Vacuum Energy and the Topology of the Universe

Manuel Asorey, Inés Cavero-Peláez and José M. Muñoz-Castañeda

Abstract We analyze the dependence of the quantum vacuum energy on the space topology. In particular we point out the existence of a renormalization ambiguity in spaces with non-vanishing curvature. The ambiguity is related to the well known ambiguity of the R^2 term of the gravitational effective action. However, there are two extra universal contributions which are genuine dependent on the topological structure of the space and completely independent of the renormalization scheme. The ambiguity does not appear in flat spaces where only the topological dependent contributions are non-vanishing. We analyze the cosmological role of universal contributions to the vacuum energy and its attractive nature in the case of conformal scalar fields.

1 Introduction

The current cosmological model is consistent with a spatially flat Universe, although, most of the relevant data are compatible with a very tiny curvature $|\Omega_\kappa| \leq 10^{-4}$ [1][2]. However, the physical observations do not allow to establish a definite answer to the longstanding dilemma on the finiteness or not of the physical space or determine the characteristics of space-time topology (see [3] and references there in for an updated review). Closed spaces leave their fingerprints in small contributions to low multipoles of the Cosmic Microwave Background (CMB) and current observations show a strong suppression of low multipoles (quadrupole, octupole, etc.). They also show an strange alignment of the quadrupole and octupole multipoles associated to the appearance of Southern hemisphere cool fingers. On the other hand, it is remarkable the observed asymmetry between even and odd multipoles and the fact that the Gaussianity of likelihood estimates starts to be manifest for $l > 32$. All these data suggest a possible role of the finite size and space topology in the

Departamento de Física Teórica. Facultad de Ciencias. Universidad de Zaragoza. 50009 Zaragoza. Spain. e-mail: asorey@unizar.es, cavero@unizar.es, jositomc@gmail.com

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low modes behavior of the CMB [4]. A compact space will imply, depending on its topology, the existence of several circles in the sky which will correspond to the mirror images of the last scattering surface where the radiation decouple from matter. The latest results do not allow to determine their existence which will be an unequivocal proof of a non-trivial space topology. However, presumably the new observational programs will be able to discriminate among the different space-time topologies. In this note we analyze the quantum implications of a non-trivial space-time topology.

2 Vacuum energy in cosmological backgrounds

Quantum fields contribute to the background space-time energy because of vacuum fluctuations. For conformal invariant fields this energy depends on the topology of the space. If the space has boundaries it also depends on the boundary conditions.

The cosmological implications of this energy are not very clear. First, the divergent nature of the leading contributions raises some questions about the validity of the renormalization philosophy in the presence of gravitational interactions. On the other hand, finite Casimir corrections encode the quantum back-reaction to the cosmological expansion of the Universe, but this is very tiny to be detected in the present Universe, although it might have played a relevant role in the early Universe. In this note we analyze the structure of such contributions in different cosmological backgrounds. This problem has been considered by Emilio Elizalde for a long time [5] [6] [7].

Although the background cosmological FRW metric evolves in time its variation is so slow in comparison with the leading quantum fluctuations that one can use adiabatic approximations to estimate the vacuum energy induced by these fluctuations. In this approximation the space-time metric can be considered as a homogeneous isotropic static on a space-time of the form $\mathbb{R} \times \mathscr{M}$.

There are three types of constant curvature spaces: hyperbolic ($R < 0$), elliptic ($R > 0$) or Euclidean ($R = 0$). If we assume that the space is compact and has no boundaries the number of candidates is reduced considerably. The hyperbolic case presents an infinite number of possibilities and has been the most analyzed in the literature [8][9][10]. We will restrict ourselves to the less analyzed cases of elliptic and flat spaces.

Spaces with constant positive curvature and no boundaries are compact manifolds and belongs to one of the following six familes. If $\mathscr{M}$ is simply connected it has to be isometric to the three-dimensional sphere S^3, because of Poincaré theorem. Multiple connected spaces belong to one of the following five families:

- Lens spaces $S^3/\mathbb{Z}_q$, with first homotopy group the cyclic group $\mathbb{Z}_q$ of order q.
- Dihedral spaces S^3/D_q^*, with first homotopy group D_q^* of order $4q$. order 24.
- Tetrahedral space S^3/T^* with $\pi_1(S^3/T^*) = T^*$ of order 24.

- Octahedral space S^3/O^* with $\pi_1(S^3/O^*) = O^*$ of order 48.
- Poincaré Dodecahedral space S^3/Y^* with $\pi_1(S^3/Y^*) = Y^*$ of order 120.

