Conformal Equivalence in Classical Gravity: the Example of "Veiled" General Relativity

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Abstract In the theory of General Relativity, gravity is described by a metric which couples minimally to the fields representing matter. We consider here its "veiled" versions where the metric is conformally related to the original one and hence is no longer minimally coupled to the matter variables. We show on simple examples that observational predictions are nonetheless exactly the same as in General Relativity, with the interpretation of this "Weyl" rescaling "à la Dicke", that is, as a spacetime dependence of the inertial mass of the matter constituents.

1 Introduction

Many extensions of General Relativity which are under current investigation (for example $f(R)$ gravity, see e.g. [1], or quintessence models, see e.g. [2]) fall in the class of scalar-tensor theories (see e.g. [3]) where gravity is represented by a scalar field $\tilde{\phi}$ together with a metric $\tilde{\boldsymbol{g}}$ which minimally couples to the matter variables. Now, as is well-known (see [4] where references to the earlier literature can also be found), the "Jordan frame" variables $\tilde{\phi}$ and $\tilde{\boldsymbol{g}}$ can be traded for the "Einstein frame" variables $(\phi_*, \boldsymbol{g}_*)$ with $\tilde{\boldsymbol{g}} = e^{2\Omega}\boldsymbol{g}_*$, the conformal factor Ω being chosen so that the action for gravity becomes Einstein-Hilbert's, the "price to pay" being that the matter fields no longer minimally couple to the metric $\boldsymbol{g}_*$.

Although there seems to be an agreement in the recent literature about the mathematical equivalence of these two "frames" (as long as Ω does not blow up) there is still some debate about their "physical" equivalence, the present trend (see e.g. [1] and references therein) being that calculations may be performed in the Einstein frame but interpretation should be done in the Jordan frame (for the opposite

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_23,

view see e.g. [5], where a comprehensive review of the earlier literature can also be found).

It should be clear however, see [6], that, just as one can formulate and interpret a theory using any coordinate system (proper account being taken of inertial accelerations if need be), one should be able to formulate and interpret (classical) gravity using any conformally related metric, proper account being taken of non-minimal coupling if need be. (For recent papers supporting this view, see e.g. [7, 8, 9].)

In this paper we shall try to make this equivalence "crystal clear" by showing that some familiar predictions of General Relativity can equivalently be made in its "veiled" versions where the metric is conformally related to the original one and hence is no longer minimally coupled to the matter variables.

2 Conformal transformations and "veiled" General Relativity

In the theory of General Relativity:

- Events are represented by the points P of a 4-dimensional manifold $\mathscr{M}$ equipped with a Riemannian metric $\boldsymbol{g}$, with components $g_{\mu\nu}(x^\alpha)$ in the (arbitrary) coordinate system x^α labelling the points P.
- Matter is represented by a collection of tensorial fields on $\mathscr{M}$, denoted $\psi_{(a)}(P)$.
- Gravity is encoded in the metric $\boldsymbol{g}$ which couples minimally to the fields $\psi_{(a)}$. This means that the action for matter is obtained from the form it takes in flat spacetime in Minkowskian coordinates by replacing $\eta_{\mu\nu}$ by $g_{\mu\nu}$.
- Finally the action for gravity is postulated to be Einstein-Hilbert's.

Hence the familiar total action:

$$S[g_{\mu\nu},\psi_{(a)}]=\frac{1}{16\pi}\int d^4x\sqrt{-g}R+S_m[g_{\mu\nu},\psi_{(a)}]\,, \tag{1}$$

where g is the determinant of the metric components $g_{\mu\nu}$ and R the scalar curvature. Our conventions are: signature $(-+++)$, $R=g^{\mu\nu}R_{\mu\nu}$, $R_{\mu\nu}=R^{\sigma}{}_{\mu\sigma\nu}$, $R^{\mu}{}_{\nu\rho\sigma}=\partial_\rho\Gamma^{\mu}{}_{\nu\sigma}+\cdots$. We use Planck units where $c=\hbar=G=1$.

The field equations are obtained by extremising S with respect to the metric $g_{\mu\nu}$ and the matter fields $\psi_{(a)}$, which yields the equally familiar Einstein equations,

$$G_{\mu\nu}=8\pi T_{\mu\nu}\,,\qquad \frac{\delta S_m}{\delta\psi_{(a)}}=0\,, \tag{2}$$

where $G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor and where $T_{\mu\nu}=-\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}$ is the total stress-energy tensor. As is well-known $T_{\mu\nu}$ is constrained by the Bianchi identity to be divergence-less,

$$D_\nu T^{\mu\nu}=0\,, \tag{3}$$

D being the covariant derivative associated with $\boldsymbol{g}$. Recall that this conservation law implies that the worldline of uncharged test particles are represented by geodesics of the metric $\boldsymbol{g}$.

