Hierarchies and Optima

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Abstract. This paper is devoted to hierarchies of partitions, on which
all criteria are proved to be connective. Optimisations are addressed by
minimizing energies that satisfy the condition of hierarchical increasing-
ness. The optimal cuts through the hierarchies are found. It is shown
that many of the classical techniques are variants of what is proposed.

1 Introduction: Optimum Optimorum

Various segmentation algorithms base their approach on a stack of increasing
partitions depending on a positive index. The stack, which has been obtained
by a first wave of processing, serves as a framework for a second step, some-
times followed by a third one, and aims to lead to a final optimal partition of
segmentation. There are several reasons to do so. Some regions of the image un-
der study may require a finer treament than others. Then one starts from over
segmentations which are recomposed [2], [18], [19]. Sometimes, one wants to re-
duce the size of an image while keeping its major features [16]. Another reason
appears with multivariate data, when one has to merge partitions coming from
different bands [1]. Sometimes also, the hierarchy is a direct consequence of the
partitioning algorithm; a topological watershed, for example, produces a series
of edges of a constant value, their saliencies [9] [11].

Most of these algorithms yield a unique final partition, though the meaning of
the underlying optimization - if it exists- is rather unclear. A significant advance
to clarify this point comes from L. Guigues et Al.[5] who introduced a linear
decomposition of some energy over partitions. We will extend below the princi-
ple of their approach. After having decribed the structure of the hierarchies of
partitions (section 2), we develop some theory about minimum cuts (section 3),
which is illustrated by a few examples in section 4.

2 Hierarchies of Partitions

From now on we suppose that a first step of segmentation already led to a
finite hierarchy, i.e. to some finite sequence of increasing finite partitions. We
will now intend to reorganize the classes of these partitions for extracting their
quintessence, namely a more synthetic partition. Does the chain structure of
the initial partitions provide us with particular assets for the second step of
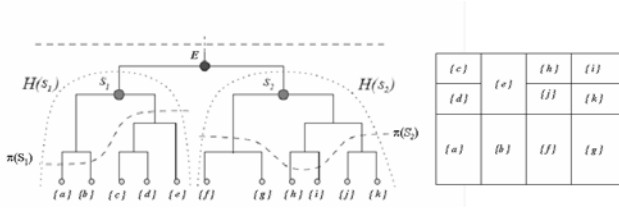optimization?

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Fig. 1. Left, hierarchical tree; right, the corresponding space structure. S_1 and S_2 are the nodes sons of E, and $H(S_1)$ and $H(S_1)$ are the associated sub-hierarchies. π_1 and π_2 are cuts of $H(S_1)$ and $H(S_1)$ respectively, and $\pi_1 \sqcup \pi_2$ is a cut of E.

2.1 Reminder

Finite hierarchies appeared initially in taxonomy for classifying objects. One can quote in particular the works of J.P. Benzécri [3] and of E. Diday [4]. We owe to the first author the theorem linking ultrametrics with hierarchies, which is involved below in Theorem 5. In image processing, hierarchical structures hold sometimes on stacks of images, but more often of operators, such as the classical families of Gaussian convolutions of D. Marr [7], or again the granulometries of G. Matheron [8]. The hierarchies indicated in the introduction occupy an intermediary position, since they start from a given chain H of segmentations of a fixed function f on set E, i.e. from a stack of scalar or vector images, and this chain is then used as the framework for further operators. Here, for the sake of simplicity, we consider function f and hierarchy H as two starting points, possibly independent. This results in the following definition:

Definition 1. *Let $\mathcal{D}$ be the set of all partitions of a finite set E, equipped with the refinement odrering $\ll$. A hierarchy H of partitions π_i is a finite chain in $\mathcal{D}$, i.e.*

$$H = \{\pi_i, 0 \le i \le n, \pi_i \in \mathcal{D} \mid i \le j \Rightarrow \pi_i \ll \pi_j\}, \tag{1}$$

of extremities the extrema of $\mathcal{D}$, namely $\pi_0 = \{\{x\}, x \in E\}$ and $\pi_n = \{E\}$.

