Springer Proceedings in Physics 137

Sergey D. Odintsov
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Sebastiàn Xambó-Descamps *Editors*

Cosmology, Quantum Vacuum and Zeta Functions

Springer

SPRINGER PROCEEDINGS IN PHYSICS

Emilio Elizalde in Moscow, May 20, 2009, during the Fourth International Sakharov Conference on Physics.

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Editors

Cosmology, Quantum Vacuum and Zeta Functions

In Honor of Emilio Elizalde

Springer

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Springer Proceedings in Physics

ISSN 0930-8989 e-ISSN 1867-4941

ISBN 978-3-642-19759-8 e-ISBN 978-3-642-19760-4

DOI 10.1007/978-3-642-19760-4

Springer Heidelberg Dordrecht London New York

Library of Congress Control Number: 2011928373

Production: SPI Publisher Services
Cover concept: eStudio Calamar Steinen
Cover design: SPI Publisher Services

SPIN: 80035686 41/3180/SPI
Printed on acid-free paper
9 8 7 6 5 4 3 2 1
Springer is a part of Springer Science+Business Media (www.springer.com)

Preface

On the occasion of the 60th Birthday of Professor Emilio Elizalde, a conference was organized on March 8-10, 2010, to discuss current progress in the main areas of his research: cosmology, quantum vacuum fluctuations and zeta functions.[1] The conference was planed to take place at the Universitat Autònoma de Barcelona, in Bellaterra, but due to an unexpected snow storm during the afternoon and evening of Monday 8, the venue had to be rescheduled: it continued on Tuesday and Wednesday at the Hotel 1898, in Barcelona's Ramblas.

The following is a more detailed list of the topics dealt with at the symposium:

- Dark energy and dark matter
- Modified gravity
- Cosmological evolution
- Cosmology and string theory
- Quantum vacuum fluctuations
- Zeta functions in physics and mathematics

Since the workshop was a success from the point of view of the quality of the speakers and the research works presented, and also by the quantity of participants, it was decided that a volume of proceedings would be published. The subsequent call for papers had a very positive response and we are please to present the result in this volume.

The papers have been grouped into three main areas: Quantum field theory (QFT) and the Casimir effect, Gravity and Cosmology, and Zeta functions in Physics and Mathematics. Written by highly qualified specialists in the different specific fields, they cover some major developments of Physics in the last three decades and a wealth of applications. A number of closely related issues are also considered, such as the nature of dark energy, modified gravity models ($f(R)$ and Gauss-Bonnet, for example), Hořava-Lifshitz gravity, and a couple of non-standard approaches. The

[1] We refer the reader to the recent volume

http://www.ieec.fcr.es/english/recerca/ftc/eli/book2010.pdf,

which gathers a selection of Elizalde's papers. This material is also available as a book.

cosmological applications of these theories play a crucial role and are at the very heart of the book. In particular, the possibility to explain in a unified way the whole history of the evolution of the Universe, from primordial inflation to accelerated expansion, one of the landmark discoveries of the last century. Further, a nice and rigorous description of the mathematical background underlying many of the physical theories considered above is provided. This includes the uses of zeta functions in physics, as in the regularization problems in QFT already mentioned, specifically in curved space-time, and in Casimir problems as, e.g., those involving pistons, which are now very fashionable.

The prerequisites to read this book are some good background knowledge of quantum physics, relativity, and basic functional analysis. Many of the articles give a detailed description of their subject and they try to be as pedagogical as possible.

Acknowledgements. We want to thank all the institutions that made possible the organization of the conference:

- Consejo Superior de Investigaciones Científicas (CSIC)
- Instituto de Ciencias del Espacio (ICE)
- Institut d'Estudis Espacials de Catalunya (IEEC)
- Institució Catalana de Recerca i Estudis Avançats (ICREA)
- Universitat de Barcelona (UB)
- Universitat Autònoma de Barcelona (UAB)
- Universitat Politècnica de Catalunya (UPC)
- Ministerio de Ciencia y Tecnología (MICINN)

It is also a very pleasant duty to thank Springer-Verlag's editorial staff for all the help in preparing the proceedings and most especially Dr. Catriona Byrne, Dr. Marina Reizakis and Mrs. Federica Corradi Dell'Acqua.

THE EDITORS

Barcelona
10/10/2010

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Emili Elizalde. Perspectives on his Life and Work

Sebastiàn Xambó-Descamps

Abstract The goal of this paper is to present a broad picture of Emili[1] Elizalde's unfolding as a person and as a researcher in physics and in mathematics. In addition to biographical information, we include his answers to a number of questions on his experience as a researcher and his role as a leading figure in his fields of expertise.

1 Prelude

The "Facultat de Matemàtiques i Estadística" (FME) of the "Universitat Politècnica de Catalunya" (UPC) dedicated the academic year 2003-2004 to Henri Poincaré. This started a practice that was followed by Albert Einstein (2004-2005), Carl F. Gauss (2005-2006), Leonhard Euler (2006-2007), Bernhard Riemann (2007-2008) and Emmy Noether (2008-2009). At the end of each of these years, the FME published a volume with the lectures delivered by the invited speakers (see [10] for more details).[2] All these names were on top of the mathematics and the physics of their times and thus it should not be a surprise, especially by those that know him, that Emili Elizalde was one of the very few that were invited twice. The first time was for the Einstein year and the second time for the Riemann year. The titles of the lectures he delivered were, respectively, *On the cosmological constant, the vacuum energy, and divergent series* and *Riemann and Physics* (see [1, 2]).

The close relation of these lectures with the topics of this symposium are obvious in the case of the first lecture. In the case of the second lecture, they become manifest

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[1] The real scientific name is Emilio, but I shall use Emili instead, the Catalan version, as this is the way old friends have been calling Elizalde since he was a freshman at the University of Barcelona. His nickname in those years was 'Eli'.

[2] The year 2009-2010 was devoted to John von Neumann and the year 2010-2011 will be dedicated to Paul Erdős, but there are no plans to publish the corresponding volumes of proceedings.

S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_1,

if we bear in mind that Riemann's approach to geometry was a key ingredient in the development of General Relativity and that the Riemann Hypothesis (on the zeros of Riemann's ζ function) is one of the deepest unsolved problems in mathematics. The aim ot this lecture was to show that "the importance of the influence in Physics of Riemann's purely mathematical works exceeds by far that of his papers that were directly devoted to physical issues", and this was accomplished by stressing:

a) The influence of Riemann's work on the zeta function to the regularization of QFT's in curved space-time (in particular, quantum vacuum fluctuations).
b) The uses of the Riemann tensor in general relativity and in very recent generalizations of this celebrated theory, which aim at understanding the presently observed acceleration of the universe expansion (the dark energy issue).

Q1. To which of the other four years (Poincaré, Gauss, Euler and Noether) would you have liked to be invited and which would have been, for each of your choices, the subject of your proposed lecture?

A1. I definitely would have liked to be invited to *Euler's* year too. Part of my research on zeta functions owes so much to Euler. I would have used the opportunity, in preparing my talk, to learn a lot more about Euler's contribution to the subject. To wit, the starting point of several of my explicit derivations of new zeta functions—that are actually useful for the analytic continuation of some divergent series in Quantum Field Theory—is to be found in Riemann's work and I am afraid I can have missed some of Euler's original insight, purely based on real calculus. The title of my talk could have been: *The zeta function: from Euler to Riemann, Selberg, and beyond*.

A mathematician all physicists most admire is Emmy Noether. Her work has had, and still has, a tremendous influence in the interface mathematics/theoretical physics. The difficulties she encountered as a female mathematician also move me a lot. I have been involved in some translations concerning her life and work, from German to Spanish and Catalan and I would have been ready to talk on this historical perspective and on some specific work I did in Group and Quantum Field Theory in my early years as a scientist that uses her celebrated theorem. A possible title could have been: *Emmy Noether and her perennial influence to modern Physics*. It goes without saying that my research has been much influenced by Poincaré and Gauss too and I could have delivered some related talks, but I understand I should not abuse. I really enjoyed to take part in the corresponding sessions as a simple participant.

The idea of writing about Elizalde's life and work took shape toward the end of the Riemann year and it was triggered by the wake of a rather special event. On April 25, 2008, Sir Michael Atiyah was awarded an honorary degree by the UPC. For that occasion, in addition to the usual *laudatio* [11], a poster exhibit on Atiyah's life and work was produced [12].[3] The reception of that work by the visitors convinced me that it might be useful to have other distinguished biographical studies of living mathematicians and physicists and in that mood the name of Emili Elizalde, with his rather high scientific impact, was a natural candidate. Since then, I have kept the project in mind. An occasion in point was the celebration of the symposium *Cosmology, the Quantum Vacuum, and Zeta Functions* to celebrate Elizalde's 60th Birthday. I prepared a set of slides [14], but I could not present them due to the snow

[3] Later this was expanded into a long paper [13].

storm (8 March 2010) that forced the authorities to close the Universitat Autònoma de Barcelona (UAB) on the afternoon of the very first day of the workshop. This text is essentially an elaboration of those slides.

2 A biographical sketch

Emili Elizalde was born on March 8, 1950. That was the year in which the comet cloud hypothesis was formulated (Jan H. Oort).[4] It was also the year in which Alan Turing introduced the concept of what would thenceforth be called the Turing machine. The family name is also intriguing, but quite common in the Basque region (specially in Navarra): Elizalde is a Basque town, in Gipuzkoa, and it is formed out of *eleiz* 'church' and the suffix *alde* 'by' or 'near'.

Balaguer, the town where Elizalde was born, and where he lived until he was seventeen, is located about 25 km northeast of Lleida, the province capital.[5] Today it has a population approaching seventeen thousand, but in 1950 it had only a little over six thousand. The river Segre, a tributary of the Ebre river, is one of its most cherished features. Gaspar de Portolà i Rovira (1716-1784), explorer and founder of San Diego and Monterrey, epitomizes the industrious nature that is attributed to the people born in La Noguera.

The following table summarizes the main discoveries in the period 1951-1959 that have a significant relation to Elizalde's future work:

1951	21cm H radiation, predicted by Van der Hulst.
	Structure of our galaxy.
1955	Galaxy explosions. Birth of new stars.
1956	Antineutrinos.
1957	Sputnik. Jodrell Bank.
1958	Mössbauer effect.
1959	Pound–Rebka experiment

From primary school, age six to ten, the event that he remembers most vividly is the launching of the Sputnik, in the fall of 1957. As he recalls now:

> By age 10 I had long decided (had not the least doubt about it) that I would become an astronaut. It was such a clear and strong feeling! Nothing on Earth could be compared to the pleasure of flying through the skies towards other worlds. The Sputnik trips propelled me towards the whole Universe. *I myself was up there, flying on the Sputnik!* This is maybe the strongest, the more lively remembrance I keep of my whole childhood. Eventually I would make a job of my most precious dream.

[4] Oort's contribution is actually an independent discovery of an idea postulated in 1932 by the Estonian astronomer Ernst Öpik, so it would be better named as the Öpik–Oort hypothesis.

[5] Balaguer itself is the capital of La Noguera county ('comarca' in Catalan).

This impression was much reinforced at the end of next decade, when he was a sophomore, by the landing on the Moon.

From age ten to age seventeen, Elizalde attended secondary school at the *Instituto Laboral de Balaguer*. His quiet ways caused that at first he went largely unnoticed by peers and teachers. But this changed suddenly at the end of his second year, as he surprised everybody when he got the highest grade, in 1962, in a school problem contest on mathematics.

Asked on his recollections on how he felt in that period at the *Instituto*, he says:

> Those seven years were very important in my life. This does not mean at all that they were happy years. My family had to go through rather hard times and I actually suffered from that. I had no money to buy say an ice cream, or to go to the cinema on Sundays, as most of my schoolmates actually did. Nevertheless, I was quite happy at the school. I liked learning things, mathematics in particular. But when I was not reading my books, on weekends, I did not know how to spend my free time. It did not help that I was not a very friendly person and so I remember many boring Sundays there.

Fig. 1 Emili Elizalde at age 12

Here are a few important discoveries produced during the secondary school years of Elizalde:

1961	Quark eight-fold way.
1963	Quasars. Arecibo radio-telescope. X-rays sources.
	Atiyah-Singer index theorem
1964	Cosmic micro-wave radiation.
1967	Pulsars.

It would be wrong to conclude, as it is plain from the answer to next question, that Elizalde's ties to Balaguer and to his peers were weak or inconsequential. The answer also hints at an interesting literary bent of Elizalde's character.

> **Q2.** What kind of ties have you maintained with Balaguer all along? What impact have they had on your career?
>
> **A2.** My ties to Balaguer have been mainly related with the *Instituto*, which now is called *Institut de Batxillerat Ciutat de Balaguer*. I have given several talks there and also at the Town Hall Auditorium. Recently we celebrated the 50th Anniversary of the *Institut*. I wrote some poems and a short story for the occasion, which you may find on my webpage. Some of my former classmates are now ruling the Town, some are Member of Parliament, in Barcelona and Madrid. Others are medical doctors or teachers in Balaguer itself. Some are really skillful in agricultural research —they are also based there and yet they lead some important company well known at European level. I am proud that our generation —that grew up in the scorned Franco time— gave rise to such brazen good professionals in so very different domains. Public education was extremely good then, I must say, and,

in a way, I do owe my whole scientific career (the possibility to pursue it later) to these years at the *Instituto*.

In contrast, I do not think any of the ties I have kept with my hometown has had any direct, significant impact in my professional life or career later. Maybe only at the psychological level: more than once I have found myself recalling Balaguer's river and its porched streets when I have felt depressed. Also, its famous Saint Christ Sanctuary, which I have always considered to be one of the most peaceful places one can be in anywhere, only matched (but not surpassed) by some selected Tibetan or Japanese shrines. Anyhow, I am extremely proud to be a *Balaguerí*.

The undergraduate years at the university were also very intense. In 1967 he began the five-year degree in Physics offered by the University of Barcelona. This institution had also a five-year degree in Mathematics and Elizalde enrolled in it in 1969. He finished these degrees in 1972 and 1973, respectively. He was a systematic and thorough student, in all subjects. His retrospective view is that his education in physics and mathematics was very good. This is a quite remarkable assessment, as those years were, from a political and social point of view, rather difficult, and more so at the university, where the student unrest and the clashes with the police were the rule rather than the exception. In all, he was especially strong in analysis and differential geometry. With regard to physics, he ended with a good knowledge of classical mechanics, thermodynamics, statistical physics and quantum mechanics.

1968	Electroweak theory. Solar neutrinos defect.
1969	Landing on the Moon.
1970	Black-body radiation.
1971	Black-body X-1 in the Swan constellation.
1972	QCD
1973	Universe, a quantum fluctuation of the vacuum?

At the end of his undergraduate studies of mathematics (1973), Elizalde widened his education in physics with a master thesis on the solar neutrino problem[6] that earned him the extraordinary distinction of his class. His advisor was Pere Pascual (1934-2006), who at that time was Full Professor of Quantum Mechanics at the Department of Theoretical Physics of the Universitat de Barcelona.

[6] That problem had been discovered in the late 1960s by John N. Bahcall (1934-2005) and Raymond Davis (1914-2006) and its satisfactory solution three decades later was the result of a sustained effort by many theoretical and experimental physicists. At the start, in the late 1960's, the experimental design of Davis to measure the flux of solar neutrinos reaching the Earth found a value that was only one third of the theoretical quantity calculated by Bahcall. These observations were confirmed later by several other experimental designs. The solution came from experimental work in the 1990s that was sensitive enough to find not only electron neutrinos, as in Davis approach, but also muon and tau neutrinos. The 2002 Nobel Prize in Physics recognized these researches by awarding Raymon Davis and Masatoshi Koshiba (Kamiokande experiments) a share of the Prize.

Immediately after his master thesis, Elizalde began his doctoral program in physics, advised by Joaquim Gomis and Pere Pascual.[7] Another decisive event in his life also happened around this time: Emili met Maria Carmen Torrent (Carme), his future wife, at the Faculty of Physics of the University of Barcelona, where she was working for her master thesis.

1976	Idea of cosmic strings.
1977	Inflationary universe.
	Atiyah begins his work on gauge theories

Elizalde defended his thesis in 1976, with the title *Galilean equations and gyromagnetic ratio in the light-cone system*. It was qualified *summa cum laude* and later obtained the doctoral extraordinary distinction of his year. As a result of his research for the master and doctoral thesis, Elizalde published his first five papers in the years 1976 and 1977. These were the initial steps of a scientist that soon would appear to be a prolific mathematical physicist.

The year 1977 was very special for Emili Elizalde from another side: he married Carme on the 23rd of April.[8] Carme has pursued her own career in Physics and at present she is a Professor of Applied Physics at the Technical University of Catalonia (UPC). Her research is focused on semiconductor lasers.

Carme and Emili have two sons: Sergi (1979) and Aleix (1982) —see Fig. 5 at the end. Sergi is a well-known mathematician. He won the Spanish Mathematical Olympiad and also medals at mathematical international competions. He got PhD degrees from UPC and MIT, with Marc Noy and Richard Stanley as respective advisors. After a postdoctoral stay at MSRI, Berkeley, with Bernd Sturmfels, he is a Professor at Dartmouth College with Peter Winkler. He married Helen, a lawyer, in 2008, and now they have a son, Guillem, born in 2009.

Aleix displayed early impressive dexterities, like mono-cycle riding. He studied internal medicine and at present he is about to finish his internship specialty at the Valle de Hebron Hospital in Barcelona. Recently he was in the news of the Spanish Medical Gazzette for having displayed one of the best performances in the MIR examinations. His girlfriend, Laia, is a pediatrist working in a different Hospital.

Sergi, Aleix and Laia are also professional pianists, with a degree from the famous Liceo Conservatory in Barcelona. As for many educated people of his generation, this touches a deep chord in Elizalde's feelings:

> They have made true my old dream of becoming a musician, which I could never fulfill, first for lack of money and later for lack of time.

---

[7] Officially, Elizalde was the first PhD student of Joaquim Gomis. Even if he also discussed his progress with Pere Pascual, at that time it was not permitted to have co-advisors and as a result the academic records do not reflect the role of the latter.

[8] The choice of the date, St. George's Day, is particularly significant in Catalonia. If it does not fall on a Sunday, it is a working day, but a rather special one. Everybody goes to work, but the general mood is that of a joyful and cherished festivity. Very early in the morning the streets everywhere are invaded by swarms of improvised sellers of roses, by bookstore stands taken out of the shop, and by the crowds that are eager to follow the ritual of buying roses and books for the loved ones.

And then, displaying a concerned fatherly look, whispers:

> It is doubtful that Sergi is coming back to Spain any soon. And Aleix and Laia have a very hard life right now, with a lot of work on night shifts at their respective hospitals. We are very proud of them since they do an important service to society, but this prevents them from having a family life at all.

For Carme and Emili, however, not all were roses, especially during the first few years after getting married. Franco's dictatorship was over, but the uncertainty about the future was overwhelming. The life in the universities was rather chaotic and seemingly with few perspectives. As a consequence, many gifted young graduates with no financial backup tried to secure a means of living outside the academic circles. In the case of Elizalde, the detour was to take the 1974 competitive examinations required to earn a post as an upper level high school mathematics teacher ('catedrático'). This is the way he came to hold offices in the high schools of Tàrrega (1975-7), a town not far from Balaguer, and Bellvitge (1977-8), in the Barcelona area.

Fig. 2 Carme and Emili at Villa Hammerschmidt (Germany), on the occasion of the 1980 Humbolt fellows reception by the President of the Federal Republic of Germany.

The escape path, for a vocational researcher as Elizalde, was provided by a prestigious scholarship of the Juan March Foundation and later by an even more illustrious Alexander von Humboldt fellowship. These grants allowed him to take a leave of absence from the high school (at that time this was still possible) and spend the academic year 1978-9 and part of next (in fact 16 months in a row) doing postdoctoral work with Rudolf Haag in the II. Institut für Theoretische Physik of the Hamburg University. The fellowship was for two years, but it was flexible and generous enough to allow the splitting (and extension) of the second year in four three-months Summer visits. These visits took place in 1981, 1985, 1987 and 1989, the latter in the Freie Universität Berlin.

> **Q3.** What would you underline of your stay in Hamburg and the research facilities there?
>
> **A3.** The Institut is in the middle of the Deutsches Elektronen Synchrotron installment (DESY), an impressive research center only second to CERN in Europe and one of the world's leading centers for the investigation of the structure of matter. The theoretical group consisted at the time of several of the most prominent QFT physicists alive, as Harry Lehmann, Kurt Symanzik, Rudolf Haag, Hans Joos, T.T. Wu, who was visiting, and several others, and a handful of younger people who did impressive careers later. The Nobel laureate Sam Ting, Wu's wife, Gustav Kramer, etc. and an extraordinary number of particle physicists, engineers and technicians (over two thousand, I think) worked for DESY.

Q4. How was your first encounter with Professor Haag?

A4. I remember very well my first encounter with him. He had told me on the telephone, repeatedly, how to arrive to his office but I got lost in that impressive place, I entered from one side, the Notkestrasse, and after forty minutes of near random walking I found myself at the other exit, on the Luruper Chaussee, almost opposite to the first. I was too shy in order to keep asking on each turn, but finally I had no other choice and so I was able to reach room 501, on the 5th floor of Building 2a. On the door it was written plainly "Rudolf Haag". In spite of arriving one hour late, Prof. Haag was utmost kind and helpful to me. During the months that followed I could discover that most of the geniuses there were extremely normal people and, what impressed me even more, that many of them had such a good sense of humor (what contradicted all I had previously heard about northern Germany). This included also the prominent visitors (several of them Nobel laureates, as C.N. Yang and later S. Weinberg) who passed by for short visits, to deliver talks on the very famous Friday seminar. The coffee served there was excellent too.

1983	W^+, W^-, Z_0.
1987	Supernova in the Magellanic cloud.
1990	The Hubble telescope. Internet
1991	WWW
1992	COBE findings on CMB

That time in Germany, and the position he subsequently took as an assistant professor ('adjunto interino') at the Department of Theoretical Physics of the University of Barcelona, meant a full recovery of Elizalde for research. This recovery was consolidated in 1984, the year that his position was made permanent as an associate professor ('profesor titular'). This happened after being classified as number one, *ex aequo* with Joaquim Gomis, in the extraordinary state-wide massive habilitation process called on the preceding year. This also meant the end of the four-year struggle sustained by the 'interinos' in favor of 'work contracts' and against the old promotion system through competitive examinations ('oposiciones').

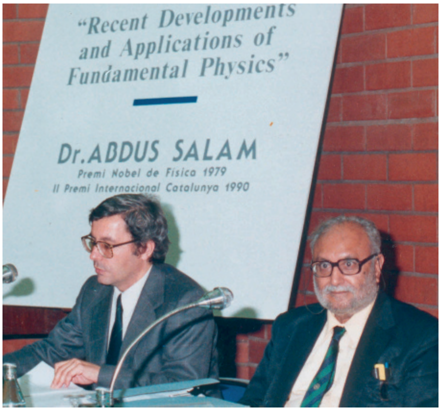

Fig. 3 Emilio Elizalde introducing Abdus Salam on the occasion of having been awarded the "II Premi Internacional Catalunya".

If we needed an image to visualize Elizalde's academic welfare in the 1980's, there is one that is especially eloquent. In 1979, Abdus Salam shared the Nobel Prize in Physics with Sheldon L. Glashow and Steven Weinberg "for their contributions to the theory of the unified weak and electromagnetic interaction between elementary particles, including, *inter alia*, the prediction of the weak neutral current" (Nobel Prize citation). Over a decade later, Salam was distinguished with the "II Premi Internacional Catalunya 1990" and the striking picture is that Elizalde, aged 40, was the person appointed to introduce the distinguished scientist.

The next promotion occurred in 1992, when he won, again as number one, a research position at the Spanish Higher Research Council (CSIC, "Consejo Superior de Investigaciones Científicas"). The level of this position was between associate professor and full professor and for several years he was allowed to combine the new job, attached to the "Centre d'Estudis Avançats de Blanes" (CEAB), with collaborative research and teaching at the University of Barcelona. His celebrated book *Ten Physical Applications of Spectral Zeta Functions* (LNP, Springer-Verlag, 1995) was written during the first two years of his association with the CEAB.

The mission of CEAB, created in 1985 as a CSIC unit, was to foster interdisciplinary research groups. It was an ambitious initiative, together with many others,[9] that was mainly driven by the desire to make up for the gloomy years that were lost before the advent of democracy. The "Nuclear Astrophysics Group" was formed in that atmosphere, as early as 1988. This group grew steadily in the next ten years into the larger group of "Cosmos Sciences". In the year 1996, it moved to Barcelona as the starting unit of the newly founded "Institut d'Estudis Espacials de Catalunya" (IEEC), a research structure created by the the Catalan Government ("Generalitat de Catalunya"). Emili Elizalde played a prominent role in that move, and he is proud of "having been one of the scientists involved in the creation of IEEC".

Q5. In retrospect, how do you remember and assess the CEAB initiative?

A5. It was very advanced for Spain at that epoch. The CEAB was an attempt to make the dreams of interdisciplinarity come true by putting together, in one of the most beautiful spots of the Costa Brava, a collection of specialists in artificial intelligence, cosmologists, biologists, and sea experts who put up the first Spanish base on the Antarctica. Artists were absent, however, and, even more regretfully, the success of the attempt was rather limited. Still, after the splitting of two big groups to create separate Institutes in Barcelona, the CEAB has become a reference Center in the domain of marine ecology and environmental sciences.

In the structure of the IEEC, the CSIC participated as one of the partners. In order to provide administrative support to this relation, in 1999 the CSIC created the Institute of Space Sciences (ICE, "Instituto de Ciencias del Espacio"). Elizalde, who played again a key role in setting up this arrangement, was attached to ICE from the very beginning. It is thus that he has been able to combine his adscription to CSIC with his leading research role in the IEEC in the areas of "Theoretical Cosmology" and "Fundamental Physics".

The ICE-IEEC is the roof under which Elizalde has been working in the last decade. In the year 2003, he was promoted to Senior Research Professor, the highest possible level at the CSIC (again as number one, in the yearly national appointments in his subject).

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[9] For example, the creation of CEAB coincided in time with the celebrated meeting "Culture and Science: Determinism and Freedom". Inspired by Salvador Dalí, already seriously ill, and organized by the Faculty of Physics of the University of Barcelona, it gathered in Figueres the Nobel laureate Ilya Prigogine, the Fields medalist René Thom and other distinguished scientists of very different fields (see [8]).

At present the ICE is housed at the Autonomous University of Barcelona, where it moved about ten years ago, but at the start it was housed at the Nexus-1 building of UPC.

1993	GPS becomes operative
1994	Black hole of 3 billion solar masses in M-87
	Top quark
1998	Acceleration of the Universe expansion
	Neutrino mass
2002	Hubble estimate age of Universe (13-14 Giga years)
2003	WMAP: 4% matter, 23% dark matter, 73% dark energy
2004	First binary system of pulsars

3 Scientific achievements

Table 1 provides a rough quantitative summary of the Elizalde's scientific production to this day. Without counting the items of types T (Technical notes and other articles) and B (Books), it gives an average of about ten papers per year, a fact that lends a strong support to our earlier appraisal that he qualifies as a prolific mathematical physicist.

Types	Description	Total
J	Papers in international journals (SCI)	235
P	Proceedings and Other journals	86
C	Book chapters	14
B	Books and Monographies	12
T	Technical notes and other articles	22
R	Reviews (Zbl and MR)	200
A	Articles in newspapers and magazines	60
O	Other original contributions	32

Table 1 Quantitative overview of Elizalde's publications. Of the 14 titles of type B, 5 are in English and 7 in Spanish. Type O includes 12 original contributions to encyclopedia and dictionary entries and translations of books and articles (20 in all) from English/German into Spanish/Catalan.