Let us now equip our manifold $\mathscr{M}$ with another metric $\bar{\boldsymbol{g}}$, with components $\bar{g}_{\mu\nu}$ in the same coordinate system x^α, which is conformally related to the original one:

$$g_{\mu\nu} = \Phi\, \bar{g}_{\mu\nu}\,, \tag{4}$$

$\Phi(x^\alpha)$ being an arbitrary function of the coordinates, that we shall restrict to be everywhere positive.[1]

Using the fact that $\sqrt{-g} = \Phi^2\sqrt{-\bar{g}}$ and that the Ricci tensors and scalar curvatures are related as

$$R_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{\bar{D}_{\mu\nu}\Phi}{\Phi} - \frac{\bar{g}_{\mu\nu}}{2}\frac{\bar{\Box}\Phi}{\Phi} + \frac{3}{2}\frac{\partial_\mu\Phi\partial_\nu\Phi}{\Phi^2}\,, \quad R = \frac{1}{\Phi}\left(\bar{R} - 3\frac{\bar{\Box}\Phi}{\Phi} + \frac{3}{2}\frac{(\bar{\partial}\Phi)^2}{\Phi^2}\right), \tag{5}$$

($\bar{R}_{\mu\nu}$, $\bar{R}$ and $\bar{D}$ being the Ricci tensor, the scalar curvature and the covariant derivative associated with the metric $\bar{\boldsymbol{g}}$), it is easy to find the "veiled" version of Einstein's equations (2),

$$\Phi\,\bar{G}_{\mu\nu} - \bar{D}_{\mu\nu}\Phi + \bar{g}_{\mu\nu}\bar{\Box}\Phi + \frac{3}{2\Phi}\left(\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}\bar{g}_{\mu\nu}(\bar{\partial}\Phi)^2\right) = 8\pi\bar{T}_{\mu\nu}\,, \quad \frac{\delta S_m}{\delta\psi_{(a)}} = 0\,, \tag{6}$$

where S_m is now expressed in terms of $\bar{g}_{\mu\nu}$, $S_m[g_{\mu\nu},\psi_{(a)}] = S_m[\Phi\,\bar{g}_{\mu\nu},\psi_{(a)}]$, and where $\bar{T}_{\mu\nu} = -\frac{2}{\sqrt{-\bar{g}}}\frac{\delta S_m}{\delta\bar{g}^{\mu\nu}}$ so that $\bar{T}_{\mu\nu} = \Phi\,T_{\mu\nu}$, with $g_{\alpha\beta}$ replaced by $\Phi\,\bar{g}_{\alpha\beta}$ in $T_{\mu\nu}$. As for the Bianchi identity (3), it translates into

$$\bar{D}_\nu\bar{T}^{\mu\nu} = \frac{\bar{\partial}^\mu\Phi}{2\Phi}\,\bar{T}\,. \tag{7}$$

The total stress-energy tensor is no longer conserved.

Equations (6), (7) can also be straightforwardly obtained from the Einstein-Hilbert action (1). Indeed it reads, using (4) and up to a boundary term,

$$S[\bar{g}_{\mu\nu},\Phi,\psi_{(a)}] = \frac{1}{16\pi}\int d^4x\sqrt{-\bar{g}}\left(\Phi\,\bar{R} + \frac{3}{2}\frac{(\bar{\partial}\Phi)^2}{\Phi}\right) + S_m[\Phi\,\bar{g}_{\mu\nu},\psi_{(a)}]\,. \tag{8}$$

Extremisation with respect to $\bar{g}_{\mu\nu}$ and $\psi_{(a)}$ yields (6). As for the extremisation with respect to Φ it is redundant since it turns out to be equivalent to the trace of equation (6). This reflects the fact that, $\bar{g}_{\mu\nu}$ remaining unconstrained, Φ is an arbitrary

[1] $(\mathscr{M},\mathrm{g})$ or $(\mathscr{M},\bar{\mathrm{g}})$ are often called, rather improperly, "frames", when a more accurate wording would be "representations" of space and time, see [6].

function and not a dynamical field.[2]

Let us now be more specific about the matter action S_m.

As an example (others are considered in the Appendix), take matter to be an electron characterized by its inertial mass m and charge q interacting with the electromagnetic field A_μ created by an infinitely massive proton, so that S_m is the Lorentz action where $\eta_{\mu\nu} \to g_{\mu\nu}$:[3]

$$S_m[g_{\mu\nu}, L] = -m \int_L \sqrt{-g_{\mu\nu} dx^\mu dx^\nu} + q \int_L A_\mu dx^\mu \,, \tag{9}$$

where L is a path determined by $x^\mu = x^\mu(\lambda)$. The equation of motion of the electron, $\frac{\delta S_m}{\delta L} = 0$, is the familiar Lorentz equation,

$$m u^\nu D_\nu u^\mu = q F^\mu{}_\nu u^\nu \,, \tag{10}$$

where $u^\mu = dx^\mu/d\tau$ with $g_{\mu\nu} u^\mu u^\nu = -1$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Equivalently, S_m reads, in terms of the metric $\bar{g}_{\mu\nu}$,

