Brane Cosmology with an $f(R)$ contribution

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Abstract A generalised induced gravity brane-world model is proposed. The brane action contains an arbitrary $f(R)$ term, R being the scalar curvature of the brane while the brane is 5-dimensional and is described by a Hilbert-Einstein action. It can be shown that the effect of the $f(R)$ term on the dynamics of a homogeneous and isotropic brane is twofold: (i) an evolving induced gravity parameter and (ii) a shift on the energy density of the brane. This new shift term, which is absent on the Dvali, Gabadadze and Porrati (DGP) model, plays a crucial role to self-accelerate the generalised normal DGP branch of our model. The stability of de Sitter solutions is analysed under homogeneous perturbations. These results are compared with the standard 4-dimensional one.

1 Introduction

Understanding the recent acceleration of the universe is one of the most challenging task nowadays in physics. The first evidence for the acceleration of the universe was provided by the analysis of the Hubble diagram of SNe Ia more than a decade ago [1]. This amusing discovery, together with (i) measurements of the fluctuations in the cosmic microwave background radiation (CMB) which implied that the universe is (quasi) spatially flat and (ii) that the amount of matter which clusters gravitationally is much less than the critical energy density, implied the existence of some stuff usually dubbed the *dark energy component* that drives the late-time acceleration of the universe. Afterwards, more precised measurements of the CMB anisotropy by WMAP [2] and the power spectrum of galaxy clustering by the 2dFGRS and SDSS surveys [3, 4] have confirmed this discovery.

A possible approach to describing the late-time acceleration of the universe is to consider a modified theory of gravity, such that a weakening of this interaction on

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the appropriate scales induces the recent speed up of the universe (cf. [5, 6, 7, 8]). In other words, the weakening of gravity on large scales would provide an *effective negative pressure* that would induce the late-time acceleration of the universe.

A possible approach is the Dvali, Gabadadze and Porrati (DGP) scenario [9, 10, 11], which corresponds to a five-dimensional (5D) brane-world model. In this model, our universe is a brane; i.e. a 4D hyper-surface, embedded in a Minkowski space-time. Matter is trapped on the brane and only gravity experiences the full bulk. The DGP model has two sets of solutions: the self-accelerating branch and the normal one. The self-accelerating brane, as its name suggests, speeds up at late-time without invoking any unknown dark energy component. On the other hand, the normal branch requires a dark energy component to accommodate the current observations [12, 13]. From a geometrical point of view, the two branches are embedded in a completely different way in the bulk [10]. Despite the very nice features of the self-accelerating DGP branch, it suffers from serious theoretical problems like the ghost issue [14]. The main aim of this paper is to consider a mech anism to self-accelerate the normal branch which is known to be free from the ghost issue [14].

This mechanism will be based on a modified Hilbert-Einstein action on the brane and the simplest gravitational option is to invoke an $f(R)$ term. Extended theories of gravity based on 4D $f(R)$ scenarios have gathered a lot of attention in the last years (cf. for example and reference cited there [5, 6, 7]). It has been shown that these 4D models should follow more or less the expansion of a LCDM universe [15, 16, 17] and could have distinctive signatures on the large scale structure of the universe [18, 19]. On the other hand, several methods have been invoked to reconstruct the shape of $f(R)$ from observations [20, 21, 22], for example, by using the dependence of the Hubble parameter with redshift which can be retrieved from astrophysical observations or a cosmographyc approach. We will show that an $f(R)$ term on the brane action can induce naturally self-accel eration on the normal DGP branch.

2 Induced gravity with an f(R) contribution on the brane action

We start considering a brane, described by a 4D hyper-surface (h with metric g), embedded in a 5D bulk space-time ($\mathscr{B}$, metric $g^{(5)}$), whose action is described by

$$\mathscr{S} = \int_{\mathscr{B}} d^5X \sqrt{-g^{(5)}} \left\{ \frac{1}{2\kappa_5^2} R[g^{(5)}] + \mathscr{L}_5 \right\} + \int_h d^4X \sqrt{-g} \left\{ \frac{1}{\kappa_5^2} K + \mathscr{L}_4 \right\}, \quad (1)$$

where κ_5^2 is the 5D gravitational constant, $R[g^{(5)}]$ is the scalar curvature in the bulk and K the extrinsic curvature of the brane in the higher dimensional bulk, corresponding to the surface boundary term [23]. We will assume that the bulk contains only a cosmological constant; i.e. $\mathscr{L}_5 = -U$. Consequently, the bulk space-time geometry is described by an Einstein space-time

$$G_{AB}[g^{(5)}] = -\kappa_5^2 U g_{AB}^{(5)}. \tag{2}$$

The 4D Lagrangian corresponds to

$$\mathscr{L}_4 = \alpha f(R) + \mathscr{L}_m, \tag{3}$$

where R is the scalar curvature of the induced metric on the brane, g, and α is a constant that measures the strength of the generalised induced gravity term $f(R)$ and has mass square units. Therefore, the function $f(R)$ has mass square units. On the other hand, $\mathscr{L}_m$ corresponds to the matter Lagrangian of the brane which in particular may include a brane tension. The previous action, includes as a particular case the DGP scenario [9, 10] when the bulk is flat, $f(R) = R$ and $\alpha = 1/2\kappa_4^2$ where κ_4^2 is proportional to the 4D gravitational constant.