These findings are even more eloquent when we consider the following evidence for its scientific impact. The overall number of accumulated citations is above 6,000, and of these more than 4,500 appear in SCI. The book [3], of which Elizalde is the single author, has accumulated more than 250 citations and the book [7], in collaboration with four coauthors, more than 460. As far as the citations of papers, there are more than twenty with 50 or more citations. Table 2 shows the number of citations for the 10 best-cited papers.

#	Year	#Co	Area
350	2004	2	Cosmology
200	2005	3	Quantum vacuum
180	1996	2	Zeta functions
170	2005	4	$f(R)$-gravity
150	2006	4	Modified gravity
130	2008	5	$f(R)$-gravity
120	1997	3	Quantum vacuum
110	2006	4	Modified gravity
100	2003	3	Quantum vacuum
100	1998	2	Quantum vacuum

Table 2 Elizalde's ten best-cited papers. The first column contains a lower bound of the number of citations, in decreasing order. The second column indicates the publication year and the third the number of co-authors (besides Elizalde) of the corresponding paper. The fourth column gives a rough indication of the main area of the paper. The most cited paper so far is *Late-time cosmology in (phantom) scalar-tensor theory: dark energy and the cosmic speed-up* (with S. Nojiri and S. D. Odintsov), Physical Review **D70**, 043539 [1-20] (2004).

Areas	#J	#P
Cosmology	19	22
Gravity	89	25
Mathematics	57	24
QFT	165	72

Table 3 The second and third column refer to the number of papers published in indexed journals and in proceedings, respectively. About one third of the papers appear in more than one area.

Another general perspective is given by the Table 3, where we can see the distribution of the contributions according to four wide areas: Cosmology, Gravity, Mathematics and QFT. We observe that QFT is the dominant area, as its weight is roughly equal to the weight of the other three areas.

If we now look more closely to each of the four areas, by subdividing them into subareas, we get the tables 4, 5, 6 and 7.

Cosmology		
Subarea	**#J**	**#P**
Cosmological constant	11	13
Large scale	8	9

Table 4 Distribution of the Cosmology papers into two subareas.

Gravity		
Subarea	**#J**	**#P**
Quantum gravity (semiclassical)	34	15
Modified gravity	19	2
General relativity	17	6
String theories	17	1

Table 5 Distribution of the Gravity papers into four subareas. Here we may also include a J-paper in the subarea of Classical gravity and two papers is the Braneworlds subarea.

Mathematics		
Subarea	**#J**	**#P**
General	2	0
Lie theory	8	5
Neural networks	4	1
Statistics (information theory)	11	3
Chowla-Selberg formula	1	2
Heat kernel	3	1

Table 6 Distribution of the Mathematics papers into six subareas.

Quantum Field Theory		
Subarea	**#J**	**#P**
Multiplicative anomaly	22	6
Casimir effect	20	19
Curved space-time	35	11
Equations (Dirac, KG, Proca, ...)	13	4
QCD	27	5
QED, neutrinos, magnetic fields	7	2
Regularization and renormalization	26	10
Vacuum energy	3	5
Yang-Mills	3	5
Quantum mechanics	6	3

Table 7 Distribution of the QFT papers into ten subareas.

As we said earlier, the dominant area in Elizalde's research has been QFT, which is the reason why we have subdivided it into more subareas (ten) than the others (see Table 7).

One of the characteristic features of Elizalde's work is that it often is carried out in collaboration with colleagues. Table 8 is like and X-ray image of this fact.

To round the picture of Elizalde's collaborations, see Table 9.

Collaboration statistics							
	0	**1**	**2**	**3**	**4**	**5**	
J	51	80	62	31	9	2	**235**
P	40	22	12	7	5	0	**86**
T	8	7	2	3	1	1	**22**
B	7	1	2	0	2	0	**12**
C	12	1	1	0	0	0	**14**
	118	**111**	**79**	**41**	**17**	**3**	**369**

Table 8 Columns headed by a number k in 0..5 indicate the number of papers with k co-authors. Thus $k = 0$ indicates the number of works published with no co-authors (118 in total). At the other end, there are 3 papers published with 5 co-authors: 2 in indexed journals and 1 as a technical report.

Collaborators		
Name/s	**J**	**P**
Sergei Odintsov	82	21
August Romeo	29	6
Sergio Zerbini	22	4
Sin'ichi Nojiri	17	1
Guido Cognola	15	5
Andrei Bytsenko, Klaus Kirsten	10	1, 5
Enrique Gaztañaga	9	4
Sergei Naftulin	8	4
Jaume Haro	7	0
Leseduarte, Yuri Shil'nov	6	1, 6
Gomis, Hildebrandt, Soto	5	0, 2, 1

Table 9 This table shows that Elizalde's main collaborator is Sergei Odintsov. The number of papers they have published jointly is about the same as the number of those written with the next four collaborators together (August Romeo, Sergio Zerbini, Sin'ichi Nojiri and Guido Cognola). The collaborations with Bytsenko, Kirsten, Gaztañaga and Naftulin have similar magnitudes, and Haro, Leseduarte, Shil'nov, Gomis, Hildebrandt and Soto follow a little behind.

To finish this overview, we include a list of the PhD thesis supervised by Elizalde:[10]

1985, Joan Soto
Effective Action of QCD and the Confinement Problem.
1989, Enrique Gaztañaga
Statistical Models for the Description of the Large Scale Structure of the Universe.
1990, August Romeo
New Aspects of Zeta Function Regularization Procedures with Incidence on QFT Vacuum Effects.
1994, Sergio Gómez
Models of Learning in Artificial Neural Networks and Applications.
1996, S. Leseduarte
Applications of the Zeta Regularization Procedure in Quantum Field Theory.
1998, Pablo Fosalba-Vela
Cosmological Perturbation Theory and the Spherical Collapse Model (co-advisor).
2001, Sergi R. Hildebrandt
Kerr-Schild and Generalized Metric Groups, with some Applications to Regularized Black Holes (co-advisor).
2002, José Barriga
Mathematical Analysis of Microwave Density Fluctuations (co-advisor).
2008, Miguel Tierz
Random Matrix Models in Chern-Simons Theory.

Q6. You belong to the 'publish-or-perish' scientific generation, whose influence was particularly striking in Spain. How did you experience that move?

A6. The pressure to publish (or perish!) was particularly strong at our University, in my generation. I remember how everything started. When I was an undergraduate, most of the Professors (*Catedráticos* in Spanish) in the Departments of Physics and Mathematics of Barcelona University had not published even a single paper in their lives. One of them, a mathematician, having published two papers, was more respected there, at that time, than Nobel Prize winners are at Harvard or MIT (I know of that through unforgettable talks with John Bardeen, who had won two of them, and thus I can compare). It is easy to imagine that the following step taken by the Spanish authorities, in order to close this gap, was to make us publish at any price. And this we did like mad (impact was not a concept then).

Q7. You were born the same year as the Turing machine and it seems that it has not been until the last years that you have become also an explorer of the worlds that were discovered after that breakthrough. How did this evolution occur?

A7. I already heard something about Alan Turing and the Turing machine when I was a University student in Barcelona, in the late sixties and early seventies. However, complexity theory and computer skills were not among my strongholds, and my real discovery of Turing occurred much more recently, about ten years ago, when I began to connect all these many different pieces of knowledge I had accumulated in my head for years and years. But the publish-or-perish pressure is the reason why during a large period of my scientific life I had no time to pause, recollect, relate, and try to explain in a unified way

[10] In addition, there are four PhD thesis in progress: Diego Sáez Gómez: *Fluid models for the dark energy*; Antonio Jesús López Revelles: *Viable models for alternative gravities*; Gloria García Cuadrado: *Gravitational wave detection with orbiting satellites*; and Roger Oliva: *Observational effects of rotating black holes with XMM-Newton*. In the case of Sáez and López, Elizalde acts as co-advisor.

> all these pieces of knowledge I was gathering. When I finally did, some time ago, I was able for the first time to put Turing's work in the very prominent place it has in the History of Mathematics. In particular, in its key role to bypass (in a practical, down-to-earth way) the terrible impact of Gödel's incompleteness theorem, which removed in a blow the very foundations of the construction by Hilbert of the entire building of Mathematics. And this can be also connected (I do that now) with the great revolutions in last century Physics, and so forth. I have never stopped to recommend Penrose's *The Emperor's New Mind* to my students as one of the best books they could ever read, together with the *The First Three Minutes* by Weinberg. Anyway, other than that, the Turing machine has not had such a direct influence on my research, I must say.

Fig. 4 Lecture on the occasion of the Honorary Professorship Award by the TSPU (Russia, 2010).

Emili Elizalde's has been a leading researcher of about eighty projects, endowed with an average of close to 300 K$ per year. These resources have included over 65 research grants or post-doc positions and have allowed his group to hire six Full Professors and one ICREA researcher.

He serves on the Editorial Board of five international journals, does referee work for forty journals and has been evaluator of scientific projects for a dozen national agencies of different European and American countries. He is a Fellow of the Institute of Physics (UK) and member of several societies, including the AMS and APS. Elizalde has participated in countless international meetings (conferences, schools and workshops), serving for over twenty occasions on the organizing committee and being himself the organizer of six conferences. The participation has been as a plenary speaker for over sixty times and as a chairman for over fifty.

Emili Elizalde has received many distinguished awards in recognition to his contributions to science. During the present year (2010), for example, he has been awarded an Honorary Professorship and the Gold Medal of the Tomsk State Pedagogical University (Russia). He has also been appointed Secretary General of the Alexander von Humboldt Association of Spain and, moreover, he has been invited, 'as internationally recognized figure in the area', to the key Conferences of the Spanish Presidency of the EU Council: ERA Board, Science Against Powerty, and Biotech for a Complex World.

4 Sources

Emili Elizalde maintains an audacious Web page [4] in which the visitors can access a wealth of materials about many aspects of his professional and personal life. From the point of view of finding out about Elizalde's scientific trajectory, one of the most valuable pieces is the book [5], whose goal, as stated at the beginning, is to be

> [...] a compendium of the more outstanding contributions of Prof. Emilio Elizalde and several of his collaborators as they have appeared in international journals during the last thirty years. A good number of original results can be here found on zeta function regularization, the extension of the Chowla-Selberg series formula, heat-kernel coefficients, fluctuations of the quantum vacuum and the Casimir effect in different configurations, as the bag model, its thermal properties, quantum gravity and black hole physics, large scale structure of the universe, and alternative cosmological models that deal with the dark energy issue from a rigorous theoretical perspective, which seeks its roots in fundamental theories and physical phenomena.

In addition to a large collection of pictures, and a short preface by Professors V.V. Obukhov and S.D. Odintsov, it contains the much informative three-page introduction "Some personal remembrances of my scientific life". This piece, together with the complete list of publications included at the end, should deserve, considering its polyhedral nature, extensive analysis from different viewpoints.

Table 10 summarizes the six parts in which the papers in [5] are grouped and the years in which they were published.[11]

Q8. It seems to me that your book [5] is also a recognition of your former students and other collaborators.

A8. Indeed, I am extremely proud of the fact that a good number of my former students are now quite well known scientists, university professors, and qualified professionals in different countries worldwide. A success accomplished, moreover, in a wide spectrum of different fields: from heavy quark physics to informatics engineering, from observational cosmology to sport physics, from financial mathematics to large-scale structure, from data compression to Casimir effect applications.
As for my other collaborators, I am also proud not only for the work done jointly, but also for their excellent independent accomplishments.

Quantum Field Theory	1984, 2002, 2003, 2004 (4)
Zeta Functions and Heat Kernels	1989-3, 1996, 1998-2, 1999, 2001 (8)
Vacuum Fluctuations, Casimir Effect	1991, 1994-2, 1997, 1998, 2001, 2006-2 (8)
Gravitation and Black Holes	1994, 2002, 2006, 2008 (4)
Statistics and Large Scale Structure	1992, 1998 (2)
Theoretical Cosmology	2003-2, 2004-2, 2005, 2006, 2007-2 (8)

Table 10 Summary of the Table of Contents of [5]. 1989-3 means that three papers are included that were published in 1989. The numbers in parenthesis are the total number of papers in each section (34 in all). This amounts to 18 papers belonging to the last decade and 15 to the preceding one.

[11] Four articles are revised versions of the published papers. The ten best-cited papers so far (cf. Table 2) were published, ordered by decreasing number of citations, on the years 2004, 2005, 1996, 2005, 2006, 2008, 1997, 2006, 2003 and 1998, respectively. We note that this amounts to seven in the last seven years and three in the preceding seven-year period.

In the case of Zeta functions, we already mentioned the monograph [3], which is “a commented guide that invites the reader to plunge into the thrilling world of zeta functions and their applications in physics”. This quote is from the Preface, in which we are also told that “the level is elementary”, that “everything is explained in full detail, in particular the mathematical difficulties and tricky points”, and that it is “to be considered as a basic introduction and exercise collection for other books that have appeared recently” (say like [7], published in 1994). Several original ‘zeta-function regularization techniques’ are presented, including ‘The zeta-function regularization theorem’ (Section 2.2). “Physical applications [...] include the proper definition of the vacuum energy, the Casimir effect, spontaneous compactification in quantum gravity, stability analysis of strings and membranes, etc., and embrace also very recent experiments of solid state and condensed matter physics employing liquid helium (those will be described in the following chapters)” (p. 28). One of the highest points in the book is an important generalization (formula 4.32) of the Chowla–Selberg formula [9]. Currently the monograph seems to be out of print and it would be interesting to have a second edition, which by now it should be supplemented, perhaps as a second volume, by many other applications obtained since its publication fifteen years ago. Meanwhile, readers interested in a quick overview of the main issues involved in this domain could start with the excellent survey [6].

> **Q9.** For some the vastness of the Universe is apprehended by degrees, from the small to the very large scale. How did it happen in your case? Under what circumstances did you begin research in Cosmology?

> **A9.** In my school years, I did not realize that my understanding was only the very local universe. The big jump was in 1986, when the first map of the universe was published, including some three thousand galaxies and clusters (De Lapparent, Huchra). This was a breakthrough: the presence of large voids, the clustering of points into large pictures, one of which seemed to be a human being, another, kind of God’s finger, was something astonishing and this put some of the best physicists in the world down to work to explain such structures as coming naturally from fundamental theories (which in fact was only partially accomplished). In my case this was the birth day of the celebrated *Barcelona School on Large Scale Structure* whose creation I started this very same day, by putting my student Enrique Gaztañaga to work on the analysis and explanation of the matter distribution in this map. COBE, the CMB map, WMAP, and the PLANCK satellite followed, the thousand points became many millions, and the expansion of our universe turned out to be accelerating (the most important discovery in physics of the last decades). By the way, Enrique is now a leading figure in cosmology at international level and two more of my former students, namely Pablo Fosalba and Sergi R. Hildebrandt, are scientists belonging to the core team of PLANCK.

Acknowledgements. It is very pleasing to thank Emili for his unfailing patience in answering my questions, for his generosity in allowing me to use his archive materials and for having led me to discover many aspects of a scientific milieu that have turned out to be even richer than what I imagined in setting up to write these notes. Thanks also to the coeditors of this volume, Sergei Odintsov and Diego Sáez-Gómez, for many fruitful interactions and discussions, and to Jaume Puigbó and Cristina España for having pointed out several corrections.

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List of acronyms

AMS	American Mathematical Society
APS	American Physical Society
CEAB	Centre d'Estudis Avançats de Blanes
CERN	European Organization for Nuclear Research
COBE	Cosmic Background Explorer
CMB	Cosmic Microwave Background
CSIC	Consejo Superior de Investigaciones Científicas
DESY	Deutsches Elektronen Synchrotron
ERA	European Research Area
EU	European Union
GPS	Global Positioning System
ICE	Instituto de Ciencias del Espacio (CSIC)
ICREA	Institució Catalana de Recerca i Estudis Avançats

IEEC	Institut d'Estudis Espacials de Catalunya
IOP	Institute of Physics
FME	Facultat de Matemàtiques i Estadística (UPC)
MIT	Massachusetts Institute of Technology
MR	Mathematical Reviews
MSRI	Mathematical Sciences Research Institute
QCD	Quantum Chromodynamics
QFT	Quantum Field Theory
SCI	Science Citation Index
TSPU	Tomks State Pedagogical University
UPC	Universitat Politècnica de Catalunya
WMAP	Wilkinson Microwave Anisotropy Probe
ZBl	Zentralblatt für Mathematik

Fig. 5 Emili, Carme, Sergi, Helen, Aleix and Laia celebrating New Year's Eve 2008.

PART I
QFT and the Casimir Effect

Colliding Hadrons as Cosmic Membranes and Possible Signatures of Lost Momentum

Irina Ya. Aref'eva

Abstract We argue that in the TeV-gravity scenario high energy hadrons colliding on the 3-brane embedded in $D = 4+n$-dimensional spacetime, with n dimensions smaller than the hadron size, can be considered as cosmic membranes. In the 5-dimensional case these cosmic membranes produce effects similar to cosmic strings in the 4-dimensional world. We calculate the corrections to the eikonal approximation for the gravitational scattering of partons due to the presence of effective hadron cosmic membranes. Cosmic membranes dominate the momentum lost in the longitudinal direction for colliding particles that opens new channels for particle decays.

1 Introduction

In recent years the study of transplanckian scattering[1] within the TeV-gravity scenario [1] has attracted significant theoretical and phenomenological interest. Within the TeV-gravity scenario [1] transplanckian scattering could be observed at the LHC and other future colliders [2, 3, 4, 5, 6, 7, 8], as well as in collisions of high-energy cosmic neutrinos with atmospheric nucleons [9, 10].

Different physical pictures are expected for different ranges of impact parameters b. For impact parameters b of the order of the Schwarzschild radius R_S of a black hole of mass $\sqrt{s}$, microscopic black hole formation and its subsequent evaporation is expected [11, 12, 13, 14] [2], while for large impact parameters $b \gg R_S$ the eikonal picture given by eikonalized single-graviton exchange is expected [19, 20, 21, 22].

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[1] Scattering at center-of-mass (CM) energies exceeding the quantum gravity scale.

[2] See also [15] and references therein; there are also proposals concerning the production of more complicated objects such as wormholes, or time machines [16, 17, 18].

S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_2,

Corrections in R_S/b to the elastic eikonal scattering have been studied [23, 24, 25, 26].

To study high-energy scattering of the hadrons one usually deals with the parton picture. In the case of a 3-brane embedded in $D = 4 + n$-dimensional spacetime for large impact parameters graviton exchanges dominate in parton amplitudes [19, 21, 23, 22, 3]. In all D dimensions the graviton is supposed to be propagated freely. Since D-dimensional gravity is strong it would be interesting to calculate the modification of the graviton propagator due to a presence of matter. This is difficult problem, however it can be solved in 2+1 gravity, where we know analytically the modification of the spacetime due to the presence of pointlike matter. We know also the modification of the spacetime metric by a cosmic string in 4-dimensional spacetime and by a cosmic membrane in 5-dimensional spacetime.

Due to Lorentz contraction we can treat colliding hadrons in the laboratory frame as membranes with the transversal characteristic scale of order of the hadron and a negligible thickness. These membranes are located on our 3-brane. Since $4+n$ gravity is strong enough we can expect that hadron membranes modify the $4+n$-spacetime metric.

Only for the case of $n = 1$ we know explicitly the modified metric and we can estimate explicitly an influence of this modification on the parton and other particle scattering. It is known that the 5-dimensional ADD model with $M_{Pl,5} \sim$ TeV is not phenomenologically acceptable and we can deal with the RS2 model [27] or with the DGP model [28]. In all these cases we treat a moving hadron as an infinite moving membrane in the 5-dimensional world with location on the 3-brane (our world). In other words, we deal with an effective 3-dimensional picture in the high-energy scattering (compare with the usual effective 2-dimensional picture in 4-dimensional spacetime, see [29, 30] and references therein).

In the framework of the picture described above, we can consider the influence of the matter on graviton propagation. Due to the presence of the hadron membrane the gravitational background is nontrivial and describes a flat spacetime with a conical singularity located on the hadron membrane. This picture is a generalization of the cosmological string picture in the 4-dimensional world to the 5-dimensional world. The deficit angle is proportional to the product of the hadron matter density on the membrane and the 5-dimensional gravitational coupling. This is a rather small number [3], $\delta_{h_0} \sim \frac{1}{M^3_{Pl,5}} \frac{M_{hadron}}{l^2_{hadron}} \sim 10^{-9}$. Since the hadrons collide with Lorentz boost factor, $\gamma = 1/\sqrt{1-v^2}$, about $\gamma \sim 10^4$, we have $\delta_h \sim 10^{-5}$. For heavy ions composed of A hadrons, this number is near $\delta_{Ion} \sim A^{1/3}_{Ion} \delta_h$.

We can take into account corrections to the graviton propagation. A study of these corrections and their physical consequences is the subject of the present letter. A more detailed discussion of the topological defects in TeV-gravity including the RS2 and DGP models and will be presented in [33]. As to higher dimensional cases we can just expect that numerical calculations could exhibit similar qualitative results.

[3] One can compare this number with an estimate of the deficit angle $\delta_{cs} \sim 10^{-6}$ for a cosmic string in 4-dimensional spacetime with the Newtonian gravitational constant $G_{N,4}$ and the density $\rho = \frac{m}{l} = 10^{33} GeV^2$, that corresponds to the Earth mass distributed on a length of about $l = 9$ km.

The paper is organized as follows. In Section 2 we present our setup and argue why in the TeV-gravity scenario the high energy hadrons colliding on the 3-brane embedded in $4+n$-dimensional spacetime with n dimensions smaller than the hadrons size, can be considered as cosmic membranes in the $4+n$-dimensional world. We recall basic facts about eikonalization of graviton exchanges and the form of the spacetime metric with a cosmic membrane. In Section 3 we present corrections to the eikonal phase due to a conical singularity. We restrict ourself here to a flat bulk for simplicity. The AdS case corresponding to the RS2 model can be investigated in a similar way. Others possible effects related with cosmic membranes and their signatures are briefly discussed in the conclusion.

2 Setup

It is known that for large impact parameters $b \gg R_S$ (elastic small-angle scattering) the transplanckian amplitude is dominated by eikonalized single-graviton exchange [19],[20],[21],[22]. The eikonal amplitude has been used in [10] to compute the differential cross section for neutrino-nucleon scattering and in [4] to compute the close to beam jet-jet production at the LHC. For small impact parameters $b \ll R_S$ the nonlinear effect are important and within the classical gravity one can expect the black hole formation.

2.1 Hadron as a membrane in 5-dimensional world

The graviton exchange is supposed to take place in the $4+n$-dimensional spacetime. In the total transplanckian cross section there is a factor, describing dependence on n and on the form of the background in the extra dimensional spacetime. In all previous considerations [10, 4, 8] the graviton is supposed to propagate freely in extra dimensions. It would be interesting to be able to calculate the modification of the propagator due to the presence of the hadron matter. This can be done for example in the 2+1 gravity, where we know analytically the modification of the spacetime due to the present of pointlike matter.

In $2+1$ dimensions, solutions to Einstein's equation with point masses are flat metrics except conical singularities at the location of the masses. In $3+1$ dimensions, there are solutions with singularities on the worldsheets of the strings. The deficit angle of the conical singularity is proportional to the mass in the $2+1$ case and the mass per length μ in the $3+1$ case [36]. In $4+1$ dimensions, there is a solution with singularity on the worldsheet of the membrane. One can imagine this membrane as high velocity moving hadron, that in the rest frame is tried as a ball. If we have extra dimensions, they are not available for the hadron and the hadron membrane cannot stretch in these dimensions. Hence, we get the 2-dimensional hadron membrane propagated on the 3-brane embedded in $4+n$-dimensional spacetime.

We know explicit solutions to Einstein's equations with the hadron membrane in the 5-dimensional ADD and RS2 models. The first case is simpler and in spite of it is not phenomenologically acceptable, we consider this case for simplicity [4].

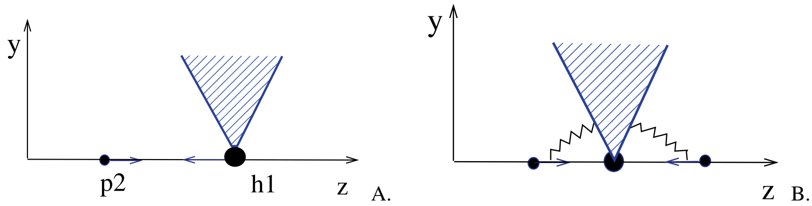

Fig. 1 A. Ultra relativistic colliding hadron h_1 as it seen by the parton p_2. B. The graviton exchange with modified propagator between partons.

2.2 *Bulk with conical singularities*

In the ADD model the metric in the bulk is flat,

$$ds^2 = -dt^2 + dx_\perp^2 + d\rho^2 + \rho^2 d\Omega^2,\ \rho^2 = \sum_1^n y_i^2 + z^2,\ x_\perp = (x_1, x_2) \tag{1}$$

here x_1, x_2, y_i, z are coordinates in the bulk and $d\Omega^2$ is the metric on the unit sphere S^n. However, the hadron membrane produces a nontrivial background. We know this background explicitly for the case of $n = 1$. In this case the bulk metric remains locally flat, $d\Omega^2 = d\phi^2$ and the hadron membrane produces only the conical singularity, i.e. the range of the angle is $0 < \phi < \alpha$. The angle α defines the deficit angle δ

$$\alpha = 2\pi - \delta, \tag{2}$$

where

$$\delta = 8\pi G_5 \frac{m_h}{S_h} = \frac{32}{M_{Pl,5}^3} \frac{m_h}{l_h^2}. \tag{3}$$

Here m_h is the hadron mass and l_h is the hadron size, $S_h = \pi l_h^2/4$. The top of the cone is located on the brane.

The gravitational effect of the hadron membrane in the RS2 model is convenient to present in the Poincaré coordinates. Starting from the metric

$$ds^2 = a^2(y)\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \tag{4}$$

[4] One can assume an anisotropic compactification with essentially suppressed $n-1$ dimensions (in this case $M_{Pl,D} \sim$ TeV and $M_{Pl,5} \sim 10^3$ TeV), or just consider a toy model with $M_{Pl,5toy} \sim 10^3$ TeV.

where $\eta_{\mu\nu}$ is the 4-dimensional Minkowski metric and the warp factor $a(z)$ has the form [27]

$$a(y) = e^{-k|y|}, \tag{5}$$

$1/k$ is the radius of 5-dimensional AdS spacetime, we get the metric in the Poincaré coordinates after the following change of variable, $y \to w$, $w = r_0 e^{y/r_0}$,

$$ds^2 = \frac{r_0^2}{w^2}(\eta_{\mu\nu}dx^\mu dx^\nu + dw^2) \tag{6}$$

[31]. According to the usual prescription to incorporate a membrane we cut a wedge. This can be done by reducing the range of a suitable angular coordinate. For example, for AdS_5

$$\frac{R_5^2}{w^2}\left[dw^2 - dt^2 + dz^2 + d\rho^2 + \rho^2 d\phi^2\right] , \tag{7}$$

and the range of the angle is $0 < \phi < 2\pi - \delta$ where δ is given by (3).