$$S_m[\bar{g}_{\mu\nu}, \Phi, L] = -\int_L \bar{m} \sqrt{-\bar{g}_{\mu\nu} dx^\mu dx^\nu} + q \int_L \bar{A}_\mu dx^\mu \,, \tag{11}$$

where $\bar{A}_\mu = A_\mu$ and

$$\bar{m} = \sqrt{\Phi}\, m \,. \tag{12}$$

As for the Lorentz equation (10), it becomes

$$\bar{m} \left[\bar{u}^\nu \bar{D}_\nu \bar{u}^\mu + \frac{1}{2\Phi} \partial_\nu \Phi \, (\bar{g}^{\mu\nu} + \bar{u}^\mu \bar{u}^\nu) \right] = q \bar{F}^\mu{}_\nu \bar{u}^\nu \,, \tag{13}$$

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[2] One notes the resemblance of the action (8) and the field equations (6) with the Brans-Dicke action and field equations [10] when their parameter ω is $\omega = -3/2$, see e.g. [11]. The difference (which makes $\omega = -3/2$ Brans-Dicke theory different from General Relativity) is that, in Brans-Dicke theory, matter is minimally coupled to the metric $\bar{\boldsymbol{g}}$ (not $\boldsymbol{g}$),

$$S_{\rm BD}^{\omega=-3/2} = \frac{1}{16\pi} \int d^4x \sqrt{-\bar{g}} \left(\Phi \bar{R} + \frac{3}{2} \frac{(\bar{\partial}\Phi)^2}{\Phi} \right) + S_m[\bar{g}_{\mu\nu}, \psi_{(a)}] \,.$$

In the spirit of [12], one could therefore introduce a "detuned" version of General Relativity based on the action,

$$S_{\rm detunedGR} = \frac{1}{16\pi} \int d^4x \sqrt{-\bar{g}} \left(\Phi \bar{R} + \frac{3}{2} \frac{(\bar{\partial}\Phi)^2}{\Phi} \right) + S_m[\Phi F(\Phi) \bar{g}_{\mu\nu}, \psi_{(a)}] \,,$$

which reduces to "veiled" General Relativity if $F(\Phi) = 1$ and to $\omega = -3/2$ Brans-Dicke theory if $F(\Phi) = \Phi^{-1}$. We shall not pursue this idea any further here.

[3] In Planck units m and q are two dimensionless numbers which are determined in a local inertial frame where gravity is "effaced"[13] and where the laws of Special Relativity hold.

with $\bar{u}^\mu = dx^\mu/d\bar{\tau}$ and $\bar{g}_{\mu\nu}\bar{u}^\mu\bar{u}^\nu = -1$.

In locally Minkowskian coordinates X^μ in the neighbourhood of some point P where $\bar{g}_{\mu\nu} \approx \eta_{\mu\nu}$ and if Φ is approximately constant, this equation takes the form,

$$\bar{m}\frac{dU^\mu}{d\tau_M} \approx q F^\mu{}_\nu U^\nu , \tag{14}$$

with $U^\mu = dX^\mu/d\tau_M$ and $\eta_{\mu\nu}U^\mu U^\nu = -1$. This equation is the same as the one governing the motion of the electron in Special Relativity apart from the fact that its mass is rescaled by the factor $\sqrt{\Phi(P)}$, see [6].[4]

As an illustration of the consequences of the rescaling of the mass in veiled General Relativity, consider for example a transition between the levels n and n' of, say, the hydrogen atom. Its frequency is given by Bohr's formula,

$$\bar{\nu}(P) = \left(\frac{1}{n'^2} - \frac{1}{n^2}\right)\frac{\bar{m}(P)q^4}{2} \qquad \text{with} \qquad \bar{m}(P) = \sqrt{\Phi(P)}\,m . \tag{15}$$

It depends on P, that is, on when and where it is measured. Hence the frequency $\bar{\nu}(P) \equiv \bar{\nu}$ of the transition measured at point P ("there and then") and the frequency $\bar{\nu}(P_0) \equiv \bar{\nu}_0$ of the same transition measured at P_0 ("here and now") are related by:[5]

$$\bar{\nu} = \sqrt{\frac{\Phi(P)}{\Phi(P_0)}}\,\bar{\nu}_0 . \tag{16}$$

3 Conformal equivalence in cosmology

Let us show here on a few examples that the standard cosmological models of General Relativity or its conformally related sister theories all lead to the same physical predictions and hence are observationally indistinguishable.

The field equations to solve are the veiled Einstein equations (6)-(7) for $\bar{g}_{\mu\nu}$ and Φ.