Denote by $\mathcal{S}$ the set of all classes $S_i(x)$ of all partitions π_i of H, i.e. $\mathcal{S} = \{S_i(x), x \in E, 0 \le i \le n\}$. The expression (1) means that at each point $x \in E$ the family of those classes $S_i(x)$ of $\mathcal{S}$ that contain x forms a finite chain $\mathcal{S}_x$ in $\mathcal{P}(E)$, of nested elements from $\{x\}$ to E :

$$\mathcal{S}_x = \{S_i(x), 0 \le i \le n\}.$$

The finiteness E is not really necessary. It mainly serves to ensure that the border lengths between classes are finite, but in 2D the Euclidean Voronoi model grants it as well. On the other hand, the search for minimal cuts by an induction on the hierarchical levels (Proposition 7), requires that their number is finite. Remark also that the parameter i of the level can be replaced by any strictly increasing function $t(i)$, which defines another *indexing* of the hierarchy.
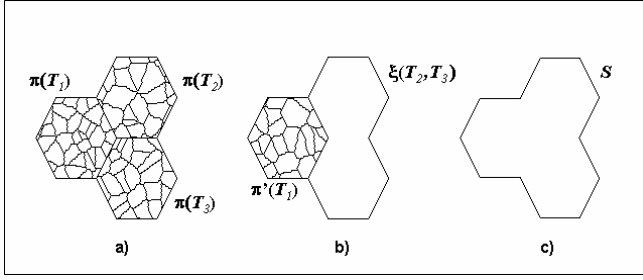
Fig. 2. The three hexagons a) are the three sons T_1, T_2, T_3 of summit S (in c)). The concatenation $\pi(T_1) \sqcup \pi(T_2) \sqcup \pi(T_3)$ of three of their cuts is a cut of S (a). Another cut of S is obtained by taking $T_1 \cup T_2 \cup T_3 = S$ (in c). However, $\pi'(T_1) \sqcup \xi(T_2, T_3)$, in b), is not a valid cut of S.

According to a classical result, a family $\{S_i(x), x \in E, 0 \leq i \leq n\}$ of indexed sets generates the classes of a hierarchy iff

$$i \leq j \quad \text{and} \quad x, y \in E \quad \Rightarrow \quad S_i(x) \subseteq S_j(y) \tag{2}$$
$$\text{or} \quad S_i(x) \supseteq S_j(y) \quad \text{or} \quad S_i(x) \cap S_j(y) = \emptyset.$$

Conventionally, a hierarchy is represented by a tree where each node of bifurcation is a class S. The classes of π_{i-1} at level $i - 1$ which are included in $S_i(x)$ are said to be the sons of $S_i(x)$. Clearly, the sets of the descendants of each S forms in turn a hierarchy $H(S)$ of summit S, which is included in the complete hierarchy $H = H(E)$. Figure 1 depicts two examples of such sub hierarchies, by the two zones $H(S_1)$ and $H(S_2)$ in small dotted lines.

2.2   Cuts in a Hierarchy

Any partition π of E whose classes are taken in $\mathcal{S}$ defines a *cut* in hierarchy H. The set of all cuts of E is denoted by $\Pi(E) = \Pi$. Every "horizontal" section $\pi_i(H)$ at level i is obviously a cut, but several levels can cooperate in a same cut, such as $\pi(S_1)$ and $\pi(S_2)$, drawn with thick dotted lines in Figure 1. Similarly, the partition $\pi(S_1) \sqcup \pi(S_2)$ generates a cut of $H(E)$. The symbol $\sqcup$ is used here for expressing that groups of classes are concatenated. Each class S may also be the matter of cuts in the sub-hierarchy $H(S)$ whose $\mathcal{S}$ is the summit. Let $\Pi(S)$ be the family of all cuts of $H(S)$; put

$$\widetilde{\Pi} = \cup\{\Pi(S), S \in \mathcal{S}\}. \tag{3}$$

The set $\widetilde{\Pi}$ does not contain all possible partial partitions with classes in $\mathcal{S}$. For example, the partial partition $\{a\} \sqcup \{f\}$ of Figure 1 does not belong to $\widetilde{\Pi}$, though its two terms are members of $\mathcal{S}$. Moreover, all unions of classes are not in $\mathcal{S}$. If for example, S is made by union of the three hexagonal sons T_1, T_2, T_3

(Figure 2), then the hierarchy $H(S)$ may accept a cut of type 2a, but surely not of type 2b, because the union of two hexagons is not a class of $\mathcal{S}$. However, $\widetilde{\Pi}$ contains the family $\Pi(E)$ of all cuts of $H(E)$.

