrix

$$\mathrm{Adj}\,[I_n - A_0 - A_1 - A_2]$$

is strictly positive, i.e.

$$(\mathrm{Adj}\,[I_n - A_0 - A_1 - A_2])\,\mathbb{1}_n > 0 \quad \textit{or} \quad \mathbb{1}_n^T\,\mathrm{Adj}\,[I_n - A_0 - A_1 - A_2] > 0. \tag{8.21}$$

Proof. If the positive system (8.10) is asymptotically stable then the matrix

$$I_n - A_0 - A_1 - A_2,$$

is invertible and the condition (8.15)is satisfied and this implies (8.21). If the condition (8.21) is satisfied then from the equality

$$\mathrm{Adj}\,[I_n - A_0 - A_1 - A_2]\,\mathbb{1}_n = \mathbb{1}_n \det\,[I_n - A_0 - A_1 - A_2]$$

and (8.20) we have

$$\lambda = (\mathrm{Adj}\,[I_n - A_0 - A_1 - A_2])\,\mathbb{1}_n > 0,$$

By Theorem 8.3 we obtain

$$[I_n - A_0 - A_1 - A_2]\,\lambda = -\mathbb{1}_n \det\,[I_n - A_0 - A_1 - A_2] < 0,$$

what implies the asymptotic stability of the positive system (8.10).
In the case of columns the proof is similar. □

Example 8.2. Consider the positive system (8.10) with the matrices

$$A_0 = \begin{bmatrix} 0.1 & 0.4 \\ 0 & 0.1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0.1 \\ 0 & 0.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.2 & 0.5 \\ 0 & 0.2 \end{bmatrix}. \tag{8.22}$$

Using (8.15) we obtain

$$x_e = [I_n - A_0 - A_1 - A_2]^{-1}\,\mathbb{1}_n = \begin{bmatrix} 0.7 & -1 \\ 0 & 0.6 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{0.42} \begin{bmatrix} 1.6 \\ 0.7 \end{bmatrix} > \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{8.23}$$

For $\lambda = x_e$ we obtain

$$[A_0 + A_1 + A_2 - I_n]\,\lambda = \begin{bmatrix} -0.7 & 1 \\ 0 & -0.6 \end{bmatrix} \frac{1}{0.42} \begin{bmatrix} 1.6 \\ 0.7 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}. \tag{8.24}$$

The condition (8.16) is satisfied and the positive system is asymptotically stable. The same result follows from Theorem 8.4, since

$$(\mathrm{Adj}\,[I_n - A_0 - A_1 - A_2])\,\mathbb{1}_n = \begin{bmatrix} 0.6 & 1 \\ 0 & 0.7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.6 \\ 0.7 \end{bmatrix} > \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{8.25}$$

and

$$\mathbb{1}_n^T\,\mathrm{Adj}\,[I_n - A_0 - A_1 - A_2] = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0.6 & 1 \\ 0 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.6 & 1.7 \end{bmatrix} > \begin{bmatrix} 0 & 0 \end{bmatrix}. \tag{8.26}$$

These considerations can be extended to positive 2D linear systems with delays.

Theorem 8.5. *The positive 2D linear system with delays*

$$x_{i+1,j+1} = \sum_{k=0}^{p}\sum_{l=0}^{q}\left(A_{kl}^{0}x_{i-k,j-l} + A_{kl}^{1}x_{i-k+1,j-l} + A_{kl}^{2}x_{i-k,j-l+1}\right), \quad i,j \in \mathbb{Z}_+, \tag{8.27}$$

is asymptotically stable if and only if the 1D positive system

$$x_{i+1} = \sum_{k=0}^{p}\sum_{l=0}^{q}\left(A_{kl}^{0} + A_{kl}^{1} + A_{kl}^{2}\right)x_i \quad for \quad x_i \in \mathbb{R}_+^n, \quad i \in \mathbb{Z}_+, \tag{8.28}$$

is asymptotically stable, where $x_{ij} \in \mathbb{R}_+^n$ is the state vector and $A_{kl}^t \in \mathbb{R}_+^{n\times n}$, $k = 0,\dots,p$; $l = 0,\dots,q$; $t = 0,1,2$.

8.1.3 Relationship between Asymptotic Stability of 1D and 2D Linear Systems

We shall show that the positive 2D system (8.10) is asymptotically stable if and only if the positive 1D system

$$x_{i+1} = (A_0 + A_1 + A_2)x_i, \quad i \in \mathbb{Z}_+, \tag{8.29}$$

is asymptotically stable.

Lemma 8.1. *[97] Let $P = [p_{ij}] \in \mathbb{R}^{n\times n}$ and $Q = [q_{ij}] \in \mathbb{C}^{n\times n}$ be a complex matrix such that $|Q| = [|q_{ij}|] \le P$ (entries of the matrix P are greater than or equal to the corresponding entries of the matrix Q). Then*

$$\rho(Q) \le \rho(P).$$

where $\rho(M)$ is spectral radius of the matrix M.

Theorem 8.6. *The positive 2D system (8.10) is asymptotically stable if and only if the positive 1D system (8.29) is asymptotically stable.*

Proof. If the positive 1D system (8.29) is asymptotically stable then by Theorem 8.1

$$\rho\left(A_0 + A_1 + A_2\right) < 1. \tag{8.30}$$

Note that for any complex z_1 and z_2 such that $|z_1| \le 1, |z_2| \le 1$ we have

$$|A_0 z_1 z_2| + |A_1 z_1| + |A_2 z_2| \le \rho\left(A_0 + A_1 + A_2\right). \tag{8.31}$$

Using Lemma 8.1, (8.30) and (8.31) we obtain

$$\rho\left(A_0 z_1 z_2 + A_1 z_1 + A_2 z_2\right) \le \rho\left(A_0 + A_1 + A_2\right) < 1, \tag{8.32}$$

what by (8.12) implies the asymptotic stability of the system (8.10).

Theorem 8.6 can be also proved as follows. From condition $f)$ of Theorem 8.1 it follows that the positive 2D system (8.29) is asymptotically stable if and only if the condition (8.16) is satisfied. By Theorem 8.3 this is necessary and sufficient condition for the asymptotic stability of the positive system (8.10). □

For $A_0 = 0$ the equation (8.29) has the form

$$x_{i+1} = (A_1 + A_2)x_i, \quad i \in \mathbb{Z}_+ \ . \tag{8.33}$$

By Theorem 8.6 for $A_0 = 0$ we have the following theorem.

Theorem 8.7. *The following statements are equivalent:*

a) The positive 2D second Fornasini-Marchesini model SF-MM (3.44) is asymptotically stable.
b) The positive 1D system (8.33) is asymptotically stable.

Example 8.3. Consider the positive 2D model (3.44) with the matrices

$$A_1 = \begin{bmatrix} a & 0.1 \\ 0 & 0.2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.3 & 0.6 \\ 0 & b \end{bmatrix}, \quad a, b \geq 0. \tag{8.34}$$

Find values of the parameters a and b, for which the SF-MM (8.34) is asymptotically stable. By Theorem 8.7 the problem can be reduced to finding a, b satisfying the condition

$$\begin{aligned} \det[I_n - z(A_1 + A_2)] &= \begin{bmatrix} 1 - z(a + 0.3) & -z0.7 \\ 0 & 1 - z(b + 0.2) \end{bmatrix} \\ &= \left[1 - z(a + 0.3)\right]\left[1 - z(b + 0.2)\right] \neq 0, \end{aligned} \tag{8.35}$$

for $|z| \leq 1$. It is easy to check that the condition (8.35) is satisfied for $a < 0.7$ and $b < 0.8$.

Theorem 8.8. *The positive 2D model(8.10) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:*

a) All coefficients of the polynomial

$$w_A(z) = \det[I_n(z+1) - (A_0 + A_1 + A_2)] = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0, \tag{8.36}$$

are positive.

b) All principal minors of the matrix

$$[I_n - (A_0 + A_1 + A_2)] = [a_{ij}], \quad i = 1, \ldots, n; \quad j = 1, \ldots, n \tag{8.37}$$

$$M_1 = |a_{11}|, \quad M_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \ldots, M_n = \det[I_n - (A_0 + A_1 + A_2)] > 0,$$

are positive.

Example 8.4. Check the asymptotic stability of the system (8.10) with the matrices (8.22).

Using the condition $a)$ of Theorem 8.8 we obtain

$$\det[I_n(z+1)-(A_0+A_1+A_2)] = \det\begin{bmatrix} z+0.7 & -1 \\ 0 & z+0.6 \end{bmatrix} = z^2+1.3z+0.42.$$

All coefficients of the polynomial are positive and the system is asymptotically stable.

The condition $b)$ of Theorem 8.8 is also satisfied since

$$a_{11}=0.7>0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0.42>0,$$

and the system is asymptotically stable.

Lemma 8.2. *The positive 2D system*

$$x_{i+1,j+1} = A_0 x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1}, \tag{8.38}$$

is asymptotically stable if and only if one of the following positive SF-MM systems:

$$\overline{x}_{i+1,j+1} = \begin{bmatrix} A_1 & A_0 \\ 0 & 0 \end{bmatrix} \overline{x}_{i+1,j} + \begin{bmatrix} A_2 & 0 \\ I_n & 0 \end{bmatrix} \overline{x}_{i,j+1}, \tag{8.39a}$$

$$\hat{x}_{i+1,j+1} = \begin{bmatrix} A_1 & 0 \\ I_n & 0 \end{bmatrix} \hat{x}_{i+1,j} + \begin{bmatrix} A_2 & A_0 \\ 0 & 0 \end{bmatrix} \hat{x}_{i,j+1}, \tag{8.39b}$$

is asymptotically stable.

Proof. Defining

$$\overline{x}_{ij} = \begin{bmatrix} x_{ij} \\ x_{i-1,j} \end{bmatrix}, \quad \hat{x}_{ij} = \begin{bmatrix} x_{ij} \\ x_{i,j-1} \end{bmatrix},$$

we may write (8.38) in the form (8.39). From (8.11) it follows that that positive 2D system (8.38) is asymptotically stable if and only if every of the systems SF-MM (8.39) is asymptotically stable. □

Theorem 8.9. *The positive 2D system (8.38) is asymptotically stable if and only if the positive 1D system*

$$x_{i+1} = \begin{bmatrix} A_1+A_2 & A_0 \\ I_n & 0 \end{bmatrix} x_i, \quad i \in \mathbb{Z}_+, \tag{8.40}$$

is asymptotically stable.

Proof. The proof follows immediately from Theorem 8.6. □

From (3.43a) it follows that

$$A_1 + A_2 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{8.41}$$

and we have the following theorem.

Theorem 8.10. *The positive Roesser model (3.40) is asymptotically stable if and only if the positive 1D system*

$$x_{i+1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} x_i, \quad i \in \mathbb{Z}_+, \tag{8.42}$$

is asymptotically stable.

Remark 8.2. To check the asymptotic stability of the positive 1D system (8.42) the conditions of Theorem 8.1 are recommended.

8.1.4 Positive Fractional Discrete-Time Linear Systems with Delays

Consider the positive fractional system with h delays

$$x_{i+1} = \sum_{j=1}^{i+1} (-1)^{j+1} \binom{\alpha}{j} x_{i-j+1} + \sum_{k=0}^{h} (A_k x_{i-k} + B_k u_{i-k}), \; i \in \mathbb{Z}_+, \tag{8.43a}$$

$$y_i = Cx_i + Du_i, \tag{8.43b}$$

Definition 8.2. The positive fractional system (8.43) is called practical stable if the system

$$x_{i+1} = \sum_{j=1}^{L+1} (-1)^{j+1} \binom{\alpha}{j} x_{i-j+1} + \sum_{k=0}^{h} (A_k x_{i-k} + B_k u_{i-k}), \quad i \in \mathbb{Z}_+, \tag{8.44a}$$

$$y_i = Cx_i + Du_i, \tag{8.44b}$$

is asymptotically stable for any finite number L.

Definition 8.3. The positive fractional system (8.43) is called asymptotically stable if the system (8.44) is practically stable for $L \to \infty$.

Remark 8.3. Note that the positive fractional system (8.43) is a system with increasing number of delays.It is well-known [20, 118] that the asymptotic stability of the positive discrete-time linear systems with delays is independent of the numbers and values of the delays and it depends only of the sum of state matrices of the system.

Theorem 8.11. *The positive fractional system with delays (8.43) is asymptotically stable if and only if the positive system*

$$x_{i+1} = \overline{A}x_i, \quad \overline{A} = I_n + \sum_{k=0}^{h} A_k, \tag{8.45}$$

is asymptotically stable.

Proof. From the Maclaurin series of the function $(1-z)^\alpha$ we have

$$(1-z)^\alpha = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha}{j} z^j = \sum_{j=0}^{\infty} c_j z^j,$$

where

$$c_j = (-1)^j \binom{\alpha}{j},$$

Substituting $z = 1$ we obtain

$$\sum_{j=0}^{\infty} c_j = 0. \tag{8.46}$$

Taking into account that $c_0 = 1$ from (8.46) we obtain

$$\sum_{j=1}^{\infty} c_j = -1. \tag{8.47}$$

Substitution of (8.47) into

$$\overline{A} = \sum_{k=0}^{h} A_k - \sum_{j=1}^{\infty} c_j I_n, \tag{8.48}$$

yields (8.45). □

Theorem 8.12. *The positive fractional system with delays (8.43) is asymptotically stable if and only if one of the following equivalent conditions is satisfied*

a) All coefficients of the polynomial $A = \overline{A} - I_n$

$$\det\left[I_n(z+1) - \overline{A}\right] = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \tag{8.49}$$

are positive.

b) There exists a strictly positive vector $\lambda > 0$ *such that*

$$A\lambda < 0, \quad A = \sum_{k=0}^{h} A_k. \tag{8.50}$$

Proof. Proof of the conditions a), b) follows immediately from the condition d) and f) of Theorem 8.1. □

Remark 8.4. The remaining conditions of Theorem 8.1 can be also used to check the asymptotic stability of the positive fractional system with delays (8.43).

The practical stability of positive fractional discrete-time linear systems without delays has been addressed in [31].

8.1.5 Positive Fractional 2D Linear Systems

Consider the positive fractional 2D linear system

$$\begin{aligned} \Delta^{\alpha,\beta} x_{i+1,j+1} &= A_0 x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1} \\ &\quad + B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1}, \qquad (8.51a) \\ y_{ij} &= C x_{ij} + D u_{ij}, \qquad (8.51b) \end{aligned}$$

where $x_{ij} \in \mathbb{R}_+^n$, $u_{ij} \in \mathbb{R}_+^m$, $y_{ij} \in \mathbb{R}_+^p$ are the state, input and output vectors and $A_k \in \mathbb{R}_+^{n\times n}$, $B_k \in \mathbb{R}_+^{n\times m}$, $k = 0,1,2$, $C \in \mathbb{R}_+^{p\times n}$, $D \in \mathbb{R}_+^{p\times m}$.

Definition 8.4. The positive factional 2D linear system (8.51) is called asymptotically stable if it is practically stable for $L_1 \to \infty$ and $L_2 \to \infty$.

Lemma 8.3. *If* $0 < \alpha < 1$ *and* $1 < \beta < 2$ *(or* $1 < \alpha < 2$ *and* $0 < \beta < 1$*), then*

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} c_{\alpha\beta}(k,l) = 0. \qquad (8.52)$$

Proof. From Maclaurin series of the function $(1-z)^\alpha$, we have

$$(1-z)^\alpha = \sum_{i=0}^{\infty} (-1)^i \binom{\alpha}{i} z^i, \qquad (8.53)$$

Substituting $z = 1$ we obtain

$$\sum_{i=0}^{\infty} (-1)^i \binom{\alpha}{i} = \sum_{i=0}^{\infty} (-1)^i \frac{\alpha(\alpha-1)\cdots(\alpha-i+1)}{i!} = 0 \quad \text{for} \quad \alpha > 0. \qquad (8.54)$$

Using (3.1b) and (8.54) we obtain

$$\begin{aligned} 0 &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} c_{\alpha\beta}(k,l) \\ &= \left(\sum_{k=0}^{\infty} (-1)^k \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \right) \left(\sum_{l=0}^{\infty} (-1)^l \frac{\beta(\beta-1)\cdots(\beta-l+1)}{l!} \right). \end{aligned}$$

□

Theorem 8.13. *The positive fractional 2D system (8.51) is asymptotically stable if and only if the positive 1D system*

$$x_{i+1} = \left[\hat{A} + I_n\right] x_i, \quad \hat{A} = A_0 + A_1 + A_2, \quad x_i \in \mathbb{R}_+^n, \quad i \in \mathbb{Z}_+, \qquad (8.55)$$

is asymptotically stable.

Proof. From (8.51) for $B_0 = B_1 = B_2 = 0$ and (3.79a) we have

$$x_{i+1,j+1} = A_0 x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1} - \sum_{k=0}^{i+1} \sum_{\substack{l=0 \\ k+l>0}}^{j+1} c_{kl} x_{i-k+1,j-l+1}, \tag{8.56}$$

where $c_{kl} = c_{\alpha\beta}(k,l)$.

By Theorem 8.5 the positive 2D linear system with delays is asymptotically stable if and only if the positive 1D system

$$x_{i+1} = \left[\hat{A} - \sum_{k=0}^{\infty} \sum_{\substack{l=0 \\ k+l>0}}^{\infty} c_{kl} I_n \right] x_i, \quad x_i \in \mathbb{R}_+^n, \quad i \in \mathbb{Z}_+, \tag{8.57}$$

is asymptotically stable. From (3.76b) we have $c_{00} = c_{\alpha\beta}(0,0) = 1$ and from (8.52) we obtain

$$\sum_{k=0}^{\infty} \sum_{\substack{l=0 \\ k+l>0}}^{\infty} c_{kl} I_n = -I_n. \tag{8.58}$$

Substituting of (8.58) into (8.57) yields (8.55). □

Theorem 8.14. *The positive fraction 2D system (8.51) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:*

a) The eigenvalues $z_1, \dots, z_n$ of the matrix $\hat{A} + I_n$ have module less than 1, i.e $|z_k| < 1$, $k = 1, \dots, n$.
b) All coefficients of the characteristic polynomial of the matrix $\hat{A}$ are positive.
c) All principal minors of the matrix $-\hat{A}$ are positive.

Proof. The proof follows immediately from Theorem 8.1. □

Theorem 8.15. *The positive fractional 2D system (8.51) is unstable if at least one entry on the diagonal of the matrix $\hat{A}$ is positive.*

Proof. If at least one entry of the diagonal of $\hat{A}$ is positive then at least one entry of the diagonal of the matrix $\hat{A} + I_n$ is greater than 1 and this by Theorem 8.2 implies the instability of the system (8.55). □

Example 8.5. Using Theorem 8.14, check the asymptotic stability of the positive fractional system (8.51), for $\alpha = 0.3$ and $\beta = 1.2$, with the matrices:

$$A_0 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ 0.2 & -1.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.2 & 0 \\ 0.2 & 0.1 \end{bmatrix}. \tag{8.59}$$

The fractional system is positive since the matrices

$$\overline{A}_0 = A_0 - \alpha\beta I_n = \begin{bmatrix} 0.04 & 0 \\ 0 & 0.14 \end{bmatrix}, \tag{8.60a}$$

$$\overline{A}_1 = A_1 + \beta I_n = \begin{bmatrix} 0.2 & 0 \\ 0.2 & 0.1 \end{bmatrix}, \tag{8.60b}$$

$$\overline{A}_2 = A_2 + \alpha I_n = \begin{bmatrix} 0.1 & 0 \\ 0.2 & 0.4 \end{bmatrix}, \tag{8.60c}$$

(8.60d)

have nonnegative entries. In this case

$$\hat{A} = A_0 + A_1 + A_2 = \begin{bmatrix} -0.8 & 0 \\ 0.4 & -0.5 \end{bmatrix}, \tag{8.61}$$

the condition $a)$ of Theorem 8.14 is satisfied since the eigenvalues of the matrix

$$\hat{A} + I_n = \begin{bmatrix} 0.2 & 0 \\ 0.4 & 0.5 \end{bmatrix}, \tag{8.62}$$

have module less than 1, i.e. $z_1 = 0.2$, $z_2 = 0.5$.

The condition $b)$ of Theorem 8.14 is also satisfied since the coefficients of the polynomial (8.61)

$$\det\left[I_n z - \hat{A}\right] = \begin{bmatrix} z+0.8 & 0 \\ -0.4 & z+0.5 \end{bmatrix} = z^2 + 1.3z + 0.4\,, \tag{8.63}$$

are positive.

All principal minors of the matrix

$$-\hat{A} = \begin{bmatrix} 0.8 & 0 \\ -0.4 & 0.5 \end{bmatrix} \tag{8.64}$$

are positive since $\Delta_1 = 0.8$ and $\Delta_2 = 0.4$.

The conditions of Theorem 8.14 are satisfied and the positive fractional system (8.51) with (8.59) is asymptotically stable.

Example 8.6. Using Theorem 8.15 we shall show that the positive fractional 2D system (8.51) for $\alpha = 0.5$ and $\beta = 1.2$ with the matrices:

$$A_0 = \begin{bmatrix} 0.6 & 0 \\ 0.1 & 0.7 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.1 & 0.3 \\ 0 & -0.2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.4 & 0.2 \\ 0 & -0.5 \end{bmatrix}, \tag{8.65}$$

is unstable.

In this case the matrix

$$\hat{A} = A_0 + A_1 + A_2 = \begin{bmatrix} 0.1 & 0.5 \\ 0.1 & 0 \end{bmatrix}, \tag{8.66}$$

has one positive diagonal entry. By Theorem 8.15 the positive fractional system is unstable. The same result follows from Theorem 8.14.

8.2 Practical Stability of Fractional Systems

8.2.1 Positive Fractional 1D Systems

Consider the positive fractional discrete-time linear system

$$x_{k+1} = A_\alpha x_k + \sum_{j=1}^{k} (-1)^j \binom{\alpha}{j+1} x_{k-j} + Bu_k, \quad k \in \mathbb{Z}_+, \tag{8.67a}$$

$$y_k = Cx_k + Du_k, \tag{8.67b}$$

where $A_\alpha = A + \alpha I_n$ and the system

$$x_{k+1} + \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} x_{k-j+1} = Ax_k + Bu_k, \quad k \in \mathbb{Z}_+, \tag{8.68a}$$

$$y_k = Cx_k + Du_k, \tag{8.68b}$$

where $x_k \in \mathbb{R}_+^n$, $u_k \in \mathbb{R}_+^m$, $y_k \in \mathbb{R}_+^p$ are the state, input and output vectors and $A \in \mathbb{R}_+^{n\times n}$, $B \in \mathbb{R}_+^{n\times m}$, $C \in \mathbb{R}_+^{p\times n}$, $D \in \mathbb{R}_+^{p\times m}$.

Definition 8.5. The positive fractional system (8.68) is called practically stable if the system (8.67) is asymptotically stable for any finite h.

Defining the new state vector

$$\tilde{x}_k = \begin{bmatrix} x_k \\ x_{k-1} \\ \vdots \\ x_{k-h} \end{bmatrix}, \tag{8.69}$$

we may write the equations (8.67) in the form:

$$\tilde{x}_{i+1} = \tilde{A}\tilde{x}_i + \tilde{B}u_i, \quad k \in \mathbb{Z}_+, \tag{8.70a}$$

$$y_i = \tilde{C}x_i + \tilde{D}u_i, \tag{8.70b}$$

where

$$\tilde{A} = \begin{bmatrix} A_\alpha & c_1 I_n & \dots & c_h I_n \\ I_n & 0 & \dots & 0 \\ 0 & I_n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}_+^{\tilde{n} \times \tilde{n}}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}_+^{\tilde{n} \times m}, \tag{8.71a}$$

$$\tilde{C} = \begin{bmatrix} C \; 0 \dots 0 \end{bmatrix} \in \mathbb{R}_+^{p \times \tilde{n}}, \quad \tilde{D} = D \in \mathbb{R}_+^{p \times m}, \quad \tilde{n} = (1+h)n, \tag{8.71b}$$

$$c_j = (-1)^j \binom{\alpha}{j+1} \quad \text{for} \quad j = 1, \dots .$$

Theorem 8.16. *The positive fractional system(8.68) is practically stable if and only if one of the following equivalent conditions is satisfied:*

a) All eigenvalues $\tilde{z}_1, \dots, \tilde{z}_{\tilde{n}}$ *of the matrix* $\tilde{A}$ *have module less than* 1*, i.e.* $|\tilde{z}_k| < 1$ *for* $k = 1, \dots, \tilde{n}$*;*
b) $\det[zI_{\tilde{n}} - \tilde{A}] \neq 0$ *for* $|z| \geq 1$*;*
c) $\rho(\tilde{A}) < 1$ *where* $\rho(\tilde{A})$ *is the spectral radius of the matrix* $\tilde{A}$ *defined by*

$$\rho(\tilde{A}) = \max_{1 \leq k \leq \tilde{n}} |\tilde{z}_k|.$$

d) All coefficients $\tilde{a}_i$*,* $i = 0, 1, \dots, \tilde{n} - 1$ *of the characteristic polynomial*

$$p_{\tilde{A}(z)} = \det\left[I_{\tilde{n}}(z+1) - \tilde{A}\right] = z^{\tilde{n}} + \tilde{a}_{\tilde{n}-1} z^{\tilde{n}-1} + \dots + \tilde{a}_1 z + \tilde{a}_0, \tag{8.72}$$

of the matrix $[\tilde{A} - I_{\tilde{n}}]$ *are positive.*
e) All principal minors of the matrix

$$I_{\tilde{n}} - \tilde{A} = \begin{bmatrix} \tilde{a}_{11} & \dots & \tilde{a}_{1\tilde{n}} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{\tilde{n}1} & \dots & \tilde{a}_{\tilde{n}\tilde{n}} \end{bmatrix}, \tag{8.73}$$

are positive, i.e.

$$|\tilde{a}_{11}| > 0, \quad \begin{vmatrix} \tilde{a}_{11} & \tilde{a}_{12} \\ \tilde{a}_{21} & \tilde{a}_{22} \end{vmatrix} > 0, \quad \dots, \quad \det\left[I_{\tilde{n}} - \tilde{A}\right] > 0. \tag{8.74}$$

f) There exist strictly positive vectors $\overline{x}_i \in \mathbb{R}_+^n$*,* $i = 0, \dots, h$*, satisfying the condition*

$$\overline{x}_0 < \overline{x}_1 < \dots < \overline{x}_{h-1} < \overline{x}_h, \tag{8.75a}$$

such that

$$A_\alpha \overline{x}_0 + c_1 \overline{x}_1 + \dots + c_h \overline{x}_h < \overline{x}_0. \tag{8.75b}$$

Proof. The first *a*)-*e*) conditions follows immediately from the corresponding conditions of Theorem 8.1. Using (8.7) for the matrix $\tilde{A}$ we obtain:

$$\begin{bmatrix} A_\alpha & c_1 I_n & \dots & c_h I_n \\ I_n & 0 & \dots & 0 \\ 0 & I_n & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \overline{x}_0 \\ \overline{x}_1 \\ \overline{x}_2 \\ \vdots \\ \overline{x}_h \end{bmatrix} < \begin{bmatrix} \overline{x}_0 \\ \overline{x}_1 \\ \overline{x}_2 \\ \vdots \\ \overline{x}_h \end{bmatrix}. \tag{8.76}$$

From (8.76) we obtain the conditions (8.75). □

Theorem 8.17. *If the positive fractional system (8.68) is practically stable then the sum of entries of every row (column) of the adjoint matrix* $\mathrm{Adj}[I_{\tilde{n}} - \tilde{A}]$ *is strictly positive*

$$\mathrm{Adj}\left[I_{\tilde{n}} - \tilde{A}\right] \mathbb{1}_{\tilde{n}} > 0 \quad \left(\mathbb{1}_{\tilde{n}}^T \,\mathrm{Adj}\left[I_{\tilde{n}} - \tilde{A}\right] > 0\right). \tag{8.77}$$

where $\mathbb{1}_{\tilde{n}} = [1, \dots, 1]^T$.

Proof. If the system (8.70) is asymptotically stable then the vector

$$\overline{x} = \left[I_{\tilde{n}} - \tilde{A}\right]^{-1} \mathbb{1}_{\tilde{n}} > 0 \tag{8.78}$$

is the strictly positive equilibrium point of the system for $\tilde{B}u = \mathbb{1}_{\tilde{n}}$. For the positive system

$$\det\left[I_{\tilde{n}} - \tilde{A}\right] > 0, \tag{8.79}$$

since $\tilde{a}_0 = \det\left[I_{\tilde{n}} - \tilde{A}\right]$ in (8.72). The conditions (8.78) and (8.79) implies (8.77). □

Example 8.7. Check the practical stability of the positive fractional system

$$\Delta^\alpha x_{k+1} = 0.1 x_k, \quad k \in \mathbb{Z}_+, \tag{8.80}$$

for $\alpha = 0.5$ and $h = 2$.

Using

$$c_j(\alpha) = (-1)^j \binom{\alpha}{j+1}, \tag{8.81}$$

and (1.8), (8.71) we obtain:

$$c_1 = \frac{1}{8}, \quad c_2 = \frac{1}{16}, \quad A_\alpha = 0.6,$$

and

$$\tilde{A} = \begin{bmatrix} 0.6 & \frac{1}{8} & \frac{1}{16} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{8.82}$$

In this case the characteristic polynomial (8.72) has the form

$$p_{\tilde{A}} = \left[I_{\tilde{n}}(z+1) - \tilde{A}\right] = z^3 + 2.4z^2 + 1.675z + 0.2125. \tag{8.83}$$

All coefficients of the polynomial (8.83) are positive and by Theorem 8.16 the system is practically stable. Using (8.77) we obtain

$$\text{Adj}\left[I_{\tilde{n}} - \tilde{A}\right] \mathbb{1}_{\tilde{n}} = \left(\text{Adj}\begin{bmatrix} 0.4 & -\frac{1}{8} & -\frac{1}{16} \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.2500 \\ 1.4625 \\ 1.6750 \end{bmatrix}. \tag{8.84}$$

It is easy to check that the condition (8.78) is satisfied.

Theorem 8.18. *The positive fractional system (8.68) is practically stable if and only if the positive 1D system*

$$x_{k+1} = A_\alpha x_k, \quad k \in \mathbb{Z}_+, \tag{8.85}$$

is asymptotically stable.

Proof. From (8.75b) we have

$$(A_\alpha - I_n)\overline{x}_0 + c_1\overline{x}_1 + \cdots + c_h\overline{x}_h < 0. \tag{8.86}$$

Note that the equality (8.86) is satisfied only if there exists a strictly positive vector $\overline{x}_0 \in \mathbb{R}_+^n$ such that

$$(A_\alpha - I_n)\overline{x}_0 < 0, \tag{8.87}$$

where $c_1\overline{x}_1 + \cdots + c_h\overline{x}_h > 0$. □

Corollary 8.1. *The positive fractional system (8.68) is unstable for any finite h if the positive system (8.85) is unstable.*

Theorem 8.19. *The positive fractional system (8.68) is unstable if at least one diagonal entry of the matrix A_α is greater than* 1.

Proof. The proof follows immediately from Theorems 8.2 and 8.18. □

Example 8.8. Consider the positive fractional system

$$\Delta^\alpha x_{k+1} = \begin{bmatrix} -0.5 & 1 \\ 2 & 0.5 \end{bmatrix} x_k, \quad k \in \mathbb{Z}_+, \tag{8.88}$$

for $\alpha = 0.8$ and any finite h.

In this case $n = 2$ and

$$A_\alpha = A + \alpha I_n = \begin{bmatrix} 0.3 & 1 \\ 2 & 1.3 \end{bmatrix}. \tag{8.89}$$

By Theorem 8.19 the positive fractional system is unstable for any finite h since the matrix (8.89) has one diagonal entry greater than 1. The same result follows from the condition $d)$ of Theorem 8.1, since the polynomial

$$p_{\tilde{A}} = z^2 + 0.4z - 2.21,$$

has one negative coefficient $\tilde{a}_0 = -2.21$.

The practical stability of the positive system has been addressed in [31].

8.2.2 Positive Fractional 2D Systems

Consider the fractional 2D linear system

$$x_{i+1,j+1} + \sum_{k=0}^{i+1} \sum_{\substack{l=0 \\ k+l>0}}^{j-k+1} c_\alpha(k,l) x_{i-k+1,j-l+1} = A_0 x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1}. \tag{8.90}$$

Defining the new state vector

$$\tilde{x}_{ij} = \left[x_{ij}^T \; x_{i-1,j}^T \; \dots \; x_{i-L_1,j}^T \; x_{i,j-1}^T \; \dots \; x_{i-L_1,j-1}^T \; \dots \; x_{i,j-L_2}^T \; \dots \; x_{i-L_1,j-L_2}^T \right]$$
$$\tilde{x}_{ij} \in \mathbb{R}^{\tilde{N}}, \quad \tilde{N} = (L_1+1)(L_2+1)n, \quad i,j \in \mathbb{Z}_+ \tag{8.91}$$

we may write the equation

$$x_{i+1,j+1} + \sum_{k=0}^{L_1+1} \sum_{\substack{l=0 \\ k+l>0}}^{L_2-k+1} c_\alpha(k,l) x_{i-k+1,j-l+1} = A_0 x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + B_0 u_{ij} + B_1 u_{i+1,j} + B_2 u_{i,j+1}. \tag{8.92}$$

in the form

$$\tilde{x}_{i+1,j+1} = \tilde{A}_0 \tilde{x}_{ij} + \tilde{A}_1 \tilde{x}_{i+1,j} + \tilde{A}_2 \tilde{x}_{i,j+1}, \quad i,j \in \mathbb{Z}_+, \tag{8.93}$$

where

$$\tilde{A}_0 = \begin{bmatrix} \overline{A}_0 & I_n c_{11} & \dots & I_n c_{L_1+1,1} & \dots & I_n c_{1,L_2+1} & \dots & I_n c_{L_1+1,L_2+1} \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \end{bmatrix}, \tag{8.94a}$$

$$\tilde{A}_1 = \begin{bmatrix} \overline{A}_1 & 0 & \dots & 0 & I_n c_{02} & \dots & 0 & I_n c_{03} & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ I_n & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & I_n & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I_n & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix}, \tag{8.94b}$$

$$\tilde{A}_2 = \begin{bmatrix} \overline{A}_2 & I_n c_{20} & \dots & I_n c_{L_1 0} & I_n c_{L_1+1,0} & \dots & 0 & 0 & \dots & 0 & 0 \\ I_n & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & I_n & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I_n & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & I_n & 0 \end{bmatrix}, \qquad (8.94c)$$

Theorem 8.20. *The 2D system (8.93) is positive if and only if*

$$\overline{A}_k = A_k - \alpha I_n \in \mathbb{R}_+^{n\times n}, \quad k = 0,1,2. \qquad (8.95)$$

Proof. The proof follows from (8.93) and (8.94) and the fact that the system is positive if and only if the conditions (8.95) are met. □

Definition 8.6. The positive fractional 2D system (8.51) is called practically stable if the system (8.92) is asymptotically stable.

Theorem 8.21. *The positive fractional 2D linear system (8.51) is practically stable if and only if one of the following equivalent conditions is satisfied:*

a)

$$\det\left[I_{\tilde{N}} - \tilde{A}_0 z_1 z_2 - \tilde{A}_1 z_1 - \tilde{A}_2 z_2\right] \neq 0 \qquad (8.96a)$$

for

$$\forall (z_1, z_2) \in \{(z_1,z_2) : |z_1| \leq 1, |z_2| \leq 1\}. \qquad (8.96b)$$

b) There exists a strictly positive vector $\lambda \in \mathbb{R}_+^{\tilde{N}}$ such that

$$\left[\tilde{A}_0 + \tilde{A}_1 + \tilde{A}_2\right]\lambda < 0. \qquad (8.97)$$

c) The positive 1D system

$$x_{i+1} = \left(\tilde{A}_0 + \tilde{A}_1 + \tilde{A}_2 - I_{\tilde{N}}\right) x_i, \quad i \in \mathbb{Z}_+, \qquad (8.98)$$

is asymptotically stable.

d) The positive 1D system

$$x_{i+1} = \begin{bmatrix} \left(\tilde{A}_1 + \tilde{A}_2\right) & \tilde{A}_0 \\ I_{\tilde{N}} & 0 \end{bmatrix} x_i, \quad i \in \mathbb{Z}_+, \qquad (8.99)$$

is asymptotically stable.

Theorem 8.22. *The positive fractional 2D system (8.51) is practically stable if the positive 2D system*

$$\tilde{x}_{i+1,j+1} = \tilde{A}_0\tilde{x}_{ij} + \tilde{A}_1\tilde{x}_{i+1,j} + \tilde{A}_2\tilde{x}_{i,j+1}, \tag{8.100}$$

is asymptotically stable.

Proof. The proof follows immediately from (8.93) and (8.94). □

Corollary 8.2. *The positive fractional system (8.51) is unstable for any L_1 and L_2 if the positive 2D system (8.100) is unstable.*

Theorem 8.23. *The positive fractional 2D system (8.51) is unstable if at least one diagonal entry of the matrix $\overline{A}_1 + \overline{A}_2$ is greater than* 1.

Proof. The proof follows from the structure of the matrices (8.94).

It is well-known that the system (8.99) is unstable if at least one diagonal entry of the matrix $\tilde{A}_1 + \tilde{A}_2$ is greater than 1. From structure of the matrices $\tilde{A}_1$ and $\tilde{A}_2$ defined by (8.94) it follows that at least one diagonal entry of the matrix $\tilde{A}_1 + \tilde{A}_2$ is greater than 1 if and only if at least one diagonal entry of the matrix $\overline{A}_1 + \overline{A}_2$ is greater than 1. By Theorem 8.16 the positive fractional 2D system (8.51) is practically stable if and only if the positive 1D system (8.99) is asymptotically stable. □

Theorem 8.24. *The positive fractional 2D system(8.51) is unstable if*

$$A_k \in \mathbb{R}_+^{n\times n}, \quad k = 1, 2. \tag{8.101}$$

Proof. By Theorem 3.17 the fractional 2D system (8.51) for $0 < \alpha < 1$ and $1 < \beta < 2$ (or $1 < \alpha < 2$ and $0 < \beta < 1$) is positive if and only if the conditions (3.99) are satisfied. From (8.90) it follows that the matrix

$$\overline{A}_1 + \overline{A}_2 = A_1 + A_2 + (\alpha + \beta)I_n, \tag{8.102}$$

has all diagonal entries greater than 1 if the condition (8.101) is satisfied. In this case by Theorem 8.23 the positive fractional 2D system (8.51) is unstable. □

8.3 Asymptotic Stability of Continuous-Time Linear Systems

8.3.1 Positive Continuous-Time Linear Systems

Consider the autonomous continuous-time linear system

$$\dot{x}(t) = Ax(t), \tag{8.103}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $A = [a_{ij}] \in \mathbb{R}^{n\times n}$.

The system (8.103) is called (internally) positive if $x(t) \in \mathbb{R}_+^n$, $t \geq 0$ for any initial conditions $x(0) \in \mathbb{R}_+^n$.
The system (8.103) is positive if and only if A is a Metzler matrix [77].

It is assumed that all diagonal entries a_{ii}, $i = 1, \ldots, n$ of the Metzler matrix are negative, otherwise the positive system (8.103) is unstable [77].

Theorem 8.25. *The positive system (8.103) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:*

a) All coefficients of the characteristic polynomial

$$\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0, \tag{8.104}$$

are positive, i.e. $a_i > 0$, $i = 0, 1, \ldots, n-1$.

b) All principal minors M_i, $i = 1, \ldots, n$ *of the matrix* $-A$ *are positive, i.e.*

$$M_1 = |-a_{11}| > 0, \quad M_2 = \begin{vmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{vmatrix} > 0, \quad \ldots, \quad M_n = \det[-A] > 0. \tag{8.105}$$

c) The diagonal entries of the matrices

$$A_{n-k}^{(k)} \quad for \quad k = 1, \ldots, n-1\,, \tag{8.106}$$

are negative, where $A_{n-k}^{(k)}$ *are defined as follows*

$$\begin{aligned} A_n^{(0)} &= A = \begin{bmatrix} a_{11}^{(0)} & \ldots & a_{1n}^{(0)} \\ \vdots & \ddots & \vdots \\ a_{n1}^{(0)} & \ldots & a_{nn}^{(0)} \end{bmatrix} = \begin{bmatrix} A_{n-1}^{(0)} & b_{n-1}^{(0)} \\ c_{n-1}^{(0)} & a_{nn}^{(0)} \end{bmatrix}, \\ A_{n-1}^{(0)} &= \begin{bmatrix} a_{11}^{(0)} & \ldots & a_{1,n-1}^{(0)} \\ \vdots & \ddots & \vdots \\ a_{n-1,1}^{(0)} & \ldots & a_{n-1,n-1}^{(0)} \end{bmatrix}, \\ b_{n-1}^{(0)} &= \begin{bmatrix} a_{1n}^{(0)} \\ \vdots \\ a_{n-1,n}^{(0)} \end{bmatrix}, \quad c_{n-1}^{(0)} = \begin{bmatrix} a_{n1}^{(0)} & \ldots & a_{n,n-1}^{(0)} \end{bmatrix}, \end{aligned} \tag{8.107}$$

and

$$\begin{aligned} A_{n-k}^{(k)} &= A_{n-k}^{(k-1)} - \frac{b_{n-k}^{(k-1)} c_{n-k}^{(k-1)}}{a_{n-k+1,n-k+1}^{(k-1)}} = \begin{bmatrix} a_{11}^{(k)} & \ldots & a_{1,n-k}^{(k)} \\ \vdots & \ddots & \vdots \\ a_{n-k,1}^{(k)} & \ldots & a_{n-k,n-k}^{(k)} \end{bmatrix} \\ &= \begin{bmatrix} A_{n-k-1}^{(k)} & b_{n-k-1}^{(k)} \\ c_{n-k-1}^{(k)} & a_{n-k,n-k}^{(k)} \end{bmatrix}, \end{aligned}$$

$$b_{n-k-1}^{(k)} = \begin{bmatrix} a_{1,n-k}^{(k)} \\ \vdots \\ a_{n-k-1,n-k}^{(k)} \end{bmatrix}, \quad c_{n-1}^{(k)} = \begin{bmatrix} a_{n-k,1}^{(k)} \ \dots \ a_{n-k,n-k-1}^{(k)} \end{bmatrix},$$

Proof. The proof of the conditions a) and b) are given in [77]. To simplify the notation in proof of the condition c) we shall assume $n = 3$. We shall show the equivalence of the conditions b) and c) of Theorem 8.25.

From (8.105) we have

$$\begin{aligned}
(-1)^1 M_1 &= a_{11} < 0, \\
(-1)^2 M_2 &= a_{11}a_{22} - a_{12}a_{21} > 0, \\
(-1)^3 M_3 &= \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\
&= a_{33} \det \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \frac{1}{a_{33}} \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \begin{bmatrix} a_{31} & a_{32} \end{bmatrix} \right\} \\
&= a_{33} \det \left\{ \frac{1}{a_{33}} \begin{bmatrix} a_{11}a_{33} - a_{13}a_{31} & a_{12}a_{33} - a_{13}a_{32} \\ a_{21}a_{33} - a_{23}a_{31} & a_{22}a_{33} - a_{23}a_{32} \end{bmatrix} \right\} < 0.
\end{aligned} \tag{8.108}$$

By the condition c) the diagonal entries of the matrices

$$\begin{aligned}
A_2^{(1)} &= A_2^{(0)} - \frac{b_2^{(0)} c_2^{(0)}}{a_{33}^{(0)}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \frac{1}{a_{33}} \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \begin{bmatrix} a_{31} & a_{32} \end{bmatrix} \\
&= \frac{1}{a_{33}} \begin{bmatrix} a_{11}a_{33} - a_{13}a_{31} & a_{12}a_{33} - a_{13}a_{32} \\ a_{21}a_{33} - a_{23}a_{31} & a_{22}a_{33} - a_{23}a_{32} \end{bmatrix} = \begin{bmatrix} \overline{a}_{11} & \overline{a}_{12} \\ \overline{a}_{21} & \overline{a}_{22} \end{bmatrix}, \\
A_1^{(2)} &= A_1^{(1)} - \frac{b_1^{(1)} c_1^{(1)}}{a_{22}^{(1)}} = \overline{a}_{11} - \frac{\overline{a}_{12}\overline{a}_{21}}{\overline{a}_{22}} = \frac{\overline{a}_{11}\overline{a}_{22} - \overline{a}_{12}\overline{a}_{21}}{\overline{a}_{22}},
\end{aligned} \tag{8.109}$$

are negative.

Note that the condition (8.108) and (8.109) are equivalent since $a_{ii} < 0$, $i = 1, 2, 3$ and the inequations

$$\begin{aligned}
\overline{a}_{11} &= \frac{a_{11}a_{33} - a_{13}a_{31}}{a_{33}} < 0, \\
\overline{a}_{22} &= \frac{a_{22}a_{33} - a_{23}a_{32}}{a_{33}} < 0, \\
&\frac{a_{11}a_{22} - a_{12}a_{21}}{a_{22}} < 0,
\end{aligned}$$

are satisfied if and only if

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} > 0,$$

and

$$\det\left\{\frac{1}{a_{33}}\begin{bmatrix} a_{11}a_{33}-a_{13}a_{31} & a_{12}a_{31}-a_{13}a_{32} \\ a_{21}a_{33}-a_{23}a_{31} & a_{22}a_{33}-a_{13}a_{32} \end{bmatrix}\right\} < 0.$$

The proof can be also accomplished by induction with respect to n. □

An other proof of Theorem 8.25 is given in [211].

Example 8.9. Using Theorem 8.25 check the asymptotic stability of the positive system (8.103) with the matrix

$$A = \begin{bmatrix} -0.5 & 0.1 \\ 0.2 & -0.6 \end{bmatrix}. \tag{8.110}$$

Using (8.104) we obtain

$$\det[I_n s - A] = \begin{vmatrix} s+0.5 & -0.1 \\ -0.2 & s+0.6 \end{vmatrix} = s^2 + 1.1s + 0.28. \tag{8.111}$$

All coefficients of the polynomial are positive and the condition *a*) is satisfied. The condition *b*) is also satisfied since

$$M_1 = 0.5, \quad M_2 = \det[-A] = \begin{vmatrix} 0.5 & -0.1 \\ -0.2 & 0.6 \end{vmatrix} = 0.28.$$

Using (8.109) for $n = 2$ we obtain

$$A_1^{(1)} = a_{11} - \frac{a_{12}a_{21}}{a_{22}} = -0.5 + \frac{0.1 \cdot 0.2}{0.6} = -\frac{0.28}{0.6} < 0.$$

Therefore, the condition of Theorem 8.25 are met and the positive system (8.103) with (8.110) is asymptotically stable.

8.3.2 *Asymptotic Stability of Positive Continuous-Time Systems with Delays*

Theorem 8.26. *The positive system*

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{q} A_k x(t-d_k) + Bu(t), \tag{8.112a}$$

$$y(t) = Cx(t) + Du(t), \tag{8.112b}$$

is asymptotically stable if and only if there exists a strictly positive vector $\lambda \in \mathbb{R}_+^n$ *satisfying the condition*

$$A\lambda < 0, \quad A = \sum_{k=0}^{q} A_k \ . \tag{8.113}$$

Proof. First we shall show that if the system (8.112) is asymptotically stable then there exists a strictly positive vector $\lambda \in \mathbb{R}_+^n$ satisfying the condition (8.113). Integrating in the interval $[0,\infty]$ the equation (8.112a) for $B = 0$ we obtain

$$\int_0^\infty \dot{x}(t)dt = A_0 \int_0^\infty x(t)dt + \sum_{k=1}^q A_k \int_0^\infty x(t-d_k)dt$$

and

$$x(\infty) - x(0) - \sum_{k=1}^q A_k \int_{-d_k}^0 x(t)dt = A \int_0^\infty x(t)dt \quad . \tag{8.114}$$

For asymptotically stable positive system

$$x(\infty) = 0, \quad x(0) + \sum_{k=1}^q A_k \int_{-d_k}^0 x(t)dt > 0, \quad \int_0^\infty x(t)dt > 0,$$

and (8.114) we have (8.113) for

$$\lambda = \int_0^\infty x(t)dt$$

If the condition (8.113) is satisfied then positive system (8.112) is asymptotically stable. It is well-known that the system (8.112) is asymptotically stable if and only if the corresponding transposed system

$$\dot{x}(t) = A_0^T x(t) + \sum_{k=1}^q A_k^T x(t-d_k) \tag{8.115}$$

is asymptotically stable. As a Lapunnov function for the positive system (8.115) we choose the function

$$V(x) = x^T(t)\lambda + \sum_{k=1}^q \int_{t-d_k}^t x^T(\tau)d\tau A_k \lambda \quad , \tag{8.116}$$

which is positive for any nonzero $x(t) \in \mathbb{R}_+^n$. Using (8.115) and (8.116) we obtain

$$\begin{aligned}\dot{V}(x) &= \dot{x}^T(t)\lambda + \sum_{k=1}^q \left(x^T(t) - x^T(t-d_k)\right) A_k \lambda \\ &= x^T(t)A_0\lambda + \sum_{k=1}^q x^T(t-d_k)A_k\lambda + \sum_{k=1}^q \left(x^T(t) - x^T(t-d_k)\right) A_k \lambda \\ &= x^T(t)A\lambda. \end{aligned} \tag{8.117}$$

If the condition (8.113) is satisfied then from (8.117) we have $\dot{V}(x) < 0$ and the system (8.112) is asymptotically stable. □

Remark 8.5. As the strictly positive vector λ we may choose the equilibrium point

$$x_e = -A^{-1}Bu \tag{8.118}$$

since

$$A\lambda = A\left(-A^{-1}Bu\right) = -Bu < 0 \quad \text{for} \quad Bu > 0 \ . \tag{8.119}$$

Theorem 8.27. *The positive system with delays (8.112) is asymptotically stable if and only if the positive system without delays*

$$\dot{x} = Ax, \quad A = \sum_{k=0}^{q} A \in M_n \tag{8.120}$$

is asymptotically stable.

