A survey of Torelli and monodromy results for holomorphic-symplectic varieties

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Abstract We survey recent results about the Torelli question for holomorphic-symplectic varieties. Following are the main topics. A Hodge theoretic Torelli theorem. A study of the subgroup W_{Exc}, of the isometry group of the weight 2 Hodge structure, generated by reflection with respect to exceptional divisors. A description of the birational Kähler cone as a fundamental domain for the W_{Exc} action on the positive cone. A proof of a weak version of Morrison's movable cone conjecture. A description of the moduli spaces of polarized holomorphic symplectic varieties as monodromy quotients of period domains of type IV.

Keywords Torelli Theorem, Holomorphic symplectic varieties, Moduli spaces, Movable cone
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Contents

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1 Introduction

An *irreducible holomorphic symplectic manifold* is a simply connected compact Kähler manifold such that $H^0(X,\Omega^2_X)$ is one-dimensional, spanned by an everywhere non-degenerate holomorphic 2-form [Be1]. There exists a unique non-degenerate symmetric integral and primitive bilinear pairing $(\bullet,\bullet)$ on $H^2(X,\mathbb{Z})$ of signature $(3,b_2(X)-3)$, with the following property. There exists a positive rational number λ_X, such that the equality

$$(\alpha,\alpha)^n = \lambda_X \int_X \alpha^{2n}$$

holds for all $\alpha \in H^2(X,\mathbb{Z})$, where $2n = \dim_{\mathbb{C}}(X)$ [Be1]. If $b_2(X) = 6$, then we require[1] further that $(\alpha,\alpha) > 0$, for every Kähler class α. The pairing is called the *Beauville-Bogomolov pairing* and (α,α) is called the *Beauville-Bogomolov degree* of the class α.

Let S be a $K3$ surface. Then the Hilbert scheme (or Douady space, in the Kähler case) $S^{[n]}$, of length n zero-dimensional subschemes of S, is an irreducible holomorphic symplectic manifold. If $n \geq 2$, then $b_2(S^{[n]}) = 23$ [Be1]. If X is deformation equivalent to $S^{[n]}$, we will say that X is of $K3^{[n]}$*-type.*

Let T be a complex torus with an origin $0 \in T$. Denote by $T^{(n)}$ the n-th symmetric product. Let $T^{(n)} \to T$ be the addition morphism. The composite morphism

$$T^{[n+1]} \longrightarrow T^{(n+1)} \longrightarrow T$$

is an isotrivial fibration. Each fiber is a $2n$-dimensional irreducible holomorphic symplectic manifold, called a *generalized Kummer variety*, and denoted by $K^{[n]}(T)$ [Be1]. If $n \geq 2$, then $b_2\left(K^{[n]}(T)\right) = 7$.

O'Grady constructed two additional irreducible holomorphic symplectic manifolds, a 10-dimensional example X with $b_2(X) = 24$, and a 6-dimensional example Y with $b_2(Y) = 8$ [O'G2, O'G3, R].

We recommend Huybrechts' excellent survey of the subject of irreducible holomorphic symplectic manifolds [Hu3]. The aim of this note is to survey developments related to the Torelli problem, obtained by various authors since Huybrechts' survey was written. The most important, undoubtedly, is Verbitsky's proof of his version of the Global Torelli Theorem [Ver2, Hu6].

1.1 Torelli Theorems

We hope to convince the reader that the concepts of monodromy and parallel-transport operators are essential for any discussion of the Torelli problem.

Definition 1.1 Let X, X_1, and X_2 be irreducible holomorphic symplectic manifolds.

[1] The condition is satisfied automatically by the assumption that the signature is $(3, b_2(X) - 3)$, if $b_2 \neq 6$.

(1) An isomorphism $f : H^*(X_1,\mathbb{Z}) \to H^*(X_2,\mathbb{Z})$ is said to be a *parallel-transport operator*, if there exist a smooth and proper family[2] $\pi : \mathscr{X} \to B$ of irreducible holomorphic symplectic manifolds, over an analytic base B, points $b_i \in B$, isomorphisms $\psi_i : X_i \to \mathscr{X}_{b_i}$, $i = 1,2$, and a continuous path $\gamma : [0,1] \to B$, satisfying $\gamma(0) = b_1$, $\gamma(1) = b_2$, such that the parallel transport in the local system $R\pi_*\mathbb{Z}$ along γ induces the homomorphism $\psi_{2_*} \circ f \circ \psi_1^* : H^*(\mathscr{X}_{b_1},\mathbb{Z}) \to H^*(\mathscr{X}_{b_2},\mathbb{Z})$. An isomorphism $g : H^k(X_1,\mathbb{Z}) \to H^k(X_2,\mathbb{Z})$ is said to be a *parallel-transport operator*, if it is the k-th graded summand of a parallel-transport operator f as above.

(2) An automorphism $f : H^*(X,\mathbb{Z}) \to H^*(X,\mathbb{Z})$ is said to be a *monodromy operator*, if it is a parallel transport operator.

(3) The *monodromy group* $Mon(X)$ is the subgroup[3] of $GL[H^*(X,\mathbb{Z})]$ consisting of all monodromy operators. We denote by $Mon^2(X)$ the image of $Mon(X)$ in $O[H^2(X,\mathbb{Z})]$.

(4) Let H_i be an ample line bundle on X_i, $i = 1,2$. An isomorphism $f : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ is said to be a *polarized parallel-transport operator* from (X_1,H_1) to (X_2,H_2), if there exists a family $\pi : \mathscr{X} \to B$, satisfying all the properties of part (1), as well as a flat section h of $R^2\pi_*\mathbb{Z}$, such that $h(b_i) = \psi_{i_*}(c_1(H_i))$, $i = 1,2$, and $h(b)$ is an ample class in $H^{1,1}(\mathscr{X}_b,\mathbb{Z})$, for all $b \in B$.

(5) Given an ample line bundle H on X, we denote by $Mon(X,H)$ the subgroup of $Mon(X)$, consisting of polarized parallel transport operators from (X,H) to itself. Elements of $Mon(X,H)$ will be called *polarized monodromy operators* of (X,H).

Following is a necessary condition for an isometry $g : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ to be a parallel transport operator. Denote by $\widetilde{\mathscr{C}}_X \subset H^2(X,\mathbb{R})$ the cone

$$\{\alpha \in H^2(X,\mathbb{R}) \ : \ (\alpha,\alpha) > 0\}.$$

Then $H^2(\widetilde{\mathscr{C}}_X,\mathbb{Z}) \cong \mathbb{Z}$ and it comes with a canonical generator, which we call the *orientation class* of $\widetilde{\mathscr{C}}_X$ (section 4). Any isometry $g : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ induces an isomorphism $\bar{g} : \widetilde{\mathscr{C}}_X \to \widetilde{\mathscr{C}}_Y$. The isometry g is said to be *orientation preserving* if

[2] Note that the family may depend on the isomorphism f.

[3] If $f \in Mon(X)$ is associated to a family $\pi' : \mathscr{X}' \to B'$ via an isomorphism $X \cong \mathscr{X}'_{b'}$, and $g \in Mon(X)$ is associated to a family $\pi'' : \mathscr{X}'' \to B''$ via an isomorphism $X \cong \mathscr{X}''_{b''}$, then fg is easily seen to be associated to the family $\pi : \mathscr{X} \to B$, obtained by "gluing" $\mathscr{X}'$ and $\mathscr{X}''$ via the isomorphism $\mathscr{X}'_{b'} \cong X \cong \mathscr{X}''_{b''}$ and connecting B' and B'' at the points b' and b'' to form the (reducible) base B.

$\bar{g}$ is. A parallel transport operator $g : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ is orientation preserving. When X and Y are $K3$ surfaces, every orientation preserving isometry is a parallel transport operator. This is no longer the case for higher dimensional irreducible holomorphic symplectic varieties [Ma5, Nam2]. A necessary and sufficient criterion for an isometry to be a parallel transport operator is provided in the $K3^{[n]}$-type case, for all $n \geq 1$ (Theorem 9.8).

A *marked pair* (X,η) consists of an irreducible holomorphic symplectic manifold X and an isometry $\eta : H^2(X,\mathbb{Z}) \to \Lambda$ onto a fixed lattice Λ. Let $\mathfrak{M}^0_\Lambda$ be a connected component of the moduli space of isomorphism classes of marked pairs (see section 2). There exists a surjective period map $P_0 : \mathfrak{M}^0_\Lambda \to \Omega_\Lambda$ onto a period domain ([Hu1], Theorem 8.1). Each point $p \in \Omega_\Lambda$ determines a weight 2 Hodge structure on $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$, such that the marking η is an isomorphism of Hodge structures. The positive cone $\mathscr{C}_X$ of X is the connected component of the cone $\{\alpha \in H^{1,1}(X,\mathbb{R}) \ : \ (\alpha,\alpha) > 0\}$, containing the Kähler cone $\mathscr{K}_X$. Following is a concise version of the *Global Torelli Theorem* ([Ver2], or Theorem 2.2 below).

Theorem 1.2 *If $P_0(X,\eta) = P_0(\widetilde{X},\tilde{\eta})$, then X and $\widetilde{X}$ are bimeromorphic. A pair (X,η) is the unique point in a fiber of P_0, if and only if $\mathscr{K}_X = \mathscr{C}_X$. This is the case, for example, if the sublattice $H^{1,1}(X,\mathbb{Z})$ is trivial, or of rank 1, generated by an element λ, with $(\lambda,\lambda) \geq 0$.*

The following theorem combines the Global Torelli Theorem with results on the Kähler cone of irreducible holomorphic symplectic manifolds [Hu2, Bou1].

Theorem 1.3 *(A Hodge theoretic Torelli theorem) Let X and Y be irreducible holomorphic symplectic manifolds, which are deformation equivalent.*

(1) X and Y are bimeromorphic, if and only if there exists a parallel transport operator $f : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$, which is an isomorphism of integral Hodge structures.

(2) Let $f : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ be a parallel transport operator, which is an isomorphism of integral Hodge structures. There exists an isomorphism $\tilde{f} : X \to Y$, such that $f = \tilde{f}_*$, if and only if f maps some Kähler class on X to a Kähler class on Y.

The theorem is proven in section 3.2. It generalizes the Strong Torelli Theorem of Burns and Rapoport [BR] or ([LP], Theorem 9.1).

Given a bimeromorphic map $f : X \to Y$, of irreducible holomorphic symplectic manifolds, denote by $f_* : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ the homomorphism induced by the closure in $X \times Y$ of the graph of f. The homomorphism f_* is known to be an isometry ([O'G1], Proposition 1.6.2). Set $f^* := (f^{-1})_*$.

The *birational Kähler cone* $\mathscr{BK}_X$ of X is the union of the cones $f^*\mathscr{K}_Y$, as f ranges through all bimeromorphic maps from X to irreducible holomorphic symplectic manifolds Y. Let $Mon^2_{Hdg}(X)$ be the subgroup of $Mon^2(X)$ preserving the Hodge structure. Results of Boucksom and Huybrechts, on the Kähler and birational Kähler cones, are surveyed in section 5. We use them to define a chamber decomposition of the positive cone $\mathscr{C}_X$, via $Mon^2_{Hdg}(X)$-translates of cones of the form $f^*\mathscr{K}_Y$ (Lemma 5.11). These chambers are said to be of *Kähler type*.

Let $\mathfrak{M}^0_\Lambda$ be a connected component of the moduli space of marked pairs. A detailed form of the Torelli theorem provides a description of $\mathfrak{M}^0_\Lambda$ as a moduli space of Hodge theoretic data as follows. A point $p \in \Omega_\Lambda$ determines a Hodge structure on Λ, and so a real subspace $\Lambda^{1,1}(p,\mathbb{R})$ in $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$, such that a marking η restricts to an isometry $H^{1,1}(X,\mathbb{R}) \to \Lambda^{1,1}(p,\mathbb{R})$, for every pair (X,η) in the fiber $P_0^{-1}(p)$.

Theorem 1.4 *(Theorem 5.16) The map $(X,\eta) \mapsto \eta(\mathscr{K}_X)$ establishes a one-to-one correspondence between points (X,η) in the fiber $P_0^{-1}(p)$ and chambers in the Kähler type chamber decomposition of the positive cone in $\Lambda^{1,1}(p,\mathbb{R})$.*

1.2 The fundamental exceptional chamber

The next few results are easier to understand when compared to the following basic fact about $K3$ surfaces. Let S be a $K3$ surface and κ_0 a Kähler class on S. The effective cone in $H^{1,1}(S,\mathbb{Z})$ is spanned by classes α, such that $(\alpha,\alpha) \geq -2$, and $(\alpha,\kappa_0) > 0$ ([BHPV], Ch. VIII Proposition 3.6). Set[4]

$$\mathcal{S}pe := \{e \in H^{1,1}(S,\mathbb{Z}) \ : \ (\kappa_0, e) > 0, \text{ and } (e,e) = -2\},$$
$$\mathcal{P}ex := \{[C] \in H^{1,1}(S,\mathbb{Z}) \ : \ C \subset S \text{ is a smooth connected rational curve}\}.$$

Clearly, $\mathcal{P}ex$ is contained in $\mathcal{S}pe$. Then the Kähler cone admits the following two characterizations ([BHPV], Ch. VIII Proposition 3.7 and Corollary 3.8).

$$\mathscr{K}_S = \{\kappa \in \mathscr{C}_S \ : \ (\kappa, e) > 0, \text{ for all } e \in \mathcal{S}pe\}. \tag{1.1}$$
$$\mathscr{K}_S = \{\kappa \in \mathscr{C}_S \ : \ (\kappa, e) > 0, \text{ for all } e \in \mathcal{P}ex\}. \tag{1.2}$$

[4] $\mathcal{P}ex$ stands for prime exceptional classes, and $\mathcal{S}pe$ stands for stably prime exceptional classes, as will be explained below.

Equality (1.1) is the simpler one, depending only on the Hodge structure and the intersection pairing. Equality (1.2) expresses the fact that a class $e \in \mathcal{S}pe$ represents a smooth rational curve, if and only if $\overline{\mathcal{K}_S} \cap e^\perp$ is a co-dimension one face of the closure of $\mathcal{K}_S$ in $\mathcal{C}_S$.

Let X be a projective irreducible holomorphic symplectic manifold. A *prime exceptional divisor* on X is a reduced and irreducible effective divisor E of negative Beauville-Bogomolov degree. The *fundamental exceptional chamber* of the positive cone is the set

$$\mathcal{F}\mathcal{E}_X := \{\alpha \in \mathcal{C}_X : (\alpha,[E]) > 0, \text{ for every prime exceptional divisor } E\}. \quad (1.3)$$

When X is a $K3$ surface, a prime exceptional divisor is simply a smooth rational curve. Furthermore, the cones $\mathcal{K}_X$, $\mathcal{B}\mathcal{K}_X$, and $\mathcal{F}\mathcal{E}_X$ are equal. If $\dim(X) > 2$, the cone $\mathcal{B}\mathcal{K}_X$ need not be convex. The following is thus a generalization of equality (1.2) in the $K3$ surface case.

Theorem 1.5 *(Theorem 6.17 and Proposition 5.6) $\mathcal{F}\mathcal{E}_X$ is an open cone, which is the interior of a closed generalized convex polyhedron in $\mathcal{C}_X$ (Definition 6.13). The birational Kähler cone $\mathcal{B}\mathcal{K}_X$ is a dense open subset of $\mathcal{F}\mathcal{E}_X$.*

Let E be a prime exceptional divisor on a projective irreducible holomorphic symplectic manifold X. In section 6 we recall that the reflection

$$R_E : H^2(X,\mathbb{Z}) \longrightarrow H^2(X,\mathbb{Z}),$$

given by $R_E(\alpha) := \alpha - \frac{2(\alpha,[E])}{([E],[E])}[E]$, is an element of $Mon^2_{Hdg}(X)$ ([Ma7], Corollary 3.6, or Proposition 6.2 below). Let $W_{Exc}(X) \subset Mon^2_{Hdg}(X)$ be the subgroup generated[5] by the reflections R_E, of all prime exceptional divisors in X. In section 6.4 we prove the following analogue of a well known result for $K3$ surfaces ([BHPV], Ch. VIII, Proposition 3.9).

Theorem 1.6 *$W_{Exc}(X)$ is a normal subgroup of $Mon^2_{Hdg}(X)$. Let X_1 and X_2 be projective irreducible holomorphic symplectic manifolds and $f : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ a parallel-transport operator, which preserves the weight 2 Hodge structure. Then there exists a unique element $w \in W_{Exc}(X_2)$ and a birational map $g : X_1 \to X_2$, such that $f = w \circ g_*$. The map g is determined uniquely, up to composition with an automorphism of X_1, which acts trivially on $H^2(X_1,\mathbb{Z})$.*

[5] Definition 6.8 of W_{Exc} is different. The two definitions will be shown to be equivalent in Theorem 6.18.

Let us emphasis the special case $X_1 = X_2 = X$ of the theorem. Denote by $Mon^2_{Bir}(X) \subset O[H^2(X,\mathbb{Z})]$ the subgroup of isometries induced by birational maps from X to itself. Then $Mon^2_{Hdg}(X)$ is the semi-direct product of $W_{Exc}(X)$ and $Mon^2_{Bir}(X)$, by Theorem 6.18 part 5. Theorem 1.6 is proven in section 6.4. The proof relies on a second $Mon^2_{Hdg}(X)$-equivariant chamber decomposition of the positive cone $\mathscr{C}_X$. We call these the *exceptional chambers* (Definition 5.10). $W_{Exc}(X)$ acts simply-transitively on the set of exceptional chambers, one of which is the fundamental exceptional chamber. The walls of a general exceptional chamber are hyperplanes orthogonal to classes of *stably prime-exceptional* line bundles. The latter are higher-dimensional analogues of effective line bundles of degree -2 on a $K3$ surface. Roughly, a line bundle L on X is stably prime-exceptional, if a generic small deformation (X',L') of (X,L) satisfies $L' \cong \mathscr{O}_{X'}(E')$, for a prime exceptional divisor E' on X' (Definition 6.4).

Let X be a projective irreducible holomorphic symplectic manifold. Denote by $\mathrm{Bir}(X)$ the group of birational self-maps of X. The intersection of $\mathscr{F}\mathscr{E}_X$ with the subspace $H^{1,1}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ is equal to the interior of the movable cone of X (Definition 6.21 and Lemma 6.22). We prove a weak version of Morrison's movable cone conjecture, about the existence of a rational convex polyhedron, which is a fundamental domain for the action of $\mathrm{Bir}(X)$ on the movable cone (Theorem 6.25). We use it to prove the following result. When X is a $K3$ surface, $\mathrm{Bir}(X) = \mathrm{Aut}(X)$. Hence the following is an analogue of a result of Looijenga and Sterk ([St], Proposition 2.6).

Theorem 1.7 *For every integer $d \neq 0$, the number of $Bir(X)$-orbits of complete linear systems, which contain an irreducible divisor of Beauville-Bogomolov degree d, is finite. For every positive integer k there is only a finite number of $Bir(X)$-orbits of complete linear systems, which contain some irreducible divisor D of Beauville-Bogomolov degree zero, such that the class $[D]$ is k times a primitive class in $H^2(X,\mathbb{Z})$.*

Theorem 1.7 is proven in section 6.5. The proof follows an argument of Looijenga and Sterk, adapted via an analogy between results on the ample cone of a projective $K3$ surface and results on the movable cone of a projective irreducible holomorphic symplectic manifold.

The following is an analogue of the characterization of the Kähler cone of a $K3$ surface given in equation (1.1).

Proposition 1.8 *(Proposition 6.10) The fundamental exceptional chamber $\mathscr{F}\mathscr{E}_X$, defined in equation (1.3), is equal to the set*

$$\{\alpha \in \mathscr{C}_X \ : \ (\alpha,\ell) > 0, \textit{ for every stably prime exceptional class } \ell\}.$$

The significance of Proposition 1.8 stems from the fact that one has an explicit characterization of the set of stably prime-exceptional classes, in terms of the weight 2 Hodge structure and a certain discrete monodromy invariant, at least in the $K3^{[n]}$-type case (Theorem 9.17). Theorem 1.5 and Proposition 1.8 thus yield an explicit description of the closure of the birational Kähler cone and of the movable cone.

1.3 Torelli and monodromy in the polarized case

In sections 7 and 8 we consider Torelli-type results for polarized irreducible holomorphic symplectic manifolds. Another corollary of the Global Torelli Theorem is the following.

Proposition 1.9 *$Mon^2(X,H)$ is equal to the stabilizer of $c_1(H)$ in $Mon^2(X)$.*

The above proposition is proven in section 7 (see Corollary 7.4).

Coarse moduli spaces of polarized projective irreducible holomorphic symplectic manifolds were constructed by Viehweg as quasi-projective varieties [Vieh]. Given a polarized pair (X,H) representing a point in such a coarse moduli space $\mathscr{V}$, the monodromy group $\Gamma := Mon^2(X,H)$ is an arithmetic group, which acts on a period domain $\mathscr{D}$ associated to $\mathscr{V}$. The quotient $\mathscr{D}/\Gamma$ is a quasi-projective variety [BB]. The following Theorem is a slight sharpening of Corollary 1.24 in [Ver2].

Theorem 1.10 *(Theorem 8.4) The period map $\mathscr{V} \to \mathscr{D}/\Gamma$ embeds each irreducible component $\mathscr{V}$, of the coarse moduli space of polarized irreducible holomorphic symplectic manifolds, as a Zariski open subset of the quasi-projective monodromy-quotient of the corresponding period domain.*

The above theorem provides a bridge between the powerful theory of modular forms, used to study the quotient spaces $\mathscr{D}/\Gamma$, and the theory of projective holomorphic symplectic varieties. The interested reader is referred to the excellent recent survey [GHS2] for further reading on this topic.

1.4 The $K3^{[n]}$-type

In section 9 we specialize to the case of varieties X of $K3^{[n]}$-type and review the results of [Ma2, Ma5, Ma7]. We introduce a Hodge theoretic *Torelli data*, consisting of the weight 2 Hodge structure of X and a certain discrete monodromy invariant (Corollary 9.5). We provide explicit computations, for many of the concepts introduced above, in terms of this Torelli data. We enumerate the connected components of the moduli space of marked pairs of $K3^{[n]}$-deformation type (Corollary 9.10). We determine the monodromy group $Mon^2(X)$, as well as a necessary and sufficient condition for an isometry $g : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ to be a parallel transport operator (Theorems 9.1 and 9.8). We provide a numerical characterization of the set of stably prime-exceptional line bundles on X (Theorem 9.17). The latter, combined with the general Theorem 1.5 and Proposition 1.8, determines the closure of the birational Kähler cone of X in terms of its Torelli data.

In section 10 we list a few open problems.

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2 The Global Torelli Theorem

Fix a positive integer $b_2 > 3$ and an even lattice Λ of signature $(3, b_2 - 3)$. Let X be an irreducible holomorphic symplectic manifold, such that $H^2(X,\mathbb{Z})$, endowed with its Beauville-Bogomolov pairing, is isometric to Λ. A *marking* for X is a choice of an isometry $\eta : H^2(X,\mathbb{Z}) \to \Lambda$. Two marked pairs (X_1, η_1), (X_2, η_2) are isomorphic, if there exists an isomorphism $f : X_1 \to X_2$, such that $\eta_1 \circ f^* = \eta_2$. There exists a coarse moduli space $\mathfrak{M}_\Lambda$ parametrizing isomorphism classes of marked pairs [Hu1]. $\mathfrak{M}_\Lambda$ is a smooth complex manifold of dimension $b_2 - 2$, but it is non-Hausdorff.