2.3 Eikonalization of graviton exchanges

The parton-parton elastic forward scattering amplitude for a large center of mass energy is given by the eikonal technique [34],[35]. In the transplanckian regime the graviton exchanges [22, 4] dominate and define the amplitude

$$\mathscr{A}_{\rm eik}(\mathbf{q}) = \mathscr{A}_{\rm Born} + \mathscr{A}_{1-loop} + \ldots = -2is\int d^2\mathbf{b}\, e^{-i\mathbf{q}\cdot\mathbf{b}}(e^{i\chi(\mathbf{q})} - 1), \tag{8}$$

where the eikonal phase χ is given by the Fourier transform of the Born amplitude in the transverse plane

$$\chi(\mathbf{b}) = \frac{1}{2s}\int \frac{d^2\mathbf{q}}{(2\pi)^2} e^{i\mathbf{q}\cdot\mathbf{b}} A_{Born}(s,\mathbf{q})\,. \tag{9}$$

The $4+n$-dimensional Born amplitude for the exchange of the graviton, which does not get any transferred momenta in the direction transversal to the brane, is given by

$$\mathscr{A}_{Born}(s,q) = \frac{-s^2}{M_D^{n+2}}\int \frac{d^n l}{\mathbf{q}^2 + l^2},\ |\mathbf{q}| = q. \tag{10}$$

The expression for the eikonal amplitude [10, 4] is

$$\mathscr{A}_{\rm eik} = 4\pi s b_c^2 F_n(b_c q), \tag{11}$$

$$F_n(y) = -i\int_0^\infty dx x J_0(xy)\left(e^{ix^{-n}} - 1\right), \tag{12}$$

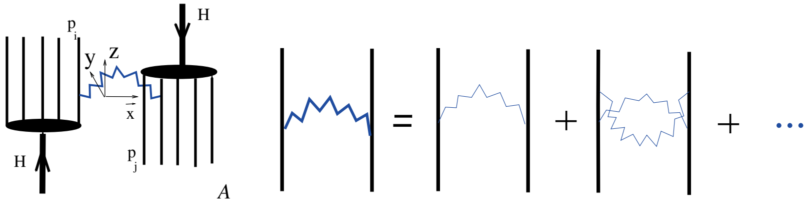

Fig. 2 A. Collision of hardons with a large impact parameter in $(x_\perp, z)$ coordinates is presented as an elastic scattering between partons due to a free graviton exchange. y-coordinate schematically presents extra dimensions. B. The $2 \to 2$ small angle T-scattering amplitude is given by a sum of crossed-ladder graviton exchanges.

where the integration variable is related with the impact parameter, $x = b/b_c$ and in (12) we take into account that the eikonal phase has the power dependence on the impact parameter

$$\chi(b) = \left(\frac{b_c}{b}\right)^n, \text{ where } b_c \equiv \left[\frac{(4\pi)^{\frac{n}{2}-1} s\Gamma(n/2)}{2M_D^{n+2}}\right]^{1/n}. \tag{13}$$

Functions F_n, $n > 1$, when $y \gg 1$ oscillate around their asymptotic values given by $F_{n,as}(y) = \frac{-in^{\frac{1}{n+1}} y^{-\frac{n+2}{n+1}}}{\sqrt{n+1}} \exp\left[-i(n+1)\left(\frac{y}{n}\right)^{\frac{n}{n+1}}\right]$ [4]. Within the TeV-gravity scenario [1] the total transplanckian cross section is finite, grows with energy, and is dominated by small-angle scattering between partonic constituents [10],[4].

The real and imaginary parts of the function F_1 are shown in Fig. 3.A, and we also see the oscillations of the real part of the function F_1.

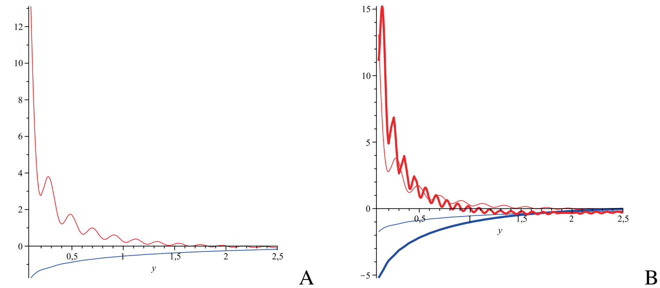

Fig. 3 A. The real (red) and imaginary (blue) parts of the eikonal amplitude F_1. B. Thick lines represent the real and imaginary parts of the eikonal amplitude with doubling eikonal phase in the toy model with the deficit angle equal to π.

3 Eikonal in the conical spacetime

The goal of this section is to estimate the influence of the hadron membrane on the forward scattering of the partons.

3.1 Graviton exchange with modified graviton propagator

The tree level 2 partons → 2 partons S-matrix element corresponding to one graviton exchange in the $C_\alpha \times M^3$ spacetime,

$$< p_1, p_2|S|p_3, p_4 >_{1\text{graviton}} \equiv \mathscr{S}_{\text{graviton},\alpha}(p_1, p_2, p_3, p_4), \tag{14}$$

is given by the linearization of gravity [22] and in $s \gg t$ regime is

$$\mathscr{S}_{\text{graviton},\alpha}(p_1, p_2, p_3, p_4) \approx -16\pi G\gamma(s)\,\mathscr{S}_{scalar,\alpha}, \tag{15}$$

here $\approx$ means that we ignore the recoil of the matter field and take the prefactor $\gamma(s)$ the same as for the flat case, $\gamma(s) = ((s-2m^2)^2 - 2m^4)/2$.

In the flat spacetime

$$\mathscr{S}_{\text{graviton,flat}}(p_1, p_2, p_3, p_4) = i(2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4)\,\mathscr{A}_{\text{Born}}(s,t), \quad t \approx -\mathbf{q}^2. \tag{16}$$

In what follows, $\mathscr{S}_{scalar,\alpha} \equiv \mathscr{S}_\alpha$ is the Born amplitude for the scalar particles scattering due to the scalar exchange in the $C_\alpha \times M^3$ spacetime. It can be written (after the Euclidean rotation) in the Schwinger representation as

$$\mathscr{S}_\alpha = \int d^4X d^4X' e^{i(p_1-p_3)X + i(p_2-p_4)X'} \int d\tau e^{-m^2\tau} K(t, x_\perp; t', x'_\perp; \tau) K_\alpha(z, 0; z', 0; \tau),$$

here $X = (t, x_\perp, z) \equiv (x^{\check{\mu}}, z)$ and $K(t, x_\perp; t', x'_\perp; \tau)$ is the heat kernel on the 3-dimensional plane and $K_\alpha(z, y; z', y'; \tau)$ is the heat kernel on the 2-dimensional cone C_α. K_α has a representation [37, 38, 39]

$$K_\alpha(z, y; z', y'; \tau) = \frac{i}{2\alpha} \int_\gamma dw \operatorname{ctg}\left(\frac{\pi w}{\alpha}\right) K(z(w), y(w); z', y'; \tau). \tag{17}$$

Here $(z(w), y(w)) = (r\cos(\theta + w), r\sin(\theta + w))$, (r, θ) are related with coordinates (z, y) as $(z, y) = (r\cos(\theta), r\sin(\theta))$, $K(z, y; z', y'; \tau)$ is the heat kernel on the 2-dimensional plane

$$K(z, y; z', y'; \tau) = \frac{1}{4\pi\tau} \exp\{-\frac{(z-z')^2 + (y-y')^2}{4\tau}\}, \tag{18}$$

and γ is a characteristic contour presented in Fig. 4, where $\Delta\theta = \theta' - \theta$ and θ' is related with z'.

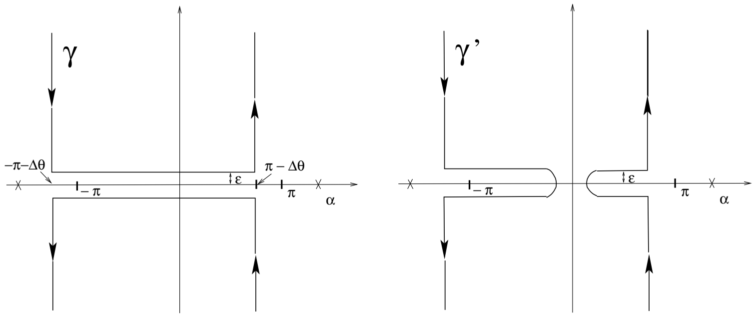

Fig. 4 Contours γ and γ'.

Under assumption that we are on the brane, $\theta = 0$ and $\theta' = 0$ (or $\theta, \theta' = \pi$) we have

$$(z(w), y(w))|_{\text{on brane}} = (z\cos(w), z\sin(w)), \ \ (z', y')|_{\text{on brane}} = (z', 0) \tag{19}$$

and

$$K_\alpha(z, 0; z', 0; \tau) = \frac{i}{2\alpha} \int_\gamma dw \, \text{ctg}\left(\frac{\pi w}{\alpha}\right) \mathscr{K}_w(z, z'; \tau) \tag{20}$$

where

$$\mathscr{K}_w(z, z'; \tau) \equiv \frac{1}{4\pi\tau} \exp\{-\frac{z^2 + z'^2 - 2zz'\cos w}{4\tau}\} \tag{21}$$

We can define the Fourier transformation of the propagator associated with (21) as

$$\mathscr{D}(r, v) = \int\int e^{ir(z-z') + iv(z+z')} e^{-m^2\tau} \mathscr{K}_w(z, z'; \tau) dz dz' \frac{d\tau}{4\pi\tau} \tag{22}$$

and find

$$\mathscr{D}(r, v) = \frac{2}{\sin w} \frac{1}{\frac{r^2}{\sin^2 \frac{w}{2}} + \frac{v^2}{\cos^2 \frac{w}{2}} + m^2}. \tag{23}$$

Finally, we get

$$\mathscr{S}_\alpha = i(2\pi)^3 \delta^3 \left((p_1 + p_2 - p_3 - p_4)_{\check{\mu}}\right) \mathscr{M}_\alpha,$$

$$\mathscr{M}_\alpha = \frac{i}{2\alpha} \int_\gamma dw \, \text{ctg}\left(\frac{\pi w}{\alpha}\right) \frac{2}{\sin w} \frac{1}{\frac{Q^2}{\sin^2 \frac{w}{2}} + \frac{P^2}{\cos^2 \frac{w}{2}} + q_{\check{\mu}}^2 + m^2}, \tag{24}$$

here and below $q_{\check{\mu}} = (q_0, q_1, q_2)$, $\check{\mu} = 0, 1, 2$, $q = (q_{\check{\mu}}, q_z)$, $q_\perp = (q_1, q_2)$,

$$Q = \frac{1}{2}(p_1 - p_2 - p_3 + p_4)_z, \ P = \frac{1}{2}(p_1 + p_2 - p_3 - p_4)_z, \ q_{\check{\mu}} = (p_1 - p_3)_{\check{\mu}}. \tag{25}$$

Q and P are related as $Q = q_z - P$. In the eikonal regime $Q \approx -P$ and this gives a simplification of (24)

$$\mathscr{M}_\alpha \approx \frac{i}{2\alpha}\int_\gamma dw\,\mathrm{ctg}\left(\frac{\pi w}{\alpha}\right)\mathscr{B}_w(q_\perp, P), \tag{26}$$

where

$$\mathscr{B}_w(q_\perp, P) = \frac{2}{\sin w}\,\frac{1}{q_\perp^2 + m^2 + \frac{4P^2}{\sin^2 w}}. \tag{27}$$

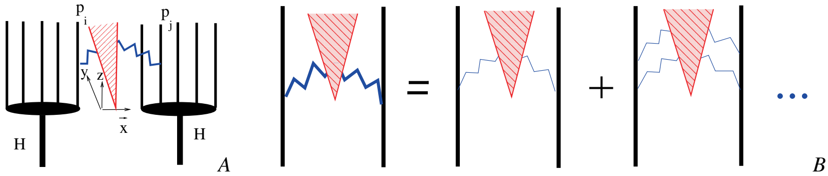

Fig. 5 A.Collision of hadrons with a large impact parameter is presented as an elastic scattering between partons due to a graviton exchange in the space $(x_\perp, z, y)$ with the conic point in the (z, y) section. B. The $2 \to 2$ small angle T-scattering amplitude is given by a sum of crossed-ladder graviton exchanges in the space with the hadron membrane.

Let now define the w-eikonal phase χ as the Fourier transform of (27)

$$\mathscr{X}_w(\mathbf{b}, P) = \frac{1}{2s}\int \frac{d^2\mathbf{q}}{(2\pi)^2} e^{i\mathbf{q}.\mathbf{b}}\mathscr{B}_w(q_\perp, P)\,. \tag{28}$$

The total eikonal phase is given by the integral over the contour γ

$$\chi_\alpha(\mathbf{b}, P) = \frac{i}{2\alpha}\int_\gamma dw\,\mathrm{ctg}\left(\frac{\pi w}{\alpha}\right)\mathscr{X}_w(\mathbf{b}, P) \tag{29}$$

Using the explicit expression for the eikonal phase for a massive particle we get

$$\mathscr{X}_w(\mathbf{b}, P) = \frac{1}{2\tau}\frac{1}{\pi \sin w} K_0\left(|\mathbf{b}|\sqrt{m^2 + P^2\frac{4}{\sin^2 w}}\right). \tag{30}$$

In the case of $m \approx 0$

$$\mathscr{X}_w(\mathbf{b}, P) = \frac{1}{2\tau}\frac{1}{\pi \sin w} K_0(2|\mathbf{b}||\frac{P}{\sin w}|). \tag{31}$$

and for small w we have

$$\mathscr{X}_w(\mathbf{b},P) \approx \frac{1}{4\tau}\frac{e^{-2|\mathbf{b}||\frac{P}{\sin w}|}}{\sqrt{\pi|\mathbf{b}||P\sin w|}} \tag{32}$$

It is known that the propagator in the conic space can be present as a sum of two terms [38, 39]

$$K_\alpha(z,y;z',y';\tau) = K(z,y;z',y';\tau) + K'_\alpha(z,y;z',y';\tau), \tag{33}$$

where

$$K'_\alpha(z,y;z',y';s) = \frac{i}{2\alpha}\int_{\gamma'} dw \operatorname{ctg}\left(\frac{\pi w}{\alpha}\right) K(z(w),y(w);z',y';s), \tag{34}$$

with a modified contour γ' presented in Fig. 4.

Therefore, the eikonal matrix element can be written as

$$\begin{aligned}\mathscr{S}_{\text{eik},\alpha}(p_1,p_2,p_3,p_4) &= i(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\mathscr{A}_{\text{eik,flat}}\\ &\quad + i(2\pi)^3\delta^3\left((p_1+p_2-p_3-p_4)_{\check{\mu}}\right)\mathscr{M}_{\text{eik},\alpha},\end{aligned} \tag{35}$$

where

$$\mathscr{M}_{\text{eik},\alpha} = -2i\tau\int d^2b_\perp e^{iq_\perp b_\perp} e^{i\chi_{plane}(b_\perp)}\left[e^{\Delta\chi_\alpha(b_\perp,P)}-1\right], \tag{36}$$

where $\chi_{\text{plane}}(b_\perp)$ is given by (13) for $n=1$ and

$$\Delta\chi_\alpha(b_\perp,P) = \frac{1}{2s}\int\frac{d^2q_\perp}{(2\pi)^2}e^{-iq_\perp b_\perp}B_\alpha(q_\perp^2,P), \tag{37}$$

where

$$B_\alpha(q_\perp,P) = \frac{i}{2\alpha}\int_{\gamma'} dw \operatorname{ctg}\left(\frac{\pi w}{\alpha}\right)\mathscr{B}_w(q_\perp,P). \tag{38}$$

Now if we take this correction perturbatively we get

$$\mathscr{M}_{\text{eik},\alpha} \approx -2is\int d^2b_\perp\,\Delta\chi_\alpha(b_\perp,P)\,e^{iq_\perp b_\perp + i\chi_{\text{plane}}(b_\perp)}. \tag{39}$$

We can analyze the correction for arbitrary angle α only numerically.

3.2 Correction to the eikonal amplitude for toy model $\alpha = \pi/N$

It is known, that the propagator in the conic space with $\alpha = \pi/N$ can be present as a finite sum of propagators

$$K_{\pi/N}(z,z',\tau) = \sum_{n=0}^{N}\mathscr{K}_{n\pi/N}(z,z',\tau), \tag{40}$$

where

$$\mathcal{K}_{n\pi/N}(z,z',\tau) \equiv \frac{1}{4\pi\tau}\exp\{-\frac{z^2+z'^2-2zz''\cos(\frac{n\pi}{N})}{4\tau}\}. \tag{41}$$

We can calculate the contour integral in (24) explicitly to get

$$\mathcal{S}_{\pi/N} = i(2\pi)^3\delta^3\left((p_1+p_2-p_3-p_4)_{\check{\mu}}\right)\left[\sum{}' \frac{2}{\sin\frac{\pi n}{N}} \frac{1}{\frac{Q^2}{\sin^2\frac{\pi n}{2N}}+\frac{P^2}{\cos^2\frac{\pi n}{2N}}+q_{\check{\mu}}^2+m^2}\right.$$
$$\left. +\ \delta(Q)\frac{\pi}{\sqrt{P^2+q_{\check{\mu}}^2+m^2}}+\delta(P)\frac{\pi}{2\sqrt{q_{\check{\mu}}^2+m^2}}\right]. \tag{42}$$

Here the prime in the sum means that we do not take into account $n=0$ and $n=N$.

If we consider $N=1$ we get just one new term as a correction to the usual Born amplitude

$$\mathcal{S}_{\pi} = \mathcal{S}_{\text{flat}}+\Delta\mathcal{S}_{\pi}, \tag{43}$$

$$\mathcal{S}_{\text{flat}} = \delta^4(p_1+p_2-p_3-p_4)\frac{i(2\pi)^4}{2\sqrt{q_{\check{\mu}}^2+m^2}}, \tag{44}$$

$$\Delta\mathcal{S}_{\pi} = \delta^3\left((p_1+p_2-p_3-p_4)_{\check{\mu}}\right)\delta\left((p_1-p_3-p_2+p_4)_z\right)\frac{i(2\pi)^4}{2\sqrt{q^2+m^2}}. \tag{45}$$

In the eikonal regime $Q\approx P$ and both terms (44) and (45) give the same contribution and we get a doubling of the eikonal phase.

4 Conclusion and Discussion

In this paper we have argued that in the TeV-gravity scenario high energy hadrons colliding on the 3-brane embedded in $D=4+n$-dimensional spacetime, with n dimensions smaller than the hadrons size, can be considered as cosmic membranes. In the 5-dimensional case this consideration leads to the 3-dimensional effective model of high energy collisions of hadrons. The cosmic membranes in the 5-dimensional case are similar to cosmic strings in the 4-dimensional world.

It is well known that, the cosmic strings give rise to remarkable classical gravitational and quantum phenomena. In particular, the cosmic string acts as a gravitational lens [31]. This effect becomes manifest when two particles move along opposite sides of the string. Also there is a self-force acting on a test charged particle around the cosmic string [40] and a freely moving charged particle radiates near the cosmic string [41, 42]. This is an analogue of the radiation by the charged particle when it suffers the Aharonov-Bohm scattering [43] and this radiation occurs due to the fall down of the Huygens principle in curved spacetime.

There are also quantum effects. The presence of the cosmic string allows effects such as particle-antiparticle pair production by a single photon and bremsstrahlung radiation from charged particles [44, 45] which are not possible in empty Minkowski space, due to conservation of linear momentum. The conical structure of the cosmic string spacetime is the source of momentum non-conservation in the plane perpendicular to the string, which permits pair production by a single photon. The gravitational mechanism that permits pair production by a single photon around a cosmic string has common topological features with the Aharonov-Bohm effect [43]. The absence of global momentum conservation was already stressed for gravity in 2+1 dimensions by Henneaux [46] and Deser [48]. It is worth also to mention that the string polarizes the vacuum around it, in a way similar to the Casimir effect between two conducting planes forming a wedge [49, 50]. The study of quantum field theories the spasetime with conic singularities requires a regularization [51]. Among possible regularizations the zeta-function regularization is more convenient [52].

Our specifics is that not all process mentioned above can be realized for particles attached to the 3-brane. In particular, to see the lens effect we have to deal with the motion of particles in the 2-plane that is perpendicular to the hadron membrane. But only gravitons can move in this plane in any direction. However one can estimate the self-force effect.

The same concerns also the quantum effects. From one side, only the graviton can propagate in the 2-plane perpendicular to the hadron membrane and feel the deficit angle. From other side, the above mentioned quantum processes are available for other particles if their have not to abandon the 3-brane to participate in the processes.

In this paper we have estimated corrections to the eikonal scattering amplitude due to the hadron membrane.

Similar to the case of cosmic string [44], one can also estimate the decay of a light ultra-relativistic particle on two heavy particles with mass M. For large longitudinal momentum of the light particle, $k_z >> 2M\delta^{-1}$, the cross-section does not depend on k_z and is defied only by the coupling g of these 3 particles and heavy mass

$$\sigma_{1\,\text{light}\to 2\,\text{heavy}} \approx \frac{g^2}{M^3} \tag{46}$$

To realize the condition $k_z >> 2M\delta$ it is enough to take $k_z \sim 1TeV$ and M of the order of the few MeV's.

Other processes we are going to estimate in the separate work [33].

Acknowledgements It is my pleasure to present this paper to Proceedings of Workshop: "Cosmology, the Quantum Vacuum, and Zeta Functions" dedicated to Emilio Elizalde's sixtieth birthday.

I am grateful to A.V. Radyushkin, I.V.Volovich and A.A.Bagrov for useful discussions. This work is supported in part by RFBR grants 08-01-00798 and 09-01-12179-ofi$_m$ and by state contract of Russian Federal Agency for Science and Innovations 02.740.11.5057.

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Vacuum Energy and the Topology of the Universe

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Abstract We analyze the dependence of the quantum vacuum energy on the space topology. In particular we point out the existence of a renormalization ambiguity in spaces with non-vanishing curvature. The ambiguity is related to the well known ambiguity of the R^2 term of the gravitational effective action. However, there are two extra universal contributions which are genuine dependent on the topological structure of the space and completely independent of the renormalization scheme. The ambiguity does not appear in flat spaces where only the topological dependent contributions are non-vanishing. We analyze the cosmological role of universal contributions to the vacuum energy and its attractive nature in the case of conformal scalar fields.

1 Introduction

The current cosmological model is consistent with a spatially flat Universe, although, most of the relevant data are compatible with a very tiny curvature $|\Omega_\kappa| \leq 10^{-4}$ [1][2]. However, the physical observations do not allow to establish a definite answer to the longstanding dilemma on the finiteness or not of the physical space or determine the characteristics of space-time topology (see [3] and references there in for an updated review). Closed spaces leave their fingerprints in small contributions to low multipoles of the Cosmic Microwave Background (CMB) and current observations show a strong suppression of low multipoles (quadrupole, octupole, etc.). They also show an strange alignment of the quadrupole and octupole multipoles associated to the appearance of Southern hemisphere cool fingers. On the other hand, it is remarkable the observed asymmetry between even and odd multipoles and the fact that the Gaussianity of likelihood estimates starts to be manifest for $l > 32$. All these data suggest a possible role of the finite size and space topology in the

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_3,

low modes behavior of the CMB [4]. A compact space will imply, depending on its topology, the existence of several circles in the sky which will correspond to the mirror images of the last scattering surface where the radiation decouple from matter. The latest results do not allow to determine their existence which will be an unequivocal proof of a non-trivial space topology. However, presumably the new observational programs will be able to discriminate among the different space-time topologies. In this note we analyze the quantum implications of a non-trivial space-time topology.

2 Vacuum energy in cosmological backgrounds

Quantum fields contribute to the background space-time energy because of vacuum fluctuations. For conformal invariant fields this energy depends on the topology of the space. If the space has boundaries it also depends on the boundary conditions.

The cosmological implications of this energy are not very clear. First, the divergent nature of the leading contributions raises some questions about the validity of the renormalization philosophy in the presence of gravitational interactions. On the other hand, finite Casimir corrections encode the quantum back-reaction to the cosmological expansion of the Universe, but this is very tiny to be detected in the present Universe, although it might have played a relevant role in the early Universe. In this note we analyze the structure of such contributions in different cosmological backgrounds. This problem has been considered by Emilio Elizalde for a long time [5] [6] [7].

Although the background cosmological FRW metric evolves in time its variation is so slow in comparison with the leading quantum fluctuations that one can use adiabatic approximations to estimate the vacuum energy induced by these fluctuations. In this approximation the space-time metric can be considered as a homogeneous isotropic static on a space-time of the form $\mathbb{R} \times \mathscr{M}$.

There are three types of constant curvature spaces: hyperbolic ($R < 0$), elliptic ($R > 0$) or Euclidean ($R = 0$). If we assume that the space is compact and has no boundaries the number of candidates is reduced considerably. The hyperbolic case presents an infinite number of possibilities and has been the most analyzed in the literature [8][9][10]. We will restrict ourselves to the less analyzed cases of elliptic and flat spaces.

Spaces with constant positive curvature and no boundaries are compact manifolds and belongs to one of the following six familes. If $\mathscr{M}$ is simply connected it has to be isometric to the three-dimensional sphere S^3, because of Poincaré theorem. Multiple connected spaces belong to one of the following five families:

- Lens spaces $S^3/\mathbb{Z}_q$, with first homotopy group the cyclic group $\mathbb{Z}_q$ of order q.
- Dihedral spaces S^3/D^*_q, with first homotopy group D^*_q of order $4q$. order 24.
- Tetrahedral space S^3/T^* with $\pi_1(S^3/T^*) = T^*$ of order 24.

- Octahedral space S^3/O^* with $\pi_1(S^3/O^*) = O^*$ of order 48.
- Poincaré Dodecahedral space S^3/Y^* with $\pi_1(S^3/Y^*) = Y^*$ of order 120.

The last space S^3/Y^* has been recently considered as a possible candidate for the global structure of the Universe [11] [12] [13] by considerantions based on the observed anomalies of CMB.

For simplicity we shall restrict ourselves to the case of conformal scalar free fields. The analysis of higher spin fields is very similar. The vacuum energy of free conformal scalar field is given by the renormalized sum of the eigenvalues of the operator $\frac{1}{2}\sqrt{-\Delta + \frac{1}{6}R}$, where R is the scalar curvature of $\mathscr{M}$. $R = \frac{6}{a}$ for a three-dimensional sphere $\mathscr{M} = S^3$ of radius a.

The eigenvalues of the operator $-\Delta + \frac{1}{6}R$ on $\mathscr{M}$ are of the form $\lambda_k = \frac{1}{a^2}(k+1)^2$ with $k \in \mathbb{Z}$, with the following degeneracies d_k [14][15]:

$\mathbb{I}$	$d_k(\mathbb{I}) = (k+1)^2$
$\mathbb{Z}_{2q+1}$	$d_k(Z_{2q+1}) = (k+1)\left(k+1-[(k+1)/(2q+1)](2q+1)+(1+(-1)^{k-[(k+1)/(2q+1)](2q+1)})/2\right)$
$\mathbb{Z}_{2q}$	$d_{2l}(Z_{2q}) = (2l+1)(2[(2l+1)/(2q)]+1)$
$\mathbb{D}^*_q$	$d_{2l}(D^*_q) = (2l+1)([l/q]+1/2(1+(-1)^l))$
$\mathbf{T}^*_q$	$d_{2l}(T^*) = (2l+1)([l/3]+2[l/2]+1-l);\ l \neq 1,2,5$
$\mathbb{O}^*_q$	$d_{2l}(O^*) = (2l+1)([l/4]+[l/3]+[l/2]+1-l);\ l \neq 1,2,3,5,7,11$
$\mathbf{Y}^*_q$	$d_{2l}(Y^*) = (2l+1)([l/5]+[l/3]+[l/2]+1-l);\ l \neq 1,2,3,4,5,7,8,9,11,13,14,17,19,23,29$

Table 1 Degeneracies of the eigenvalues of the Laplacian operator Δ for spherical factor spaces.