Proof. The positive system (8.120) is asymptotically stable if and only if there exists a strictly positive vector $\lambda \in \mathbb{R}_+^n$ such that the condition (8.113) is satisfied [84]. By Theorem 8.26 the system (8.112) is asymptotically stable if and only if the positive system (8.120) is asymptotically stable. By Theorem 8.27 the asymptotic stability problem of the positive system with delays (8.112) can be reduced to the asymptotic stability problem of the corresponding positive system without delays (8.120). □

To check the asymptotic stability of the positive system (8.112) the following Theorem can be used [77, 96].

Theorem 8.28. *The positive system with delays (8.112) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:*

a) All eigenvalues $s_1, s_2, \ldots, s_n$ of the matrix A has negative real parts, i.e. $\operatorname{Re} s_k < 0$, $k = 1, \ldots, n$.
b) All coefficients of the characteristic polynomial of the matrix A are positive.
c) All principal minors of the matrix

$$-A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \tag{8.121}$$

are positive, i.e.

$$|a_{11}| > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \dots \quad , \det[-A] > 0. \tag{8.122}$$

Example 8.10. Using the conditions $b)$ and $c)$ of Theorem 8.28 check the asymptotic stability of the positive system (8.112) for $q = 1$ with the matrices

$$A_0 = \begin{bmatrix} -1 & 0.3 \\ 0.2 & -1.4 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}. \tag{8.123}$$

The characteristic polynomial of the matrix

$$A = A_0 + A_1 = \begin{bmatrix} -0.5 & 0.4 \\ 0.4 & -0.6 \end{bmatrix}$$

has the form

$$\det[I_n s - A] = \begin{vmatrix} s+0.5 & -0.4 \\ -0.4 & s+0.6 \end{vmatrix} = s^2 + 1.1s + 0.14. \tag{8.124}$$

All coefficients of the polynomial (8.124) are positive.

All principal minors of the matrix

$$-A = \begin{bmatrix} 0.5 & -0.4 \\ -0.4 & 0.6 \end{bmatrix}$$

are positive, i.e.

$$\Delta_1 = 0.5, \quad \det[-A] = 0.14 \ .$$

The conditions $b)$ and $c)$ of Theorem 8.28 are satisfied and the positive system (8.112) with the matrices (8.123) is asymptotically stable.

Chapter 9
Stability Analysis of Fractional Linear Systems in Frequency Domain

In this chapter the stability analysis of fractional linear systems in frequency domain based on Busłowicz papers [14, 17, 30, 19, 21, 27, 23] will be presented.

9.1 Fractional Continuous-Time Systems

Consider SISO continuous-time fractional system described by the

$$\sum_{i=0}^{n} a_i \frac{d^{\alpha_i}}{dt^{\alpha_i}} y(t) = \sum_{k=0}^{m} b_k \frac{d^{\beta_k}}{dt^{\beta_k}} u(t), \tag{9.1}$$

where $u(t)$ is the input, $y(t)$ is the output, $\alpha_n > \alpha_{n-1} > \cdots > \alpha_1 > \alpha_0 \geq 0$ and $\beta_m > \beta_{m-1} > \cdots > \beta_1 > \beta_0 \geq 0$ are arbitrary real numbers, a_i $(i = 0, 1, \ldots, n)$ and b_k $(k = 0, 1, \ldots, m)$ are real coefficients.

Applying the Laplace transform to the equation (9.1) with zero initial conditions, we obtain the fractional transfer function

$$G(s) = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \cdots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \cdots + a_0 s^{\alpha_0}}. \tag{9.2}$$

The fractional linear system with the transfer function (9.2) is of [307]:

a) commensurate order if

$$\alpha_i = i\alpha, \quad i = 0, 1, \ldots, n\,, \quad \beta_k = k\alpha, \quad k = 0, 1, 2, \ldots, m\,, \tag{9.3}$$

where $\alpha > 0$ is a real number,

b) rational order if it is a commensurate order and $\alpha = 1/q$, where q is a positive integer,

c) non-commensurate order if (9.3) does not hold.

T. Kaczorek: Selected Problems of Fractional Systems Theory, LNCIS 411, pp. 189–220.
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The transfer function of fractional system of commensurate order can be written in the form

$$G(s) = \frac{b_m s^{m\alpha} + b_{m-1} s^{(m-1)\alpha} + \cdots + b_0}{a_n s^{n\alpha} + a_{n-1} s^{(n-1)\alpha} + \cdots + a_0}. \tag{9.4}$$

Substituting $\lambda = s^\alpha$ in (9.4), we obtain the associated natural order transfer function

$$\tilde{G}(\lambda) = \frac{b_m \lambda^m + b_{m-1} \lambda^{m-1} + \cdots + b_0}{a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_0}. \tag{9.5}$$

For example, if

$$G(s) = \frac{s^{0.25} + 1}{s - 2s^{0.5} + 1.25}, \tag{9.6a}$$

then for $\lambda = s^{0.25}$ we obtain the associated natural order transfer function

$$\tilde{G}(\lambda) = \frac{\lambda + 1}{s^4 - 2\lambda^2 + 1.25}. \tag{9.6b}$$

Characteristic polynomial of the fractional system (9.1) has the form

$$D(s) = a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \cdots + a_0 s^{\alpha_0}. \tag{9.7}$$

The polynomial (9.7) is a multivalued function whose domain is a Riemann surface. In general case, this surface has an infinite number of sheets and the fractional polynomial (9.7) has an infinite number of zeros. Only a finite number of which will be in the main sheet of the Riemann surface. For stability reasons only the main sheet defined by $-\pi < \arg s < \pi$ can be considered [307].

The fractional system (9.1) is called bounded-input bounded-output (BIBO) stable (shortly stable) if for any bounded input $u(t)$ its output $y(t)$ is also bounded.

Theorem 9.1. *[198, 199, 307]. The fractional system with the transfer function (9.2) is stable if and only if the fractional degree characteristic polynomial (9.7) is stable, i.e. this polynomial has no zeros in the closed right-half of the Riemann complex surface, that is*

$$D(s) \neq 0 \quad \textit{for} \quad \text{Re } s \geq 0. \tag{9.8}$$

The Riemann surface has a finite number of sheets only in the case of fractional polynomials (9.7) of commensurate degree, i.e. for

$$\alpha_i = i\alpha, \quad i = 0, 1, \ldots, n. \tag{9.9}$$

If (9.9) holds, then the fractional degree characteristic polynomial (9.7) can be written in the form

$$D(s) = a_n s^{n\alpha} + a_{n-1} s^{(n-1)\alpha} + \cdots + a_0. \tag{9.10}$$

Hence, for $\lambda = s^\alpha$ from (9.10) we obtain the associated natural number degree polynomial

$$\tilde{D}(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0. \tag{9.11}$$

For example, if $D(s) = s^{1.5} + as^{0.5} + b$ (a and b are real numbers) then $\alpha = 1/2$, $\lambda = s^{1/2}$ and the associated natural number degree polynomial has the form $\tilde{D}(\lambda) = \lambda^3 + a\lambda + b$.

Theorem 9.2. *[307]. The fractional commensurate degree characteristic polynomial (9.10) is stable if and only if all zeros of this polynomial satisfy the condition (9.8) or, equivalently, all zeros λ_i of the associated natural degree polynomial (9.11) satisfy the condition*

$$|\arg \lambda_i| > \alpha \frac{\pi}{2}, \quad i = 1, 2, \ldots, n. \tag{9.12}$$

If $0 < \alpha \leq 1$ then from (9.12) we obtain the stability region shown in Fig. 9.1.

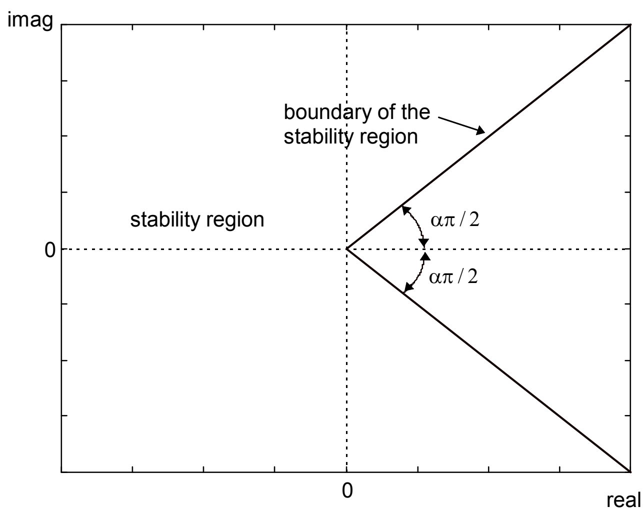

Fig. 9.1 Stability region of fractional order polynomial (9.10) in the complex λ-plane ($\lambda = s^\alpha$ with $0 < \alpha \leq 1$).

Parametric description of the boundary of the stability region is given by

$$(j\omega)^\alpha = |\omega|^\alpha e^{j\pi\alpha/2}, \quad \omega \in (-\infty, \infty). \tag{9.13}$$

The polynomial (9.10) for $\alpha = 1$ is a natural number degree polynomial and from (9.13) it follows that the imaginary axis of the complex plane is the boundary of the stability region.

From the above and Theorem 9.2 we have the following sufficient condition for stability of fractional degree polynomial (9.10) for $0 < \alpha \leq 1$.

Lemma 9.1. *The fractional commensurate degree characteristic polynomial (9.10) for $0 < \alpha \leq 1$ is stable if the associated natural number degree polynomial (9.11) is asymptotically stable, i.e. the condition (9.12) holds for $\alpha = 1$, i.e. $|\arg \lambda_i| > \pi/2$ for all zeros λ_i of (9.11).*

From Theorem 9.2 it follows that the fractional polynomial (9.10) may be stable even if the associated natural degree polynomial (9.11) is not asymptotically stable. Checking of the stability the fractional degree polynomial (9.10) using Theorem 9.2 is a difficult problem in general, because the degree of the associated polynomial (9.11) may be very large. For example, if

$$D(s) = s^{127/105} + 0.4s^{77/105} + 0.3s^{71/105} + 0.1s^{56/105} + 1,$$

then for $\lambda = s^\alpha = s^{1/105}$ we obtain the associated polynomial of natural degree [61]

$$\tilde{D}(\lambda) = \lambda^{127} + 0.4\lambda^{77} + 0.3\lambda^{71} + 0.1\lambda^{56} + 1.$$

The polynomial has degree equal to 127 and only five non-zero coefficients.

To avoid this difficulty, a method for determination of the multi-variate natural degree polynomial, associated with the fractional commensurate degree polynomial has been proposed [61]. To stability analysis of multi-variate degree polynomials, the LMI technique has been also proposed in [61].

Following [22] in this section a new frequency domain methods for stability analysis of fractional polynomials of commensurate degree will be presented. Extension of these methods to the case of non-commensurate degree fractional polynomials is given in [17]. The proposed methods are based on the Mikhailov stability criterion and the modified Mikhailov stability criterion, known from the theory of systems of natural number order (see [14, 75, 311], for example).

In the stability theory of natural degree characteristic polynomials of linear continuous-time systems, the following kinds of stability are considered (see [14], for example):

a) asymptotic stability (all zeros of the characteristic polynomial have negative real parts),
b) D-stability (all zeros of the characteristic polynomial lie in the open region D in the left half-plane of complex plane).

From Theorem 9.2 we have the following lemma.

Lemma 9.2. *The fractional degree polynomial* (9.10) *is stable if and only if the associated natural degree polynomial* (9.11) *is D-stable, where the parametric description the boundary of the region D has the form* (9.13)*. In particular, for* $0 < \alpha \leq 1$ *the D-stability region is shown in Fig. 9.1.*

It is easy to see that if $\alpha = 1$ then the fractional degree polynomial (9.10) is reduced to the natural degree polynomial (9.11) with $\lambda = s$. In such case from (9.13) it follows that boundary of the stability region is the imaginary axis of the complex plane.

Theorem 9.3. *The fractional degree characteristic polynomial* (9.10) *is stable if and only if*

$$\underset{0\leq\omega<\infty}{\Delta} \arg D(j\omega) = n\pi/2, \tag{9.14}$$

where $D(j\omega) = D(s)$ *for* $s = j\omega$.

Proof. It is easy to see that $\tilde{D}((j\omega)^{\alpha}) = D(j\omega)$. This means that (9.14) is the necessary and sufficient condition for D-stability of the natural degree polynomial (9.11) [14]. Hence, the proof follows from Lemma 9.2.

Plot of the function $D(j\omega)$, where $D(j\omega) = D(s)$ for $s = j\omega$ will be called the generalized (to the class of fractional degree polynomials) Mikhailov plot.

From (9.14) it follows that the generalized Mikhailov plot starts for $\omega = 0$ in the point $D(j0) > 0$ on positive real axis and with ω increasing from 0 to ∞ turns strictly counter-clockwise and goes through n quadrants of the complex plane.

Checking the condition (9.14) of Theorem 9.3 is a difficult task (for large values of n), because $D(j\omega)$ quickly tends to infinity as ω grows to ∞.

To remove this difficulty, we consider the rational function

$$\psi(s) = \frac{D(s)}{w_r(s)}, \tag{9.15}$$

instead of the polynomial (9.10), where $w_r(s)$ is the reference fractional polynomial of the same degree as the polynomial (9.10).

We will assume that the reference fractional polynomial $w_r(s)$ is stable, i.e.

$$w_r(s) \neq 0 \quad \text{for} \quad \operatorname{Re} s \geq 0. \tag{9.16}$$

Theorem 9.4. *The fractional degree polynomial* (9.10) *is stable if and only if*

$$\underset{\omega \in (-\infty,\infty)}{\Delta \arg} \psi(j\omega) = 0, \tag{9.17}$$

where $\psi(j\omega) = \psi(s)$ *for* $s = j\omega$ *and* $\psi(s)$ *is defined by* (9.15).

Proof. If the reference polynomial $w_r(s)$ is stable then from Theorem 9.3 we have

$$\underset{\omega \in (-\infty,\infty)}{\Delta \arg} w_r(j\omega) = n\pi. \tag{9.18}$$

From (9.15) for $s = j\omega$ it follows that

$$\Delta \arg \psi(j\omega) = \Delta \arg D(j\omega) - \Delta \arg w_r(j\omega). \tag{9.19}$$

The fractional degree polynomial (9.10) is stable if and only if

$$\underset{\omega \in (-\infty,\infty)}{\Delta \arg} D(j\omega) = \underset{\omega \in (-\infty,\infty)}{\Delta \arg} w_r(j\omega) = n\pi,$$

which holds if and only if (9.17) is satisfied. □

The reference fractional polynomial $w_r(s)$ for the polynomial (9.10) can be chosen in the form

$$w_r(s) = a_n(s+c)^{\alpha n}, \quad c > 0. \tag{9.20}$$

Note that for $c > 0$ the reference polynomial (9.20) is stable.

Plot of the function $\psi(j\omega)$, $\omega \in (-\infty, \infty)$ ($\psi(s)$ is defined by (9.15)) we will call the generalized modified Mikhailov plot.

The condition (9.17) of Theorem 9.4 holds if and only if the generalized modified Mikhailov plot does not encircle the origin of the complex plane as ω runs from $-\infty$ to ∞.

From (9.10), (9.15) and (9.20) we have

$$\lim_{\omega \to \pm\infty} \psi(j\omega) = \lim_{\omega \to \pm\infty} \frac{D(j\omega)}{w_r(j\omega)} = 1, \tag{9.21}$$

and

$$\psi(0) = \frac{D(0)}{w_r(0)} = \frac{a_0}{a_n c^{\alpha n}}. \tag{9.22}$$

From (9.22) it follows that $\psi(0) \leq 0$ if $a_0/a_n \leq 0$. Hence, from Theorem 9.4 we have the following important lemma.

Lemma 9.3. *The fractional degree polynomial* (9.10) *is unstable if* $a_0/a_n \leq 0$.

Now we consider the case in which the condition (9.17) of Theorem 9.4 is not satisfied.

Theorem 9.5. *The fractional characteristic polynomial* (9.10) *of commensurate degree has* $k \geq 0$ *zeros in the right-half of the Riemann complex surface if and only if as* ω *runs from* $-\infty$ *to* $+\infty$ *the plot of* $\psi(j\omega)$ *k times encircle in the negative direction the origin of the complex plane. In such a case*

$$\underset{\omega \in (-\infty,\infty)}{\Delta \arg} \psi(j\omega) = -k2\pi. \tag{9.23}$$

Proof. As in [75] in the case of natural degree polynomials we can show that if the fractional degree polynomial (9.10) has $k \geq 0$ zeros with positive real parts, then

$$\underset{\omega \in (-\infty,\infty)}{\Delta \arg} D(j\omega) = (n - 2k)\pi. \tag{9.24}$$

Hence, from (9.19) and (9.18), (9.24) it follows that (9.23) holds. If (9.23) is satisfied then from (9.19) and (9.18) we have (9.24). □

It is easy to see that Theorem 9.4 follows from Theorem 9.5 for $k = 0$.

Example 9.1. Consider a linear fractional order system with characteristic polynomial of commensurate degree of the form [61]

$$\begin{aligned} D(s) &= 134.7955988 s^{11/15} + 17.49138877 s^{14/15} + 7.5619 s^{7/5} + 18.60416827 s^{6/5} \\ &+ s^{8/5} + 13.68686363 s^{3/5} + 276.0731421 s^{1/3} + 269.6615050 s^{1/5} \\ &+ 218.5809037 s^{2/5} + 338.6269398 s^{8/15} + 7.3225 s^{19/15} + 55.921984 s^{16/15} \\ &+ 139.1374509 s^{13/15} + 14.79208246 s + 221.9590294. \end{aligned} \tag{9.25}$$

For $\alpha = 1/15$ and $\lambda = s^\alpha = s^{1/15}$ from the fractional commensurate degree polynomial (9.25) we obtain the associated natural degree polynomial

$$\begin{aligned}\tilde{D}(\lambda) = {} & \lambda^{24} + 7.5619\lambda^{21} + 7.3225\lambda^{19} + 18.60416827\lambda^{18} + 55.92198403\lambda^{16} \\ & + 14.79208246\lambda^{15} + 17.49138877\lambda^{14} + 139.1374509\lambda^{13} + 134.79559\lambda^{11} \\ & + 13.68686363\lambda^{9} + 338.6269398\lambda^{8} + 218.5809037\lambda^{6} + 276.0731421\lambda^{5} \\ & + 269.6615050\lambda^{3} + 221.9590294. \end{aligned} \tag{9.26}$$

By Theorem 9.2 the fractional degree polynomial (9.25) is stable if and only if the associated natural degree polynomial (9.26) has no zeros in the cone shown in Fig. 9.1 with $\alpha\pi/2 = \pi/30 = 0.1047$ rad.

Plot of the function (9.15) with $w_r(s) = (s+10)^{8/5}$ is shown in Fig. 9.2. According to (9.21) and (9.22) we have $\lim_{\omega\to\pm\infty} \psi(j\omega) = 1$, $\psi(j0) = a_0/10^{8/5} = 221.9590294/10^{8/5} = 5.5754$.

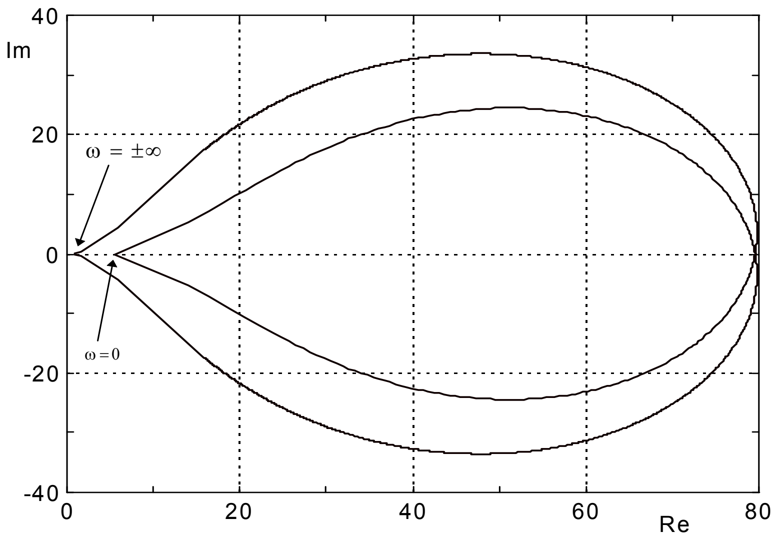

Fig. 9.2 Plot of the function (9.15) with $s = j\omega$, $\omega \in (-\infty, \infty)$. Illustration to Example 9.1.

From Fig. 9.2 it follows that the generalized modified Mikhailov plot $\psi(j\omega)$ does not encircle the origin of the complex plane and by Theorem 9.4 the system is stable.

Now we consider the fractional degree polynomial (9.25) and associated natural degree polynomial (9.26) in the case when the free term has negative sign, i.e. is -221.959029 instead of $+221.959029$. In such case $a_0/a_n = a_0 = -221.959029 < 0$ and by Lemma 9.3 the fractional system is unstable.

In this case, the generalized modified Mikhailov plot with the reference polynomial $w_r(s) = (s+10)^{8/5}$ is shown in Fig. 9.3, where $\lim_{\omega\to\pm\infty} \psi(j\omega) = 1$, $\psi(j0) = -5.5754$.

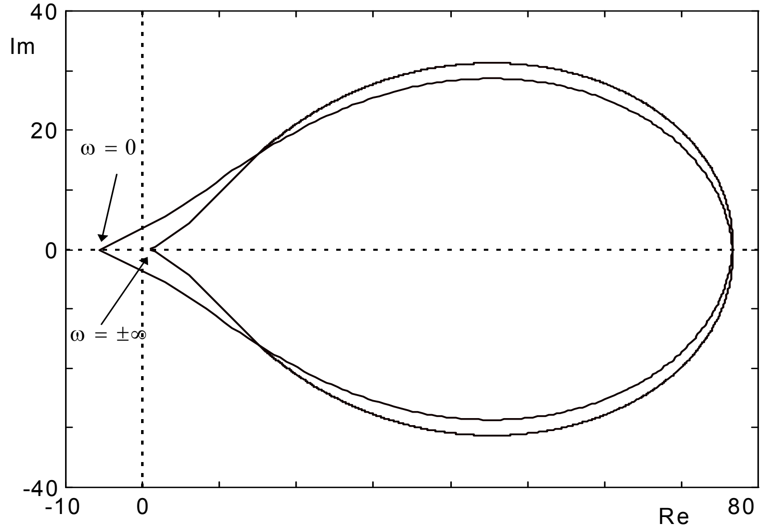

Fig. 9.3 Plot of the function (9.15) with $s = j\omega$, $\omega \in (-\infty, \infty)$, $D(s)$ of the form (9.25) with negative free term

Zeros of natural degree polynomial (9.26) with negative free term and the boundary of the stability region are shown in Fig. 9.4.

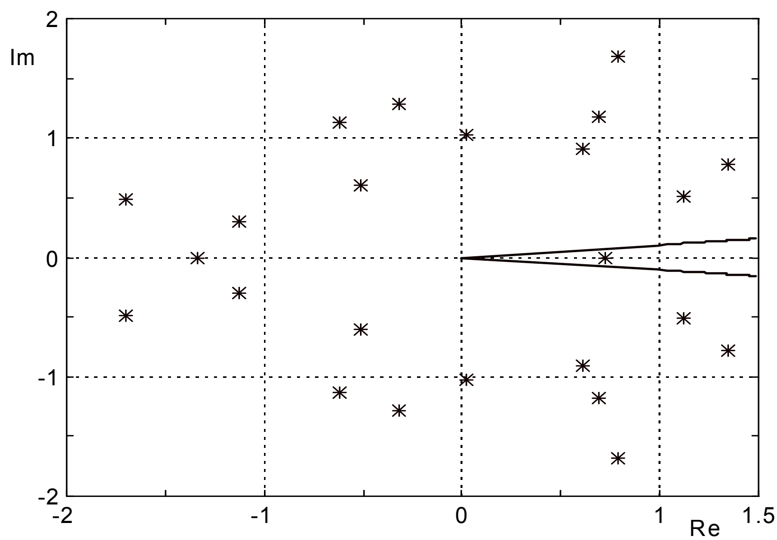

Fig. 9.4 Zeros of polynomial (9.26) with negative free term and boundary of the stability region

From Fig. 9.3 it follows that the generalized modified Mikhailov plot $\psi(j\omega)$ ones encircles the origin of the complex plane in negative direction. By Theorem 9.5, the system is unstable and the characteristic polynomial has one unstable zero. The zero lies in the instability region shown in Fig. 9.4.

Example 9.2. Consider the control system with the fractional order plant described the transfer function [259, 313]

$$G_0(s) = \frac{1}{0.8s^{2.2} + 0.5s^{0.9} + 1} = \frac{1}{D_0(s)}, \tag{9.27}$$

and the fractional PD controller

$$C(s) = 20.5 + 3.7343s^{1.15}. \tag{9.28}$$

Characteristic polynomial of the closed loop system with the plant (9.27) and controller (9.28) has the form

$$D(s) = D_0(s) + C(s) = 0.8s^{2.2} + 3.7343s^{1.15} + 0.5s^{0.9} + 21.5\,. \tag{9.29}$$

Substituting $\alpha = 1/20$ and $\lambda = s^{\alpha} = s^{1/20}$ in (9.29), we obtain the associated polynomial of natural degree

$$\tilde{D}(\lambda) = 0.8\lambda^{44} + 3.7343\lambda^{23} + 0.5\lambda^{18} + 21.5\,. \tag{9.30}$$

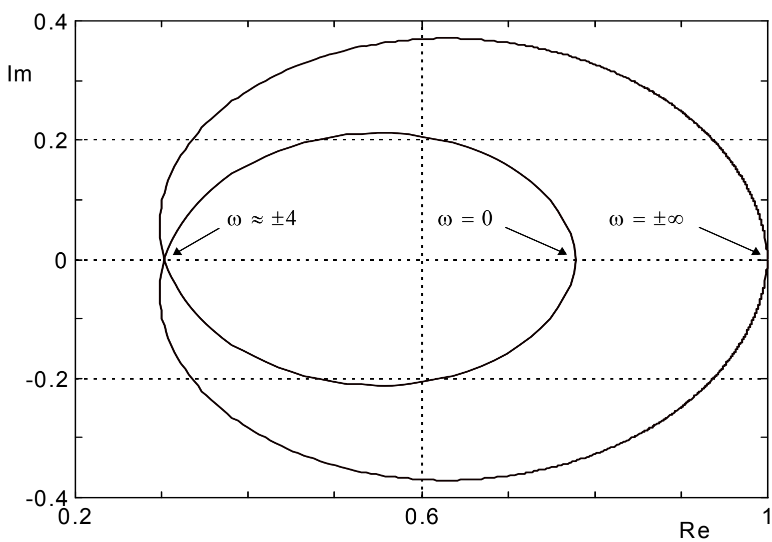

Fig. 9.5 Plot of the function $\psi(j\omega)$. Illustration to Example 9.2.

The control system is stable if and only if all zeros of polynomial (9.30) lie in the stability region shown in Fig. 9.1 with $\alpha = 1/20$.

To check stability of fractional polynomial (9.29) we use Theorem 9.4.

Plot of the function $\psi(j\omega) = D(j\omega)/w_r(j\omega)$, where $D(s)$ has the form (9.29) and $w_r(s) = 0.8(s+5)^{2.2}$ is the reference fractional polynomial, is shown in Fig. 9.5.

From (9.21) and (9.22) we have

$$\lim_{\omega\to\pm\infty} \psi(j\omega) = 1; \quad \psi(0) = \frac{D(0)}{w_r(0)} = \frac{21.5}{0.8 \cdot 5^{2.2}} = 0.7791 .$$

From Fig. 9.5 it follows that the generalized modified Mikhailov plot $\psi(j\omega)$ does not encircle the origin of the complex plane. Therefore, by Theorem 9.4 the fractional control system is stable.

9.2 Fractional Continuous-Time Systems with Delays of the Retarded Type

Following [21] consider a linear fractional system with delays described by the transfer function

$$P(s) = \frac{q_0(s) + \sum_{j=1}^{m_2} q_j(s)\exp(-s^r\beta_j),}{p_0(s) + \sum_{i=1}^{m_1} p_i(s)\exp(-s^r h_i),} = \frac{N(s)}{D(s)}, \tag{9.31}$$

where r is a real number such that $0 < r \leq 1$, the fractional order polynomials $p_i(s)$ and $q_j(s)$ with real coefficients have the forms

$$p_i(s) = \sum_{k=0}^{n} a_{ik}s^{\alpha_k}, \quad i = 0,1,\ldots,m_1 , \tag{9.32}$$

$$q_j(s) = \sum_{k=0}^{m} b_{jk}s^{\delta_k}, \quad j = 0,1,\ldots,m_2 , \tag{9.33}$$

where α_k and δ_k are real non-negative numbers and $a_{0n} \neq 0$, $b_{0m} \neq 0$.

Without loss of generality we will assume that $\alpha_n > \alpha_{n-1} > \cdots > \alpha_1 > \alpha_0 = 0$; $\delta_m > \delta_{m-1} > \cdots > \delta_1 > \delta_0 \geq 0$ and the delays β_j and h_i satisfy the inequalities $\beta_{m_2} > \beta_{m_2-1} > \cdots > \beta_1$, $h_{m_1} > h_{m_1-1} > \cdots > h_1$.

The fractional degree characteristic quasi-polynomial of the system (9.31) has the form

$$D(s) = p_0(s) + \sum_{i=1}^{m_1} p_i(s)\exp(-s^r h_i). \tag{9.34}$$

The fractional degree characteristic quasi-polynomial (9.34) is:

a) of the retarded type if $\deg p_0(s) > \deg p_i(s)$ for all $i = 1,2,\ldots,m_1$,
b) of the neutral type if $\deg p_0(s) = \deg p_i(s)$ for at least one $i = 1,2,\ldots,m_1$.

We will consider the time-delay systems of the retarded type, i.e. the systems satisfying the assumption $\deg p_0(s) > \deg p_i(s)$ for all $i = 1,2,\ldots,m_1$.

Moreover, we assume that $\deg q_0(s) > \deg q_j(s)$, $j = 1,2,\ldots,m_2$, and $\deg p_0(s) > \deg q_0(s)$ in order to deal with strictly proper systems. Suppose that $N(s)$ and $D(s)$ have no common zeros in $\{Re\ s \geq 0\} \backslash \{0\}$.

Theorem 9.6. *[9] The fractional system with the transfer function (9.31) is stable if and only if the fractional degree characteristic quasi-polynomial (9.34) is stable, i.e. all its zeros have negative real parts, that is*

$$D(s) \neq 0 \quad \textit{for} \quad \text{Re } s \geq 0. \tag{9.35}$$

Similarly as in the case of fractional order systems without delays [307], we introduce the following classification of the fractional order systems with delays.

The fractional order system with delays described by the transfer function (9.31) is:

a) of commensurate order if

$$\alpha_k = k\alpha \quad (k = 0,1,\ldots,n) \quad \text{and} \quad \delta_k = k\alpha \quad (k = 0,1,\ldots,m), \tag{9.36}$$

where $\alpha > 0$ is a real number,

b) of rational order if it is a commensurate order and $\alpha = 1/q$, where q is a positive integer (in such a case $0 < \alpha \leq 1$),

c) of non-commensurate order if (9.36) does not hold.

A numerical algorithm for stability testing of fractional order systems with delays (of non-commensurate order, in general) has been given in [70]. This algorithm is based on the use of the Cauchy integral theorem and solving an initial-value problem.

In this section following [21] new necessary and sufficient conditions for stability in frequency domain of fractional characteristic quasi-polynomials (9.34) will be proposed. First, the characteristic quasi-polynomial of commensurate degree will be analyzed and frequency domain method for stability will be given. Next, the frequency domain method for stability analysis of the characteristic quasi-polynomial of non-commensurate degree will be proposed.

The system with delays of fractional commensurate order is described by the transfer function (9.31) with

$$p_i(s) = \sum_{k=0}^{n} a_{ik} s^{k\alpha}, \quad i = 0,1,\ldots,m_1, \tag{9.37}$$

$$q_j(s) = \sum_{k=0}^{m} b_{jk} s^{k\alpha}, \quad j = 0,1,\ldots,m_2. \tag{9.38}$$

In such a case, applying substitution $\lambda = s^\alpha$ in (9.37), (9.38) and (9.31) we obtain the associated transfer function of natural order of the form

$$\tilde{P}(\lambda) = \frac{\tilde{q}_0(\lambda) + \sum_{j=1}^{m_2} \tilde{q}_j(\lambda) \exp(-\lambda^{r/\alpha}\beta_j),}{\tilde{p}_0(\lambda) + \sum_{i=1}^{m_1} \tilde{p}_i(\lambda) \exp(-\lambda^{r/\alpha}h_i),} = \frac{\tilde{N}(\lambda)}{\tilde{D}(\lambda)}, \tag{9.39}$$

where

$$\tilde{p}_i(\lambda) = \sum_{k=0}^{n} a_{ik}\lambda^k, \quad i = 0, 1, \ldots, m_1, \tag{9.40a}$$

$$\tilde{q}_j(\lambda) = \sum_{k=0}^{m} b_{jk}\lambda^k, \quad j = 0, 1, \ldots, m_2, \tag{9.40b}$$

are natural number degree polynomials.

Hence, in the case of system with delays of fractional commensurate order we can consider the natural degree quasi-polynomial

$$\tilde{D}(\lambda) = \tilde{p}_0(\lambda) + \sum_{i=1}^{m_1} \tilde{p}_i(\lambda) \exp(-\lambda^{r/\alpha}h_i), \tag{9.41}$$

associated with the characteristic quasi-polynomial (9.34) of fractional order.

Lemma 9.4. *All zeros of the fractional quasi-polynomial* (9.34) *of commensurate degree satisfy the condition* (9.35) *if and only if all zeros of the associated natural degree quasi-polynomial* (9.41) *satisfy the condition*

$$|\arg \lambda| > \alpha \frac{\pi}{2}. \tag{9.42}$$

Proof. From Theorem 9.6 it follows that boundary of the stability region of fractional quasi-polynomial (9.34) is the imaginary axis of complex s-plane with the parametric description $s = j\omega$, $\omega \in (-\infty, \infty)$. Zeros of fractional quasi-polynomial $D(s)$ of the form (9.34) and the associated natural degree quasi-polynomial $\tilde{D}(\lambda)$ of the form (9.41) satisfy the relationship $\lambda = s^\alpha$. Hence, boundary of the stability region in the complex λ-plane of the natural degree quasi-polynomial (9.41) has the parametric description

$$\lambda = (j\omega)^\alpha = |\omega|^\alpha e^{j\alpha\pi/2}, \quad \omega \in (-\infty, \infty). \tag{9.43}$$

All zeros of quasi-polynomial (9.41) lie in the stability region with the boundary (9.43) if and only if (9.42) holds. This completes the proof. □

It is easy to see that for $0 < \alpha \leq 1$ the condition (9.42) holds for zeros of quasi-polynomial (9.41) lying in the stability region shown in Fig. 9.1. This region is reduced to the open left half-plane of the complex λ-plane for $\alpha = 1$.

From (9.42) and Fig. 9.1 it follows that if $1 < \alpha < 2$ then the "stability region" is a cone in the open left half-plane.

From the fundamental properties of distribution of zeros of quasi-polynomials (see [6, 15, 64, 66, 65], for example) it follows that natural degree quasi-polynomial (9.41) of the retarded type always has at least one chain of asymptotic zeros satisfying the conditions

$$\lim_{|\lambda|\to\infty} Re\lambda = -\infty, \quad \lim_{|\lambda|\to\infty} Im\lambda = \pm\infty.$$

Therefore, the condition (9.42) with $\alpha > 1$ does not hold for the asymptotic zeros of quasi-polynomial (9.41). Therefore, we have the following important lemma.

Lemma 9.5. *The fractional quasi-polynomial* (9.34) *of commensurate degree (the condition* (9.36) *holds) is unstable for any* $\alpha > 1$.

In the stability theory of polynomials or quasi-polynomials of natural degree, the asymptotic stability (all zeros have negative real parts) and more general case of stability, namely the D-stability (all zeros lie in the open region D in the open left half-plane of complex plane), are considered [14, 15].

From the above and Lemma 9.4 we have the following lemmas.

Lemma 9.6. *The fractional quasi-polynomial* (9.34) *of commensurate degree is stable if and only if the associated natural degree quasi-polynomial* (9.41) *is D-stable, where parametric description of boundary of the region D has the form* (9.43) *with* $0 < \alpha \leq 1$ *(see Fig. 9.1).*

Lemma 9.7. *The fractional quasi-polynomial* (9.34) *of commensurate degree with* $0 < \alpha \leq 1$ *is stable if the associated natural degree quasi-polynomial* (9.41) *is asymptotically stable, i.e. all zeros of this quasi-polynomial have negative real parts (the condition* (9.42) *holds for* $\alpha = 1$*).*

By generalization of the Mikhailov theorem (see [14, 15, 75], for example) to the fractional degree characteristic quasi-polynomial (9.34) of commensurate degree we obtain the following theorem.

Theorem 9.7. *The fractional characteristic quasi-polynomial* (9.34) *of commensurate degree is stable if and only if*

$$\underset{0\leq\omega<\infty}{\Delta \arg} D(j\omega) = n\pi/2, \tag{9.44}$$

which means that plot of $D(j\omega)$ *with* ω *increasing from* 0 *to* ∞ *runs in the positive direction by n quadrants of the complex plane, missing the origin of this plane.*

Proof. Because $\tilde{D}((j\omega)^{\alpha}) = D(j\omega)$, the condition (9.44) is necessary and sufficient for D-stability of natural degree quasi-polynomial (9.41) [15]. The proof follows immediately from Lemma 9.6. □

Plot of the function $D(j\omega)$, where $D(j\omega) = D(s)$ for $s = j\omega$ ($D(s)$ has the form (9.34)) will be called the generalized (to the class of fractional degree quasi-polynomials) Mikhailov plot.

In general case checking the condition of Theorem 9.7 is difficult task, since

a) $D(j\omega)$ quickly tends to infinity as ω grows to ∞,
b) the delay terms in $D(s)$ generate an infinite number of spiral for $s = j\omega$ and $\omega \in (-\infty, \infty)$.

Therefore, Theorem 9.7 is not practically reliable. Moreover, this theorem is true only in the case of commensurate degree fractional quasi-polynomials.

To remove the difficulty *a*), we introduce the rational function

$$\psi(s) = \frac{D(s)}{w_r(s)} \tag{9.45}$$

instead of fractional degree quasi-polynomial $D(s)$ of the form (9.34).

In (9.45) $w_r(s)$ is the reference fractional polynomial (fractional quasi-poly- nomial) of the same degree α_n as quasi-polynomial (9.34). We will assume that this polynomial is stable, i.e.

$$w_r(s) \neq 0 \quad \text{for} \quad Re\ s \geq 0. \tag{9.46}$$

The reference fractional polynomial $w_r(s)$ can be chosen in the form

$$w_r(s) = a_{0n}(s+c)^{\alpha_n}, \quad c > 0, \tag{9.47}$$

where a_{0n} is the coefficient of s^{α_n} in polynomial $p_0(s)$ of the form (9.32) for $i = 0$.

Note that the reference polynomial (9.47) is stable for $c > 0$.

Theorem 9.8. *The fractional characteristic polynomial* (9.34) *(of commensurate or non-commensurate degree) is stable if and only if*

$$\underset{\omega\in(-\infty,\infty)}{\Delta \arg} \psi(j\omega) = 0, \tag{9.48}$$

where $\psi(j\omega) = \psi(s)$ *for* $s = j\omega$ *and* $\psi(s)$ *is defined by* (9.45).

Proof. From (9.45) for $s = j\omega$ it follows that

$$\underset{\omega\in(-\infty,\infty)}{\Delta \arg} \psi(j\omega) = \underset{\omega\in(-\infty,\infty)}{\Delta \arg} D(j\omega) - \underset{\omega\in(-\infty,\infty)}{\Delta \arg} w_r(j\omega). \tag{9.49}$$

By the assumption the reference polynomial $w_r(s)$ of the same fractional degree as quasi-polynomial (9.34) is stable. Therefore, the fractional quasi-polynomial (9.34) is stable if and only if

$$\underset{\omega\in(-\infty,\infty)}{\Delta \arg} D(j\omega) = \underset{\omega\in(-\infty,\infty)}{\Delta \arg} w_r(j\omega), \tag{9.50}$$

which holds if and only if (9.48) is satisfied. □

Plot of the function $\psi(j\omega)$, $\omega \in (-\infty, \infty)$ ($\psi(s)$ is defined by (9.45)) and will be called the generalized modified Mikhailov plot.

The condition (9.48) of Theorem 9.8 holds if and only if the generalized modified Mikhailov plot $\psi(j\omega)$ does not encircle the origin of the complex plane as ω runs from $-\infty$ to ∞.

Form (9.45), (9.34) and (9.47) we have

$$\lim_{\omega\to\pm\infty}\psi(j\omega)=\lim_{\omega\to\pm\infty}\frac{D(j\omega)}{w_r(j\omega)}=1, \tag{9.51a}$$

and

$$\psi(0)=\frac{D(0)}{w_r(0)}=\frac{a_{00}+a_{10}+\cdots+a_{m_10}}{a_{0n}c^{\alpha_n}}. \tag{9.51b}$$

From the above it follows that the generalized modified Mikhailov plot encircles or cross the origin of the complex plane if $\psi(j0)\le 0$. Hence, we have the following lemma.

Lemma 9.8. *The fractional characteristic quasi-polynomial* (9.34) *of commensurate or non-commensurate degree is unstable if*

$$\frac{a_{00}+a_{10}+\cdots+a_{m_10}}{a_{0n}}\le 0.$$

Example 9.3. Check the stability of fractional order system with delays with characteristic quasi-polynomial of the form

$$D(s)=s^{3/2}-1.5s-1.5s\exp(-sh)+4s^{1/2}+8. \tag{9.52}$$

For $\alpha=1/2$ and $\lambda=s^{\alpha}=s^{1/2}$ from (9.52) one obtains the following associated fractional quasi-polynomial of natural degree

$$\tilde{D}(\lambda)=\lambda^3-1.5\lambda^2-1.5\lambda^2\exp(-\lambda^2h)+4\lambda+8. \tag{9.53}$$

From Lemma 9.6 it follows that fractional quasi-polynomial (9.52) of commensurate degree is stable (all its zeros satisfy the condition (9.35)) if and only if the natural degree quasi-polynomial (9.53) is D-stable, where the region D is shown in Fig. 9.1 with $\alpha=1/2$.

Substituting $h=0$ in (9.52) and (9.53) we obtain, respectively, the fractional and natural degree polynomials $D(s)=s^{3/2}-3s+4s^{1/2}+8$ and $\tilde{D}(\lambda)=\lambda^3-3\lambda^2+4\lambda+8$. Polynomial $\tilde{D}(\lambda)$ has the following zeros: $\lambda_1=-1$, $\lambda_{2,3}=2\pm j2$. Zeros λ_2 and λ_3 lie on the boundary of D-stability region which means that polynomial $\tilde{D}(\lambda)$ is not D-stable and the fractional quasi-polynomial (9.52) is unstable for $h=0$. In [255] it was shown (see also [70]) that the fractional quasi-polynomial (9.52) is stable for a few intervals of values of the delay h, where $H_1=(0.04986, 0.78539)$ is the first interval of stability.

We check stability of the fractional characteristic quasi-polynomial (9.52) with $h=0.1$.

Plot of the function (9.45) for $s=j\omega$ and $\omega\in[0,500]$, where $D(s)$ has the form (9.52) for $h=0.1$ and $w_r(s)=(s+5)^{3/2}$, is shown in Fig. 9.6. According to (9.51) we have $\psi(0)=8/5^{3/2}=0.7155$, $\lim\limits_{\omega\to\infty}\psi(j\omega)=1$. The plot is symmetrical with respect to the real axis for negative values of frequency ω. This plot does not encircle

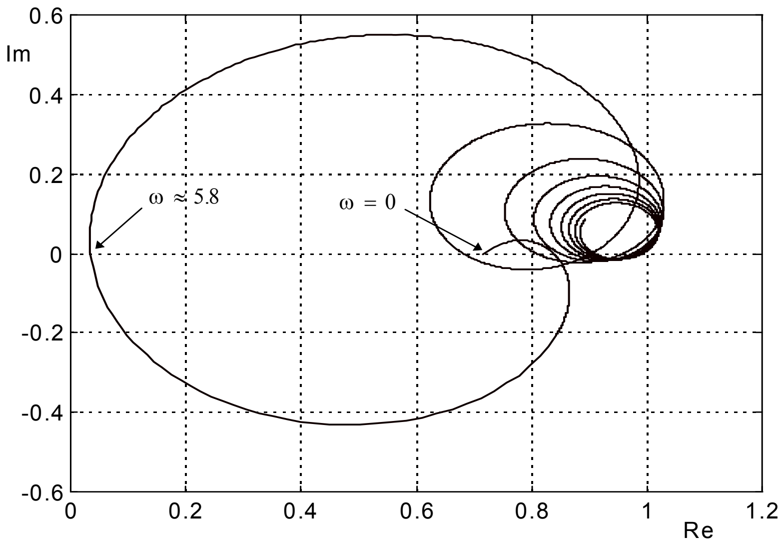

Fig. 9.6 Plot of (9.45) for $s = j\omega$, $\omega \in [0, 500]$, $w_r(s) = (s+5)^{3/2}$

the origin of the complex plane, and by Theorem 9.8 the fractional system with characteristic quasi-polynomial (9.52) with $h = 0.1$ is stable.

Example 9.4. Consider the control system shown in Fig. 9.7 with fractional order plant described by the transfer function

$$P(s) = \frac{e^{-0.5s}}{1 + s^{0.5}}. \tag{9.54}$$

and fractional PID controller

$$C(s) = K + \frac{I}{s^{\lambda}} + Ds^{\mu}, \tag{9.55}$$

where $\lambda = 1.1011$, $\mu = 0.1855$, $K = 1.4098$, $I = 1.6486$, $D = -0.2139$ [305].

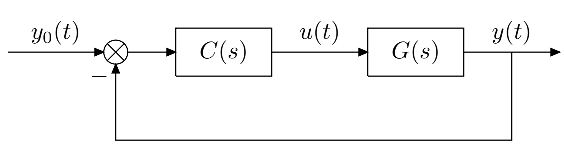

Fig. 9.7 The feedback control system. Illustration to Example 9.4.

Characteristic quasi-polynomial of the closed-loop system has the form

$$\begin{aligned} D(s) &= s^{\lambda+1/2} + s^{\lambda} + (Ks^{\lambda} + Ds^{\lambda+\mu} + I)e^{-0.5s} \\ &= s^{1.6011} + s^{1.1011} + (1.4098s^{1.1011} - 0.2139s^{1.2866} + 1.6486)e^{-0.5s}. \end{aligned} \tag{9.56}$$

The control system with characteristic quasi-polynomial (9.56) is stable if and only if all zeros of (9.56) have negative real parts.

To check the stability we apply Theorem 9.8 with the reference polynomial $w_r(s) = (s+10)^{1.6011}$. In such a case the function (9.45) has the form

$$\psi(s) = \frac{s^{1.6011} + s^{1.1011} + (1.4098s^{1.1011} - 0.2139s^{1.2866} + 1.6486)e^{-0.5s}}{(s+10)^{1.6011}}. \quad (9.57)$$

Plot of the function (9.57) for $s = j\omega$, $\omega \in (-\infty, \infty)$, is shown in Fig. 9.8, where

$$\lim_{\omega\to\pm\infty} \psi(j\omega) = 1, \quad \psi(0) = 1.6486/10^{1.6011} = 0.0413\,. \quad (9.58)$$

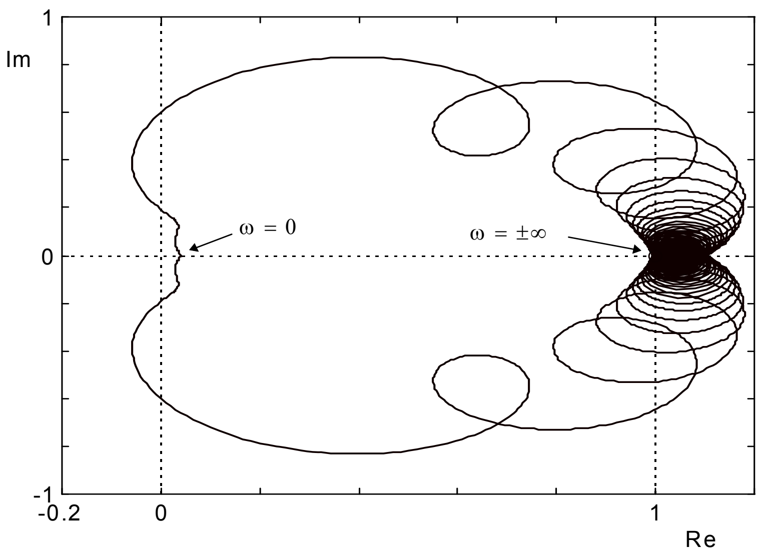

Fig. 9.8 Plot of the function (9.57) for $s = j\omega$, $\omega \in (-\infty, \infty)$

From Fig. 9.8 it follows that the generalized modified Mikhailov plot $\psi(j\omega)$ does not encircle the origin of the complex plane. Therefore, by Theorem 9.8 the fractional control system with characteristic quasi-polynomial (9.56) is stable.

9.3 Fractional Discrete-Time Systems

Consider a fractional linear discrete-time dynamical system with the transfer function

$$G(z) = \frac{b_m z^{\beta_m} + b_{m-1} z^{\beta_{m-1}} + \cdots + b_0 z^{\beta_0}}{a_n z^{\alpha_n} + a_{n-1} z^{\alpha_{n-1}} + \cdots + a_0 z^{\alpha_0}}, \quad (9.59)$$

where: $\alpha_n > \alpha_{n-1} > \cdots > \alpha_0 \geq 0$, $\beta_m > \beta_{m-1} > \cdots > \beta_0 \geq 0$ are real numbers, a_i $(i = 0, 1, \ldots, n)$ and b_k $(k = 0, 1, \ldots, m)$ are real coefficients.