We look for simplicity for spatially flat Robertson-Walker metrics,

$$d\bar{s}^2 = \bar{a}^2(t)(-dt^2 + dr^2) , \tag{17}$$

[4] This space-time dependence of the (inertial) mass can be interpreted as a local rescaling of the unit of mass, see [6]. It can also be interpreted as the result of the "interaction" of the "scalar field" Φ with matter. It must be remembered however that this "interaction" is an artefact of the introduction of the metric $\bar{\boldsymbol{g}}$, and that the "scalar force" which appears in (13) or (7) can be globally effaced by returning to the original metric $\boldsymbol{g}$, just like an inertial force can be effaced by going to an inertial frame.

[5] This difference between the two numbers $\bar{\nu}$ and $\bar{\nu}_0$ can be interpreted as simply due to the fact that they are expressed using a different unit of time at P and P_0, see [6].

where the scale factor $\bar{a}$ and the scalar field Φ depend on t only. By construction equations (6)-(7) are undetermined and we shall choose here, to make our point more strikingly, Φ to be the dynamical field describing gravity by imposing

$$\bar{a}(t) = 1\,. \tag{18}$$

Therefore the metric $\bar{\boldsymbol{g}}$ is flat.[6]

Matter is represented by the stress-energy tensor of a perfect fluid (see Appendix): $\bar{T}_{\mu\nu} = (\bar{\rho} + \bar{p})\bar{u}_\mu \bar{u}_\nu + \bar{p}\,\bar{g}_{\mu\nu}$ that we choose to be at rest with respect to the Minkowskian coordinate grid (t,r):[7] $\bar{u}^\mu = (1,0)$; as for the (veiled) density and pressure $\bar{\rho}$ and $\bar{p}$ they depend on t.

The equations of motion (6)-(7) for Φ then reduce to, a prime denoting derivation with respect to t,

$$\frac{3}{4\Phi}\Phi'^2 = 8\pi\bar{\rho}\,, \qquad \bar{\rho}' = \frac{\Phi'}{2\Phi}(\bar{\rho} - 3\bar{p})\,, \tag{19}$$

which can be solved once an equation of state is given. For $\bar{p} = w\bar{\rho}$ for example,

$$\Phi = \left(\frac{t}{t_0}\right)^{4/(1+3w)}\,, \quad \bar{\rho} = \frac{3}{2\pi(1+3w)t_0^2}\left(\frac{t}{t_0}\right)^{2(1-3w)/(1+3w)}\,. \tag{20}$$

Let us now turn to the relation between the luminosity distance D and redshift z that the model predicts.

As usual, we focus on a given atomic transition line in the spectrum of a distant galaxy at point $P = (t,r)$. The observer is at point $P_0 = (t_0, 0)$, and the atomic line emitted by this galaxy is observed at frequency ν_0. As given in (16), if $\bar{\nu}$ is the frequency of this transition measured at point P, the frequency of the same transition measured at point P_0 will be $\bar{\nu}_0 = \sqrt{\Phi(P_0)/\Phi(P)}\,\bar{\nu}$. Therefore the observed redshift is given by

$$1 + z = \frac{\bar{\nu}_0}{\nu_0} = \sqrt{\frac{\Phi(t_0)}{\Phi(t)}}\,\frac{\bar{\nu}}{\nu_0}\,. \tag{21}$$

The luminosity distance is given, by definition, as

$$D = \sqrt{\frac{L}{4\pi\ell}}\,, \tag{22}$$

where L is the absolute luminosity of the galaxy and ℓ is the apparent luminosity per unit area observed at point P_0. Since the mass of the electron in veiled General Relativity varies according to $\bar{m} = \sqrt{\Phi}\,m$, it is crucial here to recall that the absolute

[6] This does not mean that t and r represent time and position in an inertial frame since the world-lines of free particles are not straight lines. They rather solve, see (13): $\bar{u}^\nu \bar{D}_\nu \bar{u}^\mu = -\frac{1}{2\Phi}(\bar{\partial}^\mu \Phi + \bar{u}^\mu \bar{u}^\nu \partial_\nu \Phi)$, whose solution is, C being three constants: $\bar{V} \equiv \bar{u}/\bar{u}^0 = C/\sqrt{C^2 + \Phi(t)} \neq$ const..

[7] This is the familiar "Weyl postulate".

luminosity is *not* equal to the luminosity measured at the point of emission P (where the frequency of the transition is $\bar{\nu}$) but is defined as if the galaxy were at the point of reception P_0 (where the frequency of the transition is $\bar{\nu}_0$) so that we have [8]

$$L = N\frac{\bar{\nu}_0}{\Delta t} = N\bar{\nu}_0^2\,, \tag{23}$$

where N is the number of photons emitted by this transition during a period $\Delta t = 1/\bar{\nu}_0$. The apparent luminosity is given by

$$\ell = N\frac{\nu_0^2}{S} = N\frac{\nu_0^2}{4\pi r^2}\,, \tag{24}$$

where $S = 4\pi r^2$ is the surface area of a sphere of radius r since the metric $\bar{\boldsymbol{g}}$ is flat. Inserting (23) and (24) into (22), we find, using (21),

$$D = \frac{\bar{\nu}_0}{\nu_0}\,r = \sqrt{\frac{\Phi(t_0)}{\Phi(t)}}\,\frac{\bar{\nu}}{\nu_0}\,r\,. \tag{25}$$