The zeta function regularization method provides the following values for the vacuum energy $E_c = \frac{1}{a}C_{\mathscr{M}}$[16][17][18]:

- Sphere S^3

$$E_{S^3} = \frac{1}{240}\frac{1}{a}$$

- Lens spaces $S^3/\mathbb{Z}_q$

$$E_{Z_q} = -\frac{q^4 + 10q^2 - 14}{720q}\frac{1}{a}$$

- Dihedral spaces $S^3/\mathbb{D}^*_q$

$$E_{D^*_q} = -\frac{20q^4 + 8q^2 + 180q - 7}{1440q}\frac{1}{a}$$

- Polyhedral spaces $S^3/\mathbf{T}^*$, $S^3/\mathbb{O}^*$, $S^3/\mathbf{Y}^*$

$$E_{T^*} = -\frac{3761}{8640}\frac{1}{a} \qquad E_{O^*} = -\frac{11321}{17280}\frac{1}{a} \qquad E_{Y^*} = -\frac{43553}{43200}\frac{1}{a}$$

Group \ Order	24	48	120
Cyclic $\mathbb{Z}_q$	$C_{S^3/Z_{24}} = -\frac{168761}{8640}$	$C_{S^3/Z_{48}} = -\frac{2665721}{17280}$	$C_{S^3/Z_{120}} = -\frac{103751993}{43200}$
Dihedral $\mathbb{D}_q^*$	$C_{S^3/D_6^*} = -\frac{11081}{4320}$	$C_{S^3/D_{12}^*} = -\frac{168761}{8640}$	$C_{S^3/D_{30}^*} = -\frac{6497993}{21600}$
$\mathbf{T}^*$ $\mathbb{O}^*$ $\mathbf{Y}^*$	$C_{S^3/T^*} = -\frac{3761}{8640}$	$C_{S^3/O^*} = -\frac{11321}{17280}$	$C_{S^3/Y^*} = -\frac{43553}{43200}$

Table 2 Casimir energies of confomal scalar fields on spaces of compact constant curvature with group factors of order 24, 48 and 120. Notice that lens spaces tend to have larger negative energies than dihedral or polyhedral spaces with the same volumes.

These energies generate attractive forces except for the case of the three-dimensional sphere S^3, which is the only case with repulsive behaviour.

The nature of this attractive behaviour is stronger for spaces with the same volume in the cases of dihedral and lens spaces as the Table 2 points out.

3 Vacuum energy ambiguites

The values of vacuum energy shown in the previous section are not universal. In general the vacuum energy has three components

$$E(g) = E_{\text{loc}}(g) + E_{\text{anom}}(g) + E_{\text{top}}(g).$$

which are in one-to-one correspondence with the three components of the effective action

$$S(g) = S_{\text{loc}}(g) + S_{\text{anom}}(g) + S_{\text{top}}(g).$$

The first two components depend on the Riemann curvature tensor $R_{\mu\nu\alpha\sigma}$ either locally

$$S_{\text{loc}}(g) = \int d^4x\sqrt{-g}\,\{\alpha_1 C^2 + \alpha_2 E + \alpha_3 \Box R\}$$

or non-locally [19]

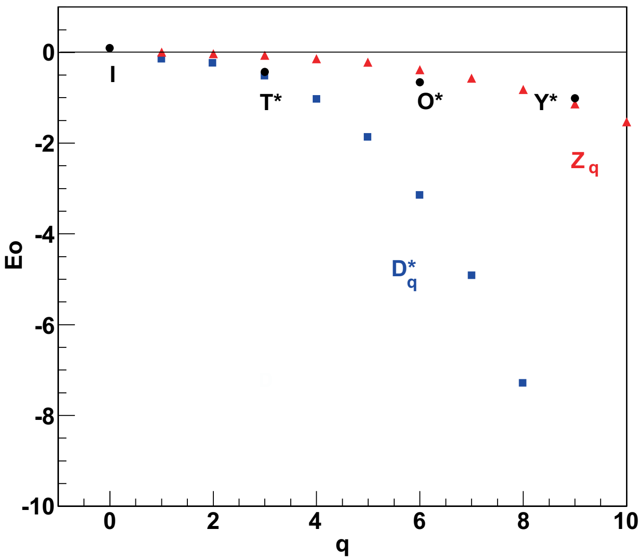

Figure 1 Casimir energies of confomal scalar fields on compact spaces of constant curvature. The only positive value appears in pure spherical spaces S^3. The dihedral factor S^3/D_q^* seem to have higher repulsive energies than lens spaces S^3/Z_q. However, this is an artifact of the different volumes weight of the respective spaces, Vol(S^3/Z_{4q})= Vol(S^3/D_q^*). The volumes of the polyhedral factors S^3/T_q^*, S^3/O_q^*, S^3/Y^* are identical to those of S^3/Z_{24}, S^3/Z_{48} and S^3/Z_{120}, respectively. However, they generate milder attractive energies.

$$S_{\rm anom}(g\,) = \frac{b}{8(4\pi)^2}\int d^4x \int d^4x' \sqrt{-g}\left(E+\frac{2}{3}\Box R\right)(x)\Box_4^{-1}(x,x')$$
$$\sqrt{-g}\left[\left(E+\frac{2}{3}\Box R\right)\right](x') + \left(c-\frac{2}{3}b\right)\frac{1}{12(4\pi)^2}\int d^4x\sqrt{-g}\,R^2$$

in terms of the Green function of the operator

$$\Box_4 \equiv \Box^2 - 2R^{\mu\nu}\nabla_\mu\nabla_\nu + \frac{2}{3}R\Box - \frac{1}{3}(\nabla^\mu R)\nabla_\mu\,,$$

the Weyl tensor $C_{\mu\nu\alpha\sigma}$ and the Euler density E.

However, the third component $S_{\rm top}(g)$ cannot be expressed in terms of local tensor densities because only depends on global properties of the space $\mathscr{M}$ like the length of minimal closed geodesic. This component is only present in multiple connected spaces.

The coefficients of the local part are ambiguous and depend on the renormalization scheme. However the b coefficient of $S_{\text{anom}}(g)$ and those of $S_{\text{top}}(g)$ are universal [20] and independent of the regularization method. The coefficient c of $S_{\text{anom}}(g)$ is ambiguous because corresponds to a local term which cannot be disentangled from a similar term of $S_{\text{loc}}(g)$ [21]. In particular, for a conformal scalar field $b = 1/360$, and although most of the regularization methods yield $c = -1/180$, there are other methods which give $c = -1/180 + \delta$, with an arbitrary contribution δ which depends on the parameters of the regulatization [21, 22].

However, not all the terms of the action are relevant for the calculation of the Casimir energy in spherical factor manifolds. It can be shown that in the case of S^3 the contribution of the non-local component of $S_{\text{anom}}(g)$, $E^{u_1}_{S^3} = \frac{1}{480}\frac{1}{a}$, is half of the total contribution [23] in zeta function regularization. The other half comes from the R^2 term and the genuine topological contribution vanishes. However, as it has been shown the R^2 contribution is arbitrary and, therefore, the total Casimir energy in such a background is also arbitrary [22].

For multiple connected spherical factor spaces the two universal contributions have a very different behaviour due to its different origin. The contribution coming from the non-local terms of $S_{\text{anom}}(g)$ is

$$E^{u_1}_{\mathscr{M}} = \frac{Vol(\mathscr{M})}{480(2\pi^2)}\frac{1}{a},$$

which is equal to the similar contribution of the sphere, up to the ratio of volumes

$$\frac{Vol(S^3)}{Vol(\mathscr{M})} = \#\pi_1(\mathscr{M}),$$

which is given by the order of the first homotopy group of the physical space $\mathscr{M}$. The contribution of $S_{\text{top}}(g)$ to the vacuum energy is non vanishing and depends on the topology of the spherical factor space. This contribution is given by

- Sphere S^3

$$E^{\text{top}}_{S^3} = 0$$

- Lens spaces $S^3/\mathbb{Z}_q$

$$E^{\text{top}}_{Z_q} = -\frac{2q^4 + 20q^2 - 25}{1440q}\frac{1}{a}$$

- Dihedral spaces $S^3/\mathbb{D}^*_q$

$$E^{\text{top}}_{D^*_q} = -\frac{40q^4 + 16q^2 + 360q - 11}{2880q}\frac{1}{a}$$

- Polyhedral spaces $S^3/\mathbf{T}^*$, $S^3/\mathbb{O}^*$, $S^3/\mathbf{Y}^*$

$$E_{T^*}^{\rm top} = -\frac{1505}{3456}\frac{1}{a} \qquad E_{O^*}^{\rm top} = -\frac{4529}{6912}\frac{1}{a} \qquad E_{Y^*}^{\rm top} = -\frac{87109}{86400}\frac{1}{a}.$$

However, in all cases there is an extra contribution coming from the R^2 term of the action whose arbitrary contribution δ makes the calculation of the vacuum energy completely ambiguous.

The same ambiguity appears in hyperbolic spaces with constant negative curvature. However, for flat spaces the behaviour is different.

4 Flat compact spaces

In the case of flat spaces the extra ambiguous contribution is absent due to the vanishing of all curvature tensors. In this case the only non-vanishing contribution arises from the $S_{\rm top}(g)$ terms of the effective action. There are six orientable compact flat manifolds: Torus (T^3), Half-Turn Space (E_2), Quarter-Turn Space (E_3),Third-Turn Space (E_4), Sixth-Turn Space (E_5) and Hantzsche-Wendt Space (E_6). These spaces correspond to different factors of the Euclidean space $\mathbb{R}^3$ by discrete subgroups of the Euclidean group ISO(3)= $T_3 \circ$ O(3). They are classified according to their rotational part, $\mathbb{Z}_1$ for E_2, $\mathbb{Z}_2$ for E_3, $\mathbb{Z}_4$ for E_4, $\mathbb{Z}_3$ for E_4, $\mathbb{Z}_6$ for E_5 and $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ for E_6.

However, due to the fact that the group factors are not normal, the vacuum energy density is not uniformly distributed, which implies the existence of space anisotropies that should be observed in the dark energy component (see Fig. 3).

The corresponding vacuum energies for compact factors of a symmetric torus of size a are given by [24] [25]

• Torus T^3

$$E_{T^3} = -\frac{1}{2\pi^2 a}\int_0^\infty dt\, t(\theta_3^3(\mathrm{e}^{-t}) - 1) = -\frac{0.8375}{a}$$

• Twisted Sixth-Turn Torus E_5

$$E_5 = -0.99\frac{1}{a}$$

• Hantzsche-Wendt Space E_6

$$E_6 = -0.32\frac{1}{a},$$

and show the same trend as in positive curvature case. The corresponding Casimir energies are negative which correspond to attractive forces [25][26]. This seems to be the generic behaviour associated to the topological contributions to vacuum energy.

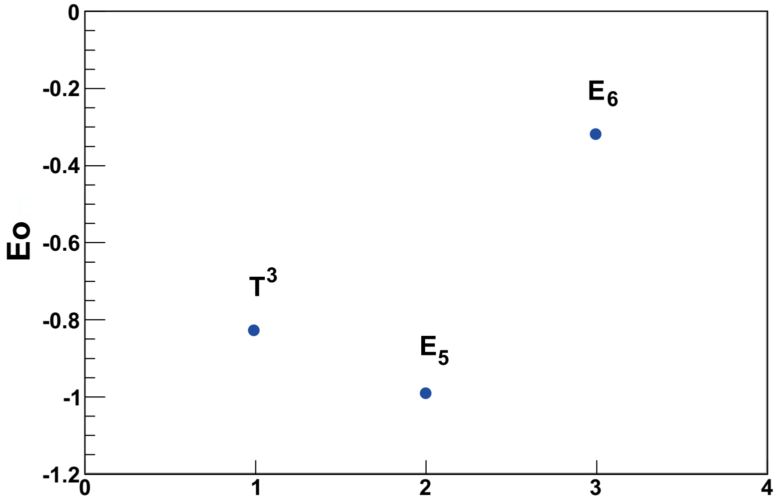

Figure 2 Casimir energies for the flat three-dimensional torus T^3, twisted sixth-turn torus E_5 and Hantzsche-Wendt Space E_6.

5 Cosmological implications

If we consider the time evolution of the space-time structure, the conformal factor a evolves in an accelerated manner, according to the current cosmological LCDM model. This implies that the quantum vacuum energy of conformal scalar fields also increases because in most of the cases the Casimir energy is negative (for higher spins the topological Casimir energy is positive for some topologies). The gravitational back-reaction to this increase of energy results in a tiny deceleration of the cosmological expansion. However, this quantum contribution is very tiny in the current Universe, although it could have played a relevant role in the early stages of the Universe evolution. The form of the Casimir energy density is very similar to the radiation component of the total energy density of the Universe. However, the pressure components are very different.

Now, because of the ambiguity which appears in the renormalization of vacuum energy it can always be chosen to be in a repulsive regime resulting into an extra acceleration of space metric. However, the renormalization origin of this behaviour is masking the real gravitational effect of quantum field fluctuations.

The decrease of energy can be compensated by particle creation [27]. Although the Zeldovich-Starobinsky condition prevents pair creation for conformally invariant theories [28], in the case of compact spaces if the size of the space is smaller than

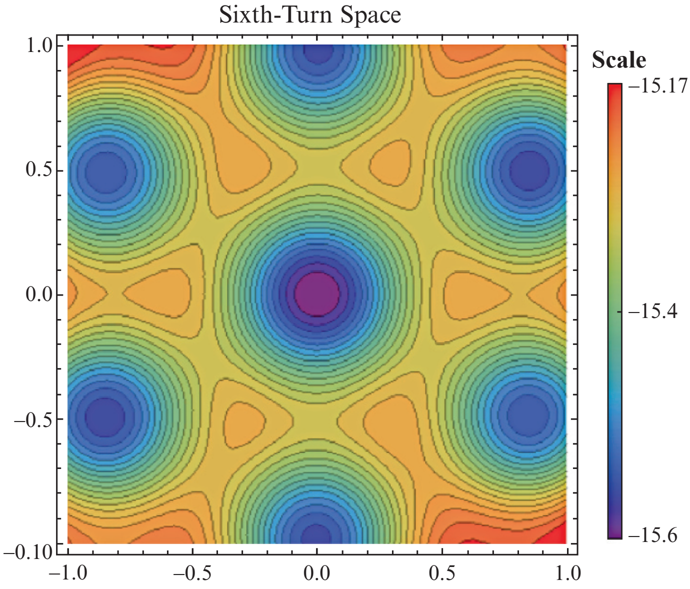

Figure 3 Casimir energies density of confomal scalars on sixth-turn flat space E_5 restricted to the fundamental domain.

the Hubble radius the phenomenon can occur [29]. The spectrum of the corresponding radiation is given by the thermal Gibbons-Hawking spectrum with temperature $T = \hbar H/2\pi k_B$, in terms of the Hubble constant and Bolthmann parameter. A realistic scenario compatible with current observations requires that the size of space is slightly smaller than the Hubble radius, in order to fit close to the Hubble horizon and still allow for pair particle creation.

As we have shown only in the case of flat compact topologies the quantum contribution to vacuum energy is universal. In those topologies this vacuum energy is anisotropic and correlated to the locations of CMB circles in the sky. Only in that case the new cosmological observations will provide crucial clues to understand the topological structure of the Universe.

Acknowledgements This work has been partially supported by the Spanish CICYT grant FPA2009-09638 and DGIID-DGA (grant 2009-E24/2).

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The Low Temperature Corrections to the Casimir Force Between a Sphere and a Plane

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Abstract We calculate the low temperature corrections to the free energy for a sphere in front of a plane. First, the scalar field obeying Dirichet or Neumann boundary conditions is considered. Second, the electromagnetic field is studied, the sphere being perfectly conducting and being a dielectric ball with both, constant permittivity and permittivity of the plasma model.

1 Introduction

During the past decade significant attention was payed to the Casimir effect at finite temperature. The interest was triggered by the desire to measure the temperature dependent part of the force (see [KMM09], section 4.D, for a review) and by the conceptual problems arising for $T \to 0$ with some thermodynamic quantities (see [KMM09], section 2.D). Recent interest came also from the interplay between temperature and geometry investigated in [GW10, WG10] using world line methods. For a scalar field with Dirichlet boundary conditions on the interacting surfaces, a generic behavior $\sim T^4$ of the temperature dependent part of the force was found and attributed to open geometry.

In our previous paper [BP09] we investigated the interaction between a ball and a plane for both, the scalar and the electromagnetic fields. We used the exact functional determinant method (also called scattering approach or 'TGTG'-formula) and focused on the limit of small separation. We showed that the Proximity Force Approximation (PFA) is reproduced exactly for medium and high temperature. For low temperature, the leading order of the free energy is the vacuum energy (for which

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_4,

the PFA is reproduced, of course) and the temperature dependent part is a small addendum for which the PFA does not hold.

In the present paper we calculate the leading behavior of the temperature dependent part $\Delta_T\mathscr{F}$ of the free energy and that of the force, $\Delta_T f$, for $T \to 0$. Again, we consider the scalar and the electromagnetic fields. For the latter we allow for dispersion including fixed permittivity and permittivity following from the plasma model (see (30) below). We use the functional determinant method and truncate the orbital momentum sum at some finite l_m. It turns out that the limit of $l_m \to \infty$ shows a sensitive dependence on the separation a for $a \to 0$. Due to the space limitations we will be quite brief concerning technical details. These will be given in a separate paper.

We mention that the interplay of geometry and temperature in sphere-plane geometry was also studied in [CDMNLR10] and related papers using the scattering approach. In a numerical analysis the ambient temperature was fixed, while the radius of the sphere and plasma frequency varied.

Throughout the paper we use units with $\hbar = c = 1$.

2 The free energy at finite temperature

In this section we collect the basic formulas for the free energy in functional determinant representation at finite temperature. We follow closely the notations used in [BP09]. At finite temperature, the free energy is given by

$$\mathscr{F} = \frac{T}{2} \sum_{n=-\infty}^{\infty} \mathrm{Tr} \ln \left(1 - \mathbf{M}(\xi_n)\right), \tag{1}$$

where $\xi_n = 2\pi n T$ are the Matsubara frequencies. The matrix $\mathbf{M}$ results from the scattering on the sphere and will be described below together with the meaning of the trace. The sum over the Matsubara frequencies can be transformed into integrals using the well known Abel-Plana formula. The free energy separates into two pieces,

$$\mathscr{F} = E_0 + \Delta_T \mathscr{F}, \tag{2}$$

where

$$E_0 = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} \, \mathrm{Tr} \ln \left(1 - \mathbf{M}(\xi)\right) \tag{3}$$

is the vacuum energy, i.e., the free energy at zero temperature, and

$$\Delta_T \mathscr{F} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} \, n_T(\xi) \, i \, \mathrm{Tr} \left[\ln(1 - \mathbf{M}(i\xi)) - \ln(1 - \mathbf{M}(-i\xi))\right] \tag{4}$$

is the temperature dependent part of the free energy containing the Boltzmann factor $n_T(\xi) = 1/(\exp(\xi/T) - 1)$.

For the scalar field, $\mathbf{M}(\xi)$ in (1) is a matrix in the orbital momentum indices l and l' with matrix elements

$$M_{l,l'}(\xi) = d_l(\xi R)\sqrt{\frac{\pi}{4\xi L}} \sum_{l''=|l-l'|}^{l+l'} K_{\nu''}(2\xi L)\, H_{ll'}^{l''} \,. \tag{5}$$

Here, the function $d_l(x)$ results from the T-matrix for the scattering on the sphere. For Dirichlet and Neumann boundary conditions on the sphere we note

$$d_l^{\mathrm{D}}(x) = \frac{I_\nu(x)}{K_\nu(x)}, \quad d_l^{\mathrm{N}}(x) = \frac{(I_\nu(x)/\sqrt{x})'}{(K_\nu(x)/\sqrt{x})'} \,. \tag{6}$$

In these formulas, R is the radius of the sphere, L is the separation between the plane and the center of the sphere, $I_\nu(x)$ and $K_\nu(x)$ are the modified Bessel functions. We introduced the notations $\nu = l+1/2$, $\nu' = l'+1/2$ and $\nu'' = l''+1/2$, which will be used throughout the paper.

The factors $H_{ll'}^{l''}$ in (5) result from the translation formulas. Their explicit form is

$$H_{ll'}^{l''} = \sqrt{(2l+1)(2l'+1)}(2l''+1) \begin{pmatrix} l & l' & l'' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & l' & l'' \\ m & -m & 0 \end{pmatrix}, \tag{7}$$

where the parentheses denote the $3j$-symbols.

The above formulas are for Dirichlet boundary conditions on the plane. For Neumann boundary conditions on the plane we have to reverse the sign in the logarithm in (4) or, equivalently, to change the sign of $\mathbf{M}$. The trace in (1),

$$\mathrm{Tr} = \sum_{m=-l_m}^{l_m} \sum_{l=m}^{l_m} \,, \tag{8}$$

is over the orbital momenta truncated at some l_m. Of course, the final expression appears for $l_m \to \infty$.

For the electromagnetic field, the matrix $\mathbf{M}$ is in addition a matrix in the two polarizations. These correspond to the TE and the TM modes in spherical geometry and we can represent the corresponding matrix elements $\mathbb{M}_{ll'}$ as matrixes (2x2),

$$\mathbb{M}_{l,l'} = \sqrt{\frac{\pi}{4\xi L}} \sum_{l''=|l-l'|}^{l+l'} K_{\nu''}(2\xi L) H_{ll'}^{l''} \begin{pmatrix} \Lambda_{l,l'}^{l''} & \tilde{\Lambda}_{l,l'} \\ \tilde{\Lambda}_{l,l'} & \Lambda_{l,l'}^{l''} \end{pmatrix} \begin{pmatrix} d_l^{\mathrm{TE}}(\xi R) & 0 \\ 0 & -d_l^{\mathrm{TM}}(\xi R) \end{pmatrix} \tag{9}$$

with the factors

$$\Lambda_{ll'}^{l''} = \frac{\frac{1}{2}\left[l''(l''+1) - l(l+1) - l'(l'+1)\right]}{\sqrt{l(l+1)l'(l'+1)}}, \quad \tilde{\Lambda}_{ll'} = \frac{2m\xi L}{\sqrt{l(l+1)l'(l'+1)}}, \tag{10}$$

which follow from the translation formulas for the vector field. The factors resulting from the scattering T-matrices are

$$d_l^{\rm TE}(x)=\frac{I_\nu(x)}{K_\nu(x)}, \quad d_l^{\rm TM}(x)=\frac{(I_\nu(x)\sqrt{x})'}{(K_\nu(x)\sqrt{x})'}. \tag{11}$$

When inserting these expressions into (1) or (4), the trace must be taken also over the polarizations and the orbital momentum sum is restricted by $l \geq \max(1,|m|)$.

3 The low temperature expansion

Due to the Boltzmann factor in (4), the low temperature expansion emerges from the expansion, for $\xi \to 0$, of

$$\mathbf{M}(\xi)=\mathbf{M}_0+\mathbf{M}_1\,(L\xi)^1+\mathbf{M}_2\,(L\xi)^2+\mathbf{M}_3\,(L\xi)^3+\dots. \tag{12}$$

The coefficients $\mathbf{M}_i=\mathbf{M}_i(\rho)$ are dimensionless functions of the ratio

$$\rho=\frac{R}{L}. \tag{13}$$

Inserting the expansion (12) into the trace of the logarithm and keeping only the first two odd orders we get

$$\operatorname{Tr}\ln(1-\mathbf{M}(\xi))=N_1(\rho)L\xi+N_3(\rho)(L\xi)^3+\dots \tag{14}$$

with

$$\begin{aligned} N_1 &= -\operatorname{Tr}\left[(1-M_0)^{-1}M_1\right], \qquad (15)\\ N_3 &= -\operatorname{Tr}\left[(1-M_0)^{-1}M_3\right]\\ &\quad -\operatorname{Tr}\left[(1-M_0)^{-1}M_1\,(1-M_0)^{-1}M_2\right]-\frac{1}{3}\operatorname{Tr}\left[\left((1-M_0)^{-1}M_1\right)^3\right], \end{aligned}$$

which are functions of ρ like the $\mathbf{M}_i$'s.

It must be mentioned that inserting (15) into (4) we interchange the orders of the limits $T\to 0$ and $l_m\to\infty$. Below we will see in which cases this is justified and in which it is not. With the expansion (15), the low-T contributions to the free energy (4) are

$$\Delta_T\mathscr{F}=-\frac{\pi}{6}N_1(\rho)\,LT^2+\frac{\pi^3}{15}N_3(\rho)\,L^3T^4+\dots. \tag{16}$$

The corresponding contributions to the force are

$$\Delta_T f\equiv-\frac{\mathrm{d}}{\mathrm{d}L}\Delta_T\mathscr{F}=\frac{\pi}{6}\frac{\mathrm{d}\,(LN_1(\rho))}{\mathrm{d}L}T^2-\frac{\pi^3}{15}\frac{\mathrm{d}\,(L^3N_3(\rho))}{\mathrm{d}L}T^4+\dots. \tag{17}$$

As it will turn out there is only one contribution $\sim T^2$ to the force (Section 4.2, below) and in all other examples considered in this paper the low-T expansion starts from T^4. This is in agreement with the findings of [WG10]. In order to compare the results we expand (17) for small separation $a = L - R$,

$$\Delta_T f = \left(c_2 R^3 + c_3 a R^2\right) T^4 + \dots, \tag{18}$$

where the coefficients c_2 and c_3 were introduced in the same way as in [WG10]

4 Results for hard boundary conditions on the sphere

In this section we consider hard boundary conditions on both, the plane and the sphere. We start with the scalar field. Here we have Dirichlet (D) or Neumann (N) boundary conditions and we denote the four combinations by (X,Y), where X stands for the sphere and Y stands for the plane. For example, (D,N) denotes Dirichlet boundary condition on the sphere and Neumann boundary conditions on the plane. We remind that Neumann boundary conditions on the plane appear from reversing the sign of $\mathbf{M}$.

4.1 The case DD

In this case we have a non-zero

$$N_1 = \rho\,, \tag{19}$$

which is independent on the truncation l_m. This is the only case where one of the function N_1 or N_3 considered in this paper does not depend on the truncation.

The function N_1, (19), delivers the T^2-contribution to the free energy which was found in [BP09]. It does not contribute to the force since the dependence on L drops out. The next-order contribution is N_3. It is a rational function of ρ. The orders of the

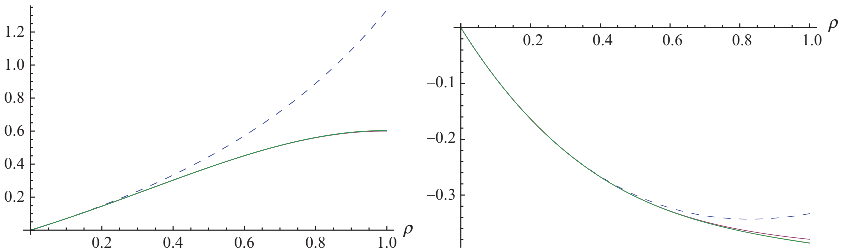

Fig. 1 The functions $N_3(\rho)$ for the case (D,D) (left panel) and $N_1(\rho)$ for the case (D,N) (right panel) for several values of the truncation l_m. The limit of small separation corresponds to $\rho = 1$. The dashed line corresponds to $l_m = 0$, i.e., to the pure s-wave contribution. Already for $l_m \geq 1$ there is nearly no dependence on l_m.

polynomials in numerator and denominator grow with the order l_m of the truncation. This is a general feature and holds for all functions N_1 and N_3 considered below except for those which vanish.