In general case, transfer function (9.59) describes the fractional systems of non-commensurate order with characteristic function

$$w(z) = a_n z^{\alpha_n} + a_{n-1} z^{\alpha_{n-1}} + \cdots + a_0 z^{\alpha_0}. \tag{9.60}$$

The function (9.60) will be called the polynomial of fractional degree or fractional polynomial.

Similarly as in the case of fractional continuous-time systems [307], we have the following classification.

The system described by transfer function (9.59) is of non-commensurate order.

This system is of a commensurate order if $\alpha_i = i\alpha$ $(i = 0, 1, \ldots, n)$ and $\beta_k = k\alpha$ $(k = 0, 1, \ldots, m)$, moreover

a) if $\alpha = 1/q$ ($q > 1$ is a positive integer), then system (9.59) is called the system of rational commensurate order,
b) if does not exist a positive integer $q > 1$ such that $\alpha = 1/q$ then system (9.59) is called the system of fractional commensurate non-rational order.

The system of fractional commensurate order is described by the transfer function

$$G(z) = \frac{b_m z^{m\alpha} + b_{m-1} z^{(m-1)\alpha} + \cdots + b_1 z^{\alpha} + b_0}{a_n z^{n\alpha} + a_{n-1} z^{(n-1)\alpha} + \cdots + a_1 z^{\alpha} + a_0} \tag{9.61}$$

with the characteristic polynomial of fractional degree

$$w(z) = a_n z^{n\alpha} + a_{n-1} z^{(n-1)\alpha} + \cdots + a_1 z^{\alpha} + a_0, \tag{9.62}$$

where $\alpha \in (0, 1]$ is a real number.

Following [60] we can formulate the following theorem.

Theorem 9.9. *The discrete-time system of fractional order is stable if and only if its characteristic polynomial* (9.60) *(or* (9.62)*) of fractional degree is stable, i.e.*

$$w(z) \neq 0 \quad \textit{for} \quad |z| \geq 1. \tag{9.63}$$

Substituting $\lambda = z^{\alpha}$ in (9.61) we obtain the associated transfer function of natural order

$$G(\lambda) = \frac{b_m \lambda^m + b_{m-1} \lambda^{m-1} + \cdots + b_1 \lambda + b_0}{a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0}. \tag{9.64}$$

From (9.64) it follows that with the fractional degree characteristic polynomial (9.62) is associated the natural degree polynomial

$$\tilde{w}(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0. \tag{9.65}$$

The characteristic polynomial (9.60) of the system described by transfer function (9.59) is a multivalued function whose domain is a Riemann surface [256, 307]. In general, this surface has an infinite number of sheets. The number of sheets is finite only if the polynomial (9.60) is of rational commensurate degree, i.e. it has the form (9.62) with $\alpha = 1/q$, where $q > 1$ is an integer.

Note that the transfer function (9.59) almost always can be written in the form (9.61) with $\alpha = 1/q$ and $q > 1$.

If $z = |z|e^{j\varphi}$, then with $\alpha = 1/q$ we have

$$z^{\alpha} = \sqrt[q]{z} = \sqrt[q]{|z|}e^{j(\varphi+2k\pi)/q}, \quad k = 0,1,2,\dots,q-1. \tag{9.66}$$

From (9.66) and the relation $\lambda = z^{\alpha}$ it follows that $|\lambda| < 1$ if and only if $|z| < 1$. Therefore, we have the following theorem.

Theorem 9.10. *The discrete-time system of fractional rational commensurate order is stable if and only if the condition (9.63) is met, or equivalently, all zeros λ_i of the associated polynomial (9.65) of natural degree satisfy the condition*

$$|\lambda_i| < 1, \quad i = 1,2,\dots,n. \tag{9.67}$$

By Theorem 9.10 the stability checking of discrete-time system (9.59) of fractional rational commensurate order can be reduced to checking of location of zeros of associated polynomial (9.65) in the unit circle.

Investigation of stability of the fractional system with transfer function (9.59) by checking the condition (9.67) for all zeros of natural degree polynomial (9.65) can be inconvenient with regard on high degree of this polynomial.

If, for example, the characteristic polynomial of fractional degree is of the form [60]

$$w(z) = z^{61/35} + 0.2z^{22/35} + 0.1z^{3/5} + 0.4z^{4/7} + 1\ , \tag{9.68}$$

then for $\alpha = 1/35$ and $\lambda = z^{\alpha} = z^{1/35}$ we obtain the associated polynomial of natural degree

$$\tilde{w}(\lambda) = \lambda^{61} + 0.2\lambda^{22} + 0.1\lambda^{21} + 0.4\lambda^{20} + 1\ , \tag{9.69}$$

which has degree equal to 61 and only 5 non-zero components.

To avoid this inconvenience, in [60] has been proposed the determination of the associated polynomial of natural degree not in the form of polynomial of one variable but in the form of polynomial of several independent variables. To investigation of stability of such polynomial, the LMI methods can be used.

In this section following [23] a computer method for stability analysis of linear discrete-time systems of fractional rational commensurate order which characteristic polynomial has the form (9.62) with $\alpha = 1/q$ and $q > 1$ will be presented. The method proposed is based on the modified Mikhailov criterion. This criterion will be applied to the associated polynomial (9.65) of natural degree.

In the case of discrete-time systems (of natural or fractional orders) the stability region is the open unit circle which boundary has the parametric description $\exp(j\omega\pi)$, $\omega \in [0,2]$. We can consider the interval $\omega \in [0,1]$ in the case of polynomials with real coefficients.

Applying the Mikhailov criterion to stability investigation of the associated polynomial (9.65) and using Theorem 9.10 we obtain the following theorem.

Theorem 9.11. *The discrete-time system of fractional rational commensurate order is stable if and only if*

$$\underset{0\le\omega\le 1}{\Delta}\arg\,\tilde{w}(\exp(j\omega\pi)) = n\pi, \tag{9.70}$$

i.e. if the plot of $\tilde{w}(\exp(j\omega\pi))$ *(the Mikhailov plot) with* ω *increasing from* 0 *to* 1 *encircles in the positive direction* $n/2$ *times the origin of the complex plane, where* $\tilde{w}(\exp(j\omega\pi)) = \tilde{w}(\lambda)$ *for* $\lambda = \exp(j\omega\pi)$.

Application of Theorem 9.11 to the investigation of stability of the fractional system is usually enough difficult since the degree of associated polynomial (9.65) is high.

The essential problems are connected with determination of number of encirclement of the origin of the complex plane by the Mikhailov plot. For example, for stable polynomial of natural order 61 the number of encirclement is equal to 30.5.

To avoid this inconvenience, we will apply the modified Mikhailov criterion [14] to investigation of the stability of associated polynomial (9.65) of natural degree.

Let $w_0(\lambda)$ be any stable polynomial of the same degree n as polynomial (9.65). We will call this polynomial the reference polynomial.

The reference polynomial can be chosen in the form

$$w_0(\lambda) = a_n\lambda^n, \tag{9.71}$$

where a_n is the coefficient of polynomial (9.62) (and (9.65)).

We will consider the rational function

$$\psi(\lambda) = \tilde{w}(\lambda)/w_0(\lambda), \tag{9.72}$$

instead of fractional degree polynomial (9.65) (or (9.62)), where $w_0(\lambda)$ is the reference polynomial.

Theorem 9.12. *The discrete-time system of fractional rational commensurate order is stable if and only if*

$$\underset{0\le\omega\le 1}{\Delta}\arg\,\psi(\exp(j\omega\pi)) = 0, \tag{9.73}$$

where $\psi(\exp(j\omega\pi)) = \psi(\lambda)$ *for* $\lambda = \exp(j\omega\pi)$, *i.e. the plot of* $\psi(\exp(j\omega\pi))$ *(called the modified Mikhailov plot) does not encircle the origin of the complex plane as* ω *runs from* 0 *to* 1.

Proof. By assumption the reference polynomial $w_0(\lambda)$ is stable and the condition (9.70) holds for this polynomial.

From (9.72) it follows that $\Delta\arg\psi(.) = \Delta\arg\tilde{w}(.) - \Delta\arg w_0(.)$. Hence, (9.70) holds if and only if (9.73) is satisfied. □

If the reference polynomial is of the form (9.71), then from (9.65) and (9.72) we have

$$\psi(e^{j0}) = \psi(1) = \tilde{w}(1)/a_n = \frac{1}{a_n}\sum_{i=0}^{n} a_i, \tag{9.74}$$

$$\psi(e^{j\pi}) = \psi(-1) = (-1)^n \tilde{w}(-1)/a_n = \frac{1}{a_n}\sum_{i=0}^{n} (-1)^{i+n} a_i. \tag{9.75}$$

Note that the plot of $\psi(\exp(j\omega\pi))$ encircles or crosses the origin of the complex plane if

$$(-1)^n\,\tilde{w}(-1)\,\tilde{w}(1) \le 0. \tag{9.76}$$

In this case the condition (9.73) is not satisfied. Hence, we have the following lemma.

Lemma 9.9. *The discrete-time system of fractional rational commensurate order with characteristic polynomial* (9.62) *is unstable if the condition* (9.76) *is met.*

Example 9.5. Check stability of the fractional system with characteristic fractional polynomial (9.68) and the associated natural degree polynomial (9.69).

We apply Theorem 9.12 with the reference polynomial of the form $w_0(\lambda) = \lambda^{61}$.

Plot of the function

$$\psi(\exp(j\omega\pi)) = \frac{\tilde{w}(\exp(j\omega\pi))}{w_0(\exp(j\omega\pi))}, \quad \omega \in [0,1], \tag{9.77}$$

is shown in Fig. 9.9. According to (9.74) and (9.75), this plot begins and finishes in the points

$$\psi(1) = \tilde{w}(1) = \sum_{i=0}^{n} a_i = 2.7\ , \tag{9.78}$$

$$\psi(-1) = (-1)^n \tilde{w}(-1) = \sum_{i=0}^{n} (-1)^{i+n} a_i = -0.5\ . \tag{9.79}$$

These points are denoted by • in Fig. 9.9.

Plot of the function (9.77) many times encloses the origin of the complex plane and by Theorem 9.12 the system is unstable.

The same result we obtain from Lemma 9.9, since

$$(-1)^n\,\tilde{w}(-1)\,\tilde{w}(1) = -1.35 < 0.$$

Example 9.6. Check stability of the fractional system with characteristic polynomial of the form [60]

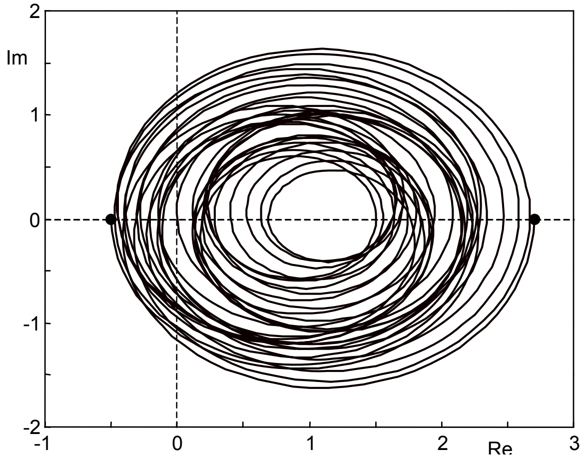

Fig. 9.9 Plot of the function (9.77). Illustration to Example 9.5.

$$
\begin{aligned}
w(z) = {} & z^{61/35} + 0.2757z^{8/5} + 0.1479z^{54/35} + 0.1653z^{51/35} \\
& - 0.2351z^{49/35} + 0.009457z^{47/35} + 0.04144z^{46/35} \\
& - 0.08902z^{44/35} + 0.0213z^{42/35} + 0.4983z^{41/35} \\
& + 0.0252z^{8/7} - 0.0493z^{39/35} + 0.05421z^{37/35} \\
& + 0.11071z^{36/35} + 0.0358z + 0.1284z^{34/35} + 0.07794z^{32/35} \\
& + 0.0288z^{31/35} + 0.00964z^{6/7} - 0.30901z^{29/35} +
\end{aligned}
$$

$$
\begin{aligned}
& - 0.00132z^{27/35} - 0.00347z^{26/35} - 0.03378z^{5/7} \\
& - 0.1669z^{24/35} + 0.1274z^{22/35} - 0.00725z^{3/5} \\
& + 0.1676z^{4/7} + 0.07858z^{19/35} + 0.07728z^{17/35} \\
& + 0.003193z^{3/7} - 0.014498z^{2/5} + 0.07091z^{12/35} \\
& - 0.04598z^{2/7} - 0.01123z^{1/5} + 0.0793z^{1/7} + 0.0007197\,.
\end{aligned}
$$

Assuming $q = 35$, $\alpha = 1/35$ and substituting $\lambda = z^{\alpha} = z^{1/35}$ we obtain the associated polynomial of natural degree

$$
\begin{aligned}
\tilde{w}(\lambda) = {} & \lambda^{61} + 0.2757\lambda^{56} + 0.1479\lambda^{54} + 0.1653\lambda^{51} \\
& - 0.2351\lambda^{49} + 0.009457\lambda^{47} + 0.04144\lambda^{46} \\
& - 0.08902\lambda^{44} + 0.0213\lambda^{42} + 0.4983\lambda^{41} \\
& + 0.0252\lambda^{40} - 0.0493\lambda^{39} + 0.05421\lambda^{37} \\
& + 0.11071\lambda^{36} + 0.0358\lambda^{35} + 0.1284\lambda^{34} + 0.07794\lambda^{32} \\
& + 0.0288\lambda^{31} + 0.00964\lambda^{30} - 0.30901\lambda^{29} \\
& - 0.00132\lambda^{27} - 0.00347\lambda^{26} - 0.03378\lambda^{25} \\
& - 0.1669\lambda^{24} + 0.1274\lambda^{22} - 0.00725\lambda^{21} \\
& + 0.1676\lambda^{20} + 0.07858\lambda^{19} + 0.07728\lambda^{17} \\
& + 0.003193\lambda^{15} - 0.014498\lambda^{14} + 0.07091\lambda^{12} \\
& - 0.04598\lambda^{10} - 0.01123\lambda^{7} + 0.0793\lambda^{5} + 0.0007197\,.
\end{aligned}
$$

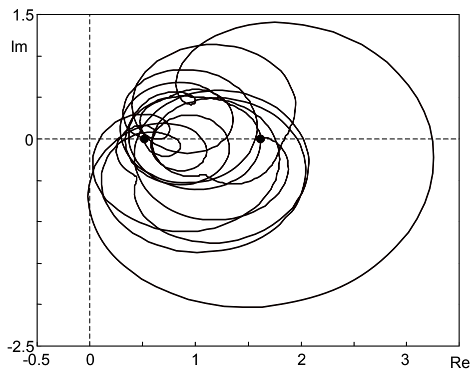

Fig. 9.10 Plot of the function (9.77). Illustration to Example 9.6.

Plot of the function (9.77) with the polynomial $\tilde{w}(\lambda)$ and the reference polynomial as in Example 9.5, is shown in Fig. 9.10. This plot begins with $\omega = 0$ in the point $\psi(1) = \tilde{w}(1) = 1.6173$ and finishes for $\omega = 1$ in the point $\psi(-1) = -\tilde{w}(-1) = 0.5177$. These points are denoted by • in Fig. 9.10.

The plot shown in Fig. 9.10 does not encircle the origin of the complex plane, and by Theorem 9.12 the fractional system is stable.

9.4 Robust Stability of Convex Combination of Two Fractional Polynomials

Consider the fractional degree polynomial

$$w(s, p) = w_1(s) + p w_2(s), \quad p \in P, \tag{9.80}$$

linearly dependent on one uncertain parameter p, where $P = [p^-, p^+]$ with $p^- < p^+$ is the value set of uncertain parameter and

$$w_1(s) = a_{1,n}s^{\alpha_n} + a_{1,n-1}s^{\alpha_{n-1}} + \cdots + a_{1,1}s^{\alpha_1} + a_{1,0}, \tag{9.81a}$$
$$w_2(s) = a_{2,m}s^{\beta_m} + a_{2,m-1}s^{\beta_{m-1}} + \cdots + a_{2,1}s^{\beta_1} + a_{2,0}, \tag{9.81b}$$

are given polynomials of fractional degrees where $\alpha_n > \alpha_{n-1} > \cdots > \alpha_1 > 0$ and $\beta_m > \beta_{m-1} > \cdots > \beta_1 > 0$ are arbitrary real numbers, $a_{1,i}$ $(i = 0,1,\ldots,n)$ and $a_{2,k}$ $(k = 0,1,\ldots,m)$ are real coefficients.

The polynomial (9.80) with uncertain parameter can be written in the form of convex combination of two fractional degrees polynomials

$$W(s,Q) = \{w(s,q) : q \in Q = [0,1]\}, \tag{9.82}$$

where

$$w(s,q) = (1-q)w_a(s) + qw_b(s), \tag{9.83}$$

with $q = (p-p^-)/(p^+ - p^-)$ and

$$\begin{aligned} w_a(s) &= w(s,p^-) = w_1(s) + p^- w_2(s) \\ &= a_n s^{\gamma_n} + a_{n-1}s^{\gamma_{n-1}} + \cdots + a_1 s^{\gamma_1} + a_0\,, \end{aligned} \tag{9.84a}$$
$$\begin{aligned} w_b(s) &= w(s,p^+) = w_1(s) + p^+ w_2(s) \\ &= b_n s^{\gamma_n} + b_{n-1}s^{\gamma_{n-1}} + \cdots + b_1 s^{\gamma_1} + b_0\,, \end{aligned} \tag{9.84b}$$

with $\gamma_n = \alpha_n$ and $\gamma_n > \gamma_{n-1} > \cdots > \gamma_1 > 0$.

In general case the family (9.82) of fractional polynomials is of the non-commensurate degree.

This family is of the commensurate degree, if polynomials (9.84) have commensurate degrees, i.e. if $\gamma_i = i\gamma$ for $i = 0,1,\ldots,n$ and $0 < \gamma < 1$. In such a case polynomials (9.84) can be written in the forms of natural degree polynomials

$$w_a(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0\,, \tag{9.85a}$$
$$w_b(\lambda) = b_n\lambda^n + b_{n-1}\lambda^{n-1} + \cdots + b_1\lambda + b_0\,, \tag{9.85b}$$

where $\lambda = s^\gamma$.

We will assume that the leading coefficient of the polynomial (9.83) is non-zero for all $q \in Q$, i. e.

$$(1-q)a_n + qb_n \neq 0, \quad \forall q \in Q.$$

If the above condition holds then the family (9.82) of fractional polynomials is degree invariant.

From the theory of stability of fractional order systems given by [198, 199] and [307], for example, we have the following theorem.

Let $w(s)$ be any fixed fractional degree polynomial.

Theorem 9.13. *The fractional order system with characteristic polynomial $w(s)$ is stable if and only if the fractional degree characteristic polynomial $w(s)$ is stable, i.e. $w(s)$ has no zeros in the closed right-half of the Riemann complex surface, i.e.*

$$w(s) \neq 0 \quad \textit{for} \quad \text{Re } s \geq 0. \tag{9.86}$$

The fractional order polynomial $w(s)$ is a multivalued function whose domain is a Riemann surface. In general case, this surface has an infinite number of sheets and the fractional polynomial $w(s)$ has an infinite number of zeros. Only a finite number of which will be in the main sheet of the Riemann surface. For stability reasons only the main sheet defined by $-\pi < \arg s < \pi$ can be considered [307].

Definition 9.1. The family (9.82) of fractional degree polynomials is called robustly stable, if polynomial $w(s,q)$ is stable for all $q \in Q$.

By generalization of Theorem 9.13 to the robust stability case we obtain the following theorem [19].

Theorem 9.14. *An uncertain system of fractional order with characteristic polynomial (9.82) is robustly stable if and only if the family (9.82) of fractional degree characteristic polynomials is robustly stable, i.e. $w(s,q)$ has no zeros in the closed right-half of the Riemann complex surface for all $q \in Q$, that is*

$$w(s,q) \neq 0 \quad \textit{for} \quad \text{Re } s \geq 0 \quad \textit{and for all} \quad q \in Q. \tag{9.87}$$

The problem of robust stability analysis of family (9.82) of fractional polynomials was considered in [29] in the case of commensurate degrees of polynomials (9.84), i.e. with $\gamma_i = i\gamma$, $i = 0, 1, \ldots, n$, $0 < \gamma < 1$. In such a case family (9.82) is robustly stable if and only if all zeros of the polynomial $w(\lambda, q) = (1-q)w_a(\lambda) + qw_b(\lambda)$ with $w_a(\lambda)$ and $w_b(\lambda)$ of the forms (9.85) satisfy the condition $|\arg \lambda| > 0.5\gamma\pi$ for all $q \in Q = [0, 1]$.

In this section following [19] the frequency domain methods for robust stability analysis of family (9.82) of fractional polynomials of non-commensurate degrees will be presented. The methods proposed are based on the Argument Principle and they are a generalization to the fractional polynomials case of the methods given in [14] in the case of natural degree polynomials.

First we consider the problem of stability analysis of fixed fractional polynomial of non-commensurate degree, in general, of the form

$$w(s) = a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \cdots + a_1 s^{\alpha_1} + a_0, \tag{9.88}$$

where $\alpha_n > \alpha_{n-1} > \cdots > \alpha_1 > 0$ are real numbers.

From [17] we have the following frequency domain method for stability analysis of the fractional polynomial (9.88).

Theorem 9.15. *The fractional degree polynomial (9.88) is stable if and only if*

$$\underset{\omega\in(-\infty,\infty)}{\Delta \arg} \quad \psi(j\omega) = 0, \tag{9.89}$$

with $\psi(j\omega) = \psi(s)$ *for* $s = j\omega$ *and*

$$\psi(s) = \frac{w(s)}{w_r(s)}, \tag{9.90}$$

where $w_r(s)$ *is the reference fractional polynomial of the same order* α_n *as (9.88) and it is stable, i.e.*

$$w_r(s) \neq 0 \quad \textit{for} \quad \mathrm{Re}\ s \geq 0. \tag{9.91}$$

Proof. From (9.90) for $s = j\omega$ it follows that

$$\Delta \arg \psi(j\omega) = \Delta \arg w(j\omega) - \Delta \arg w_r(j\omega).$$

By assumption the reference polynomial $w_r(s)$ of the same fractional degree as polynomial (9.88) is stable. Therefore, the fractional polynomial (9.88) is stable if and only if

$$\underset{\omega\in(-\infty,\infty)}{\Delta \arg} \quad w(j\omega) = \underset{\omega\in(-\infty,\infty)}{\Delta \arg} \quad w_r(j\omega),$$

which holds if and only if (9.89) is satisfied. □

The reference fractional polynomial $w_r(s)$ can be chosen in the form

$$w_r(s) = a_n(s+c)^{\alpha_n}, \quad c > 0. \tag{9.92}$$

Note that for $c > 0$ the reference polynomial (9.92) is stable.

Condition (9.89) of Theorem 9.15 holds if and only if the plot of $\psi(j\omega)$ does not encircle the origin of the complex plane as ω runs from $-\infty$ to $+\infty$.

From (9.90), (9.88) and (9.92) we have

$$\lim_{\omega\to\pm\infty} \psi(j\omega) = \lim_{\omega\to\pm\infty} \frac{w(j\omega)}{w_r(j\omega)} = 1, \tag{9.93}$$

and

$$\psi(0) = \frac{w(0)}{w_r(0)} = \frac{a_0}{a_n c^{\alpha_n}}. \tag{9.94}$$

By Theorem 9.15 the fractional degree characteristic polynomial (9.88) is unstable if $a_0/a_n \leq 0$.

Now we consider the robust stability problem of the family (9.82) of fractional polynomials.

Without loss of generality we will assume that $w_a(s)$ is the nominal polynomial of this family and that $w_a(s)$ is stable, i.e. $w_a(s) \neq 0$ for $Res \geq 0$. Theorem 9.15 can by used for stability analysis of this polynomial.

Let ω be any fixed real number. Substituting $s = j\omega$ in (9.84) one obtain complex numbers $w_a(j\omega)$ and $w_b(j\omega)$ - values of polynomials $w_a(s)$ and $w_b(s)$ for $s = j\omega$.

Definition 9.2. For any fixed complex number $s = j\omega$ the set defined by

$$w(j\omega, Q) = \{w(j\omega, q) : q \in Q = [0,1]\}, \tag{9.95}$$

where $w(j\omega, q)$ has the form (9.83) for $s = j\omega$, is called the value set of the family (9.82) of fractional polynomials.

The value set (9.95) is the straight line segment joining the points $w_a(j\omega)$ and $w_b(j\omega)$ in the complex plane.

Theorem 9.16. *Let the polynomial $w_a(s)$ be stable. Family (9.82) of fractional polynomials is robustly stable if and only if the following condition (called as the Zero Exclusion Condition)*

$$0 \notin w(j\omega, Q), \quad \forall \omega \in \Omega = [0, \infty), \tag{9.96}$$

holds, where $w(j\omega, Q)$ is defined by (9.95).

Proof. If the condition (9.96) does not hold, then there exist $\omega = \overline{\omega} \in \Omega$ and $q = \overline{q} \in Q$ such that $w(j\overline{\omega}, \overline{q}) = 0$. This means that polynomial $w(s, \overline{q}) \in W(s, Q)$ has zero $s = j\overline{\omega}$ on the imaginary axis and the family (9.82) is robustly unstable.

Now we assume that the family (9.82) of fractional polynomials is robustly unstable. Then in this family exists at least one unstable polynomial $w(s, \tilde{q})$ with $\tilde{q} > 0$. This follows from the fact that the nominal polynomial $w_a(s) = w(s, 0)$ by assumption is stable.

From the above and continuous dependence of coefficients of the polynomial $w(s, q)$ on uncertain parameter q it follows that there exists $\overline{q} \in (0, \tilde{q})$ such that polynomial $w(s, \overline{q})$ has at least one zero on the imaginary axis, i.e. $w(j\omega, \overline{q}) = 0$ for a some fixed $\omega \in \Omega$ and the condition (9.96) is not satisfied. □

If the condition (9.96) holds then the origin of the complex plane is excluded from the value set (9.95) for all $\omega \in \Omega = [0, \infty)$. Therefore, the condition (9.96) is called as the Zero Exclusion Condition, see [5], for example.

It is easy to see that $w_a(j\omega)$ and $w_b(j\omega)$ (endpoints of the value set (9.95) for fixed ω) quickly tend to infinity as $\omega \to \infty$. Therefore, application of Theorem 9.16 is in general case a difficult problem.

To remove this difficulty, similarly as in [14] in the case of natural degree polynomials, we will consider the normalized value set instead of the value set (9.95).

Definition 9.3. Let the polynomial $w_a(s)$ be stable. For the fixed complex number $s = j\omega$ the value set defined by

$$w_{nor}(j\omega, Q) = \{w_{nor}(j\omega, q) : q \in Q = [0,1]\}, \tag{9.97a}$$

with

$$w_{nor}(j\omega,q) = w(j\omega,q)/w_a(j\omega), \quad w_a(j\omega) \neq 0, \tag{9.97b}$$

where $w(j\omega,q)$ has the form (9.83) for $s = j\omega$ is called the normalized value set of the family (9.82) of fractional polynomials.

For any fixed complex number $s = j\omega$ the normalized value set (9.97a) is the straight line segment with endpoints $w_{nor}(j\omega,0) = 1 + j0$ and $w_{nor}(j\omega,1) = w_b(j\omega)/w_a(j\omega)$. Because $w_{nor}(j\omega,0) = 1 + j0$ for all $\omega \in \Omega$, the normalized value set (9.97a) always lies near of the origin of the complex plane.

From the above and Theorem 9.16 it follows that the Zero Exclusion Condition for the normalized value set (9.97a) can be formulated as follows.

Theorem 9.17. *Let the nominal polynomial $w_a(s)$ be stable. Family of polynomials (9.82) is robustly stable if and only if the following condition is satisfied*

$$0 \notin w_{nor}(j\omega,Q), \quad \forall\omega \in \Omega = [0,\infty). \tag{9.98}$$

The parametric description of the boundary of stability region, i.e. of the imaginary axis of the complex plane, has the form $s = j\omega$, $\omega \in (-\infty,\infty)$. Zeroes of fractional polynomials with real coefficients are complex conjugate. Therefore, in the Zero Exclusion Conditions (9.96) and (9.98) we can consider only the interval $\Omega = [0,\infty)$ of the parameter ω.

Satisfaction of the condition (9.98) can be checked directly by plotting the normalized value set (9.97a) (straight line segment) for all fixed $\omega = i \cdot \Delta\omega$, $i = 0,1,\dots$, where $\Delta\omega$ is the sufficiently small step.

Now we consider the methods for checking of the Zero Exclusion Condition (9.98) without plotting the normalized value set (9.97a).

It is easy to see that if for fixed $\omega = \overline{\omega} \in \Omega$ the straight line segment (9.97a) crosses the origin of the complex plane then

$$w_{nor}(j\overline{\omega},1) = w_b(j\overline{\omega})/w_a(j\overline{\omega}) < 0,$$

since $w_{nor}(j\overline{\omega},0) = 1 + j0$. In such case the following condition holds

$$|\arg(w_a(j\overline{\omega})) - \arg(w_b(j\overline{\omega}))| = \pi, \tag{9.99}$$

where $\arg(\cdot) \in [-\pi,\pi)$.

From the above it follows that the condition (9.98) of Theorem 9.17 holds if and only if

$$\varphi(\omega) \neq 0, \quad \forall\omega \in \Omega, \tag{9.100}$$

where

$$\varphi(\omega) = \pi - |\arg(w_a(j\omega)) - \arg(w_b(j\omega))| \tag{9.101}$$

is the testing function.

Hence, we have the following lemma.

Lemma 9.10. *Let the nominal polynomial* $w_a(s)$ *be stable. Family of polynomials* (9.82) *is robustly stable if and only if the condition* (9.100) *is satisfied.*

Now we shall prove the following theorem.

Theorem 9.18. *Let the nominal polynomial* $w_a(s)$ *be stable. Family of polynomials* (9.82) *is robustly stable if and only if plot of the function*

$$\vartheta(j\omega) = \frac{w_b(j\omega)}{w_a(j\omega)}, \quad \omega \in \Omega, \tag{9.102}$$

does not cross the non-positive part $(-\infty, 0]$ *of the real axis in the complex plane.*

Proof. From the above considerations it follows that for any fixed $\omega \in \Omega$ the straight line segment $w_{nor}(j\omega, Q)$ with one endpoint $w_{nor}(j\omega, 0) = 1 + j0$ does not cross the origin of the complex plane for all $\omega \in \Omega$ if and only if plot of the function $w_b(j\omega)/w_a(j\omega)$, $\omega \in \Omega$, does not cross the non-positive part of the real axis. □

From (9.102) and (9.84) it follows that

$$\vartheta(0) = \frac{b_0}{a_0}, \quad \lim_{\omega \to \pm\infty} \vartheta(j\omega) = \frac{b_n}{a_n}. \tag{9.103}$$

Lemma 9.11. *Let the nominal polynomial* $w_a(s)$ *be stable. Family* (9.82) *of fractional polynomials is robustly unstable if* $b_0/a_0 \leq 0$ *or* $b_n/a_n \leq 0$.

Proof. If $b_0/a_0 \leq 0$ or $b_n/a_n \leq 0$ then plot of the function (9.102) crosses the non-positive part of the real axis and by Theorem 9.18 the family (9.82) of fractional polynomials is unstable. □

Example 9.7. Consider the control system shown in Fig. 9.7 with the fractional order plant described by the nominal transfer function

$$G_0(s) = \frac{1}{0.8s^{2.2} + 0.5s^{0.9} + 1} = \frac{1}{D_0(s)} \tag{9.104}$$

and fractional order PID controller

$$C(s) = k_p + \frac{k_i}{s^\lambda} + k_d s^\mu. \tag{9.105}$$

In [313] it was shown that closed loop system with the plant (9.104) is stable and it has the gain margin $A_m = 1.3$ and phase margin $\phi_m = 60°$ for the controller (9.105) with $\lambda = 0.1$, $\mu = 1.15$, $k_p = 233.4234$, $k_i = 22.3972$ and $k_d = 18.5274$, i.e. for the controller PID with the transfer function

$$C(s) = \frac{18.5274s^{1.25} + 233.4234s^{0.1} + 22.3972}{s^{0.1}} = \frac{N_c(s)}{D_c(s)}. \tag{9.106}$$

Characteristic polynomial of the closed loop system with the plant (9.104) and controller (9.106) has the form

$$\begin{aligned} w_{c0}(s) &= D_0(s)D_c(s)+N_c(s) \\ &= 0.8s^{2.3}+18.5274s^{1.25}+0.5s+234.4234s^{0.1}+22.3971. \end{aligned} \quad (9.107)$$

It is assumed that the model of the plant is not precisely known and it is described by the family of transfer functions

$$G(s,p)=\frac{1}{D_0(s)+p\Delta(s)}=\frac{1}{D(s,p)}, p\in P=[-1,1], \quad (9.108)$$

where $D_0(s)$ has the form shown in (9.104) and

$$\Delta(s)=0.4s^{2.2}+0.2s^{0.9}+0.5 \quad (9.109)$$

is the perturbation polynomial.

Characteristic polynomial of the closed loop system with the plant (9.108) and controller (9.106) has the form

$$\begin{aligned} w_c(s,p) &= D(s,p)D_c(s)+N_c(s) \\ &= [D_0(s)D_c(s)+N_c(s)]+p\Delta(s)D_c(s) \\ &= w_{c0}(s)+p\Delta(s)D_c(s)=w_{c0}(s)+pd(s), \end{aligned} \quad (9.110)$$

where $p\in P=[-1,1]$, $w_{c0}(s)$ has the form (9.107) and

$$d(s)=0.4s^{2.3}+0.2s+0.5s^{0.1}. \quad (9.111)$$

The polynomial (9.110) with uncertain parameter $p\in P=[-1,1]$ can be written in the form

$$w(s,q)=(1-q)w_a(s)+qw_b(s), q\in Q=[0,1], \quad (9.112)$$

where

$$w_a(s) = w_c(s,p^-)=w_{c0}(s)-d(s), \quad (9.113a)$$

$$w_b(s) = w_c(s,p^+)=w_{c0}(s)+d(s). \quad (9.113b)$$

For the given polynomials $w_{c0}(s)$ and $d(s)$ from (9.113) we have

$$w_a(s) = 0.4s^{2.3}+18.5274s^{1.25}+0.3s+233.9234s^{0.1}+22.3971, \quad (9.114)$$

$$w_b(s) = 1.2s^{2.3}+18.5274s^{1.25}+0.7s+234.9234s^{0.1}+22.3971. \quad (9.115)$$

First, we check stability of the polynomial (9.114).

Plot of the function

$$\psi(j\omega)=\frac{w_a(j\omega)}{w_r(j\omega)}, \quad (9.116)$$

where $w_a(s)$ has the form (9.114) and $w_r(s) = 0.4(s+10)^{2.3}$ is the reference fractional polynomial, is shown in Fig. 9.11. From (9.93), (9.94) we have

$$\psi(0) = \frac{22.3971}{0.4 \cdot 10^{2.3}} = 0.2806, \qquad \lim_{\omega \to \pm\infty} \psi(j\omega) = 1.$$

From Fig. 9.11 it follows that the plot of $\psi(j\omega)$ does not encircle the origin of the complex plane. By Theorem 9.15, the nominal polynomial (9.114) is stable.

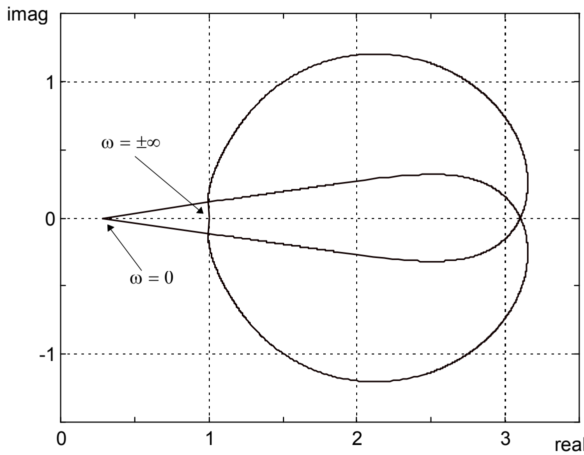

Fig. 9.11 Plot of the function (9.116)

Plot of the function (9.102) with $w_a(s)$ and $w_b(s)$ of the forms (9.114) and (9.115), respectively, is shown in Fig. 9.12. From (9.103) and (9.114), (9.115) we have

$$\vartheta(0) = \frac{w_b(0)}{w_a(s0)} = 1, \qquad \lim_{\omega \to \pm\infty} \vartheta(j\omega) = \frac{1.2}{0.4} = 3. \tag{9.117}$$

The plot of $\vartheta(j\omega)$ does not cross of the non-positive part of the real axis and by Theorem 9.18 the system is robustly stable. Therefore, the control system with the controller (9.106) and uncertain plant (9.108) is stable for all $p \in P = [-1, 1]$.

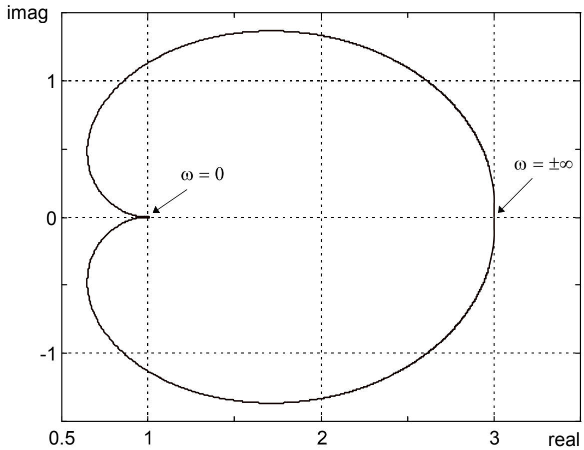

Fig. 9.12 Plot of the function $\vartheta(j\omega)$ defined by (9.102)

Chapter 10
Stabilization of Positive and Fractional Linear Systems

10.1 Fractional Discrete-Time Linear Systems with Delays

Consider the fractional discrete-time linear system with h delays:

$$x_{i+1} = \sum_{j=1}^{i+1}(-1)^{j+1}\binom{\alpha}{j}x_{i-j+1} + \sum_{k=0}^{h}\left(A_k x_{i-k} + B_k u_{i-k}\right),\ i \in \mathbb{Z}_+, \tag{10.1a}$$

$$y_i = Cx_i + Du_i, \quad 0 < \alpha < 1, \tag{10.1b}$$

and with the state-feedback

$$u_i = Kx_i, \quad i \in \mathbb{Z}_+, \tag{10.2}$$

where $K \in \mathbb{R}^{m\times n}$ is a gain matrix.

We are looking for a gain matrix K such that the closed-loop system

$$x_{i+1} = \sum_{k=0}^{h}\left(A_k + B_k K + I_n c_{k+1}\right)x_{i-k} + \sum_{j=h+2}^{i+1} c_j x_{i-j+1}, \tag{10.3}$$

$$c_j = (-1)^{j+1}\binom{\alpha}{j} \quad \text{for} \quad j = 1,\dots,i+1;$$

is positive and asymptotically stable.

Theorem 10.1. *The fractional closed-loop system (10.3) is positive and asymptotically stable if and only if there exists a diagonal matrix*

$$\Lambda = \operatorname{diag}\left[\,\lambda_1\ \dots\ \lambda_n\,\right], \tag{10.4}$$

with positive diagonal entries $\lambda_k > 0$, $k = 1,\dots,n$ and a matrix $D \in \mathbb{R}^{m\times n}$ such that

$$\left(A_k + I_n c_{k+1}\right)\Lambda + B_k D \in \mathbb{R}_+^{n\times n}, \quad k = 0,1,\dots,h \tag{10.5}$$

T. Kaczorek: Selected Problems of Fractional Systems Theory, LNCIS 411, pp. 221–243.
springerlink.com

and

$$\sum_{k=0}^{h} (A_k \Lambda + B_k D) \mathbb{1}_n < 0, \tag{10.6}$$

where $\mathbb{1}_n = [1, \ldots, 1]^T \in \mathbb{R}^{n \times n}$.

The matrix K is given by

$$K = D\Lambda^{-1}. \tag{10.7}$$

Proof. First we shall show that the closed-loop system (10.3) is positive if and only if the condition (10.5) is satisfied. Using (10.7) and (10.3) we obtain

$$\begin{aligned} A_k + B_k K + I_n c_{k+1} &= A_k + B_k D \Lambda^{-1} + I_n c_{k+1} \\ &= [(A_k + I_n c_{k+1})\Lambda + B_k D]\Lambda^{-1} \in \mathbb{R}_+^{n \times n} \end{aligned} \tag{10.8}$$

since the condition (10.5) is met.

Taking into account that

$$K\Lambda \mathbb{1}_n = D\Lambda^{-1}\Lambda \mathbb{1}_n = D\mathbb{1}_n \quad \text{and} \quad \Lambda \mathbb{1}_n = \lambda, \tag{10.9}$$

and using (8.47) we obtain

$$\begin{aligned} \left[\sum_{k=0}^{h} (A_k + B_k K) + \sum_{j=1}^{\infty} I_n c_j - I_n\right] \lambda &= \sum_{k=0}^{h} (A_k + B_k K)\Lambda \mathbb{1}_n \\ &= \sum_{k=0}^{h} (A_k \Lambda + B_k D) \mathbb{1}_n < 0, \end{aligned} \tag{10.10}$$

when the conditions (8.47) and (10.6) are satisfied.

By Theorem 8.12 the closed-loop system (10.3) is asymptotically stable if and only if the condition (10.6) is met. □

If the conditions (10.5) and (10.6) are satisfied then the gain matrix K can be found by the use of the following procedure:

Procedure 10.1

Step 1. Choose a diagonal matrix (10.4) with $\lambda_k > 0$, $k = 1, \ldots, n$ and a matrix $D \in \mathbb{R}^{m \times n}$ satisfying the conditions (10.5) and (10.6).
Step 2. Using (10.7) find the gain matrix K.

Example 10.1. Consider the fractional system (10.1) witch $\alpha = 0.5$, $h = 2$ and

$$A_0 = \begin{bmatrix} -0.4 & 0.4 \\ 0.6 & -0.3 \end{bmatrix}, \; A_1 = \begin{bmatrix} -0.1 & 0.05 \\ 0.1 & -0.1 \end{bmatrix}, \; A_2 = \begin{bmatrix} -0.04 & 0.05 \\ 0.05 & -0.05 \end{bmatrix}, \tag{10.11}$$

$$B_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; B_1 = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0.1 \\ 0.05 \end{bmatrix}. \tag{10.12}$$

Find a gain matrix $K \in \mathbb{R}^{1\times 2}$ such that the closed-loop system is positive and asymptotically stable.

The fractional system is positive but unstable since the matrices

$$\begin{aligned}
A_0 + c_1 I_n &= \begin{bmatrix} -0.4 & 0.4 \\ 0.6 & -0.3 \end{bmatrix} + 0.5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.4 \\ 0.6 & 0.2 \end{bmatrix}, \\
A_1 + c_2 I_n &= \begin{bmatrix} -0.1 & 0.05 \\ 0.1 & -0.1 \end{bmatrix} + 0.125 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.025 & 0.05 \\ 0.1 & 0.025 \end{bmatrix}, \\
A_2 + c_3 I_n &= \begin{bmatrix} -0.04 & 0.05 \\ 0.05 & -0.05 \end{bmatrix} + 0.0625 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.0225 & 0.05 \\ 0.05 & 0.0125 \end{bmatrix},
\end{aligned} \tag{10.13}$$

have positive entries and the characteristic polynomial of the matrix $A = A_0 + A_1 + A_2$

$$\det[Iz - A] = \begin{bmatrix} z + 0.54 & -0.5 \\ -0.75 & z + 0.45 \end{bmatrix} = z^2 + 0.99z - 0.132, \tag{10.14}$$

has one ($a_0 = -0.132$) negative entry. Using Procedure 10.1 we obtain the following:

Step 1. We choose:

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -0.1 & -0.2 \end{bmatrix}, \tag{10.15}$$

and we check the conditions (10.5) and (10.6)

$$\begin{aligned}
(A_0 + I_n c_1)\Lambda + B_0 D &= \begin{bmatrix} 0 & 0.2 \\ 0.5 & 0 \end{bmatrix}, \\
(A_1 + I_n c_2)\Lambda + B_1 D &= \begin{bmatrix} 0.005 & 0.01 \\ 0.09 & 0.005 \end{bmatrix}, \\
(A_2 + I_n c_3)\Lambda + B_2 D &= \begin{bmatrix} 0.0125 & 0.03 \\ 0.045 & 0.0025 \end{bmatrix},
\end{aligned}$$

and

$$\sum_{k=0}^{2} (A_k \Lambda + B_k D)\mathbb{1}_n = \begin{bmatrix} -0.43 \\ -0.045 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{10.16}$$

Thus, the conditions (10.5) and (10.6) are satisfied.

Step 2. Using (10.7), we obtain the desired gain matrix

$$K = D\Lambda^{-1} = \begin{bmatrix} -0.1 & -0.2 \end{bmatrix}. \tag{10.17}$$

The closed-loop system is positive and asymptotically stable since the matrices (10.13) have positive entries and the condition (10.16) is satisfied.

10.2 Fractional Continuous-Time Linear Systems with Delays

Consider the fractional continuous-time linear system with delays

$$\frac{d^\alpha x(t)}{dt^\alpha} = \sum_{k=0}^{q} \left[A_k x(t-d_k) + B_k u(t-d_k)\right], \tag{10.18}$$

with the state-feedback

$$u(t) = Kx(t), \tag{10.19}$$

where $K \in \mathbb{R}^{m\times n}$ is a gain matrix.

Substituting (10.19) in (10.18) we obtain the closed-loop system

$$\frac{d^\alpha x(t)}{dt^\alpha} = \sum_{k=0}^{q} (A_k + B_k K) x(t-d_k), \quad 0 < \alpha \leq 1. \tag{10.20}$$

The positive system with delays (10.20) is asymptotically stable if and only if the positive system without delays

$$\frac{d^\alpha x(t)}{dt^\alpha} = (A+BK)x(t), \quad A = \sum_{k=0}^{q} A_k, \quad B = \sum_{k=0}^{q} B_k, \tag{10.21}$$

is asymptotically stable.

We are looking for a gain matrix K such that the closed-loop system (10.20) is positive and the zeros of the characteristic polynomial

$$\det\left[I_n s^\alpha - (A+BK)\right] = (s^\alpha)^n + \overline{a}_{n-1}(s^\alpha)^{n-1} + \cdots + \overline{a}_1 s + \overline{a}_0, \tag{10.22}$$

are located in the sector $\phi = \frac{\pi}{2\alpha}$.

Theorem 10.2. *The closed-loop fractional system* (10.20) *is positive and the zeros of the polynomial* (10.22) *are located in the sector* $\phi = \frac{\pi}{2\alpha}$ *if and only if there exist a diagonal matrix*

$$\Lambda = \operatorname{diag}[\lambda_1, \dots, \lambda_n] \quad \text{with} \quad \lambda_k > 0, \quad k = 1, \dots, n; \tag{10.23}$$

and a real matrix $D \in \mathbb{R}^{m\times n}$ *such that the following conditions are satisfied*

$$A\Lambda + BD \in M_n, \tag{10.24}$$

$$(A\Lambda + BD)\mathbb{1}_n < 0. \tag{10.25}$$

The gain matrix K is given by the formula

$$K = D\Lambda^{-1}. \tag{10.26}$$

Proof. First we shall show that the closed-loop system (10.20) is positive if and only if (10.24) holds. Using (10.20), (10.21) and (10.26) we obtain

$$\sum_{k=0}^{q}(A_k + B_k K) = A + BK = A + BD\Lambda^{-1} = (A\Lambda + BD)\Lambda^{-1} \in M_n, \tag{10.27}$$

if and only if the condition (10.24) is satisfied.

Taking into account that

$$K\Lambda\mathbb{1}_n = D\Lambda^{-1}\Lambda\mathbb{1}_n = D\mathbb{1}_n \quad \text{and} \quad \Lambda\mathbb{1}_n = \lambda = [\lambda_1, \dots, \lambda_n]^T, \tag{10.28}$$

and using (10.25) we obtain

$$(A + BK)\lambda = (A + BK)\Lambda\mathbb{1}_n = (A\Lambda + BD)\mathbb{1}_n < 0. \tag{10.29}$$

Therefore, by Theorem 8.4 the zeros of the characteristic polynomial (10.22) are located in the sector $\phi = \frac{\pi}{2\alpha}$ if and only if the condition (10.25) is met. □

If the conditions of Theorem 10.2 are satisfied then the problem of stabilization can be solved by the use of the following procedure:

Procedure 10.2

Step 1. Choose a diagonal matrix (10.23) with $\lambda_k > 0$, $k = 1, \dots, n$ and a real matrix $D \in \mathbb{R}^{m \times n}$ satisfying the conditions (10.24) and (10.25).

Step 2. Using (10.26) find the gain matrix K.

Example 10.2. Given the fractional system (10.18) with $\alpha = 0.8$, $q = 2$ and the matrices

$$A_0 = \begin{bmatrix} 0.5 & 0.3 & -0.2 \\ 0.2 & -1 & 0 \\ 0 & -0.2 & 1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.3 & 0.4 & -0.3 \\ 0.1 & -0.5 & 0 \\ 0 & -0.1 & 1 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0.2 & 0.3 & -0.5 \\ 0.7 & -1.5 & 0 \\ 0 & -0.7 & 0.5 \end{bmatrix}, \tag{10.30}$$
$$B_0 = \begin{bmatrix} 0 & 0.1 \\ 0 & 0 \\ 0.2 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0.5 \\ 0 & 0 \\ 0.3 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0.4 \\ 0 & 0 \\ 0.5 & 0 \end{bmatrix}.$$

Find a gain matrix $K \in \mathbb{R}^{2 \times 3}$ such that the closed-loop system is positive and the zeros of its characteristic polynomial are located in the sector $\phi = \frac{\pi}{2\alpha}$.