The *period*, of the marked pair (X,η), is the line $\eta[H^{2,0}(X)]$ considered as a point in the projective space $\mathbb{P}[\Lambda\otimes_{\mathbb{Z}}\mathbb{C}]$. The period lies in the period domain

$$\Omega_\Lambda \;:=\; \{p \,:\, (p,p)=0 \;\text{ and }\; (p,\bar{p})>0\}. \tag{2.1}$$

Ω_Λ is an open subset, in the classical topology, of the quadric in $\mathbb{P}[\Lambda\otimes\mathbb{C}]$ of isotropic lines [Be1]. The period map

$$\begin{aligned} P \,:\, \mathfrak{M}_\Lambda &\longrightarrow \Omega_\Lambda, \\ (X,\eta) &\mapsto \eta[H^{2,0}(X)] \end{aligned} \tag{2.2}$$

is a local isomorphism, by the Local Torelli Theorem [Be1].

Given a point $p\in\Omega_\Lambda$, set $\Lambda^{1,1}(p):=\{\lambda\in\Lambda \,:\, (\lambda,p)=0\}$. Note that $\Lambda^{1,1}(p)$ is a sublattice of Λ and $\Lambda^{1,1}(p)=(0)$, if p does not belong to the countable union of hyperplane sections $\cup_{\lambda\in\Lambda\setminus\{0\}}[\lambda^\perp\cap\Omega_\Lambda]$. Given a marked pair (X,η), we get the isomorphism $H^{1,1}(X,\mathbb{Z})\cong\Lambda^{1,1}(P(X,\eta))$, via the restriction of η.

Definition 2.1 Let X be an irreducible holomorphic symplectic manifold. The cone $\{\alpha\in H^{1,1}(X,\mathbb{R}) \,:\, (\alpha,\alpha)>0\}$ has two connected components. The *positive cone* $\mathscr{C}_X$ is the connected component containing the Kähler cone $\mathscr{K}_X$.

Two points x and y of a topological space M are *inseparable*, if every pair of open subsets U, V, with $x\in U$ and $y\in V$, have a non-empty intersection $U\cap V$. A point $x\in M$ is a *Hausdorff point*, if there does not exist any point $y\in[M\setminus\{x\}]$, such that x and y are inseparable.

Theorem 2.2 *(The Global Torelli Theorem) Fix a connected component $\mathfrak{M}^0_\Lambda$ of $\mathfrak{M}_\Lambda$.*

(1) ([Hu1], Theorem 8.1) The period map P restricts to a surjective holomorphic map $P_0:\mathfrak{M}^0_\Lambda\to\Omega_\Lambda$.

(2) ([Ver2], Theorem 1.16) The fiber $P_0^{-1}(p)$ consists of pairwise inseparable points, for all $p\in\Omega_\Lambda$.

(3) ([Hu1], Theorem 4.3) Let (X_1,η_1) and (X_2,η_2) be two inseparable points of $\mathfrak{M}_\Lambda$. Then X_1 and X_2 are bimeromorphic.

(4) The marked pair (X,η) is a Hausdorff point of $\mathfrak{M}_\Lambda$, if and only if $\mathscr{C}_X=\mathscr{K}_X$.

(5) The fiber $P_0^{-1}(p)$, $p\in\Omega_\Lambda$, consists of a single Hausdorff point, if $\Lambda^{1,1}(p)$ is trivial, or if $\Lambda^{1,1}(p)$ is of rank 1, generated by a class α satisfying $(\alpha,\alpha)\geq 0$.

Proof Part (4) of the theorem is due to Huybrechts and Verbitsky. See Proposition 5.14 for a more general description of the fiber $P_0^{-1}[P_0(X,\eta)]$ in terms of the Kähler-type chamber decomposition of the positive cone $\mathscr{C}_X$, and for further details about part (4).

Part (5): $\mathscr{C}_X = \mathscr{K}_X$, if $H^{1,1}(X,\mathbb{Z})$ is trivial, or if $H^{1,1}(X,\mathbb{Z})$ is of rank 1, generated by a class α of non-negative Beauville-Bogomolov degree, by ([Hu1], Corollaries 5.7 and 7.2). The statement of part (5) now follows from part (4). □

Remark 2.3 Verbitsky states part (2) of Theorem 2.2 for a connected component of the Teichmüller space, but Theorem 1.16 in [Ver2] is a consequence of the two more general Theorems 4.22 and 6.14 in [Ver2], and both the Teichmüller space and the moduli space of marked pairs $\mathfrak{M}_\Lambda$ satisfy the hypothesis of these theorems. A complete proof of part (2) of Theorem 2.2 can be found in Huybrechts excellent Bourbaki seminar paper [Hu6].

3 The Hodge theoretic Torelli Theorem

In section 3.1 we review two theorems of Huybrechts, which relate bimeromorphic maps and parallel-transport operators. The Hodge theoretic Torelli Theorem 1.3 is proven in section 3.2.

3.1 Parallel transport operators between inseparable marked pairs

Let X_1 and X_2 be two irreducible holomorphic symplectic manifolds of dimension $2n$. Denote by π_i the projection from $X_1 \times X_2$ onto X_i, $i = 1,2$. Given a correspondence Z in $X_1 \times X_2$, of pure complex co-dimension $2n+d$, denote by $[Z]$ the cohomology class Poincaré dual to Z and by $[Z]_* : H^*(X_1) \to H^{*+2d}(X_2)$ the homomorphism defined by $[Z]_*\alpha := \pi_{2_*}(\pi_1^*(\alpha) \cup [Z])$. The following are two fundamental results of Huybrechts.

Assume that X_1 and X_2 are bimeromorphic. Denote the graph of a bimeromorphic map by $Z \subset X_1 \times X_2$.

Theorem 3.1 *([Hu2], Corollary 2.7) There exists an effective cycle $\Gamma := Z + \sum Y_j$ in $X_1 \times X_2$, of pure dimension $2n$, with the following properties.*

(1) The correspondence $[\Gamma]_* : H^*(X_1,\mathbb{Z}) \to H^*(X_2,\mathbb{Z})$ *is a parallel-transport operator.*

(2) The image $\pi_i(Y_j)$ *has codimension* ≥ 2 *in* X_i*, for all* j*. In particular, the correspondences* $[\Gamma]_*$ *and* $[Z]_*$ *coincide on* $H^2(X_1,\mathbb{Z})$*.*

Let (X_1,η_1), (X_2,η_2) be two marked pairs corresponding to inseparable points of $\mathfrak{M}_\Lambda$.

Theorem 3.2 *([Hu1], Theorem 4.3 and its proof) There exists an effective cycle* $\Gamma := Z + \sum_j Y_j$ *in* $X_1 \times X_2$*, of pure dimension* $2n$*, satisfying the following conditions.*

(1) Z *is the graph of a bimeromorphic map from* X_1 *to* X_2*.*

(2) The correspondence $[\Gamma]_* : H^*(X_1,\mathbb{Z}) \to H^*(X_2,\mathbb{Z})$ *is a parallel-transport operator. Furthermore, the composition*

$$\eta_2^{-1} \circ \eta_1 : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$$

is equal to the restriction of $[\Gamma]_*$*.*

(3) ([Hu2], Theorem 2.5 and its proof) The codimensions of $\pi_1(Y_j)$ *in* X_1 *and of* $\pi_2(Y_j)$ *in* X_2 *are equal and positive.*

(4) If $\pi_i(Y_j)$ *has codimension* 1*, then it is supported by a uniruled divisor.*

The statement that the isomorphisms $[\Gamma]_*$ in Theorems 3.1 and 3.2 are parallel transport operators is implicit in Huybrechts proofs, so we clarify that point next. In each of the proofs Huybrechts shows that there exist two smooth and proper families $\mathscr{X} \to B$ and $\mathscr{X}' \to B$, over the same one-dimensional disk B, a point b_0 in B, isomorphisms $X_1 \cong \mathscr{X}_{b_0}$ and $X_2 \cong \mathscr{X}'_{b_0}$, and an isomorphism $\tilde{f} : \mathscr{X}_{|B\setminus\{b_0\}} \to \mathscr{X}'_{|B\setminus\{b_0\}}$, compatible with projections to B. The cycle $\Gamma \subset X_1 \times X_2$ is the fiber over b_0 of the closure in $\mathscr{X} \times_B \mathscr{X}'$ of the graph of $\tilde{f}$. Choose a point b_1 in $B \setminus \{b_0\}$ and let γ be a continuous path in B from b_0 to b_1. Let $g_1 : H^*(\mathscr{X}_{b_0},\mathbb{Z}) \to H^*(\mathscr{X}_{b_1},\mathbb{Z})$ and $g_2 : H^*(\mathscr{X}'_{b_0},\mathbb{Z}) \to H^*(\mathscr{X}'_{b_1},\mathbb{Z})$ be the two parallel transport operators along γ. Then the isomorphism $g_2^{-1} \circ g_1 : H^*(\mathscr{X}_{b_0},\mathbb{Z}) \to H^*(\mathscr{X}'_{b_0},\mathbb{Z})$ is induced by the correspondence $[\Gamma]_*$. Furthermore, $g_2^{-1} \circ g_1$ is a parallel transport operator, being a composition of such operators (parallet transport operators form a groupoid, by an argument similar to that used in footnote [3]).

The reader may wonder why the image in X_i of a component Y_j of Γ has codimension ≥ 2 in Theorem 3.1, while the codimension is only ≥ 1 in Theorem 3.2. The reason is that in the proof of Theorem 3.2 one does not have control on the choice of the above mentioned families $\mathscr{X}$ and $\mathscr{X}'$, beyond the condition that $\eta_2 \circ [\Gamma]_* = \eta_1$. In the proof of Theorem 3.1, given a bimeromorphic map $f : X_1 \to X_2$, Huybrechts constructs the above two families $\mathscr{X}$ and $\mathscr{X}'$ in such a way that the following two properties hold. (1) The bimeromorphic map $\tilde{f}$ from $\mathscr{X}$ to $\mathscr{X}'$ restricts to the bimeromorphic map f between the fibers X_1 and X_2 over b_0. (2) $[\Gamma]_*$ restricts to the isometry $f_* : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ (see Theorem 2.5 in [Hu2] and its proof).

3.2 *Proof of the Hodge theoretic Torelli Theorem 1.3*

Proof of part 1: If X and Y are bimeromorphic, then there exists a parallel-transport operator $f : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$, which is an isomorphism of Hodge structures, by Theorem 3.1. Conversely, assume that such f is given. Let $\eta_Y : H^2(Y,\mathbb{Z}) \to \Lambda$ be a marking. Set $\eta_X := \eta_Y \circ f$. The assumption that f is a parallel transport operator implies that (X,η_X) and (Y,η_Y) belong to the same connected component $\mathfrak{M}^0_\Lambda$ of $\mathfrak{M}_\Lambda$. Both have the same period

$$P(X,\eta_X) = \eta_X(H^{2,0}(X)) = \eta_Y(f(H^{2,0}(X))) = \eta_Y(H^{2,0}(Y)) = P(Y,\eta_Y),$$

where the third equality follows from the assumption that f is an isomorphism of Hodge structures. Hence, (X,η_X) and (Y,η_Y) are inseparable points of $\mathfrak{M}^0_\Lambda$, by Theorem 2.2 part 2. X and Y are thus bimeromorphic, by Theorem 2.2 part 3.

Proof of part 2: Let η_X and η_Y be the markings constructed in the proof of part 1. Note that $f = \eta_Y^{-1} \circ \eta_X$. There exists an *effective* correspondence $\Gamma = Z + \sum_{i=1}^N W_i$ of pure dimension $2n$ in $X \times Y$, such that Z is the graph of a bimeromorphic map, W_i is irreducible, but not necessarily reduced, the images of the projections $W_i \to X$, $W_i \to Y$ have positive co-dimensions, and $[\Gamma]_* : H^*(X,\mathbb{Z}) \to H^*(Y,\mathbb{Z})$ is a parallel transport operator, which is equal to f in degree 2, by Theorem 3.2 and the assumption that the two points (X,η_X) and (Y,η_Y) are inseparable.

Assume that $\alpha \in \mathscr{K}_X$ is a Kähler class, such that $f(\alpha)$ is a Kähler class. The relationship between f and Γ yields:

$$f(\alpha) \;=\; [\Gamma]_*(\alpha) \;=\; [Z]_*(\alpha) + \sum_{i=1}^N [W_i]_*(\alpha).$$

Each class $[W_i]_*(\alpha)$ is either zero or a multiple $c_i[D_i]$ of the class of a prime divisor D_i, where c_i is a positive[6] real number.

We prove next that $[W_i]_*(\alpha) = 0$, for $1 \leq i \leq N$. Write $f(\alpha) = [Z]_*(\alpha) + \sum_{i=1}^N c_i[D_i]$, where c_i are all positive real numbers, and D_i is either a prime divisor, or zero. Set $D := \sum_{i=1}^N c_i D_i$. We need to show that all D_i are equal to zero. The Beauville-Bogomolov degree of α satisfies

$$(\alpha,\alpha) = (f(\alpha), f(\alpha)) = ([Z]_*\alpha, [Z]_*\alpha) + 2\sum_{i=1}^N c_i([Z]_*\alpha, [D_i]) + ([D],[D]).$$

The homomorphism $[Z]_*$, induced by the graph of the bimeromorphic map, is an isometry, by [O'G1], Proposition 1.6.2 (also by the stronger Theorem 3.1). Furthermore, if D_i is non-zero, then D_i is the strict transform of a prime divisor D'_i on X, such that $[Z]_*([D'_i]) = [D_i]$. Set $D' := \sum_{i=1}^N c_i D'_i$. We get the equalities

$$([D],[D]) = -2(\alpha,[D']), \tag{3.1}$$

$$[D] = [Z]_*[D'], \tag{3.2}$$

and

$$([D], f(\alpha)) = ([D], [Z]_*\alpha) + ([D],[D]) \overset{(3.2)}{=} ([D'],\alpha) + ([D],[D]) \overset{(3.1)}{=} -(\alpha,[D']).$$

Now $(\alpha,[D'_i])$ is zero, if $D_i = 0$, and positive, if $D_i \neq 0$, since α is a Kähler class. Hence, the right hand side above is ≤ 0. The left hand side is ≥ 0, due to the assumption that the class $f(\alpha)$ is a Kähler class. Hence, $D'_i = 0$, for all i. We conclude that $[W_i]_*(\alpha) = 0$, for $1 \leq i \leq N$, as claimed.

The equality $[Z]_*(\alpha) = f(\alpha)$ was proven above. Consequently, Z is the graph of a bimeromorphic map, which maps a Kähler class to a Kähler class. Hence, Z is the graph of an isomorphism, by a theorem of Fujiki [F]. □

4 Orientation

Let Ω_Λ be the period domain (2.1). Following are two examples, in which spaces arise with two connected components.

(1) Fix a primitive class $h \in \Lambda$, with $(h,h) > 0$. The hyperplane section

[6] The coefficient c_i is positive since Γ is effective and α is a Kähler class.

$$\Omega_{h^\perp} \;:=\; \Omega_\Lambda \cap h^\perp$$

has two connected components.

(2) Let $p \in \Omega_\Lambda$. Set $\Lambda_{\mathbb{R}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ and $\Lambda^{1,1}(p,\mathbb{R}) := \{\lambda \in \Lambda_{\mathbb{R}} : (\lambda, p) = 0\}$. Then the cone $\mathscr{C}'_p := \{\lambda \in \Lambda^{1,1}(p,\mathbb{R}) \ : \ (\lambda,\lambda) > 0\}$ has two connected components.

We recall in this section that a connected component $\mathfrak{M}^0_\Lambda$, of the moduli space of marked pairs, determines a choice of a component of $\Omega_{h^\perp}$ and of $\mathscr{C}'_p$, for all $h \in \Lambda$, with $(h,h) > 0$, and for all $p \in \Omega_\Lambda$. Let us first relate the choice of one of the two components in the two examples above. The relation can be explained in terms of the following larger cone. Set

$$\widetilde{\mathscr{C}_\Lambda} := \{\lambda \in \Lambda_{\mathbb{R}} \ : \ (\lambda,\lambda) > 0\}.$$

A subspace $W \subset \Lambda_{\mathbb{R}}$ is said to be *positive*, if the pairing of $\Lambda_{\mathbb{R}}$ restricts to W as a positive definite pairing.

Lemma 4.1

(1) $H^2(\widetilde{\mathscr{C}_\Lambda},\mathbb{Z})$ is a free abelian group of rank 1.

(2) Let $e \in \Lambda$ be an element with $(e,e) \neq 0$ and $R_e : \Lambda_{\mathbb{R}} \to \Lambda_{\mathbb{R}}$ the reflection given by $R_e(\lambda) = \lambda - \frac{2(e,\lambda)}{(e,e)} e$. Then R_e acts on $H^2(\widetilde{\mathscr{C}_\Lambda},\mathbb{Z})$ by -1, if $(e,e) > 0$, and trivially if $(e,e) < 0$.

(3) Let W be a positive three dimensional subspace of $\Lambda_{\mathbb{R}}$. Then $W \setminus \{0\}$ is a deformation retract of $\widetilde{\mathscr{C}_\Lambda}$.

Proof (3) Set $I := [0,1]$. We need to construct a continuous map $F : \widetilde{\mathscr{C}_\Lambda} \times I \to \widetilde{\mathscr{C}_\Lambda}$ satisfying

$$\begin{aligned} F(\lambda,0) &= \lambda, && \text{for all } \lambda \in \widetilde{\mathscr{C}_\Lambda}, \\ F(\lambda,1) &\in W \setminus \{0\}, && \text{for all } \lambda \in \widetilde{\mathscr{C}_\Lambda}, \\ F(w,t) &= w, && \text{for all } w \in W \setminus \{0\}. \end{aligned}$$

Choose a basis $\{e_1, e_2, e_3, \dots, e_{b_2}\}$ of $\Lambda_{\mathbb{R}}$, so that $\{e_1,e_2,e_3\}$ is a basis of W, and for $\lambda = \sum_{i=1}^{b_2} x_i e_i$, we have $(\lambda,\lambda) = x_1^2 + x_2^2 + x_3^2 - \sum_{i=4}^{b_2} x_i^2$. Then $\widetilde{\mathscr{C}_\Lambda}$ consists of λ satisfying $x_1^2 + x_2^2 + x_3^2 > \sum_{i=4}^{b_2} x_i^2$. Set $F\left(\sum_{i=1}^{b_2} x_i e_i, t\right) = \sum_{i=1}^{3} x_i e_i + (1-t)\sum_{i=4}^{b_2} x_i e_i$. Then F has the above properties of a deformation retract of $\widetilde{\mathscr{C}_\Lambda}$ onto $W \setminus \{0\}$.

Part (1) follows immediately from part (3).

(2) If $(e,e) > 0$, we can choose a positive 3 dimensional subspace W containing e, and if $(e,e) < 0$ we can choose W to be orthogonal to e. Then $W \setminus \{0\}$ is R_e invariant and R_e acts as stated on $H^2(W \setminus \{0\}, \mathbb{Z})$, hence also on $H^2(\widetilde{\mathscr{C}}_\Lambda, \mathbb{Z})$, by part (3). □

The character $H^2(\widetilde{\mathscr{C}}_\Lambda, \mathbb{Z})$ of $O(\Lambda)$ is known as the *spinor norm*.

A point $p \in \Omega_{h^\perp}$ determines the three dimensional positive definite subspace $W_p := \mathrm{Re}(p) \oplus \mathrm{Im}(p) \oplus \mathrm{span}_\mathbb{R}\{h\}$, which comes with an orientation associated to the basis $\{\mathrm{Re}(\sigma), \mathrm{Im}(\sigma), h\}$, for some choice of a non-zero element $\sigma \in p \subset \Lambda_\mathbb{C}$. The orientation of the basis is independent of the choice of σ. Consequently, an element $p \in \Omega_{h^\perp}$ determines a generator of $H^2(\widetilde{\mathscr{C}}_\Lambda, \mathbb{Z})$. The two components of $\Omega_{h^\perp}$ are distinguished by the two generators of the rank 1 free abelian group $H^2(\widetilde{\mathscr{C}}_\Lambda, \mathbb{Z})$. We refer to each of the two generators as an *orientation class* of the cone $\widetilde{\mathscr{C}}_\Lambda$.

A point $\lambda \in \mathscr{C}'_p$ determines an orientation of $\widetilde{\mathscr{C}}_\Lambda$ as follows. Choose a class $\sigma \in p$. Again we get the three dimensional positive definite subspace $W_\lambda := \mathrm{Re}(p) \oplus \mathrm{Im}(p) \oplus \mathrm{span}_\mathbb{R}\{\lambda\}$, which comes with an orientation associated to the basis $\{\mathrm{Re}(\sigma), \mathrm{Im}(\sigma), \lambda\}$. Consequently, λ determines an orientation of $\widetilde{\mathscr{C}}_\Lambda$. The orientation remains the same as λ varies in a connected component of $\mathscr{C}'_p$. Hence, a connected component of $\mathscr{C}'_p$ determines an orientation of $\widetilde{\mathscr{C}}_\Lambda$.

Let X be an irreducible holomorphic symplectic manifold. Recall that the positive cone $\mathscr{C}_X \subset H^{1,1}(X, \mathbb{R})$ is the distinguished connected component of the cone $\mathscr{C}'_X := \{\lambda \in H^{1,1}(X, \mathbb{R}) \ : \ (\lambda, \lambda) > 0\}$, which contains the Kähler cone (Definition 2.1). Denote by $\widetilde{\mathscr{C}}_X$ the positive cone in $H^2(X, \mathbb{R})$. We conclude that $\widetilde{\mathscr{C}}_X$ comes with a distinguished orientation.

Let $\mathfrak{M}^0_\Lambda$ be a connected component of the moduli space of marked pairs and $P_0 : \mathfrak{M}^0_\Lambda \to \Omega_\Lambda$ the period map. A marked pair (X, η) in $\mathfrak{M}^0_\Lambda$ determines an orientation of $\widetilde{\mathscr{C}}_\Lambda$, via the isomorphism $\widetilde{C}_X \cong \widetilde{\mathscr{C}}_\Lambda$ induced by the marking η. This orientation of $\widetilde{\mathscr{C}}_\Lambda$ is constant throughout the connected component $\mathfrak{M}^0_\Lambda$. In particular, for each class $h \in \Lambda$, with $(h,h) > 0$, we get a choice of a connected component

$$\Omega^+_{h^\perp} \tag{4.1}$$

of $\Omega_{h^\perp}$, compatible with the orientation of $\widetilde{\mathscr{C}}_\Lambda$ induced by $\mathfrak{M}^0_\Lambda$.

Let $\mathrm{Orient}(\Lambda)$ be the set of two orientations of the positive cone $\widetilde{\mathscr{C}}_\Lambda$. Let

$$\mathrm{orient} \ : \ \mathfrak{M}_\Lambda \quad \to \quad \mathrm{Orient}(\Lambda) \tag{4.2}$$

be the natural map constructed above.

5 A modular description of each fiber of the period map

We provide a modular description of the fiber of the period map $\mathfrak{M}_\Lambda^0 \to \Omega_\Lambda$ from a connected component $\mathfrak{M}_\Lambda^0$ of the moduli space of marked pairs (Theorem 5.16). Throughout this section X is an irreducible holomorphic symplectic manifold, which need not be projective.

5.1 Exceptional divisors

A reduced and irreducible effective divisor $D \subset X$ will be called a *prime* divisor.

Definition 5.1

(1) A set $\{E_1, \ldots, E_r\}$ of prime divisors is *exceptional*, if and only if its Gram matrix $\left(([E_i],[E_j])\right)_{ij}$ is negative definite.

(2) An effective divisor E is *exceptional*, if the support of E is an exceptional set of prime divisors.

Definition 5.2 The *fundamental exceptional chamber* $\mathscr{F}\mathscr{E}_X$ is the cone of classes α, such that $\alpha \in \mathscr{C}_X$, and $(\alpha,[E]) > 0$, for every prime exceptional divisor E.

5.1.1 The fundamental exceptional chamber versus the birational Kähler cone

Huybrechts and Boucksom stated an important result (Theorem 5.4 below) in terms of another chamber, which we introduce next.

Definition 5.3 ([Bou2], Section 4.2.2)

(1) A *rational effective* 1*-cycle* C is a linear combination, with positive integral coefficients, of irreducible rational curves on X.