We display $N_3(\rho)$ as a function of ρ in Fig. 1 (left) for several l_m. It is seen that already for $l_m \geq 1$ the curves cease to change. In this way the free energy and the force have a well defined limit for $l_m \to \infty$. The coefficients c_2 and c_3 defined in (18) are shown in Table 4.1 and it is seen that these fit well to those found in [WG10].

At large separation, i.e., for a small sphere, only the lower orbital momenta are on work. The function N_1 is given by (19), the function N_3 by

$$N_3 = \frac{2}{3}\rho + \frac{1}{3}\rho^2 - \frac{1}{6}\rho^3 + O(\rho^4)\,. \tag{20}$$

In this way, the leading order temperature correction to the free energy and to the force do not depend on separation, while the subleading, $\sim T^4$, correction does.

l_m	0	1	2	3	4	5	6	7	8
c_2	-2.756	-3.748	-3.770	-3.772	-3.772	-3.772	-3.772	-3.772	-3.772
c_3	-5.512	-2.910	-2.500	-2.429	-2.426	-2.427	-2.426	-2.425	-2.425

Table 1 The values of the coefficients c_2 and c_3 defined in (18) for the case (D,D) for several values of the truncation l_m. The corresponding values found in [WG10], (25), are $c_2 = -3.96$ and $c_3 = -2.7$.

4.2 The case DN

In this case, as in the previous one, the dominating contribution is N_1. However, now it depends on the order l_m of truncation. The first two orders are

$$N_1(\rho)_{|_{l_m=0}} = \frac{\rho(-2+\rho)}{2+\rho}\,, \quad N_1(\rho)_{|_{l_m=1}} = \frac{\rho(-16+8\rho-4\rho^3+\rho^4)}{16+8\rho+4\rho^3+\rho^4}\,. \tag{21}$$

In Fig. 1 (right panel), N_1 is shown as function of ρ for several values of l_m. It is seen that there is a rapid convergence for large l_m. The sign is reversed as compared to the case (D,D) like for parallel plates.

Because of the more involved dependence on ρ as compared to (19), N_1 contributes to the force. Hence, for these boundary conditions, according to (17), we have a contribution $\sim T^2$ to the force.

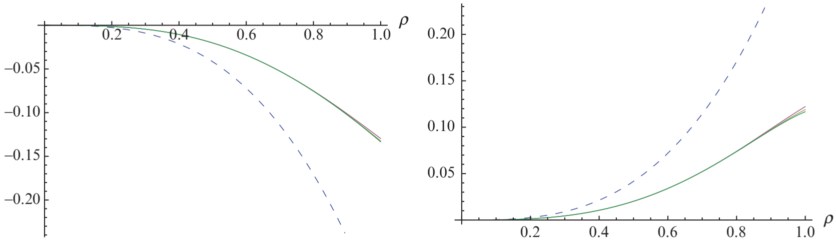

Fig. 2 The functions $N_3(\rho)$ for the cases (N,D) (left panel) and (N,N) (right panel) for several values of the truncation l_m. The limit of small separation corresponds to $\rho = 1$. The dashed line corresponds to $l_m = 0$, i.e., to the pure s-wave contribution. Already for $l_m \geq 1$ there is nearly no dependence on l_m.

4.3 The cases ND and NN

In these two cases, which have Neumann boundary conditions on the sphere, the contribution of N_1 is zero for all l_m and the expansion starts with N_3. These functions share the common features discussed above. We displayed both cases in Fig. 2.

We mention that in [BP09] we considered only the contribution from $l_m = 0$ and missed the higher order terms.

The expansions for large separation is

$$\begin{aligned} N_3^{(\mathrm{ND})} &= -\frac{1}{6}\rho^3 + \frac{1}{24}\rho^6 - \frac{1}{192}\rho^9 + O(\rho^{11}), \\ N_3^{(\mathrm{NN})} &= \frac{1}{6}\rho^3 - \frac{1}{24}\rho^6 - \frac{1}{384}\rho^9 + O(\rho^{11}). \end{aligned} \tag{22}$$

The first two terms are the same (up to the sign), higher orders are different.

4.4 The electromagnetic field with conductor boundary conditions

For the electromagnetic field, the matrix elements $\mathbb{M}_{ll'}$ are given by (9). We expand them in powers of ξ as before and obtain an expansion in parallel to (12),

$$\mathbb{M}(\xi) = \mathbb{M}_0 + \mathbb{M}_1\,(L\xi)^1 + \mathbb{M}_2\,(L\xi)^2 + \mathbb{M}_3\,(L\xi)^3 + \dots. \tag{23}$$

It turns out that the matrixes $\mathbb{M}_i$ are diagonal in the polarizations for $i = 0$ and $i = 2$ and anti-diagonal for $i = 1$. Therefore in (14), the first contribution, N_1, vanishes and from N_3 only the term in (15) does not vanish. From this structure it follows also that the contributions from the two polarizations do not mix in N_3 and we can consider these separately. Now the further calculations go in the same way as for the scalar field and we have calculated the functions $N_3(\rho)$ for both polarizations.

We displayed them in Fig. 3 While the TE mode gives a result similar to the scalar

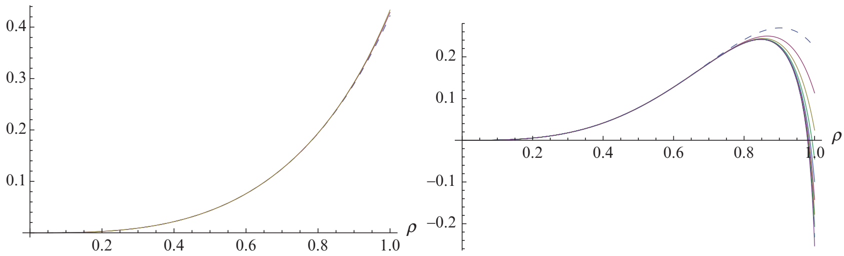

Fig. 3 The functions $N_3(\rho)$ for the electromagnetic field with conductor boundary conditions for the TE polarization (left panel) and for the TM polarization (right panel) for several values of the truncation l_m. The limit of small separation corresponds to $\rho = 1$. The dashed line corresponds to $l_m = 1$, i.e., to the pure p-wave contribution which is the lowest one in the electromagnetic case. In the TE case, already for $l_m \geq 1$, there is nearly no dependence on l_m. For the TM case, at $\rho \lesssim 1$, there is no convergence for growing l_m. We displayed until $l_m = 10$.

case in the sense that the limit $l_m \to \infty$ is approached very fast, the picture for the TM mode is different. Here we observe, for ρ close to unity, $\rho \lesssim 1$, contributions growing with l_m. This must be interpreted as a noncommutativity of the limits $T \to 0$ and $l_m \to \infty$. As a consequence, at small separation, we have to expect contributions decreasing for $T \to 0$ slower than T^4.

For large separation we find the following expansions,

$$\begin{aligned} N_3^{\mathrm{TE}} &= \frac{1}{3}\rho^3 + \frac{1}{12}\rho^6 + \frac{1}{192}\rho^9 + O(\rho^{11}), \\ N_3^{\mathrm{TM}} &= \frac{2}{3}\rho^3 - \frac{1}{3}\rho^6 - \frac{1}{12}\rho^9 + O(\rho^{11}). \end{aligned} \tag{24}$$

According to (16), the leading order contribution to the free energy is

$$\Delta_T \mathscr{F} = \frac{\pi^3}{15} R^3 T^4 - \frac{\pi^3}{60} \frac{R^6}{L^3} T^4 + \dots \tag{25}$$

and we observe a T^4-contribution to the force (from the second term). The first term coincides with the corresponding term in (6) in [CDNLR10] while the second is beyond of what is displayed there.

5 Results for a dielectric ball in front of a conducting plane

For a dielectric ball the formulas of section 2 remain valid except for the functions $d_l^{\mathrm{TX}}(x)$, (11). These must be substituted by

$$d_l^{\mathrm{TE}}(z) = \frac{2}{\pi} \frac{\sqrt{\varepsilon}\, s_l(x) s_l'(nx) - \sqrt{\mu}\, s_l'(x) s_l(nx)}{\sqrt{\varepsilon}\, e_l(x) s_l'(nx) - \sqrt{\mu}\, e_l'(x) s_l(nx)} \tag{26}$$

with the refraction index $n = \sqrt{\varepsilon\mu}$ and $s_l(x) = \sqrt{\pi x/2}\, I_\nu(x)$ and $e_l(x) = \sqrt{2x/\pi}\, K_\nu(x)$ are the modified spherical Bessel functions. The function $d_l^{\mathrm{TM}}(z)$ can be obtained from (26) by interchanging ε and μ.

Inserting these formulas into (9) and calculating the entries in (23) we see that the structure of the matrixes $\mathbb{M}_i$ remains the same. In this way we have a separation into TE and TM modes as before. As a consequence, we have only T^4 contributions.

Now we calculate the function $N_3(\rho)$ for the case of a fixed ε and for an ε taken from the plasma model.

5.1 Fixed permittivity ε

For fixed ε we consider two cases. First we put $\mu = 1$. In this case it turns out that $N_3(\rho) = 0$ for the TE mode. This means that the corresponding low-T expansion starts from a higher power in T which we do not consider here. For the TM mode the function $N_3(\rho)$ is shown in Fig. 4 (left panel). It depends on the truncation. For ε close to unity it stabilizes rapidly, for higher ε slower.

At large distances, $\rho \to 0$, we found

$$N_3^{\mathrm{TM}} = \frac{2(\varepsilon-1)}{3(\varepsilon+2)}\rho^3 - \frac{(\varepsilon-1)^2}{3(\varepsilon+2)^2}\rho^6 - \frac{(\varepsilon-1)^3\rho^9}{12(\varepsilon+2)^3}\rho^9 + O\left(\rho^{11}\right) \tag{27}$$

In dilute approximation, $\varepsilon = 1+\delta$, $\delta \ll 1$, only the lowest orbital momenta contribute until the order quadratic in δ,

$$N_3^{\mathrm{TM}} = \frac{2}{9}\rho^3\delta - \frac{1}{27}\rho^3(2+\rho^3)\delta^2 + \mathscr{O}(\delta^3), \tag{28}$$

higher orders are more complicated to obtain.

As the second case we consider $\varepsilon = 1$. Here the contribution of the TM mode to $N_3(\rho)$ is zero and we are left with the TE contribution. This function is very similar to that in the first case, however, different in details. It is shown in Fig. 4 (right panel). It stabilizes much faster when lifting the truncation as in the previous case. For large separations it reads

$$N_3^{\mathrm{TE}} = -\frac{2(\mu-1)}{3(\mu+2)}\rho^3 + \frac{(\mu-1)^2}{3(\mu+2)^2}\rho^6 - \frac{(\mu-1)^3}{24(\mu+2)^3}\rho^9 + O\left(\rho^{10}\right). \tag{29}$$

The difference (up to the sign) starts with order ρ^9. In dilute approximation we found in the first two orders the same expression as in (28) with reversed sign. Differences show up starting from the third order.

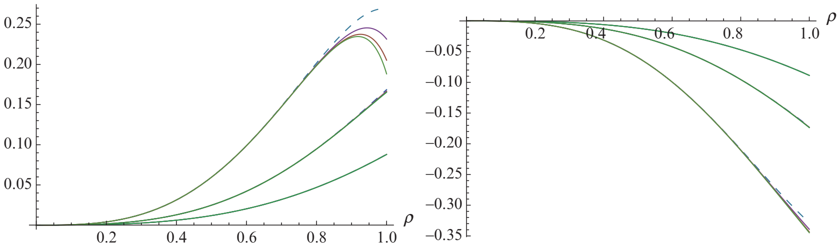

Fig. 4 The functions $N_3^{\mathrm{TM}}(\rho)$ for the dielectric ball (left panel) with $\mu = 1$, $\varepsilon = 1.5$ (lower curve), $\varepsilon = 2.3$ and $\varepsilon = 10$, and $N_3^{\mathrm{TE}}(\rho)$ (right panel) and with $\varepsilon = 1$, $\mu = 1.8$ (upper curve), $\mu = 2.3$ and $\mu = 10$. The dashed line is for $l_m = 1$.

5.2 Plasma model permittivity

The permittivity derived within the plasma model for metals is

$$\varepsilon = 1 + \frac{\omega_p^2}{\xi^2}, \tag{30}$$

where ω_p is the plasma frequency and ε is taken for imaginary frequency ξ. Inserting (30) into (26), the calculation runs as in the previous cases. The results are shown in Fig. 5

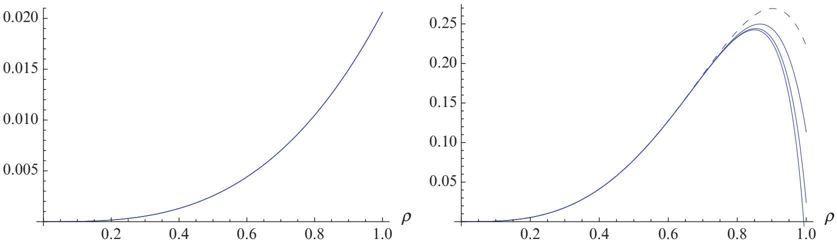

Fig. 5 The functions $N_3(\rho)$ for the dielectric ball with dispersion (30) of the plasma model for $\omega_p = 1$. For the TE mode (left panel) the convergence for $l_m \to \infty$ is rapid for all separations. For the TM mode (right panel), there is no convergence for $l_m \to \infty$ at small separation, i.e., for $\rho \lesssim 1$. The dashed curve corresponds to $l_m = 1$.

For the TE mode, the curves cease to change already for $l_m = 2$ within the precision of the plot. The analytic expressions are rational function of ρ and of hyperbolic functions of ω_p. For small ω_p we observe

$$N_3^{\mathrm{TE}}(\rho) = \frac{\rho^3 \omega_p^2}{45} + O(\omega_p^3). \tag{31}$$

A different picture we observe for the TM mode. Here the truncation can be removed for $\rho < 1$ only. At close separation, i.e., for $\rho \lesssim 1$, the contributions do not tend to a finite limit for $l_m \to \infty$. Again, we have to interpret this a non-commutativity of the limits $t \to 0$ and $l_m \to \infty$. Hence we expect for small separation lower powers in T. Also the behavior for $\omega_p \to 0$ is nonanalytic in the sense, that $N_3(\rho)^{\mathrm{TM}}$ does not vanish in this limit,

$$N_3^{\mathrm{TM}}(\rho) = \frac{\rho^3(-4+3\rho^2)}{3(-4+\rho^3)} + O(\omega_p)\,. \tag{32}$$

At large distances, the leading correction to the free energy for the TM mode does not depend on the plasma frequency,

$$N_3^{\mathrm{TM}} = \frac{2\rho^3}{3} + \mathscr{O}(\rho^5).$$

While for the TE mode the correction is sensitive to small plasma frequencies,

$$N_3^{TE} = \left(\frac{1}{3} + \frac{1}{\omega_p^2} - \frac{\coth(\omega_p)}{\omega_p}\right)\rho^3 + \mathscr{O}(\rho^5)\,,$$

but saturates at $1/3$ when $\omega_p \to \infty$.

6 Conclusions

In the forgoing sections we calculated the low temperature expansion of the free energy for a sphere or a dielectric ball in front of a plane. For the temperature dependent part $\Delta_T\mathscr{F}$ of the free energy we used the representation (4) involving the Boltzmann factor. Further we used a truncation of the orbital momentum sum, $l \le l_m$, and interchanged the limits $T \to 0$ and $l_m \to \infty$. After that, the low-T expansion is obtained simply by expanding the matrices $\mathbf{M}$ into powers of ξ and taking the lowest odd one. In this way, the low-T expansion takes the generic form

$$\Delta_T\mathscr{F} = \mathscr{F}_2 T^2 + \mathscr{F}_4 T^4 + \dots\,. \tag{33}$$

The coefficient $\mathscr{F}_2$ is present for the cases (D,D), section 4.1 (but independent on the separation) and (D,N), section 4.2. It is zero in all other cases where $\mathscr{F}_4$ is the leading order contribution.

For all examples considered in this paper, at finite separation, $\mathscr{F}_2$ and $\mathscr{F}_4$ have a finite limit for $l_m \to \infty$. Hence the generic low-T behavior is given by (33). This holds, for instance, at large separation. A different picture appears for small separation, $\rho \to 1$. In some cases, the closer the separation, the worse the convergence for $l_m \to \infty$. In these cases we do not have a result for $T \to 0$. However, we can expect lower powers of T to appear. This is a topic of future investigations.

We delegate this to a separate e-print. Also, a number of questions, especially on the non-convergence for $l_m \to \infty$, is left opened for future research.

Acknowledgements This work was supported by the Heisenberg-Landau program. The authors benefited from exchange of ideas by the ESF Research Network CASIMIR. I.P. acknowledges partial financial support from FRBR grants 09-02-12417-ofi-m and 10-02-01304-a.

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Casimir Effect for the Piecewise Uniform String

Iver Brevik

Abstract The Casimir energy for the transverse oscillations of a piecewise uniform closed string is calculated. In its simplest version the string consists of two parts I and II having in general different tension and mass density, but is always obeying the condition that the velocity of sound is equal to the velocity of light. The model, first introduced by Brevik and Nielsen in 1990, possesses attractive formal properties implying that it becomes easily regularizable by several methods, the most powerful one being the contour integration method. We also consider the case where the string is divided into $2N$ pieces, of alternating type-I and type-II material. The free energy at finite temperature, as well as the Hagedorn temperature, are found. Finally, we make some remarks on the relationship between this kind of theory and the theory of quantum star graphs, recently considered by Fulling *et al.*.

1 Introduction

Standard theory of closed strings - whatever the string is situated in Minkowski space or in superspace - assumes the string to be homogeneous, i.e. that the tension T is the same everywhere. The composite string model, in which the string is taken to consist of two or more separately different pieces, is a generalization of the usual model. An important condition that we will impose, is that the composite string is relativistic in the sense that the velocity v_s of transverse sound is everywhere assumed to be equal to the velocity of light,

$$v_s = \sqrt{T/\rho} = c = 1. \tag{1}$$

Here T, as well as the mass density ρ, refer to the string piece under consideration. At each junction there are two boundary conditions, namely (i) the transverse

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_5,

displacement $\psi = \psi(\sigma, \tau)$ is continuous, and (ii) the transverse force $T\partial\psi/\partial\sigma$ is continuous. Using the equation of motion

$$\left(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2}\right)\psi = 0 \tag{2}$$

one can calculate the eigenvalue spectrum and the Casimir energy of the string.

The simplest string model of this type is when there are only two pieces, of length L_I and L_{II}, such that the total length is $L = L_I + L_{II}$; see Fig. 1. This model

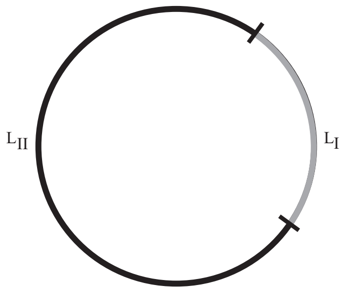

Fig. 1 The two-piece string, with piece lengths L_I and L_{II}.

was introduced in 1990 [1]; cf. also the related paper [2]. The Casimir energy was calculated for various length ratios of the pieces. It is convenient to introduce a symbol s for the length ratio, and also a symbol x for the tension ratio,

$$s = \frac{L_{II}}{L_I}, \quad x = \frac{T_I}{T_{II}}. \tag{3}$$

With moreover the function $F(x)$ defined as

$$F(x) = \frac{4x}{(1-x)^2}, \tag{4}$$

the dispersion relation becomes

$$F(x)\sin^2\left(\frac{\omega L}{2}\right) + \sin\omega L_I \sin\omega L_{II} = 0, \tag{5}$$

and the Casimir energy, describing the deviation from homogeneity, can be written formally as

$$E = E_{I+II} - E_{\text{uniform}} = \frac{1}{2}\sum\omega_n - E_{\text{uniform}}. \tag{6}$$

Since Eq. (5) is invariant under the substitution $x \to 1/x$, we can simply assume $x \leq 1$ in the following.

From a physical point of view, there is well-founded hope that this simple string model can help us to understand the issue of the energy of the vacuum state in two-dimensional quantum field theories in general. The system is strikingly easy to regularize, this being due to the relativistic property of the model. The mentioned paper [1] made use of a cutoff regularization method, whereby a function $f = \exp(-\alpha\omega)$ with α a small positive parameter was introduced. A second regularization method - the one to be dealt with in this paper - is the complex contour integration method. To our knowledge this method was first applied to the composite string model by Brevik and Elizalde [3]. A separate chapter is devoted to this model in Elizalde's book on zeta functions [4]. The great advantage of this method is that the *multiplicities* of the zeros of the dispersion function are automatically taken care of. There exists also a third convenient regularization method implying the use of the Hurwitz zeta function.

Instead of assuming only two pieces in the composite string, one can imagine that the string is composed of $2N$ pieces, all of the same length, such that the type I materials and the type II materials are alternating. Maintaining the same relativistic property as before, one will find that also this kind of system is easily regularizable and tractable analytically in general. There are by now several papers devoted to the study of the composite string in its various facets; cf. [5, 6, 7, 8, 9, 10, 11, 12] (the last of these references gives a review). As for possible applications of the model, we may also mention the paper of Lu and Huang [13], discussing the Casimir energy for a composite Green-Schwarz superstring.

In the following we review briefly the main properties of the composite string model, at zero, and also at finite, temperature, making use of the contour regularization method as mentioned. The convenience of the recursion formula in the $2N$-case is in our opinion worth attention. The quantum theory of the two-piece string for the simplifying limiting case of very small tension ratio x between the two pieces is highlighted, and the Hagedorn temperature is given for this kind of model. Finally, we comment upon the connection between the theory of the piecewise uniform string and the theory of quantum star graphs, recently developed by Fulling and others.

2 The two-piece string

According to the argument principle one has for any meromorphic function $g(\omega)$:

$$\frac{1}{2\pi i}\oint \omega \frac{d}{d\omega} \ln g(\omega) d\omega = \sum \omega_0 - \sum \omega_\infty, \tag{7}$$

where ω_0 are the zeros and ω_∞ are the poles of $g(\omega)$ inside the integration contour. As usual the contour is a semicircle of large radius R in the right half complex ω plane, closed by a straight line from $\omega = iR$ to $\omega = -iR$. A convenient choice for the dispersion function $g(\omega)$ is

$$g(\omega) = \frac{F(x)\sin^2[(s+1)\omega L_I/2] + \sin(\omega L_I)\sin(s\omega L_I)}{F(x)+1}. \tag{8}$$

The final result at zero temperature becomes [3]

$$E = \frac{1}{2\pi}\int_0^\infty \ln\left|\frac{F(x) + \frac{\sinh(\xi L_I)\sinh(s\xi L_I)}{\sinh^2[(s+1)\xi L_I/2]}}{F(x)+1}\right| d\xi, \tag{9}$$

with $\omega = i\xi$. This expression holds for all values of s, not necessarily integers. Since it is invariant under the interchange $s \to 1/s$, we can consider only the interval $s \geq 1$ without any loss of generality. If the tension ratio $x \to 0$, we find the simple formula

$$E = -\frac{\pi}{24L}\left(s + \frac{1}{s} - 2\right). \tag{10}$$

At finite temperatures, where $\xi_n = 2\pi nT$ with $n = 0,1,2,3..$ are the Matsubara frequencies, we get the corresponding expression

$$E(T) = T\sum_{n=0}^{\infty}{}' \ln\left|\frac{F(x) + \frac{\sinh(\xi_n L_I)\sinh(s\xi_n L_I)}{\sinh^2[(s+1)\xi_n L_I/2]}}{F(x)+1}\right|, \tag{11}$$

where the prime means that the case $n = 0$ is counted with half weight.

We may define two characteristic frequencies in the problem: (i) the thermal frequency $\omega_T = T = \xi_1/(2\pi)$, and (ii) the geometric frequency $\omega_{\text{geom}} = 2\pi/L_I$. The case of high temperatures corresponds to $\frac{\omega_T}{\omega_{\text{geom}}} \geq 1$, whereby we can approximate

$$E(T) = \frac{1}{2}T\ln\left|\frac{F(x)+4s/(s+1)^2}{F(x)+1}\right|. \tag{12}$$

Thus, if "our" universe (I) is small and the "mirror" universe (II) is large ($s \to \infty$), we have

$$E(T) = -\frac{1}{2}\ln\left|1+F(x)^{-1}\right|. \tag{13}$$

In the case of low temperatures, $\frac{\omega_T}{\omega_{geom}} \ll 1$, a large number of Matsubara frequencies becomes necessary.

3 The $2N$-piece string

Assume now that the string of length L is divided into $2N$ pieces of equal length, of alternating type I/type II material. The basic formalism for arbitrary integers N was set up in [5], although a full calculation was not worked out until [8, 9]. A key point in [8] was the derivation of a new recursion formula for the matrix of the dispersion function, applicable for general integers N.

In addition to the tension ratio $x = T_I/T_{II}$ we define two new symbols,

$$p_N = \frac{\omega L}{N}, \quad \alpha = \frac{1-x}{1+x}. \tag{14}$$

The eigenfrequencies are determined from

$$\mathrm{Det}\,[\mathbf{M}_{2N}(x, p_N) - \mathbf{1}] = 0, \tag{15}$$

where the system determinant satisfies the following recursion relation

$$\mathbf{M}_{2N}(x, p_N) = \left[\frac{(1+x)^2}{4x}\right]^N \Lambda^N(\alpha, p_N), \tag{16}$$

with

$$\Lambda(\alpha, p) = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix}, \tag{17}$$

$$a = e^{-ip} - \alpha^2, \quad b = \alpha(e^{-ip} - 1). \tag{18}$$

This property greatly facilitates the handling of the formalism. The way to proceed now is to calculate the eigenvalues of the matrix Λ, and express the elements of $\mathbf{M}_{2N}$ as powers of these. One finds

$$\lambda_\pm(iq) = \cosh q - \alpha^2 \pm [(\cosh q - \alpha^2)^2 - (1-\alpha^2)^2]^{1/2}, \tag{19}$$

where $\lambda_\pm$ are the eigenvalues of Λ for imaginary arguments iq of the dispersion equation.

The contour integration method gives for the Casimir energy ($T = 0$):

$$E_N(x) = \frac{N}{2\pi L}\int_0^\infty \ln\left|\frac{2(1-\alpha^2)^N - [\lambda_+^N(iq) + \lambda_-^N(iq)]}{4\sinh^2(Nq/2)}\right| dq. \tag{20}$$

It is seen that $E_N(x) < 0$, $|E_N(x)|$ increasing with increasing N. Division into a larger number of pieces thus diminishes the Casimir energy.

If $x \to 0$,

$$E_N(0) = -\frac{\pi}{6L}(N^2 - 1). \tag{21}$$

We could alternatively use zeta function regularization here. That would necessitate, however, solution of the eigenvalue spectrum. Degeneracies would have to be put in by hand. The latter method is therefore most convenient for low integers N.