The hierarchical structure of the data induces a relation between the family $\Pi(S)$ of the cuts of node S and the families $\Pi(T_1), .., \Pi(T_p)$ of the sons $T_1, .., T_p$ of S. Since all expressions of the form $\sqcup\{\pi(T_k); 1 \leq k \leq p\}$ define cuts of S, $\Pi(S)$ contains the whole family

$$\Pi'(S) = \{\pi(T_1) \sqcup ..\pi(T_k).. \sqcup \pi(T_p); \quad \pi(T_1) \in \Pi(T_1)... \pi(T_p) \in \Pi(T_p)\},$$

plus the cut of S into a unique class, i.e. S itself, which is not a member of $\Pi'(S)$. And as the other unions of several T_k are not classes listed in $\mathcal{S}$, there is no other possible cut, hence

$$\Pi(S) = \Pi'(S) \cup S. \tag{4}$$

2.3   Properties of the Hierarchies of Partitions

The chain structure of the partitions of hierarchy H gives rise to a few nice properties about lattices, connections and ultrametrics.

Lattices. The lattice structure results from the following property [17]:

Proposition 2. *The family $\Pi(E)$ of all cuts of H is a complete lattice of partitions for the ordering of refinement, where the class of the infimum of J cuts at point x is the intersection of the classes in x of the J operands, and where the supremum is their union.*

The fact that the union of the classes represents the class of the supremum derives from Proposition 13 in [13]. The lattice $\Pi(E)$ is atomic. Its atoms are the partitions whose all classes but one are singletons, the last one being a pair of singletons.

Each increasing sequence$\{\pi(i), 0 \leq i \leq n\}$ of cuts whose classes come from S, and whose extremities are π_0 and π_n, defines in turn a new hierarchy H', different form H, but of the same lenght, and made from classes of $\mathcal{S}$, just as H is. Let $\mathcal{H}$ stands for the family of all hierarchies of this type. The refinement ordering on the partitions induces a product ordering on the families of increasing cuts, i.e. on the hierarchies of $\mathcal{H}$, according to which H_1 is smaller than H_2 iff at every level k the cut $\pi_1(k)$is smaller than $\pi_2(k)$. We can state the following:

Corollary 3. *Proposition 2 shows that the set $\mathcal{H}$ of all hierarchies having their classes in $\mathcal{S}$ is itself a lattice. All these lattices are finite, hence complete.*

Connections. Concerning connective segmentation, the family $\mathcal{S}$ of all classes satisfies Proposition 4 and Theorem 5 [17]:

Proposition 4. *Every binary criterion $\sigma : (\mathcal{F}, \mathcal{S} \cup \varnothing) \rightarrow \{0, 1\}$ is partially connective, or connective when it is satisfied by the singletons.*

In other words, every subset of $\mathcal{S}$ containing $\emptyset$ is a partial connection. A similar result appears in the example 21 of [14].

Ultrametrics. A distance is said to be ultrametric when the triangular inequality is replaced by the more severe following axiom [4],[3]:

$$d(x, z) \leq max\{d(x, y), d(y, z)\}. \tag{5}$$

Two ultrametric balls can only be disjoint or concentric. Therefore equation (5) implies that the set of all balls of radius r induces a Voronoi tesselation π_r on E, whose classes are the open balls on the one hand, and the set of the frontiers on the other hand. The same equation (5) shows also that this Voronoi tesselation does not depend explicitly on a set of sites in space E, but only on radius r. When the range of variation of r is finite, then the family $H = \{\pi_r, r \geq 0\}$ forms an indexed hierarchy. Conversely, every hierarchy of partitions determines an ultrametric over the set S of the classes of H [3]. Therefore, we can equivalently give ourselves an ultrametric or a hierarchy of partitions. This aspect is studied in detail by P. Arbelaez and L. Cohen in [2], by C. Ronse in [14], and by F. Meyer and L. Najman in [9].