The last space S^3/Y^* has been recently considered as a possible candidate for the global structure of the Universe [11] [12] [13] by considerantions based on the observed anomalies of CMB.

For simplicity we shall restrict ourselves to the case of conformal scalar free fields. The analysis of higher spin fields is very similar. The vacuum energy of free conformal scalar field is given by the renormalized sum of the eigenvalues of the operator $\frac{1}{2}\sqrt{-\Delta + \frac{1}{6}R}$, where R is the scalar curvature of $\mathscr{M}$. $R = \frac{6}{a}$ for a three-dimensional sphere $\mathscr{M} = S^3$ of radius a.

The eigenvalues of the operator $-\Delta + \frac{1}{6}R$ on $\mathscr{M}$ are of the form $\lambda_k = \frac{1}{a^2}(k+1)^2$ with $k \in \mathbb{Z}$, with the following degeneracies d_k [14][15]:

$\mathbb{I}$	$d_k(\mathbb{I}) = (k+1)^2$
$\mathbb{Z}_{2q+1}$	$d_k(Z_{2q+1}) = (k+1)\left(k+1-[(k+1)/(2q+1)](2q+1)+(1+(-1)^{k-[(k+1)/(2q+1)](2q+1)})/2\right)$
$\mathbb{Z}_{2q}$	$d_{2l}(Z_{2q}) = (2l+1)(2[(2l+1)/(2q)]+1)$
$\mathbb{D}^*_q$	$d_{2l}(D^*_q) = (2l+1)([l/q]+1/2(1+(-1)^l))$
$\mathbf{T}^*_q$	$d_{2l}(T^*) = (2l+1)([l/3]+2[l/2]+1-l);\ l \neq 1,2,5$
$\mathbb{O}^*_q$	$d_{2l}(O^*) = (2l+1)([l/4]+[l/3]+[l/2]+1-l);\ l \neq 1,2,3,5,7,11$
$\mathbf{Y}^*_q$	$d_{2l}(Y^*) = (2l+1)([l/5]+[l/3]+[l/2]+1-l);\ l \neq 1,2,3,4,5,7,8,9,11,13,14,17,19,23,29$

Table 1 Degeneracies of the eigenvalues of the Laplacian operator Δ for spherical factor spaces.

The zeta function regularization method provides the following values for the vacuum energy $E_c = \frac{1}{a}C_{\mathscr{M}}$[16][17][18]:

- Sphere S^3

$$E_{S^3} = \frac{1}{240}\frac{1}{a}$$

- Lens spaces $S^3/\mathbb{Z}_q$

$$E_{Z_q} = -\frac{q^4 + 10q^2 - 14}{720q}\frac{1}{a}$$

- Dihedral spaces $S^3/\mathbb{D}^*_q$

$$E_{D^*_q} = -\frac{20q^4 + 8q^2 + 180q - 7}{1440q}\frac{1}{a}$$

- Polyhedral spaces $S^3/\mathbf{T}^*$, $S^3/\mathbb{O}^*$, $S^3/\mathbf{Y}^*$

$$E_{T^*} = -\frac{3761}{8640}\frac{1}{a} \qquad E_{O^*} = -\frac{11321}{17280}\frac{1}{a} \qquad E_{Y^*} = -\frac{43553}{43200}\frac{1}{a}$$

Group \ Order	24	48	120
Cyclic $\mathbb{Z}_q$	$C_{S^3/Z_{24}} = -\frac{168761}{8640}$	$C_{S^3/Z_{48}} = -\frac{2665721}{17280}$	$C_{S^3/Z_{120}} = -\frac{103751993}{43200}$
Dihedral $\mathbb{D}_q^*$	$C_{S^3/D_6^*} = -\frac{11081}{4320}$	$C_{S^3/D_{12}^*} = -\frac{168761}{8640}$	$C_{S^3/D_{30}^*} = -\frac{6497993}{21600}$
$\mathbf{T}^*$ $\mathbb{O}^*$ $\mathbf{Y}^*$	$C_{S^3/T^*} = -\frac{3761}{8640}$	$C_{S^3/O^*} = -\frac{11321}{17280}$	$C_{S^3/Y^*} = -\frac{43553}{43200}$

Table 2 Casimir energies of confomal scalar fields on spaces of compact constant curvature with group factors of order 24, 48 and 120. Notice that lens spaces tend to have larger negative energies than dihedral or polyhedral spaces with the same volumes.