A rather unexpected property of the system is that of *scaling invariance*. This is seen by examining the behavior of the function $f_N(x)$ defined by

$$f_N(x) = \frac{E_N(x)}{E_N(0)}, \quad 0 < f_N(x) < 1. \tag{22}$$

Numerically it turns out that the curve for $f_N(x)$ is practically the same, irrespective of the value of N, as long as $N \geq 2$. The simple analytic form

$$f_N(x) \to f(x) = (1-\sqrt{x})^{5/2} \tag{23}$$

fits the numerical values accurately, in particular in the region $0 < x < 0.45$. The reason for this behavior is not known.

At finite temperature the expression for the Casimir energy becomes

$$E_N^T(x) = T\sum_{n=0}^{\infty}{}' \ln\left|\frac{2(1-\alpha^2)^N - [\lambda_+^N(i\xi_n L/N) + \lambda_-^N(i\xi_n L/N)]}{4\sinh^2(\xi_n L/2)}\right|, \tag{24}$$

where $\lambda_\pm(i\xi_n L/N)$ are given by Eq. (19) with $q \to q_n = \xi_n L/N$. It is here useful to note that

$$\lambda_+(iq_n) + \lambda_-(iq_n) = 2(\cosh q_n - \alpha^2). \tag{25}$$

There are several special cases of interest. First, if the string is uniform ($x = 1$), we get $E_N^T(1) = 0$. This is as expected, as the Casimir energy is a measure of the string's inhomogeneity. If $N = 1$, x arbitrary, we also get a vanishing result, $E_1^T(x) = 0$. In particular, if $x \to 0$ we get the simple formula

$$E_N(0) = 2T\sum_{n=0}^{\infty}{}' \ln\left|\frac{2^N\sinh^N(\xi_n L/2N)}{2\sinh(\xi_n L/2)}\right|. \tag{26}$$

4 Oscillations of the two-piece string in D-dimensional spacetime. Quantization

We will now aim at sketching the essentials of the quantum theory of the composite string, in the case when $N = 1$ (the two-piece string). To allow for a correspondence to the superstring, we allow the number of flat spacetime dimensions D to be an arbitrary integer. In accordance with usual practice, we put now $L = L_I + L_{II} = \pi$. The theory will be based on two simplifying assumptions:

(i) The string tension ratio $x \to 0$. The dispersion relation (5) leads in this case to two different branches of solutions, namely the first branch obeying

$$\omega_n(s) = (1+s)n, \tag{27}$$

and the second branch obeying

$$\omega_n(s^{-1}) = (1+s^{-1})n, \tag{28}$$

with $n = \pm1, \pm2, \pm3, \ldots$.

(ii) The second assumption is that the length ratio s is an integer, $s = 1,2,3,....$

Let now $X^\mu(\sigma,\tau)$ with $\mu = 0,1,2,..(D-1)$ be the coordinates on the world sheet. For each branch

$$X^\mu = x^\mu + \frac{p^\mu \tau}{\pi \bar{T}(s)} + \theta(L_I - \sigma)X_I^\mu + \theta(\sigma - L_I)X_{II}^\mu, \tag{29}$$

where $\theta(x)$ is the step function, x^μ the center of mass position, and p^μ the total momentum of the string. The mean tension in the actual limit is $\bar{T}(s) = T_{II}s/(1+s)$ (we assume T_{II} finite). The string's translational energy is $p^0 = \pi\bar{T}(s)$. In the following we consider the first branch only.

In region I we make the expansion

$$X_I^\mu = \frac{i}{2\sqrt{\pi T_I}} \sum_{n\neq 0} \frac{1}{n} \left[\alpha_n^\mu(s)e^{i(1+s)n(\sigma-\tau)} + \tilde{\alpha}_n^\mu(s)e^{-i(1+s)n(\sigma+\tau)}\right], \tag{30}$$

where $\alpha_{-n}^\mu = (\alpha_n^\mu)^*$, $\tilde{\alpha}_{-n}^\mu = (\tilde{\alpha}_n^\mu)^*$. The action can be expressed as

$$S = -\frac{1}{2}\int d\tau d\sigma T(\sigma)\eta^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu, \tag{31}$$

where $T(\sigma) = T_I + (T_{II} - T_I)\theta(\sigma - L_I)$. As the conjugate momentum is $P^\mu(\sigma) = T(\sigma)\dot{X}^\mu$, we obtain the Hamiltonian

$$H = \int_0^\pi [P_\mu(\sigma)\dot{X}^\mu - L]d\sigma = \frac{1}{2}\int_0^\pi T(\sigma)(\dot{X}^2 + X'^2)d\sigma. \tag{32}$$

The fundamental condition is that $H = 0$ when applied to physical states.

The corresponding expansion of the first branch in region II is

$$X_{II}^\mu = \frac{i}{2\sqrt{\pi T_I}} \sum_{n\neq 0} \frac{1}{n}\gamma_n^\mu(s)e^{-i(1+s)n\tau}\cos[(1+s)n\sigma], \tag{33}$$

with

$$\gamma_n^\mu(s) = \alpha_n^\mu(s) + \tilde{\alpha}_n^\mu(s), \quad n \neq 0. \tag{34}$$

The condition $x \to 0$ means that there are only standing waves in region II.

We may now introduce light-cone coordinates $\sigma^- = \tau - \sigma$, $\sigma^+ = \tau + \sigma$. Some calculation shows that the total Hamiltonian can be written as a sum of two parts,

$$H = H_I + H_{II}, \tag{35}$$

where

$$H_I = \frac{1+s}{4}\sum_{-\infty}^{\infty}[\alpha_{-n}(s)\cdot\alpha_n(s) + \tilde{\alpha}_{-n}(s)\cdot\tilde{\alpha}_n(s)], \tag{36}$$

$$H_{II} = \frac{s(1+s)}{8x}\sum_{-\infty}^{\infty}\gamma_{-n}(s)\cdot\gamma_n(s). \tag{37}$$

The mass M of the string determined by $M^2 = -p^\mu p_\mu$,

$$M^2 = \pi T_{II} s \sum_{n=1}^{\infty} \left[\alpha_{-n}(s) \cdot \alpha_n(s) + \tilde{\alpha}_{-n}(s) \cdot \tilde{\alpha}_n(s) + \frac{s}{2x} \gamma_{-n}(s) \cdot \gamma_n(s) \right]. \tag{38}$$

Recall that this is the contribution from first branch only. The expression is valid for even/odd values of s.

Consider now the quantization of the first branch modes. We impose the conditions

$$T_I[\dot{X}^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = -i\delta(\sigma-\sigma')\eta^{\mu\nu} \tag{39}$$

in region I, and

$$T_{II}[\dot{X}^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = -i\delta(\sigma-\sigma')\eta^{\mu\nu} \tag{40}$$

in region II (the other commutators vanish). Then introducing creation and annihilation operators via

$$\alpha_n^\mu(s) = \sqrt{n} a_n^\mu(s), \quad \alpha_{-n}^\mu(s) = \sqrt{n} a_n^{\mu\,\dagger}(s), \tag{41}$$

$$\gamma_n^\mu(s) = \sqrt{4nx} c_n^\mu(s), \quad \gamma_{-n}(s) = \sqrt{4nx} c_n^{\mu\,\dagger}(s), \tag{42}$$

one arrives at the conventional commutation relations

$$[a_n^\mu(s), a_m^{\nu\,\dagger}(s)] = \delta_{nm}\eta^{\mu\nu}, \quad [c_n^\mu(s), c_m^{\nu\,\dagger}(s)] = \delta_{nm}\eta^{\mu\nu}, \tag{43}$$

for $n,m \geq 1$.

Now introduce $t(s)$ as

$$t(s) = \pi \bar{T}(s), \tag{44}$$

and put $D = 26$, the usual dimension for the bosonic string. The condition $H = H_I + H_{II} = 0$ leads to

$$M^2 = t(s) \sum_{i=1}^{24} \sum_{n=1}^{\infty} \omega_n(s) [a_{ni}{}^\dagger(s) a_{ni}(s) + \tilde{a}_{ni}^\dagger(s) \tilde{a}_{ni}(s) - 2]$$

$$+2st(s) \sum_{i=1}^{24} \sum_{n=1}^{\infty} \omega_n(s) [c_{ni}{}^\dagger(s) c_{ni}(s) - 1], \tag{45}$$

and the free energy becomes

$$F = -\frac{1}{24}(s + \frac{1}{s} - 2) - 2^{-40}\pi^{-26} t(s)^{-13} \int_0^\infty \frac{d\tau_2}{\tau_2^{14}} \int_{-1/2}^{1/2} d\tau_1$$

$$\times \left[\theta_3 \left(0 \middle| \frac{i\beta^2 t(s)}{8\pi^2 \tau_2} \right) - 1 \right] \left| \eta[(1+s)\tau] \right|^{-48} \eta[s(1+s)(\tau - \bar{\tau})]^{-24} \tag{46}$$

Here $\tau = \tau_1 + i\tau_2$ is the Teichmüller parameter,

$$\eta(\tau) = e^{\pi i\tau/12} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n\tau}\right) \tag{47}$$

is the Dedekind η-function, and

$$\theta_3(v|x) = \sum_{n=-\infty}^{\infty} e^{ixn^2+2\pi ivn} \tag{48}$$

is the Jacobi θ_3- function. From this the thermodynamic quantities such as internal energy U and entropy S can be calculated,

$$U = \frac{\partial(\beta F)}{\partial\beta}, \quad S = \beta^2 \frac{\partial F}{\partial\beta}. \tag{49}$$

Finally, it is of interest to write down the Hagedorn temperature $T_c = 1/\beta_c$, as the free energy $F \to \infty$ for $T > T_c$. We get

$$\beta_c = \frac{4}{s}\sqrt{\frac{\pi(1+s)}{T_{II}}} \quad (T_{II} \ \text{assumed finite}). \tag{50}$$

In the point mass limit the formalism simplifies somewhat. For dimensional reasons we must in that case have

$$F \propto 1/L_I = (1+s)/\pi \approx s/\pi. \tag{51}$$

Readers interested in a more detailed exposition of this theory may consult [10, 11, 14].

5 Final Remarks

The piecewise uniform string model is a natural generalization of the conventional uniform string. The adaptability of the formalism to various regularization schemes, in particular the contour integration method, should be emphasized. Of course, an important factor here is the assumption about relativistic invariance, as illustrated already by Eq. (1). If this assumption were removed, the formalism would be difficult to handle.

Another point worth noticing is the close connection between the relativistic invariance property and the theory of an electromagnetic field propagating in a so-called isorefractive medium meaning that the refractive index is equal to one, or at least a constant everywhere in the material system. Recent works in this direction are, for instance, [15, 16]. Again, if the isorefractive (or relativistic) condition were removed, the regularization procedure would be rather difficult to deal with, as the contact term to be subtracted off would then depend on which of the media one chooses for this purpose.

As a proposal for future work, we mention that there may be a connection between the phases of the piecewise uniform (super) string and the Bekenstein-Hawking entropy associated with this string. The entropy, as known, can be derived by counting black hole microstates, and it is natural to expect that the deviation from spatial homogeneity infaced in the composite string model could influence that sort of calculations.

Finally, we mention the interesting analogy that seems to exist between the composite string model and the so-called quantum star graph model. Fulling *et al.* [17] recently studied vacuum energy and Casimir forces in one-dimensional quantum graphs (pistons), and found that the piston force could be attractive or repulsive depending on the number of edges. It may be that the mathematical similarities between these two kinds of theories reflect a deeper physical similarity also. This remains to be explored.

Acknowledgements I thank Stephen A. Fulling for information about [17].

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$\mathscr{N} = 2$ and $\mathscr{N} = 4$ Supersymmetric Low-Energy Effective Actions in Three Dimensions

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Abstract We study the general structure of superconformal effective actions for $\mathscr{N} = 2$ and $\mathscr{N} = 4$ gauge superfields in the $\mathscr{N} = 2$, $d = 3$ superspace. Such actions may appear as the low-energy effective actions in various $\mathscr{N} = 2$ and $\mathscr{N} = 4$ supergauge models such as three-dimensional supersymmetric electrodynamics, supersymmetric matter in the gauge superfield background as well as extended SYM and Chern-Simons-matter theories. In particular, for the models of the $\mathscr{N} = 2$ and $\mathscr{N} = 4$ matter in the Abelian gauge superfield background we give the explicit calculations of the low-energy effective actions within the perturbation theory.

Dedicated to the 60 year Jubilee of Professor E. Elizalde.

1 Introduction

Three-dimensional supersymmetric models of matter and gauge superfields appear as the worldvolume field theories of D2 and M2 branes in string theory. Recent interest to such theories was inspired by the progress in constructing and studying the $\mathscr{N} = 6$ and $\mathscr{N} = 8$ supersymmetric Chern-Simons-matter models which play the role of worldvolume degrees of freedom of multiple M2 branes. Such theories are usually referred to as the Bagger-Lambert-Gustavsson (BLG) model in the $\mathscr{N} = 8$ supersymmetric case [2, 3, 4, 14, 15] and the Aharony-Bergman-Jafferis-Maldacena (ABJM) model for the case of $\mathscr{N} = 6$ supersymmetry [1].

One of the general problems for the ABJM and BLG theories is the study of the effective actions which would describe the effective dynamics of multiple M2

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_6,

branes. The effective action may receive contributions both in the sector of gauge and matter superfields. Here we consider the possible contributions in the gauge superfield sector in the Abelian case which may be induced either by the gauge or by matter superfields. As soon as the ABJM model is superconformal, the problem is reduced to the classification of superconformal invariants constructed from the gauge superfields in the $\mathcal{N}=2$, $d=3$ superspace.

The are various superspace formulations of the ABJM and BLG theories. Among them, we mark out the $\mathcal{N}=2$, $d=3$ superspace [5] and $\mathcal{N}=3$, $d=3$ harmonic superspace [6] approaches because they are based on the off-shell superfields and are best suited for quantization. In the present work we concentrate on the $\mathcal{N}=2$ superspace approach leaving the $\mathcal{N}=3$ superspace considerations for future research.

In the present work we study the general structure of the effective actions for $\mathcal{N}=2$ gauge superfield subject to the constraint of the superconformal invariance. We show that the gauge invariance and the superconformal symmetry fix the leading terms in the effective action uniquely, up to the coefficients while the higher-order terms are encoded in a single function of the $\mathcal{N}=2$ quasi-primary superfields. These considerations are very similar to the ones made in [9] for the $\mathcal{N}=2$, $d=4$ supergauge theories.

Our analysis applies not only to the effective actions for the ABJM and BLG theories, but also to other supergauge theories with $\mathcal{N}=2$ and $\mathcal{N}=4$ supersymmetry such as the effective action for matter superfields on the gauge superfield background and the extended supersymmetric electrodynamics. In particular, we give explicit computations of the Euler-Heisenberg-type effective actions for case of $\mathcal{N}=2$ chiral superfield and $\mathcal{N}=4$ hypermultiplet interacting with background Abelian gauge superfield.

The present contribution is a review of the results of our recent work [10].

2 General structure of the effective action for $\mathcal{N}=2$ gauge superfield

The $\mathcal{N}=2$, $d=3$ gauge multiplet consists of one real scalar field ϕ, one complex spinor λ_α, one vector field $A_{\alpha\beta}$ and one real auxiliary field D. In the Abelian case they appear in the component decomposition of the real gauge superfield V which in the Wess-Zumino gauge reads

$$V=\theta^\alpha\bar\theta^\beta A_{\alpha\beta}+i\theta^\alpha\bar\theta_\alpha\phi+i\theta^2\bar\theta^\alpha\bar\lambda_\alpha-i\bar\theta^2\theta^\alpha\lambda_\alpha+\theta^2\bar\theta^2 D\,. \tag{1}$$

This gauge superfield serves as the prepotential for the three superfield strengths

$$G=\frac{i}{2}\bar D^\alpha D_\alpha V\,,\quad W_\alpha=-\frac{i}{4}\bar D^2 D_\alpha V\,,\quad \bar W_\alpha=-\frac{i}{4}D^2\bar D_\alpha V\,. \tag{2}$$

The superfield G is real while W_α and $\bar W_\alpha$ are mutual conjugated.

The superfield strengths (2) obey a number of identities which should be taken into account in constructing the effective actions. First of all, the superfield strengths W_α and $\bar{W}_\alpha$ are (anti)chiral,

$$\bar{D}_\alpha W_\beta = 0\,, \qquad D_\alpha \bar{W}_\beta = 0\,, \tag{3}$$

and are intertwined by

$$D^\alpha W_\alpha = \bar{D}^\alpha \bar{W}_\alpha\,. \tag{4}$$

Next, they are expressed in terms of G as

$$W_\alpha = \bar{D}_\alpha G\,, \qquad \bar{W}_\alpha = D_\alpha G\,, \tag{5}$$

which is a linear superfield,

$$D^2 G = 0\,, \qquad \bar{D}^2 G = 0\,. \tag{6}$$

The gauge transformation for V has the standard form, $\delta V = \frac{i}{2}(\bar{\Lambda} - \Lambda)$. The strength superfields G, W_α and $\bar{W}_\alpha$ are gauge invariant in the Abelian case.

In general, the effective Lagrangian depends on the gauge superfield V, its superfield strengths G, W_α, $\bar{W}_\alpha$ and their derivatives. The only gauge invariant term with explicit dependence on the gauge superfield V and which cannot be rewritten in terms of the superfield strengths is the Chern-Simons term [27, 13, 17],

$$S_{\rm CS} = \frac{k}{2\pi}\int d^3x d^4\theta\, V G = \frac{k}{2\pi}\int d^3x (\frac{1}{2}\varepsilon^{mnp} A_m \partial_n A_p + i\lambda^\alpha \bar{\lambda}_\alpha - 2\phi D)\,, \tag{7}$$

where k is the Chern-Simons level. All other terms in the effective Lagrangian depend only on the superfield strengths and their derivatives.

Within the derivative expansion, the effective action can be represented as a series over the the superfield strengths with various number of covariant derivatives acting on them. We will restrict ourself to the long-wave approximation which means that we omit all terms with space-time derivatives of superfields, but the covariant spinor derivatives can appear in the effective Lagrangian. This is taken into account by the following constraints on the superfield strengths

$$\partial_m G = 0\,, \quad \partial_m W_\alpha = 0\,, \quad \partial_m \bar{W}_\alpha = 0\,. \tag{8}$$

Moreover, we assume that the superfield strengths obey the free Maxwell equations of motion,

$$D^\alpha W_\alpha = 0\,, \qquad \bar{D}^\alpha \bar{W}_\alpha = 0\,. \tag{9}$$

In this approximation there is very limited number of building blocks, i.e., the superfield combinations which the effective action can depend on. First of all, it depends on the superfield strength G as well as on W_α and $\bar{W}_\alpha$ which involve first covariant spinor derivatives of G, (5). Next, there are the objects with two covariant spinor derivatives of G,

$$N_{\alpha\beta} \equiv D_{(\alpha}W_{\beta)}\,, \qquad \bar{N}_{\alpha\beta} \equiv -(N_{\alpha\beta})^* = \bar{D}_{(\alpha}\bar{W}_{\beta)}\,. \tag{10}$$

One can show that $\bar{N}_{\alpha\beta}$ coincides with $N_{\alpha\beta}$ up to a sign, $N_{\alpha\beta} = -\bar{N}_{\alpha\beta}$. Finally, it is clear that any further spinor derivatives of the superfield strengths vanish in the long-wave approximation (8), e.g.,

$$\bar{D}_\alpha D_\beta W_\gamma = -2i\partial_{\alpha\beta}W_\gamma = 0\,, \qquad D^2 W_\alpha = -4i\partial_{\alpha\beta}\bar{W}^\beta = 0\,. \tag{11}$$

We conclude that the general structure of the gauge invariant effective action is given by

$$\Gamma_{\mathcal{N}=2} = \int d^3x d^4\theta [c_0 VG + \mathscr{L}_{\rm eff}(G, W_\alpha, \bar{W}_\alpha, N_{\alpha\beta})]\,, \tag{12}$$

where c_0 is an arbitrary coefficient and $\mathscr{L}_{\rm eff}$ is an effective Lagrangian being a real scalar superfield. Further restrictions on the structure of the function $\mathscr{L}_{\rm eff}$ come from the requirement of the superconformal invariance.

Let us consider the superconformal transformations of the gauge superfield V and its superfield strength G,

$$\delta_{\rm sc}V = \xi V\,, \quad \delta_{\rm sc}G = (\rho + \xi)G\,, \tag{13}$$

where ξ is a superform, $\xi = \xi^m\partial_m + \xi^\alpha D_\alpha - \bar{\xi}^\alpha\bar{D}_\alpha$. Here $(\xi^m, \xi^\alpha, \bar{\xi}_\alpha)$ is the superconformal Killing vector and ρ is a superfield constructed from the parameters of the superconformal group [10],

$$\rho = a + k_{\alpha\beta}x^{\alpha\beta} + 2i\theta^\alpha\eta_\alpha + 2i\bar{\theta}^\alpha\bar{\eta}_\alpha\,. \tag{14}$$

By construction, there are the following important identities

$$D^2\rho = \bar{D}^2\rho = D^\alpha\bar{D}_\alpha\rho = 0\,. \tag{15}$$

Note also that it can be represented as the sum of chiral and antichiral parts,

$$\rho = \frac{1}{2}(\sigma + \bar{\sigma})\,, \qquad \bar{D}_\alpha\sigma = 0\,, \quad D_\alpha\bar{\sigma} = 0\,. \tag{16}$$

Using the following properties

$$D^{(\alpha}\xi^{\beta)} + \bar{D}^{(\alpha}\bar{\xi}^{\beta)} = 0\,, \qquad D^\alpha\xi_\alpha - \bar{D}^\alpha\bar{\xi}_\alpha = -\frac{1}{3}\partial_{\alpha\beta}\xi^{\alpha\beta} = -2\rho\,, \tag{17}$$

one can easily check the superconformal invariance of the Chern-Simons action (7),

$$\delta_{\rm sc}S_{\rm CS} = \frac{k}{2\pi}\int d^3x d^4\theta(\rho + \xi)VG = 0\,. \tag{18}$$

Hence, the superconformal invariance imposes only constraints on the function $\mathscr{L}_{\rm eff}$ in (12).

In general, the effective Lagrangian contains the effective potential term $\mathscr{F}(G)$,

$$\mathscr{L}_{\rm eff} = \mathscr{F}(G) + \tilde{\mathscr{L}}_{\rm eff}(G, W_\alpha, \bar{W}_\alpha, N_{\alpha\beta}), \tag{19}$$

where $\mathscr{F}(G)$ is a holomorphic function of G only while $\tilde{\mathscr{L}}_{\rm eff}$ takes into account the superfield strength with covariant spinor derivatives. The superconformal invariance restricts the form of the effective potential $\mathscr{F}(G)$ uniquely, up to a constant. Indeed, the general condition of superconformal invariance applied to the effective potential reads

$$\delta_{\rm sc}\mathscr{F}(G) = (\rho + \xi)\mathscr{F}(G) + \sigma\mathscr{K}(G) + \bar{\sigma}\bar{\mathscr{K}}(G), \tag{20}$$

where the function $\mathscr{K}(G)$ should be linear,

$$D^2\bar{\mathscr{K}}(G) = \bar{D}^2\mathscr{K}(G) = 0 \quad \Rightarrow \quad \mathscr{K}(G) = \alpha + \beta G, \quad \bar{\mathscr{K}}(G) = \bar{\alpha} + \bar{\beta} G, \tag{21}$$

with α and β being some (complex) constants. Up to the terms vanishing under integral over full $\mathscr{N} = 2$ superspace, the general solution of (20) is given by

$$\mathscr{F}(G) = c_1 G \ln G, \tag{22}$$

where c_1 is some constant. This effective potential is responsible for a superconformal generalization of the Maxwell term in its component decomposition,

$$\int d^3x d^4\theta\, G \ln G = \frac{1}{8}\int d^3x \frac{1}{\phi} F^{mn} F_{mn} + \ldots, \tag{23}$$

where dots stand for other component terms. Note that the Lagrangian (22) being considered in the $\mathscr{N} = 1$, $d = 4$ superspace is responsible for the classical action of the improved tensor multiplet model [8].

It is much more difficult to make general analysis of the admissible form of the function $\tilde{\mathscr{L}}_{\rm eff}$ in (19) subject to the superconformal invariance of the corresponding action. The problem is that the superfields W_α, $\bar{W}_\alpha$ and $N_{\alpha\beta}$ are not quasi-primary, e.g.,

$$\delta_{\rm sc} W_\alpha = (\frac{1}{2}\rho + \sigma + \xi) W_\alpha + \omega_{\alpha\beta} W^\beta + (\bar{D}_\alpha \rho) G, \tag{24}$$

where $\omega_{\alpha\beta} = \bar{D}_{(\alpha}\bar{\xi}_{\beta)} = -D_{(\alpha}\xi_{\beta)}$ are the parameters of 'local' Lorentz transformations. Equation (24) shows that W_α transforms inhomogeneously because of the last term in (24). This is a new feature of three-dimensional supergauge models as compared to the $\mathscr{N} = 1$, $d = 4$ ones in which the superfield strengths are chiral quasi-primary, [8, 24, 25, 26]. Therefore the superfields W_α and $\bar{W}_\alpha$ are rather inconvenient for constructing superconformal actions and we are forced to introduce the following quasi-primary superfields

$$\Psi = \frac{i}{G}\bar{D}^\alpha D_\alpha \ln G, \qquad \Omega^2 = \frac{1}{8}(\frac{1}{G}\bar{D}^\alpha D_\alpha)^2 \ln G. \tag{25}$$

Indeed, using (13) and the relations (17) one can readily check that both these superfields are quasi-primary with zeroth scaling dimension,

$$\delta_{\rm sc}\Psi=\xi\Psi\,,\qquad \delta_{\rm sc}\Omega^2=\xi\Omega^2\,. \tag{26}$$

This allows us to construct a superconformal action with these superfields,

$$S_1=\int d^3xd^4\theta\,G\,\mathscr{U}(\Psi,\Omega^2)\,,\qquad \delta_{\rm sc}S_1=0\,, \tag{27}$$

where $\mathscr{U}(\Psi,\Omega^2)$ is an arbitrary function.

Neither the gauge invariance nor the superconformal symmetry impose any restrictions on possible form of the function $\mathscr{U}(\Psi,\Omega^2)$ in (27). However, for the background gauge superfield under considerations (9,8) the form of this functions can be further reduced. Indeed, for such a background there are the following equivalent representations for Ψ and Ω^2,

$$\Psi=-i\frac{W^\alpha\bar W_\alpha}{G^3}\,, \tag{28}$$

$$\Omega^2=\frac{1}{8}\frac{N^\alpha_\beta N^\beta_\alpha}{G^4}+\frac{3}{4}\frac{N^{\alpha\beta}W_\alpha\bar W_\beta}{G^5}+\frac{3}{4}\frac{W^2\bar W^2}{G^6}\,. \tag{29}$$

Owing to the odd statistics of superfield strengths W_α and $\bar W_\alpha$, the power expansion of $\mathscr{U}(\Psi,\Omega^2)$ over Ψ terminates at the second order,

$$\mathscr{U}(\Psi,\Omega^2)=\mathscr{U}_0(\Omega^2)+\Psi\mathscr{U}_1(\Omega^2)+\Psi^2\mathscr{U}_2(\Omega^2)\,. \tag{30}$$

Under the integral over $\mathscr{N}=2$ superspace the first two terms in the rhs of (30) can be brought to the form of the last term,

$$\int d^3xd^4\theta\,G[\mathscr{U}_0(\Omega^2)+\Psi\mathscr{U}_1(\Omega^2)]=\int d^3xd^4\theta\,G\Psi^2\tilde{\mathscr{U}}_2(\Omega^2)\,, \tag{31}$$

where $\tilde{\mathscr{U}}_2$ is some function. These considerations show that in the long-wave approximation the superconformal action (27) simplifies

$$S_1=\int d^3xd^4\theta\,G\Psi^2\mathscr{H}(\Omega^2)\,, \tag{32}$$

such that it is described by a single function $\mathscr{H}(\Omega^2)$ of one real variable. There are no any more constraints on the form of this function.

