

# Finite-Time Singularities in Modified $\mathcal{F}(R, G)$ -Gravity and Singularity Avoidance

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**Abstract** We study finite-time future singularities in  $\mathcal{F}(R, G)$ -gravity, where  $R$  and  $G$  are the Ricci scalar and the Gauss-Bonnet invariant, respectively. In particular, we reconstruct the  $F(G)$ -gravity and  $\mathcal{F}(R, G)$ -gravity models realizing the finite-time future singularities. We discuss a possible way to cure the finite-time future singularities in  $\mathcal{F}(R, G)$ -gravity by taking into account higher-order curvature corrections or effects of viscous fluids.

## 1 Introduction

Among the possible alternatives in order to explain the Dark Energy Issue, are the so called Modified Theories of Gravity[1, 2].

We would like to consider modified  $\mathcal{F}(R, G)$ -gravity, where the action is described by a function of the Ricci scalar  $R$  and the Gauss-Bonnet invariant  $G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma}$ .

Many of modified gravity models bring the future universe evolution to finite-time singularities. Some of these singularities are softer than other and not all physical quantities (scale factor, effective energy density and pressure) necessarily diverge at this finite future time. Note that singular solutions correspond to accelerated universe, and often appear as the final evolution of unstable de Sitter space.

The presence of finite-time singularities may cause serious problems in the black holes or stellar astrophysics[3]. Thus, it is of some interest to explore the  $\mathcal{F}(R, G)$ -gravity models realizing singularities and if any natural scenario to cure such singularities exists.

We use units of  $k_B = c = \hbar = 1$  and denote the gravitational constant  $8\pi G_N$  by  $\kappa^2 \equiv 8\pi/M_{\text{Pl}}^2$  with the Planck mass  $M_{\text{Pl}} = G_N^{-1/2} = 1.2 \times 10^{19}\text{GeV}$ .

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## 2 The model

The action of  $\mathcal{F}(R, G)$ -gravity is given by

$$S = \int d^4x \sqrt{-g} \left[ \frac{\mathcal{F}(R, G)}{2\kappa^2} + \mathcal{L}_{\text{matter}} \right], \quad (1)$$

where  $g$  is the determinant of the metric tensor  $g_{\mu\nu}$  and  $\mathcal{L}_{\text{matter}}$  is the matter Lagrangian. The spatially-flat FRW space-time is described by the metric

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2, \quad (2)$$

where  $a(t)$  is the scale factor of the universe.

From the action in (1), the FRW-equations of motion (EOM) are derived as

$$\rho_{\text{eff}} = \frac{3}{\kappa^2} H^2, \quad p_{\text{eff}} = -\frac{1}{\kappa^2} (2\dot{H} + 3H^2), \quad (3)$$

where  $\rho_{\text{eff}}$  and  $p_{\text{eff}}$  are the effective energy density and pressure of the universe, respectively, and these are defined as

$$\rho_{\text{eff}} = \frac{1}{\mathcal{F}'_R} \left\{ \rho + \frac{1}{2\kappa^2} [(\mathcal{F}'_R R - \mathcal{F}) - 6H\mathcal{F}'_R + G\mathcal{F}'_G - 24H^3\mathcal{F}'_G] \right\}, \quad (4)$$

$$p_{\text{eff}} = \frac{1}{\mathcal{F}'_R} \left\{ p + \frac{1}{2\kappa^2} \left[ -(\mathcal{F}'_R R - \mathcal{F}) + 4H\mathcal{F}'_R + 2\mathcal{F}''_R - G\mathcal{F}'_G + 16H(\dot{H} + H^2)\mathcal{F}'_G + 8H^2\mathcal{F}''_G \right] \right\}. \quad (5)$$

Here,  $H = \dot{a}(t)/a(t)$  is the Hubble parameter and the dot denotes the time derivative.  $\rho$  and  $p$  are the energy density and pressure of matter, whereas  $\mathcal{F}'_R = \partial_R \mathcal{F}(R, G)$  and  $\mathcal{F}'_G = \partial_G \mathcal{F}(R, G)$ . For general relativity with  $\mathcal{F}(R, G) = R$ ,  $\rho_{\text{eff}} = \rho$  and  $p_{\text{eff}} = p$  and therefore (4) and (5) are the Friedmann equations. Consequently, (4) and (5) imply that the contribution of modified gravity can formally be included in the effective energy density and pressure of the universe.