We can try and compare the disjunctive behavior of the ultrametric with the property of the classes of a hierarchy (Equation (2)). Indeed, the three concepts of an indexed hierarchy, an ultrametric, and a universal connective criterion turn out to be three aspects of a same notion. More precisely, we have:

Theorem 5. *The three following statements are equivalent:*

1. *H is an indexed hierarchy,*
2. *the set S of all classes of the partition π_i of H forms an ultrametric space, of distance the indexing parameter,*
3. *every binary criterion $\sigma : (\mathcal{F}, S \cup \varnothing) \to \{0, 1\}$ is partially connective.*

Proof. the implication $(1) \Rightarrow (3)$ is proved by proposition 4, and the equivalence $(1) \Leftrightarrow (2)$ is a classical result [3]. We have to prove that $(3) \Rightarrow (1)$. We observe firstly that, given two sets B_1 and B_2 , the criterion "$\sigma^*(f, A) = 1$ iff $A \subseteq B_1$ or $A \subseteq B_2$" is connective iff $B_2 \subseteq B_1$ or $B_1 \subseteq B_2$. Let then H be a family of partitions. If it is not hierarchical then there exists at least one point x included in two classes B_1 and B_2 of partitions of H such that none of the two inclusions $B_2 \subseteq B_1$ and $B_1 \subseteq B_2$ is true, which implies that criterion σ^* is not connective, which achieves the proof.

This result shows that the connective approach is inefficient for hierarchies, and orients us towards the alternative method, namely the optimization of an energy.

3   Cuts of Minimum Energy in $\Pi(E)$

We set the problem as it was formulated by L.Guigues [6] [5], and solved by him in the framework of additive energies and binary hierarchies. Below, the first of these two conditions is replaced by the more general relation (6) of hierarchical increasingness of the energy, and the second condition is shown to be cumbersome. In addition, the lack of unicity will lead us to analyze the structure of the solutions.

3.1   Minimization under Hierarchical Increasingness

Allocate an energy ω over he set $\widetilde{\Pi}$ of partial partitions (Equation (3)), i.e. a positive numerical function $\omega : \widetilde{\Pi} \to \mathbb{R}$. We propose to characterize the cut(s) of $\Pi(E) \subseteq \widetilde{\Pi}$ of minimum energy. As family $\Pi(E)$ is finite, there is always at least one cut $\pi^*(E)$ of smallest energy. More generally, for each node S, the family $\Pi(S)$ of all cuts of the hierarchy of summit S admits at least one minimum cut $\pi^*(S)$.

The search for minimum cuts becomes easier when we relate the energies of the fathers, S say, to those of their sons, T say, which can be obtained by means of hierarchical increasingness:

Definition 6. *Let H be a finite hierarchy, let S be one of its nodes, and T be one of the sons of S. An energy ω on the family $\widetilde{\Pi}$ of the cuts of H satisfies the condition of* hierarchical increasingness *in $\widetilde{\Pi}$ when*

$$\omega(\pi_1(T)) \le \omega(\pi_2(T)) \quad \Rightarrow \quad \omega[(\pi_1(T) \sqcup \pi(S\backslash T)] \le \omega[(\pi_2(T) \sqcup \pi(S\backslash T)], \quad (6)$$

where $\pi_1(T)$ and $\pi_2(T)$ are two cuts of the sub-hierarchy of summit T, and where π is an arbitrary cut of the set difference $S\backslash T$.

The implication (6) extends an inequality relative to the partitions of set T to another inequality holding on the partitions of set S, larger than T. In that, it is a matter for partial connective segmentation, in Ch. Ronse's sense [13].

Proposition 7. *Let H be a finite hierarchy, and ω be a hierarchically increasing energy on the cuts of H. If S is a node of H with p sons $T_1..T_p$ of minimum cuts $\pi_1^*, ..\pi_p^*$, then one of the two cuts*

$$\pi_1^* \sqcup \pi_2^* .. \sqcup \pi_p^*, \tag{7}$$

or the partition of S into a unique class, is a minimum cut of S.

Proof. The hierarchical increasingness of the energy implies that cut (7) has the lowest energy among all the cuts of type $\Pi'(S) = \sqcup\{\pi(T_k); 1 \le k \le p\}$ (it does not follow that it is unique). Now, from the decomposition (4), every cut of S is either an element of $\Pi'(S)$, or S itself. Therefore, the set formed by the cut (7) and S contains at least one minimum cut of S .