(2) A *uniruled divisor* D is an effective divisor each of which irreducible components D_i is covered by rational curves.

(3) The *fundamental uniruled chamber* $\mathscr{F}\mathscr{U}_X$ is the subset of $\mathscr{C}_X$ consisting of classes $\alpha \in \mathscr{C}_X$, such that $(\alpha, D) > 0$, for every non-zero uniruled divisor D.

(4) The *birational Kähler cone* $\mathcal{BK}_X$ of X is the union of $f^*\mathcal{K}_Y$, as f ranges over all bimeromorphic maps $f : X \to Y$ to an irreducible holomorphic symplectic manifold Y.

Note that the birational Kähler cone is not convex in general.

Theorem 5.4 *([Hu2] and [Bou2], Theorem 4.3)*

(1) The Kähler cone $\mathcal{K}_X$ is equal to the subset of $\mathcal{C}_X$ consisting of classes $\alpha \in \mathcal{C}_X$, such that $\int_C \alpha > 0$, for every non-zero rational effective 1-cycle C.

(2) Let $\alpha \in \mathcal{C}_X$ be a class, such that $\int_C \alpha \neq 0$, for every rational 1-cycle. Then α belongs to $\mathcal{FU}_X$, if and only if α belongs to the birational Kähler cone $\mathcal{BK}_X$.

(3) ([Bou2], Theorem 4.3 part ii, and [Hu1], Corollary 5.2) Let $\alpha \in \mathcal{C}_X$ be a class, which does not belong to $\mathcal{FU}_X$. Assume that $\int_C \alpha \neq 0$, for every rational 1-cycle. Then there exists an irreducible holomorphic symplectic manifold Y, and a bimeromorphic map $f : X \to Y$, such that $f_*(\alpha) = \beta + D'$, where β is a Kähler class on Y and D' is a non-zero linear combination of finitely many uniruled reduced and irreducible divisors with positive real coefficients.

Remark 5.5 Let X be an irreducible holomorphic symplectic manifold. Part (2) of the theorem asserts that if a class α satisfies the assumptions stated, then α is contained in $\mathcal{FU}_X$, if and only if it is contained in $\mathcal{BK}_X$. The 'only if' direction of part (2) is stated in ([Bou2], Theorem 4.3). The 'if' part is the obvious direction. Indeed, let $f : X \to Y$ be a birational map, such that $f_*(\alpha)$ is a Kähler class on Y. Let D be an effective uniruled reduced and irreducible divisor in X, and D' its strict transform in Y. We have $([D], \alpha) = ([D'], f_*(\alpha)) > 0$. Hence, α is in the fundamental uniruled chamber.

Let $\overline{\mathcal{BK}}_X$ be the closure of the birational Kähler cone $\mathcal{BK}_X$ in $\mathcal{C}_X$.

Proposition 5.6 *The following inclusions and equality hold:*

$$\mathcal{BK}_X \subset \mathcal{FU}_X = \mathcal{FE}_X \subset \overline{\mathcal{BK}}_X.$$

Proof An exceptional divisor is uniruled, by ([Bou2], Proposition 4.7). The inclusion $\mathcal{FU}_X \subset \mathcal{FE}_X$ follows. We prove next the inclusion $\mathcal{FE}_X \subset \mathcal{FU}_X$. Let α be a class in $\mathcal{FE}_X$ and D a prime uniruled divisor. If $[D]$ belongs to the closure $\overline{\mathcal{C}}_X$ of the positive cone, then $(\alpha, [D]) > 0$, since α belongs to $\mathcal{C}_X$. Otherwise, $[D]$ is a prime exceptional divisor, and so $(\alpha, [D]) > 0$. The inclusion $\mathcal{FE}_X \subset \mathcal{FU}_X$ follows.

The inclusion $\mathscr{BK}_X \subset \mathscr{FU}_X$ follows from the 'if' direction of Theorem 5.4 part 2, and the inclusion $\mathscr{FE}_X \subset \overline{\mathscr{BK}}_X$ follows from the 'only if' direction. □

The notation $\mathscr{FE}_X$ will replace $\mathscr{FU}_X$ from now on, in view of Proposition 5.6. A class $\alpha \in \mathscr{C}_X$ is said to be *very general*, if $\alpha^\perp \cap H^{1,1}(X,\mathbb{Z}) = 0$.

Corollary 5.7 *Let X_1 and X_2 be irreducible holomorphic symplectic manifolds, $g : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ a parallel transport operator, which is an isomorphism of Hodge structures, and $\alpha_1 \in \mathscr{FE}_{X_1}$ a very general class. Then $g(\alpha_1)$ belongs to $\mathscr{FE}_{X_2}$, if and only if there exists a bimeromorphic map $f : X_1 \to X_2$, such that $g = f_*$.*

Proof The 'if' part is clear, since f_* induces a bijection between the sets of exceptional divisors on X_i, $i = 1,2$. Set $\alpha_2 := g(\alpha_1)$. There exist irreducible holomorphic symplectic manifolds Y_i and bimeromorphic maps $f_i : X_i \to Y_i$, such that $f_{i_*}(\alpha_i)$ is a Kähler class on Y_i, by part (2) of Theorem 5.4. The homomorphisms $f_{i_*} : H^2(X_i,\mathbb{Z}) \to H^2(Y_i,\mathbb{Z})$ are parallel transport operators, by Theorem 3.1. Thus $(f_2^{-1})^* \circ g \circ f_1^* : H^2(Y_1,\mathbb{Z}) \to H^2(Y_2,\mathbb{Z})$ is a parallel transport operator and a Hodge-isometry, mapping the Kähler class $f_{1_*}(\alpha_1)$ to the the Kähler class $f_{2_*}(\alpha_2)$. Hence, there exists an isomorphism $h : Y_1 \to Y_2$, such that $h_* = (f_2^{-1})^* \circ g \circ f_1^*$, by Theorem 1.3. Thus, $g = [(f_2)^{-1} h f_1]_*$. □

5.1.2 The divisorial Zariski decomposition

The following fundamental result of Bouksom will be needed in section 6.2. The *effective cone* of X is the cone in $H^{1,1}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ generated by the classes of effective divisors. The *algebraic pseudo-effective cone* $\mathscr{P}\mathrm{eff}_X$ is the closure of the effective cone. Boucksom defines a larger transcendental analogue, a cone in $H^{1,1}(X,\mathbb{R})$, which he calls the *pseudo-effective cone* ([Bou2], section 2.3). We will not need the precise definition, but only the fact that the pseudo-effective cone contains $\mathscr{C}_X$ ([Bou2], Theorem 4.3 part (i)). The sum $\mathscr{C}_X + \mathscr{P}\mathrm{eff}_X$ is thus a sub-cone of Boucksom's pseudo-effective cone in $H^{1,1}(X,\mathbb{R})$. Denote by $\overline{\mathscr{FE}}_X$ the closure of the fundamental exceptional chamber in $H^{1,1}(X,\mathbb{R})$.

Theorem 5.8

(1) ([Bou2], Theorem 4.3 part (i), Proposition 4.4, and Theorem 4.8). Let X be an irreducible holomorphic symplectic manifold and α a class in $\mathscr{C}_X + \mathscr{P}\mathrm{eff}_X$. Then there exists a unique decomposition

$$\alpha \;=\; P(\alpha)+N(\alpha),$$

where $(P(\alpha),N(\alpha))=0$*,* $P(\alpha)$ *belongs to* $\overline{\mathscr{F}\mathscr{E}}_X$*, and* $N(\alpha)$ *is an exceptional* $\mathbb{R}$*-divisor.*

(2) *([Bou2], Corollary 4.11). Let L be a line bundle with* $c_1(L)\in\mathscr{C}_X+\mathscr{P}\mathrm{eff}_X$*. Set* $\alpha:=c_1(L)$*. Then the classes* $P(\alpha)$ *and* $N(\alpha)$ *correspond to* $\mathbb{Q}$*-divisors classes, which we denote by* $P(\alpha)$ *and* $N(\alpha)$ *as well. Furthermore, the homomorphism*

$$H^0\left(X,\mathscr{O}_X(kP(\alpha))\right)\to H^0(X,L^k)$$

is surjective, for every non-negative integer k, such that $kP(\alpha)$ *is an integral class.*

Remark 5.9 The class $P(\alpha)$ is stated as a class in the *modified nef cone* in ([Bou2], Theorem 4.8), but the modified nef cone is equal to the closure of the birational Kähler cone, by ([Bou2], Proposition 4.4), and hence also to $\overline{\mathscr{F}\mathscr{E}}_X$.

Part (2) of the above Theorem implies that the exceptional divisor $N(kc_1(L))$ is the fixed part of the linear system $|L^k|$. In particular, if $c_1(L)=N(c_1(L))$, then the linear system $|L^k|$ is either empty, or consists of a single exceptional divisor. Exceptional divisors are thus rigid.

5.2 A Kähler-type chamber decomposition of the positive cone

Let X be an irreducible holomorphic symplectic manifold. Denote the subgroup of $Mon^2(X)$ preserving the weight 2 Hodge structure by $Mon^2_{Hdg}(X)$. Note that the positive cone $\mathscr{C}_X$ is invariant under $Mon^2_{Hdg}(X)$, since the orientation class of $\widetilde{\mathscr{C}}_X$ is invariant under the whole monodromy group $Mon^2(X)$ (see section 4).

Definition 5.10

(1) An *exceptional chamber* of the positive cone $\mathscr{C}_X$ is a subset of the form $g[\mathscr{F}\mathscr{E}_X]$, $g\in Mon^2_{Hdg}(X)$.

(2) A *Kähler-type chamber* of the positive cone $\mathscr{C}_X$ is a subset of the form $g[f^*(\mathscr{K}_Y)]$, where $g\in Mon^2_{Hdg}(X)$, and $f:X\to Y$ is a bimeromorphic map to an irreducible holomorphic symplectic manifold Y.

Let $Mon^2_{Bir}(X) \subset Mon^2_{Hdg}(X)$ be the subgroup of monodromy operators induced by bimeromorphic maps from X to itself (see Theorem 3.1).

Lemma 5.11

(1) Every very general class $\alpha \in \mathscr{C}_X$ belongs to some Kähler-type chamber.

(2) Every Kähler-type chamber is contained in some exceptional chamber.

(3) If two Kähler-type chambers intersect, then they are equal.

(4) If two exceptional chambers $g_1[\mathscr{F}\mathscr{E}_X]$ and $g_2[\mathscr{F}\mathscr{E}_X]$ contain a common very general class α, then they are equal.

(5) $Mon^2_{Hdg}(X)$ acts transitively on the set of exceptional chambers.

(6) The subgroup of $Mon^2_{Hdg}(X)$ stabilizing $\mathscr{F}\mathscr{E}_X$ is equal to $Mon^2_{Bir}(X)$.

Proof Part (1): There exists an irreducible holomorphic symplectic manifold $\widetilde{X}$ and a correspondence $\Gamma := Z + \sum_i Y_i$ in $X \times \widetilde{X}$, such that Z is the graph of a bimeromorphic map $f : X \to \widetilde{X}$, the restriction $g : H^2(X,\mathbb{Z}) \to H^2(\widetilde{X},\mathbb{Z})$ of $[\Gamma]_*$ is a parallel transport operator, and $g(\alpha)$ is a Kähler class of $\widetilde{X}$, by ([Hu1], Corollary 5.2). Set $h := f^* \circ g$. Then h belongs to $Mon^2_{Hdg}(X)$, by Theorem 3.1, $h(\alpha) = (f^* \circ g)(\alpha)$ belongs to $f^*\mathscr{K}_{\widetilde{X}}$, and $f^*\mathscr{K}_{\widetilde{X}}$ is a Kähler-type chamber, by Definition 5.3. Consequently, $h^{-1}(f^*\mathscr{K}_{\widetilde{X}})$ is a Kähler-type chamber containing α.

Part (2): Let Ch be the Kähler-type chamber $g[f^*(\mathscr{K}_Y)]$, where f, g, and Y are as in Definition 5.10. Then $f^*(\mathscr{F}\mathscr{E}_Y) = \mathscr{F}\mathscr{E}_X$, by Corollary 5.7, and so Ch is contained in the exceptional chamber $g[\mathscr{F}\mathscr{E}_X]$.

Part (3): Let Y_i be irreducible holomorphic symplectic manifolds, $f_i : X \to Y_i$ bimeromorphic maps, $g_i \in Mon^2_{Hdg}(X)$, $i = 1,2$, and α a class in $g_1[f_1^*(\mathscr{K}_{Y_1})] \cap g_2[f_2^*(\mathscr{K}_{Y_2})]$. The composition $\varphi := f_{2_*} \circ g_2^{-1} \circ g_1 \circ f_1^* : H^2(Y_1,\mathbb{Z}) \to H^2(Y_2,\mathbb{Z})$ is a parallel-transport operator, which maps the Kähler class $f_{1_*}(g_1^{-1}(\alpha))$ to the Kähler class $f_{2_*}(g_2^{-1}(\alpha))$. Hence, φ is induced by an isomorphism $\tilde{\varphi} : Y_1 \to Y_2$, by Theorem 1.3. We get the equality $g_1^{-1}g_2f_2^*(\mathscr{K}_{Y_2}) = f_1^*\tilde{\varphi}^*(\mathscr{K}_{Y_2}) = f_1^*(\mathscr{K}_{Y_1})$. Consequently, $g_1[f_1^*(\mathscr{K}_{Y_1})] = g_2[f_2^*(\mathscr{K}_{Y_2})]$.

Part (4): Set $g := g_2^{-1}g_1$ and $\beta := g_2^{-1}(\alpha)$. Then β belongs to the intersection $g[\mathscr{F}\mathscr{E}_X] \cap \mathscr{F}\mathscr{E}_X$. So $g^{-1}(\beta)$ and β both belong to $\mathscr{F}\mathscr{E}_X$ and g maps the former to the latter. Hence, g is induced by a birational map from X to itself, by Corollary 5.7. Thus, $g[\mathscr{F}\mathscr{E}_X] = \mathscr{F}\mathscr{E}_X$ and so $g_1[\mathscr{F}\mathscr{E}_X] = g_2[\mathscr{F}\mathscr{E}_X]$.

Part (5): The action is transitive, by definition.

Part (6) is an immediate consequence of Corollary 5.7. □

Lemma 5.12 *Let X_1 and X_2 be irreducible holomorphic symplectic manifolds and $g : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ a parallel transport operator, which is an isomorphism of Hodge structures.*

(1) g maps each exceptional chamber in $\mathscr{C}_{X_1}$ onto an exceptional chamber in $\mathscr{C}_{X_2}$.

(2) g maps each Kähler-type chamber in $\mathscr{C}_{X_1}$ onto a Kähler-type chamber in $\mathscr{C}_{X_2}$.

Proof There exists a bimeromorphic map $h : X_1 \to X_2$, by Theorem 1.3. The homomorphism $h_* : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ is a parallel transport operator, and an isomorphism of Hodge structures, by Theorem 3.1.

Part (1): Let f be an element of $Mon^2_{Hdg}(X_1)$. We need to show that $g(f[\mathscr{F}\mathscr{E}_{X_1}])$ is an exceptional chamber in $\mathscr{C}_{X_2}$. Indeed, we have the equalities

$$g(f[\mathscr{F}\mathscr{E}_{X_1}]) = (gfh^*)\{h_*[\mathscr{F}\mathscr{E}_{X_1}]\} = (gfh^*)[\mathscr{F}\mathscr{E}_{X_2}],$$

and gfh^* belongs to $Mon^2_{Hdg}(X_2)$.

Part (2): Any Kähler-type chamber of $\mathscr{C}_{X_1}$ is of the form $f[\tilde{h}^*(\mathscr{K}_{Y_1})]$, where $\tilde{h} : X_1 \to Y_1$ is a bimeromorphic map to an irreducible holomorphic symplectic manifold Y_1, and f is an element of $Mon^2_{Hdg}(X_1)$. We have the equality

$$gf[\tilde{h}^*(\mathscr{K}_{Y_1})] \quad = \quad (gfh^*)\left\{(h\tilde{h}^{-1})_*(\mathscr{K}_{Y_1})\right\},$$

$(h\tilde{h}^{-1})_*(\mathscr{K}_{Y_1})$ is a Kähler-type chamber of X_2 and gfh^* belongs to $Mon^2_{Hdg}(X_2)$, by Theorem 3.1. Thus $gf[\tilde{h}^*(\mathscr{K}_{Y_1})]$ is a Kähler-type chamber of X_2. □

Corollary 5.13 *Let (X_1,η_1), (X_2,η_2) be two inseparable points in $\mathfrak{M}^0_\Lambda$.*

(1) The composition $\eta_2^{-1} \circ \eta_1$ maps each Kähler-type chamber in $\mathscr{C}_{X_1}$ onto a Kähler-type chamber in $\mathscr{C}_{X_2}$. Similarly, $\eta_2^{-1} \circ \eta_1$ maps each exceptional chamber in $\mathscr{C}_{X_1}$ onto an exceptional chamber in $\mathscr{C}_{X_2}$.

(2) $(\eta_2^{-1} \circ \eta_1)(\mathscr{F}\mathscr{E}_{X_1}) = \mathscr{F}\mathscr{E}_{X_2}$, if and only if there exists a bimeromorphic map f from X_1 to X_2, such that $\eta_2^{-1} \circ \eta_1 = f_*$.

Proof The composition $\eta_2^{-1} \circ \eta_1$ is a parallel-transport operator, and a Hodge-isometry, by Theorem 3.2 part 2. Part (1) follows from Lemma 5.12. Part (2) follows from Corollary 5.7. □

5.3 $\mathfrak{M}_\Lambda$ as the moduli space of Kähler-type chambers

Consider the period map $P_0 : \mathfrak{M}^0_\Lambda \to \Omega_\Lambda$ from the connected component $\mathfrak{M}^0_\Lambda$ containing the isomorphism class of the marked pair (X,η). Denote by $\mathscr{KT}(X)$ the set of Kähler-type chambers in $\mathscr{C}_X$. Let

$$\rho \ : \ P_0^{-1}[P_0(X,\eta)] \ \longrightarrow \ \mathscr{KT}(X) \tag{5.1}$$

be the map given by $\rho(\widetilde{X},\tilde{\eta}) = (\eta^{-1}\tilde{\eta})(\mathscr{K}_{\widetilde{X}})$. The map ρ is well defined, by Corollary 5.13. $Mon^2_{Hdg}(X)$ acts on $\mathscr{KT}(X)$, by Lemma 5.12.

Note that each period $P(X,\eta) \in \Omega_\Lambda$ is invariant under the subgroup

$$Mon^2_{Hdg}(X)^\eta \ := \ \{\eta g \eta^{-1} \ : \ g \in Mon^2_{Hdg}(X)\} \tag{5.2}$$

of $O(\Lambda)$. Consequently, $Mon^2_{Hdg}(X)$ acts on the fiber $P_0^{-1}[P_0(X,\eta)]$ of the period map by

$$g(\widetilde{X},\tilde{\eta}) \ := \ (\widetilde{X}, \eta g \eta^{-1}\tilde{\eta}).$$

Proposition 5.14

(1) The map ρ is a $Mon^2_{Hdg}(X)$-equivariant bijection.

(2) The marked pair (X,η) is a Hausdorff point of $\mathfrak{M}_\Lambda$, if and only if $\mathscr{C}_X = \mathscr{K}_X$.

(3) ([Hu1], Corollaries 5.7 and 7.2) $\mathscr{C}_X = \mathscr{K}_X$, if $H^{1,1}(X,\mathbb{Z})$ is trivial, or if $H^{1,1}(X,\mathbb{Z})$ is of rank 1, generated by a class α of non-negative Beauville-Bogomolov degree.

Proof Part (1): Assume that $\rho(X_1,\eta_1) = \rho(X_2,\eta_2)$. Then $\eta_2^{-1}\eta_1(\mathscr{K}_{X_1}) = \mathscr{K}_{X_2}$. Hence, $\eta_2^{-1}\eta_1 = f_*$, for an isomorphism $f : X_1 \to X_2$, by Theorem 1.3. Thus, (X_1,η_1) and (X_2,η_2) are isomorphic, and ρ is injective.

Given a Kähler-type chamber Ch in $\mathscr{C}_X$ and a very general class α in Ch, there exists an element $g \in Mon^2_{Hdg}(X)$, such that $g(\alpha)$ belongs to $\mathscr{FE}_X$, by Lemma 5.11 part 5. There exists an irreducible holomorphic symplectic manifold Y and a bimeromorphic map $h : X \to Y$, such that $h_*(g(\alpha))$ belongs to $\mathscr{K}_Y$, by Theorem 5.4 part 2. Thus, $(h_* \circ g)(Ch) = \mathscr{K}_Y$, by Lemma 5.12. We conclude that $\rho(Y, \eta \circ g^{-1} \circ h^*) = g^{-1}h^*(\mathscr{K}_Y) = Ch$ and ρ is surjective.

Part (2) follows from part (1). □

Fix a connected component $\mathfrak{M}^0_\Lambda$ of the moduli space of marked pairs. We get the following modular description of the fiber $P_0^{-1}(p)$ in terms of the period p. Set

$\Lambda^{1,1}(p,\mathbb{R}) := \{\lambda \in \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \ : \ (\lambda, p) = 0\}$. Let $\mathscr{C}_p$ be the connected component, of the cone $\mathscr{C}'_p$ in $\Lambda^{1,1}(p,\mathbb{R})$, which is compatible with the orientation of the positive cone $\widetilde{\mathscr{C}}_\Lambda$ determined by $\mathfrak{M}^0_\Lambda$ (see section 4).

Definition 5.15 A *Kähler-type chamber* of $\mathscr{C}_p$ is a subset of the form $\eta(Ch) \subset \mathscr{C}_p$, where (X,η) is a marked pair $\mathfrak{M}^0_\Lambda$ and $Ch \subset \mathscr{C}_X$ is a Kähler-type chamber of X.

Denote by $\mathscr{K}\mathscr{T}(p)$ the set of Kähler-type chambers in $\mathscr{C}_p$. The map

$$\eta \ : \ \mathscr{K}\mathscr{T}(X) \ \longrightarrow \ \mathscr{K}\mathscr{T}(p),$$

sending a Kähler-type chamber $Ch \in \mathscr{K}\mathscr{T}(X)$ to $\eta(Ch)$, is a bijection, for every marked pair (X,η) in the fiber $P_0^{-1}(p)$, by Corollary 5.13 and Proposition 5.14. $Mon^2_{Hdg}(X)^\eta$, given in equation (5.2), is the same subgroup of $O(\Lambda)$, for all $(X,\eta) \in P_0^{-1}(p)$, and we denote it by $Mon^2_{Hdg}(p)$. The following statement is an immediate corollary of Proposition 5.14.

Theorem 5.16 *The map*

$$\rho \ : \ P_0^{-1}(p) \ \longrightarrow \ \mathscr{K}\mathscr{T}(p),$$

given by $\rho(X,\eta) := \eta(\mathscr{K}_X)$, is a $Mon^2_{Hdg}(p)$-equivariant bijection.

Remark 5.17 Compare Theorem 5.16 with the more detailed analogue for $K3$ surfaces, which is provided in ([LP], Theorem 10.5). Ideally, one would like to have a description of the set $\mathscr{K}\mathscr{T}(p)$, depending only on the period p, the deformation type of X, and possibly some additional discrete monodromy invariant of X (see the invariant ι_X introduced in Corollary 9.5). Such a description would depend on the determination of the Kähler-type chambers in $\mathscr{C}_X$. In particular, it requires a determination of the Kähler cone of an irreducible holomorphic symplectic variety, in terms of the Hodge structure of $H^2(X,\mathbb{Z})$, the Beauville-Bogomolov pairing, and the discrete monodromy invariants of X. The determination of the Kähler cone $\mathscr{K}_X$ in terms of such data is a very difficult problem addressed in a sequence of papers of Hassett and Tschinkel [HT1, HT2, HT3, HT4]. Precise conjectures for the determination of the Kähler cones in the $K3^{[n]}$-type, for all n, and for generalized Kummer fourfolds, are provided in [HT4], Conjectures 1.2 and 1.4. The determination of the birational Kähler cone, in terms of such data, is the subject of section 6.