We will be mainly interested in the cosmology of a homogeneous and isotropic brane, therefore, it is quite useful to follow the approach introduced by Shiromizu, Maeda and Sasaki in[1] [24]. Then, the projected Einstein equation on the brane reads, where we have assumed a mirror symmetry across the brane,

$$G_{\mu\nu}[g] = -\frac{1}{2}\kappa_5^2 U g_{\mu\nu} + \kappa_5^4 \Pi_{\mu\nu} - E_{\mu\nu}. \tag{4}$$

Here, $\Pi_{\mu\nu}$ corresponds to the quadratic energy momentum tensor [24]

$$\Pi_{\mu\nu} = -\frac{1}{4}\tau_{\mu\sigma}\tau_\nu{}^\sigma + \frac{1}{12}\tau\tau_{\mu\nu} + \frac{1}{8}g_{\mu\nu}(\tau_{\rho\sigma}\tau^{\rho\sigma} - \frac{1}{3}\tau^2), \tag{5}$$

and $E_{\mu\nu}$ is the (trace-free) projected Weyl tensor on the brane.

The total energy momentum on the brane can be defined as

$$\tau_{\mu\nu} \equiv -2\frac{\delta\mathscr{L}_4}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathscr{L}_4. \tag{6}$$

It is useful to split the previous energy momentum tensor into two terms

$$\tau_{\mu\nu} = \tau_{\mu\nu}^{(m)} + \tau_{\mu\nu}^{(f)}. \tag{7}$$

The first term $\tau_{\mu\nu}^{(m)}$ corresponds to the energy momentum tensor of matter (which include in particular the brane tension) on the brane. The second term

$$\tau_{\mu\nu}^{(f)} = -2\alpha\left\{\frac{df}{dR}R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu}\left[g_\mu{}^\alpha g_\nu{}^\beta - g_{\mu\nu}g^{\alpha\beta}\right]\nabla_\alpha\nabla_\beta\left(\frac{df}{dR}\right)\right\}. \tag{8}$$

[1] For a different approach to deduce the equations of evolution of a DGP brane with curvature modifications on the brane action see the references[25, 26]. See as well [27] for a brane-world model with an $f(R)$ term.

corresponds to the energy momentum tensor due to the generalised induced gravity term, $f(R)$, on the brane. Now, if f is proportional to the scalar curvature of the brane, then $\tau_{\mu\nu}^{(f)}$ is proportional to the Einstein tensor of the brane; i.e. the standard induced gravity brane-world scenario is recovered:

$$\tau_{\mu\nu}^{(f)} = -2\alpha G_{\mu\nu}. \tag{9}$$

Using the 5D Codacci equation, the bulk Einstein equation, and the junction condition at the brane, it turns out that the total energy momentum tensor of the brane is conserved $\tau_{\mu\nu}$ [24], i.e.

$$\nabla^{\nu}\tau_{\mu\nu} = 0. \tag{10}$$

On the other hand, because[2]

$$\nabla^{\nu}\tau_{\mu\nu}^{(f)} = 0, \tag{11}$$

we can conclude that the energy momentum tensor of matter on the brane is conserved

$$\nabla^{\nu}\tau_{\mu\nu}^{(m)} = 0. \tag{12}$$

3 Dynamics of a homogeneous and isotropic brane

In what follows, we consider a homogeneous and isotropic brane. The matter sector on the brane can be described by a perfect fluid with energy density $\rho^{(m)}$ and pressure $p^{(m)}$, where $\rho^{(m)}$ is conserved as we have pointed above. On the other hand, an effective energy density and an effective pressure associated to the energy momentum tensor coming from the $f(R)$ term on action can be defined as follows [29]

$$\rho^{(f)} = -2\alpha\left[3\left(H^2+\frac{k}{a^2}\right)f' - \frac{1}{2}(Rf'-f) + 3H\dot{R}f''\right], \tag{13}$$

$$p^{(f)} = 2\alpha\left\{\left(2\dot{H}+3H^2+\frac{k}{a^2}\right)f' - \frac{1}{2}(Rf'-f)\left[\ddot{R}f''+(\dot{R})^2f'''+2H\dot{R}f''\right]\right\} \tag{14}$$

Notice that the definition of $\rho^{(f)}$ and $p^{(f)}$ is different from the standard 4D definition in $f(R)$ models [29]. On the other hand, the energy density is conserved on the brane.