In order finally to relate $\bar{\nu}$ to ν_0 and r to T we must study the propagation of light from P to P_0. Light follows the null cones of $(\mathscr{M}, \bar{g}_{\mu\nu} = \eta_{\mu\nu})$ so that r is the time light takes to propagate from P to P_0, and the frequency $\bar{\nu}$ measured at P is the same as the frequency ν_0 observed at P_0:

$$(\bar{\nu} = \nu_0\,,\quad r = t_0 - t) \qquad \Longrightarrow \qquad z = \sqrt{\frac{\Phi(t_0)}{\Phi(t)}} - 1\,,\quad D = \sqrt{\frac{\Phi(t_0)}{\Phi(t)}}\,(t_0 - t)\,. \tag{26}$$

Let us, for cosmetics, trade an integration on t by an integration on z:

$$t_0 - t = \int_t^{t_0} dt = -\int_0^z \frac{dz}{dz/dt} = \int_0^z \frac{2\Phi^{3/2}}{\Phi'}dz\,. \tag{27}$$

This leads us to the relationship between the luminosity-distance and redshift that our cosmological model in veiled General Relativity predicts:

$$D = (1+z)\int_0^z \frac{dz}{H}\,, \tag{28}$$

where $H \equiv \Phi'/(2\Phi^{3/2})$ must be expressed in terms of $z = \sqrt{\frac{\Phi(t_0)}{\Phi(t)}} - 1$ after integration of the equations of motion (19) for Φ.

Now, in General Relativity, that is, in the "unveiled frame", $d\bar{s}^2 = a^2\,ds^2$ with $a = \sqrt{\Phi}$, where matter is minimally coupled to the metric $g_{\mu\nu} = a^2\eta_{\mu\nu}$, H is nothing but the "Hubble parameter":

[8] This (crucial) coupling of the inertial masses to the scalar field Φ is forgotten in some papers, see e.g. [14], which hence (wrongly) conclude to the inequivalence of the Jordan and Einstein frames.

$$H \equiv \frac{\Phi'}{2\Phi^{3/2}} = \frac{a'}{a^2} = \frac{1}{a}\frac{da}{d\tau}, \tag{29}$$

with $d\tau \equiv a\,dt$. Moreover the equations of motion (19) for Φ are identical to the standard Friedmann-Lemaître equations,

$$3H^2 = 8\pi\rho\,, \quad \dot{\rho} + 3H(\rho + p) = 0\,, \tag{30}$$

with $\rho \equiv \bar{\rho}/\Phi^2$ and $p \equiv \bar{p}/\Phi^2$ (see Appendix). Finally, the text-book derivation of the relation luminosity-distance versus redshift yields (28). Therefore the predicted relationship between the observables z and D is the same, whether we represent gravity by a curved Robertson-Walker metric $g_{\mu\nu} = a^2\eta_{\mu\nu}$ minimally coupled to matter as in General relativity, or by a flat metric $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ together with a scalar field Φ coupled to matter, in its "veiled" version.

The physical *interpretation* of (28) is however different. Indeed, in the particular version of veiled General Relativity that we considered here:

- The evolution of the universe is not interpreted by cosmic expansion. Since we chose $\Phi = a^2$ there is in fact no cosmic expansion at all: $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$; but we defined on this flat manifold a scalar field Φ which evolves in time and describes the interaction of gravity and matter.
- There is no redshifting of photons, since the frequency of an atomic transition *measured* at P is equal to the frequency of that same transition as *observed* at P_0 ($\bar{\nu} = \nu_0$).
- However the interaction of Φ with matter implies that the mass $\bar{m}$ of the electron varies in time ($\bar{m} = \sqrt{\Phi}\,m = a\,m$). Therefore the frequency of an atomic transition as measured in a lab there and then at P is not the same as the frequency measured here and now at P_0: $\bar{\nu} = \sqrt{\Phi(P)/\Phi(P_0)}\,\bar{\nu}_0 = (a/a_0)\bar{\nu}_0$. This redshifting due to a varying mass is exactly the same as the one due to a cosmological redshift in General Relativity.

Pursuing the above interpretation, the temperature of the cosmic microwave background can be considered constant, since photons are not redshifted, and chosen to be the present temperature $T_0 = 2.725\mathrm{K}$, throughout the whole history of the universe (that is, during the whole time-evolution of the gravitational field Φ). The universe was in thermal equilibrium when the electron mass was smaller by a factor of more than 10^3 compared to the mass today, that is when the ground state binding energy of the hydrogen was less than $0.0136\mathrm{eV}$. The "Big-Bang" is flat space at time $t = 0$ when the masses of the matter constituents are zero.