The condition of hierarchical increasingness (6) involved in Proposition 7 is not compulsory, but it opens a broad range of energies, and is easy to check. The analysis presented in [6] and [5] focuses on the case of the separable energies, i.e. of those that satisfy the additivity relation

$$\omega(T_1 \sqcup T_2 .. \sqcup T_p) = \sum_{1 \le k \le p} \omega(T_k) \tag{8}$$

between the energy of a partition and those of its classes. It is the concern of a particular hierarchical increasingness. But the algorithm proposed in [6] p.142, for finding minimum cuts in case of separable energies applies indeed to any hierarchically increasing energy ω. It can be stated as follows:

Proposition 8. *Guigues'algorithm:*

- scan in one pass all nodes of H according to an ascending lexicographic order ;

- determine at each node S a temporary minimum cut of H by comparing the energy of S to that of the concatenation of the temporary minimum cuts of the (already scanned) sons T_k of S.

Remark that from Proposition 7, the temporary minimum holds always on two terms only, at each comparison, whatever the number of sons of S.

3.2   Lattice of the Minimum Cuts

The previous results never impose unicity for the minimum cuts. Each node S involves its own series of comparisons, so that it can perfectly happen that in the family $\Pi(S)$ of Relation (4) a minimum cut of $\Pi'(S)$ has the same energy as that of S. This event introduces two solutions which are then carried over the whole induction. And since such a doublet can occur regarding any node S of hierarchy H, the number of minimal cuts is *a priori* indefinite, and may be large. However, one can reduce it by some additional condition, the simplest one being the ordering of refinement itself. Indeed, at each node S, among the two possible solutions of Proposition 7, the partition of S into a single class is always the larger one for the refinement order. By ordering the solutions at each step, we thus structure them in a complete lattice whose cardinal increases according to thickness of quantification of the energy.

What advantage can we draw from this lattice structure? First of all, it solves the unicity problem. The question "find the cut that minimizes the energy" is replaced by "find the largest cut that minimizes the energy". This largest solution is characterized by the following proposition:

Proposition 9. *Let $\{x\}$ be an element of the base of a hierarchy H with $n+1$ levels, and let $S_1(x), S_2(x), ..S_n(x)$ be the sequence of the nodes anterior to $\{x\}$. Let ω be hierarchically increasing energy for the hierarchy on $\Pi(E)$, and $\pi^*(E)$ be a minimum cut $H(E)$ for this energy. The class of $\pi^*(E)$ that contains $\{x\}$ is then $S_i(x)$ such that*

$$\omega[\pi^*(S_{i-1}(x))] \geq \omega[\pi^*(S_i(x))] \tag{9}$$

$$i \leq j < p \Rightarrow \omega[\pi^*(S_j(x))] < \omega[\pi^*(S_{j+1}(x))] \tag{10}$$

Instead of using the refinement, we can, alternatively, introduce a second optimization. For example, for color images, ω can hold on the luminance, and the criterion for choosing between the optimal cuts can derive from the product saturation×hue.

3.3   Minimization with External Partitions

It may be sometimes advantageous to complete family $\widetilde{\Pi}$ of Relation (3) by additional partitions. For example, one can wish to suppress parasite small classes

by clustering them with their neighbor. This leads to introduce a set $\Xi(S)$ of partial partitions of S whose all classes are unions of more than one T_k, such as the class $\xi(T_2, T_3)$ made of the union of two hexagons in figure 2b. These supplementary cuts of S generate the following family (up to a permutation of the indexes)

$$\Pi''(S) = \{\pi(T_1) \sqcup ..\pi(T_k) \sqcup \xi(T_{k+1}..T_p) \qquad 1 \leq k \leq p \; ; \; \xi(T_{k+1}..T_p) \in \Xi(S)\}. \tag{11}$$

Family (11) has to be added to that, $\Pi(S)$, of the legitimate sons (Relation (4)). Proposition 7 admits now the new formulation [17]

Proposition 10. *Let S be a node in hierarchy H, of p sons $T_1..T_p$ associated with p minimum cuts $\pi_1^*, ..\pi_p^*$, for some increasing energy. The minimum cuts of the set formed by*

i) the cut

$$\pi_1^* \sqcup \pi_2^* .. \sqcup \pi_p^*, \tag{12}$$

ii) the family

$$\{\pi_1^* \sqcup \pi_2^* .. \sqcup \pi_k^* \sqcup \xi(T_{k+1}, ..T_p) \; ; \; \xi \in \Xi(S)\}, \tag{13}$$

contains the minimum cuts of S.