6 $Mon^2_{Hdg}(X)$ is generated by reflections and $Mon^2_{Bir}(X)$

Throughout this section X denotes a *projective* irreducible holomorphic symplectic manifold. Under the projectivity assumption, we can define a subgroup W_{Exc} of the Hodge-monodromy group $Mon^2_{Hdg}(X)$, which is generated by reflections with respect to classes of prime exceptional divisors (Definition 6.8 and Theorem 6.18 part 3). The fundemental exceptional chamber $\mathcal{FE}_X$, introduced in Definition 5.2, is the interior of a fundamental domain for the action of the reflection group W_{Exc} on the positive cone $\mathcal{C}_X$. Significant regularity properties follow from this description of $\mathcal{FE}_X$ (Theorem 6.17). We prove also that W_{Exc} is a normal subgroup of $Mon^2_{Hdg}(X)$ and the latter is a semi-direct-product of W_{Exc} and $Mon^2_{Bir}(X)$ (Theorem 6.18). A weak version of Morrison's movable cone conjecture follows from the above results in the special case of irreducible holomorphic symplectic manifolds (Theorems 1.7 and 6.25).

6.1 Reflections

Let X be a projective irreducible holomorphic symplectic manifold of dimension $2n$ and $E \subset X$ a prime exceptional divisor (Definition 5.1).

Proposition 6.1 *([Dr], Proposition 1.4) There exists a sequence of flops of X, resulting in a smooth birational model X' of X, such that the strict transform E' of E in X' is contractible via a projective birational morphism $\pi : X' \to Y$ onto a normal projective variety Y. The exceptional locus of π is equal to the support of E'.*

Identify $H^2(X,\mathbb{Q})^*$ with $H_2(X,\mathbb{Q})$. Set

$$[E]^\vee \quad := \quad \frac{-2([E],\bullet)}{([E],[E])} \;\in\; H_2(X,\mathbb{Q}).$$

Proposition 6.2 *([Ma7], Corollary 3.6 part 1).*

(1) There exists a Zariski dense open subset $E^0 \subset E$ and a proper holomorphic fibration $\pi : E^0 \to B$, onto a smooth holomorphic symplectic variety of dimension $2n-2$, with the following property. The class $[E]^\vee$ is the class of a generic fiber of π. The generic fiber is either a smooth rational curve, or the union of two homologous smooth rational curves meeting at one point non-tangentially. In par-

ticular, the class $[E]^\vee$ *is integral, as is the reflection* $R_E : H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$, *given by* $R_E(x) = x + (x,[E]^\vee)[E]$.

(2) *The reflection* R_E *belongs to* $Mon^2_{Hdg}(X)$.

Remark 6.3

(1) The proof of Proposition 6.2 relies heavily on Druel's result stated above in Proposition 6.1. The fact that $R_{E'}$ belongs to $Mon^2_{Hdg}(X')$ was proven earlier in ([Ma6], Theorem 1.4), using fundamental work of Namikawa [Nam1] (see [Nam3] for an alternative proof). The author does not know if the analogue of Proposition 6.1 holds for a non-projective irreducible holomorphic symplectic manifold X as well. This is the reason for the projectivity assumption throughout section 6.

(2) The variety B in part (1) of the proposition is an étale cover of a Zariski open subset of the image of E' in Y ([Nam1], section 1.8).

6.2 Stably prime-exceptional line bundles

Let X be an irreducible holomorphic symplectic manifold. Denote by $Def(X)$ the local Kuranishi deformation space of X and let $0 \in Def(X)$ be the special point corresponding to X. Let L be a line bundle on X. Set $\Lambda := H^2(X,\mathbb{Z})$. The period map $P : Def(X) \to \Omega_\Lambda$ embeds $Def(X)$ as an open analytic subset of the period domain Ω_Λ and the intersection $Def(X,L) := Def(X) \cap c_1(L)^\perp$ is the Kuranishi deformation space of the pair (X,L), i.e., it consists of deformations of the complex structure of X along which $c_1(L)$ remains of type $(1,1)$. We assume that both $Def(X)$ and the intersection $Def(X,L)$ are simply connected, possibly after replacing $Def(X)$ by a smaller open neighborhood of 0 in the Kuranishi deformation space, which we denote again by $Def(X)$.

Let $\pi : \mathscr{X} \to Def(X)$ be the universal family and denote by X_t the fiber of π over $t \in Def(X)$. Denote by ℓ the flat section of the local system $R^2\pi_*\mathbb{Z}$ through $c_1(L)$ and let $\ell_t \in H^{1,1}(X_t,\mathbb{Z})$ be its value at $t \in Def(X,L)$. Let L_t be the line bundle on X_t with $c_1(L_t) = \ell_t$.

Definition 6.4 A line bundle $L \in \mathrm{Pic}(X)$ is called *stably prime-exceptional*, if there exists a closed analytic subset $Z \subset Def(X,L)$, of positive codimension, such that the linear system $|L_t|$ consists of a prime exceptional divisor E_t, for all $t \in [Def(X,L) \setminus Z]$.

Note that a stably prime-exceptional line bundle L is effective, by the semi-continuity theorem. Furthermore, if we set $\ell := c_1(L)$ and define the reflection $R_\ell(\alpha) := \alpha - 2\frac{(\alpha,\ell)}{(\ell,\ell)}\ell$, then R_ℓ belongs to $Mon^2_{Hdg}(X)$.

Remark 6.5 Note that the linear system $|L|$, of a stably prime-exceptional line bundle L, may have positive dimension, if the Zariski decomposition of Theorem 5.8 is non-trivial. Even if $|L|$ consists of a single exceptional divisor, it may be reducible or non-reduced, i.e., the special point 0 may belong to the closed analytic subset Z in Definition 6.4.

Proposition 6.6 *Let E be a prime exceptional divisor on a projective irreducible holomorphic symplectic manifold X.*

(1) ([Ma7], Proposition 5.2) The line bundle $\mathscr{O}_X(E)$ is stably prime-exceptional.

(2) ([Ma7], Proposition 5.14) Let Y be an irreducible holomorphic symplectic manifold and $g : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ a parallel-transport operator, which is an isomorphism of Hodge structures. Set $\alpha := g([E]) \in H^{1,1}(Y,\mathbb{Z})$. Then either α or $-\alpha$ is the class of a stably prime-exceptional line bundle.

Example 6.7 Let X be a $K3$ surface. A line bundle L is stably prime-exceptional, if and only if $\deg(L) = -2$, and $(c_1(L),\kappa) > 0$, for some Kähler class κ on X.

Denote by $\mathcal{S}pe \subset H^{1,1}(X,\mathbb{Z})$ the subset of classes of stably prime-exceptional divisors.

Definition 6.8 Let $W_{Exc} \subset Mon^2_{Hdg}(X)$ be the reflection subgroup generated by $\{R_\ell \ : \ \ell \in \mathcal{S}pe\}$.

Note that $R_\ell = R_{-\ell}$.

Corollary 6.9 *The union $\mathcal{S}pe \cup -\mathcal{S}pe$ is a $Mon^2_{Hdg}(X)$-invariant subset of $H^{1,1}(X,\mathbb{Z})$. In particular, W_{Exc} is a normal subgroup of $Mon^2_{Hdg}(X)$*

Proposition 6.10 *The fundamental exceptional chamber $\mathscr{F}\mathscr{E}_X$, introduced in Definition 5.2, is equal to the subset*

$$\{\alpha \in \mathscr{C}_X \ : \ (\alpha,\ell) > 0, \textit{ for every } \ell \in \mathcal{S}pe\}. \tag{6.1}$$

Proof Denote the exceptional chamber (6.1) by Ch_0. Then $Ch_0 \subset \mathscr{F}\mathscr{E}_X$, since a prime exceptional divisor is stably prime-exceptional, by Proposition 6.6. Let α be a class in $\mathscr{F}\mathscr{E}_X$, $\ell \in \mathcal{S}pe$, and $\ell = P(\ell) + N(\ell)$ its Zariski decomposition of Theorem 5.8. Then $N(\ell)$ is a non-zero exceptional divisor, since $(\ell, \ell) < 0$ and $(P(\ell), P(\ell)) \geq 0$. Furthermore, $(\alpha, P(\ell)) \geq 0$, since α and $P(\ell)$ belong to the closure of the positive cone. Thus, $(\alpha, \ell) \geq (\alpha, N(\ell)) > 0$. We conclude that α belongs to Ch_0 and so $\mathscr{F}\mathscr{E}_X \subset Ch_0$. □

In section 9.2 we will provide a numerical determination of the set $\mathcal{S}pe$, and hence of $\mathscr{F}\mathscr{E}_X$, for X of $K3^{[n]}$-type.

6.3 Hyperbolic reflection groups

Consider the vector space $\mathbb{R}^{n+1}$, endowed with the quadratic form $q(x_0, \ldots, x_n) = x_0^2 - \sum_{i=1}^n x_i^2$. We will denote the inner product space $(\mathbb{R}^{n+1}, q)$ by V and denote by (v, w), $v, w \in V$, the inner product, such that $q(v) = (v, v)$. Let $v := (v_0, \ldots, v_n)$ be the coordinates of a vector v in V. The *hyperbolic* (or *Lobachevsky*) *space* is

$$\mathbb{H}^n \; := \; \{v \in V \; : \; q(v) = 1 \text{ and } v_0 > 0\}.$$

$\mathbb{H}^n$ has two additional descriptions. It is the set of $\mathbb{R}_{>0}$ orbits (half lines) in one of the two connected component of the cone $\mathscr{C}'_V := \{v \in V \; : \; q(v) > 0\}$. We will denote by $\mathscr{C}_V$ the chosen connected component of $\mathscr{C}'_V$ and refer to $\mathscr{C}_V$ as the *positive cone*. $\mathbb{H}^n$ also naturally embeds in $\mathbb{P}^n(\mathbb{R})$ as the image of $\mathscr{C}_V$. A *hyperplane* in $\mathbb{H}^n$ is a non-empty intersection of $\mathbb{H}^n$ with a hyperplane in $\mathbb{P}^n(\mathbb{R})$.

The first description of $\mathbb{H}^n$ above depended on the diagonal form of the quadratic form q. The last two descriptions of $\mathbb{H}^n$ produce a copy of $\mathbb{H}^n$ associated more generally to any quadratic form $q(x_0, \ldots, x_n) = \sum_{i,j=0}^n a_{ij} x_i x_j$, $a_{ij} \in \mathbb{Q}$, of signature $(1, n)$. We will consider from now on this more general set-up.

$\mathbb{H}^n$ admits a metric of constant curvature [VS]. Let $O^+(V)$ be the subgroup of the isometry group of V mapping $\mathscr{C}_V$ to itself. Then $O^+(V)$ acts transitively on $\mathbb{H}^n$ via isometries. The stabilizer $Stab_{O^+(V)}(t)$, of every point $t \in \mathbb{H}^n$, is compact, since the hyperplane $t^\perp \subset V$ is negative definite.

A subgroup $\Gamma \subset O^+(V)$ is said to be a *discrete group of motions of* $\mathbb{H}^n$, if for each point $t \in \mathbb{H}^n$, the stabilizer $Stab_\Gamma(t)$ is finite and the orbit $\Gamma \cdot t$ is a discrete subset of $\mathbb{H}^n$. The arithmetic group $O^+(V,\mathbb{Z})$ is a discrete group of motions ([VS], Ch. 1, section 2.2). Furthermore, if a subgroup $\Gamma \subset O^+(V)$ is commensurable to a discrete group of motions, then Γ is a discrete group of motions as well ([VS], Ch. 1, Proposition 1.13). Given a group homomorphism $\widetilde{\Gamma} \rightarrow O^+(V)$, we say that $\widetilde{\Gamma}$ *acts on* $\mathbb{H}^n$ *via a discrete group of motions,* if its image $\Gamma \subset O^+(V)$ is a discrete group of motions.

Lemma 6.11 *Let X be a projective irreducible holomorphic symplectic manifold. Then* $Mon^2_{Hdg}(X)$ *acts via a discrete group of motions on the hyperbolic space* $\mathbb{H}_X$ *associated to* $V := H^{1,1}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$ *as well as on the hyperbolic space* $\widetilde{\mathbb{H}}_X$ *associated to* $H^{1,1}(X,\mathbb{R})$.

Proof Let ρ be the rank of $\mathrm{Pic}(X)$. The Beauville-Bogomolov pairing restricts to $H^{1,1}(X,\mathbb{Z})$ as a non-degenerate pairing of signature $(1,\rho-1)$. The action of $Mon^2_{Hdg}(X)$ on $\mathbb{H}_X$ factors through the action of $O^+[H^{1,1}(X,\mathbb{Z})]$. The latter acts as a discrete group of motions on $\mathbb{H}_X$ (see [VS], Ch. 1, section 2.2). The statement of the lemma follows for $\mathbb{H}_X$.

Let G be the kernel of the restriction homomorphism $Mon^2_{Hdg}(X) \rightarrow O^+[H^{1,1}(X,\mathbb{Z})]$. We prove next that G is a finite group. Let $T(X)$ be the subspace of $H^2(X,\mathbb{R})$ orthogonal to $H^{1,1}(X,\mathbb{Z})$. Set $T^{1,1}(X) := T(X) \cap H^{1,1}(X,\mathbb{R})$. The Beauville-Bogomolov pairing restricts to $T^{1,1}(X)$ as a negative definite pairing. Let $T^+(X) \subset T(X)$ be the orthogonal complement of $T^{1,1}(X)$ in $T(X)$. Then $T^+(X)$ is the two-dimensional positive definite subspace of $T(X)$, spanned by the real and imaginary parts of a holomorphic 2-form on X. G acts faithfully on $T(X)$ and it embeds as a discrete subgroup of the compact group $O(T^+(X)) \times O\left(T^{1,1}(X)\right)$. We conclude that G is finite.

The linear subspace $\mathbb{P}\left(T^{1,1}(X)\right)$ of $\mathbb{P}\left(H^{1,1}(X,\mathbb{R})\right)$ is disjoint from $\widetilde{\mathbb{H}}_X$ and so the orthogonal projection $H^{1,1}(X,\mathbb{R}) \rightarrow V$ induces a well defined $Mon^2_{Hdg}(X)$-equivariant map $\pi : \widetilde{\mathbb{H}}_X \rightarrow \mathbb{H}_X$. Explicitly, a point $\tilde{v}$ in the positive cone of $H^{1,1}(X,\mathbb{R})$ can be uniquely decomposed as a sum $\tilde{v} = v + t$, with $v \in V$ and $t \in T^{1,1}(X)$, and π takes the image of $\tilde{v}$ in $\widetilde{\mathbb{H}}_X$ to the image of v in $\mathbb{H}_X$.

We show next that $Mon^2_{Hdg}(X)$ acts on $\widetilde{\mathbb{H}}_X$ via a discrete group of motions. Set $\Gamma := Mon^2_{Hdg}(X)/G$. Let $\tilde{x}$ be a point of $\widetilde{\mathbb{H}}_X$ and set $x := \pi(\tilde{x})$. The stabilizing subgroup $Stab_\Gamma(x)$ is finite, since Γ acts on $\mathbb{H}_X$ as a discrete group of motions. The preimage of $Stab_\Gamma(x)$ in $Mon^2_{Hdg}(X)$ is finite and contains the stabilizer of $\tilde{x}$ in $Mon^2_{Hdg}(X)$. Hence, the latter stabilizer is finite. Let y be a point in the orbit $\Gamma \cdot x$ in $\mathbb{H}_X$. Then $\pi^{-1}(y)$ intersects the orbit $Mon^2_{Hdg}(X) \cdot \tilde{x}$ in an orbit of a finite subgroup, namely, an orbit of the preimage of $Stab_\Gamma(y)$ in $Mon^2_{Hdg}(X)$. The orbit

$Mon^2_{Hdg}(X)\cdot\tilde{x}$ is a discrete subset of $\widetilde{\mathbb{H}}_X$, since π restricts to it as a finite map onto the discrete orbit of x in $\mathbb{H}_X$. $\square$

Given an element $e\in V$, with $q(e)<0$, we get the reflection $R_e\in O^+(V)$, given by $R_e(w)=w-2\frac{(e,w)}{(e,e)}e$.

Definition 6.12 A *hyperbolic reflection group* is a discrete group of motions of $\mathbb{H}^n$ generated by reflections.

Given a vector $e\in V$, with $q(e)<0$, set

$$H_e^+ := \{v\in\mathscr{C}_V \ : \ (v,e)>0\}/\mathbb{R}_{>0}.$$

Define H_e^- similarly using the inequality $(v,e)<0$. Set $H_e:=e^\perp\cap\mathbb{H}^n$, where $e^\perp$ is the hyperplane of $\mathbb{P}(V)$ orthogonal to e. Then $\mathbb{H}^n\setminus H_e$ is the disjoint union of its two connected components H_e^+ and H_e^-. The closures $\overline{H}_e^\pm$ are called *half-spaces*.

Definition 6.13

(1) A set $\{\Sigma_i \ : i\in I\}$, of subsets of a topological space X, is *locally finite*, if each point $x\in X$ has an open neighborhood U_x, such that the intersection $\Sigma_i\cap U_x$ is empty, for all but finitely many indices $i\in I$.

(2) A *decomposition* of $\mathbb{H}^n$ is a locally finite covering of $\mathbb{H}^n$ by closures of open connected subsets, no two of which have common interior points.

(3) A closure D of an open subset of $\mathbb{H}^n$ is said to be a *fundamental domain* of a discrete group of motions Γ, if $\{\gamma(D) \ : \ \gamma\in\Gamma\}$ is a decomposition of $\mathbb{H}^n$.

(4) ([AVS], Ch. 1, Definition 3.9) A *convex polyhedron* is an intersection of finitely many half-spaces, having a non-empty interior.

(5) ([VS], Ch 1, Definition 1.9) A closed subset $P\subset\mathbb{H}^n$ is a *generalized convex polyhedron*, if P is the closure of an open subset, and the intersection of P with every bounded convex polyhedron, containing at least one common interior point, is a convex polyhedron.

(6) A closed cone in $\mathscr{C}_V$ is a *generalized convex polyhedron*, if its image in $\mathbb{H}^n$ is a generalized convex polyhedron.

(7) A closed cone Π in $\mathscr{C}_V$ is a *rational convex polyhedron*, if its image in $\mathbb{H}^n$ is a convex polyhedron, which is the intersection of finitely many half spaces H_e^+ with $e\in\mathbb{Q}^{n+1}$.

Theorem 6.14

(1) *([VS], Ch. 1 Theorem 1.11) Any discrete group of motions of $\mathbb{H}^n$ has a fundamental domain, which is a generalized convex polyhedron.*

(2) *([VS], Ch. 2 Theorem 2.5) The action on $\mathbb{H}^n$ of any arithmetic subgroup of $O^+(V)$ has a fundamental domain, which is a convex polyhedron.*

The decomposition of $\mathbb{H}^n$, induced by translates of the fundamental domain in Theorem 6.14, is not canonical in general. A canonical decomposition exists, if the discrete group of motions is a reflection group. The hyperplanes of $n-1$ dimensional faces of a generalized convex polyhedron are called its *walls*.

Let Γ be a hyperbolic reflection group and $\mathscr{R}_\Gamma \subset \Gamma$ the subset of reflections. Given a reflection $\rho \in \mathscr{R}_\Gamma$, let $H_\rho \subset \mathbb{H}^n$ be the hyperplane fixed by ρ. Connected components of $\mathbb{H}^n \setminus \bigcup_{\rho \in \mathscr{R}_\Gamma} H_\rho$ are called *chambers*.

Theorem 6.15 *([VS], Ch. 5 Theorem 1.2 and Proposition 1.4)*

(1) *The closure of each chamber of Γ in $\mathbb{H}^n$ is a generalized convex polyhedron,*[7] *which is a fundamental domain for Γ.*

(2) *Γ is generated by reflections in the walls of any of its chambers in $\mathbb{H}^n$.*

Let Γ be any discrete group of motions of $\mathbb{H}^n$. Denote by Γ_r the subgroup of Γ generated by all reflections in Γ. We call Γ_r the *reflection subgroup of Γ*. Choose a chamber D of Γ_r. Let $\Gamma_D \subset \Gamma$ be the subgroup $\{\gamma \in \Gamma \ : \ \gamma(D) = D\}$.

Theorem 6.16 *([VS], Ch. 5 Proposition 1.5) Γ_r is a normal subgroup of Γ, and Γ is the semi-direct product of Γ_r and Γ_D.*

We refer the reader to the book [VS] and the interesting recent survey [Do] for detailed expositions of the subject of hyperbolic reflection groups.

Let X be a projective irreducible holomorphic symplectic manifold.

Theorem 6.17 *The fundamental exceptional chamber $\mathscr{F}\mathscr{E}_X$, introduced in Definition 5.2, is equal to the connected component of*

$$\mathscr{C}_X \setminus \bigcup \{\ell^\perp \ : \ \ell \in \mathcal{S}pe\} \tag{6.2}$$

[7] This polyhedron is moreover a generalized Coxeter polyhedron ([VS], Ch. 5 Definition 1.1), but we will not use this fact.

containing the Kähler cone. In particular, $\mathscr{F}\mathscr{E}_X$ is the interior of a generalized convex polyhedron (Definition 6.13).

Proof The group W_{Exc} is a hyperbolic reflection group and the set U in equation (6.2) is an open subset of $\mathscr{C}_X$, which is the union of the interiors of the fundamental chambers of the W_{Exc}-action on $\mathscr{C}_X$, by Theorem 6.15. The intersection of $\mathscr{F}\mathscr{E}_X$ and U is the union of connected components of U, by the definitions of $\mathscr{F}\mathscr{E}_X$ and W_{Exc}. $\mathscr{F}\mathscr{E}_X$ is contained in U, by Proposition 6.10. $\mathscr{F}\mathscr{E}_X$ is convex cone, hence a connected component of U. $\mathscr{F}\mathscr{E}_X$ contains $\mathscr{K}_X$, by the definition of $\mathscr{F}\mathscr{E}_X$. □

6.4 $Mon^2_{Hdg}(X)$ is a semi-direct product of W_{Exc} and $Mon^2_{Bir}(X)$

Denote by $\mathcal{P}ex$ the set of prime exceptional divisors in X. Given $E \in \mathcal{P}ex$, denote by R_E the corresponding reflection (Proposition 6.2).

Theorem 6.18

(1) The group $Mon^2_{Hdg}(X)$ acts transitively on the set of exceptional chambers, introduced in Definition 5.10, and the subgroup W_{Exc} acts simply-transitively on this set.

(2) The exceptional chambers are precisely the connected component of the open set in equation (6.2), i.e., each exceptional chamber is the interior of a fundamental domain of the W_{Exc} action on $\mathscr{C}_X$.

(3) The group W_{Exc} is generated by $\{R_e \ : \ e \in \mathcal{P}ex\}$.

(4) The subgroup of $Mon^2_{Hdg}(X)$ stabilizing the fundamental exceptional chamber $\mathscr{F}\mathscr{E}_X$ is equal to $Mon^2_{Bir}(X)$.

(5) $Mon^2_{Hdg}(X)$ is the semi-direct product of its subgroups W_{Exc} and $Mon^2_{Bir}(X)$.

When X is a $K3$ surface $Mon^2_{Hdg}(X)$ is equal to the group of Hodge isometries of $H^2(X,\mathbb{Z})$ preserving the spinor norm and $Mon^2_{Bir}(X)$ is equal to the group of biregular automorphisms of X. Furthermore, the fundamental exceptional chamber is equal to the Kähler cone of the $K3$ surface. Theorem 6.18 is well known in the case of $K3$ surfaces [BR, PS], or ([LP], Proposition 1.9).