## 3 Finite-time future singularities

We consider the case in which the Hubble parameter is expressed as

$$H = \frac{h}{(t_0 - t)^\beta} + H_0, \quad (6)$$

where  $h$ ,  $t_0$  and  $H_0$  are positive constants and  $t < t_0$  because it should be for expanding universe.  $\beta$  is a positive constant or a negative non-integer number, so that,

when  $t$  is close to  $t_0$ ,  $H$  or some derivative of  $H$  and therefore the curvature become singular.

Such choice of Hubble parameter corresponds to accelerated universe, because if (6) is a solution of the EOM (3), it is easy to see that the strong energy condition ( $\rho_{\text{eff}} + 3p_{\text{eff}} \geq 0$ ) is always violated when  $\beta > 0$ , or is violated for small value of  $t$  when  $\beta < 0$ . It means that in any case the singularity could emerge as final evolution of accelerated universe.

The finite-time future singularities can be classified in the following way[4]:

- Type I and Big Rip. It corresponds to  $\beta > 1$  and  $\beta = 1$ .  $H$  and  $R$  ( $\sim H^2$ ) diverge.
- Type II (sudden). It corresponds to  $-1 < \beta < 0$ .  $R$  ( $\sim \dot{H}$ ) diverges.
- Type III. It corresponds to  $0 < \beta < 1$ .  $H$  and  $R$  ( $\sim \dot{H}$ ) diverge.
- Type IV. It corresponds to  $\beta < -1$  but  $\beta$  is not any integer number. Some derivative of  $H$  and therefore the curvature becomes singular.

We note that in the present paper, we call singularities for  $\beta = 1$  and those for  $\beta > 1$  as the ‘‘Big Rip’’ singularities and the ‘‘Type I’’ singularities, respectively.

## 4 Reconstruction method

In order to study the finite-time singularities in  $\mathcal{F}(R, G)$ -gravity, we use the reconstruction method.

We assume that the contribute of ordinary matter and radiation in expanding singular universe is too small with respect to the modified gravity, and we study the pure gravitational action of  $\mathcal{F}(R, G)$ -gravity, i.e., the action in (1) without  $\mathcal{L}_{\text{matter}}$ . In this case, it follows from (4) and (5) that the EOM of  $\mathcal{F}(R, G)$ -gravity are given by[6]:

$$24H^3 \mathcal{F}'_G + 6H^2 \mathcal{F}'_R + 6H \dot{\mathcal{F}}'_R + (\mathcal{F} - R\mathcal{F}'_R - G\mathcal{F}'_G) = 0, \quad (7)$$

$$8H^2 \dot{\mathcal{F}}'_G + 2\ddot{\mathcal{F}}'_R + 4H \dot{\mathcal{F}}'_R + 16H \dot{\mathcal{F}}'_G (\dot{H} + H^2) + \mathcal{F}'_R (4\dot{H} + 6H^2) + \mathcal{F} - R\mathcal{F}'_R - G\mathcal{F}'_G = 0. \quad (8)$$

In the case of pure gravity, these two equations are linearly dependents.

Moreover, we have

$$R = 6(2H^2 + \dot{H}), \quad G = 24H^2(H^2 + \dot{H}). \quad (9)$$

It is easy to see that, in the case of Type I, II and III singularities,  $G$  and  $R$  tend to infinitive when  $t \rightarrow t_0$  in Equation (6), and in the case of Type IV singularities tend to zero.

By using proper functions  $Z(t)$ ,  $P(t)$  and  $Q(t)$  of a scalar field which is identified with the cosmic time  $t$ , we can rewrite the action in (1) without  $\mathcal{L}_{\text{matter}}$  to

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} (Z(t)R + P(t)G + Q(t)). \quad (10)$$