The modified Friedmann equation on the brane can be written as

[2] We have proved this equation using the 4D Bianchi identity on the brane; i.e. $\nabla^{\nu}G_{\mu\nu} = 0$, and the relation between the non commutative character of two covariant derivatives and its relation to the Riemann curvature tensor (again on the brane), see for example equation 3.2.12 of [28]. Therefore, the conservation relation (11) can be proven in analogy to how it is done in the standard 4D $f(R)$ scenario.

$$3H^2 = \frac{\kappa_5^4}{12}\rho^2. \tag{15}$$

While, the spatial component of Einstein equation can be expressed as

$$2\dot{H} + 3H^2 = -\frac{\kappa_5^4}{12}\rho(\rho + 2p), \tag{16}$$

where the energy density ρ and the pressure p are defined as

$$\rho = \rho^{(m)} + \rho^{(f)}, \quad p = p^{(m)} + p^{(f)}. \tag{17}$$

For simplicity, on equations (15) and (16) we have used the spatially flat chart of the brane. We have also assumed no dark radiation on the brane; i.e. the bulk corresponds to a 5D maximally symmetric space-time. Notice that even in more general cases the dark radiation term will have no influence on the late-time dynamics of the brane as this term is constrained to be already subdominant by the time of nucleosynthesis [30].

4 de Sitter branes

A de Sitter space-time is the simplest cosmological solution that exhibits acceleration and therefore it is worthwhile to prove the existence of this solution in our model and study its stability. This would be a first step towards describing in a realistic way the late-time acceleration of the universe in an $f(R)$ brane-world model. This approach will also enable us to look for self-accelerating solutions on the modified normal DGP branch. So, in this section, we first obtain the fixed points of the model corresponding to a de Sitter space-time and then we study their stability under homogeneous perturbations.

4.1 Background solutions

In our model, the Hubble parameter for de Sitter solutions can be expressed as[3]

$$2\kappa_5^4\alpha^2F_0^2H_0^2 = 1 + \frac{1}{3}\kappa_5^4\alpha^2F_0(R_0F_0 - f_0) + \varepsilon\sqrt{1 + \frac{2}{3}\kappa_5^4\alpha F_0\left[\alpha(R_0F_0 - f_0)\right]} \tag{18}$$

here $\varepsilon = \pm 1$, the subscript 0 stands for quantities evaluated at the de Sitter space-time, $R_0 = 12H_0^2$ and $F = df/dR$. We recover the DGP model for $f(R) = R$. In fact,

[3] For a maximally symmetric brane in our model, the matter content of the brane behaves like a cosmological constant. As such a term can always be reabsorbed in the $f(R)$ term we will disregard the matter content in our analysis of de Sitter branes.

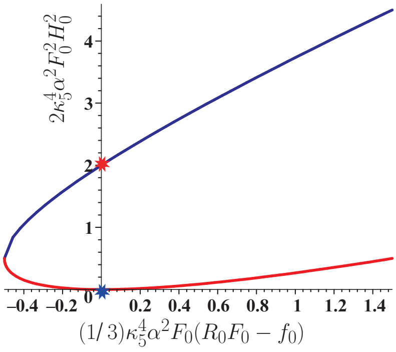

Fig. 1 The figure shows the behaviour of the rescaled squared Hubble rate $2\kappa_5^4\alpha^2F_0^2H_0^2$ for the two branches that generalise the DGP solution versus the rescaled energy density $\rho^{(c)}$ defined as $\frac{1}{3}\kappa_5^4\alpha^2F_0(R_0F_0-f_0)$. The blue star corresponds to the normal DGP branch which is flat. The red star corresponds to the self-accelerating DGP branch. On the other hand, the blue curve corresponds to the generalised (by the inclusion of the $f(R)$ term) self-accelerating branch, while the red curve corresponds to the generalised (by the inclusion of the $f(R)$ term) normal branch.

in that case, the de Sitter self-accelerating DGP branch is obtained for $\varepsilon=1$ and the normal DGP branch or the non-self-accelerating solution for $\varepsilon=-1$. When the brane action contains curvature corrections to the Hilbert-Einstein action given by the brane scalar curvature, the branch with $\varepsilon=-1$ is no longer flat and accelerates (cf. Figs. 1, 2). Consequently, an $f(R)$ term on the brane action induce in a natural way self-acceleration on the normal branch. Most importantly, it is known that such a branch is free from the ghost problem (see [14] and references therein). The reason behind the self-acceleration of the generalised normal brane is the presence of the effective energy density

$$\rho_0^{(c)}=\alpha(F_0R_0-f_0) \tag{19}$$

on the modified Friedmann equation on the brane. This can be easily shown by comparing the Friedmann equation (18) with that of modified gravity on brane world-models [31].