Note that the fractional system with (10.30) is not positive since the matrices A_0, A_1 and A_2 have negative off-diagonal entries.

In this case

$$A = \sum_{k=0}^{2} A_k = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -3 & 0 \\ 0 & -1 & 2.5 \end{bmatrix}, \quad B = \sum_{k=0}^{2} B_k = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{10.31}$$

Using Procedure and (10.31) we obtain the following

Step 1. We choose

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0.5 & 2 & -3.5 \\ -4 & 0 & 1.4 \end{bmatrix}, \tag{10.32}$$

and we check the condition (10.24)

$$A\Lambda + BD = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -3 & 0 \\ 0 & -1 & 2.5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & 2 & -3.5 \\ -4 & 0 & 1.4 \end{bmatrix}$$
$$= \begin{bmatrix} -3 & 2 & 0.4 \\ 1 & -6 & 0 \\ 0.5 & 0 & -1 \end{bmatrix} \in M_3,$$

and the condition (10.25)

$$(A\Lambda + BD)\mathbb{1}_n = \begin{bmatrix} -3 & 2 & 0.4 \\ 1 & -6 & 0 \\ 0.5 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.6 \\ -5 \\ -0.5 \end{bmatrix}.$$

Therefore, the conditions are satisfied.

Step 2. Using (10.26) we obtain the gain matrix

$$K = D\Lambda^{-1} = \begin{bmatrix} 0.5 & 2 & -3.5 \\ -4 & 0 & 1.4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.5 & 1 & -3.5 \\ -4 & 0 & 1.4 \end{bmatrix}.$$

The closed-loop system is positive since the matrix

$$A_c = A + BK = \begin{bmatrix} -3 & 1 & 0.4 \\ 1 & -3 & 0 \\ 0.5 & 0 & -1 \end{bmatrix},$$

is a Metzler matrix.

The characteristic polynomial

$$\det[I_n\lambda - A_c] = \begin{vmatrix} \lambda+3 & -1 & -0.4 \\ -1 & \lambda+3 & 0 \\ -0.5 & 0 & \lambda+1 \end{vmatrix} = \lambda^3 + 7\lambda^2 + 13.8\lambda + 7.4\,,$$

has positive coefficients. Therefore, zeros of the characteristic polynomial of the closed-loop system are located in the desired sector $\phi = \frac{5}{8}\pi$.

10.3 Application of LMI to Stabilization of Fractional Linear Systems

10.3.1 Fractional 1D Linear Systems

Definition 10.1. An inequality of the form

$$F(x) + \mathscr{F} > 0 \tag{10.33}$$

where x takes values in real vector space V, the mapping $F : V \longrightarrow S^n$ is linear, and $\mathscr{F} \in S^n$, is called the linear matrix inequality (LMI). The LMI is called feasible if there exists an $x \in V$ such that the inequality (10.33) is satisfied, otherwise the LMI is called infeasible.

A matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is called the Metzler matrix if its off-diagonal entries are nonnegative, i.e. $a_{ij} \geq 0$ for $i \neq j$, $i, j = 1, \ldots, n$. The matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is called Hurwitz matrix if it has all eigenvalues with negative real parts (the system $\dot{x} = Ax$ is asymptotically stable). The matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is called Schur matrix if it has all eigenvalues with module less than one (the system $x_{i+1} = Ax_i$ is asymptotically stable).

Lemma 10.1. *A Metzler matrix $A \in \mathbb{R}^{n\times n}$ is Hurwitz matrix if and only if the LMI*

$$\text{block diag}\left[-\left(A^T P + PA\right)\ ,\ P\right] \succ 0\ , \tag{10.34}$$

is feasible with respect to the diagonal matrix P.

Remark 10.1. It is well-known that $A \in \mathbb{R}^n_+$ is Schur matrix if and only if $(A - I_n)$ is Hurwitz matrix.

Lemma 10.2. *A nonnegative matrix $A \in \mathbb{R}^n_+$ is Schur matrix if and only if the LMI*

$$\text{block diag}\left[-\left((A - I_n)^T P + P(A - I_n)\right)\ ,\ P\right] \succ 0\ , \tag{10.35}$$

is feasible with respect to the diagonal matrix P.

Lemma 10.3. *A nonnegative matrix $A \in \mathbb{R}^n_+$ is Schur matrix if and only if the LMI*

$$\text{block diag}\left[P - A^T PA\ ,\ P\right] \succ 0\ , \tag{10.36}$$

is feasible with respect to the diagonal matrix P.

Theorem 10.3. *The positive fractional system (8.68) is practically stable if and only if one of the following equivalent conditions holds:*

a) The LMI

$$\text{block diag}\left\{\begin{bmatrix} \tilde{P}_1 - A_\alpha^T P_1 A_\alpha & -c_1 A_\alpha^T P_1 & \dots & -c_{h-1} A_\alpha^T P_1 & -c_h A_\alpha^T P_1 \\ -c_1 P_1 A_\alpha & \tilde{P}_2 - c_1^2 P_1 & \dots & -c_1 c_{h-1} P_1 & -c_1 c_h P_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_{h-1} P_1 A_\alpha & -c_1 c_{h-1} P_1 & \dots & \tilde{P}_h - c_{h-1}^2 P_1 & -c_{h-1} c_h P_1 \\ -c_h P_1 A_\alpha & -c_1 c_h P_1 & \dots & -c_{h-1} c_h P_1 & P_{h+1} - c_h^2 P_1 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} P_1 & 0 & \dots & 0 & 0 \\ 0 & P_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P_h & 0 \\ 0 & 0 & \dots & 0 & P_{h+1} \end{bmatrix} \right\} \succ 0 \ , \tag{10.37}$$
$$\tilde{P}_i = P_i - P_{i+1}, \quad i = 1, \dots, h;$$

is feasible with respect to the diagonal matrix $P_1, \dots, P_{h+1}$.

b) The LMI

$$\text{block diag}\left\{\begin{bmatrix} A_\alpha^T P_1 + P_1 A_\alpha - 2P_1 & P_2 + c_1 P_1 & \dots & c_{h-1} P_1 & c_h P_1 \\ P_2 + c_1 P_1 & -2P_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{h-1} P_1 & 0 & \dots & -2P_{h-1} & P_{h+1} \\ c_h P_1 & 0 & \dots & P_{h+1} & -2P_h \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} P_1 & 0 & \dots & 0 & 0 \\ 0 & P_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P_h & 0 \\ 0 & 0 & \dots & 0 & P_{h+1} \end{bmatrix} \right\} \succ 0 \ , \tag{10.38}$$

is feasible with respect to the diagonal matrix $P_1, \dots, P_{h+1}$.

c) The LMI

$$\text{block diag}\left\{\begin{bmatrix} P_1 & 0 & \dots & 0 & -A_\alpha^T P_1 & -P_2 & \dots & 0 \\ 0 & P_2 & \dots & 0 & -c_1 P_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_{h+1} & -c_h P_1 & 0 & \dots & -P_{h+1} \\ -P_1 A_\alpha & -c_1 P_1 & \dots & -c_h P_1 & P_1 & 0 & \dots & 0 \\ -P_2 & 0 & \dots & 0 & 0 & P_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -P_{h+1} & 0 & 0 & \dots & P_{h+1} \end{bmatrix},\right.$$
$$\left.\begin{bmatrix} P_1 & 0 & \dots & 0 & 0 \\ 0 & P_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & P_h & 0 \\ 0 & 0 & \dots & 0 & P_{h+1} \end{bmatrix}\right\} \succ 0 \;, \tag{10.39}$$

is feasible with respect to the diagonal matrix $P_1, \dots, P_{h+1}$.

Proof. Proof is given in [121]. □

Example 10.3. Using the LMI approaches check the practical stability of the positive fractional system

$$\Delta^\alpha x_{k+1} = 0.1 x_k, \quad k \in \mathbb{Z}_+, \tag{10.40}$$

for $\alpha = 0.5$ and $h = 2$.
In this case we have:

$$c_1 = \frac{1}{8}, \quad c_2 = \frac{1}{16}, \quad A_\alpha = 0.6 \;,$$

and

$$\tilde{A} = \begin{bmatrix} A_\alpha & c_1 & c_2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0.6 & \frac{1}{8} & \frac{1}{16} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Applying Theorem 10.3 and using MATLAB environment together with SEDUMI solver and YALMIP parser we obtain for the LMI (10.37)

$$\text{block diag}\left\{\begin{bmatrix} P_1 - P_2 - A_\alpha^T P_1 A_\alpha & -c_1 A_\alpha^T P_1 & -c_2 A_\alpha^T P_1 \\ -c_1 P_1 A_\alpha & P_2 - P_3 - c_1^2 P_1 & -c_1 c_2 P_1 \\ -c_2 P_1 A_\alpha & -c_1 c_2 P_1 & P_3 - c_2^2 P_1 \end{bmatrix}, \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix}\right\} \succ 0$$

where

$$\text{block diag}\begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix} = \text{diag}\begin{bmatrix} 7.8921 & 3.5026 & 2.1132 \end{bmatrix}.$$

For LMI (10.38)

$$\text{block diag}\left\{\begin{bmatrix} A_\alpha^T P_1 + P_1 A_\alpha - 2P_1 & P_2 + c_1 P_1 & c_2 P_1 \\ P_2 + c_1 P_1 & -2P_1 & P_3 \\ c_2 P_1 & P_3 & -2P_2 \end{bmatrix}, \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix}\right\} \succ 0$$

where

$$\text{block diag}\begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix} = \text{diag}\begin{bmatrix} 6.9266 & 3.1156 & 2.6096 \end{bmatrix},$$

and for LMI (10.39)

$$\text{block diag}\left\{\begin{bmatrix} P_1 & 0 & 0 & -A_\alpha^T P_1 & -P_2 & 0 \\ 0 & P_2 & 0 & -c_1 P_1 & 0 & -P_3 \\ 0 & 0 & P_3 & -c_2 P_1 & 0 & 0 \\ -P_1 A_\alpha & -c_1 P_1 & -c_2 P_1 & P_1 & 0 & 0 \\ -P_2 & 0 & 0 & 0 & P_2 & 0 \\ 0 & -P_3 & 0 & 0 & 0 & P_3 \end{bmatrix}, \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & 0 \\ 0 & 0 & P_3 \end{bmatrix}\right\} \succ 0 \tag{10.41}$$

where

$$\text{block diag}\begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix} = \text{diag}\begin{bmatrix} 7.7203 & 3.6738 & 2.2765 \end{bmatrix}.$$

Therefore, the LMIs are feasible with respect to the matrices P_1, P_2, P_3 and the positive fractional system (10.40) is practically stable.

Example 10.4. Using the LMI approaches check the practical stability of the positive fractional system

$$\Delta^\alpha x_{k+1} = \begin{bmatrix} -0.2 & 1 \\ 0.1 & b \end{bmatrix} x_k, \quad k \in \mathbb{Z}_+, \tag{10.42}$$

for $\alpha = 0.8$ and $h = 2$ and the following two values of the coefficient b:

$$a) \quad b = -0.5, \qquad b) \quad b = 0.5.$$

In this case we have:

$$c_1 = 0.08, \quad c_2 = 0.032.$$

In Case a) we have:

$$A_{\alpha_1} = \begin{bmatrix} 0.6 & 1 \\ 0.1 & 0.3 \end{bmatrix},$$

$$\tilde{A}_1 = \begin{bmatrix} A_\alpha & c_1 I_2 & c_2 I_2 \\ I_2 & 0 & 0 \\ 0 & I_2 & 0 \end{bmatrix} = \begin{bmatrix} 0.6 & 1 & 0.08 & 0 & 0.032 & 0 \\ 0.1 & 0.3 & 0 & 0.08 & 0 & 0.032 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

In Case b) we have:

$$A_{\alpha_2} = \begin{bmatrix} 0.6 & 1 \\ 0.1 & 1.3 \end{bmatrix},$$

$$\tilde{A}_2 = \begin{bmatrix} A_\alpha & c_1 I_2 & c_2 I_2 \\ I_2 & 0 & 0 \\ 0 & I_2 & 0 \end{bmatrix} = \begin{bmatrix} 0.6 & 1 & 0.08 & 0 & 0.032 & 0 \\ 0.1 & 1.3 & 0 & 0.08 & 0 & 0.032 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

In Case a) applying Theorem 10.3 and using MATLAB environment together with SEDUMI solver and YALMIP parser we obtain for the LMI (10.37)

$$\text{block diag}\begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix} =$$
$$\text{block diag}\left\{\begin{bmatrix} 16.0915 & 0 \\ 0 & 84.368 \end{bmatrix}, \begin{bmatrix} 4.2540 & 0 \\ 0 & 16.3556 \end{bmatrix}, \begin{bmatrix} 2.5726 & 0 \\ 0 & 8.6007 \end{bmatrix}\right\},$$

for LMI (10.38)

$$\text{block diag}\begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix} =$$
$$\text{block diag}\left\{\begin{bmatrix} 8.8848 & 0 \\ 0 & 35.5971 \end{bmatrix}, \begin{bmatrix} 2.5601 & 0 \\ 0 & 7.2962 \end{bmatrix}, \begin{bmatrix} 2.2771 & 0 \\ 0 & 5.2364 \end{bmatrix}\right\}.$$

In Case b) for LMI (10.39) we obtain

$$\text{block diag}\begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix} =$$
$$\text{block diag}\left\{\begin{bmatrix} -0.0834 & 0 \\ 0 & -0.3933 \end{bmatrix}, \begin{bmatrix} 0.4152 & 0 \\ 0 & 0.316 \end{bmatrix}, \begin{bmatrix} 0.4417 & 0 \\ 0 & 0.6885 \end{bmatrix}\right\}.$$

In Case a) the positive fractional system (10.42) is practically stable. In Case b) the positive fractional system (10.42) is unstable for any h (not only for $h = 2$) since the matrix A_{α_2} has one diagonal entry grater then 1.

The characteristic polynomial of the matrix $A_{\alpha_2} - I_n$

$$p(z) = \det[I_n(z+1) - A_{\alpha_2}] = \begin{vmatrix} z-0.4 & -1 \\ -0.1 & z-0.3 \end{vmatrix} = z^2 - 0.7z - 0.22\,,$$

has two negative coefficients. Therefore, the system (10.42) is also unstable for any h.

10.3.2 Positive 2D Linear Systems

Theorem 10.4. *The positive Roesser model (3.40) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:*

a) LMI

$$\text{block diag} \left\{ \begin{bmatrix} 2P_1 - A_{11}^T P_1 - P_1 A_{11} & -A_{21}^T P_2 - P_1 A_{12} \\ -A_{12}^T P_1 - P_2 A_{21} & 2P_2 - A_{22}^T P_2 - P_2 A_{22} \end{bmatrix}, \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \right\} \succ 0 \ , \tag{10.43}$$

is feasible with respect to the diagonal matrices P_1 and P_2.

b) LMI

$$\text{block diag} \left\{ \begin{bmatrix} P_1 - A_{11}^T P_1 A_{11} - A_{21}^T P_2 A_{21} & -A_{11}^T P_1 A_{12} - A_{21}^T P_2 A_{22} \\ -A_{12}^T P_1 A_{11} - A_{22}^T P_2 A_{21} & P_2 - A_{12}^T P_1 A_{12} - A_{22}^T P_2 A_{22} \end{bmatrix}, \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \right\} \succ 0, \tag{10.44}$$

is feasible with respect to the diagonal matrices P_1 and P_2.

Proof. By Theorem 8.10 the positive Roesser model (3.40) is asymptotically stable if and only if the equivalent 1D system (8.42) is asymptotically stable. Using Lemma 10.2 to the system (8.42) we obtain LMI (10.43) since

$$\begin{aligned} &\text{block diag} \left\{ \begin{bmatrix} I_{n_1} - A_{11}^T & -A_{21}^T \\ -A_{12}^T & I_{n_2} - A_{22}^T \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \right. \\ &\quad \left. + \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} I_{n_1} - A_{11} & -A_{12} \\ -A_{21} & I_{n_2} - A_{22} \end{bmatrix} \right\} \\ &\quad = \text{block diag} \left\{ \begin{bmatrix} 2P_1 - A_{11}^T P_1 - P_1 A_{11} & -A_{21}^T P_2 - P_1 A_{12} \\ -A_{12}^T P_1 - P_2 A_{21} & 2P_2 - A_{22}^T P_2 - P_2 A_{22} \end{bmatrix}, \right. \\ &\quad \left. \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \right\} \succ 0 \ . \end{aligned}$$

Similarly, using Lemma 10.3 to the system (8.42) we obtain LMI (10.44) since

$$\begin{aligned} &\text{block diag} \left\{ \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} - \begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}, \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \right\} \\ &= \text{block diag} \left\{ \begin{bmatrix} P_1 - A_{11}^T P_1 A_{11} - A_{21}^T P_2 A_{21} & -A_{11}^T P_1 A_{12} - A_{21}^T P_2 A_{22} \\ -A_{12}^T P_1 A_{11} - A_{22}^T P_2 A_{21} & P_2 - A_{12}^T P_1 A_{12} - A_{22}^T P_2 A_{22} \end{bmatrix}, \right. \\ &\quad \left. \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \right\} \succ 0 \ . \end{aligned}$$

□

Theorem 10.5. *The positive (general model) system (8.10) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:*

a) LMI

$$\text{block diag}\left[2P - \sum_{k=0}^{2}\left(A_k^T P + PA_k\right)\ ,\ P\right] \succ 0\ , \tag{10.45}$$

is feasible with respect to the diagonal matrix P.

b) LMI

$$\text{block diag}\left[P - \sum_{k=0}^{2}\sum_{l=0}^{2}\left(A_k^T PA_l\right)\ ,\ P\right] \succ 0\ , \tag{10.46}$$

is feasible with respect to the diagonal matrix P.

c) LMI

$$\text{block diag}\ \left\{\begin{bmatrix} 2P_1 - \left(A_1^T + A_2^T\right)P_1 - P_1\left(A_1 + A_2\right) & -P_2 - P_1A_0 \\ -P_2 - A_0^T P_1 & 2P_2 \end{bmatrix},\right.$$
$$\left.\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\right\} \succ 0\ , \tag{10.47}$$

is feasible with respect to the diagonal matrices P_1 and P_2.

d) LMI

$$\text{block diag}\ \left\{\begin{bmatrix} P_1 - (A_1 + A_2)^T P_1\left(A_1 + A_2\right) - P_2 & -\left(A_1 + A_2\right)^T P_1A_0 \\ -A_0^T P_1\left(A_1 + A_2\right) & P_2 - A_0^T P_1 A_0 \end{bmatrix},\right.$$
$$\left.\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\right\} \succ 0\ , \tag{10.48}$$

is feasible with respect to the diagonal matrices P_1 and P_2.

Corollary 10.1. *The positive 2D SF-MM is asymptotically stable if and only if one of the following equivalent conditions is satisfied:*

a) LMI

$$\text{block diag}\left[2P - \sum_{k=1}^{2}\left(A_k^T P + PA_k\right)\ ,\ P\right] \succ 0, \tag{10.49}$$

is feasible with respect to the diagonal matrix P.

b) LMI

$$\text{block diag}\left[P - \sum_{k=1}^{2}\sum_{l=1}^{2}\left(A_k^T PA_l\right)\ ,\ P\right] \succ 0, \tag{10.50}$$

is feasible with respect to the diagonal matrix P.

10.3.3 Positive 2D Linear Systems with Delays

Consider the positive 2D Roesser model with q delays

$$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \sum_{k=0}^{q} A_k \begin{bmatrix} x_{i-k,j}^h \\ x_{i,j-k}^v \end{bmatrix}, \quad i, j \in \mathbb{Z}_+, \tag{10.51}$$

where $x_{ij}^h \in \mathbb{R}_+^{n_1}$, $x_{ij}^v \in \mathbb{R}_+^{n_2}$ are the horizontal and vertical state vectors in the point (i, j) and

$$A_k = \begin{bmatrix} A_{11}^k & A_{12}^k \\ A_{21}^k & A_{22}^k \end{bmatrix}, \quad k = 1, \ldots, q \ . \tag{10.52}$$

Theorem 10.6. *The positive Roesser model (10.51) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:*

a) LMI

$$\text{block diag}\left\{ \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \right\} \succ 0 \ , \tag{10.53}$$

where

$$P_{11} = \begin{bmatrix} 2P_1^0 - (A_{11}^0)^T P_1^0 - P_1^0 A_{11}^0 & -P_1^1 - P_1^0 A_{11}^1 & \ldots & -P_1^0 A_{11}^{q-1} & -P_1^0 A_{11}^q \\ -(A_{11}^1)^T P_1^0 - P_1^1 & 2P_1^1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -(A_{11}^{q-1})^T P_1^0 & 0 & \ldots & 2P_1^{q-1} & -P_1^q \\ -(A_{11}^q)^T P_1^0 & 0 & \ldots & -P_1^q & 2P_1^q \end{bmatrix}, \tag{10.54a}$$

$$P_{12} = P_{21}^T = -\begin{bmatrix} (A_{21}^0)^T P_2^0 + P_1^0 A_{21}^0 & P_1^0 A_{12}^1 & \ldots & P_1^0 A_{12}^{q-1} & P_1^0 A_{12}^q \\ (A_{21}^1)^T P_2^0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (A_{21}^{q-1})^T P_2^0 & 0 & \ldots & 0 & 0 \\ (A_{21}^q)^T P_2^0 & 0 & \ldots & 0 & 0 \end{bmatrix}, \tag{10.54b}$$

$$P_{22} = \begin{bmatrix} 2P_2^0 - (A_{22}^0)^T P_2^0 - P_2^0 A_{22}^0 & -P_2^1 - P_2^0 A_{22}^1 & \ldots & -P_2^0 A_{22}^{q-1} & -P_2^0 A_{22}^q \\ -(A_{22}^1)^T P_2^0 - P_2^1 & 2P_2^1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -(A_{22}^{q-1})^T P_2^0 & 0 & \ldots & 2P_2^{q-1} & -P_2^q \\ -(A_{22}^q)^T P_2^0 & 0 & \ldots & -P_2^q & 2P_2^q \end{bmatrix}, \tag{10.54c}$$

$$P_k = \text{block diag}\left[P_k^0 \ P_k^1 \ \ldots \ P_k^q \right], \quad k = 1, 2 \ , \tag{10.54d}$$

is feasible with respect to the diagonal matrices P_1 and P_2.

b) LMI

$$\text{block diag}\left\{ \begin{bmatrix} \overline{P}_{11} & \overline{P}_{12} \\ \overline{P}_{21} & \overline{P}_{22} \end{bmatrix}, \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \right\} \succ 0 \ , \tag{10.55}$$

where

$$\overline{P}_{11} = \begin{bmatrix} \tilde{P}_{11}^1 \\ \tilde{P}_{11}^2 \\ \vdots \\ \tilde{P}_{11}^{R_{11}} \end{bmatrix}, \quad \overline{P}_{12} = \overline{P}_{21}^T = -\begin{bmatrix} \tilde{P}_{12}^1 \\ \tilde{P}_{12}^2 \\ \vdots \\ \tilde{P}_{12}^{R_{12}} \end{bmatrix}, \quad \overline{P}_{22} = \begin{bmatrix} \tilde{P}_{22}^1 \\ \tilde{P}_{22}^2 \\ \vdots \\ \tilde{P}_{22}^{R_{22}} \end{bmatrix}, \tag{10.56}$$

$$
\begin{aligned}
\tilde{P}_{11}^{1} &= \big[P_1^0 - P_1^1 - (A_{11}^0)^T P_1^0 A_{11}^0,\ -(A_{11}^0)^T P_1^0 A_{11}^1, \dots,\ -(A_{11}^0)^T P_1^0 A_{11}^q, \\
&\quad -(A_{21}^0)^T P_2^0 A_{21}^0\ ,\ -(A_{21}^0)^T P_2^0 A_{21}^1\ ,\ \dots\ ,\ -(A_{21}^0)^T P_2^0 A_{21}^q\big], \\
\tilde{P}_{11}^{2} &= \big[-(A_{11}^1)^T P_1^0 A_{11}^0,\ P_1^1 - P_1^2 - (A_{11}^1)^T P_1^0 A_{11}^1, \dots,\ -(A_{11}^1)^T P_1^0 A_{11}^q\ , \\
&\quad -(A_{21}^1)^T P_2^0 A_{21}^0\ ,\ -(A_{21}^1)^T P_2^0 A_{21}^1\ ,\ \dots\ ,\ -(A_{21}^1)^T P_2^0 A_{21}^q\big], \\
&\ \vdots \\
\tilde{P}_{11}^{R_{11}} &= \big[-(A_{11}^q)^T P_1^0 A_{11}^0\ ,\ -(A_{11}^q)^T P_1^0 A_{11}^1\ ,\ \dots\ ,\ P_1^q - (A_{11}^q)^T P_1^0 A_{11}^q\ , \\
&\quad -(A_{21}^q)^T P_2^0 A_{21}^0\ ,\ -(A_{21}^q)^T P_2^0 A_{21}^1\ ,\ \dots\ ,\ -(A_{21}^q)^T P_2^0 A_{21}^q\big], \\
\tilde{P}_{12}^{1} &= \big[(A_{11}^0)^T P_1^0 A_{12}^0\ ,\ (A_{11}^0)^T P_1^0 A_{12}^1\ ,\ \dots\ ,\ (A_{11}^0)^T P_1^0 A_{12}^q\ , \\
&\quad (A_{21}^0)^T P_2^0 A_{22}^0\ ,\ (A_{21}^0)^T P_2^0 A_{22}^1\ ,\ \dots\ ,\ (A_{21}^0)^T P_2^0 A_{22}^q\big], \\
\tilde{P}_{12}^{2} &= \big[(A_{11}^1)^T P_1^0 A_{12}^0\ ,\ (A_{11}^1)^T P_1^0 A_{12}^1\ ,\ \dots\ ,\ (A_{11}^1)^T P_1^0 A_{12}^q\ , \\
&\quad (A_{21}^1)^T P_2^0 A_{22}^0\ ,\ (A_{21}^1)^T P_2^0 A_{22}^1\ ,\ \dots\ ,\ (A_{21}^1)^T P_2^0 A_{22}^q\big], \\
&\ \vdots \\
\tilde{P}_{12}^{R_{12}} &= \big[(A_{11}^q)^T P_1^0 A_{12}^0\ ,\ (A_{11}^q)^T P_1^0 A_{12}^1\ ,\ \dots\ ,\ (A_{11}^q)^T P_1^0 A_{12}^q\ , \\
&\quad (A_{21}^q)^T P_2^0 A_{22}^0\ ,\ (A_{21}^q)^T P_2^0 A_{22}^1\ ,\ \dots\ ,\ (A_{21}^q)^T P_2^0 A_{22}^q\big], \\
\tilde{P}_{22}^{1} &= \big[P_2^0 - (A_{22}^0)^T P_2^0 A_{22}^0\ ,\ -(A_{22}^0)^T P_2^0 A_{22}^1\ ,\ \dots\ ,\ -(A_{22}^0)^T P_2^0 A_{22}^q\ , \\
&\quad -(A_{12}^0)^T P_1^0 A_{12}^0\ ,\ -(A_{12}^0)^T P_1^0 A_{12}^1\ ,\ \dots\ ,\ -(A_{12}^0)^T P_1^0 A_{12}^q\big], \\
\tilde{P}_{22}^{2} &= \big[-(A_{22}^1)^T P_2^0 A_{22}^0\ ,\ P_2^1 - (A_{22}^1)^T P_2^0 A_{22}^1\ ,\ \dots\ ,\ -(A_{22}^1)^T P_2^0 A_{22}^q\ , \\
&\quad -(A_{12}^1)^T P_1^0 A_{12}^0\ ,\ -(A_{12}^1)^T P_1^0 A_{12}^1\ ,\ \dots\ ,\ -(A_{12}^1)^T P_1^0 A_{12}^q\big], \\
&\ \vdots \\
\tilde{P}_{22}^{R_{22}} &= \big[-(A_{22}^q)^T P_2^0 A_{22}^0\ ,\ -(A_{22}^q)^T P_2^0 A_{22}^1\ ,\ \dots\ ,\ P_2^q - (A_{22}^q)^T P_2^0 A_{22}^q\ , \\
&\quad -(A_{12}^q)^T P_1^0 A_{12}^0\ ,\ -(A_{12}^q)^T P_1^0 A_{12}^1\ ,\ \dots\ ,\ -(A_{12}^q)^T P_1^0 A_{12}^q\big],
\end{aligned}
$$

is feasible with respect to the diagonal matrices P_1 and P_2.

Proof. The positive Roesser model (10.51) is asymptotically stable if and only if the reduced model (3.69) is asymptotically stable. Applying to the reduced model (3.69) Theorem 10.4 and using (3.70) and (10.43) we obtain (10.54). Similarly, using (3.70) and (10.44) we obtain (10.56). □

Example 10.5. Using LMI check the asymptotic stability of the positive Roesser model (10.51) for $q = 1$ with the matrices:

$$A_0 = \begin{bmatrix} 0.1 & 0.2 & | & 0 \\ 0 & 0.1 & | & 0.3 \\ -- & -- & -|- & -- \\ 0 & 0 & | & 0.2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.2 & 0.1 & | & 0.2 \\ 0 & 0.1 & | & 0.2 \\ -- & -- & -|- & -- \\ 0 & 0 & | & 0.2 \end{bmatrix}. \tag{10.57}$$

In this case the matrix (3.70) of the reduced positive Roesser model has the form

$$A = \begin{bmatrix} A_{11}^0 & A_{11}^1 & A_{12}^0 & A_{12}^1 \\ I_2 & 0 & 0 & 0 \\ A_{21}^0 & A_{21}^1 & A_{22}^0 & A_{22}^1 \\ 0 & 0 & I_1 & 0 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 & 0.2 & 0.1 & 0 & 0.2 \\ 0 & 0.1 & 0 & 0.1 & 0.3 & 0.2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{10.58}$$

Using Theorem 10.6 and using MATLAB environment together with SEDUMI solver and YALMIP parser we obtain LMI (10.53).

$$\text{block diag}\left[\, P_1 \; P_2 \,\right] = \text{diag}\left[\, 0.7799 \; 0.7883 \; 0.5625 \; 0.5615 \; 0.9452 \; 0.5931 \,\right],$$

and for LMI (10.55)

$$\text{block diag}\left[\, P_1 \; P_2 \,\right] = \text{diag}\left[\, 1.5526 \; 1.5897 \; 0.8374 \; 0.8074 \; 1.8736 \; 0.9290 \,\right].$$

Moreover, LMI is feasible with respect to the diagonal matrices P_1 and P_2 and the positive Roesser model is asymptotically stable.

The above considerations can be easily extended for the positive Roesser model with delays of the form

$$\begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = \sum_{k=0}^{q_1} \sum_{l=0}^{q_2} A_{kl} \begin{bmatrix} x_{i-k,j}^h \\ x_{i,j-l}^v \end{bmatrix}, \quad i, j \in \mathbb{Z}_+,$$

where $x_{ij}^h \in \mathbb{R}_+^{n_1}$, $x_{ij}^v \in \mathbb{R}_+^{n_2}$ are the horizontal and vertical state vectors in the point (i, j) and $A_{kl} \in \mathbb{R}_+^{(n_1+n_2)\times(n_1+n_2)}$.

Theorem 10.7. *The positive 2D (general model) system with q delays (3.72) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:*

a) LMI

$$\text{block diag}\left[\, 2P - \left(\hat{P}_0 + \hat{P}_1 + \hat{P}_2\right) \,,\, P \,\right] \succ 0 \,, \tag{10.59}$$

where

$$P = \text{block diag}\left[P_0\ P_1\ \dots\ P_q\right],$$

$$\hat{P}_0 = \begin{bmatrix} (A_0^0)^T P_0 + P_0 A_0^0 & P_1 + P_0 A_1^0 & \dots & P_0 A_{q-1}^0 & P_0 A_q^0 \\ (A_1^0)^T P_0 + P_1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (A_{q-1}^0)^T P_0 & 0 & \dots & 0 & P_q \\ (A_q^0)^T P_0 & 0 & \dots & P_q & 0 \end{bmatrix}, \quad (10.60a)$$

$$\hat{P}_k = \begin{bmatrix} (A_0^k)^T P_0 + P_0 A_0^k & P_0 A_1^k & \dots & P_0 A_{q-1}^k & P_0 A_q^k \\ (A_1^k)^T P_0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (A_{q-1}^k)^T P_0 & 0 & \dots & 0 & 0 \\ (A_q^k)^T P_0 & 0 & \dots & 0 & 0 \end{bmatrix}, \quad (10.60b)$$

is feasible with respect to the diagonal matrix P.

b) *LMI*

$$\text{block diag}\left[P - \hat{P}\ ,\ P\right] \succ 0\ . \quad (10.61)$$

where

$$P = \text{block diag}\left[P_0\ P_1\ \dots\ P_q\right],$$

$$\hat{P} = \begin{bmatrix} (A_0)^T P_0 A_0 + P_1 & (A_0)^T P_0 A_1 & \dots & (A_0)^T P_0 A_{q-1} & (A_0)^T P_0 A_q \\ (A_1)^T P_0 A_0 & (A_1)^T P_0 A_1 + P_2 & \dots & (A_1)^T P_0 A_{q-1} & (A_1)^T P_0 A_q \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (A_{q-1})^T P_0 A_0 & (A_{q-1})^T P_0 A_1 & \dots & (A_{q-1})^T P_0 A_{q-1} + P_q & (A_{q-1})^T P_0 A_q \\ (A_q)^T P_0 A_0 & (A_q)^T P_0 A_1 & \dots & (A_q)^T P_0 A_{q-1} & (A_q)^T P_0 A_q \end{bmatrix}$$

$$A_k = A_k^0 + A_k^1 + A_k^2, \quad k = 0, 1, \dots, q\ , \quad (10.62)$$

is feasible with respect to the diagonal matrix P.

Proof. The positive 2D system (3.72) is asymptotically stable if and only if the reduced system (3.75) is asymptotically stable. Applying to the reduced system (3.75) LMI (10.45), we obtain LMI (10.60). Similarly, applying to the reduced system (3.75) LMI (10.46), we obtain LMI (10.62). □

Remark 10.2. In a similar way using LMI (10.47) and (10.48) to the reduced system (3.75), we obtain LMI for the positive system (3.72).

Remark 10.3. Substituting $A_k^0 = 0$, $k = 0, 1, \dots, q$ in Theorem 10.7, we obtain the corresponding LMI conditions for the positive 2D SF-MM.

Example 10.6. Using LMI check the asymptotic stability of the positive 2D system (3.72) for $q = 1$ with the matrices:

$$A_0^0 = \begin{bmatrix} 0.6 & 0.1 \\ 0 & 0.2 \end{bmatrix}, \quad A_0^1 = \begin{bmatrix} 0.1 & 0.2 \\ 0 & 0.21 \end{bmatrix}, \quad A_0^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$
$$A_1^1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1^1 = \begin{bmatrix} 0.2 & 0.1 \\ 0 & 0.1 \end{bmatrix}, \quad A_1^2 = \begin{bmatrix} 0 & 0.2 \\ 0 & 0.4 \end{bmatrix}.$$

The matrices (3.74) of the reduced system (3.75) have the form:

$$\overline{A}_0 = \begin{bmatrix} 0.6 & 0.1 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \overline{A}_1 = \begin{bmatrix} 0.1 & 0.2 & 0.2 & 0.1 \\ 0 & 0.21 & 0 & 0.1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \overline{A}_2 = \begin{bmatrix} 0 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Using Theorem 10.7 and using MATLAB environment together with SEDUMI solver and YALMIP parser we obtain LMI (10.59)

$$P = \operatorname{diag}\begin{bmatrix} 0.1382 & 2.0346 & 0.0414 & 1.0731 \end{bmatrix},$$

and for LMI (10.61)

$$P = \operatorname{diag}\begin{bmatrix} 0.2681 & 3.5981 & 0.0647 & 1.8976 \end{bmatrix}.$$

Moreover, LMI is feasible with respect to the diagonal matrix P and the positive 2D system is asymptotically stable.

The consideration can be easily extended for the positive 2D system of the form

$$x_{i+1,j+1} = \sum_{k=0}^{q_1} \sum_{l=0}^{q_2} \left(A_{kl}^0 x_{i-k,j-l} + A_{kl}^1 x_{i+1-k,j-l} + A_{kl}^2 x_{i-k,j+1-l} \right), \quad i,j \in \mathbb{Z}_+, \tag{10.63}$$

where $x_{ij} \in \mathbb{R}_+^n$ is the state vector in the point (i,j) and $A_{kl}^t \in \mathbb{R}^{n\times n}$, $k = 0,1,\dots,q_1$; $l = 0,1,\dots,q_2$; $t = 0,1,2$.

10.3.4 Fractional 2D Roesser Model

Consider the positive fractional Roesser model (3.49) with the state-feedback

$$u_{ij} = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix}, \tag{10.64}$$

where $K = [K_1, K_2] \in \mathbb{R}^{m\times n}$, $K_j \in \mathbb{R}^{m\times n}$, $j = 1,2$ is a gain matrix.

We are looking for a gain matrix K such that the closed-loop system

$$\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = \begin{bmatrix} \overline{A}_{11}+B_1K_1 & A_{12}+B_1K_2 \\ A_{21}+B_2K_1 & \overline{A}_{22}+B_2K_2 \end{bmatrix} \begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} - \begin{bmatrix} \sum_{k=2}^{i+1} c_\alpha(k) x^h_{i-k+1,j} \\ \sum_{l=2}^{j+1} c_\beta(l) x^v_{i,j-l+1} \end{bmatrix}, \quad (10.65)$$

is positive and asymptotically stable [127].

Theorem 10.8. *The positive fractional closed-loop system (10.65) is positive and asymptotically stable if and only if there exist a block diagonal matrix*

$$\Lambda = \text{block diag}\left[\Lambda_1 \ \Lambda_2\right], \quad \Lambda_k = \text{diag}\left[\lambda_{k1},\ldots,\lambda_{kn_k}\right], \quad \lambda_{kj} > 0, \quad (10.66)$$

$$k = 1,2; \quad j = 1,\ldots,n_k;$$

and a real matrix

$$D = \left[D_1 \ D_2\right], \quad D_k \in \mathbb{R}^{m\times n_k}, \quad k = 1,2; \quad (10.67)$$

satisfying conditions

$$\begin{bmatrix} \overline{A}_{11}\Lambda_1 + B_1D_1 & A_{12}\Lambda_2 + B_1D_2 \\ A_{21}\Lambda_1 + B_2D_1 & \overline{A}_{22}\Lambda_2 + B_2D_2 \end{bmatrix} \in \mathbb{R}^{n\times n}_+ \quad (10.68)$$

and

$$\begin{bmatrix} A_{11}\Lambda_1 + B_1D_1 & A_{12}\Lambda_2 + B_1D_2 \\ A_{21}\Lambda_1 + B_2D_1 & A_{22}\Lambda_2 + B_2D_2 \end{bmatrix} \begin{bmatrix} \mathbb{1}_{n_1} \\ \mathbb{1}_{n_2} \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (10.69)$$

where $\mathbb{1}_{n_k} = [1,\ldots,1]^T \in \mathbb{R}^{n_k}_+$, $k = 1,2$. *The gain matrix is given by the formula*

$$K = \left[K_1 \ K_2\right] = \left[D_1 \ D_2\right]\Lambda^{-1} = \left[D_1\Lambda_1^{-1} \ D_2\Lambda_2^{-1}\right] \quad (10.70)$$

Proof. Proof of this Theorem is similar to the proof of Theorem 10.1 [166]. □

It is well-known that the positive closed-loop system (10.65) is asymptotically stable if and only if the positive 1D system with the matrix

$$\begin{bmatrix} \overline{A}_{11}+B_1K_1 & A_{12}+B_1K_2 \\ A_{21}+B_2K_1 & \overline{A}_{22}+B_2K_2 \end{bmatrix} - \sum_{k=2}^{\infty} \begin{bmatrix} I_{n_1}c_\alpha(k) & 0 \\ 0 & I_{n_2}c_\beta(k) \end{bmatrix}, \quad (10.71)$$

is asymptotically stable.

Taking into account that

$$\sum_{k=2}^{\infty} c_\alpha(k) = \alpha - 1, \quad \sum_{l=2}^{\infty} c_\beta(l) = \beta - 1, \quad (10.72)$$

and $\overline{A}_{11} = A_{11} + I_{n_1}\alpha$, $\overline{A}_{22} = A_{22} + I_{n_2}\beta$, we may write the matrix (10.71) in the form

$$\begin{bmatrix} \hat{A}_{11} + B_1K_1 & A_{12} + B_1K_2 \\ A_{21} + B_2K_1 & \hat{A}_{22} + B_2K_2 \end{bmatrix} = A + BK, \tag{10.73}$$

where $\hat{A}_{11} = A_{11} + I_{n_1}$, $\hat{A}_{22} = A_{22} + I_{n_2}$ and

$$A = \begin{bmatrix} \hat{A}_{11} & A_{12} \\ A_{21} & \hat{A}_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \tag{10.74}$$

Theorem 10.9. *The fractional closed-loop system (10.65) is positive and asymptotically stable if and only if there exist a positive definite block diagonal matrix (10.66) and a real matrix (10.67)such that the condition (10.68) is satisfied and the LMI*

$$\begin{bmatrix} -\Lambda & A\Lambda + BD \\ (A\Lambda + BD)^T & -\Lambda \end{bmatrix} \prec 0 \ , \tag{10.75}$$

is feasible with respect to the positive definite diagonal matrix Λ.

Proof. The closed-loop system (10.65) is positive if and only if the condition (10.68) is satisfied since the condition

$$\begin{aligned} \begin{bmatrix} \overline{A}_{11} + B_1K_1 & A_{12} + B_1K_2 \\ A_{21} + B_2K_1 & \overline{A}_{22} + B_2K_2 \end{bmatrix} &= \begin{bmatrix} \overline{A}_{11} + B_1D_1\Lambda_1^{-1} & A_{12} + B_1D_2\Lambda_2^{-1} \\ A_{21} + B_2D_1\Lambda_1^{-1} & \overline{A}_{22} + B_2D_2\Lambda_2^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \overline{A}_{11}\Lambda_1 + B_1D_1 & A_{12}\Lambda_2 + B_1D_2 \\ A_{21}\Lambda_1 + B_2D_1 & \overline{A}_{22}\Lambda_2 + B_2D_2 \end{bmatrix} \\ &\times \begin{bmatrix} \Lambda_1^{-1} & 0 \\ 0 & \Lambda_2^{-1} \end{bmatrix}, \end{aligned}$$

is equivalent to (10.68).

The closed-loop system (10.65) is asymptotically stable if and only if the LMI

$$P - (A + BK)^T P (A + BK) \succ 0 \ , \tag{10.76}$$

is feasible with respect to a positive definite diagonal matrix P.

Using the Schur complement we can write the condition (10.76) in the form

$$\begin{bmatrix} -P & P(A + BK) \\ (A + BK)^T P & -P \end{bmatrix} \prec 0 \ . \tag{10.77}$$

Substituting of (10.70) and $P = \Lambda^{-1}$ into (10.77) yields

$$\begin{aligned} \begin{bmatrix} -\Lambda^{-1} & \Lambda^{-1}\left(A + BD\Lambda^{-1}\right) \\ \left(A + BD\Lambda^{-1}\right)^T \Lambda^{-1} & -\Lambda^{-1} \end{bmatrix} &= \begin{bmatrix} -\Lambda^{-1} & 0 \\ 0 & -\Lambda^{-1} \end{bmatrix} \\ \times \begin{bmatrix} -\Lambda & A\Lambda + BD \\ (A\Lambda + BD)^T & -\Lambda \end{bmatrix} \begin{bmatrix} -\Lambda^{-1} & 0 \\ 0 & -\Lambda^{-1} \end{bmatrix} &\prec 0 \ . \end{aligned} \tag{10.78}$$

Applying the congruent transformation with the matrix block diag$[\Lambda,\Lambda]$ we obtain the condition (10.75). □

Example 10.7. Given the fractional 2D Roesser model with $\alpha = 0.4$ $\beta = 0.5$ and

$$A_{11} = \begin{bmatrix} -0.5 & -0.1 \\ 0.1 & 0.01 \end{bmatrix}, A_{12} = \begin{bmatrix} -0.1 & -0.1 \\ 0.2 & 0.1 \end{bmatrix}, A_{21} = \begin{bmatrix} -0.3 & -0.1 \\ 0.2 & 0.1 \end{bmatrix}, \quad (10.79a)$$

$$A_{22} = \begin{bmatrix} -1 & -0.1 \\ 0.4 & 0.1 \end{bmatrix}, B_1 = \begin{bmatrix} -0.2 \\ 0.1 \end{bmatrix}, B_2 = \begin{bmatrix} -0.3 \\ 0.2 \end{bmatrix}. \quad (10.79b)$$

Find a gain matrix $K = [K_1, K_2]$, $K_i \in \mathbb{R}^{1\times 2}$, $i = 1,2$ such that the closed-loop system is positive and asymptotically stable.

The fractional 2D Roesser model with (10.79a) is not positive since the matrices

$$\begin{bmatrix} \overline{A}_{11} & A_{12} \\ A_{21} & \overline{A}_{22} \end{bmatrix} = \begin{bmatrix} -0.1 & -0.1 & -0.1 & -0.1 \\ 0.1 & 0.41 & 0.2 & 0.1 \\ -0.3 & -0.1 & -0.5 & -0.1 \\ 0.2 & 0.1 & 0.4 & 0.6 \end{bmatrix}$$

have negative entries. The model is also unstable since the matrix

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} -0.5 & -0.1 & -0.1 & -0.1 \\ 0.1 & 0.01 & 0.2 & 0.1 \\ -0.3 & -0.1 & -1 & -0.1 \\ 0.2 & 0.1 & 0.4 & 0.1 \end{bmatrix} \quad (10.80)$$

has positive diagonal entries.

We choose:

$$D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}, \quad D_1 = D_2 = \begin{bmatrix} -0.4 & -0.2 \end{bmatrix}. \quad (10.81)$$

Applying Theorem 10.9 and using MATLAB environment together with SEDUMI solver and YALMIP parser for the LMI (10.75) we obtain:

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}, \quad \Lambda_1 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix}, \quad \Lambda_2 = \begin{bmatrix} 0.2258 & 0 \\ 0 & 0.2413 \end{bmatrix}. \quad (10.82)$$

Therefore, the LMI is feasible with respect to the diagonal matrix Λ.

Using (10.70) we obtain the gain matrix

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix} = \begin{bmatrix} D_1\Lambda_1^{-1} & D_2\Lambda_2^{-1} \end{bmatrix} = \begin{bmatrix} -1 & -0.5 & -1.7712 & -0.8289 \end{bmatrix}. \quad (10.83)$$

The closed-loop system is positive since matrices:

$$\overline{A}_{11}+B_1K_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.36 \end{bmatrix}, \quad A_{12}+B_1K_2 = \begin{bmatrix} 0.2542 & 0.0658 \\ 0.0229 & 0.0171 \end{bmatrix},$$
$$A_{21}+B_2K_1 = \begin{bmatrix} 0 & 0.05 \\ 0 & 0 \end{bmatrix}, \quad \overline{A}_{22}+B_2K_2 = \begin{bmatrix} 0.0313 & 0.1487 \\ 0.0458 & 0.4342 \end{bmatrix},$$

have all nonnegative entries.

The closed-loop system is asymptotically stable since its characteristic polynomial

$$\det \begin{bmatrix} I_{n_1}z-(A_{11}+B_1K_1) & -(A_{12}+B_1K_2) \\ -(A_{21}+B_2K_1) & I_{n_2}z-(A_{22}+B_2K_2) \end{bmatrix}$$
$$= z^4+0.8744z^3+0.2166z^2+0.0141z+0.0003,$$

has positive coefficients.

Example 10.8. Given the positive fractional 2D Roesser model with $\alpha = 0.4$, $\beta = 0.9$ and

$$A_{11} = \begin{bmatrix} -0.4 & 0.01 \\ 0.03 & 0.001 \end{bmatrix}, A_{12} = \begin{bmatrix} 0.01 & 0.01 \\ 0.01 & 0.2 \end{bmatrix}, A_{21} = \begin{bmatrix} 0.01 & 0.2 \\ 0 & 0.01 \end{bmatrix}, \tag{10.84a}$$
$$A_{22} = \begin{bmatrix} -0.9 & 0.01 \\ 0.01 & -0.8 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0.001 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ 0.002 \end{bmatrix}. \tag{10.84b}$$

Find a gain matrix $K = [K_1, K_2]$, $K_i \in \mathbb{R}^{1\times 2}$, $i = 1,2$ such that the closed-loop system is positive and asymptotically stable.

The fractional 2D Roesser model with (10.84) is unstable since the matrix

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} -0.4 & 0.01 & 0.01 & 0.01 \\ 0.03 & 0.001 & 0.01 & 0.2 \\ 0.01 & 0.2 & -0.9 & 0.01 \\ 0 & 0.01 & 0.01 & -0.8 \end{bmatrix} \tag{10.85}$$

has positive diagonal entries.

We choose:

$$D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.13 & -0.37 \end{bmatrix}, \quad D_2 = \begin{bmatrix} -3.19 & -0.11 \end{bmatrix}. \tag{10.86}$$

Applying Theorem 10.9 and using MATLAB environment together with SEDUMI solver and YALMIP parser for the LMI (10.75) we obtain:

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}, \Lambda_1 = \begin{bmatrix} 0.0554 & 0 \\ 0 & 0.0755 \end{bmatrix}, \Lambda_2 = \begin{bmatrix} 0.8659 & 0 \\ 0 & 0.0032 \end{bmatrix}. \tag{10.87}$$

Therefore, the LMI is feasible with respect to the diagonal matrix Λ.