Summing up all together, we conclude that the general form of the superconformal effective action in the long-wave approximation is given by

$$\Gamma_{\mathscr{N}=2}=\Gamma_{\rm CS}+\Gamma_{\rm Maxweel}+\Gamma_{\rm higher}\,, \tag{33}$$

where

$$\Gamma_{\rm CS} = c_0 \int d^3x d^4\theta\, V G\,, \tag{34}$$

$$\Gamma_{\rm Maxwell} = c_1 \int d^3x d^4\theta\, G \ln G\,, \tag{35}$$

$$\Gamma_{\rm higher} = \int d^3x d^4\theta\, G \Psi^2 \mathscr{H}(\Omega^2)\,. \tag{36}$$

In components, this action contains the Chern-Simons term (7), the Maxwell F^2 term (23) and all higher order terms F^{2n} with $n \geq 2$. The undefined coefficients c_0, c_1 and the arbitrary function $\mathscr{H}$ will be found in the next section by explicit quantum computations for the model of $\mathscr{N}=2$ chiral superfield on the Abelian gauge superfield background.

3 Effective actions for $\mathscr{N}=2$ and $\mathscr{N}=4$ matter on gauge superfield background

3.1 Effective action of chiral superfield in the background gauge superfield

The classical action of the $\mathscr{N}=2$ chiral superfield Q interacting with the Abelian gauge superfield V is given by

$$S_{\mathscr{N}=2} = -\int d^3x d^4\theta\, \bar{Q} e^{2V} Q\,, \tag{37}$$

The effective action in the model (37) can be written schematically as

$$\Gamma_{\mathscr{N}=2} = \frac{i}{4} {\rm Tr}_+ \ln \square_+ + c.c.\,, \tag{38}$$

where $\square_+$ is the covariantly chiral box operator,

$$\square_+ = \nabla^m \nabla_m + G^2 + \frac{i}{2}(D^\alpha W_\alpha) + i W^\alpha \nabla_\alpha\,. \tag{39}$$

Here ∇_m and ∇_α are gauge covariant generalizations of ∂_m and D_α. To calculate the trace of the logarithm of this operator one has to specify the background gauge superfield. Here we consider the superfield strength which is constant with respect to the space-time coordinates (8) and obey the free Maxwell equations (9). In this approxiamtion the methods of quantum computations in superspace are very well elaborated in [21, 20, 22]. As a result, we get (see [10] for details)

$$\Gamma_{\mathscr{N}=2} = \frac{1}{4\pi} \int d^3x d^4\theta\, G \left[V + \ln G + \frac{1}{4} \int_0^\infty \frac{ds}{\sqrt{i\pi s}} e^{isG^2} \frac{W^2 \bar{W}^2}{G B^2} \left(\frac{\tanh(sB/2)}{sB/2} - 1 \right) \right], \tag{40}$$

where $B^2 = \frac{1}{2}N_\alpha^\beta N_\beta^\alpha$. We stress that this effective action is obtained in the on-shell approximation (9). However, there is the possibility to go off shell if one expresses this action in terms on superconformal invariants (34,35,36) (see [9] for analogous trick for the $\mathcal{N}=2$, $d=4$ superconformal theories). This allows us to rewrite the effective action in the form (33) with

$$\Gamma_{\rm CS} = \frac{1}{4\pi}\int d^3x d^4\theta\, V G\,, \tag{41}$$

$$\Gamma_{\rm Maxwell} = \frac{1}{4\pi}\int d^3x d^4\theta\, G\ln G\,, \tag{42}$$

$$\Gamma_{\rm higher} = \frac{1}{32\pi}\int d^3x d^4\theta\, G\frac{\Psi^2}{\Omega^2}\int_0^\infty \frac{dt\, e^{it}}{\sqrt{i\pi t}}\left(\frac{\tanh(t\Omega)}{t\Omega}-1\right). \tag{43}$$

Here Ψ and Ω are quasi-primary $\mathcal{N}=2$ superfields (25).

3.2 *Effective action of charged hypermultiplet in the background gauge superfield*

Let us consider the model of charged hypermultiplet interacting with the Abelian $\mathcal{N}=4$ gauge superfield (V,Φ) [1]

$$S_{\mathcal{N}=4} = -\int d^3x d^4\theta\left(\bar{Q}_+ e^{2V} Q_+ + \bar{Q}_- e^{-2V} Q_-\right) - \left(\int d^3x d^2\theta\, Q_+\Phi Q_- + c.c.\right). \tag{44}$$

For the constant background, $D_\alpha\Phi = 0$, the one-loop effective action is given schematically by

$$\Gamma_{\mathcal{N}=4} = \frac{i}{2}\mathrm{Tr}_+\ln(\Box_+ + \bar{\Phi}\Phi) + c.c. \tag{45}$$

The result of evaluation of this expression reads

$$\begin{aligned}\Gamma_{\mathcal{N}=4} = \frac{1}{2\pi}\int d^3x d^4\theta\Big[&-\sqrt{G^2+\bar{\Phi}\Phi} + G\ln(G+\sqrt{G^2+\bar{\Phi}\Phi})\\ &+\frac{1}{8}\frac{\Theta^2}{\Xi^2}\sqrt{G^2+\bar{\Phi}\Phi}\int_0^\infty\frac{dt\, e^{it}}{\sqrt{i\pi t}}\left(\frac{\tanh(t\Xi)}{t\Xi}-1\right)\Big],\end{aligned} \tag{46}$$

where

$$\begin{aligned}\Theta &= \frac{i}{G}\bar{D}^\alpha D_\alpha\ln(G+\sqrt{G^2+\bar{\Phi}\Phi})\,,\\ \Xi^2 &= \frac{1}{8}\frac{1}{\sqrt{G^2+\bar{\Phi}\Phi}}\bar{D}^\alpha D_\alpha\frac{1}{G}\bar{D}^\beta D_\beta\ln(G+\sqrt{G^2+\bar{\Phi}\Phi})\,.\end{aligned} \tag{47}$$

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[1] The chiral superfield Φ is the $\mathcal{N}=4$ superpartner of the $\mathcal{N}=2$ gauge superfield V.

are the $\mathcal{N}=4$ supersymmetric generalizations of the $\mathcal{N}=2$ quasi-primary superfields (25). The action (46) is superconformal as well.

It is interesting to note that the terms in the first line in (46) being written in four-dimensional space-time are known as the action of the improved tensor multiplet [11, 23, 16] which has recently been studied in [19]. The corresponding three-dimensional action was considered in the recent publication [18] as a dual representation of the Gaiotto-Witten model [12]. Here we derive this action as the leading contribution in the charged hypermultiplet model.

4 Conclusions

In this work we studied the general structure of the effective action in superconformal three-dimensional $\mathcal{N}=2$ supersymmetric gauge theories. We showed that the gauge and superconformal invariance fixes the form of the leading terms in the effective action uniquely, up the the coefficients while the higher order terms with respect to the Maxwell field strength are described by one function of quasi-primary $\mathcal{N}=2$ superfields.

The obtained form of the effective action can appear in various $\mathcal{N}=2$ supergauge models including $\mathcal{N}=2$ supersymmetric electrodynamics, SYM theory and chiral matter on the Abelian gauge superfield background. For the latter model we explicitly compute the low-energy effective action within the perturbation theory and fix the freedom which remained after the symmetry analysis. This effective action is also generalized to the case of $\mathcal{N}=4$ charged hypermultiplet interacting with Abelian gauge superfield.

It wound be interesting to apply the obtained results to study the effective action in the supergauge models with reacher supersymmetry. In particular, it may appear as a part of the effective action in the ABJM and BLG theories describing the dynamics of multiple M2 branes. Another interesting direction of the further research is the study of the effective action in the ABJM and BLG theories in the $\mathcal{N}=3$, $d=3$ harmonic superspace [6, 7].

Acknowledgements The present work is supported by RFBR grant, project No 09-02-00078 and by a grant for LRSS, project No 3558.2010.2. I.L.B. and I.B.S. acknowledge the support from the RFBR grants No 10-02-90446 and No 09-02-91349 as well as from a DFG grant, project No 436 RUS/113/669. The work of I.B.S. is also supported by the fellowship of the Dynasty foundation. N.G.P. acknowledges the support from RFBR grant, project No 08-02-00334.

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Colour Confinement, the Goto-Imamura-Schwinger Term and Renormalization Group*

Masud Chaichian

Abstract In connection with the question of colour confinement the origin of the Goto-Imamura-Schwinger term has been studied with the help of renormalization group. An emphasis has been laid on the difference between theories with and without a cut-off.

Foreword

It is a great honour for me to take part in the celebration of the dear colleague and friend, Professor Emilio Elizalde, on the occasion of his sixtieth anniversary, and I feel a sense of priviledge in dedicating this article to him. Emilio and myself share a common interest in understanding several problems in theoretical physics, so that I shall concentrate my attention to the exploration of the connection between the colour confinement and the Goto-Imamura-Schwinger term in this article.

1 Introduction

Field theory is full of ghosts and bugs, and we have to bring divergences, anomalies and ambiguities under control. Among others we shall concentrate on the origin of the so-called Goto-Imamura-Schwinger (GIS) term [9, 20] in field theory, since it bears a close connection with the question of colour confinement [13–15].

* Work originally done in collaboration with the late Professor Kazuhiko Nishijima, Nishina Memorial Foundation, 2-28-45 Honkomagome, Bunkyo-ku, Tokyo 113-8941, Japan.

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_7,

In evaluating the equal-time commutator (ETC) between two local operators we sometimes encounter a result in conflict with that obtained by a naive application of the canonical commutation relations (CCRs). The deviation from the naive expectation is referred to as the Goto-Imamura-Schwinger (GIS) term hereafter. Such a term does not arise, however, when we evaluate the ETC between two fundamental fields, and it indicates that the origin of the GIS term must be sought in the definition of the singular product of field operators at the same space-time point.

In many examples it is possible to find a renormalization group (RG) equation controlling the GIS term in question, but then the next question is raised of how to formulate the initial or boundary condition for this equation. In the RG approach we introduce running parameters such as the running coupling constant and they tend to the bare or nonrenormalized ones in the high energy limit provided that we introduce a cut-off in the unrenormalized version of the theory as we shall see in Sec. 2. Then we can introduce boundary conditions in the high energy limit into the cut-off theory by assuming the CCRs. In some cases it is possible to formulate the boundary condition kinematically, namely, without reference to the dynamics of the system but often it is necessary to refer to the dynamics of the system by evaluating higher order corrections. In Sec. 3 we shall illustrate these statements in quantum electrodynamics (QED). Then, we find that the origin of the GIS terms may be attributed to one of the following causes: (1) operator-mixing under renormalization [13, 15], (2) non-local character of the product of field operators at the same space-time point and (3) divergences induced by lifting the cut-off. In Sec. 4 we shall proceed to quantum chromodynamics (QCD) in connection with the question of colour confinement.

2 Renormalization group

In introducing the RG approach [2, 8, 21] we shall employ the neutral scalar theory for illustration. We assume the quartic interaction of the scalar field $\phi(x)$ with the coupling constant g. The unrenormalized Green function is given by

$$G_0^{(n)}(x_1,...,x_n) = \langle 0|T\left[\phi^{(0)}(x_1)\cdots\phi^{(0)}(x_n)\right]|0\rangle \ , \tag{1}$$

where the subscript 0 and the superscript (0) denote unrenormalized quantities. The Fourier transform of the renormalized n-point Green function is denoted by

$$G^{(n)}(p_1,...,p_n;g(\mu),\mu) \ , \tag{2}$$

where μ denotes the renormalization point defined below and $g(\mu)$ the running coupling constant defined at the renormalization point as seen from

$$(p^2+m^2)G^{(2)}(p^2;g(\mu),\mu) = 1 \ , \quad \text{for } p^2 = \mu^2 \ , \tag{3}$$

$$G^{(4)}_{conn}(p_1,...,p_4;g(\mu),\mu)$$
$$= g(\mu)\prod_{i=1}^{4} G^{(2)}(p_i^2;g(\mu),\mu)\cdot \Gamma(p_1,\cdots,p_4;g(\mu),\mu)\ , \tag{4}$$

$$\Gamma(p_1,\cdots,p_4;g(\mu),\mu) = 1\ , \quad \text{for } p_i\cdot p_j = \frac{\mu^2}{3}(4\delta_{ij}-1)\ , \tag{5}$$

where $G^{(4)}_{conn}$ denotes the 4-point Green function for connected Feynmann diagrams alone. These are the normalization conditions for the Green functions and specify the renormalization point in the Pauli metric.

The generator of the RG is given by

$$\mathscr{D} = \mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g}\ , \tag{6}$$

and the RG equation for the n-point Green function is given by

$$\left[\mathscr{D} + n\gamma_\phi(g)\right] G^{(n)}(p_1,...,p_n;g,\mu) = 0\ , \tag{7}$$

where we write g for $g(\mu)$ and γ_ϕ denotes the anomalous dimension of the scalar field ϕ. For the two-point Green function or the propagator we may assume the Lehmann representation [11],

$$G^{(2)}(p^2;g,\mu) = \int d\kappa^2 \frac{\rho(\kappa^2;g,\mu)}{p^2+\kappa^2-i\varepsilon}\ , \tag{8}$$

and we have

$$\left[\mathscr{D} + 2\gamma_\phi(g)\right] \rho(\kappa^2;g,\mu) = 0\ . \tag{9}$$

Then Eq. (3) in the limit $\mu\to\infty$ yields

$$\lim_{\mu\to\infty}(\mu^2+m^2)G^{(2)}(\mu^2;g,\mu) = \int d\kappa^2\rho(\kappa^2;g(\infty),\infty) = 1\ , \tag{10}$$

in the cut-off theory where m denotes the mass of the quantum of the scalar field.

Lehmann's theorem [11] on the ETC for the field operator normalized at μ readily yields the relation

$$\delta(x_0-y_0)\left[\phi(x;g,\mu),\dot\phi(y;g,\mu)\right] = i\delta^4(x-y)\int d\kappa^2\rho(\kappa^2;g,\mu)\ , \tag{11}$$

and Eq. (10) then implies that the field operators are identified with the unrenormalized ones in the limit $\mu\to\infty$ since they satisfy the CCR. At the same time we can show that $g(\mu)$ also tends to the bare coupling constant g_0 in the same limit.

In order to define the running parameters we introduce

$$R(\rho) = exp(\rho\mathscr{D})\ , \tag{12}$$

where ρ denotes the parameter of the RG, then $R(\rho)$ obeys the composition law

$$R(\rho_1)\cdot R(\rho_2) = R(\rho_1+\rho_2)\;, \tag{13}$$

and the RG is literally a group identified with $GL(1,R)$.

The running parameters in the scalar theory are defined by

$$\overline{g}(\rho) = R(\rho)\cdot g\;, \tag{14}$$
$$\overline{\mu}(\rho) = R(\rho)\cdot \mu = \mu\; exp(\rho)\;, \tag{15}$$

then we readily obtain

$$R(\rho)\; G^{(n)}(p_1,...,p_n;g,\mu) = G^{(n)}(p_1,...,p_n;\overline{g}(\rho),\overline{\mu}(\rho))\;. \tag{16}$$

We differentiate this equation with respect to ρ and combine it with Eq. (7) to obtain

$$\begin{aligned}\frac{\partial}{\partial\rho}G^{(n)}(p_1,...,p_n;\overline{g}(\rho),\overline{\mu}(\rho)) &= R(\rho)\,\mathscr{D}G^{(n)}(p_1,...,p_n;g,\mu)\\ &= -nR(\rho)\gamma_\phi(\rho)\; G^{(n)}(p_1,...,p_n;g,\mu)\\ &= -n\gamma_\phi(\overline{g}(\rho))\; G^{(n)}(p_1,...,p_n;\overline{g}(\rho),\overline{\mu}(\rho))\;.\end{aligned} \tag{17}$$

We have to introduce a boundary condition to this differential equation. In a cut-off theory we may set

$$\lim_{\mu\to\infty} G^{(n)}(p_1,...,p_n;g(\mu),\mu) = G_0^{(n)}(p_1,...,p_n;g_0)\;, \tag{18}$$

where g_0 denotes the bare coupling constant.

By integrating Eq. (17) we find

$$\begin{aligned}G^{(n)}(p_1,...,p_n;g,\mu) &= \exp\left[n\int_0^\rho d\rho\,\gamma_\phi(\overline{g}(\rho))\right]\\ &\quad\cdot\; G^{(n)}(p_1,...,p_n;\overline{g}(\rho),\overline{\mu}(\rho))\;.\end{aligned} \tag{19}$$

In the limit $\rho\to\infty$ and consequently $\overline{\mu}(\rho)\to\infty$ we have

$$G^{(n)}(p_1,...,p_n;g,\mu) = \exp\left[n\int_0^\infty d\rho\,\gamma_\phi(\overline{g}(\rho))\right]\cdot G_0^{(n)}(p_1,...,p_n;g_0)\;. \tag{20}$$

In a cut-off theory all the vertex corrections to $g(\mu)$ for $\mu\to\infty$ tend to vanish leaving only the bare one, namely,

$$\lim_{\mu\to\infty} g(\mu) = \lim_{\rho\to\infty}\overline{g}(\rho) = g_0\;. \tag{21}$$

The fundamental field ϕ is multiplicatively renormalized as

$$\phi^{(0)}(x) = Z_\phi^{1/2}\phi(x) , \tag{22}$$

where Z_ϕ is the renormalization constant of the scalar field ϕ, and it is a function of g. Comparison of Eqs. (20) and (22) yields

$$Z_\phi^{-1} = \exp\left[2\int_0^\infty d\rho\,\gamma_\phi(\overline{g}(\rho))\right] . \tag{23}$$

The running renormalization constant is given by

$$\begin{aligned} Z_\phi^{-1}(\rho) &= R(\rho) Z_\phi^{-1}(g) \\ &= \exp\left[2\int_\rho^\infty d\rho'\,\gamma_\phi(\overline{g}(\rho'))\right] . \end{aligned} \tag{24}$$

When Z_ϕ depends not only on g but also on μ, $\gamma_\phi(\rho)$ must be replaced by $\gamma_\phi(\rho,\mu)$.

In *a cut-off theory* we have

$$\lim_{\rho\to\infty} Z_\phi^{-1}(\rho) = 1 , \tag{25}$$

but this is not true when the integral in the exponent of Eq. (23) does not converge and as we shall see later this feature is a possible cause of the emergence of the GIS terms.

Although the RG approach has been introduced for the scalar theory we can easily extend it to gauge theories. In QED the generator of the RG is given by

$$\mathscr{D} = \mu\frac{\partial}{\partial\mu} + \beta(e)\frac{\partial}{\partial e} - 2\alpha\gamma_V(e)\frac{\partial}{\partial\alpha} , \tag{26}$$

where α denotes the gauge parameter. The $\gamma_V(e)$ denotes the anomalous dimension of the electromagnetic field and is related to $\beta(e)$ through the Ward identity

$$\beta(e) = e\gamma_V(e) . \tag{27}$$

Furthermore in QCD the generator is given by

$$\mathscr{D} = \mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - 2\alpha\gamma_V(g,\alpha)\frac{\partial}{\partial\alpha} , \tag{28}$$

where g denotes the gauge coupling constant and γ_V the anomalous dimension of the colour gauge field. The running parameters in QCD satisfy the following equations:

$$\frac{d\overline{g}}{d\rho} = \beta(\overline{g}) , \tag{29}$$

$$\frac{d\overline{\alpha}}{d\rho} = -2\overline{\alpha}\gamma_V(\overline{g},\overline{\alpha}) . \tag{30}$$

Then we introduce their asymptotic values by

$$g_\infty = \lim_{\rho\to\infty} \overline{g}(\rho) \ , \quad \alpha_\infty = \lim_{\rho\to\infty} \overline{\alpha}(\rho) \ . \tag{31}$$

This is possible since the RG is a group $GL(1,R)$ but not $U(1)$. Asymptotic freedom [10, 19] of QCD implies

$$g_\infty = 0 \ . \tag{32}$$

By integrating Eq. (30) we immediately find a sum rule,

$$2\int_0^\infty d\rho \overline{\gamma}_V(\rho) = ln(\frac{\alpha}{\alpha_\infty}) \tag{33}$$

and hence we also have [14, 15, 17]

$$Z_3^{-1} = \exp\left[2\int_0^\infty d\rho \overline{\gamma}_V(\rho)\right] = \frac{\alpha}{\alpha_\infty} \ , \tag{34}$$

where $\overline{\gamma}_V(\rho) \equiv \gamma_V(\overline{g}(\rho), \overline{\alpha}(\rho))$.

In QCD it is known that α_∞ can take three possible values [14, 15, 17]

$$\alpha_\infty = 0 \ , \ \alpha_0 \ , \ -\infty \ , \tag{35}$$

where α_0 is a constant which depends only on the number of quark flavors. These three values are related to the integral of γ_V as

$$\int_0^\infty d\rho \overline{\gamma}_V(\rho) = \begin{cases} \infty \ , & \text{for } \alpha_\infty = 0 \\ \text{finite} \ , & \text{for } \alpha_\infty = \alpha_0 \\ -\infty \ , & \text{for } \alpha_\infty = -\infty \end{cases} \tag{36}$$

and Z_3^{-1} vanishes when $\alpha_\infty = -\infty$.

3 Quantum electrodynamics

Quantum electrodynamics is a suitable ground to exercise the analysis of the GIS terms. The Lagrangian density for QED is given by

$$\mathscr{L} = \mathscr{L}_{em} + \mathscr{L}_{matter} \ , \tag{37}$$

where the unrenormalized version of the Lagrangian density for the electromagnetic field is given by

$$\mathscr{L}_{em} = -\frac{1}{4} F^{(0)}_{\mu\nu} \cdot F^{(0)}_{\mu\nu} + \partial_\mu B^{(0)} \cdot A^{(0)}_\mu + \frac{\alpha_0}{2} B^{(0)} \cdot B^{(0)} \ , \tag{38}$$

where B denotes the Nakanishi-Lautrup auxiliary field [12] and the interactions are included in the matter Lagrangian. The resulting field equations are given by

$$\partial_\mu F^{(0)}_{\mu\nu} + \partial_\nu B^{(0)} = -J^{(0)}_\nu \ , \tag{39}$$
$$\partial_\mu A^{(0)}_\mu = \alpha_0 B^{(0)} \ , \tag{40}$$

and the renormalized version of these equations can be expressed as

$$\partial_\mu F_{\mu\nu} + \partial_\nu B = -J_\nu \ , \tag{41}$$
$$\partial_\mu A_\mu = \alpha B \ . \tag{42}$$

The fundamental fields A_μ and B as well as the gauge parameter α are renormalized multiplicatively,

$$A^{(0)}_\mu = Z_3^{1/2} A_\mu \ , \tag{43a}$$
$$B^{(0)} = Z_3^{-1/2} B \ , \tag{43b}$$
$$\alpha_0 = Z_3 \alpha \ . \tag{43c}$$

Apparently renormalization of the composite current operator J_ν is not multiplicative, but its execution requires operator mixing [13, 15] as illustrated by

$$J^{(0)}_\nu = Z_3^{1/2}\left[J_\nu + (1 - Z_3^{-1})\partial_\nu B\right] \ , \tag{44a}$$
$$J_\nu = Z_3^{-1/2}\left[J^{(0)}_\nu + (1 - Z_3)\partial_\nu B^{(0)}\right] \ . \tag{44b}$$

Operator mixing is one of the sources of the GIS terms, and in order to illustrate this statement we shall evaluate the ETC

$$\delta(x_0 - y_0)\,[A_j(x),\, J_0(y)] \tag{45}$$

for $j = 1, 2, 3$. In the unrenormalized version we have

$$\delta(x_0 - y_0)\left[A^{(0)}_j(x),\, J^{(0)}_0(y)\right] = 0 \ . \tag{46}$$

As has been mentioned before we can rely on the ETCs only between two fundamental fields, so that we shall express J in terms of A and B by using Eqs. (41) and (43),

$$\begin{aligned}[A_j(x),\, J_4(y)] &= -\left[A_j(x),\, \partial_\mu F_{\mu 4}(y) + \partial_4 B(y)\right] \\ &= -Z_3^{-1}\left[A^{(0)}_j(x),\, \partial_k F^{(0)}_{k4}(y)\right] - \left[A^{(0)}_j(x),\, \partial_4 B^{(0)}(y)\right] \\ &= (-Z_3^{-1} + 1)\partial_j \delta^3(x - y)\end{aligned}$$

for $x_0 = y_0$. Thus we have

$$\begin{aligned}\delta(x_0 - y_0)\,[A_j(x),\, J_0(y)] &= i(Z_3^{-1} - 1)\partial_j \delta^4(x - y) \\ &\equiv is\partial_j \delta^4(x - y) \ , \end{aligned} \tag{47}$$

where s is the coefficient of the GIS term. In this case it is clear that the origin of the GIS term is the operator mixing. Then s satisfies the RG equation

$$[\mathscr{D}+2\gamma_V(e)]\,(s+1)=[\mathscr{D}+2\gamma_V(e)]\,Z_3^{-1}=0\,, \tag{48}$$

where $\mathscr{D}$ is given by Eq. (26), and the running GIS coefficient $\bar{s}(\rho)$ satisfies the differential equation

$$\left[\frac{\partial}{\partial\rho}+2\gamma_V(\bar{e}(\rho))\right](\bar{s}(\rho)+1)=0\,. \tag{49}$$

In a cut-off theory the GIS term is absent in the unrenormalized version as expressed by Eq. (46), and the boundary condition for $\bar{s}(\rho)$ is given by

$$\bar{s}(\infty)=0\,. \tag{50}$$

By combining the boundary condition (50) with Eq. (49) we find the solution

$$Z_3^{-1}(\rho)=1+\bar{s}(\rho)=\exp\left[2\int_\rho^\infty d\rho'\gamma_V(\bar{e}(\rho'))\right]\,. \tag{51}$$

In the absence of the cut-off we do not know what kind of boundary condition we should impose on $\bar{s}(\rho)$ so that we take this solution (51) for granted even in this case.

In QED we assume that $Z_3^{-1}=Z_3^{-1}(0)$ is divergent so that we have

$$1+\bar{s}(\infty)=\lim_{\rho\to\infty}\exp\left[2\int_\rho^\infty d\rho'\gamma_V(\bar{e}(\rho'))\right]=\infty\,, \tag{52}$$

and the boundary condition (50) is no longer satisfied in the absence of the cut-off. This is another source of the GIS terms, and the field operators do not necessarily tend to the unrenormalized ones in the limit $\rho\to\infty$ and hence $\mu\to\infty$ when the cut-off is lifted.