By the variation with respect to  $t$ , we obtain

$$Z'(t)R + P'(t)G + Q'(t) = 0, \quad (11)$$

from which in principle it is possible to find  $t = t(R, G)$ . Here, the prime denotes differentiation with respect to  $t$ . By substituting  $t = t(R, G)$  into (10), we find the action in terms of  $\mathcal{F}(R, G)$

$$\mathcal{F}(R, G) = Z(R, G)R + P(R, G)G + Q(R, G). \quad (12)$$

We describe the scale factor as

$$a(t) = a_0 \exp(g(t)), \quad (13)$$

where  $a_0$  is a constant and  $g(t)$  is a function of time. By using the (7), (8) and (13), the matter conservation law  $\dot{\rho} + 3(\rho + p)$  and then neglecting the contribution from matter, we get the differential equation

$$Z''(t) + 4\dot{g}^2(t)P''(t) - \dot{g}(t)Z'(t) + (8\dot{g}\ddot{g} - 4\dot{g}^3(t))P'(t) + 2\ddot{g}(t)Z(t) = 0, \quad (14)$$

where the Hubble parameter is  $H(t) = \dot{g}(t)$ . By using (7),  $Q(t)$  becomes

$$Q(t) = -24\dot{g}^3(t)P'(t) - 6\dot{g}^2(t)Z(t) - 6\dot{g}(t)Z'(t). \quad (15)$$

It means that in principle, by solving (14) on the singular solution (6), it is possible to reconstruct  $\mathcal{F}(R, G)$  producing finite-time future singularities.

In general, if  $Z(t) \neq 0$ ,  $\mathcal{F}(R, G)$  can be written in the following form:

$$\mathcal{F}(R, G) = Rg(R, G) + f(R, G), \quad (16)$$

where  $g(R, G) \neq 0$  and  $f(R, G)$  are generic functions of  $R$  and  $G$ . From (7) and (8), we obtain

$$\rho_{\text{eff}} = -\frac{1}{2\kappa^2 g(R, G)} \left[ 24H^3 \mathcal{F}'_G + 6H^2 \left( R \frac{dg(R, G)}{dR} + \frac{df(R, G)}{dR} \right) + 6H \mathcal{F}'_R \right. \\ \left. + (\mathcal{F} - R\mathcal{F}'_R - G\mathcal{F}'_G) \right], \quad (17)$$

$$p_{\text{eff}} = \frac{1}{2\kappa^2 g(R, G)} \left[ 8H^2 \mathcal{F}'_G + 2\mathcal{F}'_R + 4H \mathcal{F}'_R + 16H \mathcal{F}'_G (\dot{H} + H^2) \right. \\ \left. + \left( R \frac{dg(R, G)}{dR} + \frac{df(R, G)}{dR} \right) (4\dot{H} + 6H^2) + \mathcal{F} - R\mathcal{F}'_R - G\mathcal{F}'_G \right], \quad (18)$$

where  $\rho_{\text{eff}}$  and  $p_{\text{eff}}$  are given by the expressions in (3).

The modification of gravity could be included into the Equation of State (EoS) of an inhomogeneous dark fluid with energy density  $\rho_{\text{eff}}$  and pressure  $p_{\text{eff}}$

$$p_{eff} = \omega p_{eff} + \mathcal{G}(H, \dot{H}...), \quad (19)$$

where  $\omega$  is the constant EoS parameter of matter and  $\mathcal{G}(H, \dot{H}...)$  is a viscosity term given by

$$\begin{aligned} \mathcal{G}(H, \dot{H}...) = & \frac{1}{2\kappa^2 g(R, G)} \left\{ (1 + \omega)(\mathcal{F} - R\mathcal{F}'_R - G\mathcal{F}'_G) \right. \\ & + \left( R \frac{dg(R, G)}{dR} + \frac{df(R, G)}{dR} \right) [6H^2(1 + \omega) + 4\dot{H}] \\ & \left. + H\mathcal{F}'_R(4 + 6\omega) + 8H\mathcal{F}'_G [2\dot{H} + H^2(2 + 3\omega)] + 2\ddot{\mathcal{F}}'_R + 8H^2\ddot{\mathcal{F}}'_G \right\} \quad (20) \end{aligned}$$

The use of this equation requires that  $g(R, G) \neq 0$  on the solution.

By combining the two equations in (3), we obtain

$$\mathcal{G}(H, \dot{H}...) = -\frac{1}{\kappa^2} [2\dot{H} + 3(1 + \omega)H^2], \quad (21)$$

where (19) has been used.