Proposition 10 reduces the number of comparisons to do for finding the minimum cut of $H(S)$ from those of the sons of S. To illustrate that, we can calculate how many comparisons are necessary when S has $2, 3$ or 4 sons. For two sons T_1 and T_2, it suffices to compare the energies of the two terms $\pi_1^* \sqcup \pi_2^*$ and $S = T_1 \cup T_2$. When a third son T_3 is added, we go to five terms $\pi_1^* \sqcup \pi_2^* \sqcup \pi_3^*$, $T_1 \cup T_2 \cup T_3$, plus the three cuts of type $\pi_1^* \sqcup \xi(T_2, T_3)$ (see Figure 2). For the nodes with four sons one finds, similarly, $1 + C_4^2 + C_4^3 + 4 = 15$ possibilities. Note that all these values can only reduce when we require also that the classes of partitions ξ must be connected. Finally, the previous algorithm in one pass remains valid, but more comparisons must be performed at each node. But now the solutions no longer form a lattice, which is an obvious drawback.

4   Examples of Hierarchical Minimizations

We go back over the studies quoted in introduction, and try and interpret them in the framework that has just been developed. These studies are obviously richer than their specifics aspects of hierarchical optimization on which we concentrate here.

4.1   Suprema of Energies

We firstly contemplate energies whose law of composition is similar to the additivity relation (8), but written for the supremum. If $T_1, ..T_k, ..T_p$, stand for the classes of partition π, we thus have

$$\omega(T_1 \sqcup T_2, .. \sqcup T_p) = \sup\{\omega(T_k), 1 \leq k \leq p\}. \tag{14}$$

The energies involved in the law (14) increase hierarchically, hence lend themselves to the above optimizations. Here are two examples, where we suppose also that the energy is an increasing function of the class, i.e.

$$T \subseteq S \quad \Rightarrow \quad \omega(T) \leq \omega(S) \qquad T, S \in \mathcal{S}. \tag{15}$$

This last condition may seem to contradict the hierarchical rise, since the partition into singletons, from which we start, becomes a solution. But it makes sense when we quantify the energy in 0 and 1, if we look for the largest element in the lattice of all solutions.

Lasso. This algorithm, due to F. Zanoguera et Al. [20], appears also in [9]. The initial image is first segmented into α-flat zones, which generate a hierarchy as the slope α increases. But the optimization itself applies to any hierarchy of segmentations. It consists in drawing manually a closed curve around the object to segment. If A designates the inside of this lasso, then we take the following function

$$\omega(S) = 0 \quad \text{when} \ \ S \subseteq A \ ; \ \ S \in \mathcal{S}$$
$$\omega(S) = 1 \quad \text{when not},$$

for energy, and we go from classes to partitions by the law (14) of $\vee$-composition of the energies. The largest cut that minimizes ω is depicted in Figure 3c. We see that the resulting contour follows the edges of the petals. Indeed, a segmented class can jump over this high gradient only for α large enough, and then this class is rejected because it spreads out beyond the limit of the lasso.
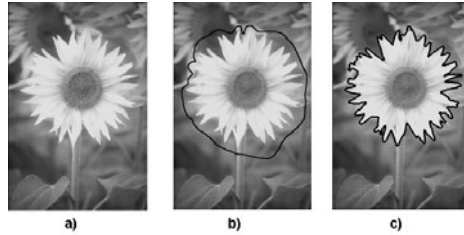

Fig. 3. a) Initial image; b) manual lasso; c) contour of the union of the classes inside the lasso

Segmentation under constraint. This method was proposed by P. Soille and J. Grazzini in [18] and [19] with several variants; it is re-formulated by C. Ronse in a more general framework in [15]. We now take for energy ω of the classes the function

$$\omega(S) = 0 \quad \text{when} \ \ \sup\{f(x), x \in S\} - \inf\{f(x), x \in S\} \leq \omega_0$$
$$\omega(S) = 1 \quad \text{when not},$$

where ω_0 is a given threshold, and we go to partitions by $\vee$-composition. In the examples of [18] and [19], the hierarchy is obtained by α-flat zones, and,

for multi-spectral images, by the intersection of the α-flat zones of the various bands. Sometimes the criterion "there is at least one extremum of f inside S" is added. The class at point x of the largest partition of minimum energy is given by the largest $S \in \mathcal{S}$, that contains x, and such that the amplitude of variation of f inside S be $\leq \omega_0$.