These energies generate attractive forces except for the case of the three-dimensional sphere S^3, which is the only case with repulsive behaviour.

The nature of this attractive behaviour is stronger for spaces with the same volume in the cases of dihedral and lens spaces as the Table 2 points out.

3 Vacuum energy ambiguites

The values of vacuum energy shown in the previous section are not universal. In general the vacuum energy has three components

$$E(g) = E_{\text{loc}}(g) + E_{\text{anom}}(g) + E_{\text{top}}(g).$$

which are in one-to-one correspondence with the three components of the effective action

$$S(g) = S_{\text{loc}}(g) + S_{\text{anom}}(g) + S_{\text{top}}(g).$$

The first two components depend on the Riemann curvature tensor $R_{\mu\nu\alpha\sigma}$ either locally

$$S_{\text{loc}}(g) = \int d^4x\sqrt{-g}\,\{\alpha_1 C^2 + \alpha_2 E + \alpha_3 \Box R\}$$

or non-locally [19]

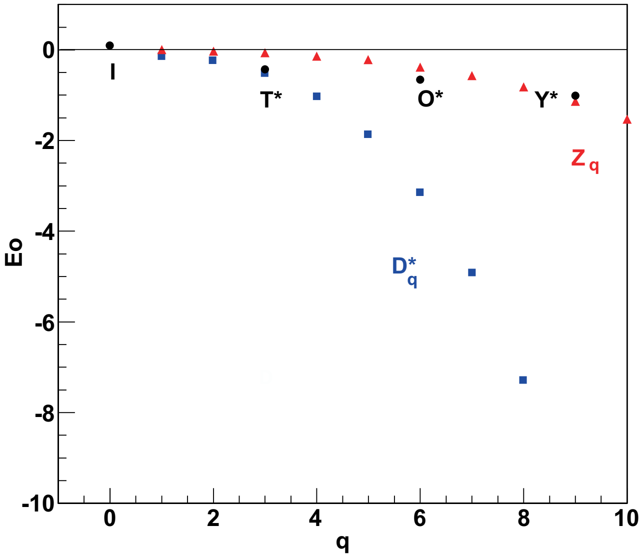

Figure 1 Casimir energies of confomal scalar fields on compact spaces of constant curvature. The only positive value appears in pure spherical spaces S^3. The dihedral factor S^3/D_q^* seem to have higher repulsive energies than lens spaces S^3/Z_q. However, this is an artifact of the different volumes weight of the respective spaces, Vol(S^3/Z_{4q})= Vol(S^3/D_q^*). The volumes of the polyhedral factors S^3/T_q^*, S^3/O_q^*, S^3/Y^* are identical to those of S^3/Z_{24}, S^3/Z_{48} and S^3/Z_{120}, respectively. However, they generate milder attractive energies.

$$S_{\rm anom}(g\ ) = \frac{b}{8(4\pi)^2}\int d^4x \int d^4x' \sqrt{-g}\left(E+\frac{2}{3}\Box R\right)(x)\Box_4^{-1}(x,x')$$
$$\sqrt{-g}\left[\left(E+\frac{2}{3}\Box R\right)\right](x') + \left(c-\frac{2}{3}b\right)\frac{1}{12(4\pi)^2}\int d^4x\sqrt{-g}\,R^2$$

in terms of the Green function of the operator

$$\Box_4 \equiv \Box^2 - 2R^{\mu\nu}\nabla_\mu\nabla_\nu + \frac{2}{3}R\Box - \frac{1}{3}(\nabla^\mu R)\nabla_\mu\,,$$

the Weyl tensor $C_{\mu\nu\alpha\sigma}$ and the Euler density E.

However, the third component $S_{\rm top}(g)$ cannot be expressed in terms of local tensor densities because only depends on global properties of the space $\mathscr{M}$ like the length of minimal closed geodesic. This component is only present in multiple connected spaces.

The coefficients of the local part are ambiguous and depend on the renormalization scheme. However the b coefficient of $S_{\rm anom}(g)$ and those of $S_{\rm top}(g)$ are universal [20] and independent of the regularization method. The coefficient c of $S_{\rm anom}(g)$ is ambiguous because corresponds to a local term which cannot be disentangled from a similar term of $S_{\rm loc}(g)$ [21]. In particular, for a conformal scalar field $b = 1/360$, and although most of the regularization methods yield $c = -1/180$, there are other methods which give $c = -1/180 + \delta$, with an arbitrary contribution δ which depends on the parameters of the regulatization [21, 22].