## 5 Singularities in $\mathcal{F}(R, G)$ -gravity

### Big Rip singularity in $F(G)$ -gravity

As a simple example of reconstruction method, we examine the Big Rip singularity in Gauss-Bonnet  $F(G)$ -gravity, where  $\mathcal{F}(R, G) = R + F(G)$  and  $F(G)$  is a function of Gauss-Bonnet invariant only. In this case, by putting  $Z(t) = 1$ , the action of (10) can be written in terms of two proper functions  $P(t)$  and  $Q(t)$  and the variation with respect to  $t$  yields

$$P'(t)G + Q'(t) = 0, \quad (22)$$

from which we can find  $t = t(G)$  and the action in terms of  $R$  and  $F(G)$

$$F(G) = P(G)G + Q(G). \quad (23)$$

Equations (14) and (15) read

$$2 \frac{d}{dt} \left( \dot{g}^2(t) \frac{dP(t)}{dt} \right) - 2\dot{g}^3(t) \frac{dP(t)}{dt} + \ddot{g}(t) = 0, \quad (24)$$

$$Q(t) = -24\dot{g}^3(t) \frac{dP(t)}{dt} - 6\dot{g}^2(t). \quad (25)$$

For the Big Rip singularity,  $\beta = 1$  in (6). If we assume  $H_0 = 0$  (the constant is negligible in the asymptotic singular limit  $t \rightarrow t_0$ ),  $\dot{g}(t) = h/(t_0 - t)$  and the most

general solution of (24) is given by

$$P(t) = \frac{1}{4h(h-1)}(2t_0-t)t + c_1 \frac{(t_0-t)^{3-h}}{3-h} + c_2, \tag{26}$$

where  $c_1$  and  $c_2$  are generic constants. From (25), we get

$$Q(t) = -\frac{6h^2}{(t_0-t)^2} - \frac{24h^3 \left[ \frac{(t_0-t)}{2h(h-1)} - c_1(t_0-t)^{2-h} \right]}{(t_0-t)^3}. \tag{27}$$

Furthermore, from (22) we obtain  $t$  in terms of  $G$  and, by solving (23), we find the most general form of  $F(G)$  which realizes the Big Rip singularity

$$F(G) = \frac{\sqrt{6h^3(1+h)}}{h(1-h)}\sqrt{G} + c_1 G^{\frac{1+h}{4}} + c_2 G. \tag{28}$$

This is an exact solution of the EOM in the case of Big Rip. The term  $c_2 G$  is a topological invariant. In general, if for large values of  $G$ ,  $F(G) \sim \alpha G^{1/2}$ , where  $\alpha (\neq 0)$  is a constant, the Big Rip singularity could appear for any value of  $h \neq 1$ . Note that  $c_2 G^{(1+h)/4}$  is an invariant with respect to the Big Rip solution.

**Other types of singularities and more general  $\mathcal{F}(R, G)$ -gravity case**

In a similar way, it is possible to reconstruct  $F(G)$ -gravity models in which the other types of singularities could appear, when  $\beta \neq 1$  in (6) and the scale factor, when  $H_0 = 0$ , behaves as

$$a(t) = \exp \left[ \frac{h(t_0-t)^{1-\beta}}{\beta-1} \right]. \tag{29}$$

We give some results.

The asymptotic solution (in the limit  $t \rightarrow t_0$ ) of  $F(G)$  when  $\beta > 1$  is expressed as

$$F(G) = -12\sqrt{\frac{G}{24}}. \tag{30}$$

Hence, if for large values of  $G$ ,  $F(G) \sim -\alpha\sqrt{G}$  with  $\alpha > 0$ , a Type I singularity could appear.

When  $\beta < 1$ , the asymptotic solution of  $F(G)$  becomes

$$F(G) \sim \alpha|G|^\gamma, \quad \gamma = \frac{2\beta}{3\beta+1}, \tag{31}$$

where  $\alpha$  is a constant. If for large values of  $G$ ,  $F(G)$  has this form with  $0 < \gamma < 1/2$ , we find  $0 < \beta < 1$  and a Type III singularity could emerge. If for  $G \rightarrow -\infty$ ,  $F(G)$  has the form in (31) with  $-\infty < \gamma < 0$ , we find  $-1/3 < \beta < 0$  and a Type II (sudden)

singularity could appear. Moreover, if for  $G \rightarrow 0^-$ ,  $F(G)$  has the form in (31) with  $1 < \gamma < \infty$ , we obtain  $-1 < \beta < -1/3$  and a Type II singularity could occur. If for  $G \rightarrow 0^-$ ,  $F(G)$  has the form in (31) with  $2/3 < \gamma < 1$ , we obtain  $-\infty < \beta < -1$  and a Type IV singularity could appear. We also require that  $\gamma \neq 2n/(3n-1)$ , where  $n$  is a natural number.