Using (10.70) we obtain the gain matrix

$$\begin{aligned} K &= \begin{bmatrix} K_1 & K_2 \end{bmatrix} = \begin{bmatrix} D_1 \Lambda_1^{-1} & D_2 \Lambda_2^{-1} \end{bmatrix} \\ &= \begin{bmatrix} 2.3460 & -4.9035 & -3.6840 & -34.1058 \end{bmatrix}. \end{aligned} \tag{10.88}$$

The closed-loop system is positive since matrices:

$$\begin{aligned} \overline{A}_{11} + B_1 K_1 &= \begin{bmatrix} 0 & 0.01 \\ 0.0323 & 0.3961 \end{bmatrix}, \quad A_{12} + B_1 K_2 = \begin{bmatrix} 0.01 & 0.01 \\ 0.0063 & 0.1659 \end{bmatrix}, \\ A_{21} + B_2 K_1 &= \begin{bmatrix} 0.01 & 0.2 \\ 0.0047 & 0.0002 \end{bmatrix}, \quad \overline{A}_{22} + B_2 K_2 = \begin{bmatrix} 0 & 0.01 \\ 0.0026 & 0.0318 \end{bmatrix}, \end{aligned}$$

have all nonnegative entries.

The closed-loop system is asymptotically stable since its characteristic polynomial

$$\begin{aligned} &\det \begin{bmatrix} I_{n_1} z - (A_{11} + B_1 K_1) & -(A_{12} + B_1 K_2) \\ -(A_{21} + B_2 K_1) & I_{n_2} z - (A_{22} + B_2 K_2) \end{bmatrix} \\ &= z^4 + 2.1721 z^3 + 1.4953 z^2 + 0.3159 z + 0.0004, \end{aligned}$$

has positive coefficients.

These considerations can be extended to the closed-loop systems with poles located in desired sectors of the left half complex plane [187].

Chapter 11
Singular Fractional Linear Systems

11.1 Singular Fractional Continuous-Time Linear Systems

Consider singular fractional linear system described by the state equations

$$E\frac{d^\alpha}{dt^\alpha}x(t) = Ax(t) + Bu(t), \quad (11.1a)$$

$$y(t) = Cx(t) + Du(t), \quad (11.1b)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the state, input and output vectors and $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

The initial condition for (11.1a) is given by

$$x(0) = x_0. \quad (11.2)$$

It is assumed that the pencil of the pair (E, A) is regular, i.e.

$$\det[Es - A] \neq 0, \quad (11.3)$$

for some $z \in \mathbb{C}$ (the field of complex numbers). It is well-known [62, 89] that if the pencil is regular then there exists a pair of nonsingular matrices $P, Q \in \mathbb{R}^{n\times n}$ such that

$$P[Es - A]Q = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} s - \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}, \quad (11.4)$$

where n_1 is equal to degree of the polynomial $\det[Es - A]$, $A_1 \in \mathbb{R}^{n_1\times n_1}$, $N \in \mathbb{R}^{n_2\times n_2}$ is a nilpotent matrix with the index μ (i.e. $N^\mu = 0$ and $N^{\mu-1} \neq 0$) and $n_1 + n_2 = n$.

Applying to the equation (11.1a) with zero initial conditions $x_0 = 0$ the Laplace transform ($\mathscr{L}$) we obtain

$$[Es^\alpha - A]X(s) = BU(s), \quad (11.5)$$

where $X(s) = \mathscr{L}[x(t)] = \int_0^\infty x(t)e^{-st}dt$ and $U(s) = \mathscr{L}[u(t)]$. By assumption (11.3) the pencil $[Es^\alpha - A]$ is regular and we may apply the decomposition (11.4) to equation (11.1a).

T. Kaczorek: Selected Problems of Fractional Systems Theory, LNCIS 411, pp. 245–266.
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Premultiplying the equation (11.1a) by the matrix $P \in \mathbb{R}^{n\times n}$ and introducing the new state vector

$$\bar{x}(t) = Q^{-1}x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad x_1(t) \in \mathbb{R}^{n_1}, \quad x_2(t) \in \mathbb{R}^{n_2}, \tag{11.6}$$

we obtain

$$\frac{d^\alpha}{dt^\alpha}x_1(t) = A_1x_1(t) + B_1u(t), \tag{11.7a}$$

$$N\frac{d^\alpha}{dt^\alpha}x_2(t) = x_2(t) + B_2u(t), \tag{11.7b}$$

where

$$PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad B_1 \in \mathbb{R}^{n_1\times m}, \quad B_2 \in \mathbb{R}^{n_2\times m}. \tag{11.8}$$

Using (2.15) we obtain the solution to the equation (11.7a) in the form

$$x_1(t) = \Phi_{10}(t)x_{10} + \int_0^t \Phi_{11}(t-\tau)B_1u(\tau)d\tau, \tag{11.9}$$

where

$$\Phi_{10}(t) = \sum_{k=0}^{\infty} \frac{A_1^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \tag{11.10a}$$

$$\Phi_{11}(t) = \sum_{k=0}^{\infty} \frac{A_1^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}, \tag{11.10b}$$

and $x_{10} \in \mathbb{R}^{n_1}$ is the initial condition for (11.7a) defined by

$$\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix} = Q^{-1}x_0, \quad x_0 = x(0). \tag{11.11}$$

To find the solution of the equation (11.7b) we apply to the equation the Laplace transform and we obtain

$$Ns^\alpha X_2(s) - Ns^{\alpha-1}x_{20} = X_2(s) + B_2U(s), \tag{11.12}$$

since [36, 100] for $0 < \alpha < 1$

$$\mathscr{L}\left[\frac{d^\alpha}{dt^\alpha}x_2(t)\right] = s^\alpha X_2(s) - s^{\alpha-1}x_{20}, \tag{11.13}$$

where $X_2(s) = \mathscr{L}[x_2(t)]$. From (11.12) we have

$$X_2(s) = [Ns^\alpha - I_{n_2}]^{-1}\left(B_2U(s) + Ns^{\alpha-1}x_{20}\right). \tag{11.14}$$

It is easy to check that

$$[Ns^\alpha - I_{n_2}]^{-1} = -\sum_{i=0}^{\mu-1} N^i s^{i\alpha}, \tag{11.15}$$

since

$$[Ns^\alpha - I_{n_2}]\left(-\sum_{i=0}^{\mu-1} N^i s^{i\alpha}\right) = I_{n_2}, \tag{11.16}$$

and $N^i = 0$ for $i = \mu, \mu+1, \dots$.

Substitution of (11.15) into (11.14) yields

$$X_2(s) = -B_2U(s) - \frac{Nx_{20}}{s^{1-\alpha}} - \sum_{i=1}^{\mu-1}\left[N^iB_2s^{i\alpha}U(s) + N^{i+1}s^{(i+1)\alpha-1}x_{20}\right], \tag{11.17}$$

Using inverse Laplace transform ($\mathscr{L}^{-1}$) to (11.17) and the convolution theorem we obtain for $1-\alpha > 0$

$$\begin{aligned} x_2(t) &= \mathscr{L}^{-1}[X_2(s)] \qquad (11.18)\\ &= -B_2u(t) - Nx_{20}\frac{t^{-\alpha}}{\Gamma(1-\alpha)} - \sum_{i=1}^{\mu-1}\left[N^iB_2\frac{d^{i\alpha}}{dt^{i\alpha}}u(t) + N^{i+1}\frac{d^{(i+1)\alpha-1}}{dt^{(i+1)\alpha-1}}x_{20}\right], \end{aligned}$$

since $\mathscr{L}^{-1}\left[\frac{1}{s^{\alpha+1}}\right] = \frac{t^\alpha}{\Gamma(1+\alpha)}$ for $\alpha+1>0$.

Therefore, The following theorem has been proved.

Theorem 11.1. *The solution to the equation* (11.1a) *with the initial condition* (11.2) *has the form*

$$x(t) = Q\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{11.19}$$

where $x_1(t)$ *and* $x_2(t)$ *are given by* (11.9) *and* (11.18) *respectively.*

Knowing the solution (11.19) we can find the output $y(t)$ of the system using the formula

$$y(t) = CQ\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + Du(t). \tag{11.20}$$

11.2 Singular Fractional Electrical Circuits

Let the current $i_C(t)$ in the supercondensator with the capacity C be the α order derivative of its charge $q(t)$ [143]

$$i_C(t) = \frac{d^\alpha q(t)}{dt^\alpha}. \tag{11.21}$$

Taking into account that $q(t) = Cu_C(t)$ we obtain

$$i_C(t) = C\frac{d^\alpha u_C(t)}{dt^\alpha}, \tag{11.22}$$

where $u_C(t)$ is the voltage on the supercondensator.

Similarly, let the voltage $u_L(t)$ on the supercoil (inductor) with the inductance L be the β order derivative of its magnetic flux $\psi(t)$

$$u_L(t) = \frac{d^\beta \Psi(t)}{dt^\beta}. \tag{11.23}$$

Taking into account that $\psi(t) = Li_L(t)$ we obtain

$$u_L(t) = L\frac{d^\beta i_L(t)}{dt^\beta}, \tag{11.24}$$

where $i_L(t)$ is the current in the supercoil.

Example 11.1. Consider electrical circuit shown on Fig. 11.1 with given resistance R, capacitances C_1, C_2, C_3 and source voltages e_1 and e_2. Using the Kirchhoff's laws

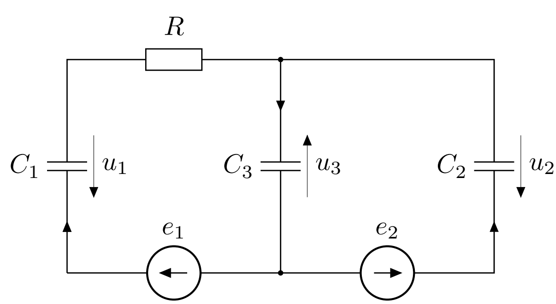

Fig. 11.1 Electrical circuit. Illustration to Example 11.1.

we can write for the electrical circuit the equations

$$e_1 = RC_1\frac{d^\alpha u_1}{dt^\alpha} + u_1 + u_3, \tag{11.25a}$$

$$0 = C_1\frac{d^\alpha u_1}{dt^\alpha} + C_2\frac{d^\alpha u_2}{dt^\alpha} - C_3\frac{d^\alpha u_3}{dt^\alpha}, \tag{11.25b}$$

$$e_2 = u_2 + u_3\ . \tag{11.25c}$$

The equations (11.25) can be written in the form

$$\begin{bmatrix} RC_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & 0 & 0 \end{bmatrix} \frac{d^\alpha}{dt^\alpha} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \tag{11.26}$$

In this case we have

$$E = \begin{bmatrix} RC_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{11.27}$$

Note that the matrix E is singular ($\det E = 0$) but the pencil

$$\det[Es^\alpha - A] = \begin{vmatrix} RC_1 s^\alpha + 1 & 0 & 1 \\ C_1 s^\alpha & C_2 s^\alpha & -C_3 s^\alpha \\ 0 & 1 & 1 \end{vmatrix} = (RC_1 s^\alpha + 1)(C_2 + C_3)s^\alpha + C_1 s^\alpha, \tag{11.28}$$

is regular. Therefore, the electrical circuit is a singular fractional linear system.

Remark 11.1. If the electrical circuit contains at least one mesh consisting of branches with only ideal supercondensators and voltage sources then its matrix E is singular since the row corresponding to this mesh is zero row. This follows from the fact that the equation written by the use of the voltage Kirchhoff's law is algebraic one.

Example 11.2. Consider electrical circuit shown on Fig. 11.2 with given resistances R_1, R_2, R_3, inductances L_1, L_2, L_3 and source voltages e_1 and e_2. Using the

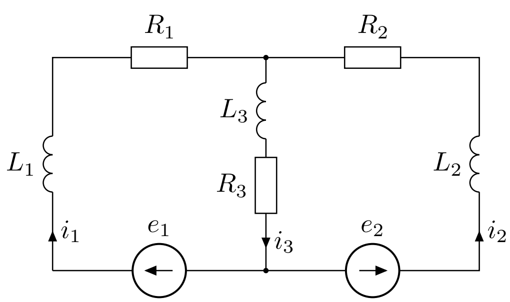

Fig. 11.2 Electrical circuit. Illustration to Example 11.2.

Kirchhoff's laws we can write for the electrical circuit the equations

$$e_1 = R_1 i_1 + L_1 \frac{d^\beta i_1}{dt^\beta} + R_3 i_3 + L_3 \frac{d^\beta i_3}{dt^\beta}, \tag{11.29a}$$

$$e_2 = R_2 i_2 + L_2 \frac{d^\beta i_2}{dt^\beta} + R_3 i_3 + L_3 \frac{d^\beta i_3}{dt^\beta}, \tag{11.29b}$$

$$0 = i_1 + i_2 - i_3. \tag{11.29c}$$

The equations (11.29) can be written in the form

$$\begin{bmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 0 & 0 & 0 \end{bmatrix} \frac{d^\beta}{dt^\beta} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} -R_1 & 0 & -R_3 \\ 0 & -R_2 & -R_3 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \tag{11.30}$$

In this case we have

$$E = \begin{bmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -R_1 & 0 & -R_3 \\ 0 & -R_2 & -R_3 \\ 1 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{11.31}$$

Note that the matrix E is singular but the pencil

$$\begin{aligned} \det[Es^\beta - A] &= \begin{vmatrix} L_1 s^\beta + R_1 & 0 & L_3 s^\beta + R_3 \\ 0 & L_2 s^\beta + R_2 & L_3 s^\beta + R_3 \\ -1 & -1 & 1 \end{vmatrix} \\ &= [L_1(L_2 + L_3) + L_2 L_3] s^{2\beta} \\ &+ [(L_2 + L_3)R_1 + (L_1 + L_3)R_2 + (L_1 + L_2)R_3] s^\beta \\ &+ R_1(R_2 + R_3) + R_2 R_3, \end{aligned} \tag{11.32}$$

is regular. Therefore, the electrical circuit is a singular fractional linear system.

Remark 11.2. If the electrical circuit contains at least one node with branches with supercoils then its matrix E is singular since it has at least one zero row. This follows from the fact that the equation written using the current Kirchhoff's law for this node is algebraic one.

In general case we have the following theorem.

Theorem 11.2. *Every electrical circuit is a singular fractional system if it contains at least one mesh consisting of branches with only ideal supercondensators and voltage source or at least one node with branches with supercoils.*

Proof. By Remark 11.1 the matrix E of the system is singular if the electrical circuit contains at least one mesh consisting of branches with only ideal supercondensators and voltage source.
Similarly, by Remark 11.2 the matrix E is singular if the electrical circuit contains at least one node with branches with supercoils. □

Using the solution (11.19) of the equation (11.1a) we may find the voltages on the supercondensators and currents in the supercoils in the transient states of the singular fractional linear electrical circuits. Knowing the voltages and currents and using (11.20) we may find also any currents and voltages in the singular fractional linear electrical circuits.

Example 11.3. (an continuation of Example 11.1) Using one of the well-known methods [36, 39, 89] we can find for the pencil (11.28) the matrices

$$P = \begin{bmatrix} \frac{1}{RC_1} & 0 & -\frac{C_2}{RC_1(C_2+C_3)} \\ -\frac{1}{R(C_2+C_3)} & \frac{1}{C_2+C_3} & \frac{C_2}{R(C_2+C_3)^2} \\ 0 & 0 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{C_3}{C_2+C_3} \\ 0 & -1 & \frac{C_2}{C_2+C_3} \end{bmatrix}, \tag{11.33}$$

which transform it to the canonical form (11.4) with

$$A_1 = \begin{bmatrix} -\frac{1}{RC_1} & \frac{1}{RC_1} \\ \frac{1}{R(C_2+C_3)} & -\frac{1}{R(C_2+C_3)} \end{bmatrix}, \quad N = [0], \quad n_1 = 2, \quad n_2 = 1. \tag{11.34}$$

Using the matrix B given by (11.27), (11.33) and (11.8) we obtain

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = PB = \begin{bmatrix} \frac{1}{RC_1} & -\frac{C_2}{RC_1(C_2+C_3)} \\ -\frac{1}{R(C_2+C_3)} & \frac{C_2}{R(C_2+C_3)^2} \\ 0 & -1 \end{bmatrix}, \tag{11.35}$$

from (11.9) we have

$$x_1(t) = \Phi_{10}(t)x_{10} + \int_0^t \Phi_{11}(t-\tau)B_1u(\tau)d\tau, \tag{11.36}$$

for any given initial condition $x_{10} \in \mathbb{R}^{n_1}$ and input $u(t)$, where

$$\Phi_{10}(t) = \sum_{k=0}^{\infty} \frac{A_1^k t^{k\alpha}}{\Gamma(k\alpha+1)},$$
$$\Phi_{11}(t) = \sum_{k=0}^{\infty} \frac{A_1^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]},$$

and $0 < \alpha < 1$.

In this case using (11.18) we obtain

$$x_2(t) = -B_2u(t), \tag{11.37}$$

since $N = [0]$.

In a similar way we may find currents in the supercoils of the singular fractional electrical circuit shown on Fig. 11.2.

11.3 Singular Fractional Discrete-Time Linear Systems

Consider the singular fractional discrete-time linear system described by the state equation

$$E\Delta^{\alpha}x_{i+1} = Ax_i + Bu_i, \quad i \in \mathbb{Z}_+ = \{0, 1, \ldots\}, \tag{11.38}$$

where, $x_i \in \mathbb{R}^n, u_i \in \mathbb{R}^m$ are the state and input vectors, $A \in \mathbb{R}^{n\times n}$, $E \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and the fractional difference of the order α is defined by

$$\Delta^\alpha x_i = \sum_{k=0}^{i}(-1)^k \binom{\alpha}{k} x_{i-k}, \quad 0<\alpha<1, \tag{11.39}$$

$$\binom{\alpha}{k} = \begin{cases} 1 & \text{for} \quad k=0 \\ \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} & \text{for} \quad k=1,2,\dots \end{cases} \tag{11.40}$$

It is assumed that

$$\det E = 0, \tag{11.41a}$$

and

$$\det[Ez-A] \neq 0, \tag{11.41b}$$

for some $z \in \mathbb{C}$ (the field of complex numbers).

Lemma 11.1. *[62, 89] If (11.41) holds then there exist nonsingular matrices $P, Q \in \mathbb{R}^{n\times n}$ such that*

$$PEQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \quad PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}, \tag{11.42}$$

where $N \in \mathbb{R}^{n_2\times n_2}$ is a nilpotent matrix with the index μ (i.e. $N^\mu = 0$ and $N^{\mu-1} \neq 0$), $A_1 \in \mathbb{R}^{n_1\times n_1}$, n_1 is equal to degree of the polynomial

$$\det[Es-A] = a_{n_1} z^{n_1} + \cdots + a_1 z + a_0, \tag{11.43}$$

and $n_1 + n_2 = n$.

A method for computation of the matrices P and Q has been given in [39].

Using Lemma 11.1 we shall derive the solution x_i to the equation (11.38) for a given initial conditions x_0 and an input vector u_i, $i \in \mathbb{Z}_+$.

Premultiplying the equation (11.38) by the matrix $P \in \mathbb{R}^{n\times n}$ and introducing the new state vector

$$\bar{x}_i = \begin{bmatrix} \bar{x}_i^{(1)} \\ \bar{x}_i^{(2)} \end{bmatrix} = Q^{-1} x_i, \quad \bar{x}_i^{(1)} \in \mathbb{R}^{n_1}, \quad \bar{x}_i^{(2)} \in \mathbb{R}^{n_2}, \quad i \in \mathbb{Z}_+, \tag{11.44}$$

we obtain

$$PEQQ^{-1}\Delta^\alpha x_{i+1} = PEQ\Delta^\alpha Q^{-1} x_{i+1} = PAQQ^{-1} x_i + PBu_i, \tag{11.45}$$

and after using (11.42) and (11.44)

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} \Delta^\alpha \begin{bmatrix} \bar{x}_{i+1}^{(1)} \\ \bar{x}_{i+1}^{(2)} \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} \begin{bmatrix} \bar{x}_i^{(1)} \\ \bar{x}_i^{(2)} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_i, \quad i \in \mathbb{Z}_+, \tag{11.46}$$

where

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = PB, \quad B_1 \in \mathbb{R}^{n_1 \times m}, \quad B_2 \in \mathbb{R}^{n_2 \times m}. \tag{11.47}$$

Taking into account (11.39) from (11.46) we obtain

$$\begin{aligned} \overline{x}_{i+1}^{(1)} &= -\sum_{k=1}^{i+1}(-1)^k \binom{\alpha}{k} \overline{x}_{i-k+1}^{(1)} + A_1 \overline{x}_i^{(1)} + B_1 u_i \\ &= A_{1\alpha} \overline{x}_i^{(1)} + \sum_{k=2}^{i+1}(-1)^{k-1} \binom{\alpha}{k} \overline{x}_{i-k+1}^{(1)} + B_1 u_i, \end{aligned} \tag{11.48}$$

and

$$N\left[\overline{x}_{i+1}^{(2)} + \sum_{k=1}^{i+1}(-1)^k \binom{\alpha}{k} \overline{x}_{i-k+1}^{(2)}\right] = \overline{x}_i^{(2)} + B_2 u_i, \tag{11.49}$$

where $A_{1\alpha} = A_1 + I_{n_1}\alpha$.

The solution $\overline{x}_i^1$ to the equation (11.48) is well-known [100, 135] and it is given by the theorem.

Theorem 11.3. *The solution $\overline{x}_i^{(1)}$ of the equation (11.48) is given by the formula*

$$\overline{x}_i^{(1)} = \Phi_i \overline{x}_0^{(1)} + \sum_{k=0}^{i-1} \Phi_{i-k-1} B_1 u_k, \quad i \in \mathbb{Z}_+, \tag{11.50}$$

where the matrices Φ_i are determined by the equation

$$\Phi_{i+1} = \Phi_i A_{1\alpha} + \sum_{k=2}^{i+1}(-1)^{k-1} \binom{\alpha}{k} \Phi_{i-k+1}, \quad \Phi_0 = I_{n_1}. \tag{11.51}$$

To find the solution $\overline{x}_i^2$ of the equation (11.49) for $N \neq 0$ it is assumed that

$$N = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \in \mathbb{R}^{n_2 \times n_2}. \tag{11.52}$$

For (11.52) the equation (11.49) can be written in the form

$$\begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \left(\sum_{j=0}^{i+1}(-1)^j \binom{\alpha}{j} \begin{bmatrix} \overline{x}_{i-j+1}^{(21)} \\ \overline{x}_{i-j+1}^{(22)} \\ \vdots \\ \overline{x}_{i-j+1}^{(2,n_2)} \end{bmatrix} \right) = \begin{bmatrix} \overline{x}_i^{(21)} \\ \overline{x}_i^{(22)} \\ \vdots \\ \overline{x}_i^{(2,n_2)} \end{bmatrix} + \begin{bmatrix} B_{21} \\ B_{22} \\ \vdots \\ B_{2,n_2} \end{bmatrix} u_i. \tag{11.53}$$

From (11.53) we have

$$\begin{aligned}
\overline{x}_i^{(21)} &= -B_{21}u_i,\\
\overline{x}_i^{(22)} &= \sum_{j=0}^{i+1}(-1)^j\binom{\alpha}{j}\overline{x}_{i-j+1}^{(21)} - B_{22}u_i\\
&= -\sum_{j=0}^{i+1}(-1)^j\binom{\alpha}{j}B_{21}u_{i-j+1} - B_{22}u_i,\\
\overline{x}_i^{(23)} &= \sum_{j=0}^{i+1}(-1)^j\binom{\alpha}{j}\overline{x}_{i-j+1}^{(22)} - B_{23}u_i\\
&= -\sum_{j=0}^{i+1}(-1)^j\binom{\alpha}{j}\sum_{k=0}^{i-j+2}(-1)^k\binom{\alpha}{k}B_{21}u_{i-j-k+2}\\
&\quad -\sum_{j=0}^{i+1}(-1)^j\binom{\alpha}{j}B_{22}u_{i-j+1} - B_{23}u_i,\\
&\ \ \vdots\\
\overline{x}_i^{(2,n_2)} &= \sum_{j=0}^{i+1}(-1)^j\binom{\alpha}{j}\overline{x}_{i-j+1}^{(2,n_2-1)} - B_{2,n_2}u_i.
\end{aligned} \tag{11.54}$$

If $N = 0$ then from (11.49) we have

$$\overline{x}_i^{(2)} = -B_2u_i, \quad i \in \mathbb{Z}_+. \tag{11.55}$$

This approach can be easily extended for

$$N = \text{block diag}[N_1\ N_2\ \dots\ N_h], \tag{11.56}$$

where $N_k \in \mathbb{R}^{n_k\times n_k}$ has the form (11.52) and $\sum_{k=1}^{h} n_k = n_2$.

If the matrix N has the form

$$N = \begin{bmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 1\\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{n_2\times n_2}, \tag{11.57}$$

the considerations are similar (dual).

Note that the matrices (11.52) and (11.57) are related by $N = S\overline{N}S$ where

$$S = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Knowing $\overline{x}_i^{(1)}$ and $\overline{x}_i^{(2)}$ we can find the desired solution of the equation (11.38) from (11.44)

$$x_i = Q \begin{bmatrix} \overline{x}_i^{(1)} \\ \overline{x}_i^{(2)} \end{bmatrix}, \quad i \in \mathbb{Z}_+. \tag{11.58}$$

Example 11.4. Find the solution x_i of the singular fractional linear system (11.38) with the matrices

$$E = \begin{bmatrix} -1 & -1 & -1 \\ 2 & 4 & 2 \\ 1 & 4 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0.8 & 1.7 & 2.8 \\ 0.4 & 0.8 & 1.4 \\ 2.2 & 4.6 & 2.2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \tag{11.59}$$

for $\alpha = 0.5$, $u_i = u$, $i \in \mathbb{Z}_+$ and $x_0 = [\,1\ 2\ -1\,]^T$ (T denotes the transpose).

It is easy to check that the matrices (11.59) satisfy the assumptions (11.41). In this case the matrices P and Q have the forms

$$P = \frac{1}{11}\begin{bmatrix} 1 & -2 & 5 \\ -2 & 4 & 1 \\ 4 & 3 & -2 \end{bmatrix}, \quad Q = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{11.60}$$

and

$$PEQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{11.61a}$$

$$PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} = \begin{bmatrix} 0.1 & 1 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{11.61b}$$

$$PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -4 \\ -3 \\ 6 \end{bmatrix}, \tag{11.61c}$$

$$A_{1\alpha} = A_1 + I_{n_1}\alpha = \begin{bmatrix} 0.6 & 1 \\ 0 & 0.7 \end{bmatrix}, \quad \begin{matrix} n_1 = 2, \\ n_2 = 1. \end{matrix} \tag{11.61d}$$

The equations (11.48) and (11.49) have the forms

$$\overline{x}_{i+1}^{(1)} = \begin{bmatrix} 0.6 & 1 \\ 0 & 0.7 \end{bmatrix} \overline{x}_i^{(1)} + \sum_{k=2}^{i+1} (-1)^{k-1} \binom{0.5}{k} \overline{x}_{i-k+1}^{(1)} - \frac{1}{11}\begin{bmatrix} 4 \\ 3 \end{bmatrix} u_i, \tag{11.62}$$

and

$$\overline{x}_i^{(2)} = -B_2 u_i = -\frac{6}{11} u_i, \quad i \in \mathbb{Z}_+. \tag{11.63}$$

The solution $\overline{x}_i^{(1)}$ of the equation (11.62) has the form

$$\overline{x}_i^{(1)} = \Phi_i \overline{x}_0^{(1)} + \sum_{k=0}^{i-1} \Phi_{i-k-1} B_1 u_k, \quad i \in \mathbb{Z}_+, \tag{11.64}$$

where

$$\Phi_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{11.65a}$$

$$\Phi_1 = A_{1\alpha} = \begin{bmatrix} 0.6 & 1 \\ 0 & 0.7 \end{bmatrix}, \tag{11.65b}$$

$$\Phi_2 = A_{1\alpha}^2 - I_{n_1} \frac{\alpha(\alpha-1)}{2!} = \begin{bmatrix} 0.485 & 1.300 \\ 0 & 0.615 \end{bmatrix}, \tag{11.65c}$$

$$\vdots$$

and

$$\overline{x}_0 = Q^{-1} x_0 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}, \quad \overline{x}_0^{(1)} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \quad \overline{x}_0^{(2)} = [-1]. \tag{11.66}$$

The desired solution of the singular fractional system with (11.59) is given by

$$x_i = Q\overline{x}_i = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \overline{x}_i^{(1)} \\ \overline{x}_i^{(2)} \end{bmatrix}, \tag{11.67}$$

where $\overline{x}_i^{(1)}$ and $\overline{x}_i^{(2)}$ are determined by (11.63) and (11.64), respectively.

Example 11.5. Find the solution x_i of the singular fractional linear system (11.38) with the matrices

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0.2 & 2 & -2 \\ 2 & 1 & 0 \\ -1.8 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ -1 & 2 \\ 2 & -1 \end{bmatrix}, \tag{11.68}$$

for $\alpha = 0.8$, arbitrary u_i, $i \in \mathbb{Z}_+$ and $x_0 = [\,1\ 1\ 1\,]^T$.

It is easy to check that the matrices (11.68) satisfy the assumptions (11.41). In this case the matrices P and Q have the forms

$$P = \begin{bmatrix} -1 & 2 & 2 \\ 1 & -1 & -1 \\ -1 & 2 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix}, \tag{11.69}$$

and

$$PEQ = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{11.70a}$$

$$PAQ = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix} = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{11.70b}$$

$$PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}, \tag{11.70c}$$

$$A_{1\alpha} = A_1 + I_{n_1}\alpha = [1], \quad \begin{matrix} n_1 = 1, \\ n_2 = 2. \end{matrix} \tag{11.70d}$$

In this case the equations (11.48) and (11.49) have the forms

$$\bar{x}_{i+1}^{(1)} = \bar{x}_i^{(1)} + \sum_{k=2}^{i+1} (-1)^{k-1} \binom{0.8}{k} \bar{x}_{i-k+1}^{(1)} + [\,1\ 0\,]u_i, \quad i \in \mathbb{Z}_+, \tag{11.71}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \left(\sum_{j=0}^{i+1} (-1)^j \binom{0.8}{j} \begin{bmatrix} \bar{x}_{i-j+1}^{(21)} \\ \bar{x}_{i-j+1}^{(22)} \end{bmatrix} \right) = \begin{bmatrix} \bar{x}_i^{(21)} \\ \bar{x}_i^{(22)} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} u_i, \tag{11.72}$$

and

$$\bar{x}_0 = Q^{-1}x_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \quad \bar{x}_0^{(1)} = [1], \quad \bar{x}_0^{(2)} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}. \tag{11.73}$$

The solution $\bar{x}_i^{(1)}$ of the equation (11.71) with $\bar{x}_0^{(1)} = 1$ can be easily found using (11.50) and (11.51).

From (11.72) we have

$$\bar{x}_i^{(21)} = [\,0\ -1\,]u_i, \quad i \in \mathbb{Z}_+, \tag{11.74a}$$

$$\bar{x}_i^{(22)} = \sum_{j=0}^{i+1} (-1)^j \binom{0.8}{j} [\,0\ -1\,]u_{i-j+1} + [\,1\ -1\,]u_i. \tag{11.74b}$$

The desired solution of the singular fractional system with (11.68) is given by

$$x_i = Q\bar{x}_i = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{x}_i^{(1)} \\ \bar{x}_i^{(21)} \\ \bar{x}_i^{(22)} \end{bmatrix}, \tag{11.75}$$

where $\bar{x}_i^{(1)}$, $\bar{x}_i^{(21)}$ and $\bar{x}_i^{(22)}$ are determined by (11.71) and (11.74), respectively.

For example for singular positive linear systems with different fractional order. The linear systems with different fractional orders are described by the equation [10].

$$\begin{bmatrix} \frac{d^{\alpha}x_1}{dt^{\alpha}} \\ \frac{d^{\beta}x_2}{dt^{\beta}} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \quad \begin{matrix} p-1 < \alpha < p; \\ q-1 < \beta < q; \\ p, q \in \mathbb{N}; \end{matrix} \tag{11.76}$$

where $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$ are the state vectors and $A_{ij} \in \mathbb{R}^{n_i \times n_j}$, $B_i \in \mathbb{R}^{n_i \times m}$, $i, j = 1, 2$; and $u \in \mathbb{R}^m$ is the input vector. Initial conditions for (11.76) have the form $x_1(0) = x_{10}$ and $x_2(0) = x_{20}$.

11.4 Reduction of Singular Fractional Systems to Equivalent Standard Fractional Systems

Consider the singular fractional discrete-time linear system described by the state equation

$$E\Delta^{\alpha}x_{i+1} = Ax_i + Bu_i, \quad i \in \mathbb{Z}_+ = \{0, 1, \ldots\}, \tag{11.77}$$

where, $x_i \in \mathbb{R}^n, u_i \in \mathbb{R}^m$ are the state and input vectors, $A \in \mathbb{R}^{n \times n}$, $E \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and the fractional difference of the order α is defined by

$$\Delta^{\alpha}x_i = \sum_{k=0}^{i} (-1)^k \binom{\alpha}{k} x_{i-k}, \quad 0 < \alpha < 1, \tag{11.78}$$

$$\binom{\alpha}{k} = \begin{cases} 1 & \text{for} \quad k = 0 \\ \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} & \text{for} \quad k = 1, 2, \ldots \end{cases} \tag{11.79}$$

It is assumed that

$$\det E = 0, \tag{11.80a}$$

and

$$\det[Ez - A] \neq 0, \tag{11.80b}$$

for some $z \in \mathbb{C}$ (the field of complex numbers).

Substituting (11.78) into (11.77) we obtain

$$\sum_{k=0}^{i+1} Ec_k x_{i-k+1} = Ax_i + Bu_i, \quad i \in \mathbb{Z}_+, \tag{11.81}$$

where

$$c_k = (-1)^k \binom{\alpha}{k}. \tag{11.82}$$

The following elementary operations on rows(columns) will be used [89] (Appendix D).

a) Multiplication of the i-th row (column) by nonzero scalar c. This operation will be denoted by $L(ixc)$ $(R(ixc))$.
b) Addition to the i-th row (column) of the j-th row (column) multiplied by nonzero scalar b. This operation will be denoted by $L(i+jxb)$ $(R(i+jxb))$.
c) Interchange of the i-th and j-th rows (columns). This operation will be denoted by $L(i,j)$ $(R(i,j))$.

Applying the row elementary operations to (11.81) we obtain

$$\sum_{k=0}^{i+1}\begin{bmatrix}E_1\\0\end{bmatrix}c_k x_{i-k+1}=\begin{bmatrix}A_1\\A_2\end{bmatrix}x_i+\begin{bmatrix}B_1\\B_2\end{bmatrix}u_i,\quad i\in\mathbb{Z}_+, \tag{11.83}$$

where $E_1\in\mathbb{R}^{n_1\times n}$ is full row rank and $A_1\in\mathbb{R}^{n_1\times n}$, $A_2\in\mathbb{R}^{(n-n_1)\times n}$, $B_1\in\mathbb{R}^{n_1\times m}$, $B_2\in\mathbb{R}^{(n-n_1)\times m}$. The equation (11.83) can be rewritten as

$$\sum_{k=0}^{i+1}E_1 c_k x_{i-k+1}=A_1x_i+B_1u_i, \tag{11.84a}$$

and

$$0=A_2x_i+B_2u_i. \tag{11.84b}$$

Substituting in (11.84b) i by $i+1$ we obtain

$$A_2x_{i+1}=-B_2u_{i+1}. \tag{11.85}$$

The equations (11.84a) and (11.85) can be written in the form

$$\begin{aligned}\begin{bmatrix}E_1\\A_2\end{bmatrix}x_{i+1}=&\begin{bmatrix}A_1-c_1E_1\\0\end{bmatrix}x_i-\begin{bmatrix}c_2E_1\\0\end{bmatrix}x_{i-1}-\cdots-\begin{bmatrix}c_{i+1}E_1\\0\end{bmatrix}x_0\\&+\begin{bmatrix}B_1\\0\end{bmatrix}u_i+\begin{bmatrix}0\\-B_2\end{bmatrix}u_{i+1}.\end{aligned} \tag{11.86}$$

If the matrix

$$\begin{bmatrix}E_1\\A_2\end{bmatrix}, \tag{11.87}$$

is nonsingular then premultiplying the equation (11.86) by the inverse matrix $\begin{bmatrix}E_1\\A_2\end{bmatrix}^{-1}$ we obtain the standard system

$$x_{i+1}=\overline{A}_0x_i+\overline{A}_1x_{i-1}+\cdots+\overline{A}_ix_0+\overline{B}_0u_i+\overline{B}_1u_{i+1}, \tag{11.88}$$

where

$$\begin{aligned}
\overline{A}_0 &= \begin{bmatrix} E_1 \\ A_2 \end{bmatrix}^{-1} \begin{bmatrix} A_1 - c_1 E_1 \\ 0 \end{bmatrix}, \quad \overline{A}_1 = -\begin{bmatrix} E_1 \\ A_2 \end{bmatrix}^{-1} \begin{bmatrix} c_2 E_1 \\ 0 \end{bmatrix}, \dots, \\
\overline{A}_i &= -\begin{bmatrix} E_1 \\ A_2 \end{bmatrix}^{-1} \begin{bmatrix} c_{i+1} E_1 \\ 0 \end{bmatrix}, \\
\overline{B}_0 &= \begin{bmatrix} E_1 \\ A_2 \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad \overline{B}_1 = \begin{bmatrix} E_1 \\ A_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -B_2 \end{bmatrix}.
\end{aligned} \tag{11.89}$$

If the matrix (11.87) is singular then applying the row elementary operations to (11.86) we obtain

$$\begin{aligned}
\begin{bmatrix} E_2 \\ 0 \end{bmatrix} x_{i+1} &= \begin{bmatrix} A_{20} \\ \overline{A}_{20} \end{bmatrix} x_i + \begin{bmatrix} A_{21} \\ \overline{A}_{21} \end{bmatrix} x_{i-1} + \cdots + \begin{bmatrix} A_{2,i} \\ \overline{A}_{2,i} \end{bmatrix} x_0 \\
&+ \begin{bmatrix} B_{20} \\ \overline{B}_{20} \end{bmatrix} u_i + \begin{bmatrix} B_{21} \\ \overline{B}_{21} \end{bmatrix} u_{i+1},
\end{aligned} \tag{11.90}$$

where $E_2 \in \mathbb{R}^{n_2 \times n}$ is full row rank with $n_2 \geq n_1$ and $A_{2,j} \in \mathbb{R}^{n_2 \times n}$, $\overline{A}_{2,j} \in \mathbb{R}^{(n-n_2) \times n}$, $j = 0, 1, \dots, i$; $B_{2,k} \in \mathbb{R}^{n_2 \times m}$, $\overline{B}_{2,k} \in \mathbb{R}^{(n-n_2) \times m}$, $k = 0, 1$.

From (11.90) we have

$$0 = \overline{A}_{20} x_i + \overline{A}_{21} x_{i-1} + \cdots + \overline{A}_{2,i} x_0 + \overline{B}_{20} u_i + \overline{B}_{21} u_{i+1}. \tag{11.91}$$

Substituting in (11.91) i by $i+1$ (in state vector x and in input u) we obtain

$$\overline{A}_{20} x_{i+1} = -\overline{A}_{21} x_i - \cdots - \overline{A}_{2,i} x_1 - \overline{B}_{20} u_{i+1} - \overline{B}_{21} u_{i+2} \tag{11.92}$$

From (11.90) and (11.92) we have

$$\begin{aligned}
\begin{bmatrix} E_2 \\ \overline{A}_{20} \end{bmatrix} x_{i+1} &= \begin{bmatrix} A_{20} \\ -\overline{A}_{21} \end{bmatrix} x_i + \begin{bmatrix} A_{21} \\ -\overline{A}_{22} \end{bmatrix} x_{i-1} + \cdots + \begin{bmatrix} A_{2,i} \\ 0 \end{bmatrix} x_0 \\
&+ \begin{bmatrix} B_{20} \\ 0 \end{bmatrix} u_i + \begin{bmatrix} B_{21} \\ -\overline{B}_{20} \end{bmatrix} u_{i+1} + \begin{bmatrix} 0 \\ -\overline{B}_{21} \end{bmatrix} u_{i+2}.
\end{aligned} \tag{11.93}$$

If the matrix

$$\begin{bmatrix} E_2 \\ \overline{A}_{20} \end{bmatrix}, \tag{11.94}$$

is nonsingular then premultiplying the equation (11.93) by its inverse we obtain the standard system

$$x_{i+1} = \hat{A}_0 x_i + \hat{A}_1 x_{i-1} + \cdots + \hat{A}_i x_0 + \hat{B}_0 u_i + \hat{B}_1 u_{i+1} + \hat{B}_2 u_{i+2}, \tag{11.95}$$

where

$$\hat{A}_0 = \begin{bmatrix} E_2 \\ \overline{A}_{20} \end{bmatrix}^{-1} \begin{bmatrix} A_{20} \\ -\overline{A}_{21} \end{bmatrix}, \quad \hat{A}_1 = \begin{bmatrix} E_2 \\ \overline{A}_{20} \end{bmatrix}^{-1} \begin{bmatrix} A_{21} \\ -\overline{A}_{22} \end{bmatrix}, \dots,$$

$$\hat{A}_i = \begin{bmatrix} E_2 \\ \overline{A}_{20} \end{bmatrix}^{-1} \begin{bmatrix} A_{2,i} \\ 0 \end{bmatrix}, \tag{11.96a}$$

$$\hat{B}_0 = \begin{bmatrix} E_2 \\ \overline{A}_{20} \end{bmatrix}^{-1} \begin{bmatrix} B_{20} \\ 0 \end{bmatrix}, \quad \hat{B}_1 = \begin{bmatrix} E_2 \\ \overline{A}_{20} \end{bmatrix}^{-1} \begin{bmatrix} B_{21} \\ -\overline{B}_{20} \end{bmatrix},$$

$$\hat{B}_2 = \begin{bmatrix} E_2 \\ \overline{A}_{20} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -\overline{B}_{21} \end{bmatrix}. \tag{11.96b}$$

If the matrix (11.94) is singular we repeat the procedure. Continuing this procedure after at most n steps we finally obtain a nonsingular matrix and the desired standard fractional system. The procedure can be justified as follows. The elementary row operations do not change the rank of the matrix $[Ez-A]$. The substitution in the equations (11.84b) and (11.91) i by $i+1$ also does not change the rank of the matrix $[Ez-A]$ since it is equivalent to multiplication of its lower rows by z and by assumption (11.80b) holds. Therefore, the following theorem has been proved.

Theorem 11.4. *The singular fractional linear system (11.81) satisfying the assumption (11.80) can be reduced to the standard fractional linear system*

$$x_{i+1} = \tilde{A}_0 x_i + \tilde{A}_1 x_{i-1} + \cdots + \tilde{A}_i x_0 + \tilde{B}_0 u_i + \tilde{B}_1 u_{i+1} + \cdots + \tilde{B}_p u_{i+p}, \tag{11.97}$$

where $\tilde{A}_j \in \mathbb{R}^{n\times n}$, $j = 0,1,\dots,i$; $\tilde{B}_k \in \mathbb{R}^{n\times m}$, $k = 0,1,\dots,p < n$ *whose dynamics depends on the future inputs* $u_{i+1},\dots,u_{i+p}$.

Example 11.6. Consider the singular fractional linear system (11.77) for $\alpha = 0.5$ with

$$E = \begin{bmatrix} 5 & 0 & 2 \\ 2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.2 & 2 & -2 \\ 2 & 1 & 0 \\ -1.8 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ -1 & 2 \\ 2 & -1 \end{bmatrix}. \tag{11.98}$$

In this case the conditions (11.80) are satisfied since

$$\det E = 0 \quad \text{and} \quad \det[Ez-A] = \begin{vmatrix} 5z-0.2 & -2 & 2z+2 \\ 2z-2 & -1 & z \\ z+1.8 & 0 & 1 \end{vmatrix} = z - 0.2\,.$$

Applying to the matrices (11.98) the following elementary row operations $L[1+2\times(-2)]$, $L[3+1\times(-1)]$ we obtain

$$\begin{bmatrix} E & A & B \end{bmatrix} = \begin{bmatrix} 5 & 0 & 2 & 0.2 & 2 & -2 & 1 & 2 \\ 2 & 0 & 1 & 2 & 1 & 0 & -1 & 2 \\ 1 & 0 & 0 & -1.8 & 0 & -1 & 2 & -1 \end{bmatrix} \tag{11.99}$$
$$\to \begin{bmatrix} 1 & 0 & 0 & -3.8 & 0 & -2 & 3 & -2 \\ 2 & 0 & 1 & 2 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 2 & 0 & 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} E_1 & A_1 & B_1 \\ Q & A_2 & B_2 \end{bmatrix},$$

and the equations (11.84) have the form

$$\sum_{k=0}^{i+1} c_k \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} x_{i-k+1} = \begin{bmatrix} -3.8 & 0 & -2 \\ 2 & 1 & 0 \end{bmatrix} x_i + \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} u_i, \tag{11.100a}$$

and

$$0 = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} x_i + \begin{bmatrix} -1 & 1 \end{bmatrix} u_i. \tag{11.100b}$$

Using (11.82) we obtain $c_1 = -\binom{\alpha}{1} = -\alpha = -0.5$, $c_2 = (-1)^2\binom{\alpha}{2} = \frac{\alpha(\alpha-1)}{2!} = -\frac{1}{8}$, $\dots, c_{i+1} = (-1)^{i-1}\frac{\alpha(\alpha-1)\cdots(\alpha-i)}{(i+1)!}\Big|_{\alpha=0.5}$ and the equation (11.86) has the form

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix} x_{i+1} &= \begin{bmatrix} -3.3 & 0 & -2 \\ 3 & 1 & 0.5 \\ 0 & 0 & 0 \end{bmatrix} x_i + \frac{1}{8}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x_{i-1} \\ &\quad - \cdots - c_{i+1}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x_0 \\ &\quad + \begin{bmatrix} 3 & -2 \\ -1 & 2 \\ 0 & 0 \end{bmatrix} u_i + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} u_{i+1}. \end{aligned} \tag{11.101}$$

The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix}$ is singular and we perform the elementary row operation $L[3 + 2 \times (-1)]$ on (11.101) obtaining the following

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x_{i+1} &= \begin{bmatrix} -3.3 & 0 & -2 \\ 3 & 1 & 0.5 \\ -3 & -1 & -0.5 \end{bmatrix} x_i + \frac{1}{8}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -2 & 0 & -1 \end{bmatrix} x_{i-1} \\ &\quad - \cdots - c_{i+1}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -2 & 0 & -1 \end{bmatrix} x_0 \\ &\quad + \begin{bmatrix} 3 & -2 \\ -1 & 2 \\ 1 & -2 \end{bmatrix} u_i + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} u_{i+1}. \end{aligned} \tag{11.102}$$

The matrix

$$\begin{bmatrix} E_2 \\ \bar{A}_{20} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -3 & -1 & -0.5 \end{bmatrix}, \tag{11.103}$$

is nonsingular and we obtain the equation (11.95) with the matrices

$$\begin{aligned}
\hat{A}_0 &= \begin{bmatrix} E_2 \\ \bar{A}_{20} \end{bmatrix}^{-1} \begin{bmatrix} A_{20} \\ -\bar{A}_{21} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -3 & -1 & -0.5 \end{bmatrix}^{-1} \begin{bmatrix} -3.3 & 0 & -2 \\ 3 & 1 & 0.5 \\ 0.25 & 0 & 0.125 \end{bmatrix} = \begin{bmatrix} -3.30 & 0 & -2 \\ 4.85 & -0.5 & 3.63 \\ 9.60 & 1 & 4.50 \end{bmatrix}, \\
&\vdots \\
\hat{A}_i &= \begin{bmatrix} E_2 \\ \bar{A}_{20} \end{bmatrix}^{-1} \begin{bmatrix} -A_{2,i} \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -3 & -1 & -0.5 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & -0.5 \\ 0 & 0 & 1 \end{bmatrix}, \\
\hat{B}_0 &= \begin{bmatrix} E_2 \\ \bar{A}_{20} \end{bmatrix}^{-1} \begin{bmatrix} B_{20} \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -3 & -1 & -0.5 \end{bmatrix}^{-1} \begin{bmatrix} 3 & -2 \\ -1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -5.5 & 3 \\ -7 & 6 \end{bmatrix}, \\
\hat{B}_1 &= \begin{bmatrix} E_2 \\ \bar{A}_{20} \end{bmatrix}^{-1} \begin{bmatrix} B_{21} \\ -\bar{B}_{20} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -3 & -1 & -0.5 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -2 \\ 0 & 0 \end{bmatrix}, \\
\hat{B}_2 &= \begin{bmatrix} E_2 \\ \bar{A}_{20} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -\bar{B}_{21} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -3 & -1 & -0.5 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}.
\end{aligned} \tag{11.104}$$

11.5 Decomposition of Singular Fractional System into Dynamic and Static Parts

Consider the singular fractional system (11.81) satisfying the assumptions (11.80). Applying the procedure presented in section 11.4 after p steps we obtain

$$\begin{bmatrix} E_p \\ 0 \end{bmatrix} x_{i+1} = \begin{bmatrix} A_{p,0} \\ \overline{A}_{p,0} \end{bmatrix} x_i + \begin{bmatrix} A_{p,1} \\ \overline{A}_{p,2} \end{bmatrix} x_{i-1} + \cdots + \begin{bmatrix} A_{pi} \\ \overline{A}_{pi} \end{bmatrix} x_0 + \begin{bmatrix} B_{p,0} \\ \overline{B}_{p,0} \end{bmatrix} u_i + \begin{bmatrix} B_{p,1} \\ \overline{B}_{p,1} \end{bmatrix} u_{i+1} + \cdots + \begin{bmatrix} B_{p,p-1} \\ \overline{B}_{p,p-1} \end{bmatrix} u_{i+p-1}, \tag{11.105}$$

where $E_p \in \mathbb{R}^{n_p \times n}$ is full row rank, $A_{pj} \in \mathbb{R}^{n_p \times n}$, $\overline{A}_{pj} \in \mathbb{R}^{(n-n_p)\times n}$, $j = 0, 1, \dots, p$; and $B_{pk} \in \mathbb{R}^{n_p \times m}$, $\overline{B}_{pk} \in \mathbb{R}^{(n-n_p)\times m}$, $k = 0, 1, \dots, p-1$ with nonsingular matrix

$$\begin{bmatrix} E_p \\ \overline{A}_{p,0} \end{bmatrix} \in \mathbb{R}^{n \times n}. \tag{11.106}$$

Using the elementary column operations we may reduced the matrix (11.106) to the form

$$\begin{bmatrix} I_{n_p} & 0 \\ A_{21} & I_{n-n_p} \end{bmatrix}, \; A_{21} \in \mathbb{R}^{(n-n_p)\times n_p}, \tag{11.107}$$

and performing the same elementary operations on the matrix I_n we can find the matrix $Q \in \mathbb{R}^{n \times n}$ such that

$$\begin{bmatrix} E_p \\ \overline{A}_{p,0} \end{bmatrix} Q = \begin{bmatrix} I_{n_p} & 0 \\ A_{21} & I_{n-n_p} \end{bmatrix}. \tag{11.108}$$

Taking into account (11.108) and defining the new state vector

$$\tilde{x}_i = Q^{-1} x_i = \begin{bmatrix} \tilde{x}_i^{(1)} \\ \tilde{x}_i^{(2)} \end{bmatrix}, \quad \tilde{x}_i^{(1)} \in \mathbb{R}^{n_p}, \quad \tilde{x}_i^{(2)} \in \mathbb{R}^{n-n_p}, \quad i \in \mathbb{Z}_+, \tag{11.109}$$

from (11.105) we obtain

$$\begin{aligned} \tilde{x}_{i+1}^{(1)} &= E_p x_{i+1} = E_p Q Q^{-1} x_{i+1} = A_{p,0} Q Q^{-1} x_i + A_{p,1} Q Q^{-1} x_{i-1} + \cdots + A_{pi} Q Q^{-1} x_0 \\ &+ B_{p,0} u_i + B_{p,1} u_{i+1} + \cdots + B_{p,p-1} u_{i+p-1} \\ &= [A_{p,0}^{(1)} \; A_{p,0}^{(2)}] \begin{bmatrix} \tilde{x}_i^{(1)} \\ \tilde{x}_i^{(2)} \end{bmatrix} + [A_{p,1}^{(1)} \; A_{p,1}^{(2)}] \begin{bmatrix} \tilde{x}_{i-1}^{(1)} \\ \tilde{x}_{i-1}^{(2)} \end{bmatrix} \\ &+ \cdots + [A_{pi}^{(1)} \; A_{pi}^{(2)}] \begin{bmatrix} \tilde{x}_0^{(1)} \\ \tilde{x}_0^{(2)} \end{bmatrix} \\ &+ B_{p,0} u_i + B_{p,1} u_{i+1} + \cdots + B_{p,p-1} u_{i+p-1} \\ &= A_{p,0}^{(1)} \tilde{x}_i^{(1)} + A_{p,0}^{(2)} \tilde{x}_i^{(2)} + \cdots + A_{pi}^{(1)} \tilde{x}_0^{(1)} + A_{pi}^{(2)} \tilde{x}_0^{(2)} \\ &+ B_{p,0} u_i + B_{p,1} u_{i+1} + \cdots + B_{p,p-1} u_{i+p-1}, \; i \in Z_+, \end{aligned} \tag{11.110}$$

and

$$\begin{aligned} \tilde{x}_i^{(2)} &= -A_{21} \tilde{x}_i^{(1)} - \overline{A}_{p,1}^{(1)} \tilde{x}_{i-1}^{(1)} - \overline{A}_{p,2}^{(2)} \tilde{x}_{i-1}^{(2)} - \cdots - \overline{A}_{pi}^{(1)} \tilde{x}_0^{(1)} - \overline{A}_{pi}^{(2)} \tilde{x}_0^{(2)} \\ &- \overline{B}_{p,0} u_i - \cdots - \overline{B}_{p,p-1} u_{i+p-1}, \quad i \in \mathbb{Z}_+, \end{aligned} \tag{11.111}$$

where

$$A_{pj}Q = [A_{pj}^{(1)}\ A_{pj}^{(2)}], \quad \overline{A}_{pj} = [\overline{A}_{pj}^{(1)}\ \overline{A}_{pj}^{(2)}], \quad j = 0, 1, \ldots, i\,. \tag{11.112}$$

Substitution of (11.111) into (11.110) yields

$$\tilde{x}_{i+1}^{(1)} = \tilde{A}_{p,0}\tilde{x}_i^{(1)} + \cdots + \tilde{A}_{pi}\tilde{x}_0^{(1)} + \tilde{B}_{p,0}u_i + \cdots + \tilde{B}_{p,p-1}u_{i+p-1}, \quad i \in \mathbb{Z}_+ \tag{11.113}$$

where

$$\tilde{A}_{p,0} = A_{p,0}^{(1)} - A_{p,0}^{(2)}A_{21}, \quad \ldots, \quad \tilde{A}_{pi} = A_{pi}^{(1)} - A_{p,0}^{(2)}A_{pi}^{(1)}, \tag{11.114a}$$

$$\tilde{B}_{p,0} = B_{p,0} - A_{p,0}^{(2)}\overline{B}_{p,0}, \quad \ldots, \quad \tilde{B}_{p,p-1} = B_{p,p-1} - A_{p,0}^{(2)}\overline{B}_{p,p-1}. \tag{11.114b}$$

The standard system described by the equation (11.113) is called the dynamic part of the system (11.81) and the system described by the equation (11.111) is called the static part of the system (11.81).