In conclusion, the above considerations show that the physical interpretation of the equations can be very different in General Relativity or its veiled versions, but the resulting relations between observables are completely independent of the conformal representation (or "frame") one chooses.

4 Conformal equivalence in local gravity

We shall see here that the tests of General Relativity in the Solar System (gravitational redshift, bending of light, perihelion advance, Shapiro effect...) can just as well be constructed using veiled General Relativity.

For definiteness let us describe the gravitational field of the Sun by the Schwarzschild solution of the vacuum Einstein equations written in Droste coordinates $x^\mu = (t, r, \theta, \phi)$,

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = -(1-2M/r)\, dt^2 + \frac{dr^2}{1-2M/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (31)$$

where M is the (active) gravitational mass of the Sun. The propagation of light and the motion of planets in the Solar System are represented by (null) geodesics of this Schwarzschild spacetime. Proper time as measured in, say, Planck units, by a clock travelling in the Solar System is represented by the length of its timelike worldline $x^\mu(\lambda)$, that is, by the number $\tau = \int \sqrt{-g_{\mu\nu} dx^\mu dx^\nu}$.

Let us now introduce the following "veiled" Schwarzschild line element $ds^2 = \Phi\, d\bar{s}^2$ with $\Phi = 1 - 2M/r$ so that

$$d\bar{s}^2 \equiv \bar{g}_{\mu\nu} dx^\mu dx^\nu = -dt^2 + \frac{dr^2}{(1-2M/r)^2} + \frac{r^2}{1-2M/r}(d\theta^2 + \sin^2\theta d\phi^2), \quad (32)$$

which solves the "veiled" vacuum Einstein equations (6) (we shall restrict our attention to the region outside the horizon, $r > 2M$).[9]

Light follows the null geodesics of $\bar{g}_{\mu\nu}$ which are the same as those of $g_{\mu\nu}$. Therefore the prediction for the bending of light is the same as in General Relativity.

Test particles do not follow geodesics of $\bar{g}_{\mu\nu}$ and their equation of motion is given by (13) (with $q = 0$). However this equation is just a rewriting of the geodesic equation in the metric $g_{\mu\nu}$. Therefore the trajectories $r = r(\phi)$ in the equatorial plane $\theta = \pi/2$ are the same in both General Relativity and its veiled version. The prediction for, say, the perihelion advance of Mercury is hence the same.

Consider now an atom at rest at r and an observer at rest at r_0. Since t is proper time, the frequency ν_0 of an atomic transition, as observed at r_0, will be the same as the frequency $\bar{\nu}$ measured at r: $\bar{\nu} = \nu_0$. However, in close analogy with the cosmological case, since the mass of the electron undergoing this transition depends on r as (12), $\bar{m} = \sqrt{\Phi(r)}\, m$, $\bar{\nu}$ is related to the frequency $\bar{\nu}_0$ of the transition as measured at r_0 by (16): $\bar{\nu} = \bar{\nu}_0 \sqrt{\Phi(r)/\Phi(r_0)}$. Hence the gravitational redshift is predicted to be

[9] One may wonder if the line elements $d\bar{s}^2 = ds^2/\Phi$, ds^2 being the Schwarzschild solution, are the only solutions of the "veiled" vacuum Einstein equations (which, beware, are not $\bar{G}_{\mu\nu} = 0$!). The answer is yes since, by construction, these equations are undetermined and Φ can be chosen at will to solve them. One then chooses $\Phi = 1$ and invokes the uniqueness of the Schwarzschild solution.

$$z \equiv \frac{\bar{\nu}_0}{\nu_0} - 1 = \sqrt{\frac{\Phi(r_0)}{\Phi(r)}} - 1 = \sqrt{\frac{1-2M/r_0}{1-2M/r}} - 1\,, \tag{33}$$

which is exactly the same as the prediction of General Relativity.

Finally let us consider predictions for the tests of General Relativity relying on time measurements (such as the Shapiro effect, GPS,...). In veiled General Relativity, the proper time interval $d\bar{\tau} = \sqrt{-\bar{g}_{\mu\nu}\,dx^\mu dx^\nu}$ between two adjacent events $x^\mu = (t, r, \theta, \phi)$ and $x^\mu + dx^\mu$ differs from that of General Relativity $d\tau$: $d\bar{\tau} = d\tau/\sqrt{\Phi}$. However, if we recall that time measurements are based on atomic clocks, that is, time intervals are counted in units of a frequency of an atomic transition, we readily find that the observed number of 'ticks' will be the same,

$$N_{\text{ticks}} = \bar{\nu}\, d\bar{\tau} = \sqrt{\Phi}\,\nu\,\frac{d\tau}{\sqrt{\Phi}} = \nu\, d\tau\,, \tag{34}$$

where $\bar{\nu}$ and ν are the frequencies of an atomic transition defined in veiled and unveiled General Relativity, respectively. Thus predictions for all the time measurements in veiled General Relativity again exactly agree with those in General Relativity.