Proof Parts (1) and (2): $Mon^2_{Hdg}(X)$ acts transitively on the set of exceptional chambers, by their definition. The subgroup W_{Exc} acts simply-transitively on the set of connected components of the set U in equation (6.2), by Theorem 6.15. One of these is $\mathscr{F}\mathscr{E}_X$, by Theorem 6.17. Hence, every connected component of U is an exceptional chamber. $Mon^2_{Hdg}(X)$ acts on the set of connected component of U, by Corollary 6.9. Hence, every exceptional chamber is a connected component of U.

Part (3): The walls in the boundary of the fundamental exceptional chamber are all of the form $[E]^\perp \cap \mathscr{C}_X$, for some prime exceptional divisor E, by definition. $\mathscr{F}\mathscr{E}_X$ is the interior of a chamber of W_{Exc}, by Theorem 6.17. We conclude that W_{Exc} is generated by $\{R_e \ : \ e \in \mathcal{P}ex\}$, by Theorem 6.15.

Part (4): $Mon^2_{Bir}(X)$ is the subgroup of $Mon^2_{Hdg}(X)$ leaving $\mathscr{F}\mathscr{E}_X$ invariant, by Lemma 5.11 part 6.

Part (5): $Mon^2_{Hdg}(X)$ is generated by W_{Exc} and $Mon^2_{Bir}(X)$, by parts (1) and (4). The intersection $W_{Exc} \cap Mon^2_{Bir}(X)$ is trivial, since the action of W_{Exc} on the set of exceptional chambers is free. W_{Exc} is a normal subgroup of $Mon^2_{Hdg}(X)$, by Corollary 6.9. □

Caution 6.19 When X is a $K3$ surface, then W_{Exc} is the reflection subgroup of $Mon^2_{Hdg}(X)$, i.e., every reflection $g \in Mon^2_{Hdg}(X)$ is of the form R_ℓ, for a class ℓ satisfying $(\ell,\ell) = -2$. This follows easily from the fact that $H^2(X,\mathbb{Z})$ is a unimodular lattice. W_{Exc} may be strictly smaller than the reflection subgroup of $Mon^2_{Hdg}(X)$, for a higher dimensional irreducible holomorphic symplectic manifold X. In other words, there are examples of elements $\alpha \in H^{1,1}(X,\mathbb{Z})$, with $(\alpha,\alpha) < 0$, such that R_α belongs to $Mon^2_{Hdg}(X)$, but neither α, nor $-\alpha$, belongs to $\mathcal{S}pe$. Instead, R_α is induced by a bimeromorphic map from X to itself (see Example 9.20 below, and section 11 of [Ma7] for additional examples).

Let L be a stably prime-exceptional line bundle and set $\ell := c_1(L)$. The hyperplane $\ell^\perp$ intersects $\overline{\mathscr{F}\mathscr{E}}_X$ in a top dimensional cone in $\ell^\perp$, only if $L = \mathscr{O}_X(E)$ for some prime exceptional divisor E, by Proposition 6.10. We show next that the condition is also sufficient.

Lemma 6.20 *Let E be a prime exceptional divisor on X. Then $E^\perp \cap \overline{\mathscr{F}\mathscr{E}}_X$ is a top dimensional cone in the hyperplane $E^\perp$. Consequently, W_{Exc} can not be generated by any proper subset of $\{R_e \ : \ e \in \mathcal{P}ex\}$.*

Proof Let e be an element of $\mathcal{P}ex$. It suffices to show that $e^\perp \cap \overline{\mathscr{F}\mathscr{E}}_X \cap \mathscr{C}_X$ contains elements, which are not orthogonal to any other $e' \in \mathcal{P}ex$. Choose $x \in \mathscr{F}\mathscr{E}_X$ and set $y := x - \frac{(x,e)}{(e,e)}e$. Then $(y,e) = 0$. Given $e' \in \mathcal{P}ex$, $e' \neq e$, then $(e,e') \geq 0$ and $(x,e') > 0$. Now $(e,e) < 0$. We get the following inequalities.

$$(e',y) = (e',x) - \frac{(x,e)}{(e,e)}(e',e) \ > \ 0.$$
$$(y,y) = (x,x) - \frac{(x,e)^2}{(e,e)} \ > \ 0.$$

We conclude that y belongs to $e^{\perp} \cap \overline{\mathscr{F}\mathscr{E}}_X \cap \mathscr{C}_X$, and y does not belong to $(e')^{\perp}$, for any $e' \in \mathcal{P}ex \setminus \{e\}$. □

Proof of Theorem 1.6: W_{Exc} is a normal subgroup of $Mon^2_{Hdg}(X)$, by Corollary 6.9. There exists a bimeromorphic map $h : X_1 \to X_2$, by Theorem 1.3, and h^* is a parallel transport operator, by Theorem 3.1. The composition $f \circ h^*$ belongs to $Mon^2_{Hdg}(X_2)$. There exists an element w of $W_{Exc}(X_2)$, such that $w^{-1} f \circ h^*$ belongs to $Mon^2_{Bir}(X_2)$, by Theorem 6.18. Let $\varphi : X_2 \to X_2$ be a bimeromorphic map, such that $\varphi_* = w^{-1} f \circ h^*$. Then $f = w(\varphi h)_*$. Set $g := \varphi h$ to obtain the desired decomposition $f = w \circ g_*$.

Assume that $\tilde{g} : X_1 \to X_2$ is a birational map and $\tilde{w}$ is an element of $W_{Exc}(X_2)$, such that $f = \tilde{w}\tilde{g}_*$. Then $w^{-1}\tilde{w} = (\tilde{g}^{-1}g)_*$ belongs to the intersection of $W_{Exc}(X_2)$ and $Mon^2_{Bir}(X_2)$, which is trivial, by Theorem 6.18. Thus, $w = \tilde{w}$ and $g_* = \tilde{g}_*$. Now, $\tilde{g} = g(g^{-1}\tilde{g})$, and $g^{-1}\tilde{g}$ is a birational map inducing the identity on $H^2(X_1, \mathbb{Z})$. In particular, $g^{-1}\tilde{g}$ maps $\mathscr{K}_{X_1}$ to itself, and hence is a biregular automorphism. □

6.5 *Morrison's movable cone conjecture*

Let X be a projective irreducible holomorphic symplectic manifold. We describe first an analogy between results on the ample cone of a projective $K3$ surface and results on the movable cone of X. Set $\mathrm{NS} := H^{1,1}(X, \mathbb{Z})$, $\mathrm{NS}_{\mathbb{R}} := \mathrm{NS} \otimes_{\mathbb{Z}} \mathbb{R}$, and $\mathrm{NS}_{\mathbb{Q}} := \mathrm{NS} \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $\mathscr{C}_{\mathrm{NS}}$ be the intersection $\mathscr{C}_X \cap \mathrm{NS}_{\mathbb{R}}$.

Definition 6.21

(1) A line bundle L on X is *movable*, if the base locus of the linear system $|L|$ has codimension ≥ 2.

(2) The *movable cone* $\mathscr{M}\mathscr{V}_X$ is the convex hull in $\mathrm{NS}_{\mathbb{R}}$ of all classes of movable line bundles.

Let $\mathscr{M}\mathscr{V}^0_X$ be the interior of $\mathscr{M}\mathscr{V}_X$ and $\overline{\mathscr{M}\mathscr{V}}_X$ the closure of $\mathscr{M}\mathscr{V}_X$ in $\mathrm{NS}_{\mathbb{R}}$.

Lemma 6.22 *The equality $\mathscr{MV}_X^0 = \mathscr{FE}_X \cap \mathrm{NS}_{\mathbb{R}}$ holds. W_{Exc} acts faithfully on $\mathscr{C}_{\mathrm{NS}}$ and the map $Ch \mapsto Ch \cap \mathrm{NS}_{\mathbb{R}}$ induces a one-to-one correspondence between the set of exceptional chambers and the chambers in $\mathscr{C}_{\mathrm{NS}}$ of the W_{Exc} action. In particular, the closure of $\mathscr{MV}_X$ in $\mathscr{C}_{\mathrm{NS}}$ is a fundamental domain for the action of W_{Exc} on $\mathscr{C}_{\mathrm{NS}}$.*

Proof The equality $\mathscr{MV}_X^0 = \mathscr{FE}_X \cap \mathrm{NS}_{\mathbb{R}}$ follows immediately from the Zariski decomposition (Theorem 5.8). The set $\mathbb{S}pe$ is contained in NS, hence the W_{Exc} action on $\mathscr{C}_{\mathrm{NS}}$ is faithful and the map $Ch \mapsto Ch \cap \mathrm{NS}_{\mathbb{R}}$ induces a bijection. □

Let $\rho : Mon^2_{Hdg}(X) \to O(\mathrm{NS})$ be the restriction homomorphism. We denote $\rho(W_{Exc})$ by W_{Exc} as well.

Lemma 6.23

(1) The image Γ of ρ is a finite index subgroup of $O^+(\mathrm{NS})$.

(2) The kernel of ρ is a subgroup of $Mon^2_{Bir}(X)$.

(3) Γ is a semi-direct product of its normal subgroup W_{Exc} and the quotient group $\Gamma_{Bir} := Mon^2_{Bir}(X)/\ker(\rho)$.

Proof (1) The positive cone $\mathscr{C}_X$ is $Mon^2_{Hdg}(X)$-invariant and $\mathscr{C}_{\mathrm{NS}} = \mathscr{C}_X \cap \mathrm{NS}$ is thus Γ-invariant. Hence, Γ is a subgroup of $O^+(\mathrm{NS})$. Let $O^+_{Hdg}\left(H^2(X,\mathbb{Z})\right)$ be the subgroup of $O^+\left(H^2(X,\mathbb{Z})\right)$ preserving the Hodge structure. Then $O^+_{Hdg}\left(H^2(X,\mathbb{Z})\right)$ maps onto a finite index subgroup of $O^+(\mathrm{NS})$. The index of $Mon^2(X)$ in $O^+H^2(X,\mathbb{Z})$ is finite, by a result of Sullivan [Su] (see also [Ver2], Theorem 3.4). Hence, $Mon^2_{Hdg}(X)$ is a finite index subgroup of $O^+_{Hdg}\left(H^2(X,\mathbb{Z})\right)$. Part (1) follows.

(2) Let g be an element of $\ker(\rho)$. Then g acts trivially on $\mathbb{S}pe$. Hence, g maps $\mathscr{FE}_X$ to itself. It follows that g belongs to $Mon^2_{Bir}(X)$, by Theorem 6.18 part 4.

Part (3) is an immediate consequence of part (2) and Theorem 6.18 part 5. □

Let $\mathscr{E}\mathrm{ff}_X \subset \mathrm{NS}_{\mathbb{R}}$ be the convex cone generated by classes of effective divisors on X. Set $\mathscr{MV}_X^e := \overline{\mathscr{MV}}_X \cap \mathscr{E}\mathrm{ff}_X$. Following is Morrison's movable cone conjecture.

Conjecture 6.24 *[Mor1, Mor2, Ka] There exists a rational convex polyhedral cone (Definition 6.13 part 7) Π, which is a fundamental domain for the action of* $\mathrm{Bir}(X)$ *on $\mathscr{MV}_X^e$.*

Morrison formulated a version of the conjecture for the ample cone as well. The two versions coincide in dimension 2 and for abelian varieties. The $K3$ surface case of the conjecture is proven by Looijenga and Sterk ([St], Lemma 2.4), the Enriques surfaces case by Namikawa ([Nam], Theorem 1.4), the case of abelian and hyperelliptic surfaces by Kawamata ([Ka], Theorem 2.1), the case of two-dimensional Calabi-Yau pairs by Totaro [Tot], and the case of abelian varieties by Prendergast-Smith [Pre]. A version of the conjectures for fiber spaces was formulated by Kawamata and proven in dimension 3 in [Ka].

The following theorem is a weaker version of Morrison's movable cone conjecture, in the special case of projective irreducible holomorphic symplectic manifolds. Let $\mathcal{M}\mathcal{V}_X^+$ be the convex hull of $\overline{\mathcal{M}\mathcal{V}}_X \cap \mathrm{NS}_{\mathbb{Q}}$. Clearly, $\mathcal{M}\mathcal{V}_X^0$ is equal to the interior of both $\mathcal{M}\mathcal{V}_X^+$ and $\mathcal{M}\mathcal{V}_X^e$. When X is a $K3$ surface the equality $\mathcal{M}\mathcal{V}_X^+ = \mathcal{M}\mathcal{V}_X^e$ holds. In the $K3$ case the inclusion $\mathcal{M}\mathcal{V}_X^+ \subset \mathcal{M}\mathcal{V}_X^e$ follows from ([BHPV], Proposition 3.6 part i) and the inclusion $\mathcal{M}\mathcal{V}_X^+ \supset \mathcal{M}\mathcal{V}_X^e$ is proven in ([Ka], Proposition 2.4).

Theorem 6.25 *There exists a rational convex polyhedral cone Π in $\mathcal{M}\mathcal{V}_X^+$, such that Π is a fundamental domain for the action of Γ_{Bir} on $\mathcal{M}\mathcal{V}_X^+$.*

Proof The proof is identical to that of Lemma 2.4 in [St], which proves the $K3$-surface case of the Theorem. When X is a $K3$ surface, $\mathcal{M}\mathcal{V}_X^0$ is the ample cone and $\mathcal{P}ex$ is the set of nodal -2 classes. The proof is lattice theoretic. Following is the dictionary translating our notation to that of Sterk.

Our notation	$\mathcal{M}\mathcal{V}_X^0$	$\mathcal{C}_{\mathrm{NS}}$	$\mathcal{M}\mathcal{V}_X^+$	$\mathcal{P}ex$	$\mathcal{S}pe$	Γ	Γ_{Bir}	W_{Exc}
Sterk's notation	K	$\mathcal{C}$	$\overline{K} \cap \mathcal{C}_+$	B	Δ^+	Γ	Γ_B	W

One slight inaccuracy in the above dictionary is that Sterk chose Γ to be the subgroup of $O^+\left(H^2(X,\mathbb{Z})\right)$ acting trivially on the transcendental lattice $\mathrm{NS}^\perp$, while we consider (in case X is a $K3$ surface) the image of $O^+_{Hdg}\left(H^2(X,\mathbb{Z})\right)$ in $O^+(\mathrm{NS})$. So Sterk's Γ is the finite index subgroup of our Γ acting trivially on the finite discriminant group $\mathrm{NS}^*/\mathrm{NS}$. Both choices satisfy the following complete list of assertions needed for the Looijenga-Sterk argument (in Sterk's notation).

(1) NS is a lattice of signature $(1,*)$ and Γ is an arithmetic subgroup of $O^+(\mathrm{NS})$.

(2) $W \subset O^+(\mathrm{NS})$ is the reflection group generated by reflections in elements of $B \subset \mathrm{NS}$.

(3) Γ_B is equal to the subgroup $\{g \in \Gamma \ : \ g(B) = B\}$.

(4) W is a normal subgroup of Γ and $\Gamma = \Gamma_B \cdot W$ is a semi-direct product decomposition.

(5) $\overline{K} \cap \mathscr{C}$ is a fundamental domain for the action of W on $\mathscr{C}$, cut-out by closed half-spaces associated to elements of B.

Assertion (1) is verified in our case in Lemma 6.23 part 1. Assertion (2) is verified in Theorem 6.18 part 3. $Mon^2_{Bir}(X) = \{g \in Mon^2_{Hdg}(X) \ : \ g(\mathcal{P}ex) = \mathcal{P}ex\}$, by Theorem 6.18 part 4 and Lemma 6.20. Assertion (3) follows from the latter equality by Lemma 6.23 part 2. Assertion (4) is verified in Lemma 6.23 part 3. Assertion (5) is verified in Lemma 6.22.

The argument proceeds roughly as follows. Choose a rational element $x_0 \in \mathscr{M}\mathscr{V}_X$ which is not fixed by any element of Γ. Let $\mathscr{C}_+$ be the convex hull of $\overline{\mathscr{C}}_{\mathrm{NS}} \cap \mathrm{NS}_{\mathbb{Q}}$ in $\mathrm{NS}_{\mathbb{R}}$. Set

$$\Pi := \{x \in \mathscr{C}_+ \ : \ (x_0, x) \leq (x_0, \gamma(x)), \text{ for all } \gamma \in \Gamma\}.$$

Then Π is a fundamental domain for the Γ action on $\mathscr{C}_+$, known as the *Dirichlet domain with center* x_0 (compare[8] with [VS], Ch. 1 Proposition 1.10). Π is shown to be a rational convex polyhedron ([St], Lemma 2.3, see also Theorem 6.14 part (2) above). The above depends only on Assertion (1). The interior of any fundamental domain for Γ can not intersect any hyperplane $e^\perp$, $e \in \mathcal{P}ex$. Hence, Π is contained in $\mathscr{M}\mathscr{V}^+_X$, by Assertions (2) and (5). $\mathscr{M}\mathscr{V}^+_X$ is a fundamental domain for the W_{Exc} action on $\mathscr{C}_+$, by Assertion (5). Hence, any fundamental domain for the Γ-action on $\mathscr{C}_+$ which is contained in $\mathscr{M}\mathscr{V}^+_X$, is a fundamental domain for the Γ_{Bir} action on $\mathscr{M}\mathscr{V}^+_X$, by Assertions (3) and (4). □

Proof (Of Theorem 1.7) Assume that D is an irreducible divisor on X. Then D is either prime exceptional, or the class $[D]$ belongs to $\overline{\mathscr{M}\mathscr{V}}_X$, by Theorem 5.8. If D is prime exceptional, the statement follows by the same argument used in the $K3$ surface case ([St], Proposition 2.5). Otherwise, $[D]$ belongs to $\mathscr{M}\mathscr{V}^+_X$, and there exists $g \in \Gamma_{Bir}$, such that $g([D])$ belongs to the rational convex polyhedron Π in Theorem 6.25. The intersection $\Pi \cap \mathrm{NS}$ is a finitely generated semi-group. Choose generators $\{x_1, \ldots, x_m\}$. Then $(x_i, x_i) \geq 0$, and $(x_i, x_j) > 0$, if x_i and x_j are linearly independent. It follows that $\Pi \cap \mathrm{NS}$ contains at most finitely many elements of any given positive Beauville-Bogomolov degree, and at most finitely many primitive isotropic classes. □

[8] The bilinear pairing (x_0, x) in the above definition of the Dirichlet domain is replaced with the hyperbolic distance $\rho(x_0, x)$ in Definition 1.8 in Ch. 1 of [VS]. However, the two definitions are equivalent, by the relation $cosh(\rho(x_0, x)) = (x_0, x)$ (see Ch. 1 section 4.2 in [AVS]).

7 The monodromy and polarized monodromy groups

In section 7.1 we prove Proposition 1.9, stating that the polarized monodromy group $Mon^2(X,H)$ is the stabilizer of $c_1(H)$ in $Mon^2(X)$. In section 7.2 we fix a lattice Λ and define the coarse moduli space of polarized Λ-marked pairs of a given deformation type.

7.1 *Polarized parallel transport operators*

Let Ω_Λ be a period domain as in equation (2.1). Choose a connected component $\mathfrak{M}^0_\Lambda$ of the moduli space of marked pairs, a class $h \in \Lambda$ with $(h,h) > 0$, and let $\Omega^+_{h^\perp}$ be the period domain given in equation (4.1). Let $P_0 : \mathfrak{M}^0_\Lambda \to \Omega_\Lambda$ be the period map. Denote the inverse image $P_0^{-1}\left(\Omega^+_{h^\perp}\right)$ by $\mathfrak{M}^+_{h^\perp}$. The discussion in section 4 provides the following modular description of $\mathfrak{M}^+_{h^\perp}$. A marked pair (X,η) belongs to $\mathfrak{M}^+_{h^\perp}$, if and only if (X,η) belongs to $\mathfrak{M}^0_\Lambda$, the class $\eta^{-1}(h)$ is of Hodge type $(1,1)$, and $\eta^{-1}(h)$ belongs to the positive cone $\mathscr{C}_X$.

Proposition 7.1 $\mathfrak{M}^+_{h^\perp}$ *is path-connected.*

Proof The proof is similar to that of Proposition 5.11 in [Ma7]. The proof relies on the Global Torelli Theorem 2.2 and the connectedness of $\Omega^+_{h^\perp}$. □

Definition 7.2 Let $Mon^2\left(\mathfrak{M}^0_\Lambda\right)$ be the subgroup $\eta \circ Mon^2(X) \circ \eta^{-1} \subset O(\Lambda)$, for some marked pair $(X,\eta) \in \mathfrak{M}^0_\Lambda$. Let $Mon^2\left(\mathfrak{M}^0_\Lambda\right)_h$ be the subgroup of $Mon^2\left(\mathfrak{M}^0_\Lambda\right)$ stabilizing h.

The subgroup $Mon^2\left(\mathfrak{M}^0_\Lambda\right)$ is independent of the choice of (X,η), since $\mathfrak{M}^0_\Lambda$ is connected, by definition. $Mon^2\left(\mathfrak{M}^0_\Lambda\right)_h$ naturally acts on $\mathfrak{M}^+_{h^\perp}$.

Let

$$\mathfrak{M}^a_{h^\perp} \tag{7.1}$$

be the subset of $\mathfrak{M}^+_{h^\perp}$, consisting of isomorphism classes of pairs (X,η), such that $\eta^{-1}(h)$ is an ample class of X. The stability of Kähler manifolds implies that $\mathfrak{M}^a_{h^\perp}$ is an open subset of $\mathfrak{M}^+_{h^\perp}$ ([Voi], Theorem 9.3.3). We refer to $\mathfrak{M}^a_{h^\perp}$ as a *connected component of the moduli space of polarized marked pairs.*

Corollary 7.3 *$\mathfrak{M}^a_{h^\perp}$ is a $Mon^2\left(\mathfrak{M}^0_\Lambda\right)_h$-invariant path-connected open Hausdorff subset of $\mathfrak{M}^+_{h^\perp}$. The period map restricts as an injective open $Mon^2\left(\mathfrak{M}^0_\Lambda\right)_h$-equivariant morphism from $\mathfrak{M}^a_{h^\perp}$ onto an open dense subset of $\Omega^+_{h^\perp}$.*

Proof Let us check first that $\mathfrak{M}^a_{h^\perp}$ is $Mon^2\left(\mathfrak{M}^0_\Lambda\right)_h$-invariant. Indeed, let (X,η) belong to $\mathfrak{M}^a_{h^\perp}$ and let g be an element of $Mon^2\left(\mathfrak{M}^0_\Lambda\right)_h$. Denote by H the line bundle with $c_1(H)=\eta^{-1}(h)$. Then $g=\eta f\eta^{-1}$, for some $f\in Mon^2(X)$ stabilizing $c_1(H)$, by definition of $Mon^2\left(\mathfrak{M}^0_\Lambda\right)_h$. The pair $(X,g\eta)=(X,\eta f)$ belongs to $\mathfrak{M}^0_\Lambda$, since f is a monodromy-operator. We have

$$(g\eta)^{-1}(h)=f^{-1}(\eta^{-1}(h))=f^{-1}(c_1(H))=c_1(H).$$

Hence, $(g\eta)^{-1}(h)$ is an ample class in $H^{1,1}(X,\mathbb{Z})$.

Let (X,η) and (Y,ψ) be two inseparable points of $\mathfrak{M}^a_{h^\perp}$. Then $\psi^{-1}\eta$ is a parallel-transport operator, preserving the Hodge structure, by Theorem 3.2. Furthermore, $\psi^{-1}\eta$ maps the ample class $\eta^{-1}(h)$ to the ample class $\psi^{-1}(h)$, by definition. Hence, there exists an isomorphism $f:X\to Y$, such that $f_*=\psi^{-1}\eta$, by Theorem 1.3 part 2. The two pairs (X,η) and (Y,ψ) are thus isomorphic. Hence, $\mathfrak{M}^a_{h^\perp}$ is a Hausdorff subset of $\mathfrak{M}^+_{h^\perp}$.