However, not all the terms of the action are relevant for the calculation of the Casimir energy in spherical factor manifolds. It can be shown that in the case of S^3 the contribution of the non-local component of $S_{\rm anom}(g)$, $E^{u_1}_{S^3} = \frac{1}{480}\frac{1}{a}$, is half of the total contribution [23] in zeta function regularization. The other half comes from the R^2 term and the genuine topological contribution vanishes. However, as it has been shown the R^2 contribution is arbitrary and, therefore, the total Casimir energy in such a background is also arbitrary [22].

For multiple connected spherical factor spaces the two universal contributions have a very different behaviour due to its different origin. The contribution coming from the non-local terms of $S_{\rm anom}(g)$ is

$$E^{u_1}_{\mathscr{M}} = \frac{Vol(\mathscr{M})}{480(2\pi^2)}\frac{1}{a},$$

which is equal to the similar contribution of the sphere, up to the ratio of volumes

$$\frac{Vol(S^3)}{Vol(\mathscr{M})} = \#\pi_1(\mathscr{M}),$$

which is given by the order of the first homotopy group of the physical space $\mathscr{M}$. The contribution of $S_{\rm top}(g)$ to the vacuum energy is non vanishing and depends on the topology of the spherical factor space. This contribution is given by

- Sphere S^3

$$E^{\rm top}_{S^3} = 0$$

- Lens spaces $S^3/\mathbb{Z}_q$

$$E^{\rm top}_{Z_q} = -\frac{2q^4 + 20q^2 - 25}{1440q}\frac{1}{a}$$

- Dihedral spaces $S^3/\mathbb{D}^*_q$

$$E^{\rm top}_{D^*_q} = -\frac{40q^4 + 16q^2 + 360q - 11}{2880q}\frac{1}{a}$$

- Polyhedral spaces $S^3/\mathbf{T}^*$, $S^3/\mathbb{O}^*$, $S^3/\mathbf{Y}^*$

$$E_{T^*}^{\rm top} = -\frac{1505}{3456}\frac{1}{a} \qquad E_{O^*}^{\rm top} = -\frac{4529}{6912}\frac{1}{a} \qquad E_{Y^*}^{\rm top} = -\frac{87109}{86400}\frac{1}{a}.$$

However, in all cases there is an extra contribution coming from the R^2 term of the action whose arbitrary contribution δ makes the calculation of the vacuum energy completely ambiguous.

The same ambiguity appears in hyperbolic spaces with constant negative curvature. However, for flat spaces the behaviour is different.

4 Flat compact spaces

In the case of flat spaces the extra ambiguous contribution is absent due to the vanishing of all curvature tensors. In this case the only non-vanishing contribution arises from the $S_{\rm top}(g)$ terms of the effective action. There are six orientable compact flat manifolds: Torus (T^3), Half-Turn Space (E_2), Quarter-Turn Space (E_3),Third-Turn Space (E_4), Sixth-Turn Space (E_5) and Hantzsche-Wendt Space (E_6). These spaces correspond to different factors of the Euclidean space $\mathbb{R}^3$ by discrete subgroups of the Euclidean group ISO(3)= $T_3 \circ$ O(3). They are classified according to their rotational part, $\mathbb{Z}_1$ for E_2, $\mathbb{Z}_2$ for E_3, $\mathbb{Z}_4$ for E_4, $\mathbb{Z}_3$ for E_4, $\mathbb{Z}_6$ for E_5 and $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ for E_6.

However, due to the fact that the group factors are not normal, the vacuum energy density is not uniformly distributed, which implies the existence of space anisotropies that should be observed in the dark energy component (see Fig. 3).

The corresponding vacuum energies for compact factors of a symmetric torus of size a are given by [24] [25]

• Torus T^3

$$E_{T^3} = -\frac{1}{2\pi^2 a}\int_0^\infty dt\, t(\theta_3^3(\mathrm{e}^{-t})-1) = -\frac{0.8375}{a}$$

• Twisted Sixth-Turn Torus E_5

$$E_5 = -0.99\frac{1}{a}$$

• Hantzsche-Wendt Space E_6

$$E_6 = -0.32\frac{1}{a},$$

and show the same trend as in positive curvature case. The corresponding Casimir energies are negative which correspond to attractive forces [25][26]. This seems to be the generic behaviour associated to the topological contributions to vacuum energy.