5 Conclusion

In 1912 Nordström proposed a theory where gravity was represented by a scalar field Φ on Minkowski spacetime with metric $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$. Of course, matter was non-minimally coupled to that field, so that its interaction to gravity be described (see e.g. [16]). In 1914 Einstein and Fokker introduced a conformally flat metric $g_{\mu\nu} = \Phi\,\eta_{\mu\nu}$ which turned Nordström's equation of motion of test particles into the geodesic equation of the metric $\boldsymbol{g}$. Hence matter was minimally coupled to $\boldsymbol{g}$. As for the Klein-Gordon field equation for Φ it became an equation relating the scalar curvature of $\boldsymbol{g}$ to the trace of the stress-energy tensor of matter. It was clear (at least to Einstein and Fokker !) that the two versions of the theory were strictly equivalent, Nordström's formulation being the "veiled" one. And if Nordström's theory was soon abandoned it was not because it had been formulated first in flat spacetime but because its predictions (deduced either from its "veiled" or "unveiled" formulations) were in contradiction with observations.

In this paper we did nothing more than what Einstein and Fokker did in 1914 but applied the idea to General Relativity itself, in order to show, in a hopefully clear way, that, even if the description of phenomena could be different in General Relativity and in its conformally related sister theories, the predictions for the relationships between (classical) observables were strictly the same.

It should then become obvious that the same conclusion holds too when dealing with extensions of General Relativity such as $f(R)$ theories, coupled quintessence or, more generally, scalar-tensor theories (even if the scalar field Φ is then truly dynamical): the Jordan frame, where matter is minimally coupled to the metric,

and the Einstein frame, where the action for gravity is Hilbert's, are equivalent, mathematically and physically, at least when dealing with classical phenomena and the motion of objects which are weakly gravitationally bound. Preferring to interpret the phenomena in the Jordan frame is somewhat similar to preferring to work in an inertial frame in Special Relativity: this allows to forget about the spacetime dependence of the inertial mass of the matter constituents just like one can forget about inertial forces in an inertial frame.

This analogy between inertial forces and non-minimal couplings points to quantum phenomena where the equivalence between the Jordan and Einstein frames may not hold.

Another point which deserves further investigation is the equivalence of conformally related frames when it comes to the motion of compact bodies whose gravitational binding energy is significant. It is known for example that a small black hole follows a geodesic in General Relativity [17]. In scalar tensor theories weakly gravitating bodies follow geodesics of the Jordan frame metric (to which matter is minimally coupled) but small black holes follow geodesics of the Einstein metric, see [18] and e.g. [19]. How this result, which is interpreted as a violation of the Strong Equivalence Principle, can be obtained using the Jordan frame exclusively remains to be elucidated. (see [20] for a recent study on the motion of small bodies in covariant classical field theories).

Acknowledgements N.D. thanks the Yukawa Institute for its hospitality and financial support during the YKIS2010 Symposium and Gravity and Cosmology 2010 Workshop. The work of M.S. is supported in part by Korea Institute for Advanced Study under the KIAS Scholar program, by the Grant-in-Aid for the Global COE Program "The Next Generation of Physics, Spun from Universality and Emergence" from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan, by JSPS Grant-in-Aid for Scientific Research (A) No. 21244033, and by JSPS Grant-in-Aid for Creative Scientific Research No. 19GS0219.

Appendix

We considered in the main text the example of matter being an electron in the field of an infinitely massive proton.

As a second example, consider matter to be a massive scalar field ψ with action,

$$S_m^{[\xi]} = -\frac{1}{2}\int d^4x\sqrt{-g}\left[(\partial\psi)^2 + \left(m^2 + \frac{\xi}{6}R\right)\psi^2\right]. \tag{35}$$

When $\xi = 0$ and $\xi = 1$, its extremisation with respect to ψ yields the familiar Klein-Gordon equations which read, respectively,

$$\Box\psi - m^2\psi = 0\,, \qquad \Box\psi - \left(m^2 + \frac{R}{6}\right)\psi = 0\,. \tag{36}$$

As for the veiled versions of (35), they are, respectively,

$$S_m^{[0]} = -\frac{1}{2}\int d^4x\sqrt{-\bar{g}}\,\Phi\left[(\bar{\partial}\psi)^2 + \bar{m}^2\psi^2\right],$$
$$S_m^{[1]} = -\frac{1}{2}\int d^4x\sqrt{-\bar{g}}\left[(\bar{\partial}\bar{\psi})^2 + \left(\bar{m}^2 + \frac{\bar{R}}{6}\right)\bar{\psi}^2\right] \tag{37}$$

where $g_{\mu\nu} = \Phi\bar{g}_{\mu\nu}$, $\bar{m} = \sqrt{\Phi}\,m$, and where $\bar{\psi} = \sqrt{\Phi}\,\psi$ in $S_m^{[1]}$. The extremisations of $S_m^{[0]}$ with respect to ψ and of $S_m^{[1]}$ with respect to $\bar{\psi}$ yield, respectively,