4.2   Additive Energies

Separable models. They are energies whose archetype is given by the classical Mumford and Shah functional. One constructs a separable energy in $\mathbb{R}^2$ or in $\mathbb{Z}^2$ by starting from a partition of S into its sons $T_1, ..T_p$ plus the set Γ of their frontiers. A first measure ω_μ holds on the T_k, and a second one, ω_ν, on their frontiers ∂T_k, and we put

$$\omega(S) = \sum_{1 \leq k \leq p} \omega_\mu(T_k) + \omega_\nu(\partial T_k), \tag{16}$$

(according to Integral Geometry, a third term, proportional to Euler-Poincaré Constant could be added). This models are said to be separable [6] [5] in that we have $\omega(S) = \omega(T_1) + ... + \omega(T_p)$, up to external edge effects. The energy ω, additive, is thus hierarchically increasing. Different instructive variants may be found in [6].

Thumbnails. The creation of digital thumbnails by Ph. Salembier and L. Garrido [16] is a matter for separable models. One aims to generate reduced versions of a given color image f from a hierarchy of its segmentations. In each class T function f is replaced by its mean value $m(T)$. The approximation is estimated in the L_2 norm, i.e.

$$\omega_\mu(T) = \sum_{x \in R} \| f(x) - m(T) \|^2 .$$

If the coding cost for a frontier element is k, that of the whole class T becomes

$$\omega_\nu(T) = 24 + \frac{k}{2} \mid \partial T \mid$$

with 24 bits for $m(T)$. The total energy of a cut is written $\omega(S) = \lambda\omega_\mu(S) + \omega_\nu(S)$. For a fixed λ, it yields a first minimization, followed by a second one relatively to the Lagrange parameter λ.

4.3   Other Laws of Composition

A number of laws of composition are compatible with hierarchically increasing energies. Instead of supremum and sum, that we just presented, one could use infimum, product, difference sup-inf, quadratic sum, etc., and all their combinations. Moreover, one can make depend ω on more than one class, on the proximity of the edges, on several elements of the lattice $\mathcal{H}$ of the hierarchies, etc...

Here is an example of another law of composition, which extends the technique developed in [1]. Start from three partition hierarchies H_l, H_s, and H_h of luminance, saturation, and hue of a given color image. The purpose is to reduce them to a unique significant cut, and the idea is that the higher (resp. lower) the saturation, the more representative is the hue (resp. luminance). Therefore, when a region has a weak (resp. strong) saturation, then its luminance (hue) is prioritary for the segmentation. Suppose that saturation s varies between 0 and 1. Provide the classes of H_s with the energy

$$\omega(T_s) = \frac{1}{area\, T_s}[1 - \int_{T_s} s(x)dx], \qquad (17)$$

and weight by areas for expressing the energies of the partitions from those of the classes. It results in the minimum cut $\overline{\pi}_s$. Repeat the process by replacing $1 - \int_{T_s} s(x)dx$ by $\int_{T_s} s(x)dx$ in (17), which results in the new energy $\omega'(T_s)$ and the new minimum cut $\overline{\pi}'_s$. Intersect then the two partitions $\overline{\pi}_s$ and $\overline{\pi}'_s$. When the class at point x of the intersection has an energy $\omega - \omega' > 0$, it is labelled "class for luminance", and when not, "class for hue". The process ends by taking the partial hierarchy of the luminance (resp.the hue) in the union of the classes for luminance (resp. for hue), and segmenting luminance and hue individually, in their own domains.

5   Conclusion

Considering the hierarchies of partitions of an image as a segmentation tool, we looked for the meaning of an optimum cut, for laws able to govern this optimum. We showed that the assumption of hierarchical increasingness (6) allows to regroup various segmentation techniques which seem to be far away from each other, and we proposed new ones.

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