As a consequence, a large class of realistic models of  $F(G)$ -gravity, which reproduce the current acceleration and the early-time inflation, could generate future time-singularities, as for example[7]:

$$F_1(G) = \frac{a_1 G^n + b_1}{a_2 G^n + b_2}, \quad F_2(G) = \frac{a_1 G^{n+N} + b_1}{a_2 G^n + b_2}, \quad F_3(G) = a_3 G^n (1 + b_3 G^m). \quad (32)$$

All this models contain power functions of  $G$  and for some choices of parameters could produce singularities.

With reconstruction method it is possible to derive also more general  $\mathcal{F}(R, G)$ -models producing finite-time singularities. For example, in the model  $\mathcal{F}(R, G) = R - \alpha G/R$ , where  $\alpha$  is a positive constant, could appear the Type I singularity, whereas in the model  $\mathcal{F}(R) = R + \alpha R^\gamma$ , where  $\alpha$  and  $\gamma$  are constants, could appear Types II, III or IV singularities (for a review, see [5]).

## 6 Curing the finite-time future singularities

We discuss a possible way to cure the finite-time future singularities in  $F(G)$ -gravity and  $\mathcal{F}(R, G)$ -gravity. In the case of large curvature, the quantum effects become important and lead to higher-order curvature corrections. It is therefore interesting to resolve the finite-time future singularities with some power function of  $G$  or  $R$ .

We consider the description of modified gravity as inhomogeneous fluid. If some singularities occur, (21) behaves as

$$\mathcal{G}(H, \dot{H} \dots) \simeq -\frac{3(1+\omega)h^2}{\kappa^2} (t_0 - t)^{-2\beta} + \frac{2\beta h}{\kappa^2} (t_0 - t)^{-\beta-1}. \quad (33)$$

One way to prevent a singularity appearing could be that the function  $\mathcal{G}(H, \dot{H} \dots)$  becomes inconsistent with the behavior of (33) in the singular limit ( $t \rightarrow t_0$ ).

Let us consider a simple example in order to cure Big Rip singularity in  $F(G)$ -gravity. Suppose that for large values of  $G$ ,

$$R + F(G \rightarrow \infty) \longrightarrow R + \gamma G^m, \quad m \neq 1, \quad (34)$$

with  $\gamma \neq 0$ . For  $H = h/(t_0 - t)$ , namely the Big Rip case, we have

$$\mathcal{G}(H, \dot{H} \dots) \simeq \frac{\alpha}{(t_0 - t)^{4m}}. \quad (35)$$

Hence, if  $m > 1/2$ ,  $\mathcal{G}(H, \dot{H} \dots)$  tends to infinity faster than (33) and we avoid this kind of singularity.

As general results, we find that the term  $\gamma G^m$  with  $m > 1/2$  and  $m \neq 1$  cure the singularities occurring when  $G \rightarrow \pm\infty$  (Type I, II and III). Moreover, the term  $\gamma G^m$  with  $m \leq 0$  cure the singularities occurring when  $G \rightarrow 0^-$  (Type II, IV).

In  $f(R)$ -gravity (namely,  $R$  plus a function of  $R$ ), by using the term  $\gamma R^m$ , the same consequences are found. The term  $\gamma R^m$  with  $m > 1$  cures the Type I, II and III singularities. The term  $\gamma R^m$  with  $m < 2$  cures the Type IV singularity.

Within the framework of  $\mathcal{F}(R, G)$ -gravity, we can use the terms such as  $G^m/R^n$  to cure the singularities. For example, we can avoid the Type I singularities if the asymptotic behavior of the model is given by  $\gamma G^m R^n$ , with  $m, n > 0$ .