$$\bar{\Box}\psi - \bar{m}^2\psi = -\bar{\partial}\psi\cdot\frac{\bar{\partial}\Phi}{\Phi}, \qquad \bar{\Box}\bar{\psi} - \left(\bar{m}^2 + \frac{\bar{R}}{6}\right)\bar{\psi} = 0 \tag{38}$$

which are nothing but a rewriting of equations (36). In locally Minkowskian coordinates X^μ in the neighbourhood of some point P where $\bar{g}_{\mu\nu} \to \eta_{\mu\nu}$ and where Φ is approximately constant they reduce to their Special Relativistic forms, where the mass of the field has to be rescaled: $m \to \bar{m} = m\sqrt{\Phi(P)}$.[10]

In the case $\xi = 0$ the coupling of ψ to Φ can be globally effaced, by returning to the metric $\boldsymbol{g}$. In the case $\xi = 1$ the Klein-Gordon equation is, as is well-known, conformally invariant.

In the conformal invariant case, one might be confused by the fact that the stability of the field ψ depends on the sign of $(m^2 + R/6)$, while one can easily change its sign by a conformal transformation. This seemingly paradoxical situation is resolved by investigating more carefully the relation between the field in two different conformal frames.

As an example, let us consider the case when $g_{\mu\nu} = \eta_{\mu\nu}/(H\eta)^2$ is the (expanding part of) de Sitter metric (with $-\infty < \eta < 0$), and $m^2 < 0$ but $m^2 + R/6 = m^2 + 2H^2 > 0$, so that the field ψ is stable. Now consider the conformal transformation to the frame $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$. Then we have $\bar{m}^2 = m^2/(H\eta)^2 < 0$. Thus the field is badly unstable because the mass-squared is not only negative but diverges at $\eta = 0$. However if we recall that $\bar{\psi} = (-H\eta)^{-1}\psi$, this instability is solely due to the ill behaviour of the conformal factor as $\eta \to -0$.

Now let us consider a converse case when $g_{\mu\nu} = \eta_{\mu\nu}$ and $m^2 < 0$, so that the field ψ is unstable: $\psi \propto e^{|m|\eta}$ diverges exponentially. Turn now to the expanding de Sitter frame $\bar{g}_{\mu\nu} = \eta_{\mu\nu}/(H\eta)^2$, with $-\infty < \eta < 0$. Then the effective mass-squared $\bar{m}^2 + \bar{R}/6 = m^2/(H\eta)^2 + 2H^2$ will eventually become positive as $\eta \to -0$, hence the field must be stable in the expanding de Sitter frame. This seeming paradox can be resolved by noting the fact that $\bar{\psi} = (-H\eta)\psi \propto H\eta e^{|m|\eta}$. Thus time is bounded from above at $\eta = 0$, and hence there is literally 'no time' for the instability to develop.[11]

[10] Note that the same rescaling of mass occurs in a conformal transformation of the Dirac equation, see e.g. [15].

[11] We thank Andrei Linde for raising this issue.

As a last example consider matter to be a perfect fluid. Its stress-energy tensor and equations of motion, deduced from their special relativistic expressions by the replacement $\eta_{\mu\nu} \to g_{\mu\nu}$ are

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}\,, \qquad D_j T^{\mu\nu} = 0 \tag{39}$$

where ρ and p are the energy density and pressure of the fluid measured in a local inertial frame and where u^i is its 4-velocity normalised as $g_{\mu\nu}u^i u^j = -1$. Now, since $T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}$ (where we need not specify S_m), we have

$$\bar{T}_{\mu\nu} \equiv -\frac{2}{\sqrt{-\bar{g}}}\frac{\delta S_m}{\delta \bar{g}^{\mu\nu}} = \Phi T_{\mu\nu}\,, \tag{40}$$

so that the "veiled" version of (39) is, cf (7),

$$\bar{T}_{\mu\nu} = (\bar{\rho} + \bar{p})\bar{u}_\mu \bar{u}_\nu + \bar{p}\,\bar{g}_{\mu\nu}\,, \qquad \bar{D}_\nu \bar{T}^{\mu\nu} = \frac{\bar{\partial}^\mu \Phi}{2\Phi}\,\bar{T}\,, \tag{41}$$

where $\bar{g}_{\mu\nu}\bar{u}^i\bar{u}^j = -1$, and with $\bar{\rho} = \Phi^2\rho$ and $\bar{p} = \Phi^2 p$.[12]

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[12] Since the dimensions of ρ and p are $[M][L]^{-3}$ their rescaling is in keeping with the local rescaling of units alluded to in a previous footnote: $[M] \to [\bar{M}] = \sqrt{\Phi}[M]$ and $[L] \to [\bar{L}] = [L]/\sqrt{\Phi}$.

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