Therefore, the following theorem has been proved.

Theorem 11.5. *The singular fractional linear system* (11.81) *satisfying the assumption* (11.80) *can be decomposed into the dynamical part* (11.113) *and static part* (11.111) *whose dynamics depend on the future inputs* $u_{i+1}, \ldots, u_{i+p-1}$.

Example 11.7. Consider the singular fractional system (11.77) for $\alpha = 0.5$ with the matrices (11.98). The matrix (11.103) is nonsingular. To reduce this matrix to the form (11.107) we perform the elementary operations $R[1+3\times(-2)]$, $R[2\times(-1)]$, $R[2,3]$. The matrix Q has the form

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} E_2 \\ \overline{A_{20}} \end{bmatrix} Q = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -3 & -1 & -0.5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -0.5 & 1 \end{bmatrix},$$

$$A_{21} = [-2\ \ -0.5], \quad n_2 = 2.$$

The new state vector (11.109) is

$$\tilde{x}_i = Q^{-1}x_i = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_{1,i} \\ x_{2,i} \\ x_{3,i} \end{bmatrix} = \begin{bmatrix} \tilde{x}_i^{(1)} \\ \tilde{x}_i^{(2)} \end{bmatrix},$$

$$\tilde{x}_i^{(1)} = \begin{bmatrix} x_{1,i} \\ 2x_{1,i} + x_{3,i} \end{bmatrix}, \tag{11.115}$$

$$\tilde{x}_i^{(2)} = -x_{2,i}.$$

In this case the equations (11.110) and (11.111) have the forms

$$\tilde{x}_{i+1}^{(1)} = \begin{bmatrix} 0.7 & -2 \\ 2 & 0.5 \end{bmatrix} \tilde{x}_i^{(1)} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tilde{x}_i^{(2)} + \frac{1}{8} \tilde{x}_{i-1}^{(1)} - \cdots - c_{i+1} \tilde{x}_0^{(1)} + \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix} u_i, \tag{11.116}$$

and

$$\tilde{x}_i^{(2)} = [2\ 0.5]\tilde{x}_i^{(1)} + [0.25\ 0]\tilde{x}_{i-1}^{(1)} + \cdots + c_{i+1}[-2\ 0]\tilde{x}_0^{(1)} - [1\ -2]u_i - [1\ -1]u_{i+1}. \tag{11.117}$$

Substituting (11.117) into (11.116) we obtain

$$\tilde{x}_{i+1}^{(1)} = \begin{bmatrix} 0.7 & -2 \\ 0 & 0 \end{bmatrix} \tilde{x}_i^{(1)} + \frac{1}{8} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tilde{x}_{i-1}^{(1)} - \cdots - c_{i+1} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \tilde{x}_0^{(1)} + \begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix} u_i + \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} u_{i+1}. \tag{11.118}$$

The dynamic part of the system is described by (11.118) and the static part by (11.117).

This considerations can be extended for singular fractional continuous-time and discrete-time linear systems with delays.

Chapter 12
Positive Continuous-Discrete Linear Systems

12.1 General Model of Continuous-Discrete Linear Systems and Its Solution

Consider the general model of linear continuous-discrete systems described by the equations

$$\begin{aligned}\dot{x}(t,i+1) &= A_0x(t,i)+A_1\dot{x}(t,i)+A_2x(t,i+1)\\ &\quad + B_0u(t,i)+B_1\dot{u}(t,i)+B_2u(t,i+1), \quad (12.1a)\\ y(t,i) &= Cx(t,i)+Du(t,i), \quad t\in\mathbb{R}_+=[0,+\infty], \quad i\in\mathbb{Z}_+, \quad (12.1b)\end{aligned}$$

where $\dot{x}(t,i)=\frac{\partial x(t,i)}{\partial t}$, $x(t,i)\in\mathbb{R}^n$, $u(t,i)\in\mathbb{R}^m$, $y(t,i)\in\mathbb{R}^p$ are the state, input and output vectors and $A_k\in\mathbb{R}^{n\times n}$, $B_k\in\mathbb{R}^{n\times m}$, $k=0,1,2$; $C\in\mathbb{R}^{p\times n}$, $D\in\mathbb{R}^{p\times m}$.

Boundary conditions for (12.1a) are given by

$$x(0,i)=x_i, \quad i\in\mathbb{Z}_+ \quad \text{and} \quad x(t,0)=x_{t0}, \quad \dot{x}(t,0)=x_{t1}, \quad t\in\mathbb{R}_+. \quad (12.2)$$

The transition matrix T_{ij} of the model (12.1) is defined as follows

$$T_{i,j}=\begin{cases} I_n & \text{for} \quad i=j=0\\ A_0T_{i-1,j-1}+A_1T_{i,j-1}+A_2T_{i-1,j} & \\ =T_{i-1,j-1}A_0+T_{i,j-1}A_1+T_{i-1,j}A_2 & \text{for} \quad i+j>0;\ i,j\in\mathbb{Z}_+\\ 0 & \text{for} \quad k<0 \quad \text{or} \quad l<0\end{cases} \quad (12.3)$$

Theorem 12.1. *The solution of the equation* (12.1a) *with boundary conditions* (12.2) *has the form*

T. Kaczorek: Selected Problems of Fractional Systems Theory, LNCIS 411, pp. 267–305.
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$$
\begin{aligned}
x(t,i) = & \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left(T_{k,i-l-1}B_0\int_0^t \frac{(t-\tau)^k}{k!}u(\tau,l)d\tau + T_{k,i-l}B_2\right. \\
& \times \int_0^t \frac{(t-\tau)^k}{k!}u(\tau,l)d\tau - T_{k,i-l-1}B_1\frac{t^k}{k!}u(0,l) \\
& \left. + T_{k,i-l}\frac{t^k}{k!}x(0,l) - T_{k,i-l-1}A_1\frac{t^k}{k!}x(0,l)\right) \\
& + \sum_{k=1}^{\infty}\sum_{l=0}^{\infty}\left(T_{k,i-l-1}B_1\int_0^t \frac{(t-\tau)^{k-1}}{(k-1)!}u(\tau,l)d\tau\right) + \sum_{l=0}^{\infty}T_{0,i-l-1}B_1u(t,l) \\
& - \sum_{k=0}^{\infty}\left(T_{k,i}B_2\int_0^t \frac{(t-\tau)^k}{k!}u(\tau,0)d\tau + T_{k,i}A_2\right. \\
& \left. \times \int_0^t \frac{(t-\tau)^k}{k!}x(\tau,0)d\tau + T_{k,i}\frac{t^k}{k!}x(0,0)\right) \\
& + \sum_{k=1}^{\infty}\left(T_{k,i}\int_0^t \frac{(t-\tau)^{k-1}}{(k-1)!}x(\tau,0)d\tau\right) + T_{0,i}x(t,0). \qquad (12.4)
\end{aligned}
$$

Proof. Let $X(s)$ be the Laplace transform of the continuous-time vector $x(t)$

$$X(s) = \mathscr{L}[x(t)] = \int_0^{\infty} x(t)e^{-st}dt, \qquad (12.5)$$

and $X(z)$ be the zet transform of the discrete-time vector $x(i)$

$$X(z) = \mathscr{Z}[x(i)] = \sum_{i=0}^{\infty} x(i)z^{-i}. \qquad (12.6)$$

Using (12.5) and (12.6) it is easy to show that

$$
\begin{aligned}
\mathscr{Z}\{\mathscr{L}[\dot{x}(t,i)]\} &= \mathscr{Z}\{sX(s,i)-x(0,i)\} = sX(s,z)-X(0,z), \\
\mathscr{Z}\{\mathscr{L}[\dot{x}(t,i+1)]\} &= \mathscr{Z}\{sX(s,i+1)-x(0,i+1)\} \\
&= szX(s,z)-szX(s,0)-zX(0,z)+zx(0,0), \qquad (12.7) \\
\mathscr{Z}\{\mathscr{L}[x(t,i+1)]\} &= \mathscr{Z}\{X(s,i)\} = zX(s,z)-zX(s,0),
\end{aligned}
$$

where $X(s,0) = \mathscr{L}[x(t,0)]$, $X(0,z) = \mathscr{Z}[x(0,i)]$.

Using (12.7) from (12.1a) we obtain

$$
\begin{aligned}
X(s,z) = & \left[I_n - A_0s^{-1}z^{-1} - A_1z^{-1} - A_2s^{-1}\right]^{-1} \\
& \times \begin{bmatrix} (B_0s^{-1}z^{-1}+B_1z^{-1}+B_2s^{-1})U(s,z) - B_1s^{-1}z^{-1}U(0,z) \\ -B_2s^{-1}U(s,0) + (I_n - A_2s^{-1})X(s,0) \\ +(I_ns^{-1} - A_1s^{-1}z^{-1})X(0,z) - s^{-1}x(0,0) \end{bmatrix}, \qquad (12.8)
\end{aligned}
$$

where $U(s,z) = \mathscr{Z}\{\mathscr{L}[u(t,i)]\}$, $U(s,0) = \mathscr{L}[u(t,0)]$, $U(0,z) = \mathscr{Z}[u(0,i)]$.

Let

$$\left[I_n - A_0 s^{-1} z^{-1} - A_1 z^{-1} - A_2 s^{-1}\right]^{-1} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} T_{i,j} s^{-i} z^{-j} \tag{12.9}$$

From the definition of inverse matrix we have

$$\begin{aligned} I_n &= \left[I_n - A_0 s^{-1} z^{-1} - A_1 z^{-1} - A_2 s^{-1}\right] \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} T_{i,j} s^{-i} z^{-j} \\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} T_{i,j} s^{-i} z^{-j} \left[I_n - A_0 s^{-1} z^{-1} - A_1 z^{-1} - A_2 s^{-1}\right]. \end{aligned} \tag{12.10}$$

Comparing the matrices at the same power of s^{-1} and z^{-1} of (12.10) we obtain (12.3).

Substitution of (12.9) into (12.8) yields

$$\begin{aligned} X(s,z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} T_{i,j} s^{-i} z^{-j} &\left[(B_0 s^{-1} z^{-1} + B_1 z^{-1} + B_2 s^{-1})U(s,z)\right. \\ &- B_2 s^{-1} U(s,0) - B_1 s^{-1} z^{-1} U(0,z) + (I_n - A_2 s^{-1})X(s,0) \\ &\left.+ (I_n s^{-1} - A_1 s^{-1} z^{-1})X(0,z) - s^{-1} x(0,0)\right]. \end{aligned} \tag{12.11}$$

Applying to (12.11) the inverse transforms and the convolution theorem we obtain (12.4). □

Knowing the matrices A_k, B_k, $k = 0,1,2$ of (12.1a), boundary conditions (12.2) and input $u(t,i)$, $t \in \mathbb{R}_+$, $i \in \mathbb{Z}_+$ we can compute the transition matrices (12.3) and using (12.4) the state vector $x(t,i)$ for $t \in \mathbb{R}_+$, $i \in \mathbb{Z}_+$. Substituting the state vector into (12.1b) we can find the output vector $y(t,i)$ for $t \in \mathbb{R}_+$, $i \in \mathbb{Z}_+$.

12.2 Positive General Model of Continuous-Discrete Linear Systems

Definition 12.1. The general model (12.1) is called positive if $x(t,i) \in \mathbb{R}_+^n$ and $y(t,i) \in \mathbb{R}_+^p$, $t \in \mathbb{R}_+$, $i \in \mathbb{Z}_+$ for any boundary conditions

$$x_{t0} \in \mathbb{R}_+^n, \quad x_{t1} \in \mathbb{R}_+^n, \quad t \in \mathbb{R}_+, \quad x_i \in \mathbb{R}_+^n, \quad i \in \mathbb{Z}_+, \tag{12.12}$$

and all inputs $u(t,i) \in \mathbb{R}_+^m$, $\dot{u}(t,i) \in \mathbb{R}_+^m$, $t \in \mathbb{R}_+$, $i \in \mathbb{Z}_+$.

Theorem 12.2. *The general model (12.1) is positive if and only if*

$$A_2 \in M_n, \tag{12.13a}$$

$$A_0, A_1 \in \mathbb{R}_+^{n\times n}, \quad A = A_0 + A_1 A_2 \in \mathbb{R}_+^{n\times n}, \tag{12.13b}$$

$$B_k \in \mathbb{R}_+^{n\times m}, \quad k = 0,1,2, \quad C \in \mathbb{R}_+^{p\times n}, \quad D \in \mathbb{R}_+^{p\times m}, \tag{12.13c}$$

where M_n is the set of $n\times n$ Metzler matrices (with nonnegative off-diagonal entries).

Proof. Necessity. From (12.1a) for $i = 0$ and $B_k = 0$, $k = 0,1,2$ we have

$$\dot{x}(t,1) = A_2 x(t,1) + F(t,0), \tag{12.14}$$

where

$$F(t,0) = A_0 x(t,0) + A_1 \dot{x}(t,0). \tag{12.15}$$

Assuming $x_{t0} = 0$, $x_{t1} = 0$ we obtain $F(t,0) = 0$ and from (12.14)

$$x(t,1) = e^{A_2 t} x(0,1). \tag{12.16}$$

Necessity of $A_0 \in \mathbb{R}_+^{n\times n}$ and $A_1 \in \mathbb{R}_+^{n\times n}$ follows immediately from (12.15) since $F(t,0) \in \mathbb{R}_+^n$, $t \in \mathbb{R}_+$ and x_{t0}, x_{t1} are arbitrary. From (12.16) it follows that $A_2 \in M_n$ since $e^{A_2 t} \in \mathbb{R}_+^{n\times n}$ only if A_2 is a Metzler matrix, $x(t,1) \in \mathbb{R}_+^n$, $t \in \mathbb{R}_+$ and $x(0,1)$ is arbitrary.

From (12.1a) for $i = 1$ and $B_k = 0$, $k = 0,1,2$ we have

$$\dot{x}(t,2) = A_2 x(t,2) + F(t,1), \tag{12.17}$$

where

$$F(t,1) = A_0 x(t,1) + A_1 \dot{x}(t,1). \tag{12.18}$$

Substitution of (12.14) into (12.18) yields

$$F(t,1) = (A_0 + A_1 A_2) x(t,1) + A_1 F(t,0). \tag{12.19}$$

From (12.19) it follows that $F(t,1) \in \mathbb{R}_+^n$, $t \in \mathbb{R}_+$ for any boundary conditions (12.12) only if $A = A_0 + A_1 A_2 \in \mathbb{R}_+^{n\times n}$. Proof of the necessity of (12.13c) is similar to the one for standard general 2D model [77].

The proof of sufficiency will be accomplished by induction with respect to i. For $i = 0$ the equation (12.1a) for $B_k \neq 0$, $k = 0,1,2$ takes the form (12.14) and

$$F(t,0) = A_0 x(t,0) + A_1 \dot{x}(t,0) + B_0 u(t,0) + B_1 \dot{u}(t,0) + B_2 u(t,1). \tag{12.20}$$

If the conditions (12.12) and (12.13) are satisfied and $u(t,i) \in \mathbb{R}_+^m$, $\dot{u}(t,i) \in \mathbb{R}_+^m$, $t \in \mathbb{R}_+$, $i = 0$ then $F(t,0) \in \mathbb{R}_+^n$, $t \in \mathbb{R}_+$. The solution of the equation (12.14) has the form

$$x(t,1) = e^{A_2 t} x(0,1) + \int_0^t e^{A_2(t-\tau)} F(\tau,0) d\tau \in \mathbb{R}_+^n, \quad t \in \mathbb{R}_+, \tag{12.21}$$

since $e^{A_2 t} \in \mathbb{R}_+^{n\times n}$, $t \in \mathbb{R}_+$ and $x(0,1) \in \mathbb{R}_+^n$.

From (12.1b) for $i = 1$ we have

$$y(t,1) = Cx(t,1) + Du(t,1) \in \mathbb{R}_+^p, \quad t \in \mathbb{R}_+. \tag{12.22}$$

Substituting (12.21) into

$$F(t,1) = A_0 x(t,1) + A_1 \dot{x}(t,1) + B_0 u(t,1) + B_1 \dot{u}(t,1) + B_2 u(t,2). \tag{12.23}$$

we obtain

$$\begin{aligned} F(t,1) &= A e^{A_2 t} x(0,1) + A \int_0^t e^{A_2(t-\tau)} F(\tau,0) d\tau + A_1 F(t,0) \\ &+ B_0 u(t,1) + B_1 \dot{u}(t,1) + B_2 u(t,2) \in \mathbb{R}_+^n, \end{aligned} \tag{12.24}$$

since $A \in \mathbb{R}_+^{n\times n}$, $A_1 \in \mathbb{R}_+^{n\times n}$, $e^{A_2 t} \in \mathbb{R}_+^{n\times n}$, $F(t,0) \in \mathbb{R}_+^n$, $u(t,i) \in \mathbb{R}_+^m$, $i = 1,2$ and $\dot{u}(t,1) \in \mathbb{R}_+^m$, $t \in \mathbb{R}_+$. Assuming that the $x(t,i) \in \mathbb{R}_+^n$, $F(t,i-1) \in \mathbb{R}_+^n$ for $t \in \mathbb{R}_+$, $i \geq 1$ we shall show that $x(t,i+1) \in \mathbb{R}_+^n$ for $t \in \mathbb{R}_+$ if the assumptions (12.13) are satisfied.

From (12.1a) we have

$$\dot{x}(t,i+1) = A_2 x(t,i+1) + F(t,i), \tag{12.25}$$

where

$$\begin{aligned} F(t,i) &= A_0 x(t,i) + A_1 \dot{x}(t,i) + B_0 u(t,i) + B_1 \dot{u}(t,i) + B_2 u(t,i+1) \\ &= A e^{A_2 t} x(0,i) + A \int_0^t e^{A_2(t-\tau)} F(\tau,i-1) d\tau + A_1 F(t,i-1) \\ &+ B_0 u(t,i) + B_1 \dot{u}(t,i) + B_2 u(t,i+1) \in \mathbb{R}_+^n, \end{aligned} \tag{12.26}$$

if the assumptions are satisfied.

The solution of (12.25) has the form

$$x(t,i+1) = e^{A_2 t} x(0,i+1) + \int_0^t e^{A_2(t-\tau)} F(\tau,i) d\tau \in \mathbb{R}_+^n \tag{12.27}$$

and it satisfies the condition $x(t,i+1) \in \mathbb{R}_+^n$, since $e^{A_2 t} \in \mathbb{R}_+^{n\times n}$, $t \in \mathbb{R}_+$ and $F(\tau,i) \in \mathbb{R}_+^n$.

From (12.1b) we have $y(t,i+1) = Cx(t,i+1) + Du(t,i+1) \in \mathbb{R}_+^p$, $t \in \mathbb{R}_+$ since $x(t,i+1) \in \mathbb{R}_+^n$, $u(t,i+1) \in \mathbb{R}_+^m$ and $C \in \mathbb{R}_+^{p\times n}$, $D \in \mathbb{R}_+^{p\times m}$. This completes the proof. □

Consider the general 2D model [77, 192]

$$\begin{aligned} x_{i+1,j+1} &= A_0 x_{i,j} + A_1 x_{i+1,j} + A_2 x_{i,j+1} \\ &+ B_0 u_{i,j} + B_1 u_{i+1,j} + B_2 u_{i,j+1}, \quad i,j \in \mathbb{Z}_+, \end{aligned} \tag{12.28}$$

where $x_{i,j} \in \mathbb{R}^n$, $u_{i,j} \in \mathbb{R}^m$ are the state and input vectors, and $A_k \in \mathbb{R}^{n\times n}$, $B_k \in \Re^{n\times m}$, $k = 0,1,2$.

The model (12.28) is called positive if $x_{i,j} \in \mathbb{R}_+^n$, $i,j \in \mathbb{Z}_+$ for all boundary conditions

$$x_{i0} \in \mathbb{R}_+^n, \quad i \in \mathbb{Z}_+, \quad x_{0j} \in \mathbb{R}_+^n, \quad j \in \mathbb{Z}_+ \tag{12.29}$$

and every input $u_{i,j} \in \mathbb{R}_+^m$, $i,j \in \mathbb{Z}_+$.

Theorem 12.3. *[77] The model (12.28) is positive if and only if*

$$A_k \in \mathbb{R}_+^{n\times n}, \quad B_k \in \mathbb{R}_+^{n\times m} \quad \textit{for} \quad k = 0,1,2. \tag{12.30}$$

It is well-known (Lemma 5.2 in [77]) that the transition matrix T_{ij} (defined also by (12.3)) of the positive model (12.28) is a positive matrix, i.e. $T_{i,j} \in \mathbb{R}_+^{n\times n}$ for $i,j \in \mathbb{Z}_+$. Note that the transition matrix T_{ij} of the positive model (12.1) may be not always a positive matrix. For example for the model (12.1) with the matrices

$$A_0 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}, \quad B_k \in \mathbb{R}_+^{2\times 2}, \quad k = 0,1,2, \tag{12.31}$$

we have

$$A = A_0 + A_1 A_2 = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \in \mathbb{R}_+^{2\times 2}. \tag{12.32}$$

Therefore, by Theorem 12.2 the model with the matrices (12.31) is positive, but the matrices

$$\begin{aligned} T_{11} &= A_0 + A_1 A_2 + A_2 A_1 = \begin{bmatrix} 4 & 5 \\ -3 & 0 \end{bmatrix}, \\ T_{20} &= A_2^2 = \begin{bmatrix} 3 & -6 \\ -3 & 6 \end{bmatrix}, \end{aligned} \tag{12.33}$$

have some negative entries.

Remark 12.1. From (12.13) it follows that if $A_2 = 0$ then the general model (12.1) of the continuous-discrete systems is positive if and only if the general 2D model (12.28) is positive.

12.3 Reachability of the Standard and Positive General Model

Definition 12.2. The model (12.1) is called reachable at the point (t_f, q) if for any given final state $x_f \in \mathbb{R}^n$ there exists an input $u(t,i)$, $0 \le t \le t_f$, $0 \le i \le q$ which steers the system form zero boundary conditions to the state x_f , i.e. $x(t_f, q) = x_f$.

Theorem 12.4. *The model (12.1) is reachable at the point (t_f, q) for $t_f > 0$ and $q = 1$ if and only if one of the following conditions is satisfied.*

a) $\operatorname{rank}\left[B_0, A_2 B_0, \ldots, A_2^{n-1} B_0\right] = n \quad \Leftrightarrow \quad \operatorname{rank}\left[I_n s - A_2, B_0\right] = n, \quad \forall s \in \mathbb{C}$

b) the rows of the matrix $e^{A_2 t} B_0$ are linearly independent over the field of complex numbers $\mathbb{C}$.

Proof. Let $B_1 = B_2 = 0$. For $i = 0$ and zero boundary conditions from (12.1a) we have

$$\dot{x}(t,1) = A_2 x(t,1) + B_0 u(t,0),$$

and

$$x_f = x(t_f,1) = \int_0^{t_f} e^{A_2(t_f-\tau)} B_0 u(\tau,0)d\tau, \tag{12.34}$$

since $x(0,1) = 0$.

From Sylvester formula we have

$$e^{A_2 t_f} = \sum_{k=0}^{n-1} A_2^k c_k(t_f). \tag{12.35}$$

Substitution of (12.35) into (12.34) yields

$$\begin{aligned} x_f &= \sum_{k=0}^{n-1} A_2^k B_0 \int_0^{t_f} c_k(t_f-\tau)u(\tau,0)d\tau \\ &= \left[B_0, A_2B_0, \ldots, A_2^{n-1}B_0\right] \begin{bmatrix} v_0(t_f) \\ v_1(t_f) \\ \vdots \\ v_{n-1}(t_f) \end{bmatrix}, \end{aligned} \tag{12.36}$$

where

$$v_k(t_f) = \int_0^{t_f} c_k(t_f-\tau)u(\tau,0)d\tau. \tag{12.37}$$

The equation (12.36) has a solution $v_k(t_f)$ for $k = 0,1,\ldots,n-1$ and any given x_f if and only if the condition a) is satisfied.

The equivalence of the condition a) and b) are known (see [75], page 131). □

Theorem 12.5. *The model (12.1) is reachable at the point (t_f, q) for $t_f > 0$ and $q = 1$ if and only if the matrix*

$$R_f = \int_0^{t_f} e^{A_2\tau} B_0 B_0^T e^{A_2^T \tau} d\tau, \quad t_f > 0, \tag{12.38}$$

is positive definite (nonsingular), (see [77], page 130).

Moreover, the input which steers the system from zero boundary conditions to x_f is given by

$$u(t,0) = B_0^T e^{A_2^T(t_f-\tau)} R_f^{-1} x_f. \tag{12.39}$$

Proof. If the matrix R_f is invertible (nonsingular) then (12.39) is well defined. We shall show that the input (12.39) steers the system from zero boundary conditions to x_f. Substituting (12.39) into (12.34) we obtain

$$x_f = x(t_f,1) = \int_0^{t_f} e^{A_2(t_f-\tau)} B_0 B_0^T e^{A_2^T(t_f-\tau)} d\tau R_f^{-1} x_f = x_f, \tag{12.40}$$

since

$$\int_0^{t_f} e^{A_2(t_f-\tau)} B_0 B_0^T e^{A_2^T(t_f-\tau)} d\tau = R_f.$$

□

Remark 12.2. Reachability is independent of the matrices A_0, A_1, B_1, B_2.

Remark 12.3. To simplify the calculation we may assume that $u(t,0)$ is piecewise constant (is the step function).

Example 12.1. Consider the general model (12.1) with the matrices

$$A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{12.41}$$

and arbitrary remaining matrices of the system.

Applying the condition *a*) of Theorem 12.4 we obtain

$$\operatorname{rank}[B_0, A_2 B_0] = \operatorname{rank} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = 2, \tag{12.42}$$

and

$$\operatorname{rank}[I_n s - A_2, B_0] = \operatorname{rank} \begin{bmatrix} s-1 & 0 & 1 \\ -1 & s-2 & 0 \end{bmatrix} = 2, \quad \forall s \in \mathbb{C}. \tag{12.43}$$

Therefore, the system (12.1) with matrices (12.41) is reachable for $q = 1$ and $t_f > 0$.

Assuming $t_f = 2$ and

$$x_f = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \tag{12.44}$$

from (12.39) and (12.38) we may find the input that steers the system form zero boundary conditions to the desired state (12.44)

$$u(t,0) = B_0^T e^{A_2^T(t_f-\tau)} R_f^{-1} x_f = 0.5519 e^{2-t} - 0.0953 e^{4-2t}. \tag{12.45}$$

The plots of the state variables for $q = 1$, $t \in [0,2]$ and input for $q = 0$ and $t \in [0,2]$ are shown on Fig. 12.1 and Fig. 12.2, respectively.

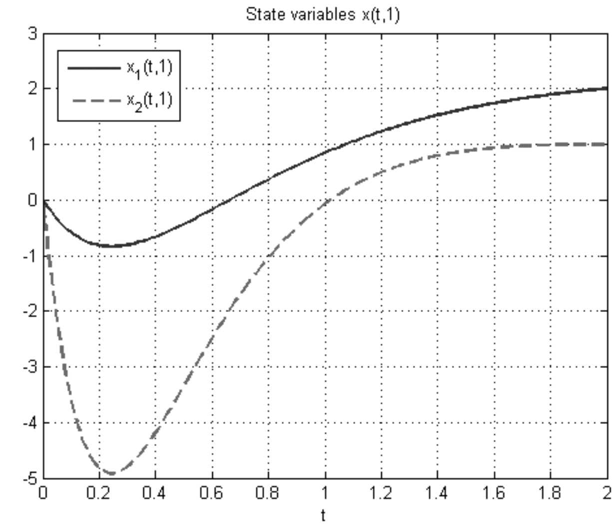


Fig. 12.1 State variables of the system. Illustration to Example 12.1.

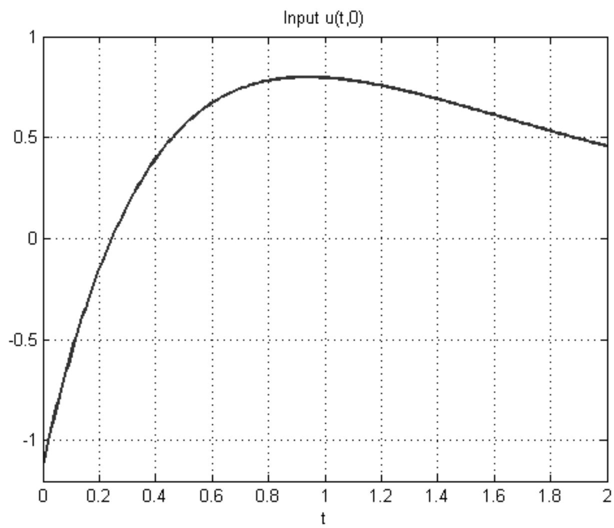


Fig. 12.2 Input of the system. Illustration to Example 12.1.

Let us assume, that the input of the system is piecewise constant, i.e.

$$u(t,0) = \begin{cases} u_1 \text{ for } 0 \le t < t_1 \\ u_2 \text{ for } t_1 \le t \le t_f \end{cases} \tag{12.46}$$

where u_1 and u_2 are constant values.

Taking into account (12.41) and (12.44) for (12.36) we obtain

$$\begin{bmatrix} v_0(t_f) \\ v_1(t_f) \end{bmatrix} = [B_0, A_2B_0]^{-1} x_f = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{12.47}$$

From (12.37) for (12.46) we have

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \int_0^{t_1} c_0(t_f-\tau)d\tau & \int_{t_1}^{t_f} c_0(t_f-\tau)d\tau \\ \int_0^{t_1} c_1(t_f-\tau)d\tau & \int_{t_1}^{t_f} c_1(t_f-\tau)d\tau \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{12.48}$$

Using (12.35) it is easy to show that

$$c_0(t) = 2e^t - e^{2t}, \quad c_1(t) = e^{2t} - e^t. \tag{12.49}$$

Using formula (12.48) we may compute values of the system input for arbitrary t_1 and t_2 $(0 < t_1 < t_f)$.

For $t_1 = 1$ and $t_f = 2$, we obtain

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -0.0481 \\ 1.2948 \end{bmatrix}. \tag{12.50}$$

The plots of state variables and input for $q = 1$ and $t \in [0,2]$ are shown on Figure 12.3 and Figure 12.4, respectively.

Definition 12.3. The positive system (12.1) is called reachable at the point (t_f, q) if for any given final state $x_f \in \mathbb{R}^n_+$ there exists a nonnegative input $u(t,i) \in \mathbb{R}^m_+$, $0 \le t \le t_f$, $0 \le i \le q$ which steers the system form zero boundary conditions to the state x_f , i.e. $x(t_f, q) = x_f$.

Theorem 12.6. *The positive model (12.1) is reachable at the point (t_f, q) for $t_f > 0$ and $q = 1$ if the matrix*

$$R_f = \int_0^{t_f} e^{A_2\tau} B_0 B_0^T e^{A_2^T \tau} d\tau, \quad t_f > 0, \tag{12.51}$$

is a monomial matrix.

The input that steers the system in time t_f from zero boundary conditions to the state x_f is given by the formula (12.39).

Proof. If R_f is a monomial matrix, then there exists the inverse matrix $R_f^{-1} \in \mathbb{R}^{n\times n}_+$ and the input (12.39) is well defined and nonnegative for $0 \le t \le t_f$. Similarly as in proof of Theorem 12.4, it can be shown that the input (12.39) steers the system from zero boundary conditions to nonnegative final state $x_f \in \mathbb{R}^n_+$. □

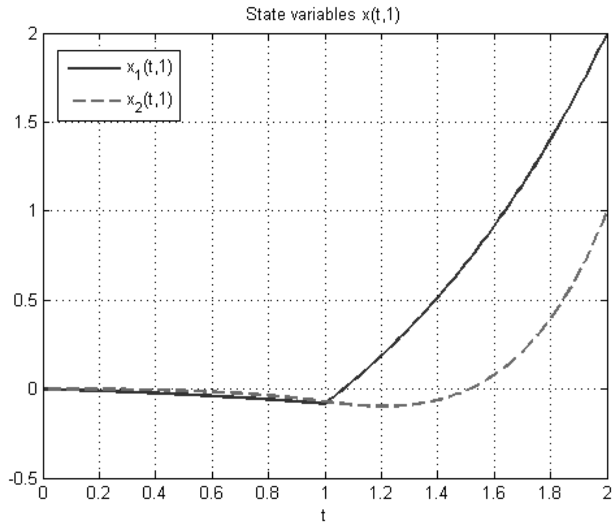


Fig. 12.3 State variables of the system. Illustration to Example 12.1.

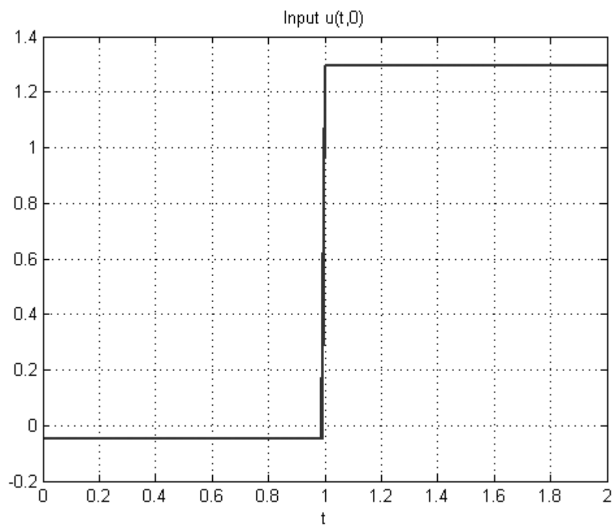


Fig. 12.4 Input of the system. Illustration to Example 12.1.

The considerations for the controllability to zero of the general model (12.1) are similar.

12.4 Stability of the Positive General Model

Consider the continuous-discrete linear 2D system [77, 76]

$$\dot{x}(t,i+1) = A_0x(t,i) + A_1\dot{x}(t,i) + A_2x(t,i+1) + Bu(t,i), \quad t \in \mathbb{R}_+, \quad i \in \mathbb{Z}_+, \tag{12.52}$$

where $\dot{x}(t,i) = \frac{\partial x(t,i)}{\partial t}$, $x(t,i) \in \mathbb{R}^n$, $u(t,i) \in \mathbb{R}^m$ $A_0, A_1, A_2 \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$.

Definition 12.4. The positive model (12.52) is called asymptotically stable if for $u(t,i) = 0$

$$\lim_{t,i\to\infty} x(t,i) = 0, \tag{12.53}$$

for bounded boundary conditions.

The matrix $A \in \mathbb{R}^{n\times n}$ is called asymptotically stable (Hurwitz) if all its eigenvalues lie in the open left half of the complex plane.

Definition 12.5. The point x_e is called equilibrium point of the asymptotically stable system (12.52) if for $Bu = 1_n = [\,1\ \dots\ 1\,]^T \in \mathbb{R}^n_+$

$$0 = A_0x_e + A_2x_e + 1_n. \tag{12.54}$$

Asymptotic stability implies $\det[A_0 + A_2] \neq 0$ and from (12.54) we have

$$x_e = -[A_0 + A_2]^{-1}1_n. \tag{12.55}$$

Remark 12.4. From (12.52) for $B = 0$ it follows that the positive system is asymptotically stable only if the matrix $A_1 - I_n$ is Hurwitz Metzler matrix [77, 52].

In what follows it is assumed that the matrix $A_1 - I_n$ is a Hurwitz Metzler matrix.

Theorem 12.7. *[109] The linear continuous-discrete positive 2D system (12.52) is asymptotically stable if and only if all coefficients of the polynomial*

$$\det[I_ns(z+1) - A_0 - A_1s - A_2(z+1)] = s^nz^n + \overline{a}_{n,n-1}s^nz^{n-1} + \overline{a}_{n-1,n}s^{n-1}z^n + \cdots + \overline{a}_{10}s + \overline{a}_{01}z + \overline{a}_{00}, \tag{12.56}$$

are positive, i.e.

$$\overline{a}_{k,l} > 0 \quad \textit{for} \quad k,l = 0,1,\dots,n; \quad (\overline{a}_{n,n} = 1). \tag{12.57}$$

Theorem 12.8. *Let the matrix $A_1 - I_n$ be a Hurwitz Metzler matrix. The positive continuous-discrete linear 2D system (12.52) is asymptotically stable if and only if there exists a strictly positive vector $\lambda \in \mathbb{R}^n_+$ (all components of the vectors are positive) such that*

$$(A_0 + A_2)\lambda < 0. \tag{12.58}$$

Proof. Integrating the equation (12.52) with $B = 0$ in the interval $(0, +\infty)$ for $i \to +\infty$ we obtain

$$x(+\infty, +\infty) - x(0, +\infty) = A_0 \int_0^{+\infty} x(\tau, +\infty) d\tau + A_1 x(+\infty, +\infty) - A_1 x(0, +\infty) + A_2 \int_0^{+\infty} x(\tau, +\infty) d\tau. \quad (12.59)$$

If the system is asymptotically stable then by (12.53) from (12.59) we obtain

$$(A_1 - I_n) x(0, +\infty) = (A_0 + A_2) \int_0^{+\infty} x(\tau, +\infty) d\tau. \quad (12.60)$$

If the matrix $A_1 - I_n$ is Hurwitz Metzler matrix then for every $x(0, +\infty) > 0$ such that $(A_1 - I_n)x(0, +\infty)$ is a strictly negative vector, $\lambda = \int_0^{+\infty} x(\tau, +\infty) d\tau$ is a strictly positive vector and (12.58) holds.

Now we shall show that if there exists a strictly positive vector λ such that (12.58) holds then the positive system (12.52) is asymptotically stable. It is well-known that the positive system (12.52) with $B = 0$ is asymptotically stable if and only if the corresponding transpose positive system

$$\dot{x}(t, i+1) = A_0^T x(t, i) + A_1^T \dot{x}(t, i) + A_2^T x(t, i+1), \quad t \in \mathbb{R}_+, \quad i \in \mathbb{Z}_+, \quad (12.61)$$

is asymptotically stable. As a candidate for a Lapunov function for the positive system (12.61) we choose

$$V(t, x(i)) = x^T(t, i)\lambda, \quad \lambda > 0, \quad (12.62)$$

which is positive for every nonzero $x(t, i) \in \mathbb{R}_+^n$. Using (12.62) and (12.61) we obtain

$$\begin{aligned} \Delta \dot{V}(t, x(i)) &= \dot{V}(t, x(i+1)) - \dot{V}(t, x(i)) = \dot{x}^T(t, i+1)\lambda - \dot{x}^T(t, i)\lambda \\ &= \dot{x}^T(t, i)[A_1 - I_n]\lambda + x^T(t, i)A_0\lambda + x^T(t, i+1)A_2\lambda \quad (12.63) \\ &\le \begin{cases} x^T(t, i)(A_0 + A_2)\lambda & \text{for} \quad x(t, i) \ge x(t, i+1) \\ x^T(t, i+1)(A_0 + A_2)\lambda & \text{for} \quad x(t, i) < x(t, i+1) \end{cases} \end{aligned}$$

since by assumption $[A_1 - I_n]\lambda < 0$. If (12.58) holds then from (12.63) we have $\Delta \dot{V}(t, x(i)) < 0$ and the positive system is asymptotically stable. □

Remark 12.5. As the strictly positive vector λ we may choose the equilibrium point (12.55) since for $\lambda = x_e$ we have

$$(A_0 + A_2)\lambda = -(A_0 + A_2)(A_0 + A_2)^{-1} 1_n = -1_n. \quad (12.64)$$

Theorem 12.9. *The positive system* (12.52) *is asymptotically stable if and only if both matrices*

$$A_1 - I_n, \quad A_0 + A_2, \quad (12.65)$$

are Hurwitz Metzler matrices.

Proof. From Remark 12.4 it follows that the positive system (12.52) is asymptotically stable only if the matrix $A_1 - I_n$ is Hurwitz Metzler matrix. By Theorem 12.8 the positive system is asymptotically stable if and only if there exists a strictly positive vector λ such that (12.58) holds but this is equivalent that the matrix $A_0 + A_2$ is Hurwitz Metzler matrix. □

To test of the matrices (12.65) are Hurwitz Metzler matrices the following theorem is recommended [211, 77].

Theorem 12.10. *The matrix $A \in \mathbb{R}^{n\times n}$ is a Hurwitz Metzler matrix if and only if one of the following equivalent conditions is satisfied:*

a) all coefficients $a_0, \dots, a_{n-1}$ of the characteristic polynomial

$$\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0, \tag{12.66}$$

are positive, i.e. $a_i \geq 0$, $i = 0, 1, \dots, n-1$,

b) the diagonal entries of the matrices

$$A_{n-k}^{(k)} \quad \textit{for} \quad k = 1, \dots, n-1, \tag{12.67}$$

are negative, where

$$A_n^{(0)} = A = \begin{bmatrix} a_{11}^{(0)} & \dots & a_{1,n}^{(0)} \\ \vdots & \ddots & \vdots \\ a_{n,1}^{(0)} & \dots & a_{n,n}^{(0)} \end{bmatrix} = \begin{bmatrix} A_{n-1}^{(0)} & b_{n-1}^{(0)} \\ c_{n-1}^{(0)} & a_{n,n}^{(0)} \end{bmatrix}, \tag{12.68}$$

$$A_{n-1}^{(0)} = \begin{bmatrix} a_{11}^{(0)} & \dots & a_{1,n-1}^{(0)} \\ \vdots & \ddots & \vdots \\ a_{n-1,1}^{(0)} & \dots & a_{n-1,n-1}^{(0)} \end{bmatrix},$$

$$b_{n-1}^{(0)} = \begin{bmatrix} a_{1,n}^{(0)} \\ \vdots \\ a_{n-1,n}^{(0)} \end{bmatrix}, \quad c_{n-1}^{(0)} = [\,a_{n,1}^{(0)} \ \dots \ a_{n,n-1}^{(0)}\,],$$

$$A_{n-k}^{(k)} = A_{n-k}^{(n-1)} - \frac{b_{n-k}^{(k-1)} c_{n-k}^{(k-1)}}{a_{n-k+1,n-k+1}^{(k-1)}} = \begin{bmatrix} a_{11}^{(k)} & \dots & a_{1,n-k}^{(k)} \\ \vdots & \ddots & \vdots \\ a_{n-k,1}^{(k)} & \dots & a_{n-k,n-k}^{(k)} \end{bmatrix}$$

$$= \begin{bmatrix} A_{n-k-1}^{(k)} & b_{n-k-1}^{(k)} \\ c_{n-k-1}^{(k)} & a_{n-k,n-k}^{(k)} \end{bmatrix},$$

$$b_{n-k-1}^{(k)} = \begin{bmatrix} a_{1,n-k}^{(k)} \\ \vdots \\ a_{n-k-1,n-k}^{(k)} \end{bmatrix}, \quad c_{n-k-1}^{(k)} = [\,a_{n-k,1}^{(k)} \ \dots \ a_{n-k,n-k-1}^{(k)}\,],$$

for $k = 0, 1, \dots, n-1$.

To check the stability of the positive system (12.52) the following procedure can be used.

Procedure 12.1

Step 1. Check if at least one diagonal entry of the matrix $A_1 \in \mathbb{R}_+^{n\times n}$ is equal or greater than 1. If this holds then positive system (12.52) is unstable [77].

Step 2. Using Theorem 12.10 check if the matrix $A_1 - I_n$ is Hurwitz Metzler matrix. If not the positive system (12.52) is unstable.

Step 3. Using Theorem 12.10 check if the matrix $A_0 + A_2$ is Hurwitz Metzler matrix. If yes the positive system (12.52) is asymptotically stable.

Example 12.2. Consider the positive system (12.52) with the matrices

$$A_0 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.4 & 0.2 \\ 0.1 & 0.3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.5 & 0.1 \\ 0.2 & -0.6 \end{bmatrix}. \tag{12.69}$$

By Theorem 12.2 the system is positive since $A_2 \in M_n$, $A_0, A_1 \in \mathbb{R}_+^{n\times n}$ and

$$A_0 + A_1 A_2 = \begin{bmatrix} 0.04 & 0.02 \\ 0.11 & 0.13 \end{bmatrix} \in \mathbb{R}_+^{2\times 2}.$$

Using Procedure 12.1 we obtain the following

Step 1. All diagonal entries of the matrix A_1 are less than 1.