$\mathfrak{M}^a_{h^\perp}$ is the complement of a countable union of closed complex analytic subsets of $\mathfrak{M}^+_{h^\perp}$. Hence, $\mathfrak{M}^a_{h^\perp}$ is path-connected (see, for example, [Ver2], Lemma 4.10).

The period map restricts to an injective map on any Hausdorff subset of a connected component of the moduli space of marked pairs, by Theorem 2.2. The image of $\mathfrak{M}^a_{h^\perp}$ contains the subset of $\Omega^+_{h^\perp}$, consisting of points p, such that $\Lambda^{1,1}(p)=\mathrm{span}_{\mathbb{Z}}\{h\}$, by Huybrechts' projectivity criterion [Hu1], and Theorem 2.2. Hence, the image of $\mathfrak{M}^a_{h^\perp}$ is dense in $\Omega^+_{h^\perp}$. The image is open, since $\mathfrak{M}^a_{h^\perp}$ is an open subset and the period map is open, being a local homeomorphism. □

Let (X_i,H_i), $i=1,2$, be two pairs, each consisting of a projective irreducible holomorphic symplectic manifold X_i, and an ample line bundle H_i. Set $h_i:=c_1(H_i)$.

Corollary 7.4 *A parallel transport operator $f:H^2(X_1,\mathbb{Z})\to H^2(X_2,\mathbb{Z})$ is a polarized parallel transport operator from (X_1,H_1) to (X_2,H_2) (Definition 1.1), if and only if $f(h_1)=h_2$.*

Proof The 'only if' part is clear. We prove the 'if' part. Assume that $f(h_1)=h_2$. Choose a marking $\eta_2:H^2(X_2,\mathbb{Z})\to\Lambda$, and set $\eta_1:=\eta_2\circ f$. Then $\eta_1(h_1)=\eta_2(h_2)$. Denote both $\eta_i(h_i)$ by h. Let $\mathfrak{M}^0_\Lambda$ be the connected component of (X_1,η_1). Then (X_2,η_2) belongs to $\mathfrak{M}^0_\Lambda$, by the assumption that f is a parallel transport operator.

Consequently, $P_0(X_i,\eta_i)$, $i=1,2$, both belong to the same connected component of $\Omega_{h^\perp}$. We may choose η_2, so that this connected component is $\Omega^+_{h^\perp}$. Then (X_1,η_1) and (X_2,η_2) both belong to $\mathfrak{M}^a_{h^\perp}$.

Choose a path $\gamma\colon[0,1]\to\mathfrak{M}^a_{h^\perp}$, with $\gamma(0)=(X_1,\eta_1)$ and $\gamma(1)=(X_2,\eta_2)$. This is possible, by Corollary 7.3. For each $t\in[0,1]$, there exists a simply-connected open neighborhood U_t of $\gamma(t)$ in $\mathfrak{M}^a_{h^\perp}$ and a semi-universal family $\pi_t:\mathscr{X}_t\to U_t$. The covering $\{U_t\}_{t\in[0,1]}$ of $\gamma([0,1])$ has a finite sub-covering $\{V_j\}_{j=1}^k$, for some integer[9] $k\geq 1$, with the property that $\gamma\left(\left[\frac{j-1}{k},\frac{j}{k}\right]\right)$ is contained in V_j. Consider the analytic space B, obtained from the disjoint union of V_j, $1\leq j\leq k$, by gluing V_j to V_{j+1} at the single point $\gamma\left(\frac{j}{k}\right)$ with transversal Zariski tangent spaces. Let $\pi_j:\mathscr{X}_j\to V_j$ be the universal family and denote its fiber over $v\in V_j$ by $\mathscr{X}_{j,v}$. Endow each fiber $\mathscr{X}_{j,v}$, of π_j over $v\in V_j$, with the marking $H^2(\mathscr{X}_{j,v},\mathbb{Z})\to\Lambda$ corresponding to the point v. For $1\leq j\leq k$, choose an isomorphism of $\mathscr{X}_{j,\gamma\left(\frac{j}{k}\right)}$ with $\mathscr{X}_{j+1,\gamma\left(\frac{j}{k}\right)}$ compatible with the marking chosen, and use it to glue the family π_j to the family π_{j+1}. We get a family $\pi:\mathscr{X}\to B$. The paths $\gamma:\left[\frac{j-1}{k},\frac{j}{k}\right]\to V_j$ can now be reglued to a path $\tilde{\gamma}\colon[0,1]\to B$. Parallel transport along $\tilde{\gamma}$ induces the isomorphism $\eta^{-1}_{\tilde{\gamma}(1)}\circ\eta_{\tilde{\gamma}(0)}=\eta^{-1}_{\gamma(1)}\circ\eta_{\gamma(0)}=\eta_2^{-1}\circ\eta_1=f$. Hence, f is a polarized parallel transport operator from (X_1,H_1) to (X_2,H_2). □

7.2 Deformation types of polarized marked pairs

Fix an irreducible holomorphic symplectic manifold X_0 and let Λ be the lattice $H^2(X_0,\mathbb{Z})$, endowed with the Beauville-Bogomolov pairing. Let τ be the set of connected components of $\mathfrak{M}_\Lambda$, consisting of pairs (X,η), such that X is deformation equivalent to X_0.

Lemma 7.5 *The set τ is finite. The group $O(\Lambda)$ acts transitively on τ and the stabilizer of a connected component $\mathfrak{M}^0_\Lambda\in\tau$ is the subgroup $Mon^2(\mathfrak{M}^0_\Lambda)$, introduced in Definition 7.2.*

Proof The set $O[H^2(X,\mathbb{Z})]/Mon^2(X)$ is finite, by a result of Sullivan [Su] (see also [Ver2], Theorem 3.4). The rest of the statement is clear. □

Denote by $\mathfrak{M}^\tau_\Lambda$ the disjoint union of connected components parametrized by the set τ. We refer to $\mathfrak{M}^\tau_\Lambda$ as *the moduli space of marked pairs of deformation type τ.*

[9] We could take $k=1$, if there exists a universal family over $\mathfrak{M}^a_{h^\perp}$, but such a family need not exist.

An example would be the moduli space of marked pairs of $K3^{[n]}$-type. Given a point $t \in \tau$, denote by $\mathfrak{M}^t_\Lambda$ the corresponding connected component of $\mathfrak{M}^\tau_\Lambda$.

Remark 7.6 If $Mon^2(X)$ is a normal subgroup of $O[H^2(X,\mathbb{Z})]$, then the subgroup $Mon^2(\mathfrak{M}^t_\Lambda)$ of $O(\Lambda)$ is equal to a fixed subgroup $Mon^2(\tau,\Lambda) \subset O(\Lambda)$, for all $t \in \tau$. This is the case when X is of $K3^{[n]}$-type (Theorem 9.1). The set τ is an $O(\Lambda)/Mon^2(\tau,\Lambda)$-torsor, by Lemma 7.5. We will identify the torsor τ with an explicit lattice theoretic $O(\Lambda)/Mon^2(\tau,\Lambda)$-torsor in Corollary 9.10.

We get the *refined period map*

$$\widetilde{P} : \mathfrak{M}^\tau_\Lambda \longrightarrow \Omega_\Lambda \times \tau, \tag{7.2}$$

sending a marked pair (X,η) to the pair $(P(X,\eta),t)$, where $\mathfrak{M}^t_\Lambda$ is the connected component containing (X,η). Then $\widetilde{P}$ is $O(\Lambda)$-equivariant with respect to the diagonal action of $O(\Lambda)$ on $\Omega_\Lambda \times \tau$.

Given $h \in \Lambda$, with $(h,h) > 0$, denote by $\Omega^{t,+}_{h^\perp}$ the period domain associated to $\mathfrak{M}^t_\Lambda$ in equation (4.1). Set $\mathfrak{M}^{t,+}_{h^\perp} := \widetilde{P}^{-1}(\Omega^{t,+}_{h^\perp})$. Let $\mathfrak{M}^{t,a}_{h^\perp} \subset \mathfrak{M}^{t,+}_{h^\perp}$ be the open subset of polarized pairs introduced in equation (7.1).

We construct next a polarized analogue of the refined period map. Given an $O(\Lambda)$-orbit $\bar{h} \subset \Lambda \times \tau$, of pairs (h,t) with $(h,h) > 0$, consider the disjoint unions

$$\mathfrak{M}^+_{\bar{h}} := \bigcup_{(h,t)\in\bar{h}} \mathfrak{M}^{t,+}_{h^\perp},$$
$$\Omega^+_{\bar{h}} := \bigcup_{(h,t)\in\bar{h}} \Omega^{t,+}_{h^\perp},$$

and let

$$\widetilde{P} : \mathfrak{M}^+_{\bar{h}} \longrightarrow \Omega^+_{\bar{h}} \tag{7.3}$$

be the map induced by the refined period map on each connected component. Then $\widetilde{P}$ is $O(\Lambda)$-equivariant and surjective. The disjoint union

$$\mathfrak{M}^a_{\bar{h}} := \bigcup_{(h,t)\in\bar{h}} \mathfrak{M}^{t,a}_{h^\perp} \tag{7.4}$$

is an $O(\Lambda)$-invariant open subset of $\mathfrak{M}^+_{\bar{h}}$. This open subset will be called *the moduli space of polarized marked pairs of deformation type* $\bar{h}$. Indeed, $\mathfrak{M}^a_{\bar{h}}$ coarsely represents a functor from the category of analytic spaces to sets, associating to a complex analytic space T the set of all equivalence classes of families of marked polarized triples (X,L,η), where X is of deformation type τ, L is an ample line bundle, and $\eta : H^2(X,\mathbb{Z}) \to \Lambda$ is an isometry, such that the pair $[\eta(c_1(L)),t]$ belongs to the $O(\Lambda)$-orbit $\bar{h}$, where $\mathfrak{M}^t_\Lambda$ is the connected component of (X,η). A family $(\pi : \mathscr{X} \to T, \mathscr{L}, \tilde{\eta})$ consists of a family π, an element $\mathscr{L}$ of Pic$(\mathscr{X}/T)$ and a trivialization $\tilde{\eta} : R^2\pi_*\mathbb{Z} \to (\Lambda)_T$, via isometries. Two families $(\mathscr{X} \to T, \mathscr{L}, \tilde{\eta})$ and $(\mathscr{X}' \to T, \mathscr{L}', \tilde{\eta}')$ are equivalent, if there exists a T-isomorphism $f : \mathscr{X} \to \mathscr{X}'$, such that $f^*\mathscr{L}' \cong \mathscr{L}$ and $\tilde{\eta}' = \tilde{\eta} \circ f^*$. We omit the detailed definition of this functor, as well as the proof that $\mathfrak{M}^a_{\bar{h}}$ coarsely represents it, as we will not use the latter fact below.

8 Monodromy quotients of type IV period domains

Fix a connected component $\mathfrak{M}^a_{h^\perp}$ of the moduli space $\mathfrak{M}^a_{\bar{h}}$ of polarized marked pairs of polarized deformation type $\bar{h}$. In the notation of section 7.2, $\mathfrak{M}^a_{h^\perp} := \mathfrak{M}^{t,a}_{h^\perp}$, for some $(h,t) \in \bar{h}$. Let $\mathfrak{M}^0_\Lambda$ be the connected component of $\mathfrak{M}_\Lambda$ containing $\mathfrak{M}^a_{h^\perp}$. Set $\Gamma := Mon^2\left(\mathfrak{M}^0_\Lambda\right)_h$ (Definition 7.2). The period domain $\Omega^+_{h^\perp}$ is a homogeneous domain of type IV ([Sa], Appendix, section 6). Γ is an arithmetic group, by ([Ver2], Theorem 3.5). The quotient $\Omega^+_{h^\perp}/\Gamma$ is thus a normal quasi-projective variety [BB].

Lemma 8.1 *There exist natural isomorphisms of complex analytic spaces*

$$\begin{aligned} \mathfrak{M}^+_{\bar{h}}/O(\Lambda) &\longrightarrow \mathfrak{M}^+_{h^\perp}/\Gamma, \\ \mathfrak{M}^a_{\bar{h}}/O(\Lambda) &\longrightarrow \mathfrak{M}^a_{h^\perp}/\Gamma, \\ \Omega^+_{\bar{h}}/O(\Lambda) &\longrightarrow \Omega^+_{h^\perp}/\Gamma. \end{aligned}$$

Furthermore, the period map descends to an open embedding

$$\overline{P} : \mathfrak{M}^a_{\bar{h}}/O(\Lambda) \hookrightarrow \Omega^+_{h^\perp}/\Gamma. \tag{8.1}$$

Proof We have the following commutative equivariant diagram

$$\begin{array}{ccccc} \mathfrak{M}^a_{\bar{h}} & \xrightarrow{\widetilde{P}} & \Omega^+_{\bar{h}} & \longrightarrow & \Omega^+_{\bar{h}}/O(\Lambda) \\ \uparrow & & \uparrow & & \uparrow \\ \mathfrak{M}^a_{h^\perp} & \xrightarrow{P_0} & \Omega^+_{h^\perp} & \longrightarrow & \Omega^+_{h^\perp}/\Gamma, \end{array}$$

with respect to the $O(\Lambda)$ action on the top row, the Γ-action on the bottom, and the inclusion homomorphism $\Gamma \hookrightarrow O(\Lambda)$. $O(\Lambda)$ acts transitively on its orbit $\bar{h}$, and the stabilizer of the pair $(h,\ \mathfrak{M}^{+}_{h^\perp}) \in \bar{h}$ is precisely Γ, by Lemma 7.5 and Proposition 1.9.

The morphism (8.1) is an open embedding, since the Γ-equivariant open morphism $\mathfrak{M}^{a}_{h^\perp} \to \Omega^{+}_{h^\perp}$ is injective, by Corollary 7.3. □

A *polarized irreducible holomorphic symplectic manifold* is a pair (X,L), consisting of a smooth projective irreducible holomorphic symplectic variety X and an ample line bundle L. Consider the contravariant functor F' from the category of schemes over $\mathbb{C}$ to the category of sets, which associates to a scheme T the set of isomorphism classes of flat families of polarized irreducible holomorphic symplectic manifolds (X,L) over T, with a fixed Hilbert polynomial $p(n) := \chi(L^n)$. The coarse moduli space representing the functor F' was constructed by Viehweg as a quasi-projective scheme with quotient singularities [Vieh]. Fix a connected component $\mathscr{V}$ of this moduli space. Then $\mathscr{V}$ is a quasi-projective variety. Denote by F the functor represented by the connected component $\mathscr{V}$. The universal property of a coarse moduli space asserts that there is a natural transformation $\theta : F \to \mathrm{Hom}(\bullet,\mathscr{V})$, satisfying the following properties.

(1) $\theta(\mathrm{Spec}(\mathbb{C})) : F(\mathrm{Spec}(\mathbb{C})) \to \mathscr{V}$ is bijective.
(2) Given a scheme B and a natural transformation $\chi : F \to \mathrm{Hom}(\bullet,B)$, there is a unique morphism $\psi : \mathscr{V} \to B$, hence a natural transformation $\psi_* : Hom(\bullet,\mathscr{V}) \to \mathrm{Hom}(\bullet,B)$, with $\chi = (\psi_*)\circ\theta$.

Remark 8.2 Property (2) replaces the data of a universal family over $\mathscr{V}$, which may not exist when $\mathscr{V}$ fails to be a fine moduli space. When a universal family $\mathscr{U} \in F(\mathscr{V})$ exists, then the morphism ψ is the image of $\mathscr{U}$ via $\chi : F(\mathscr{V}) \to \mathrm{Hom}(\mathscr{V},B)$.

Denote by $\bar{h}$ the deformation type of a polarized pair (X,L) in $\mathscr{V}$. We regard $\bar{h}$ both as a point in $[\Lambda \times \tau]/O(\Lambda)$ and as a subset of $\Lambda \times \tau$. Choose a point $(h,t) \in \bar{h}$ and set $\Omega^{+}_{h^\perp} := \Omega^{t,+}_{h^\perp}$.

Lemma 8.3 *There exists a natural injective and surjective morphism* $\varphi : \mathscr{V} \to \mathfrak{M}^{a}_{\bar{h}}/O(\Lambda)$ *in the category of analytic spaces.*

Proof The morphism $\Phi : \mathscr{V} \to \Omega^{+}_{\bar{h}}/O(\Lambda) \cong \Omega^{+}_{h^\perp}/\Gamma$, sending an isomorphism class of a polarized pair (X,L) to its period, is constructed in the proof of ([GHS1], Theorem 1.5). The morphism Φ is set-theoretically injective, by the Hodge theoretic Torelli Theorem 1.3. The image $\Phi(\mathscr{V})$ is the same subset as the image $P\left(\mathfrak{M}^{a}_{\bar{h}}\right)$, by definition of the two moduli spaces. The latter is the image also of the open immersion $\overline{P} : \mathfrak{M}^a_{\bar{h}}/O(\Lambda) \hookrightarrow \Omega^+_{\bar{h}}/O(\Lambda)$, by Lemma 8.1. Hence, the composition $\overline{P}^{-1} \circ \Phi : \mathscr{V} \to \mathfrak{M}^a_{\bar{h}}/O(\Lambda)$ is well defined and we denote it by φ. □

Theorem 8.4 *The composition* Φ *of*

$$\mathscr{V} \xrightarrow{\varphi} \mathfrak{M}^a_{\bar{h}}/O(\Lambda) \cong \mathfrak{M}^a_{h^\perp}/\Gamma \xrightarrow{\overline{P}} \Omega^+_{h^\perp}/\Gamma$$

is an open immersion in the category of algebraic varieties.

Proof The proof is similar to that of Theorem 1.5 in [GHS1] and Claim 5.4 in [O'G5]. If Γ happens to be torsion free, then any complex analytic morphism, from a complex algebraic variety to $\Omega^+_{h^\perp}/\Gamma$, is an algebraic morphism, as a consequence of Borel's extension theorem [Bo]. Γ need not be torsion free, but for sufficiently large positive integer N, the subgroup $\Gamma(N) \subset \Gamma$, acting trivially on $\Lambda/N\Lambda$, is torsion free, as a consequence of ([Sa], IV, Lemma 7.2). In our situation, where the domain $\mathscr{V}$ of Φ is a moduli space, one can apply Borel's extension theorem after passage to a finite cover $\widetilde{\mathscr{V}} \to \mathscr{V}$, where $\widetilde{\mathscr{V}}$ is a connected component of the moduli space of polarized irreducible holomorphic symplectic manifolds with a level-N structure, as done in the proofs of ([Has], Proposition 2.2.2) and ([GHS1], Theorem 1.5). The morphism Φ lifts to a morphism $\widetilde{\Phi} : \widetilde{\mathscr{V}} \to \Omega^+_{h^\perp}/\Gamma(N)$. $\widetilde{\Phi}$ is algebraic, by Borel's extension theorem, and a descent argument implies that so is Φ.

The morphism $P : \mathfrak{M}^a_{\bar{h}} \to \Omega^+_{\bar{h}}/O(\Lambda)$ is open. Hence, the image $\overline{P}(\mathfrak{M}^a_{h^\perp}/\Gamma)$ of Φ is an open subset of $\Omega^+_{h^\perp}/\Gamma$ in the analytic topology. The image of Φ is also a constructibe set, in the Zariski topology. The image is thus a Zariski dense open subset. Φ is thus an algebraic open immersion, by Zariski's Main Theorem. □

Remark 8.5 Theorem 8.4 answers Question 2.6 in the paper [GHS1], concerning the polarized $K3^{[n]}$-type moduli spaces. The map Φ in Theorem 8.4 is denoted by $\tilde{\varphi}$ in ([GHS1], Question 2.6) and is defined in ([GHS1], Theorem 2.3). There is a typo in the definition of $\tilde{\varphi}$ in [GHS1]; its target $\widetilde{O}^+(L_{2n-2}, h)\backslash\mathscr{D}_h$ should be replaced by $\widehat{O}^+(L_{2n-2}, h)\backslash\mathscr{D}_h$. When $n = 2$, these two quotients are the same, but for $n \geq 3$, the former is a branched double cover of the latter. Modulo this minor change, Theorem 8.4 provides an affirmative answer to ([GHS1], Question 2.6).

9 The $K3^{[n]}$ deformation type

In section 9.1 we review results about parallel-transport operators of $K3^{[n]}$-type. In section 9.2 we explicitly calculate the fundamental exceptional chamber $\mathscr{F}\mathscr{E}_X$ of a projective manifold X of $K3^{[n]}$-type.

9.1 *Characterization of parallel-transport operators of* $K3^{[n]}$*-type*

In sections 9.1.1, 9.1.2, and 9.1.3, we provide three useful characterizations of the monodromy group $Mon^2(X)$ of an irreducible holomorphic symplectic manifold of $K3^{[n]}$-type. Given X_1 and X_2 of $K3^{[n]}$-type, we state in section 9.1.4 a necessary and sufficient condition for an isometry $g : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ to be a parallel-transport operator.

9.1.1 First two characterizations of $Mon^2(K3^{[n]})$

Let X be an irreducible holomorphic symplectic manifold of $K3^{[n]}$-type. If $n = 1$, then X is a $K3$ surface. In that case it is well known that $Mon^2(X) = O^+H^2(X,\mathbb{Z})$ (see [Bor]). From now on we assume that $n \geq 2$.

Given a class $u \in H^2(X,\mathbb{Z})$, with $(u,u) \neq 0$, let $R_u : H^2(X,\mathbb{Q}) \to H^2(X,\mathbb{Q})$ be the reflection $R_u(\lambda) = \lambda - \frac{2(u,\lambda)}{(u,u)}u$. Set $\rho_u := \begin{cases} R_u & \text{if } (u,u) < 0, \\ -R_u & \text{if } (u,u) > 0. \end{cases}$ Then ρ_u belongs to $O^+H^2(X,\mathbb{Q})$. Note that ρ_u is an integral isometry, if $(u,u) = 2$ or -2. Let $\mathscr{N} \subset O^+H^2(X,\mathbb{Z})$ be the subgroup generated by such ρ_u.

$$\mathscr{N} \ := \ \langle \rho_u \ : \ u \in H^2(X,\mathbb{Z}) \text{ and } (u,u) = 2 \text{ or } (u,u) = -2 \rangle. \tag{9.1}$$

Clearly, $\mathscr{N}$ is a normal subgroup.

Theorem 9.1 *([Ma5], Theorem 1.2)* $Mon^2(X) = \mathscr{N}$.

A second useful description of $Mon^2(X)$ depends on the fact that the lattice $H^2(X,\mathbb{Z})$ is isometric to the orthogonal direct sum

$$\Lambda \ := \ E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U \oplus \mathbb{Z}\delta,$$

where $E_8(-1)$ is the negative definite (unimodular) E_8 root lattice, U is the rank 2 unimodular lattice of signature $(1,1)$, and $(\delta,\delta) = 2-2n$. See [Be1] for a proof of this fact.

Set $\Lambda^* := \mathrm{Hom}(\Lambda,\mathbb{Z})$. Then Λ^*/Λ is a cyclic group of order $2n-2$. Let $O(\Lambda^*/\Lambda)$ be the subgroup of $\mathrm{Aut}(\Lambda^*/\Lambda)$ consisting of multiplication by all elements of $t \in \mathbb{Z}/(2n-2)\mathbb{Z}$, such that $t^2 = 1$. Then $O(\Lambda^*/\Lambda)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^r$, where r is the number of distinct primes in the prime factorization $n-1 = p_1^{d_1}\cdots p_r^{d_r}$ of $n-1$ (see [Ogu]). The isometry group $O(\Lambda)$ acts on Λ^*/Λ and the image of $O^+(\Lambda)$ in $\mathrm{Aut}(\Lambda^*/\Lambda)$ is equal to $O(\Lambda^*/\Lambda)$ ([Ni], Theorem 1.14.2).

Let $\pi : O^+(\Lambda) \to O(\Lambda^*/\Lambda)$ be the natural homomorphism. The following characterization of the monodromy group follows from Theorem 9.1 via lattice theoretic arguments.