## 7 Effects of viscous fluid in singular universe

As the last point, we explore the role of perfect/viscous fluids within singular modified gravity, investigating how the singularities may change or disappear, due to the contribution of quintessence or phantom fluids.

We consider the class of modified gravity  $\mathcal{F}(R, G) = R + f(R, G)$ , where  $f(R, G)$  is a function of the Ricci scalar  $R$  and the Gauss-Bonnet invariant  $G$ , and we suppose the presence in the universe of cosmic viscous fluid, whose EoS is given by

$$p = \omega\rho - 3H\zeta, \quad (36)$$

where  $p$  and  $\rho$  are the pressure and energy density of fluid, respectively, and  $\omega$  is the EoS parameter.  $\zeta$  is the bulk viscosity and in general it could depend on  $\rho$ , but we will consider the simplest case of constant viscosity only (for more general cases, see [8]). On thermodynamical grounds, in order to have the positive sign of the entropy change in an irreversible process,  $\zeta$  has to be a positive quantity.

The FRW-equations of motion are:

$$\rho_G + \rho = \frac{3}{8\pi G_N} H^2, \quad p_G + p = -\frac{1}{8\pi G_N} (2\dot{H} + 3H^2). \quad (37)$$

The modified gravity is formally included into the modified energy density  $\rho_G$  and the modified pressure  $p_G$ , which correspond to (17)-(18) for  $g(R, G) = 1$ .

The fluid energy conservation law is a consequence of the EOM (37):

$$\dot{\rho} + 3H\rho(1 + \omega) = 9H^2\zeta. \quad (38)$$

The presence of fluid could influence the behaviour of singular  $f(R, G)$ -models (i.e. models that in absence of fluids produce some singularities). We will check the solutions of the fluid energy density when  $H$  is singular.



**Non viscous case**

In the non-viscous case  $\zeta = 0$  (perfect fluid), the solution of (38) assumes the classical form:

$$\rho = \rho_0 a(t)^{-3(1+\omega)}, \tag{39}$$

where  $\rho_0$  is a positive constant and  $a(t)$  is the scale factor of the universe. By combining (39) with (29), it is easy to see that for  $\beta > 1$  (Type I singularity),  $\rho$  grows up and diverges exponentially if  $\omega < -1$ . In the presence of phantom fluid, the EOM (37) become inconsistent with respect to the singular form of Hubble parameter in (6), and the Type I singularity is not realized in  $f(R, G)$ -gravity.

When  $0 < \beta < 1$ , the fluid energy density  $\rho$  is avoidable on the Type III singular solution, whereas for Type II and IV singular models ( $\beta < 0$ ), the presence of quintessence or phantom fluids makes the singularities worse. In particular, in the case of  $\beta < -1$ , the dynamical behaviour of (37) could become inconsistent, because  $\rho$  behaves as  $(t_0 - t)$  and it is larger than the time-dependent part of  $H^2$  ( $\sim (t_0 - t)^{-\beta}$ ) when  $t \rightarrow t_0$ .

**Constant viscosity**

Suppose to have the bulk viscosity equal to a constant,  $\zeta = \zeta_0$ , and the Hubble parameter in the general form of (6). The asymptotic solutions of (38) in the singular limit  $t \rightarrow t_0$  are:

$$\rho \simeq \frac{3h\zeta_0}{(1 + \omega)(t_0 - t)^\beta}, \quad \beta > 1, \tag{40}$$

$$\rho \simeq \frac{9\zeta_0 h^2}{(2\beta - 1)(t_0 - t)^{2\beta - 1}}, \quad 1 > \beta > 0, \tag{41}$$

$$\rho \simeq \frac{9hH_0\zeta_0}{(\beta - 1)(t_0 - t)^{\beta - 1}} + \frac{3H_0\zeta_0}{1 + \omega}, \quad 0 > \beta, H_0 \neq 0. \tag{42}$$

In the first and second cases ( $\beta > 0$ ), it is possible to see that  $\rho$  diverges more slowly than  $H^2$ , so that viscous fluid does not influence the asymptotically behaviour of Types I and III singular models in (37), due to the constant viscosity.