Step 2. The matrix $A_1 - I_n$ is Hurwitz since the coefficient of the polynomial

$$\det[I_2 s - A_1 + I_n] = \begin{vmatrix} s+0.6 & -0.2 \\ -0.1 & s+0.7 \end{vmatrix} = s^2 + 1.3s + 0.4,$$

are positive.

Step 3. The matrix

$$A = A_0 + A_2 = \begin{bmatrix} -0.3 & 0.2 \\ 0.3 & -0.3 \end{bmatrix},$$

is also Hurwitz since (using condition *b*) of Theorem 12.10)

$$A_1^{(1)} = -0.3 + \frac{0.2 * 0.3}{0.3} = -0.1 < 0.$$

By Theorem 12.9 the positive system (12.52) with (12.69) is asymptotically stable. The polynomial (12.56) for positive system has the form

$$\begin{aligned} &\det[I_2 s(z+1) - A_0 - A_1 s - A_2(z+1)] \\ &= \begin{vmatrix} s(z+1) - 0.2 - 0.4s + 0.5(z+1) & -0.1 - 0.2s - 0.1(z+1) \\ -0.1 - 0.1s - 0.2(z+1) & s(z+1) - 0.3 - 0.3s + 0.6(z+1) \end{vmatrix} \\ &= s^2 z^2 + 1.3 s^2 z + 1.1 s z^2 + 1.26 sz + 0.28 z^2 + 0.26 z + 0.4 s^2 + 0.31 s + 0.03. \end{aligned}$$

All coefficient of the polynomial are positive. Therefore, by Theorem 12.9 the positive system is also asymptotically stable.

It is well-known [77] that substituting $A_0 = 0$, $B = 0$ in (12.52) we obtain the autonomous second Fornasini-Marchesini continuous-discrete linear 2D model (system)

$$\dot{x}(t,i+1) = A_1\dot{x}(t,i) + A_2 x(t,i+1), \quad t \in \mathbb{R}_+, \quad i \in \mathbb{Z}_+. \tag{12.70}$$

The autonomous Roesser type continuous-discrete model has the form [77]

$$\begin{bmatrix} \dot{x}^h(t,i) \\ x^v(t,i+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h(t,i) \\ x^v(t,i) \end{bmatrix}, \quad t \in \mathbb{R}_+, \quad i \in \mathbb{Z}_+, \tag{12.71}$$

where $\dot{x}(t,i) = \frac{\partial x(t,i)}{\partial t}$, $x^h(t,i) \in \mathbb{R}^{n_1}$ and $x^v(t,i) \in \mathbb{R}^{n_2}$ are the horizontal and vertical vectors and $A_{kl} \in \mathbb{R}^{n_k \times n_l}$, $k,l = 1,2$. The model (12.71) is positive if and only if [77] A_{11} is a Metzler matrix and $A_{12} \in \mathbb{R}_+^{n_1 \times n_2}$, $A_{21} \in \mathbb{R}_+^{n_2 \times n_1}$, $A_{22} \in \mathbb{R}_+^{n_2 \times n_2}$. The positive model (12.71) is a particular case of the model (12.70) for [77]

$$A_1 = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}, \quad A_2 = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}. \tag{12.72}$$

Theorem 12.11. *The positive Roesser type continuous-discrete model (12.71) is asymptotically stable if and only if the coefficients of the polynomial*

$$\begin{aligned} &\det \begin{bmatrix} I_{n_1}s(z+1) - A_{11}(z+1) & -A_{12}(z+1) \\ -A_{21}s & I_{n_2}s(z+1) - A_{22}s \end{bmatrix} \\ &= s^{n_1}z^{n_2} + \hat{a}_{n_1,n_2-1}s^{n_1}z^{n_2-1} + \hat{a}_{n_1-1,n_2}s^{n_1-1}z^{n_2} \\ &+ \cdots + \hat{a}_{11}sz + \hat{a}_{10}s + \hat{a}_{01}z + \hat{a}_{00}, \end{aligned} \tag{12.73}$$

are positive.

Proof. To transform the model (12.71) to the model (12.70) we perform the following two operations:

1) In the equation
$$\dot{x}^h(t,i) = [A_{11}\ A_{12}] \begin{bmatrix} x^h(t,i) \\ x^v(t,i) \end{bmatrix},$$
we substitute i by $i+1$.
2) We differentiate with respect to t the equation
$$x^v(t,i+1) = [A_{21}\ A_{22}] \begin{bmatrix} x^h(t,i) \\ x^v(t,i) \end{bmatrix}.$$

Note that to operation 1) corresponds the multiplication of the z-transform by z and to the operation 2) the multiplication of the Laplace transform by s. These operations do not change the asymptotic stability of the positive system (model). To shift the unit circle of the complex plane in the left half of the complex plane we replace z by $z+1$.

Taking into account

$$\begin{bmatrix} I_{n_1}sz - A_{11}z & -A_{12}z \\ -A_{21}s & I_{n_2}s(z+1) - A_{22}s \end{bmatrix} = \begin{bmatrix} I_{n_1}z & 0 \\ 0 & I_{n_2}s \end{bmatrix} \begin{bmatrix} I_{n_1}s - A_{11} & -A_{12} \\ -A_{21} & I_{n_2}(z+1) - A_{22} \end{bmatrix}$$

and Theorem 12.10 we conclude that the positive Roesser type model (12.71) is asymptotically stable if and only if all coefficients of the polynomial (12.73) are positive. □

Example 12.3. Consider the positive scalar model (12.71) with [26]

$$A_1 = \begin{bmatrix} 0 & 0 \\ a_{21} & a_{22} \end{bmatrix}, \quad A_2 = \begin{bmatrix} a_{11} & a_{12} \\ 0 & 0 \end{bmatrix}, \quad \begin{matrix} a_{11} < 0, & a_{21} \geq 0, \\ a_{12} \geq 0, & a_{22} \geq 0. \end{matrix} \tag{12.74}$$

The polynomial (12.73) for (12.74) has the form

$$\begin{aligned} &\det \begin{bmatrix} s(z+1) - a_{11}(z+1) & -a_{12}(z+1) \\ -a_{21}s & s(z+1) - a_{22}s \end{bmatrix} \\ &= s^2z^2 + (2 - a_{22})s^2z - a_{11}sz^2 + (1 - a_{22})s^2 \\ &+(-2a_{11} + a_{11}a_{22} - a_{12}a_{21})sz + (a_{11}a_{22} - a_{12}a_{21} - a_{11})s, \end{aligned} \tag{12.75}$$

and its coefficients are positive if and only if $a_{11} < 0$, $0 \leq a_{22} < 1$ and $a_{11}a_{22} - a_{12}a_{21} > a_{11}$. This result is consistent with the one obtained in [26] by different method.

Theorem 12.12. *The positive linear continuous-discrete 2D system (12.52) is asymptotically stable if and only if all coefficients of the polynomial*

$$\begin{aligned} &\det[I_n s(z+1) - A_0 - A_1 s - A_2(z+1)] \\ &= s^n z^n + \overline{a}_{n,n-1}s^n z^{n-1} + \overline{a}_{n-1,n}s^{n-1}z^n \\ &+ \cdots + \overline{a}_{10}s + \overline{a}_{01}z + \overline{a}_{00}, \end{aligned} \tag{12.76}$$

are positive, i.e.

$$\overline{a}_{k,l} > 0 \quad \mathit{for} \quad k,l = 0,1,\ldots,n(\overline{a}_{n,n} = 1). \tag{12.77}$$

Proof. It is well-known that the zeros $w_1,\ldots,w_n$ of the characteristic polynomial

$$\det[I_n w - A] = w^n + a_{n-1}w^{n-1} + \cdots + a_1 w + a_0, \tag{12.78}$$

located in the unit circle in the left half of the complex plane w can be shifted into the unit circle of the complex plane z by the substitution $w = z+1$ (Fig. 12.5) i.e. the zeros $z_1,\ldots,z_n$ ($z_k = w_k + 1$, $k = 1,\ldots,n$) of the characteristic polynomial

$$\det[I_n(z+1) - A] = z^n + \hat{a}_{n-1}z^{n-1} + \cdots + \hat{a}_1 z + \hat{a}_0, \tag{12.79}$$

are located in the unit circle of the complex plane.

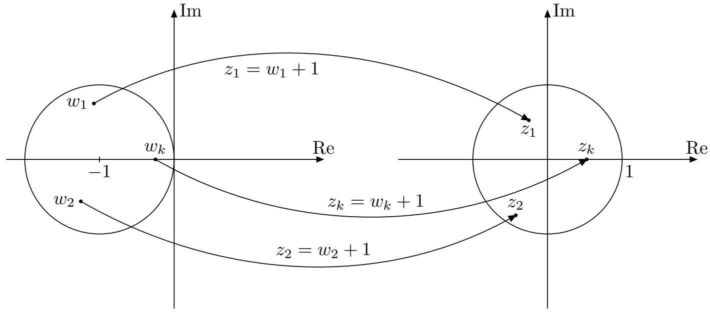


Fig. 12.5 Shifting the zeros w into unit circle of the complex plane.

Therefore, the positive continuous-discrete 2D system (12.52) is asymptotically stable if and only if the coefficients of the polynomial (12.76) are positive. □

Theorem 12.13. *[77] The positive linear system*

$$\dot{x} = Ax, A \in M_n, \tag{12.80}$$

is asymptotically stable if and only if the characteristic polynomial

$$\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0, \tag{12.81}$$

has positive coefficients, i.e. $a_k > 0$ *for* $k = 0, 1, \ldots, n-1$.

Lemma 12.1. *[52] Nonnegative matrix* $A \in \mathbb{R}_+^{n\times n}$ *is asymptotically stable (nonnegative Schur matrix) if and only if the Metzler matrix* $A - I_n$ *is asymptotically stable (Metzler Hurwitz matrix).*

Example 12.4. Consider system (12.52) with the matrices

$$A_0 = \begin{bmatrix} 0.2 & 0 \\ 0.1 & 0.1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.4 & 0 \\ 0.5 & 0.3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.3 & 0 \\ 1 & -0.2 \end{bmatrix}. \tag{12.82}$$

The matrices (12.82) satisfy the conditions (12.13) since

$$A_0 + A_1 A_2 = \begin{bmatrix} 0.08 & 0 \\ 0.25 & 0.04 \end{bmatrix} \in \mathbb{R}_+^{2\times 2}, \tag{12.83}$$

and then the system is positive.

In this case the polynomial (12.76) has the form

$$\begin{aligned} &\det[I_n s(z+1) - A_0 - A_1 s - A_2(z+1)] \qquad (12.84)\\ &= \det \begin{bmatrix} s(z+1) - 0.2 - 0.4s + 0.3(z+1) & 0 \\ -0.1 - 0.5s - (z+1) & s(z+1) - 0.1 - 0.3s + 0.2(z+1) \end{bmatrix} \\ &= s^2 z^2 + 1.3s^2 z + 0.5sz^2 + 0.42s^2 + 0.06z^2 + 0.53sz + 0.13s + 0.05z + 0.01. \end{aligned}$$

All coefficient of the polynomial (12.84) are positive. Therefore, by Theorem 12.12 the positive continuous-discrete system (12.52) with (12.82) is asymptotically stable.

Theorem 12.14. *The positive continuous-discrete 2D linear system* (12.52) *is unstable if one of the following conditions is satisfied*

a) $\det[-(A_0+A_2)] \le 0$,
b) $\det[-A_2] \le 0$,
c) $\det[I_n - A_1] \le 0$.

Proof. Substitution $s = z = 0$ into (12.76) yields

$$\det[-(A_0+A_2)] = \overline{a}_{00}. \tag{12.85}$$

If the condition *a*) is satisfied then from (12.85) we have $\overline{a}_{00} \le 0$ and by Theorem 12.13 the system (12.52) is unstable. Substituting $s = 0$ into (12.76) we obtain

$$\det[-A_2 z - (A_0+A_2)] = \overline{a}_{0,n} z^n + \cdots + \overline{a}_{01} z + \overline{a}_{00}, \tag{12.86}$$

and $\det[-A_2] = \overline{a}_{0,n}$. If the condition *b*) is met then $\overline{a}_{0n} \le 0$ and by Theorem 12.12 the system (12.52) is unstable. Similarly, substituting $z = 0$ into (12.76) we obtain

$$\det[(I_n - A_1)s - (A_0+A_2)] = \overline{a}_{n,0} s^n + \cdots + \overline{a}_{10} s + \overline{a}_{00}, \tag{12.87}$$

and $\det[(I_n - A_1)] = \overline{a}_{n,0}$. If the condition *c*) is met then $\overline{a}_{n,0} \le 0$ and by Theorem 12.13 the system (12.52) is unstable. □

Example 12.5. Consider the system (12.52) with the matrices

$$A_0 = \begin{bmatrix} 0.5 & 0.3 \\ 0.4 & 0.4 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.3 & 0.1 \\ 0.2 & -0.4 \end{bmatrix}. \tag{12.88}$$

The matrices (12.88) satisfy the conditions 12.13 since

$$A_0 + A_1 A_2 = \begin{bmatrix} 0.46 & 0.28 \\ 0.43 & 0.29 \end{bmatrix} \in \mathbb{R}_+^{2\times 2}, \tag{12.89}$$

and then the system is positive.

Using (12.88) we obtain

$$\det[-(A_0+A_2)] = \det \begin{bmatrix} -0.2 & -0.4 \\ -0.6 & 0 \end{bmatrix} = -0.24,$$
$$\det[-A_2] = \det \begin{bmatrix} 0.3 & -0.1 \\ -0.2 & 0.4 \end{bmatrix} = 0.1,$$
$$\det[I_n - A_1] \det \begin{bmatrix} 0.8 & -0.1 \\ -0.1 & 0.7 \end{bmatrix} = 0.55,$$

and the condition a) of Theorem 12.14 is satisfied. Therefore, the positive system (12.52) with (12.88) is unstable.

In this case the polynomial (12.76) has the form

$$\begin{aligned}
&\det[I_n s(z+1) - A_0 - A_1 s - A_2(z+1)] \\
&= \det\begin{bmatrix} sz+0.8s+0.3z-0.2 & -0.1s-0.1z-0.4 \\ -0.1s-0.2z-0.6 & sz+0.7s+0.4z \end{bmatrix} \\
&= s^2z^2 + 1.5s^2z + 0.7sz^2 + 0.55s^2 + 0.1z^2 + 0.3sz - 0.24s - 0.22z - 0.24,
\end{aligned} \tag{12.90}$$

and by Theorem 12.13 the system is also unstable.

12.5 Robust Stability of Linear Continuous-Discrete Linear System

Following [26] we shall consider the new general 2D model of scalar continuous-discrete linear system (for $i \in \mathbb{Z}_+$ and $t \in \mathbb{R}_+$)

$$\dot{x}_1(t,i) = a_{11}x_1(t,i) + a_{12}x_2(t,i) + b_1u(t,i), \tag{12.91a}$$

$$x_2(t,i+1) = a_{21}x_1(t,i) + a_{22}x_2(t,i) + b_2u(t,i), \tag{12.91b}$$

$$y(t,i) = c_1x_1(t,i) + c_2x_2(t,i) + du(t,i), \tag{12.91c}$$

where $\dot{x}_1(t,i) = \partial x_1(t,i)/\partial t$, $x_1(t,i) \in \mathbb{R}$, $x_2(t,i) \in \mathbb{R}$, $u(t,i) \in \mathbb{R}$, $y(t,i) \in \mathbb{R}$ and a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2, c_1, c_2 and d are constant coefficients.

The boundary conditions for (12.91a) and (12.91b) have the form

$$x_1(0,i) = x_1(i), \quad i \in \mathbb{Z}_+ \quad \text{and} \quad x_2(t,0) = x_2(t), \quad t \in \mathbb{R}_+. \tag{12.92}$$

The model (12.91) can be written in the form

$$\begin{bmatrix} \dot{x}_1(t,i) \\ x_2(t,i+1) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t,i) \\ x_2(t,i) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t,i), \tag{12.93a}$$

$$y(t,i) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1(t,i) \\ x_2(t,i) \end{bmatrix} + du(t,i). \tag{12.93b}$$

The general model (12.91) is called positive (internally) if $x_1(t,i) \geq 0$ and $x_2(t,i) \geq 0$ for all boundary conditions $x_1(i) \geq 0$, $i \in \mathbb{Z}_+$ and $x_2(t) \geq 0$, $t \in \mathbb{R}_+$, and all inputs $u(t,i) \geq 0$, $t \in \mathbb{R}_+$, $i \in \mathbb{Z}_+$.

The general model (12.91) is positive (internally) if and only if

$$a_{11} \in \mathbb{R},\ a_{12},\ a_{21},\ a_{22} \geq 0 \text{ and } b_1, b_2 \geq 0,\ c_1,\ c_2 \geq 0,\ d \geq 0. \tag{12.94}$$

Characteristic function of the model (12.91) (and (12.93)) is a polynomial in two independent variables s and z, of the form

$$w(s,z) = \det \begin{bmatrix} s-a_{11} & -a_{12} \\ -a_{21} & z-a_{22} \end{bmatrix} \\ = sz - sa_{22} - za_{11} + (a_{11}a_{22} - a_{12}a_{21}). \tag{12.95}$$

The general model (12.91) is called asymptotically stable (or Hurwitz-Schur stable) if for $u(t,i) \equiv 0$ and bounded boundary conditions (12.92) the condition $x(t,i) \to 0$ holds for $t,i \to \infty$.

From the papers [8, 67] we have the following theorem.

Theorem 12.15. *The general model (12.91) is asymptotically stable if and only if*

$$w(s,z) \neq 0, \quad Res \geq 0, \quad |z| \geq 1. \tag{12.96}$$

The polynomial (12.95) satisfying condition (12.96) is called continuous-discrete stable (C-D stable) or Hurwitz-Schur stable [8].

Now we consider the system (12.91) with uncertain coefficients $a_{11}, a_{12}, a_{21}, a_{22}$ and assume that

$$a_{ik} \in [a_{ik}^-, \quad a_{ik}^+], \quad i,k = 1,2, \tag{12.97}$$

where a_{ik}^- and a_{ik}^+ with $a_{ik}^- < a_{ik}^+$ $(i,k = 1,2)$ are given real numbers.

By generalization to the case of systems with uncertain parameters one obtains the following definition and theorems.

Definition 12.6. The general uncertain model (12.91) is called robustly stable if for $u(t,i) \equiv 0$ and bounded boundary conditions (12.92) the condition $x(t,i) \to 0$ holds for $t,i \to \infty$ and for all coefficients a_{ik}, $i,k = 1,2$, satisfying (12.97).

Theorem 12.16. *The general uncertain model (12.91), (12.97) is positive if and only if*

$$a_{11} \in [a_{11}^-, a_{11}^+] \subset \mathbb{R}, \quad a_{12}^-, a_{21}^-, a_{22}^- \geq 0, \tag{12.98a}$$

and

$$b_1, b_2 \geq 0, \quad c_1, c_2 \geq 0, \quad d \geq 0. \tag{12.98b}$$

Theorem 12.17. *The general uncertain model (12.91) is robustly stable if and only if condition (12.96) holds for all coefficients a_{ik}, $i,k = 1,2$, of the polynomial (12.95) satisfying (12.97).*

In this section following [26] a simple analytical conditions for stability and for robust stability of general model (12.91) of continuous-discrete linear systems, standard (i.e. non-positive) and positive will be presented.

Theorem 12.18. *The general model (12.91) is asymptotically stable if and only if the following two conditions hold*

$$w(s, \exp(j\omega)) \neq 0, \quad \mathrm{Re}\, s \geq 0, \quad \forall \omega \in [0, 2\pi], \tag{12.99}$$

$$w(jy, z) \neq 0, \quad |z| \geq 1, \quad \forall y \in [0, \infty). \tag{12.100}$$

Proof. From [67] it follows that (12.96) is equivalent to the conditions

$$w(s,z) \neq 0, \quad Re\, s \geq 0, \quad |z| = 1, \tag{12.101}$$

$$w(s,z) \neq 0, \quad Re\, s = 0, \quad |z| \geq 1. \tag{12.102}$$

It is easy to see that conditions (12.101) and (12.102) can be written in the forms (12.99) and (12.100), respectively. □

Solving the equation $w(s,z) = 0$ for $z = \exp(j\omega)$, where $w(s,z)$ has the form (12.95), we obtain

$$s(j\omega) = a_{11} + \frac{a_{12}a_{21}}{\exp(j\omega) - a_{22}}. \tag{12.103}$$

From (12.103) it follows that $s(j\omega)$ is a discontinuous function in the points $\omega = 0$ and $\omega = \pi$ for $a_{22} = 1$ and $a_{22} = -1$, respectively. Therefore, for excluding this discontinuity, we assume that $a_{22} \neq \pm 1$.

Substituting $\omega = 0$ and $\omega = \pi$ in (12.103) we obtain, respectively,

$$s_0 = s(j0) = a_{11} + \frac{a_{12}a_{21}}{1 - a_{22}}, \tag{12.104}$$

$$s_\pi = s(j\pi) = a_{11} - \frac{a_{12}a_{21}}{1 + a_{22}}. \tag{12.105}$$

Let $s(j\omega) = u(\omega) + jv(\omega)$, where $u(\omega) = Re\, s(j\omega)$, $v(\omega) = Im\, s(j\omega)$. It is easy to check that $[u(\omega) - s_c]^2 + v^2(\omega) = r^2$, where $s_c = 0.5(s_0 + s_\pi)$, $r = |s_0 - s_c|$. This means that the plot of $s(j\omega)$, $\omega \in [0,\ 2\pi]$, is a circle with the center s_c and radius r. Hence, the condition $Res(j\omega) < 0$ holds for all $\omega \in [0, 2\pi]$ if and only if

$$\max\left\{ a_{11} - \frac{a_{12}a_{21}}{a_{22} - 1},\ a_{11} - \frac{a_{12}a_{21}}{1 + a_{22}} \right\} < 0. \tag{12.106}$$

From the above we have the following lemma.

Lemma 12.2. *For the general model* (12.91) *the condition* (12.99) *is equivalent to* (12.106).

Now we consider the condition (12.100).

Lemma 12.3. *For the general model* (12.91) *the condition* (12.100) *is equivalent to*

$$-1 < a_{22} < 1 \quad \textit{and} \quad a_{11}^2 - (a_{11}a_{22} - a_{12}a_{21})^2 > 0. \tag{12.107}$$

Proof. From (12.95) for $s = jy$ we have that the root of the equation $w(jy,z) = 0$ has the form

$$z(jy) = \frac{jya_{22} - (a_{11}a_{22} - a_{12}a_{21})}{jy - a_{11}}. \tag{12.108}$$

The condition (12.100) holds if and only if $|z(jy)| < 1, \forall y \in \mathbb{R}$, i.e.

$$y^2(1-a_{22}^2)+a_{11}^2-(a_{11}a_{22}-a_{12}a_{21})^2 > 0, \quad \forall y \in \mathbb{R}. \tag{12.109}$$

It is easy to see that (12.109) is equivalent to (12.107). □

Theorem 12.19. *The general model (12.91) is asymptotically stable if and only if*

$$-1 < a_{22} < 1, \tag{12.110}$$

and (12.106) is satisfied, or equivalently, one of the following conditions holds:

$$a_{12}a_{21} \geq 0, a_{11} < \frac{a_{12}a_{21}}{a_{22}-1}, \tag{12.111}$$

$$a_{12}a_{21} < 0, a_{11} < \frac{a_{12}a_{21}}{1+a_{22}}. \tag{12.112}$$

Proof. It follows directly from Theorem 12.18 and Lemmas 12.2 and 12.3. □

Example 12.6. Consider the model (12.91) with $a_{12} = -1$ and $a_{21} = 1$. Check stability of the model for $a_{22} = -0.5$ and $a_{22} = 0.5$.

In this case $a_{12}a_{21} = -1 < 0$ and the necessary condition (12.110) holds. From (12.112) it follows that the model is asymptotically stable if and only if:

a) $a_{11} < -2$ for $a_{22} = -0.5$,
b) $a_{11} < -2/3$ for $a_{22} = 0.5$.

In the case of positive general model (12.91), from (12.94) and Theorem 12.19 we have the following theorem.

Theorem 12.20. *The positive general model (12.91) is asymptotically stable if and only if*

$$a_{12}, a_{21} \geq 0, \quad 0 < a_{22} < 1 \quad \textit{and} \quad a_{11} < \frac{a_{12}a_{21}}{a_{22}-1}. \tag{12.113}$$

Remark 12.6. Note that the conditions (12.113) follows immediately also from Theorem 12.17.

Example 12.7. Let us consider positive model (12.91) with $a_{12} = a_{21} = 1$. Check stability of the model for $a_{22} = 0$ and $a_{22} = 0.5$.

From Theorem 12.20 we have that the model is positive and asymptotically stable if and only if:

a) $a_{11} < -1$ for $a_{22} = 0$,
b) $a_{11} < -2$ for $a_{22} = 0.5$.

Let us consider two real interval numbers $A = [a^-, a^+]$, $a^- < a^+$ and $B = [b^-, b^+]$, $b^- < b^+$.

Recall, that real interval number $X = [x^-, x^+]$ is the set of real numbers x such that $x^- \leq x \leq x^+$.

It is well known from the interval analysis that (see [1, 7] for example)

$$A - B = \{a - b : a \in A, b \in B\} = [a^- - b^+, a^+ - b^-], \tag{12.114}$$

$$A \cdot B = \{a \cdot b : a \in A, b \in B\} = [\alpha, \beta], \tag{12.115}$$

where

$$\alpha = \min(a^- b^-, a^- b^+, a^+ b^-, a^+ b^+), \tag{12.116}$$
$$\beta = \max(a^- b^-, a^- b^+, a^+ b^-, a^+ b^+), \tag{12.117}$$

and

$$A/B = \{a/b : a \in A, b \in B\} = [a^-, a^+] \cdot [1/b^+, 1/b^-], 0 \notin B. \tag{12.118}$$

Hence, for any fixed $a_{12} \in [a_{12}^-, a_{12}^+]$ and $a_{21} \in [a_{21}^-, a_{21}^+]$ we have $a_{12}a_{21} \in [\alpha_{12}^-, \alpha_{12}^+]$, where

$$\alpha_{12}^- = \min(a_{12}^- a_{21}^-,\ a_{12}^- a_{21}^+,\ a_{12}^+ a_{21}^-,\ a_{12}^+ a_{21}^+), \tag{12.119a}$$
$$\alpha_{12}^+ = \max(a_{12}^- a_{21}^-,\ a_{12}^- a_{21}^+,\ a_{12}^+ a_{21}^-,\ a_{12}^+ a_{21}^+). \tag{12.119b}$$

From (12.110) and (12.97) it follows that the necessary condition for robust stability has the form

$$-1 < a_{22}^- < a_{22}^+ < 1. \tag{12.120}$$

Using the rules (12.115), (12.118) we obtain the following:

$$\frac{a_{12}a_{21}}{a_{22} - 1} \in [\alpha_1, \beta_1], \tag{12.121}$$

where

$$\alpha_1 = \min\left(\frac{\alpha_{12}^-}{a_{22}^+ - 1}, \frac{\alpha_{12}^-}{a_{22}^- - 1}, \frac{\alpha_{12}^+}{a_{22}^+ - 1}, \frac{\alpha_{12}^+}{a_{22}^- - 1}\right), \tag{12.122}$$
$$\beta_1 = \max\left(\frac{\alpha_{12}^-}{a_{22}^+ - 1}, \frac{\alpha_{12}^-}{a_{22}^- - 1}, \frac{\alpha_{12}^+}{a_{22}^+ - 1}, \frac{\alpha_{12}^+}{a_{22}^- - 1}\right), \tag{12.123}$$

and

$$\frac{a_{12}a_{21}}{a_{22} + 1} \in [\alpha_2, \beta_2], \tag{12.124}$$

where

$$\alpha_2 = \min\left(\frac{\alpha_{12}^-}{a_{22}^+ + 1}, \frac{\alpha_{12}^-}{a_{22}^- + 1}, \frac{\alpha_{12}^+}{a_{22}^+ + 1}, \frac{\alpha_{12}^+}{a_{22}^- + 1}\right), \tag{12.125}$$
$$\beta_2 = \max\left(\frac{\alpha_{12}^-}{a_{22}^+ + 1}, \frac{\alpha_{12}^-}{a_{22}^- + 1}, \frac{\alpha_{12}^+}{a_{22}^+ + 1}, \frac{\alpha_{12}^+}{a_{22}^- + 1}\right). \tag{12.126}$$

Theorem 12.21. *The general uncertain model* (12.91), (12.97) *is robustly stable if and only if the necessary condition* (12.120) *is satisfied and*

$$a_{11}^{+} < \min(\alpha_1, \alpha_2). \tag{12.127}$$

Proof. Using the rule (12.114) and (12.97) with $i = k = 1$, (12.121), (12.124), from (12.106) we obtain the condition $\max\{a_{11}^{+} - \alpha_1, a_{11}^{+} - \alpha_2\} < 0$, which can be written in the form (12.127). The proof follows from Theorem 12.19. □

From the above considerations the following algorithm for robust stability analysis of the standard uncertain general model (12.91), (12.97) follows.

Procedure 12.2

Step 1. Compute α_{12}^{-}, α_{12}^{+} from (12.119) and α_1, α_2 from (12.122) and (12.125) respectively.
Step 2. Check satisfaction of the conditions of Theorem 12.21.

Example 12.8. Find values of coefficient a_{11} for which the uncertain general model (12.91) with $a_{12} \in [-1, 2]$, $a_{21} \in [2, 3]$ and $a_{22} \in [-0.5, 0.5]$ is robustly stable.

According to Procedure 12.2 we have:

Step 1. From (12.119) and (12.122), (12.125) one obtains: $\alpha_{12}^{-} = -3$, $\alpha_{12}^{+} = 6$, $\alpha_1 = -12$, $\alpha_2 = -6$.
Step 2. In this case the necessary condition (12.120) holds and from (12.127) we have $a_{11}^{+} < \min(\alpha_1, \alpha_2) = -12$. This means that the model is robustly stable if and only if $a_{11} \in (-\infty, -12)$.

Now we consider the following special cases:

a) $[\alpha_{12}^{-}, \alpha_{12}^{+}] \subset [0, \infty) \Leftrightarrow \alpha_{12}^{-} \geq 0$,
b) $[\alpha_{12}^{-}, \alpha_{12}^{+}] \subset (-\infty, 0] \Leftrightarrow \alpha_{12}^{+} \leq 0$,

where α_{12}^{-} and α_{12}^{+} are computed form (12.119).

Assume that the necessary condition (12.120) holds. From (12.122) and (12.125) we obtain the following:

a) if $\alpha_{12}^{-} \geq 0$ then

$$\alpha_1 = \frac{\alpha_{12}^{+}}{a_{22}^{+} - 1} < 0, \quad \alpha_2 = \frac{\alpha_{12}^{-}}{a_{22}^{+} + 1} > 0, \tag{12.128}$$

b) if $\alpha_{12}^{+} \leq 0$ then

$$\alpha_1 = \frac{\alpha_{12}^{+}}{a_{22}^{-} - 1} > 0, \quad \alpha_2 = \frac{\alpha_{12}^{-}}{a_{22}^{-} + 1} < 0. \tag{12.129}$$

Hence, from Theorem 12.21 we have the following lemmas.

Lemma 12.4. *The standard uncertain general model* (12.91), (12.97) *with* $\alpha_{12}^- \geq 0$ *is robustly stable if and only if*

$$-1 < a_{22}^- < a_{22}^+ < 1 \quad \textit{and} \quad a_{11}^+ < \frac{\alpha_{12}^+}{a_{22}^+ - 1}. \tag{12.130}$$

Lemma 12.5. *The standard uncertain general model* (12.91), (12.97) *with* $\alpha_{12}^+ \leq 0$ *is robustly stable if and only if*

$$-1 < a_{22}^- < a_{22}^+ < 1 \quad \textit{and} \quad a_{11}^+ < \frac{\alpha_{12}^-}{a_{22}^- + 1}. \tag{12.131}$$

In the case of positive uncertain model (12.91), (12.97) the conditions (12.98) holds. In this case $\alpha_{12}^- \geq 0$. From (12.98) and Lemma 12.4 we have the following theorem.

Theorem 12.22. *The uncertain general model* (12.91), (12.97) *is positive and robustly stable if and only if*

$$0 \leq a_{22}^- < a_{22}^+ < 1 \quad \textit{and} \quad a_{11}^+ < \frac{\alpha_{12}^+}{a_{22}^+ - 1}. \tag{12.132}$$

Example 12.9. Consider the general uncertain model (12.91) with $a_{12} \in [-5, -1]$, $a_{21} \in [2, 4]$, $a_{22} \in [-0.6, 0.6]$. Find values of the coefficient a_{11} for which the model is robustly stable.

In this case from (12.119) we have $\alpha_{12}^- = -20$, $\alpha_{12}^+ = -2$. Because $\alpha_{12}^+ < 0$ and (12.120) holds, we apply condition (12.131) of Lemma 12.5. From this condition we have that the model is robustly stable if and only if $a_{11}^+ < \alpha_{12}^-/(a_{22}^- + 1) = -20/0.4 = -50$. The same result one obtains from Procedure 12.2.

Example 12.10. Find values of the coefficient a_{11} for which is robustly stable the positive uncertain general model (12.91) with $a_{12} \in [1, 4]$, $a_{21} \in [2, 6]$ and $a_{22} \in [0, 0.5]$.

From (12.119) and Theorem 12.22 we obtain $\alpha_{12}^- = 2$, $\alpha_{12}^+ = 24$ and $a_{11}^+ < -24/0.5 = -48$. This means that the positive model is robustly stable if and only if $a_{11} \in (-\infty, -48)$. The same result one obtains from Procedure 12.2.

The considerations can be extended to the matrix general model of continuous-discrete linear systems.

12.6 Positive Realization Problem for Continuous-Discrete Linear Systems

12.6.1 Problem Formulation

Consider a continuous-discrete linear system described by the 2D general model [77]

$$\dot{x}(t,i+1) = A_0x(t,i) + A_1\dot{x}(t,i) + A_2x(t,i+1) + B_0u(t,i) + B_1\dot{u}(t,i) + B_2u(t,i+1), \tag{12.133a}$$

$$y(t,i) = Cx(t,i) + Du(t,i), \tag{12.133b}$$

$$t \in \mathbb{R}_+ = [0,+\infty], \quad i \in \mathbb{Z}_+ = \{0,1,\ldots\},$$

where $\dot{x}(t,i) = \frac{\partial x(t,i)}{\partial t}$, $x(t,i) \in \mathbb{R}^n$, $u(t,i) \in \mathbb{R}^m$, $y(t,i) \in \mathbb{R}^p$ are the state, input and output vectors and

$$A_k \in \mathbb{R}^{n\times n}, \quad B_k \in \mathbb{R}^{n\times m}, \quad k = 0,1,2; \quad C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}. \tag{12.133c}$$

Boundary conditions for (12.133a) have the form

$$x(0,i), \quad i \in \mathbb{Z}_+, \quad \text{and} \quad x(t,0), \quad \dot{x}(t,0), \quad t \in \mathbb{R}_+. \tag{12.134}$$

Definition 12.7. The continuous-discrete system (12.133a) is called internally positive if $x(t,i) \in \mathbb{R}^n_+$ and $y(t,i) \in \mathbb{R}^p_+$, $t \in \mathbb{R}_+$, $i \in \mathbb{Z}_+$ for arbitrary boundary conditions

$$x(0,i) \in \mathbb{R}^n_+, \quad i \in \mathbb{Z}_+, \quad x(t,0) \in \mathbb{R}^n_+, \quad \dot{x}(t,0) \in \mathbb{R}^n_+, \quad t \in \mathbb{R}_+, \tag{12.135}$$

and any inputs

$$u(t,i) \in \mathbb{R}^m_+, \quad \dot{u}(t,i) \in \mathbb{R}^m_+, \quad t \in \mathbb{R}_+, \quad i \in \mathbb{Z}_+. \tag{12.136}$$

The transfer matrix $T(s,z)$ of the continuous-discrete system (12.133a) is given by

$$T(s,z) = C\left[I_nsz - A_0 - A_1s - A_2z\right]^{-1}(B_0 + B_1s + B_2z) + D \in \mathbb{R}^{p\times m}(s,z), \tag{12.137}$$

where $\mathbb{R}^{p\times m}(s,z)$ is the set of $p\times m$ rational matrices in s and z with real coefficients.

Theorem 12.23. *The continuous-discrete system (12.133a) is internally positive if and only if*

$$A_2 \in M_n, \tag{12.138a}$$

$$A_0, A_1 \in \mathbb{R}^{n\times n}_+, \quad A_0 + A_1A_2 \in \mathbb{R}^{n\times n}_+, \quad B_0, B_1, B_2 \in \mathbb{R}^{n\times m}_+, \quad C \in \mathbb{R}^{p\times n}_+, \quad D \in \mathbb{R}^{p\times m}_+, \tag{12.138b}$$

where M_n is the set of $n\times n$ Metzler matrices (with nonnegative off-diagonal entries).

From (12.137) we have

$$D = \lim_{s,z\to\infty} T(s,z), \tag{12.139}$$

since

$$\lim_{s,z\to\infty} \left[I_nsz - A_0 - A_1s - A_2z\right]^{-1} = 0$$

Knowing the matrix D we can find the strictly positive transfer matrix

$$T_{sp}(s,z) = T(s,z) - D. \tag{12.140}$$

Definition 12.8. The matrices (12.133c) satisfying the conditions (12.138) and (12.137) are called positive realization of the transfer matrix $T(s,z)$.

The realization problem can be stated as follows. Given a rational matrix $T(s,z) \in \mathbb{R}^{p\times m}(s,z)$, find its positive realization.

Sufficient conditions for the existence of a positive realization will be established and a procedure for computation of a positive realization for a given transfer matrix $T(s,z)$ will be given.

12.6.2 SISO Systems

First we shall solve the problem for single-input single-output (SISO) continuous-discrete systems using the state variable diagram method [77, 169].

Let a given transfer function of the SISO continuous-discrete system have the form

$$\begin{aligned} T(s,z) &= \frac{b_{q_1,q_2}s^{q_1}z^{q_2} + b_{q_1,q_2-1}s^{q_1}z^{q_2-1} + \cdots + b_{11}sz + b_{10}s + b_{01}z + b_{00}}{s^{q_1}z^{q_2} - a_{q_1,q_2-1}s^{q_1}z^{q_2-1} - \cdots - a_{11}sz - a_{10}s - a_{01}z - a_{00}} \\ &= \frac{n(s,z)}{d(s,z)} \in \mathbb{R}^{p\times m}(s,z), \end{aligned} \tag{12.141}$$

which by definition is the ratio of $Y(s,z)$ and $U(s,z)$ for zero boundary conditions, where $U(s,z) = \mathscr{Z}\{\mathscr{L}[u(t,i)]\}$, $Y(s,z) = \mathscr{Z}\{\mathscr{L}[y(t,i)]\}$ and $\mathscr{L}$ and $\mathscr{Z}$ are Laplace and z-operators.

Using (12.139) and (12.140) we can find

$$D = \lim_{s,z\to\infty} T(s,z) = b_{q_1,q_2}, \tag{12.142}$$

and the strictly proper transfer function

$$\begin{aligned} T(s,z) &= \frac{b_{q_1,q_2}s^{q_1}z^{q_2} + b_{q_1,q_2-1}s^{q_1}z^{q_2-1} + \cdots + b_{11}sz + b_{10}s + b_{01}z + b_{00}}{s^{q_1}z^{q_2} - a_{q_1,q_2-1}s^{q_1}z^{q_2-1} - \cdots - a_{11}sz - a_{10}s - a_{01}z - a_{00}} - b_{q_1,q_2} \\ &= \frac{\overline{b}_{q_1,q_2-1}s^{q_1}z^{q_2-1} + \cdots + \overline{b}_{11}sz + \overline{b}_{10}s + \overline{b}_{01}z + \overline{b}_{00}}{s^{q_1}z^{q_2} - a_{q_1,q_2-1}s^{q_1}z^{q_2-1} - \cdots - a_{11}sz - a_{10}s - a_{01}z - a_{00}}, \end{aligned} \tag{12.143}$$

where $\overline{b}_{kl} = b_{kl} + b_{q_1,q_2}a_{kl}$, $k = 0,1,\ldots,q_1$; $l = 0,1,\ldots,q_2$; $(k+l \neq q_1+q_2)$.

Multiplying the numerator and denominator of (12.143) by $s^{-q_1}z^{-q_2}$ we obtain

$$T(s,z) = \frac{Y(s,z)}{U(s,z)} = \frac{\overline{b}_{q_1,q_2-1}z^{-1} + \cdots + \overline{b}_{11}s^{1-q_1}z^{1-q_2} + \cdots + \overline{b}_{00}s^{-q_1}z^{-q_2}}{1 - a_{q_1,q_2-1}z^{-1} - \cdots - a_{11}s^{1-q_1}z^{1-q_2} - \cdots - a_0 0 s^{-q_1}z^{-q_2}}. \tag{12.144}$$

Defining

$$E(s,z) = \frac{U(s,z)}{1 - a_{q_1,q_2-1}z^{-1} - \cdots - a_{11}s^{1-q_1}z^{1-q_2} - \cdots - a_{00}s^{-q_1}z^{-q_2}}. \tag{12.145}$$

From (12.145) and (12.144) we have

$$E(s,z) = U(s,z) + \left(a_{q_1,q_2-1}z^{-1} + \cdots + a_{11}s^{1-q_1}z^{1-q_2} + \cdots + a_{00}s^{-q_1}z^{-q_2}\right)E(s,z), \tag{12.146}$$

and

$$Y(s,z) = \left[\overline{b}_{q_1,q_2-1}z^{-1} + \cdots + \overline{b}_{11}s^{1-q_1}z^{1-q_2} + \cdots + \overline{b}_{00}s^{-q_1}z^{-q_2}\right]E(s,z). \tag{12.147}$$

Using (12.146) and (12.147) we may draw the state variable diagram shown on Fig. 12.6.

The number of integration elements $1/s$ is equal to q_1 and the number of delay elements $1/z$ is equal to $2q_2$. The outputs of delay elements are chosen as the variables $x_{q_1+1}(t,i),\ldots,x_{q_1+q_2}(t,i),x_{q_1+q_2+1}(t,i),\ldots,x_{q_1+2q_2}(t,i)$. Using the state variable diagram we may write the equations

$$\begin{aligned}
\dot{x}_1(t,i) &= x_2(t,i),\\
\dot{x}_2(t,i) &= x_3(t,i),\\
&\vdots\\
\dot{x}_{q_1-1}(t,i) &= x_{q_1}(t,i),\\
\dot{x}_{q_1}(t,i) &= a_{0,q_2}x_1(t,i) + a_{1,q_2}x_2(t,i) + \cdots + a_{q_1-1,q_2}x_{q_1}(t,i)\\
&\quad + x_{q_1+1}(t,i) + u(t,i),
\end{aligned} \tag{12.148a}$$

$$\begin{aligned}
x_{q_1+1}(t,i+1) &= \overline{a}_{0,q_2-1}x_1(t,i) + \overline{a}_{1,q_2-1}x_2(t,i) + \cdots + \overline{a}_{q_1-1,q_2-1}x_{q_1}(t,i)\\
&\quad + a_{q_1,q_2-1}x_{q_1+1}(t,i) + x_{q_2+2}(t,i) + a_{q_1,q_2-1}u(t,i),\\
&\vdots\\
x_{q_1+q_2-1}(t,i+1) &= \overline{a}_{01}x_1(t,i) + \overline{a}_{11}x_2(t,i) + \cdots + \overline{a}_{q_1-1,1}x_{q_1}(t,i)\\
&\quad + a_{q_1,1}x_{q_1+1}(t,i) + x_{q_1+q_2}(t,i) + a_{q_1,1}u(t,i),\\
x_{q_1+q_2}(t,i+1) &= \overline{a}_{00}x_1(t,i) + \overline{a}_{10}x_2(t,i) + \cdots + \overline{a}_{q_1-1,0}x_{q_1}(t,i)\\
&\quad + a_{q_1,0}x_{q_1+1}(t,i) + a_{q_1,0}u(t,i),\\
x_{q_1+q_2+1}(t,i+1) &= \hat{a}_{00}x_1(t,i) + \hat{a}_{10}x_2(t,i) + \cdots + \hat{a}_{q_1-1,0}x_{q_1}(t,i)\\
&\quad + \overline{b}_{q_1,0}x_{q_1+1}(t,i) + \overline{b}_{q_1,0}u(t,i),\\
&\vdots\\
x_{q_1+2q_2}(t,i+1) &= \hat{a}_{0,q_2-1}x_1(t,i) + \hat{a}_{1,q_2-1}x_2(t,i) + \cdots + \hat{a}_{q_1-1,q_2-1}x_{q_1}(t,i)\\
&\quad + \overline{b}_{q_1,q_2-1}x_{q_1+1}(t,i) + x_{q_1+2q_2-1}(t,i) + \overline{b}_{q_1,q_2-1}u(t,i),
\end{aligned} \tag{12.148b}$$

$$y(t,i) = \overline{b}_{0,q_2}x_1(t,i) + \overline{b}_{1,q_2}x_2(t,i) + \cdots + \overline{b}_{q_1-1,q_2}x_{q_1}(t,i) + x_{q_1+2q_2}(t,i), \tag{12.148c}$$

where

$$\begin{aligned}
\overline{a}_{kl} &= a_{kl} + a_{q_1l}a_{kq_2}, \quad k = 0,1,\ldots,q_1-1;\\
\hat{a}_{kl} &= \overline{b}_{kl} + \overline{b}_{q_1l}a_{kq_2}, \quad l = 0,1,\ldots,q_2-1,
\end{aligned} \tag{12.148d}$$

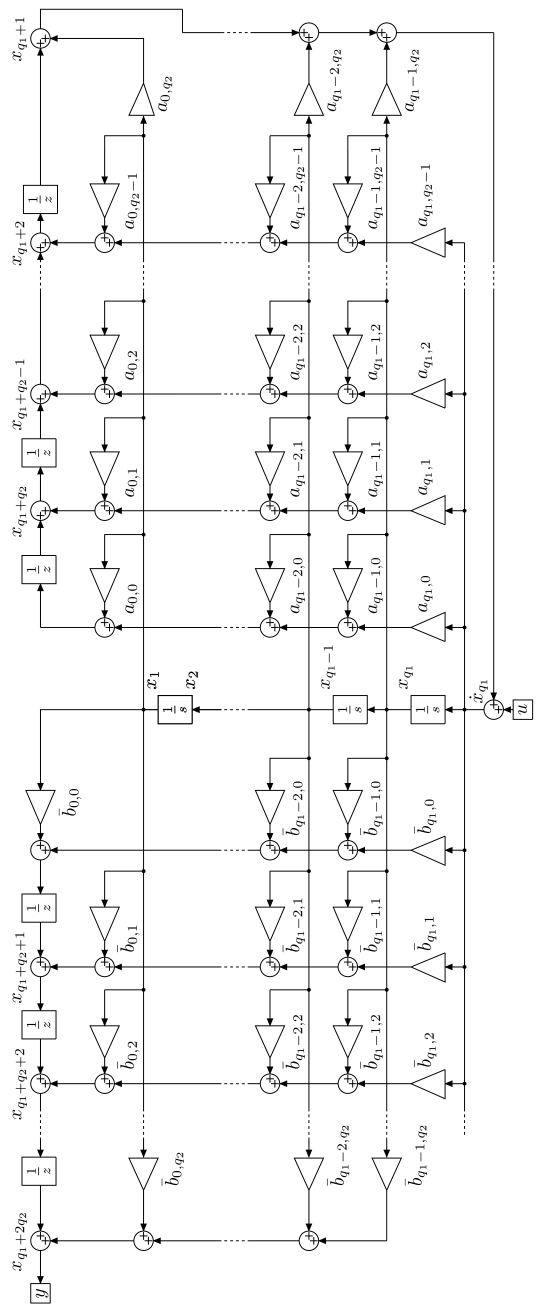


Fig. 12.6 State variable diagram for transfer function (12.144)

Substituting in the equations (12.148a) i by $i+1$ and differentiating with respect to t the equations (12.148b) we obtain the equation (12.133a) with

$$
\begin{aligned}
&A_0 = 0, \quad B_0 = 0, \quad C = \begin{bmatrix} C_1 & 0 & C_3 \end{bmatrix} \in \mathbb{R}^{1\times(q_1+2q_2)},\\
&A_1 = \begin{bmatrix} 0 & 0 & 0 \\ A_{21}^{(1)} & A_{22}^{(1)} & 0 \\ A_{31}^{(1)} & A_{32}^{(1)} & A_{33}^{(1)} \end{bmatrix} \in \mathbb{R}^{(q_1+2q_2)\times(q_1+2q_2)},\\
&A_2 = \begin{bmatrix} A_{11}^{(2)} & A_{12}^{(2)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{(q_1+2q_2)\times(q_1+2q_2)},\\
&B_1 = \begin{bmatrix} 0 \\ B_{12} \\ B_{13} \end{bmatrix} \in \mathbb{R}^{(q_1+2q_2)\times 1}, \quad B_2 = \begin{bmatrix} B_{21} \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^{(q_1+2q_2)\times 1},
\end{aligned}
$$

where

$$A_{21}^{(1)} = \begin{bmatrix} \overline{a}_{0,q_2-1} & \overline{a}_{1,q_2-1} & \dots & \overline{a}_{q_1-1,q_2-1} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{a}_{0,1} & \overline{a}_{1,1} & \dots & \overline{a}_{q_1-1,1} \\ \overline{a}_{0,0} & \overline{a}_{1,0} & \dots & \overline{a}_{q_1-1,0} \end{bmatrix} \in \mathbb{R}^{q_2 \times q_1}, \tag{12.149a}$$

$$A_{22}^{(1)} = \begin{bmatrix} a_{q_1,q_2-1} & 1 & 0 & \dots & 0 \\ a_{q_1,q_2-2} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{q_1,1} & 0 & 0 & \dots & 1 \\ a_{q_1,0} & 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{q_2 \times q_2},$$

$$A_{31}^{(1)} = \begin{bmatrix} \hat{a}_{0,0} & \hat{a}_{1,0} & \dots & \hat{a}_{q_1-1,0} \\ \hat{a}_{0,1} & \hat{a}_{1,1} & \dots & \hat{a}_{q_1-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{a}_{0,q_2-1} & \hat{a}_{1,q_2-1} & \dots & \hat{a}_{q_1-1,q_2-1} \end{bmatrix} \in \mathbb{R}^{q_2 \times q_1},$$

$$A_{32}^{(1)} = \begin{bmatrix} \overline{b}_{q_1,0} & 0 & 0 & \dots & 0 \\ \overline{b}_{q_1,1} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{b}_{q_1,q_2-2} & 0 & 0 & \dots & 0 \\ \overline{b}_{q_1,q_2-1} & 0 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{q_2 \times q_2},$$

$$A_{33}^{(1)} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \in \mathbb{R}^{q_2 \times q_2},$$

$$A_{11}^{(2)} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{0,q_2} & a_{1,q_2} & a_{2,q_2} & \dots & a_{q_1-1,q_2} \end{bmatrix} \in \mathbb{R}^{q_1 \times q_1},$$

$$A_{12}^{(2)} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{bmatrix} \in \mathbb{R}^{q_1 \times q_2},$$

$$B_{12} = \begin{bmatrix} a_{q_1,q_2-1} \\ \vdots \\ a_{q_1,1} \\ a_{q_1,0} \end{bmatrix} \in \mathbb{R}^{q_2\times 1}, \quad B_{13} = \begin{bmatrix} \overline{b}_{q_1,0} \\ \overline{b}_{q_1,1} \\ \vdots \\ a_{q_1,1} \\ \overline{b}_{q_1,q_2-1} \end{bmatrix} \in \mathbb{R}^{q_2\times 1},$$

$$B_{21} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{q_1\times 1}$$

$$C_1 = \begin{bmatrix} \overline{b}_{0,q_2} & \overline{b}_{1,q_2} & \dots & \overline{b}_{q_1-1,q_2} \end{bmatrix} \in \mathbb{R}^{1\times q_1},$$
$$C_3 = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} \in \mathbb{R}^{1\times q_2}.$$

Theorem 12.24. *There exists a positive realization (12.149) of the transfer function (12.141) if the following conditions are satisfied*

a) $a_{kl} \geq 0$ *for* $k = 0,1,\dots,q_1$*;* $l = 0,1,\dots,q_2$*;* $k+l \neq q_1+q_2$*;*
b) $b_{kl} \geq 0$ *for* $k = 0,1,\dots,q_1$*;* $l = 0,1,\dots,q_2$*;*

Proof. If the condition $b)$ is met then $D = b_{q_1,q_2} \geq 0$ and the coefficients of the strictly proper transfer function (12.143) are nonnegative. From (12.149) it follows that if the conditions $a)$ and $b)$ are satisfied then $A_k \in \mathbb{R}_+^{n\times n}$, $B_k \in \mathbb{R}_+^{n\times m}$, $C \in \mathbb{R}_+^{p\times n}$, $k = 0,1,2$; $D \geq 0$ and by Theorem 12.23 the realization (12.149) is positive. □

From (12.149) we have the following corollary.