Lemma 9.2 *([Ma5], Lemma 4.2) $Mon^2(X)$ is equal to the inverse image via π of the subgroup $\{1,-1\} \subset O(\Lambda^*/\Lambda)$.*

We conclude that the index of $Mon^2(X)$ in $O^+H^2(X,\mathbb{Z})$ is 2^{r-1}, and $Mon^2(X) = O^+H^2(X,\mathbb{Z})$, if and only if $n=2$ or $n-1$ is a prime power. If $n=7$, for example, then $Mon^2(X)$ has index two in $O^+H^2(X,\mathbb{Z})$.

9.1.2 A third characterization of $Mon^2(K3^{[n]})$

The third characterization of $Mon^2(X)$ is more subtle, as it depends also on $H^4(X,\mathbb{Z})$. It is however this third characterization that will generalize to the case of parallel transport operators.

Given a $K3$ surface S, denote by $K(S)$ the integral K-ring generated by the classes of complex topological vector bundles over S. Let $\chi : K(S) \to \mathbb{Z}$ be the Euler characteristic $\chi(x) = \int_S ch(x)td_S$. Given classes $x,y \in K(S)$, let $x^\vee$ be the dual class and set

$$(x,y) \;:=\; -\chi(x^\vee \otimes y). \tag{9.2}$$

The above yields a unimodular symmetric bilinear pairing on $K(S)$, called the *Mukai pairing* [Mu1]. The lattice $K(S)$, endowed with the Mukai pairing, is isometric to the orthogonal direct sum

$$\widetilde{\Lambda} \;:=\; E_8(-1)\oplus E_8(-1)\oplus U\oplus U\oplus U\oplus U$$

and is called the *Mukai lattice*.

Let $Q^4(X,\mathbb{Z})$ be the quotient of $H^4(X,\mathbb{Z})$ by the image of the cup product homomorphism $\cup : H^2(X,\mathbb{Z}) \otimes H^2(X,\mathbb{Z}) \to H^4(X,\mathbb{Z})$. Clearly, $Q^4(X,\mathbb{Z})$ is a $Mon(X)$-module, and it comes with a pure integral Hodge structure of weight 4. Let $q : H^4(X,\mathbb{Z}) \to Q^4(X,\mathbb{Z})$ be the natural homomorphism and set $\bar{c}_2(X) := q(c_2(TX))$.

Theorem 9.3 *([Ma5], Theorem 1.10) Let X be of $K3^{[n]}$-type, $n \geq 4$.*

(1) $Q^4(X,\mathbb{Z})$ is a free abelian group of rank 24.

(2) The element $\frac{1}{2}\bar{c}_2(X)$ is an integral and primitive class in $Q^4(X,\mathbb{Z})$.

(3) There exists a unique symmetric, even, integral, unimodular, $Mon(X)$-invariant bilinear pairing $(\bullet,\bullet)$ on $Q^4(X,\mathbb{Z})$, such that $\left(\frac{\bar{c}_2(X)}{2}, \frac{\bar{c}_2(X)}{2}\right) = 2n-2$. The resulting lattice $\left[Q^4(X,\mathbb{Z}),(\bullet,\bullet)\right]$ is isometric to the Mukai lattice $\widetilde{\Lambda}$.

(4) The $Mon(X)$-module $\mathrm{Hom}\left[H^2(X,\mathbb{Z}),Q^4(X,\mathbb{Z})\right]$ contains a unique integral rank 1 saturated $Mon(X)$-submodule

$$\mathbb{E}(X),$$

which is a sub-Hodge structure of type $(1,1)$. A generator $e \in \mathbb{E}(X)$ induces a Hodge-isometry

$$e \; : \; H^2(X,\mathbb{Z}) \;\longrightarrow\; \bar{c}_2(X)^{\perp}$$

onto the co-rank 1 sublattice of $Q^4(X,\mathbb{Z})$ orthogonal to $\bar{c}_2(X)$.

Parts (1), (3), and (4) of the Theorem are explained in the following section 9.1.3.

Denote by $O(\Lambda,\widetilde{\Lambda})$ the set of primitive isometric embeddings of the $K3^{[n]}$-lattice Λ into the Mukai lattice $\widetilde{\Lambda}$. The isometry groups $O(\Lambda)$ and $O(\widetilde{\Lambda})$ act on $O(\Lambda,\widetilde{\Lambda})$. The action on $\iota \in O(\Lambda,\widetilde{\Lambda})$, of elements $g \in O(\Lambda)$, and $f \in O(\widetilde{\Lambda})$, is given by $(g,f)\iota = f \circ \iota \circ g^{-1}$.

Lemma 9.4 *([Ma5], Lemma 4.3) $O^+(\Lambda) \times O(\widetilde{\Lambda})$ acts transitively on $O(\Lambda,\widetilde{\Lambda})$. The subgroup $\mathscr{N} \subset O^+(\Lambda)$, given in (9.1), is equal to the stabilizer in $O^+(\Lambda)$ of every point in the orbit space $O(\Lambda,\widetilde{\Lambda})/O(\widetilde{\Lambda})$.*

The lemma implies that $O(\Lambda,\widetilde{\Lambda})$ is a finite set of order $[\mathcal{N} : O^+(\Lambda)]$. The following is our third characterization of $Mon^2(X)$.

Corollary 9.5

(1) An irreducible holomorphic symplectic manifold X of $K3^{[n]}$-type, $n \geq 2$, comes with a natural choice of an $O(\widetilde{\Lambda})$-orbit ι_X of primitive isometric embeddings of $H^2(X,\mathbb{Z})$ in the Mukai lattice $\widetilde{\Lambda}$.

(2) The subgroup $Mon^2(X)$ of $O^+[H^2(X,\mathbb{Z})]$ is equal to the stabilizer of ι_X as an element of the orbit space $O\left(H^2(X,\mathbb{Z}),\widetilde{\Lambda}\right)/O(\widetilde{\Lambda})$.

Proof Part (1): If $n=2$, or $n=3$, then $O(\Lambda,\widetilde{\Lambda})$ is a singleton, and there is nothing to prove. Assume that $n \geq 4$. Let $e : H^2(X,\mathbb{Z}) \to Q^4(X,\mathbb{Z})$ be one of the two generators of $\mathbb{E}(X)$. Choose an isometry $g : Q^4(X,\mathbb{Z}) \to \widetilde{\Lambda}$. This is possible by Theorem 9.3. Set $\iota := g \circ e : H^2(X,\mathbb{Z}) \to \widetilde{\Lambda}$ and let ι_X be the orbit $O(\widetilde{\Lambda})\iota$. Then ι_X is independent of the choice of g. If we choose $-e$ instead we get the same orbit, since -1 belongs to $O(\widetilde{\Lambda})$.

Part (2): Follows immediately from Theorem 9.1 and Lemma 9.4. □

Example 9.6 Let S be a projective $K3$ surface, H an ample line bundle on S, and $v \in K(S)$ a class in the K-group. Denote by $M_H(v)$ the moduli space of Gieseker-Maruyama-Simpson H-stable coherent sheaves on S of class v. A good reference about these moduli spaces is the book [HL]. Assume that $M_H(v)$ is smooth and projective (i.e., we assume that every H-semi-stable sheaf is automatically also H-stable). Then $M_H(v)$ is known to be connected and of $K3^{[n]}$-type, by a theorem due to Mukai, Huybrechts, O'Grady, and Yoshioka. It can be found in its final form in [Y2].

Let π_i be the projection from $S \times M_H(v)$ onto the i-th factor, $i = 1,2$. Denote by $\pi_{2_!} : K[S \times M_H(v)] \to K[M_H(v)]$ the Gysin map and by $\pi_1^! : K(S) \to K[S \times M_H(v)]$ the pull-back homomorphism. Assume, further, that there exists a universal sheaf $\mathscr{E}$ over $S \times M_H(v)$. Let $[\mathscr{E}] \in K[S \times M_H(v)]$ be the class of the universal sheaf in the K-group. We get the natural homomorphism

$$u \; : \; K(S) \;\; \to \;\; K(M_H(v)), \tag{9.3}$$

given by $u(x) := \pi_{2_!}\left\{\pi_1^!(x^\vee) \otimes [\mathscr{E}]\right\}$. Let $v^\perp \subset K(S)$ be the co-rank 1 sub-lattice of $K(S)$ orthogonal to the class v and consider Mukai's homomorphism

$$\theta \; : \; v^\perp \;\; \longrightarrow \;\; H^2(M_H(v),\mathbb{Z}), \tag{9.4}$$

given by $\theta(x) = c_1[u(x)]$. Then θ is an isometry, with respect to the Mukai pairing on $v^{\perp}$, and the Beauville-Bogomolov pairing on $H^2(\mathscr{M}_H(v),\mathbb{Z})$, by the work of Mukai, Huybrechts, O'Grady, and Yoshioka [Y2]. Furthermore, the orbit $\iota_{M_H(v)}$ of Corollary 9.5 is represented by the inverse of θ

$$\iota_{M_H(v)} = O[K(S)] \cdot \theta^{-1}, \tag{9.5}$$

by ([Ma5], Theorem 1.14).

9.1.3 Generators for the cohomology ring $H^*(X,\mathbb{Z})$

Part (1) of Theorem 9.3 is a simple consequence of the following result. Consider the case, where X is a moduli space M of H-stable sheaves on a $K3$ surface S and M is of $K3^{[n]}$-type, as in Example 9.6. Choose a basis $\{x_1,x_2,\ldots,x_{24}\}$ of $K(S)$. Let $u : K(S) \to K(M)$ be the homomorphism given in equation (9.3).

Theorem 9.7 *([Ma4], Theorem 1) The cohomology ring $H^*(M,\mathbb{Z})$ is generated by the Chern classes $c_j(u(x_i))$, for $1 \le i \le 24$, and for j an integer in the range $0 \le j \le 2n$.*

The map $\tilde{\varphi} : K(S) \to H^4(M,\mathbb{Z})$, given by $\tilde{\varphi}(x) = c_2(u(x))$, is not a group homomorphism. Nevertheless, the composition $\varphi := q \circ \tilde{\varphi} : K(S) \to Q^4(M,\mathbb{Z})$, of $\tilde{\varphi}$ with the projection $q : H^4(M,\mathbb{Z}) \to Q^4(M,\mathbb{Z})$, is a homomorphism of abelian groups ([Ma4], Proposition 2.6). We note here only that 2φ is clearly a group homomorphism, since $2c_2(y) = c_1^2(y) - 2ch_2(y)$, the map $2ch_2 : K(M) \to H^4(M,\mathbb{Z})$ is known to be a group homomorphism, and the term $c_1^2(y)$ is annihilated by the projection to $Q^4(M,\mathbb{Z})$.

Part (1) of Theorem 9.3 follows from the fact that φ is an isomorphism. The homomorphism φ is surjective, by Theorem 9.7. It remains to prove that φ is injective. Injectivity would follow, once we show that $Q^4(M,\mathbb{Z})$ has rank 24. Now cup product induces an injective homomorphism $\mathrm{Sym}^2 H^2(M,\mathbb{Q}) \to H^4(M,\mathbb{Q})$, for any irreducible holomorphic symplectic manifold of dimension ≥ 4, by a general result of Verbitsky [Ver1]. When $n \ge 4$, i.e., $\dim_{\mathbb{C}}(M) \ge 8$, then $\dim H^4(M,\mathbb{Q}) - \dim \mathrm{Sym}^2 H^2(M,\mathbb{Q}) = 24$, by Göttsche's formula for the Betti numbers of $S^{[n]}$ [Gö]. Hence, the rank of $Q^4(M,\mathbb{Z})$ is 24.

The bilinear pairing on $Q^4(M,\mathbb{Z})$, constructed in part (3) of Theorem 9.3, is simply the push-forward via the isomorphism φ of the Mukai pairing on $K(S)$. We then show that this bilinear pairing is monodromy invariant, hence it defines a bilinear pairing on $Q^4(X,\mathbb{Z})$, for any X of $K3^{[n]}$-type.

The isometric embedding $e : H^2(M,\mathbb{Z}) \to Q^4(M,\mathbb{Z})$, constructed in part (4) of Theorem 9.3, is simply the composition $\varphi \circ \theta^{-1}$, where θ is given in equation (9.4). We show that the composition is $Mon(M)$-equivariant, up to sign, hence defines the $Mon(X)$-submodule $\mathbb{E}(X)$ in part (4) of Theorem 9.3, for any X of $K3^{[n]}$-type.

9.1.4 Parallel transport operators of $K3^{[n]}$-type

Let X_1 and X_2 be irreducible holomorphic symplectic manifolds of $K3^{[n]}$-type. Denote by ι_{X_i} the natural $O(\widetilde{\Lambda})$-orbit of primitive isometric embedding of $H^2(X_i,\mathbb{Z})$ into the Mukai lattice $\widetilde{\Lambda}$, given in Corollary 9.5.

Theorem 9.8 *An isometry $g : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ is a parallel-transport operator, if and only if g is orientation preserving and*

$$\iota_{X_1} = \iota_{X_2} \circ g. \tag{9.6}$$

Proof Assume first that g is a parallel-transport operator. Then g lifts to a parallel-transport operator $\tilde{g} : H^*(X_1,\mathbb{Z}) \to H^*(X_2,\mathbb{Z})$. Now $\tilde{g}$ induces a parallel-transport operators $\tilde{g}_4 : Q^4(X_1,\mathbb{Z}) \to Q^4(X_2,\mathbb{Z})$, as well as

$$Ad_{\tilde{g}} : \mathrm{Hom}\left[H^2(X_1,\mathbb{Z}), Q^4(X_1,\mathbb{Z})\right] \longrightarrow \mathrm{Hom}\left[H^2(X_2,\mathbb{Z}), Q^4(X_2,\mathbb{Z})\right],$$

given by $f \mapsto \tilde{g}_4 \circ f \circ g^{-1}$. We have the equality $Ad_{\tilde{g}}(\mathbb{E}_{X_1}) = \mathbb{E}_{X_2}$, by the characterization of the $Mon(X_i)$-module $\mathbb{E}(X_i)$ provided in Theorem 9.3. Hence, the equality (9.6) holds, by construction of ι_{X_i}.

Conversely, assume that the isometry g satisfies the equality (9.6). There exists a parallel-transport operator $f : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$, since X_1 and X_2 are deformation equivalent. Hence, the equality $\iota_{X_1} = \iota_{X_2} \circ f$ holds, as well. We get the equality $\iota_{X_1} = \iota_{X_1} \circ f^{-1}g$. We conclude that $f^{-1}g$ belongs to $Mon^2(X_1)$, by Corollary 9.5. The equality $g = f(f^{-1}g)$ represents g as a composition of two parallel-transport operators. Hence, g is a parallel-transport operator. □

The following statement is an immediate corollary of Theorems 1.3 and 9.8.

Corollary 9.9 *Let X and Y be two manifolds of $K3^{[n]}$-type.*

(1) X and Y are bimeromorphic, if and only if there exists a Hodge-isometry $f : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$, satisfying $\iota_X = \iota_Y \circ f$.

(2) X and Y are isomorphic, if and only if there exists a Hodge-isometry f as in part (1), which maps some Kähler class of X to a Kähler class of Y.

We do not require f in part (1) to be orientation preserving, since if it is not then $-f$ is, and the orbits $\iota_Y \circ f$ and $\iota_Y \circ (-f)$ are equal.

Let τ be the set of connected components of the moduli space of marked pairs (X,η), where X is of $K3^{[n]}$-type, and $\eta : H^2(X,\mathbb{Z}) \to \Lambda$ is an isometry. Denote by $\mathfrak{M}_\Lambda^\tau$ the moduli space of isomorphism classes of marked pairs (X,η), where X is of $K3^{[n]}$-type. The group $O(\Lambda)$ acts on the set τ and the stabilizer of a connected component $\mathfrak{M}_\Lambda^t$, $t \in \tau$, is the monodromy group $Mon^2(\mathfrak{M}_\Lambda^t) \subset O(\Lambda)$ (Definition 7.2). Let

$$\text{orb} : \mathfrak{M}_\Lambda^\tau \to O(\Lambda,\widetilde{\Lambda})/O(\widetilde{\Lambda})$$

be the map given by $(X,\eta) \mapsto \iota_X \circ \eta^{-1}$. Let orient : $\mathfrak{M}_\Lambda^\tau \to \text{Orient}(\Lambda)$ be the map given in equation (4.2). The characterization of the monodromy group in Corollary 9.5 yields the following enumeration of τ.

Corollary 9.10 *The map* $(\text{orb}, \text{orient}) : \mathfrak{M}_\Lambda^\tau \to O(\Lambda,\widetilde{\Lambda})/O(\widetilde{\Lambda}) \times \text{Orient}(\Lambda)$ *factors through a bijection*

$$\tau \to O(\Lambda,\widetilde{\Lambda})/O(\widetilde{\Lambda}) \times \text{Orient}(\Lambda).$$

9.2 A numerical determination of the fundamental exceptional chamber

Definition 9.11 A class $\ell \in H^{1,1}(X,\mathbb{Z})$ is called *monodromy-reflective*, if ℓ is a primitive class, $(\ell,\ell) < 0$, and R_ℓ is a monodromy operator. A holomorphic line bundle $L \in \text{Pic}(X)$ is called *monodromy-reflective*, if the class $c_1(L)$ is *monodromy-reflective*.

Let X be a manifold of $K3^{[n]}$-type, $n \geq 2$. In section 9.2.1 we classify monodromy-orbits of monodromy-reflective classes. This is done in terms of explicitly computable monodromy invariants. In section 9.2.2 we describe the values of the monodromy invariants, for which the monodromy-reflective class is stably prime-exceptional (Theorem 9.17). When X is projective Theorems 6.17 and 9.17

combine to provide a determination of the closure $\overline{\mathcal{BK}}_X$ of the birational Kähler cone in $\mathcal{C}_X$ in terms of explicitly computable invariants.

9.2.1 Monodromy-reflective classes of $K3^{[n]}$-type

Set $\Lambda := H^2(X,\mathbb{Z})$. Recall that if $\ell \in \Lambda$ is monodromy-reflective, then R_ℓ acts on Λ^*/Λ via multiplication by ± 1 (Lemma 9.2). The set of monodromy-reflective classes is determined by the following statement.

Proposition 9.12 *([Ma7], Proposition 1.5) Let $\ell \in H^2(X,\mathbb{Z})$ be a primitive class of negative degree $(\ell,\ell) < 0$. Then R_ℓ belongs to $Mon^2(X)$, if and only if ℓ has one of the following two properties.*

(1) $(\ell,\ell) = -2$.

(2) $(\ell,\ell) = 2-2n$*, and* $(n-1)$ *divides the class* $(\ell,\bullet) \in H^2(X,\mathbb{Z})^*$.

R_ℓ acts on Λ/Λ^* as the identity in case (1), and via multiplication by -1 in case (2).

Given a primitive class $e \in H^2(X,\mathbb{Z})$, we denote by $\mathrm{div}(e,\bullet)$ the largest positive integer dividing the class $(e,\bullet) \in H^2(X,\mathbb{Z})^*$. Let $\mathcal{R}_n(X) \subset H^2(X,\mathbb{Z})$ be the subset of primitive classes of degree $2-2n$, such that $n-1$ divides $\mathrm{div}(e,\bullet)$. Let $\ell \in \mathcal{R}_n(X)$ and choose an embedding $\iota : H^2(X,\mathbb{Z}) \hookrightarrow \widetilde{\Lambda}$ in the natural orbit ι_X provided by Corollary 9.5. Choose a generator $v \in \widetilde{\Lambda}$ of the rank 1 sublattice orthogonal to the image of ι. Set $e := \iota(\ell)$ and let

$$L \subset \widetilde{\Lambda} \tag{9.7}$$

be the saturation of the rank 2 sublattice spanned by e and v.

Definition 9.13 Two pairs (L_i,e_i), $i = 1,2$, each consisting of a lattice L_i and a class $e_i \in L_i$, are said to be *isometric*, if there exists an isometry $g : L_1 \to L_2$, such that $g(e_1) = e_2$.

Given a rank 2 lattice L, let $I_n(L) \subset L$ be the subset of primitive classes e with $(e,e) = 2-2n$.

Lemma 9.14 *There exists a natural one-to-one correspondence between the orbit set $I_n(L)/O(L)$ and the set of isometry classes of pairs (L',e'), such that L' is isometric to L and e' is a primitive class in L' with $(e',e') = 2-2n$.*

Proof Let $\mathscr{P}(L,n)$ be the set of isometry classes of pairs (L',e') as above. Define the map $f : \mathscr{P}(L,n) \to I_n(L)/O(L)$ as follows. Given a pair (L',e') representing a class in $\mathscr{P}(L,n)$, choose an isometry $g : L' \to L$ and set $f(L',e') := O(L)g(e')$. The map f is well defined, since the orbit $O(L)g(e')$ is clearly independent of the choice of g. The map f is surjective, since given $e \in I_n(L)$, $f(L,e) = O(L)e$. If $f(L_1,e_1) = f(L_2,e_2)$, then there exist isometries $g_i : L_i \to L$ and an element $h \in O(L)$, such that $g_2(e_2) = hg_1(e_1)$. Then $g_2^{-1}hg_1$ is an isometry from (L_1,e_1) to (L_2,e_2). Hence, the map f is injective. □

Let U be the unimodular hyperbolic plane. Let $U(2)$ be the rank 2 lattice with Gram matrix $\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$ and let D be the rank 2 lattice with Gram matrix $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$.

Proposition 9.15 *([Ma7], Propositions 1.8 and 6.2)*

(1) If $(\ell,\ell) = -2$ then the $Mon^2(X)$-orbit of ℓ is determined by $\mathrm{div}(\ell,\bullet)$.

(2) Let $\ell \in \mathscr{R}_n(X)$.

(1) The lattice L, given in (9.7), is isometric to one of the lattices U, $U(2)$, or D.

(2) Let $f : \mathscr{R}_n(X) \longrightarrow I_n(U)/O(U) \cup I_n(U(2))/O(U(2)) \cup I_n(D)/O(D)$ be the function, sending a class ℓ to the isometry class of the pair $(L,\iota(\ell))$. Then the values $\mathrm{div}(\ell,\bullet)$ *and $f(\ell)$ determine the $Mon^2(X)$-orbit of ℓ.*

The values of the function f can be conveniently enumerated and calculated as follows. Set $e := \iota(\ell) \in L$. Let ρ be the largest integer, such that $(e+v)/\rho$ is an integral class of L. Let σ be the largest integer, such that $(e-v)/\sigma$ is an integral class of L. If $\mathrm{div}(\ell,\bullet) = n-1$ and n is even, set $\{r,s\}(\ell) = \{\rho,\sigma\}$. Otherwise, set $\{r,s\}(\ell) = \{\frac{\rho}{2},\frac{\sigma}{2}\}$. The unordered pair $\{r,s\} := \{r,s\}(\ell)$ has the following properties.

Proposition 9.16 *([Ma7], Lemma 6.4)*

(1) The isometry class of the lattice L and the product rs are determined in terms of (ℓ,ℓ), $\mathrm{div}(\ell,\bullet)$, *$n$, and $\{\rho,\sigma\}$ by the following table.*

	(ℓ,ℓ)	$\mathrm{div}(\ell,\bullet)$	n	$\rho\sigma$	L	$\{r,s\}$	$r\cdot s$
1)	$2-2n$	$2n-2$	≥ 2	$4n-4$	U	$\{\frac{\rho}{2},\frac{\sigma}{2}\}$	$n-1$
2)	$2-2n$	$n-1$	*even*	$n-1$	D	$\{\rho,\sigma\}$	$n-1$
3)	$2-2n$	$n-1$	*odd*	$2n-2$	$U(2)$	$\{\frac{\rho}{2},\frac{\sigma}{2}\}$	$(n-1)/2$
4)	$2-2n$	$n-1$	$\equiv 1$ *modulo* 8	$n-1$	D	$\{\frac{\rho}{2},\frac{\sigma}{2}\}$	$(n-1)/4$

(2) The pair $\{r,s\}$ consists of relatively prime positive integers. All four rows in the above table do occur, and every relatively prime decomposition $\{r,s\}$ of the integer in the rightmost column occurs, for some $\ell\in\mathscr{R}_n(X)$.