4.2 Stability of the self-accelerating solutions

We next analyse the stability of de Sitter solutions under homogeneous perturbations up to first order on $\delta H=H(t)-H_0$. We will follow the method used in [32].

The perturbed Friedmann equation (15) implies an evolution equation for δH:

$$\delta\ddot{H}+3H_0\delta\dot{H}+m_{\rm eff}^2\delta H=0, \tag{20}$$

where $m_{\rm eff}^2$ is defined as

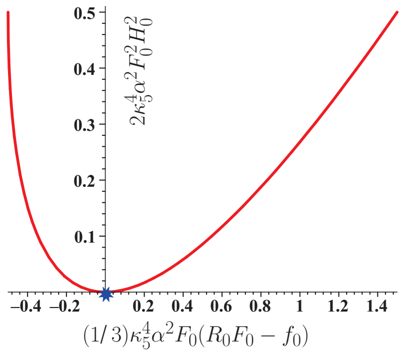

Fig. 2 The figure corresponds to a zoom of the normal branch as it appears on the figure 4.1.

$$m^2_{\rm eff} = m^2_{(4)} + m^2_{\rm shift} + m^2_{\rm pert}. \tag{21}$$

where

$$\begin{aligned}
m^2_{(4)} &= \frac{1}{3}\left(\frac{F_0}{f_{RR}} - 2\frac{f_0}{F_0}\right), \qquad (22)\\
m^2_{\rm back} &= -\frac{2}{\alpha^2\kappa_5^4F_0^2}\left[1-\sqrt{1+\frac{2}{3}\alpha^2\kappa_5^4F_0(f_0-\kappa_5^2UF_0)}\right],\\
m^2_{\rm pert} &= \frac{F_0}{3f_{RR}}\left[1-\sqrt{1+\frac{2}{3}\alpha^2\kappa_5^4F_0(f_0-\kappa_5^2UF_0)}\right]^{-1}.
\end{aligned}$$

and $f_{RR} = d^2f/dR^2$. All this quantities are evaluated at the de Sitter background solution. Any de Sitter solution is stable as long as $m^2_{\rm eff}$ is positive.

The terms defined in (22) have the following physical meaning: (i) $m^2_{(4)}$ is the analogous quantity to $m^2_{\rm eff}$ in a 4D f(R) model [32], (ii) $m^2_{\rm back}$ is a purely background effect due to the shift on the Hubble parameter respect to the standard 4D case and (iii) $m^2_{\rm pert}$ is a purely perturbative extra-dimensional effect.

If we assume that we are close to the 4D regime; i.e. the Hubble rate of the brane is close to its analogous quantity in a 4D $f(R)$ model, then $m^2_{\rm back} > 0$ and $m^2_{\rm pert} < 0$. Consequently, $m^2_{\rm back}$ tends to make the perturbation heavier. However, the perturbative effect encoded on $m^2_{\rm pert}$ would make the perturbation lighter. It can be shown that the extra-dimension has a *benigner* effect in the 4D f(R) model; i.e. $m^2_{\rm eff} > m^2_{(4)}$, as long as[4]

$$F_0^2 < 4f_0f_{RR}. \tag{23}$$

[4] We have assumed the natural condition $F_0 > 0$; i.e. the effective gravitational constant of the brane is positive. On the other hand, we have also assumed that we are slightly perturbing the Hilbert-Einstein action of the brane, i.e. $f_0 \sim R_0$. Therefore, f_0 is positive because $R_0 = 12H_0^2$.

5 Conclusions

A mechanism to self-accelerate the normal DGP branch has been presented which unlike the original self-accelerating DGP branch is known to be free of the ghost problem. The mechanism is based in including curvature modifications on the brane action. For simplicity, we choose those terms to correspond to an $f(R)$ contribution, which in addition is known to be the only higher order gravity theories that avoid the so called Ostrogradski instability in 4D models [7]. Notice as well that by embedding the DGP model in a higher dimensional space-time, the ghost issue present in the original DGP model may be cured [33] while preserving the existence of a self-accelerating solution [34]. See also [35, 36].

It is know that 4D $f(R)$ models are not free from theoretical problems, so in constructing an $f(R)$ brane-world model, we should of course try to avoid these theoretical troubles. We have just undertaken a first step towards constructing realistic self-accelerating solutions in the normal DGP branch. There are still many issues to be addressed, for example which $f(R)$ should we pick up to be in agreement with the cosmological observations and the solar system tests? We leave these interesting issues for future works.

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