Corollary 12.1. *If the conditions* $a)$ *and* $b)$ *of Theorem 12.24 are satisfied then there exists a positive realization of the transfer function (12.141) with* $A_0 = 0$ *and* $B_0 = 0$ *and* $A_2 \in \mathbb{R}_+^{n\times n}$.

Example 12.11. Find the positive realization of the transfer function

$$T(s,z) = \frac{s^2z^2 + s^2z + s^2 + z^2 + z + 2}{s^2z^2 - 2s^2z - s^2 - z^2 - 2z - 1}. \tag{12.150}$$

Using (12.139) and (12.140) we obtain

$$D = \lim_{s,z\to\infty} T(s,z) = 1, \tag{12.151}$$

and the strictly proper transfer function

$$\begin{aligned} T_{sp}(s,z) = T(s,z) - 1 &= \frac{3s^2z + 2s^2 + 2z^2 + 3z + 3}{s^2z^2 - 2s^2z - s^2 - z^2 - 2z - 1} \\ &= \frac{3z^{-1} + 2z^{-2} + 2s^{-2} + 3s^{-2}z^{-1} + 3s^{-2}z^{-2}}{1 - 2z^{-1} - z^{-2} - s^{-2} - 2s^{-2}z^{-1} - s^{-2}z^{-2}} \end{aligned} \tag{12.152}$$

In this case (12.146) and (12.147) have the form

$$E(s,z) = U(s,z) + \left(2z^{-1} + z^{-2} + s^{-2} + 2s^{-2}z^{-1} + s^{-2}z^{-2}\right)E(s,z), \qquad (12.153)$$

and

$$Y(s,z) = \left(3z^{-1} + 2z^{-2} + 2s^{-2} + 3s^{-2}z^{-1} + 3s^{-2}z^{-2}\right)E(s,z). \qquad (12.154)$$

Using (12.153) and (12.154) we may draw the state variable diagram shown on Fig. 12.7.

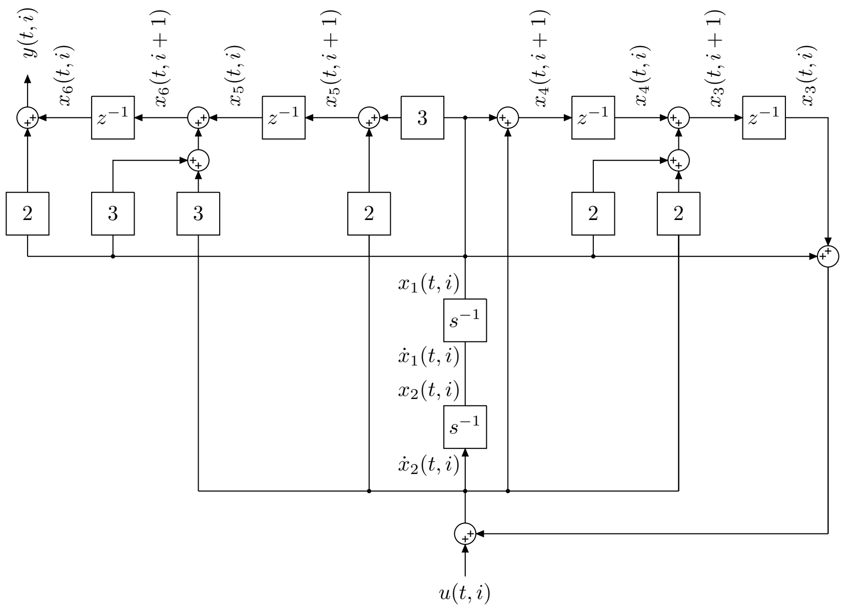


Fig. 12.7 State variable diagram for transfer function (12.152)

The outputs of the integration elements are chosen as the state variables $x_1(s,z)$, $x_2(s,z)$ and the outputs of the delay elements as the state variables $x_3(s,z)$, $x_4(s,z)$, $x_5(s,z)$, $x_6(s,z)$. From the state variable diagram we have the equations

$$\begin{aligned}
\dot{x}_1(t,i+1) &= x_2(t,i+1),\\
\dot{x}_2(t,i+1) &= x_1(t,i+1) + x_3(t,i+1) + u(t,i+1),\\
\dot{x}_3(t,i+1) &= 4\dot{x}_1(t,i) + 2\dot{x}_3(t,i) + \dot{x}_4(t,i) + 2\dot{u}(t,i),\\
\dot{x}_4(t,i+1) &= 2\dot{x}_1(t,i) + \dot{x}_3(t,i) + \dot{u}(t,i),\\
\dot{x}_5(t,i+1) &= 5\dot{x}_1(t,i) + 2\dot{x}_3(t,i) + 2\dot{u}(t,i),\\
\dot{x}_6(t,i+1) &= 6\dot{x}_1(t,i) + 3\dot{x}_3(t,i) + \dot{x}_5(t,i) + 3\dot{u}(t,i),
\end{aligned} \qquad (12.155)$$

and

$$y(t,i) = 2x_1(t,i) + x_6(t,i). \tag{12.156}$$

The equations (12.155) and (12.156) can be written in the form (12.133a), where

$$x(t,i) = \begin{bmatrix} x_1(t,i) \; x_2(t,i) \; x_3(t,i) \; x_4(t,i) \; x_5(t,i) \; x_6(t,i) \end{bmatrix}^T, \tag{12.157}$$

$$A_0 = 0, \quad A_1 = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 4&0&2&1&0&0 \\ 2&0&1&0&0&0 \\ 5&0&2&0&0&0 \\ 6&0&3&0&1&0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0&1&0&0&0&0 \\ 1&0&1&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix},$$

$$B_0 = 0, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The desired positive realization of (12.150) is given by (12.151) and (12.157).

12.6.3 MIMO Systems

First we shall consider linear continuous-discrete m-inputs and one-output systems with the transfer matrix [153]

$$T(s,z) = \begin{bmatrix} T_1(s,z) \; \dots \; T_m(s,z) \end{bmatrix} \in \mathbb{R}^{1\times m}(s,z), \tag{12.158}$$

where

$$T_k(s,z) = \frac{n_k(s,z)}{d_k(s,z)}, \quad k = 1,\dots,m. \tag{12.159}$$

It is assumed that the minimal common denominator $d(s,z)$ satisfies the assumption

$$d(s,z) = \prod_{k=1}^{m} d_k(s,z). \tag{12.160}$$

Using (12.139) and (12.140) we can find the matrix D and the strictly proper transfer matrix $T_{sp}(s,z)$. Applying the approach presented above for SISO systems to MIMO system with (12.158) we may find a realization of each transfer function (12.159). A realization of the transfer function (12.158) can be found by use of the following theorem.

Theorem 12.25. *Let*

$$A_{0k} = 0, \quad A_{1k}, \quad A_{2k}, \quad B_{0k} = 0, \quad B_{1k}, \quad B_{2k}, \quad C, \quad k = 1,\dots,m, \tag{12.161}$$

be a realization of the transfer function (12.159). Then a realization of the strictly proper transfer matrix

$$T_{\text{sp}}(s,z) = T(s,z) - D = \left[\, T_1(s,z) - D_1 \;\ldots\; T_m(s,z) - D_m \,\right], \quad (12.162a)$$
$$D_k = \lim_{s,z\to\infty} T_k(s,z), \quad (12.162b)$$

is given by

$$\begin{aligned}
&A_1 = \text{block diag}\left[\, A_{11} \;\ldots\; A_{1m} \,\right], \quad A_2 = \text{block diag}\left[\, A_{21} \;\ldots\; A_{2m} \,\right],\\
&B_1 = \text{block diag}\left[\, B_{11} \;\ldots\; B_{1m} \,\right], \quad B_2 = \text{block diag}\left[\, B_{21} \;\ldots\; B_{2m} \,\right],\\
&C = \left[\, C_1 \;\ldots\; C_m \,\right].
\end{aligned} \quad (12.163)$$

Proof. Using (12.139), (12.162a) and (12.163) we obtain

$$\begin{aligned}
T_{sp}(s,z) &= \left[\, C_1 \;\ldots\; C_m \,\right] \left\{\text{block diag}\left[\, I_n sz - A_1 s - A_2 z \,\right]\right\}^{-1}\\
&\times \left\{\text{block diag}\left[\, B_{11}s + B_{21}z \;\ldots\; B_{1m}s + B_{2m}z \,\right]\right\}\\
&= \left[\, C_1 \;\ldots\; C_m \,\right] \left\{\text{block diag}\left[\, I_n sz - A_1 s - A_2 z \,\right]^{-1}\right\}\\
&\times \left\{\text{block diag}\left[\, B_{11}s + B_{21}z \;\ldots\; B_{1m}s + B_{2m}z \,\right]\right\} =
\end{aligned}$$

$$\begin{aligned}
&= \Big[C_1 \left[\, I_n sz - A_1 s - A_2 z \,\right]^{-1} (B_{11}s + B_{21}z)\\
&\quad \ldots \quad C_m \left[\, I_n sz - A_1 s - A_2 z \,\right]^{-1} (B_{1m}s + B_{2m}z) \Big]\\
&= \left[\, T_1(s,z) - D_1 \;\ldots\; T_m(s,z) - D_m \,\right].
\end{aligned}$$

□

Theorem 12.26. *There exists a positive realization (12.163) of the transfer matrix (12.158) if all coefficients of the numerator $n_k(s,z)$, $k = 1,\ldots,m$; are nonnegative and all coefficient of the denominators $d_k(s,z)$, $k = 1,\ldots,m$; are nonnegative except the leading coefficient equal to 1.*

Theorem 12.27. *If the assumptions are satisfied then by Theorem 12.24 the realization (12.161) of the transfer function (12.158) is positive.*

Proof. From (12.163) it follows that in this case all matrices (12.163) have nonnegative entries and by Theorem 12.23 the realization of the transfer matrix is positive. □

Example 12.12. Given the transfer matrix

$$T(s,z) = \left[\, T_1(s,z) \; T_2(s,z) \,\right], \quad (12.164)$$

where $T_1(s,z)$ is given by (12.150) and

$$T_2(s,z) = \frac{2s^2z^2 + 2s^2 + 3z^2 + s + 1}{s^2z^2 - 2s^2 - z^2 - 2sz - s - 2}. \tag{12.165}$$

Using (12.139) and (12.140) from (12.164), (12.150) and (12.165) we have

$$D = \lim_{s,z\to\infty} T(s,z) = \begin{bmatrix} 1 & 2 \end{bmatrix}, \tag{12.166}$$

and

$$\begin{aligned} T_{sp} &= T(s,z) - D \\ &= \begin{bmatrix} \frac{3s^2z+2s^2+2z^2+3z+3}{s^2z^2-2s^2z-s^2-z^2-2z-1} & \frac{6s^2+5z^2+4sz+3s+5}{s^2z^2-2s^2-z^2-2sz-s-2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{3z^{-1}+2z^{-2}+2s^{-2}+3s^{-2}z^{-1}+3s^{-2}z^{-2}}{1-2z^{-1}-z^{-2}-s^{-2}-2s^{-2}z^{-1}-s^{-2}z^{-2}} & \frac{6z^{-2}+5s^{-2}+4s^{-1}z^{-1}+3s^{-1}z^{-2}+5s^{-2}z^{-2}}{1-2z^{-2}-s^{-2}-2s^{-1}z^{-1}-s^{-1}z^{-2}-2s^{-2}z^{-2}} \end{bmatrix}. \end{aligned} \tag{12.167}$$

The state variable diagram corresponding to the transfer function $T_{sp1}(s,z)$ is shown on Fig. 12.7 and the positive realization is given by (12.157) i.e.

$$A_{11} = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 4&0&2&1&0&0 \\ 2&0&1&0&0&0 \\ 5&0&2&0&0&0 \\ 6&0&3&0&1&0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0&1&0&0&0&0 \\ 1&0&1&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{bmatrix}, \tag{12.168a}$$

$$B_{11} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 2&0&0&0&0&0 \end{bmatrix}. \tag{12.168b}$$

The state variable diagram corresponding to $T_{sp2}(s,z)$ is shown on Fig. 12.8.

Using this variable diagram we can write the equations

$$\begin{aligned} \dot{x}_1(t,i+1) &= x_2(t,i+1), \\ \dot{x}_2(t,i+1) &= x_1(t,i+1) + x_3(t,i+1) + u_2(t,i+1), \\ \dot{x}_3(t,i+1) &= 2\dot{x}_2(t,i) + \dot{x}_4(t,i), \\ \dot{x}_4(t,i+1) &= 4\dot{x}_1(t,i) + \dot{x}_2(t,i) + 2\dot{x}_3(t,i) + 2\dot{u}_2(t,i), \\ \dot{x}_5(t,i+1) &= 11\dot{x}_1(t,i) + 3\dot{x}_2(t,i) + 6\dot{x}_3(t,i) + 6\dot{u}_2(t,i), \\ \dot{x}_6(t,i+1) &= 4\dot{x}_2(t,i) + \dot{x}_5(t,i), \\ y(t,i) &= 5x_1(t,i) + x_6(t,i). \end{aligned} \tag{12.169}$$

From these equations we have the realization of $T_{sp2}(s,z)$ in the form

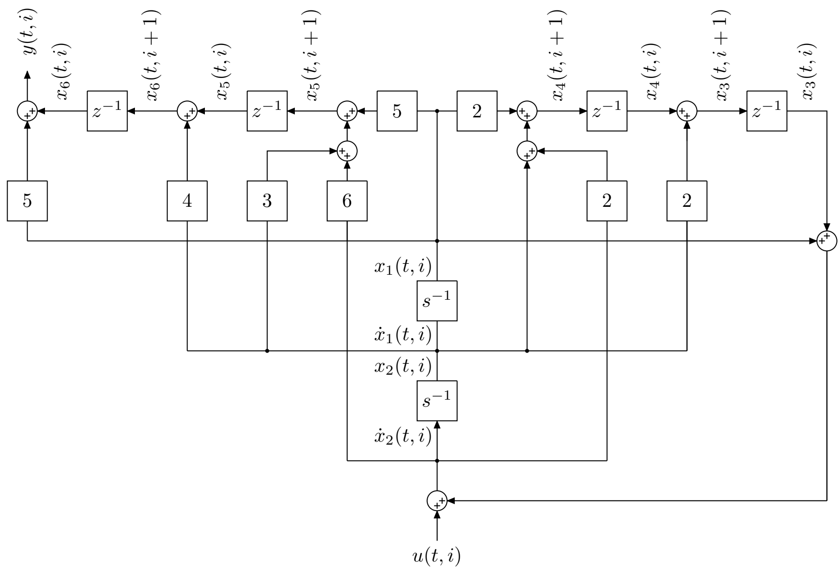


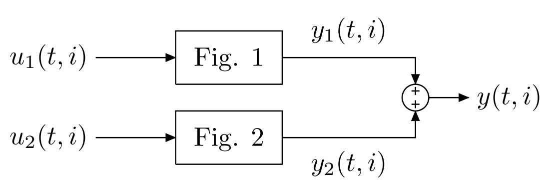
Fig. 12.8 State variable diagram for transfer function (12.167)

$$A_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 4 & 1 & 2 & 0 & 0 & 0 \\ 11 & 3 & 6 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{12.170a}$$

$$B_{11} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 6 \\ 0 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{12.170b}$$

The state variable diagram corresponding to the transfer matrix (12.167) can be obtained as the connection of the state variable diagrams shown on Fig. 12.7 and Fig. 12.8 (see Fig. 12.9).

Fig. 12.9 Connection of state variable diagrams



By Theorem 12.25 the desired realization of the transfer matrix (12.164) is given by

$$A_1 = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{12} \end{bmatrix}, \quad A_2 = \begin{bmatrix} A_{21} & 0 \\ 0 & A_{22} \end{bmatrix}, \tag{12.171a}$$

$$B_1 = \begin{bmatrix} B_{11} & 0 \\ 0 & B_{12} \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_{21} & 0 \\ 0 & B_{22} \end{bmatrix}, \tag{12.171b}$$

$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 2 \end{bmatrix}, \tag{12.171c}$$

where the submatrices A_{11}, A_{12}, B_{11}, B_{12}, C_1 are given by (12.168) and submatrices A_{21}, A_{22}, B_{21}, B_{22}, C_2 are given by (12.170). The realization is positive since all entries of the matrices (12.171) are nonnegative.

Remark 12.7. If the assumption (12.160) is not satisfied and

$$\deg_s d(s,z) < \prod_{k=1}^{m} \deg_s d_k(s,z), \quad \deg_z d(s,z) < \prod_{k=1}^{m} \deg_z d_k(s,z), \tag{12.172}$$

then to decrease the dimension of a realization of (12.158) it is recommended to find the polynomial $d(s,z)$ and write the transfer matrix (12.158) in the form

$$T(s,z) = \frac{1}{d(s,z)} \left[\overline{n}_1(s,z) \; \ldots \; \overline{n}_m(s,z) \right], \tag{12.173}$$

where $\deg_s d(s,z)$ ($\deg_z d(s,z)$) denotes the degree of the minimal common denominators with respect to s (z).

Note that the m-inputs and p-outputs system can be considered as the sequence of p m-inputs and one-output systems. In this way the presented approach can be extended for m-inputs and p-outputs linear systems.

The considerations can be extended to fractional positive continuous-discrete 2D linear systems.

Appendix A
Laplace Transforms of Continuous-Time Functions and Z-Transforms of Discrete-Time Functions

A.1 Convolutions of Continuous-Time and Discrete-Time Functions and Their Transforms

Definition A.1. The Laplace transform of a continuous-time function $f(t)$ is defined by

$$\mathscr{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt = F(s), \tag{A.1}$$

where $f(t) = 0$ for $t < 0$.

Definition A.2. The continuous-time function defined by

$$f_1(t) * f_2(t) = \int_0^t f_1(t-\tau)f_2(\tau)d\tau, \tag{A.2}$$

is called the convolution of the continuous-time functions $f_1(t)$ and $f_2(t)$.

Theorem A.1. *If*

$$F_1(s) = \mathscr{L}[f_1(t)], \quad F_2(s) = \mathscr{L}[f_2(t)], \tag{A.3}$$

then

$$\mathscr{L}\left[\int_0^t f_1(t-\tau)f_2(\tau)d\tau\right] = F_1(s)F_2(s). \tag{A.4}$$

Proof.

$$\begin{aligned}
\mathscr{L}\left[\int_0^t f_1(t-\tau)f_2(\tau)d\tau\right] &= \mathscr{L}\left[\int_0^\infty f_1(t-\tau)f_2(\tau)d\tau\right] \\
&= \int_0^\infty \left[\int_0^\infty f_1(t-\tau)f_2(\tau)\right] e^{-st}d\tau dt \\
&= \int_0^\infty f_1(u)e^{-su}du \int_0^\infty f_2(\tau)e^{-s\tau}d\tau = F_1(s)F_2(s).
\end{aligned}$$

□

Definition A.3. The discrete-time function defined by

$$f_1(i) * f_2(i) = \sum_{k=0}^{i} f_1(i-k) f_2(k), \qquad (A.5)$$

is called the convolution of the discrete-time functions $f_1(i)$ and $f_2(i)$.

Theorem A.2. *If:*

$$F_1(z) = \mathscr{Z}[f_1(i)], \quad F_2(z) = \mathscr{Z}[f_2(i)], \qquad (A.6)$$

then

$$\mathscr{Z}\left[\sum_{k=0}^{i} f_1(i-k) f_2(k)\right] = F_1(z) F_2(z). \qquad (A.7)$$

Proof. The proof is similar to the Theorem A.1. □

A.2 Laplace Transforms of Derivative-Integrals

Theorem A.3. *The Laplace transform of the function t^α has the form*

$$\mathscr{L}[t^\alpha] = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \quad \alpha \in \mathbb{R}. \qquad (A.8)$$

Proof.

$$\mathscr{L}[t^\alpha] = \int_0^\infty t^\alpha e^{-st} dt = \int_0^\infty \frac{x^\alpha}{s^{\alpha+1}} e^{-x} dx = \frac{1}{s^{\alpha+1}} \int_0^\infty x^\alpha e^{-x} dx = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}.$$

□

Theorem A.4. *The Laplace transform of the first order derivative of the function $f(t)$ has the form*

$$\mathscr{L}\left[\frac{d}{dt} f(t)\right] = sF(s) - f(0^+). \qquad (A.9)$$

Proof.

$$\begin{aligned} \mathscr{L}\left[\frac{d}{dt} f(t)\right] &= \int_0^\infty \frac{d}{dt} f(t) e^{-st} dt = \int_0^\infty e^{-st} df = e^{-st} f\Big|_0^\infty + s\int_0^\infty f(t) e^{-st} dt \\ &= sF(s) - f(0^+). \end{aligned}$$

Generalizing (A.9) for n order derivative we obtain

$$\mathscr{L}\left[\frac{d^n}{dt^n} f(t)\right] = s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0^+). \qquad (A.10)$$

□

Theorem A.5. *The Laplace transform of the fractional α-order integral has the form*

$$\mathscr{L}\left[{}_0I_t^{\alpha} f(t)\right] = \mathscr{L}\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau\right] = \frac{F(s)}{s^{\alpha}}. \tag{A.11}$$

Proof. Using (A.4) we obtain

$$\mathscr{L}\left[{}_0I_t^{\alpha} f(t)\right] = \mathscr{L}\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau\right] = \frac{1}{\Gamma(\alpha)}\mathscr{L}\left[t^{\alpha-1}\right]\mathscr{L}\left[f(t)\right] = \frac{F(s)}{s^{\alpha}}.$$

□

Theorem A.6. *The Laplace transform of the fractional α-order integral of n-order derivative of the function $f(t)$ has the form*

$$\mathscr{L}\left[{}_0I_t^{\alpha} f^{(n)}(t)\right] = \frac{F(s)}{s^{\alpha-n}} - \sum_{k=1}^{n} \frac{f^{(k-1)}(0^+)}{s^{\alpha-n+k}}. \tag{A.12}$$

Proof. Using (A.4) we obtain

$$\begin{aligned}
\mathscr{L}\left[{}_0I_t^{\alpha} f^{(n)}(t)\right] &= \mathscr{L}\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f^{(n)}(\tau)d\tau\right] \\
&= \frac{1}{\Gamma(\alpha)}\mathscr{L}\left[t^{\alpha-1}\right]\mathscr{L}\left[f^{(n)}(t)\right] \\
&= \frac{1}{\Gamma(\alpha)}\frac{\Gamma(\alpha)}{s^{\alpha}}\left(s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0^+)\right) \\
&= \frac{F(s)}{s^{\alpha-n}} - \sum_{k=1}^{n} \frac{f^{(k-1)}(0^+)}{s^{\alpha-n+k}}.
\end{aligned}$$

□

A.3 Z-Transforms of Discrete-Time Functions

Theorem A.7. *If*

$$\mathscr{Z}[x_i] = \sum_{i=0}^{\infty} x_i z^{-i}. \tag{A.13}$$

then

$$\mathscr{Z}[x_{i+1}] = zX(z) - zx_0, \tag{A.14a}$$

$$\mathscr{Z}[x_{i-p}] = z^{-p}X(z) + z^{-p}\sum_{j=-1}^{-p} x_j z^{-j}. \tag{A.14b}$$

Proof. Using (A.13) we obtain

$$\mathscr{Z}[x_{i+1}] = \sum_{i=0}^{\infty} x_{i+1} z^{-i} = \sum_{j=1}^{\infty} x_j z^{-(j-1)} = z \sum_{j=0}^{\infty} x_j z^{-j} - z x_0 = z X(z) - z x_0,$$

$$\mathscr{Z}[x_{i-p}] = \sum_{i=0}^{\infty} x_{i-p} z^{-i} = \sum_{j=-p}^{\infty} x_j z^{-(j+p)} = z^{-p} \sum_{j=-p}^{\infty} x_j z^{-j}$$

$$= z^{-p} \sum_{j=0}^{\infty} x_j z^{-j} + z^{-p} \sum_{j=-1}^{-p} x_j z^{-j} = z^{-p} X(z) + z^{-p} \sum_{j=-1}^{-p} x_j z^{-j}.$$

□

Theorem A.8. *Let $X(z_1, z_2)$ be the 2D z-transform of the function x_{ij} defined by*

$$\mathscr{Z}[x_{ij}] = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} x_{ij} z_1^{-i} z_2^{-j}. \tag{A.15}$$

Then

$$\mathscr{Z}[x_{i+1,j+1}] = z_1 z_2 [X(z_1, z_2) - X(z_1, 0) - X(0, z_2) + x_{00}], \tag{A.16a}$$

$$\mathscr{Z}[x_{i-k,j+1}] = z_1^{-k} z_2 [X(z_1, z_2) - X(z_1, 0)], \tag{A.16b}$$

$$\mathscr{Z}[x_{i+1,j-l}] = z_1 z_2^{-l} [X(z_1, z_2) - X(0, z_2)], \tag{A.16c}$$

$$\mathscr{Z}[x_{i-k,j-l}] = z_1^{-k} z_2^{-l} X(z_1, z_2), \tag{A.16d}$$

$$\mathscr{Z}[x_{i+1,j}] = z_1 [X(z_1, z_2) - X(0, z_2)], \tag{A.16e}$$

$$\mathscr{Z}[x_{i,j+1}] = z_2 [X(z_1, z_2) - X(z_1, 0)]. \tag{A.16f}$$

where

$$X(z_1, 0) = \sum_{i=0}^{\infty} x_{i0} z_1^{-i}, \quad X(0, z_2) = \sum_{j=0}^{\infty} x_{0j} z_2^{-j}.$$

Proof. Using (A.15) we obtain

$$\begin{aligned}
\mathscr{Z}[x_{i+1,j+1}] &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{i+1,j+1} z_1^{-i} z_2^{-j} = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} x_{kl} z_1^{1-k} z_2^{1-l} \\
&= z_1 z_2 \left[\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} x_{kl} z_1^{-k} z_2^{-l} - \sum_{k=0}^{\infty} x_{k0} z_1^{-k} - \sum_{l=0}^{\infty} x_{0l} z_2^{-l} + x_{00}\right], \\
\mathscr{Z}[x_{i-k,j+1}] &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{i-k,j+l} z_1^{-i} z_2^{-j} = \sum_{p=1}^{\infty}\sum_{q=-l}^{\infty} x_{pq} z_1^{1-p} z_2^{-(q+l)} \\
&= z_1 z_2^{-l} \left[\sum_{p=1}^{\infty}\sum_{q=-l}^{\infty} x_{pq} z_1^{-p} z_2^{-q}\right] \\
&= z_1 z_2^{-l} \left[\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} x_{pq} z_1^{-p} z_2^{-q} - \sum_{i=0}^{\infty} x_{i0} z_1^{-i}\right], \\
\mathscr{Z}[x_{i+1,j-l}] &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{i+1,j-l} z_1^{-i} z_2^{-j} = \sum_{p=-k}^{\infty}\sum_{q=1}^{\infty} x_{pq} z_1^{-(p+k)} z_2^{1-q} \\
&= z_1^{-k} z_2 \left[\sum_{p=-k}^{\infty}\sum_{q=1}^{\infty} x_{pq} z_1^{-p} z_2^{-q}\right] \\
&= z_1^{-k} z_2 \left[\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} x_{pq} z_1^{-p} z_2^{-q} - \sum_{j=0}^{\infty} x_{0j} z_2^{-j}\right], \\
\mathscr{Z}[x_{i-k,j-l}] &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{i-k,j-l} z_1^{-i} z_2^{-j} = \sum_{p=-k}^{\infty}\sum_{q=-l}^{\infty} x_{pq} z_1^{-(p+k)} z_2^{-(q+l)} \\
&= z_1^{-k} z_2^{-l} \left[\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} x_{pq} z_1^{-p} z_2^{-q}\right], \\
\mathscr{Z}[x_{i+1,j}] &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{i+1,j} z_1^{-i} z_2^{-j} = \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} x_{kj} z_1^{1-k} z_2^{-j} \\
&= z_1 \left[\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} x_{kj} z_1^{-k} z_2^{-j}\right] = z_1 \left[\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} x_{kj} z_1^{-k} z_2^{-j} - \sum_{j=0}^{\infty} x_{0j} z_2^{-j}\right],
\end{aligned}$$

$$\begin{aligned}
\mathscr{Z}[x_{i,j+1}] &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x_{i,j+1} z_1^{-i} z_2^{-j} = \sum_{i=0}^{\infty}\sum_{k=1}^{\infty} x_{ik} z_1^{-i} z_2^{1-k} \\
&= z_2 \left[\sum_{i=0}^{\infty}\sum_{k=1}^{\infty} x_{ik} z_1^{-i} z_2^{-k}\right] = z_2 \left[\sum_{i=0}^{\infty}\sum_{k=0}^{\infty} x_{ik} z_1^{-i} z_2^{-k} - \sum_{i=0}^{\infty} x_{i0} z_1^{-i}\right].
\end{aligned}$$

□

Appendix B
Infinite Long Cable with Zero Inductance as an Example of Fractional System

Consider the infinite long cable with zero inductance described by the equations (B.1):

$$\frac{\partial u(x,t)}{\partial x} = Ri(x,t), \tag{B.1a}$$

$$\frac{\partial i(x,t)}{\partial x} = C\frac{\partial u(x,t)}{\partial t}, \tag{B.1b}$$

where $u(x,t)$ and $i(x,t)$ are the voltage and current in the point x and time t and R and C are the resistance and capacitance of unit length cable. Differentiating the equation (B.1a) with respect x and using (B.1b) we obtain:

$$\frac{\partial u(x,t)}{\partial t} = \alpha\frac{\partial^2 u(x,t)}{\partial x^2}, \quad \alpha = \frac{1}{RC}. \tag{B.2}$$

It is assumed that the voltages $(u(0,t) = u_0(t))$ and $u(x,0)$ are given and the voltage at the end of the cable is zero $u(\infty,t) = 0$. Solution of the equation (B.2) will be derived by the use of the Laplace transform method. Let

$$U(x,s) = \mathscr{L}_t[u(x,t)] = \int_0^\infty u(x,t)e^{-st}dt, \tag{B.3a}$$

and

$$U(p,s) = \mathscr{L}_x[U(x,s)] = \int_0^\infty U(x,s)e^{-sx}dx. \tag{B.3b}$$

Applying the Laplace transform to the equation (B.2) we obtain

$$\left(p^2 - \frac{s}{\alpha}\right)U(p,s) = -\frac{1}{\alpha}U(p,0) + pU(0,s) + \overline{U}(0,s), \tag{B.4a}$$

and

$$U(p,s) = \frac{1}{p^2 - \frac{s}{\alpha}}\left[-\frac{1}{\alpha}U(p,0) + pU(0,s) + \overline{U}(0,s)\right], \tag{B.4b}$$

where

$$\overline{U}(0,s) = \int_0^\infty \frac{du(0,t)}{dx} e^{-st} dt = \frac{dU(0,s)}{dx} = \frac{dU_0(s)}{dx}. \tag{B.5}$$

Taking into account that

$$\frac{1}{p^2 - \frac{s}{\alpha}} = \frac{1}{2\sqrt{\frac{s}{\alpha}}\left(p - \sqrt{\frac{s}{\alpha}}\right)} - \frac{1}{2\sqrt{\frac{s}{\alpha}}\left(p + \sqrt{\frac{s}{\alpha}}\right)}, \tag{B.6}$$

and using the Laplace transform of the convolution to (B.4) we obtain

$$\begin{aligned} U(x,s) = &\int_0^x \frac{1}{2\sqrt{\frac{s}{\alpha}}} e^{(x-y)\sqrt{\frac{s}{\alpha}}} \left[-\frac{1}{\alpha} u(y,0)\right] dy \\ &- \int_0^x \frac{1}{2\sqrt{\frac{s}{\alpha}}} e^{-(x-y)\sqrt{\frac{s}{\alpha}}} \left[-\frac{1}{\alpha} u(y,0)\right] dy \\ &+ U(0,s)\cosh\left(x\sqrt{\frac{s}{\alpha}}\right) + \frac{\overline{U}(0,s)}{\sqrt{\frac{s}{\alpha}}} \sin\left(x\sqrt{\frac{s}{\alpha}}\right), \end{aligned} \tag{B.7a}$$

since

$$\mathscr{L}_x^{-1}\left[\frac{p}{p - \frac{s}{\alpha}}\right] = \cosh\left(x\sqrt{\frac{s}{\alpha}}\right), \quad \mathscr{L}_x^{-1}\left[\frac{1}{p - \frac{s}{\alpha}}\right] = \frac{1}{\sqrt{\frac{s}{\alpha}}} \sinh\left(x\sqrt{\frac{s}{\alpha}}\right). \tag{B.7b}$$

Substituting

$$\cosh\left(x\sqrt{\frac{s}{\alpha}}\right) = \frac{1}{2}\left(e^{x\sqrt{\frac{s}{\alpha}}} + e^{-x\sqrt{\frac{s}{\alpha}}}\right), \quad \sinh\left(x\sqrt{\frac{s}{\alpha}}\right) = \frac{1}{2}\left(e^{x\sqrt{\frac{s}{\alpha}}} - e^{-x\sqrt{\frac{s}{\alpha}}}\right) \tag{B.8}$$

to (B.7) we obtain

$$\begin{aligned} U(x,s) = &\frac{e^{x\sqrt{\frac{s}{\alpha}}}}{2}\left[U(0,s) + \frac{\overline{U}(0,s)}{\sqrt{\frac{s}{\alpha}}} - \frac{1}{\alpha\sqrt{\frac{s}{\alpha}}} \int_0^x e^{-y\sqrt{\frac{s}{\alpha}}} u(y,0) dy\right] \\ &+ \frac{e^{-x\sqrt{\frac{s}{\alpha}}}}{2}\left[U(0,s) - \frac{\overline{U}(0,s)}{\sqrt{\frac{s}{\alpha}}} + \frac{1}{\alpha\sqrt{\frac{s}{\alpha}}} \int_0^x e^{y\sqrt{\frac{s}{\alpha}}} u(y,0) dy\right]. \end{aligned} \tag{B.9}$$

Note that in (B.9) we have the product of the components

$$\frac{e^{-x\sqrt{\frac{s}{\alpha}}}}{2} \quad \text{and} \quad \frac{1}{\alpha\sqrt{\frac{s}{\alpha}}} \int_0^x e^{y\sqrt{\frac{s}{\alpha}}} u(y,0) dy.$$

The first component decreases to zero and the second component increases to infinity with $x \to \infty$. Applying the l'Hospital rule we obtain

$$\lim_{x\to\infty} \frac{\frac{1}{\alpha\sqrt{\frac{s}{\alpha}}}\int_0^x e^{y\sqrt{\frac{s}{\alpha}}}u(y,0)dy}{e^{x\sqrt{\frac{s}{\alpha}}}} = \frac{\frac{1}{s}e^{x\sqrt{\frac{s}{\alpha}}}u(x,0)}{e^{x\sqrt{\frac{s}{\alpha}}}} = \frac{u(\infty,0)}{s}. \tag{B.10}$$

By assumption $u(\infty,s) = 0$. From (B.9) for $x = \infty$ we have

$$U(\infty,s) = \lim_{x\to\infty} e^{x\sqrt{\frac{s}{\alpha}}}\left[U(0,s) + \frac{\overline{U}(0,s)}{\sqrt{\frac{s}{\alpha}}} - \frac{1}{\alpha\sqrt{\frac{s}{\alpha}}}\int_0^x e^{-y\sqrt{\frac{s}{\alpha}}}u(y,0)dy\right] = 0, \tag{B.11}$$

since the second component tends to zero for $x \to \infty$. From (B.11) we have

$$U(0,s) + \frac{\overline{U}(0,s)}{\sqrt{\frac{s}{\alpha}}} - \frac{1}{\alpha\sqrt{\frac{s}{\alpha}}}\int_0^x e^{-y\sqrt{\frac{s}{\alpha}}}u(y,0)dy = 0. \tag{B.12}$$

From (B.1a) for $x = 0$ we obtain

$$I(0,s) = \mathscr{L}_t[x(0,t)] = \frac{1}{R}\frac{dU(0,s)}{dx} = \frac{\overline{U}(0,s)}{R}. \tag{B.13}$$

From (B.12) and (B.13) we have

$$U(0,s) = \frac{1}{\alpha\sqrt{\frac{s}{\alpha}}}\int_0^x e^{-y\sqrt{\frac{s}{\alpha}}}u(y,0)dy - \frac{RI(0,s)}{\sqrt{\frac{s}{\alpha}}}. \tag{B.14}$$

Note that

$$\frac{1}{\alpha\sqrt{\frac{s}{\alpha}}}\int_0^x e^{-y\sqrt{\frac{s}{\alpha}}}u(y,0)dy = \frac{1}{\alpha\sqrt{\frac{s}{\alpha}}}U(q,0), \quad q = \sqrt{\frac{s}{\alpha}}. \tag{B.15}$$

Taking into account (B.15) from (B.14) we obtain

$$U(0,s) = \frac{1}{\alpha\sqrt{\frac{s}{\alpha}}}U(q,0) - \frac{RI(0,s)}{\sqrt{\frac{s}{\alpha}}}, \quad q = \sqrt{\frac{s}{\alpha}}. \tag{B.16}$$

By assumption $u(x,0) = 0$, $(U(p,0) = 0)$ and from (B.16) we have

$$\frac{U(0,s)}{I(0,s)} = -\frac{R\sqrt{\alpha}}{\sqrt{s}}. \tag{B.17}$$

Applying the inverse Laplace transform we obtain

$$i(0,t) = -\frac{1}{R\sqrt{\alpha}}\frac{d^{\frac{1}{2}}u(0,t)}{dt^{\frac{1}{2}}} = -\frac{1}{R\sqrt{\alpha}}\frac{d^{\frac{1}{2}}u_0(t)}{dt^{\frac{1}{2}}}, \tag{B.18a}$$

and

$$u_0(t) = u(0,t) = -R\sqrt{\alpha}\frac{d^{-\frac{1}{2}}i(0,t)}{dt^{-\frac{1}{2}}}. \tag{B.18b}$$

Therefore, the voltage $u_0(t)$ and the current $i(0,t)$ at the beginning of the cable are related by (B.18). To find the voltage $u(x,t)$ for given initial conditions we substitute (B.16) into (B.9) and we apply the inverse Laplace transform with respect to s. Other examples of fractional linear systems are presented in [292, 213, 251, 260].

Appendix C
Right Inverse of Matrices

Consider the matrix linear algebraic equation

$$Ax = b, \tag{C.1}$$

where $A \in \mathbb{R}^{n\times m}$, $b \in \mathbb{R}^n$, $x \in \mathbb{R}^m$, $m > n$.

Theorem C.1. *The equation (C.1) has the solution for every b if and only if the matrix A has full row rank.*

Proof. The equation (C.1) has a solution if and only if $\operatorname{rank}A = \operatorname{rank}[A, b]$. From this condition it follows that the solution of (C.1) for any b there exists if and only if $\operatorname{rank}A = \operatorname{rank}[A, b]$. □

For $m > n$ the equation (C.1) has many solutions.

Theorem C.2. *The set of solutions of the equation (C.1) is given by*

$$x = A_r b, \tag{C.2}$$

where A_r is the right inverse of the matrix A, satisfying the condition $AA_r = I_n$.

Proof. Substitution of (C.2) into (C.1), yields

$$AA_r b = b, \tag{C.3}$$

since $AA_r = I_n$. □

Theorem C.3. *There exists the right inverse A_r of $A \in \mathbb{R}^{n\times m}$ if*

$$\operatorname{rank}A = n.$$

The right inverse is given by one of the formule

a)

$$A_r = A^T \left[AA^T\right]^{-1} + \left(I_n - A^T \left[AA^T\right]^{-1} A\right) K, \tag{C.4}$$

b)

$$A_r = K[AK]^{-1}, \quad \det[AK] \neq 0, \quad K \in \mathbb{R}^{m\times n}, \tag{C.5}$$

c)

$$A_r = R\begin{bmatrix} I_n \\ K \end{bmatrix} A_1^{-1}, \quad AR = [A_1, 0], \quad A_1 \in \mathbb{R}^{n\times n}, \quad \det A_1 \neq 0. \tag{C.6}$$

where R is the matrix of elementary column of operations (Appendix D).

Proof.

a) If $\operatorname{rank} A = n$, then the matrix AA^T is nonsingular (positive defined). In this case there exists the inverse matrix $[AA^T]^{-1}$ and the matrix (C.4) satisfies the condition

$$AA_r = AA^T\left[AA^T\right]^{-1} + A\left[I_n - A^T\left[AA^T\right]^{-1}A\right]K = I_n.$$

b) If $\operatorname{rank} A = n$, then there exists a matrix K such that $\det[AK] \neq 0$ and the matrix (C.5) satisfies the condition

$$AA_r = AK[AK]^{-1} = I_n.$$

c) Let R be a matrix of elementary column operations satisfying the condition $AP = [A_1, 0]$, $\det A_1 \neq 0$. In this case the matrix (C.6) satisfies the condition

$$AA_r = AR\begin{bmatrix} I_n \\ K \end{bmatrix} A_1^{-1} = [A_1 0]\begin{bmatrix} I_n \\ K \end{bmatrix} A_1^{-1} = A_1A_1^{-1} = I_n.$$

□

Example C.1. Compute the right inverse of the matrix

$$A = \begin{bmatrix} 2 & 0 & -3 \\ 4 & 1 & 1 \end{bmatrix}.$$

a) Using (C.4) and taking into account that

$$AA^T = \begin{bmatrix} 2 & 0 & -3 \\ 4 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ 0 & 1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 13 & 5 \\ 5 & 18 \end{bmatrix}, \quad \left[AA^T\right]^{-1} = \frac{1}{209}\begin{bmatrix} 18 & -5 \\ -5 & 13 \end{bmatrix},$$

we obtain

$$A_r = \begin{bmatrix} 2 & 4 \\ 0 & 1 \\ -3 & 1 \end{bmatrix} \frac{1}{209} \begin{bmatrix} 18 & -5 \\ -5 & 13 \end{bmatrix} + \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right.$$
$$\left. - \begin{bmatrix} 2 & 4 \\ 0 & 1 \\ -3 & 1 \end{bmatrix} \frac{1}{209} \begin{bmatrix} 18 & -5 \\ -5 & 13 \end{bmatrix} \begin{bmatrix} 2 & 0 & -3 \\ 4 & 1 & 1 \end{bmatrix} \right\} \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \\ k_5 & k_6 \end{bmatrix}$$
$$= \frac{1}{209} \begin{bmatrix} 16+19k_1-42k_3+6k_5 & 42+9k_2-42k_4+6k_6 \\ -5-42k_1+196k_3-28k_5 & 13-42k_2+196k_4-28k_6 \\ -59+6k_1-28k_3+4k_5 & 28+6k_2-28k_4+4k_6 \end{bmatrix}.$$

b) The matrix K is chosen so that

$$\det[AK] = \det \begin{bmatrix} 2k_1-3k_5 & 2k_2-3k_6 \\ 4k_1+k_3+k_5 & 4k_2+k_4+k_6 \end{bmatrix}$$
$$= 2k_1k_4+14k_1k_6-14k_5k_2-3k_5k_4-2k_2k_3+3k_6k_3 \neq 0.$$

Using (C.5) we obtain

$$A_r = \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \\ k_5 & k_6 \end{bmatrix} \begin{bmatrix} 2k_1-3k_5 & 2k_2-3k_6 \\ 4k_1+k_3+k_5 & 4k_2+k_4+k_6 \end{bmatrix}^{-1}$$

$$= \frac{1}{\det[AK]} \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \\ k_5 & k_6 \end{bmatrix} \begin{bmatrix} 4k_2+k_4+k_6 & 3k_6-2k_2 \\ -4k_1-k_3-k_5 & 2k_1-3k_5 \end{bmatrix}$$
$$= \frac{1}{\det[AK]} \begin{bmatrix} k_1k_4+k_1k_6-k_2k_3-k_5k_2 & 3k_1k_6-3k_5k_2 \\ 4k_2k_3+k_6k_3-4k_1k_4-k_5k_4 & 3k_6k_3-2k_2k_3+2k_1k_4-3k_5k_4 \\ 4k_5k_2+k_5k_4-4k_1k_6-k_6k_3 & -2k_5k_2+2k_1k_6 \end{bmatrix}.$$

c) Using the following elementary column operations we eliminate from A its linearly dependent columns

$$\begin{bmatrix} 2 & 0 & -3 \\ 4 & 1 & 1 \end{bmatrix} \xrightarrow{R[3+2\times(-1)]} \begin{bmatrix} 2 & 0 & -3 \\ 4 & 1 & 0 \end{bmatrix} \xrightarrow[R[1+3\times(\frac{2}{3})]]{R[1+2\times(-4)]}$$
$$\begin{bmatrix} 0 & 0 & -3 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R[1,3]} \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

To find the matrix R we perform on the identity matrix I_n the elementary column operations

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R[3+2\times(-1)]} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R[1+3\times(\frac{2}{3})]]{R[1+2\times(-4)]}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{14}{3} & 1 & -1 \\ \frac{2}{3} & 0 & 1 \end{bmatrix} \xrightarrow{R[1,3]} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -\frac{14}{3} \\ 1 & 0 & \frac{2}{3} \end{bmatrix}.$$

Hence

$$R = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -\frac{14}{3} \\ 1 & 0 & \frac{2}{3} \end{bmatrix} \tag{C.7}$$

and

$$AP = \begin{bmatrix} 2 & 0 & -3 \\ 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -\frac{14}{3} \\ 1 & 0 & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \end{bmatrix}$$

Using (C.6) for

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}, \tag{C.8}$$

we obtain

$$A_r = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -\frac{14}{3} \\ 1 & 0 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ k_1 & k_2 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{k_1}{3} & k_2 \\ \frac{1}{3} + \frac{14}{9}k_1 & 1 - \frac{14}{3}k_2 \\ -\frac{1}{3} - \frac{2}{9}k_1 & \frac{2}{3}k_2 \end{bmatrix}.$$

Appendix D
Elementary Operations on Matrices

Definition D.1. The following operations are called elementary operations on a real matrix $A \in \mathbb{R}^{n \times m}$

a) Multiplication of any i-th row (column) by the number $a \neq 0$.
b) Addition to any i-th row (column) of the j-th row (column) multiplied by any number $b \neq 0$.
c) The interchange of any two rows (columns).

In this book the following notation is used.

$L[i \times a]$	multiplication of the i-th row by the number $a \neq 0$
$R[i \times a]$	multiplication of the i-th column by the number $a \neq 0$
$L[i + j \times b]$	addition to the i-th row of the j-th row multiplied by the number b
$R[i + j \times b]$	addition to the i-th column of the j-th column multiplied by the number b
$L[i, j]$	the interchange of the i-th and the j-th rows
$R[i, j]$	the interchange of the i-th and the j-th columns

The elementary operation can be extended to polynomial matrices [89].
The rank of the matrix A is invariant under the elementary operations.

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