(3) If $\ell\in\mathscr{R}_n(X)$, then $\mathrm{div}(\ell,\bullet)$ and $\{r,s\}(\ell)$ determine the $Mon^2(X)$-orbit of ℓ.

9.2.2 Stably prime-exceptional classes of $K3^{[n]}$-type

Theorem 9.17 *([Ma7], Theorem 1.12). Let $\kappa\in H^{1,1}(X,\mathbb{R})$ be a Kähler class and L a monodromy reflective line bundle. Set $\ell:=c_1(L)$. Assume that $(\kappa,\ell)>0$.*

(1) If $(\ell,\ell)=-2$, then L^k is stably prime-exceptional, where

$$k=\begin{cases}2, \text{ if } \ \mathrm{div}(\ell,\bullet)=2 \ \text{ and } n=2,\\ 1, \text{ if } \ \mathrm{div}(\ell,\bullet)=2 \ \text{ and } n>2,\\ 1 \text{ if } \ \mathrm{div}(\ell,\bullet)=1.\end{cases}$$

(2) If $\mathrm{div}(\ell,\bullet)=2n-2$ and $\{r,s\}(\ell)=\{1,n-1\}$, then L^2 is stably prime-exceptional.

(3) If $\mathrm{div}(\ell,\bullet)=2n-2$ and $\{r,s\}(\ell)=\{2,(n-1)/2\}$, then L is stably prime-exceptional.

(4) If $\mathrm{div}(\ell,\bullet)=n-1$, n is even, and $\{r,s\}(\ell)=\{1,n-1\}$, then L is stably prime-exceptional.

(5) If $\mathrm{div}(\ell,\bullet)=n-1$, n is odd, and $\{r,s\}(\ell)=\{1,(n-1)/2\}$, then L is stably prime-exceptional.

(6) In all other cases, $H^0(L^k)$ vanishes, and so L^k is not stably prime-exceptional, for every non-zero integer k.

When X is projective Proposition 9.12 and Theorem 9.17 determine the set $\mathcal{S}pe\subset H^{1,1}(X,\mathbb{Z})$, of stably prime-exceptional classes, and hence also the fundamental exceptional chamber $\mathscr{F}\mathscr{E}_X$, by Proposition 6.10.

The proof of Theorem 9.17 has two ingredients. First we deform the pair (X,L) to a pair (M,L_1), where M is a moduli space of sheaves on a projective $K3$ surface, and L_1 is a monodromy-reflective line bundle with the same monodromy invariants. Then L is stably prime-exceptional, if and only if L_1 is, by Proposition 6.6. We then laboriously check an example, one for each value of the monodromy invariants n, (ℓ,ℓ), $\mathrm{div}(\ell,\bullet)$, and $\{r,s\}(\ell)$, and show that either R_ℓ is induced by a birational map $f: M \to M$, such that $f^*(L_1) = L_1^{-1}$, or that the linear system $|L_1^k|$ consists of a single prime exceptional divisor, for the power k prescribed by Theorem 9.17.

The two possible values of the degree -2 or $2-2n$, of a prime exceptional divisor, correspond to two types of well known constructions in the theory of moduli spaces of sheaves on a $K3$ surface S. We briefly describe these constructions below.

Pairs $(M, \mathscr{O}_M(E))$, where $M := M_H(v)$ is a moduli space of H-stable coherent sheaves of class $v \in K(S)$, and E is a prime exceptional divisor of Beauville-Bogomolov degree -2, arise as follows. The Mukai isometry (9.4) associates to the line bundle $\mathscr{O}_M(E)$ a class $e \in v^\perp$, with $(e,e) = -2$. In the examples considered in [Ma7], e is the class of an H-stable sheaf F on S. Such a sheaf is necessarily rigid, i.e., $\mathrm{Ext}^1(F,F) = 0$. Indeed,

$$\dim \mathrm{Ext}^1(F,F) = \dim \mathrm{Hom}(F,F) + \dim \mathrm{Ext}^2(F,F) - \chi(F^\vee \otimes F) = 1+1-2 = 0.$$

Furthermore, the moduli space $M_H(e)$ is connected, by a theorem of Mukai, and consists of the single point $\{F\}$ (see [Mu1]). The prime exceptional divisor E is the Brill-Noether locus

$$\{V \in M_H(v) \ : \ \dim \mathrm{Ext}^1(F,V) > 0\}.$$

Specific examples are easier to describe using Mukai's notation. Recall Mukai's isomorphism

$$ch(\bullet)\sqrt{td_S} \ : \ K(S) \ \longrightarrow \ H^*(S,\mathbb{Z}), \tag{9.8}$$

sending a class $v \in K(S)$ to the integral singular cohomology group. Let $D : H^*(S,\mathbb{Z}) \to H^*(S,\mathbb{Z})$ be the automorphism acting by $(-1)^i$ on $H^{2i}(S,\mathbb{Z})$. The homomorphism (9.8) is an isometry once we endow $H^*(S,\mathbb{Z})$ with the pairing

$$(x,y) \ := \ -\int_S D(x) \cup y,$$

by the Hirzebruch-Riemann-Roch theorem and the definition of the Mukai pairing in equation (9.2). We have $ch(v)\sqrt{td_S} = (r, c_1(v), s)$, where $r = \mathrm{rank}(v)$, $s = \chi(v) - r$, and we identify $H^0(S,\mathbb{Z})$ and $H^4(S,\mathbb{Z})$ with $\mathbb{Z}$, using the classes Poincaré-dual to S and to a point. Given two classes $v_i \in K(S)$, with $\mathrm{rank}(v_i) = r_i$, $c_1(v_i) = \alpha_i$, and

$s_i := \chi(v_i) - r_i$, then

$$(v_1, v_2) \;=\; \left(\int_S \alpha_1\alpha_2\right) - r_1s_2 - r_2s_1.$$

Example 9.18 Following is a simple example in which a prime exceptional divisor E of degree -2 and divisibility $\mathrm{div}([E],\bullet) = 1$ is realized as a Brill-Noether locus. Consider a $K3$ surface S, containing a smooth rational curve C. Consider the Hilbert scheme $M := S^{[n]}$ as the moduli space of ideal sheaves of length n subschemes. Let F be the torsion sheaf $\mathscr{O}_C(-1)$, supported on C as a line bundle of degree -1. Let $v \in K(S)$ be the class of an ideal sheaf in $S^{[n]}$ and e the class of F. The Mukai vector of v is $(1,0,1-n)$, that of e is $(0,[C],0)$, and $(v,e) = 0$. Let $E \subset M$ be the divisor of ideal sheaves I_Z of subscheme Z with non-empty intersection $Z \cap C$. The space $\mathrm{Hom}(F, I_Z)$ vanishes for all $I_Z \in M$, and so $\dim \mathrm{Ext}^1(F, I_Z) = \dim \mathrm{Ext}^2(F, I_Z)$, for all $I_Z \in M$. Now, $\mathrm{Ext}^2(F, I_Z) \cong \mathrm{Hom}(I_Z, F)^*$ vanishes, if and only if $Z \cap C = \emptyset$. Hence, $\mathrm{Ext}^1(F, I_Z) \neq 0$, if and only if I_Z belongs to E. See [Ma1, Y1] for many more examples of prime exceptional divisors E of degree -2 and $\mathrm{div}([E],\bullet) = 1$. See [Ma7], Lemma 10.7 for the case $(e,e) = -2$, $\mathrm{div}(e,\bullet) = 2$, and $n \equiv 2$ modulo 4.

Jun Li constructed a birational morphism from the moduli space of Gieseker-Maruyama H-stable sheaves on a $K3$ surface to the Uhlenbeck-Yau compactification of the moduli space of H-slope-stable locally-free sheaves [Li]. The examples of prime exceptional divisors of degree $2-2n$ on a moduli space of sheaves, provided in [Ma7], were all constructed as exceptional divisors for Jun Li's morphism.

Example 9.19 The simplest example is the Hilbert-Chow morphism, from the Hilbert scheme $S^{[n]}$, $n \geq 2$, to the symmetric product $S^{(n)}$ of a $K3$ surface S, where the exceptional divisor E is the big diagonal. The Mukai vector of the ideal sheaf is $v = (1,0,1-n)$. In this case $[E] = 2\delta$, where $\delta = (1,0,n-1)$. Note that $(\delta,\delta) = 2-2n$. The second cohomology of $S^{[n]}$ is an orthogonal direct sum $H^2(S,\mathbb{Z}) \oplus \mathbb{Z}\delta$, by [Be1] or by Mukai's isometry (9.4). Hence, $\mathrm{div}(\delta,\bullet) = 2n-2$. The largest integer ρ dividing $\delta + v = (2,0,0)$ is 2 and the largest integer σ dividing $\delta - v = (0,0,2n-2)$ is $2n-2$. Hence, $\{r,s\}(\delta) = \{1, n-1\}$, by Proposition 9.16 and Equation (9.5).

Example 9.20 Consider, more generally, the moduli space $M_H(r,0,-s)$ of H-stable sheaves with Mukai vector $v = (r,0,-s)$, satisfying $s > r \geq 1$ and $\gcd(r,s) = 1$. Then $M_H(r,0,-s)$ is of $K3^{[n]}$-type, $n = rs+1$. The Mukai vector $e := (r,0,s) \in v^\perp$ maps to a monodromy-reflective class $\ell \in H^2(M_H(v),\mathbb{Z})$ of degree $(\ell,\ell) = 2-2n$, divisibility $\mathrm{div}(\ell,\bullet) = 2n-2$, and $\{r,s\}(\ell) = \{r,s\}$, by Proposition 9.16 and Equation

(9.5). When $r = 2$, ℓ is the class of the exceptional divisor E of Jun Li's morphism. E is the locus of sheaves, which are not locally-free or not H-slope-stable ([Ma7], Lemma 10.16). When $r > 2$, the exceptional locus has co-dimension ≥ 2, and no multiple of the class ℓ is effective. Instead, the reflection R_ℓ is induced by the birational map $f : M_H(r,0,-s) \to M_H(r,0,-s)$, sending a locally-free H-slope stable sheaf F of class $(r,0,-s)$ to the dual sheaf F^* ([Ma7], Proposition 11.1).

Remark 9.21 Fix an integer $n > 0$, such that $n-1$ is not a prime power, and consider all possible factorizations $n-1 = rs$, with $s > r \geq 1$ and $\gcd(r,s) = 1$. The sublattice $(r,0,-s)^\perp$ of the Mukai lattice of a $K3$ surface S is the orthogonal direct sum $H^2(S,\mathbb{Z}) \oplus \mathbb{Z}(r,0,s)$. We get the isometry

$$\theta \; : \; H^2(S,\mathbb{Z}) \oplus \mathbb{Z}(r,0,s) \;\; \longrightarrow \;\; H^2\left(M_H(r,0,-s),\mathbb{Z}\right),$$

using Mukai's isometry given in equation (9.4). Let $n-1 = r_1s_1 = r_2s_2$ be two different such factorizations. Then the two moduli spaces $M_H(r_1,0,-s_1)$ and $M_H(r_2,0,-s_2)$, considered in Example 9.20, come with a natural Hodge isometry

$$g \; : \; H^2(M_H(r_1,0,-s_1),\mathbb{Z}) \;\; \longrightarrow \;\; H^2(M_H(r_2,0,-s_2),\mathbb{Z}),$$

which restricts as the identity on the direct summand $\theta\left(H^2(S,\mathbb{Z})\right)$ and maps the class $\ell_1 := \theta(r_1,0,s_1) \in H^2(M_H(r_1,0,-s_1),\mathbb{Z})$ to the class $\ell_2 := \theta(r_2,0,s_2) \in H^2(M_H(r_2,0,-s_2),\mathbb{Z})$. The Hodge isometry g is not a parallel-transport operator, since the monodromy-invariants $\{r,s\}(\ell_i) = \{r_i,s_i\}$ are distinct. Indeed, these moduli spaces are not birational in general ([Ma5], Proposition 4.10). Furthermore, if $n-1 = rs$ is such a factorization with $r > 2$, then the birational Kähler cones $\mathscr{B}\mathscr{K}_{S^{[n]}}$ and $\mathscr{B}\mathscr{K}_{M_H(r,0,-s)}$ are not isometric in general. Indeed, $S^{[n]}$ admits a stably prime-exceptional class, while $M_H(r,0,-s)$ does not, for a $K3$ surface with a suitably chosen Picard lattice.

10 Open problems

Following is a very brief list of central open problems closely related to this survey. See [Be2] for a more complete recent survey of open problems in the subject of irreducible holomorphic symplectic manifolds.

Question 10.1 Let X be one of the known examples of irreducible holomorphic symplectic manifolds, i.e., of $K3^{[n]}$-type, a generalized Kummer variety, or one of the two exceptional examples of O'Grady [O'G2, O'G3]. Let Y be an irreducible holomorphic symplectic manifold, with $H^2(Y,\mathbb{Z})$ isometric to $H^2(X,\mathbb{Z})$. Is Y necessarily deformation equivalent to X?

Let Λ be a lattice isometric to $H^2(X,\mathbb{Z})$. At present it is only known that the number of deformation types of irreducible holomorphic symplectic manifolds of a given dimension $2n$, and with second cohomology lattice isometric to Λ, is finite [Hu4]. The moduli space $\mathfrak{M}_\Lambda$, of isomorphism classes of marked pairs (X,η), with X of dimension $2n$ and $\eta : H^2(X,\mathbb{Z}) \to \Lambda$ an isometry, has finitely many connected components, by Huybrechts' result and Lemma 7.5. O'Grady has made substantial progress towards the proof of uniqueness of the deformation type in case the dimension is 4 and the lattice Λ is of $K3^{[2]}$-type [O'G5].

Problem 10.2 *Let X be an irreducible holomorphic symplectic manifold of $K3^{[n]}$-type, $n \geq 2$. Determine the Kähler-type chamber (Definition 5.10) in the fundamental exceptional chamber $\mathscr{F}\mathscr{E}_X$ of X, containing a given very general class $\alpha \in \mathscr{F}\mathscr{E}_X$, in terms of the weight 2 integral Hodge structure $H^2(X,\mathbb{Z})$, the Beauville-Bogomolov pairing, and the orbit ι_X of isometric embeddings of $H^2(X,\mathbb{Z})$ in the Mukai lattice, given in Corollary 9.5.*

Note that the data specified in Problem 10.2 determines the isomorphism class of an irreducible holomorphic symplectic manifold Y, bimeromorphic to X, and an $\mathrm{Aut}(X) \times \mathrm{Aut}(Y)$-orbit[10] of a bimeromorphic map $f : Y \to X$, such that $f^*(\alpha)$ is a Kähler class on Y, by Corollaries 5.7 and 9.9. The homomorphism f^* takes the Kähler-type chamber in Problem 10.2 to $\mathscr{K}_Y$. Hassett and Tschinkel formulated a precise conjectural solution to problem 10.2 [HT4]. The Kähler cone, according to their conjecture, does not depend on the orbit ι_X. The birational Kähler cone does, as we saw in Remark 9.21.

Problem 10.3 *Find an explicit necessary and sufficient condition for a Hodge isometry $g : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ to be a parallel-transport operator, in the case X and Y are deformation equivalent to generalized Kummer varieties, or to O'Grady's two exceptional examples.*

Problem 10.4 *Let X be deformation equivalent to a generalized Kummer variety, or to one of O'Grady's two exceptional examples. Find an explicit necessary and sufficient condition for a class $\ell \in H^{1,1}(X,\mathbb{Z})$ to be stably prime-exceptional (Definition 6.4).*

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[10] The orbit of f is the set $\{g_1 f g_2^{-1} \ : \ g_1 \in \mathrm{Aut}(X),\ g_2 \in \mathrm{Aut}(Y)\}$.

Problem 10.4 is solved in the $K3^{[n]}$-type case (Proposition 9.12 and Theorem 9.17). A solution to problem 10.4 yields a determination of the fundamental exceptional chamber $\mathscr{F}\mathscr{E}_X$, by Proposition 1.8, and of the closure of the birational Kähler cone, by Proposition 5.6. Once solutions to Problems 10.3 and 10.4 are provided, the analogue of Problem 10.2 may be formulated as well.

Question 10.5 Is the monodromy group $Mon^2(X)$, of an irreducible holomorphic symplectic manifold X, necessarily a normal subgroup of the isometry group of $H^2(X,\mathbb{Z})$?

Let X be a generalized Kummer variety of dimension $2n$, $n \geq 2$. Then $H^2(X,\mathbb{Z})$ is isometric to the lattice $\Lambda := U \oplus U \oplus U \oplus \mathbb{Z}\delta$, where U is the unimodular rank 2 lattice of signature $(1,1)$, and $(\delta,\delta) = -2-2n$ (see [Be1, Y2]).

Conjecture 10.6 *$Mon^2(X)$ is equal to the subgroup $\mathscr{N}(X)$ of the signed isometry group $O^+H^2(X,\mathbb{Z})$, generated by products of an even number of reflections $R_{\ell_1}\cdots R_{\ell_{2k}}$, where $(\ell_i,\ell_i) = 2$, for an even number of indices i, and $(\ell_i,\ell_i) = -2$ for the rest of the indices i.*

The inclusion $\mathscr{N}(X) \subset Mon^2(X)$ was proven by the author in an unpublished work. When $n = 2$, the equality $\mathscr{N}(X) = Mon^2(X)$ follows from the Global Torelli Theorem 2.2 and Namikawa's counter example to the naive Hodge theoretic Torelli statement [Nam2].

Let X be an irreducible holomorphic symplectic manifold deformation equivalent to O'Grady's 10-dimensional exceptional example [O'G2]. Then $H^2(X,\mathbb{Z})$ is isometric to the orthogonal direct sum of $H^2(S,\mathbb{Z}) \oplus G_2$, where S is a $K3$ surface, and G_2 is the negative definite root lattice of type G_2, with Gram matrix $\begin{pmatrix} -2 & 3 \\ 3 & -6 \end{pmatrix}$ (see [R]). The isometry group $O(G_2)$ is equal to the Weyl group of G_2 and its extension to $H^2(X,\mathbb{Z})$, via the trivial action on $H^2(S,\mathbb{Z})$, is contained in $Mon^2(X)$, by ([Ma6], Lemma 5.1).

Conjecture 10.7 *$Mon^2(X) = O^+H^2(X,\mathbb{Z})$.*

There are many examples of non-isomorphic $K3$ surfaces with equivalent bounded derived categories of coherent sheaves [Or].

Question 10.8 Let X and Y be projective irreducible holomorphic symplectic manifolds, such that $H^2(X,\mathbb{Z})$ and $H^2(Y,\mathbb{Z})$ are Hodge isometric. Are their bounded derived categories of coherent sheaves necessarily equivalent?

When $X = S_1^{[n]}$ and $Y = S_2^{[n]}$, where S_1 and S_2 are $K3$ surfaces, the answer to Question 10.8 is affirmative (see the proof of [Pl], Proposition 10). See [Hu5] for a survey on the topic of question 10.8.

Recall that a class $\ell \in H^{1,1}(X,\mathbb{Z})$ is monodromy-reflective, if it is a primitive class, and the reflection R_ℓ is a monodromy operator (Definition 9.11).

Question 10.9 Let $\ell \in H^{1,1}(X,\mathbb{Z})$ be a monodromy-reflective class. Is there always some non-zero integer λ, such that the class $\lambda(\ell,\bullet) \in H^2(X,\mathbb{Z})^* \cong H_2(X,\mathbb{Z})$ corresponds to an effective one-cycle?

An affirmative answer to the above question implies that the reflection R_ℓ can not be induced by a regular automorphism[11] of X. It follows that the Kähler cone is contained in a unique chamber of the subgroup of $Mon^2_{Hdg}(X)$ generated by all reflections in $Mon^2_{Hdg}(X)$ (see Theorem 6.15).

Problem 10.10 *Prove an analogue of Proposition 6.1, about birational contractibility of a prime exceptional divisor, for non-projective irreducible holomorphic symplectic manifolds.*

Druel's proof of Proposition 6.1 relies on results in the minimal model program, which are currently not available in the Kähler category [Dr].

Question 10.11 Let X be a projective irreducible holomorphic symplectic manifold. Is the semi-group Σ, of effective divisor classes on X, equal to the semi-group Σ' generated by the prime exceptional classes and integral points on the closure $\overline{\mathscr{BK}}_X$ of the birational Kähler cone in $H^{1,1}(X,\mathbb{R})$?

The answer is affirmative for any $K3$ surface, even without the projectivity assumption ([BHPV], Ch. IIIV, Proposition 3.7). Stronger results hold true for projective $K3$ surfaces [Kov]. The inclusion $\Sigma \subset \Sigma'$ is known in general, by the divisorial Zariski decomposition (Theorem 5.8). The integral points of $\mathscr{C}_X \cap \overline{\mathscr{BK}}_X$ are known to be contained in Σ. This is seen as follows. The integral points of the positive cone

[11] A weaker version of this assertion, namely the non-existence of a fixed-point free such automorphism g, is always true. Indeed, if $g^* = R_\ell$, and g is a fixed-point-free (necessarily symplectic) automorphism, then g^2 acts trivially on $H^2(X,\mathbb{Z})$. Hence, g^2 is an isometry with respect to a Kähler metric. It follows that g has finite order, since it generates a discrete subgroup of the compact isometry group. Thus, $X/\langle g \rangle$ is a non simply connected holomorphic symplectic Kähler manifold, with $h^{k,0}(X) = 1$, for even k in the range $0 \leq k \leq \dim_{\mathbb{C}}(X)$, and $h^{k,0}(X) = 0$, otherwise. Such X does not exist, by [HN], Proposition A.1.

are known to correspond to big line bundles, by ([Hu1], Corollary 3.10). Each integral point of $\mathscr{C}_X \cap \overline{\mathscr{BK}}_X$ thus coresponds to a big and nef line bundle L on some birational irreducible holomorphic symplectic manifold Y, by Theorems 5.4 and 6.17, and so the cohomology groups $H^i(Y,L)$ vanish, for $i > 0$, by the Kawamata-Viehweg vanishing theorem. Set $\ell := c_1(L)$. If X is of $K3^{[n]}$-type or deformation equivalent to a generalized Kummer variety, then an explicit formula is known for the Euler characteristic $\chi(L)$ of a line bundle L, in terms of its Beauville-Bogomolov degree (ℓ,ℓ) ([Hu3], Examples 7 and 8). One sees, in particular, that $\chi(L) > 0$, if $(\ell,\ell) \geq 0$, and so L is effective.

An affirmative answer to Question 10.11 would thus follow, if one could prove that nef line bundles with $(\ell,\ell) = 0$ are effective. Some experts conjectured that such line bundles are related to Lagrangian fibrations ([Marku], Conjecture 1.7; [Saw], Conjecture 1, [Ver3], Conjecture 1.7). We refer the reader also to the important work of Matsushita on Lagrangian fibrations [Mat1, Mat2] and to the survey ([Be2], section 1.6).

Question 10.12 Which components, of the moduli spaces of polarized projective irreducible holomorphic symplectic manifolds, are unirational? Which are of general type?

Gritsenko, Hulek, and Sankaran had studied this question for fourfolds X of $K3^{[2]}$-type, and for primitive polarizations $h \in H^2(X,\mathbb{Z})$, with $\mathrm{div}(h,\bullet) = 2$. Let $(h,h) = 2d$. They show that for $d \geq 12$, the moduli space is of general type ([GHS1], Theorem 4.1). They use the theory of modular forms to show that the quotient of the period domain $\Omega^+_{h^\perp}$, given in equation (4.1), by the polarized monodromy group $Mon^2(X,h)$, is of general type.

On the other hand, unirational components are those likely to admit explicit and very beautiful geometric descriptions [BD, DV, IR, Mu2, O'G4].

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