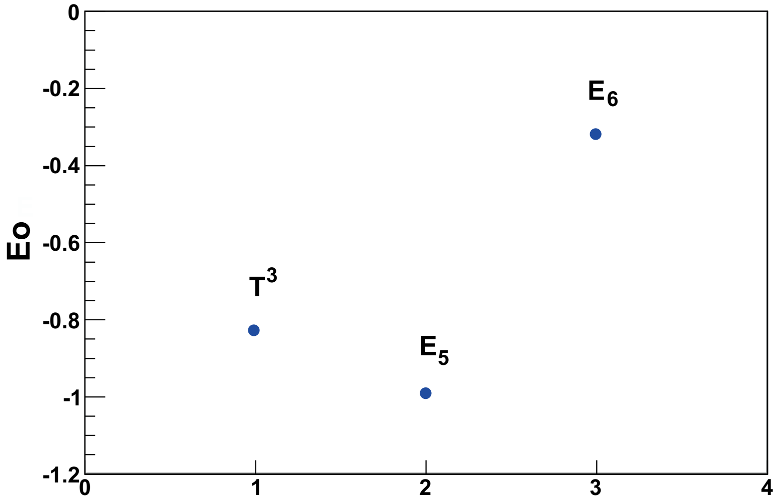

Figure 2 Casimir energies for the flat three-dimensional torus T^3, twisted sixth-turn torus E_5 and Hantzsche-Wendt Space E_6.

5 Cosmological implications

If we consider the time evolution of the space-time structure, the conformal factor a evolves in an accelerated manner, according to the current cosmological LCDM model. This implies that the quantum vacuum energy of conformal scalar fields also increases because in most of the cases the Casimir energy is negative (for higher spins the topological Casimir energy is positive for some topologies). The gravitational back-reaction to this increase of energy results in a tiny deceleration of the cosmological expansion. However, this quantum contribution is very tiny in the current Universe, although it could have played a relevant role in the early stages of the Universe evolution. The form of the Casimir energy density is very similar to the radiation component of the total energy density of the Universe. However, the pressure components are very different.

Now, because of the ambiguity which appears in the renormalization of vacuum energy it can always be chosen to be in a repulsive regime resulting into an extra acceleration of space metric. However, the renormalization origin of this behaviour is masking the real gravitational effect of quantum field fluctuations.

The decrease of energy can be compensated by particle creation [27]. Although the Zeldovich-Starobinsky condition prevents pair creation for conformally invariant theories [28], in the case of compact spaces if the size of the space is smaller than

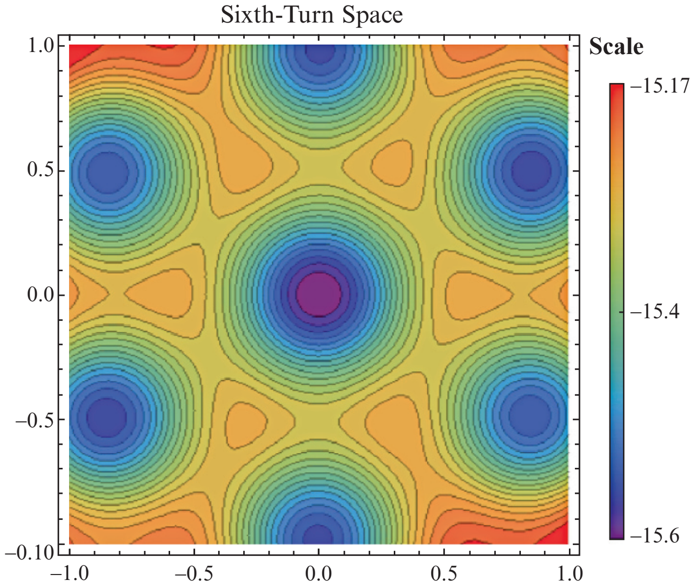

Figure 3 Casimir energies density of confomal scalars on sixth-turn flat space E_5 restricted to the fundamental domain.

the Hubble radius the phenomenon can occur [29]. The spectrum of the corresponding radiation is given by the thermal Gibbons-Hawking spectrum with temperature $T = \hbar H/2\pi k_B$, in terms of the Hubble constant and Bolthmann parameter. A realistic scenario compatible with current observations requires that the size of space is slightly smaller than the Hubble radius, in order to fit close to the Hubble horizon and still allow for pair particle creation.

As we have shown only in the case of flat compact topologies the quantum contribution to vacuum energy is universal. In those topologies this vacuum energy is anisotropic and correlated to the locations of CMB circles in the sky. Only in that case the new cosmological observations will provide crucial clues to understand the topological structure of the Universe.

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