In the third case, we consider fluids that tend to a non-negligible energy density when  $\beta < 0$ . It automatically leads to  $H_0 \neq 0$  in (6) and  $\rho$  behaves as in (42). Large bulk viscosity  $\zeta_0$  becomes relevant in the EOM. Moreover, if  $\omega < -1$ , the effective energy density (namely,  $\rho_G + \rho$ ) could be negative and avoid the Type II and IV singularities for expanding universe (where  $H_0 > 0$ ).

## 8 Conclusion

We have investigated the finite-time future singularities in  $F(G)$ -gravity and  $\mathcal{F}(R, G)$ -gravity. We can reconstruct the  $F(G)$ -gravity and  $\mathcal{F}(R, G)$ -gravity models in which the singularities could appear. Note that all types of future-time singularities could appear in  $\mathcal{F}(R, G)$ -modified gravity. In addition, we have discussed a possible way to resolve the finite-time future singularities in  $F(G)$ -gravity and  $\mathcal{F}(R, G)$ -gravity under quantum effects of higher-order curvature corrections or the presence of perfect/viscous fluid in the universe.

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## References

1. S. Nojiri and S. D. Odintsov, eConf **C0602061**, 06 (2006), Int. J. Geom. Meth. Mod. Phys. **4**, 115, arXiv:0601213 [hep-th] (2007); G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, L. Sebastiani and S. Zerbini, Phys. Rev. D **77**, 046009, arXiv:0712.4017 [hep-th] (2008); S. Nojiri and S. D. Odintsov, arXiv:0801.4843 [astro-ph] (2008); S. Nojiri and S. D. Odintsov, arXiv:0807.0685 [hep-th] (2008); T. P. Sotiriou and V. Faraoni, arXiv:0805.1726 [gr-qc] (2008); F. S. N. Lobo, arXiv:0807.1640 [gr-qc] (2008); S. Capozziello and M. Francaviglia, Gen.Rel.Grav.40:357-420, arXiv:0706.1146 [astro-ph] (2008); A. Silvestri and M. Trodden, Rept. Prog. Phys. **72**, 096901, arXiv:0904.0024 [astro-ph.CO] (2009); M. Sami, arXiv:0904.3445 [hep-th] (2009).
2. S. Nojiri and S. D. Odintsov, Phys. Lett. B **631**, 1 (2005); S. Nojiri, S. D. Odintsov and O. G. Gorbunova, J. Phys. A **39**, 6627 arXiv:0510183[hep-th] (2006); G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov and S. Zerbini, Phys. Rev. D **73**, 084007 arXiv:0601008[hep-th] (2006); S. Nojiri, S. D. Odintsov and M. Sami, *ibid.* **74**, 046004 arXiv:0605039(2006)[hep-th] (2006); S. Nojiri and S. D. Odintsov, J. Phys. Conf. Ser. **66**, 012005 arXiv:0611071[hep-th], (2007); G. Cognola, E. Elizalde, S. Nojiri, S. Odintsov and S. Zerbini, Phys. Rev. D **75**, 086002, arXiv:0611198 [hep-th] (2007); K. Bamba, C. Q. Geng, S. Nojiri and S. D. Odintsov, arXiv:0909.4397 [hep-th] (2009).
3. T.Kobayashi and K. Maeda, Phys.Rev. **D 78**, 064019, arXiv:0807.2503v2 [astro-ph] (2008).
4. S. Nojiri, S. D. Odintsov and S. Tsujikawa, Phys. Rev. D **71**, 063004, arXiv:0501025 [hep-th] (2005).
5. K. Bamba, S. Nojiri, S. D. Odintsov, J. Cosmol. Astropart. Phys. **0810**, 045, arXiv: 0807.2575 [hep-th] (2008); K. Bamba, S. D. Odintsov, L. Sebastiani, S. Zerbini, Eur. Phys. J. C67: 295-310, arXiv: 0911.4390 [hep-th] (2010).
6. G. Cognola, M. Gastaldi and S. Zerbini, Int. J. Theor. Phys. **47**, 898, gr-qc/0701138 (2008).
7. S. Nojiri, S. D. Odintsov and P. V. Tretyakov, Prog. Theor. Phys. Suppl. **172**, 81 (2008).
8. O. Gorbunova and L. Sebastiani, arXiv:1004.1505 [gr-qc], to appear in Gen.Relat.Grav. (2010).