Finally we shall turn our attention to the ETC between two components of the current density. This is precisely the original problem in which the GIS term was recognized [9, 20]. We shall make use of the field equations (41) to express the current density as a linear combination of the fundamental fields, and then we can make use of the commutativity of B with $F_{\mu\nu}$ and B itself [12],

$$\begin{aligned}\left[J_\mu(x),\,J_\nu(y)\right]&=\left[\partial_\alpha F_{\alpha\mu}(x)+\partial_\mu B(x),\,\partial_\beta F_{\beta\nu}(y)+\partial_\nu B(y)\right]\\&=\left[\partial_\alpha F_{\alpha\mu}(x),\,\partial_\beta F_{\beta\nu}(y)\right]\,,\end{aligned} \tag{53}$$

and we introduce the GIS coefficient s by

$$\delta(x_0-y_0)\langle 0|\,[J_j(x),\,J_0(y)]\,|0\rangle=is\partial_j\delta^4(x-y)\,. \tag{54}$$

As a matter of fact, the ETC on the left-hand-side of Eq. (54) is known to be a c-number before taking its vacuum expectation value in spinor electrodynamics [16]. Here we are aware of the fact that the GIS term can be expressed in terms of the ETC between derivatives of field strengths. In order to evaluate the ETC by making use of the CCRs it is necessary to express derivatives of field strengths in terms of canonical variables by making use of canonical field equations. Therefore, we are taking commutators between those operators that are non-local in time and then taking the local limit. The way in which this limit is taken is dictated in the evaluation of the higher order corrections as we shall see below.

This is in a sharp contrast to the original naive way of evaluating the commutator between two bilinear forms of the Dirac fields by making use of only the CCRs without taking the possibility of non-locality into consideration. This gap generates the GIS term.

By combining Eqs. (53) and (54) we find that the GIS coefficient s satisfies the RG equation

$$[\mathscr{D}+2\gamma_V(e)]\,s=0\,. \tag{55}$$

In this case we cannot give the boundary condition for this equation since it requires the information about the dynamics of the system such as the photon propagator. The Lehmann representation of the electromagnetic field is given in the following form:

$$\langle 0|T\left[A_\mu(x),A_\nu(y)\right]|0\rangle=\frac{-i}{(2\pi)^4}\int d^4k e^{ik\cdot(x-y)}D_{F\mu\nu}(k)\,, \tag{56}$$

$$D_{F\mu\nu}(k)=\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2-i\varepsilon}\right)\int dM^2\frac{\rho(M^2;e,\mu)}{k^2+M^2-i\varepsilon}+\alpha\frac{k_\mu k_\nu}{(k^2-i\varepsilon)^2}\,. \tag{57}$$

Then inserting this expression into Eq. (53) we find

$$s=\int dM^2\rho(M^2;e,\mu)M^2\,. \tag{58}$$

This expression certainly satisfies Eq. (55) since we have

$$[\mathscr{D}+2\gamma_V(e)]\,\rho(M^2;e,\mu)=0\,. \tag{59}$$

It is clear that Z_3^{-1} also satisfies Eq. (55) since it is given by

$$Z_3^{-1}=\int dM^2\rho(M^2;e,\mu)\,. \tag{60}$$

We may conclude that the GIS terms are controlled by RG if not completely.

4 Colour confinement in quantum chromodynamics

In QCD the GIS term plays an important role in connection with colour confinement [13–15]. The field equation in QCD corresponding to Eq. (41) is given by

$$\partial_\mu F^a_{\mu\nu} + J^a_\nu = i\delta\overline{\delta}A^a_\nu \ , \tag{61}$$

where δ and $\overline{\delta}$ denote two kinds of Becchi-Rouet-Stora (BRS) transformations [1], respectively, and the superscript a represents the colour index. Since we are not entering the subject of BRS transformations here we shall refer to other references [13–15] for their definitions.

We are interested in ETC

$$\begin{aligned}
&\partial_\mu \langle 0|T\left[i\delta\overline{\delta}A^a_\mu(x),\, A^b_j(y)\right]|0\rangle \\
&= \delta(x_0 - y_0)\langle 0|\left[i\delta\overline{\delta}A^a_0(x),\, A^b_j(y)\right]|0\rangle \\
&= i\delta_{ab} C\, \partial_j \delta^4(x-y) \ ,
\end{aligned}$$

or

$$\delta(x_0 - y_0)\langle 0|\left[\partial_k F^a_{k4}(x) + J^a_4(x),\, A^b_j(y)\right]|0\rangle = -\delta_{ab} C\, \partial_j \delta^4(x-y) \ . \tag{62}$$

The constant C is gauge-dependent, and a sufficient condition for colour confinement is the existence of a gauge in which the following equality holds:

$$C = 0 \ . \tag{63}$$

In order to determine C we have to evaluate the ETC in Eq. (62), and for that purpose we introduce the RG equation satisfied by C [3–5],

$$(\mathscr{D} - 2\gamma_{FP})C = 0 \ , \tag{64}$$

where $\mathscr{D}$ is given by Eq. (28) and γ_{FP} denotes the anomalous dimension of the Faddeev-Popov ghost fields. Then the renormalization constant of the ghost fields denoted by $\tilde{Z}_3$ also satisfies the same RG equation,

$$(\mathscr{D} - 2\gamma_{FP})\tilde{Z}_3 = 0 \ . \tag{65}$$

We are going to study the relationship between C and $\tilde{Z}_3$ in this section. They satisfy the same RG equation, but their normalizations are different.

The unrenormalized version of Eq. (61) reads as

$$i\delta\overline{\delta}A^{(0)}_\nu(x) = \partial_\mu A^{(0)}_{\mu\nu} + g_0 \partial_\mu (A^{(0)}_\mu \times A^{(0)}_\nu) + J^{(0)}_\nu \ , \tag{66}$$

where $A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ denotes the linear part of $F_{\mu\nu}$ and we have suppressed the colour index. The cross product denotes the antisymmetric product in the colour space defined in terms of the structure constants of the algebra $su(3)$. When we insert the r.h.s. of Eq. (66) into the ETC (62) in the unrenormalized version, we find that only the first term $\partial_\mu A^{(0)}_{\mu\nu}$ gives a non-vanishing canonical commutator and the rest would give only a vanishing result provided that the naive CCRs are employed. However, this is true only in a cut-off theory or in a convergent theory and in general we should not discard the possibility of a non-vanishing GIS term so that the unrenormalized constant C_0 would be given by

$$C_0 = 1 + s\,. \tag{67}$$

The first term is a result of the CCR and is equal to unity. Thus the renormalized C is given by

$$C = C_0\tilde{Z}_3 = (1+s)\tilde{Z}_3\,. \tag{68}$$

Then a question is raised of how to evaluate the GIS coefficient s. For this purpose we introduce a cut-off theory and we write $\overline{a}(\rho)$ for $Z_3^{-1}(\rho)$, and we shall rewrite Eq. (67) in the form

$$\overline{C}(\infty) = \overline{a}(\infty) \tag{69}$$

based on the argument developed in Sec. 2. In a cut-off theory the GIS coefficient s vanishes, but it does not vanish when the cut-off is lifted. The running parameters $\overline{C}(\rho)$, $\overline{a}(\rho)$ and $\tilde{Z}_3(\rho)$ satisfy the following differential equations, respectively,

$$\left[\frac{\partial}{\partial\rho} - 2\overline{\gamma}_{FP}(\rho)\right]\overline{C}(\rho) = 0\,, \tag{70}$$

$$\left[\frac{\partial}{\partial\rho} + 2\overline{\gamma}_{V}(\rho)\right]\overline{a}(\rho) = 0\,, \tag{71}$$

$$\left[\frac{\partial}{\partial\rho} - 2\overline{\gamma}_{FP}(\rho)\right]\tilde{Z}_3(\rho) = 0\,. \tag{72}$$

Among them the last two are renormalization constants, and they are immediately given by

$$\begin{aligned}\overline{a}(\rho) = Z_3^{-1}(\rho) &= \exp\left[2\int_\rho^\infty d\rho'\overline{\gamma}_V(\rho')\right]\,,\\ \tilde{Z}_3^{-1}(\rho) &= \exp\left[2\int_\rho^\infty d\rho'\overline{\gamma}_{FP}(\rho')\right]\,.\end{aligned} \tag{73}$$

We should be aware of the following relations:

$$Z_3^{-1} = Z_3^{-1}(0)\,, \quad \tilde{Z}_3^{-1} = \tilde{Z}_3^{-1}(0)\,. \tag{74}$$

Then $\overline{C}(\rho)$ should be determined by solving Eq. (70) under the boundary condition (69) and we obtain

$$\overline{C}(\rho) = \lim_{\rho'\to\infty}\exp\left[2\int_{\rho'}^\infty d\rho''\overline{\gamma}_V(\rho'') - 2\int_\rho^{\rho'} d\rho''\overline{\gamma}_{FP}(\rho'')\right]\,, \tag{75}$$

and, in particular, we have

$$C = \lim_{\rho' \to \infty} \exp \left[2 \int_{\rho'}^{\infty} d\rho'' \overline{\gamma}_V(\rho'') - 2 \int_0^{\rho'} d\rho'' \overline{\gamma}_{FP}(\rho'') \right] . \tag{76}$$

¿From now on we lift the cut-off while keeping these formulas. With recourse to Eqs. (34) and (36) we find that C vanishes when Z_3^{-1} vanishes as claimed before [13–15]. Then we may express Eq. (76) as

$$C = \lim_{\rho \to \infty} \exp \left[2 \int_{\rho}^{\infty} d\rho' \overline{\gamma}_V(\rho') \right] \cdot \tilde{Z}_3 , \tag{77}$$

and with reference to Eq. (68) we find

$$\begin{aligned} 1 + s &= \lim_{\rho \to \infty} \exp \left[2 \int_{\rho}^{\infty} d\rho' \overline{\gamma}_V(\rho') \right] \\ &= \begin{cases} \infty , \text{ for } \alpha_\infty = 0 \\ 1 , \text{ for } \alpha_\infty = \alpha_0 \\ 0 , \text{ for } \alpha_\infty = -\infty \end{cases} . \end{aligned} \tag{78}$$

Only in the case $\alpha_\infty = \alpha_0$ do we find the vanishing GIS coefficient s, and this is precisely what happens when the integration of $\overline{\gamma}_V$ converges just as in the cut-off theory. Now we shall summarize the relationship between C and $\tilde{Z}_3$ as follows:

$$C = \begin{cases} \infty , & \alpha_\infty = 0 \\ \tilde{Z}_3 , & \alpha_\infty = \alpha_0 \\ 0 & \alpha_\infty = -\infty \end{cases} . \tag{79}$$

As we have seen above we formulate the boundary condition for a given RG equation by introducing a cut-off, but when the cut-off is lifted in the solution the GIS term appears as a manifestation of the divergent character of the theory.

Further works along these lines have been persued in references [3–7, 18].

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Non-central extensions of (Super) Poincaré algebra and (Susy) Electromagnetic Backgrounds

Joaquim Gomis

Abstract We present the Maxwell and super Maxwell algebra and introduce a massive particle invariant under Maxwell symmetry. We also consider a κ-invariant massless superparticle model providing a dynamical realization of the superMaxwell algebra. This article is dedicated to Emili Elizalde, my first PhD student, for his 60 birthday.

1 Introduction

The Poincaré algebra and Poincaré group describe the symmetry of empty Minkowski space-time. Filling such a flat space-time with some background fields leads to a modification of Poincaré symmetries. An example of such a modification is the so-called Maxwell symmetries, which was obtained already in the seventies [1][2] by considering Minkowski space with an added constant electromagnetic (EM) background. The collection of arbitrary values of the constant EM field strengths provides additional degrees of freedom in Minkowski space, supplementing the Poincaré group with additional group parameters and the Poincaré algebra with new generators.

The Maxwell algebra [2] is obtained by adding to the Poincaré generators $(P_\mu, M_{\mu\nu})$ the tensorial central charges $Z_{\mu\nu}$ $(Z_{\mu\nu} = -Z_{\nu\mu})$ which modify the commutativity of the four-momenta P_μ [1]

$$[P_\mu, P_\nu] = iZ_{\mu\nu}, \tag{1}$$

where $M_{\mu\nu}$ are the Lorentz algebra generators and

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[1] The Bacry-Combe-Richard (BCR) algebra [1] is a subalgebra of the Maxwell algebra in which $Z_{\mu\nu}$ takes fixed numerical values.

S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_8,

$$\left[Z_{\mu\nu}, M_{\rho\sigma}\right] = -i\eta_{\nu[\rho}Z_{|\mu|\sigma]} + i\eta_{\mu[\rho}Z_{|\nu|\sigma]}, \tag{2}$$

$$\left[P_\mu, Z_{\rho\sigma}\right] = \left[Z_{\mu\nu}, Z_{\rho\sigma}\right] = 0. \tag{3}$$

The aim of this article is twofold: one is to present a Maxwell group-invariant particle model on the extended space-time $(x^\mu, \phi^{\mu\nu})$, [3]-[6], with the translations of $\phi^{\mu\nu}$ generated by $Z_{\mu\nu}$. The interaction term described by a Maxwell-invariant one-form introduces new tensor degrees of freedom $f_{\mu\nu}$ - momenta conjugate to $\phi^{\mu\nu}$. In the equations of motion they play the role of a background EM field which is constant on-shell.

The second goal is to introduce the supersymmetric extension of the Maxwell symmetries with new N=1 superMaxwell algebra and to investigate the corresponding superMaxwell-invariant massless superparticle model [7]. Analogously to the Maxwell case, one can introduce the generalized phase space with coordinates $(x^\mu, \theta^\alpha, \phi^{\mu\nu}, \phi^\alpha, \phi)$ and conjugate momenta $(p_\mu, \zeta_\alpha, f_{\mu\nu}, \tilde{\lambda}_\alpha, \mathrm{D})$. Since $(\phi^{\mu\nu}, \phi^\alpha, \phi)$ are cyclic coordinates the conjugate momenta $(f_{\mu\nu}, \tilde{\lambda}_\alpha, \mathrm{D})$ are constant on-shell describing the constant Abelian SUSY N=1 gauge field background.

This paper is based on the works [3]-[8].

2 Particle model with Maxwell symmetry

To formulate a relativistic particle model, invariant under the Maxwell group, it is convenient to consider the coset G/H=Maxwell/Lorentz, which locally is parametrized as $g = e^{iP_\mu x^\mu} e^{\frac{i}{2}Z_{\mu\nu}\phi^{\mu\nu}}$ [3]-[6]. The Maurer-Cartan (MC) form is

$$\Omega = -ig^{-1}dg = P_\mu L^\mu + \frac{1}{2}Z_{\mu\nu}L_Z^{\mu\nu} + \frac{1}{2}M_{\mu\nu}L_M^{\mu\nu}, \tag{4}$$

where

$$L^\mu = dx^\mu,\ L_Z^{\mu\nu} = d\phi^{\mu\nu} + \frac{1}{2}(x^\mu dx^\nu - x^\nu dx^\mu),\ L_M^{\mu\nu} = 0. \tag{5}$$

The particle action invariant under the Maxwell algebra (1) and (3) is described by the following Lagrangian:

$$\mathscr{L} = \frac{\dot{x}_\mu \dot{x}^\mu}{2e} - \frac{m^2}{2}e + \frac{1}{2}f_{\mu\nu}L_Z^{\mu\nu *}, \tag{6}$$

where e is the einbein implementing the diffeomorphism invariance, $f_{\mu\nu}$ is a tensorial variable canonically conjugate to the new coordinates $\phi^{\mu\nu}$ and $L_Z^{\mu\nu *}$ is the pullback of $L_Z^{\mu\nu}$. In the proper time gauge, one obtains from (6) the equations of motion

$$m\ddot{x}_\mu = f_{\mu\nu}\dot{x}^\nu, \quad \dot{f}_{\mu\nu} = 0, \quad \dot{\phi}^{\mu\nu} = -\frac{1}{2}(x^\mu \dot{x}^\nu - x^\nu \dot{x}^\mu). \tag{7}$$

Note that for this case the equation of motion for $f_{\mu\nu}$ does not affect the dynamics of the coordinates. This equation tells us that $\dot{\theta}^{\mu\nu}$ is proportional to the $\mu\nu$

component of the angular momentum (or magnetic moment) of the particle. In other words $\theta^{\mu\nu}$ is a non-local function of the components of the angular momenta of the particle.

Integration of the equation of motion associated to θ gives $f_{\mu\nu} = f^0_{\mu\nu} = eF^0_{\mu\nu}$. We see that this solution spontaneously breaks Lorentz symmetry. If we substitute this solution in the equation of motion for the variable x, we find that it describes the motion of a particle in a constant, fixed EM field with

$$F_{\mu\nu} = F^0_{\mu\nu} = constant. \tag{8}$$

The EM potential is described by the one-form $\mathscr{A} = \frac{1}{2} f_{\mu\nu} L^{\mu\nu}_{\mathrm{Z}}$. In the closed two-form field strength

$$\mathscr{F} = d\mathscr{A} = \frac{1}{2} f_{\mu\nu} L^{\mu} \wedge L^{\nu} + \frac{1}{2} df_{\mu\nu} \wedge L^{\mu\nu}_{\mathrm{Z}} \tag{9}$$

the second term vanishes on-shell due to (7) and the field strength components are constant $f_{\mu\nu}$.

3 From Maxwell algebra to superMaxwell algebra

In a recent paper [7] we have proposed the supersymmetric extension, denoted by $\mathscr{G}_5$, of the Maxwell algebra in 4 dimensions. The starting point for the construction of this algebra is the extension of the superPoincaré algebra in D=4 with Majorana supercharges Q_α as $(\alpha, \beta = 1,2,3,4)$

$$\{Q_\alpha, Q_\beta\} = 2(C\gamma^\mu)_{\alpha\beta} P_\mu, \quad \left[P_\mu,\, P_\nu\right] = iZ_{\mu\nu}. \tag{10}$$

The complete form of the algebra is obtained using the restrictions imposed by the Jacobi identities and the Eilenberg-Chevalley cohomology. The algebra is

$$\begin{aligned}
\left[P_\mu, P_\nu\right] &= iZ_{\mu\nu}, & \left[P_\mu, Q_\alpha\right] &= -i\Sigma_\beta (\gamma_\mu)^\beta{}_\alpha, \\
\{Q_\alpha, Q_\beta\} &= 2(C\gamma^\mu)_{\alpha\beta} P_\mu, & \{Q_\alpha, \Sigma_\beta\} &= \frac{1}{2}(C\gamma^{\mu\nu})_{\alpha\beta} Z_{\mu\nu} + (C\gamma_5)_{\alpha\beta} B, \\
[B_5, Q_\alpha] &= -i(Q\gamma_5)_\alpha, & [B_5, \Sigma_\alpha] &= i(\Sigma\gamma_5)_\alpha, \\
\left[P_\mu, M_{\rho\sigma}\right] &= -i\eta_{\mu[\rho} P_{\sigma]}, & \left[Z_{\mu\nu}, M_{\rho\sigma}\right] &= -i\eta_{\nu[\rho} Z_{|\mu|\sigma]} + i\eta_{\mu[\rho} Z_{|\nu|\sigma]}, \\
\left[M_{\rho\sigma}, Q_\alpha\right] &= -\frac{i}{2}(Q\gamma_{\rho\sigma})_\alpha, & \left[M_{\rho\sigma}, \Sigma_\alpha\right] &= -\frac{i}{2}(\Sigma\gamma_{\rho\sigma})_\alpha, \\
\left[M_{\mu\nu}, M_{\rho\sigma}\right] &= -i\eta_{\nu[\rho} M_{|\mu|\sigma]} + i\eta_{\mu[\rho} M_{|\nu|\sigma]}.
\end{aligned} \tag{11}$$

The bosonic generators $(P_\mu, M_{\mu\nu}, Z_{\mu\nu})$, linked to translations, Lorentz rotations and additional tensorial coordinates, form the bosonic Maxwell subalgebra and the fermionic generators Q_α, Σ_α, $(\alpha = 1,2,3,4)$ are two Majorana spinor charges. B

is a central charge and B_5 generates chiral transformations. We point out that D=4 Maxwell superalgebra can also be considered as an enlargement by generators $Z_{\mu\nu}$ of the algebra with 8 supercharges introduced by Green [9].

There are three subalgebras obtained by consistently removing generators B and/or B_5 from (11) (see [7]).

1) The minimal supersymmetric extension $\mathscr{G}$, with a bosonic sector consisting only of the Maxwell algebra generators, is obtained if we remove B and B_5.

2) Removing only generator B_5, we get a central extension $\widetilde{\mathscr{G}}$ of $\mathscr{G}$. The generator B is required if we wish to introduce the scalar degree of freedom describing the off-shell extension of D=4 $U(1)$ field strength supermultiplet.

3) One can consider a subalgebra with only the generator B_5 which acts on the supercharges Q_α, Σ_α as chiral generator. If B is present, B_5 is also required for the existence of the supersymmetric mass Casimir.

We add that all cases describe the supersymmetric extension of the Maxwell algebra with minimal number of supercharges (eight real or four complex) and all these supersymmetrizations describe N=1 Maxwell superalgebra. We note that four additional supercharges Σ_α are present due to the supersymmetrization of the constant electromagnetic background.

4 Massless superparticle model with Maxwell supersymmetry

We construct a massless superparticle model using a non-linear realization of the superMaxwell algebra $\mathscr{G}_5$. The supergroup element $\widetilde{g}$ is parametrized as

$$\widetilde{g} = e^{\frac{i}{2}Z_{\mu\nu}\phi^{\mu\nu}}\, e^{iP_\mu x^\mu}\, e^{i\Sigma_\alpha\phi^\alpha}\, e^{iQ_\alpha\theta^\alpha}\, e^{iB\phi} \tag{12}$$

using the supercoset $G/H{=}\mathscr{G}_5/(M\times B_5)$. Here the chiral generator B_5 is in the unbroken subgroup because we construct a massless particle. The components of the MC form $\widetilde{\Omega} = -i\widetilde{g}^{-1}d\widetilde{g}$ are

$$\begin{aligned}
\widetilde{L}^\mu &= dx^\mu + i(\overline{\theta}\gamma^\mu d\theta), \quad \widetilde{L}^\alpha = d\theta^\alpha, \quad \widetilde{L}^{\mu\nu}_M = 0,\\
\widetilde{L}^{\mu\nu}_Z &= d\phi^{\mu\nu} + i(\overline{\theta}\gamma^{\mu\nu})_\alpha d\phi^\alpha + \frac{1}{2}(x^\mu dx^\nu - x^\nu dx^\mu)\\
&\quad + \frac{i}{2}(\overline{\theta}\gamma^{\mu\nu}\gamma_\rho\theta)(dx^\rho + \frac{i}{6}(\overline{\theta}\gamma^\rho d\theta)),\\
\widetilde{L}^\alpha_\Sigma &= d\phi^\alpha + (\gamma_\rho\theta)^\alpha(dx^\rho + \frac{i}{3}(\overline{\theta}\gamma^\rho d\theta)), \quad \widetilde{L}^5 = 0,\\
\widetilde{L}_B &= d\phi + i(\overline{\theta}\gamma_5)_\alpha d\phi^\alpha + \frac{i}{2}(\overline{\theta}\gamma_5\gamma_\rho\theta)(dx^\rho + \frac{i}{6}(\overline{\theta}\gamma^\rho d\theta))
\end{aligned} \tag{13}$$

The massless super particle action invariant under the superMaxwell group is

$$\mathscr{L} = \frac{\pi_\mu^2}{2e} + \mathscr{L}^{I*}; \quad \mathscr{L}^I = \frac{1}{2} f_{\mu\nu} \widetilde{L}_Z^{\mu\nu} + i\lambda_\alpha \widetilde{L}_\Sigma^\alpha + \mathrm{D}\widetilde{L}_B, \tag{14}$$

where $\pi_\mu = \dot{x}_\mu + i\overline{\theta}\gamma_\mu\dot{\theta}$ is the pullback of $\widetilde{L}_\mu$ to the world line and e describes the einbein. Here the $(f_{\mu\nu}, \lambda_\alpha, \mathrm{D})$ are dynamical variables transforming as Lorentz tensor, Majorana spinor and scalar, respectively. The interaction Lagrangian can be written explicitly as

$$\mathscr{L}^{I*} = \frac{1}{2} f_{\mu\nu}\, \dot{\phi}^{\mu\nu} + i\tilde{\lambda}_\alpha\, \dot{\phi}^\alpha + \mathrm{D}\,\dot{\phi} + \pi^\mu A_\mu + \dot{\theta}^\alpha \tilde{A}_\alpha, \tag{15}$$

where

$$\tilde{\lambda}_\alpha = \lambda_\alpha + \mathrm{D}(\overline{\theta}\gamma_5)_\alpha + \frac{1}{2} f_{\mu\nu}(\overline{\theta}\gamma^{\mu\nu})_\alpha \tag{16}$$

and the U(1) SUSY gauge potentials are

$$\begin{aligned} \tilde{A}_\alpha &= i(\overline{\theta}\gamma^\mu)_\alpha[-\frac{1}{2} f_{\mu\nu}x^\nu + i(\frac{2}{3}\tilde{\lambda} - \frac{1}{8}\overline{\theta}\gamma_{\rho\sigma}f^{\rho\sigma} - \frac{1}{4}\mathrm{D}\overline{\theta}\gamma_5)\gamma_\mu\theta], \\ A_\mu &= -\frac{1}{2} f_{\mu\nu}x^\nu + i(\tilde{\lambda} - \frac{1}{4}\overline{\theta}\gamma_{\rho\sigma}f^{\rho\sigma} - \frac{1}{2}\mathrm{D}\overline{\theta}\gamma_5)\gamma_\mu\theta. \end{aligned} \tag{17}$$

The variation of $\mathscr{L}$ with respect to $(\phi^{\mu\nu}, \phi^\alpha, \phi)$ gives

$$\dot{f}_{\mu\nu} = \dot{\tilde{\lambda}}_\alpha = \dot{\mathrm{D}} = 0, \tag{18}$$

i.e., the U(1) potentials (17) are functions of the superspace coordinates (x^μ, θ^α) and the variables $(f_{\mu\nu}, \tilde{\lambda}_\alpha, \mathrm{D})$ which take constant values on-shell. The variation of $\mathscr{L}$ with respect to $(f_{\mu\nu}, \tilde{\lambda}_\alpha, \mathrm{D})$ gives the equations for the variables $(\phi^{\mu\nu}, \phi^\alpha, \phi)$

$$(\widetilde{L}_Z^{\mu\nu})^* = (\widetilde{L}_\Sigma^\alpha)^* = (\widetilde{L}_B)^* = 0. \tag{19}$$

The variation of $\mathscr{L}$ with respect to e puts the momenta π_μ on mass shell with vanishing mass

$$\pi^2 = 0. \tag{20}$$

Finally, the variation of $\mathscr{L}$ with respect to (x^μ, θ^α) gives, using (17)–(18), the superparticle equations of motion in superspace,

$$\frac{d}{d\tau}(\frac{\pi_\mu}{e}) = \pi^\nu F_{\mu\nu} + \dot{\theta}^\beta F_{\mu\beta}, \tag{21}$$

$$2i\,(\dot{\overline{\theta}}\gamma^\mu)_\alpha\,(\frac{\pi_\mu}{e}) = \pi^\nu F_{\nu\alpha}, \tag{22}$$

where the superfield strengths are ($D_\alpha = \partial_\alpha + i(\bar{\theta}\gamma^\mu)_\alpha\partial_\mu$)

$$\begin{aligned} F_{\mu\nu} &= (\partial_\mu A_\nu - \partial_\nu A_\mu) = f_{\mu\nu}, \\ F_{\mu\alpha} &= (\partial_\mu \tilde{A}_\alpha - D_\alpha A_\mu) = i(\lambda\gamma_\mu)_\alpha, \end{aligned} \tag{23}$$

and the superspace constraints following from (17)

$$F_{\alpha\beta} = (D_\alpha \tilde{A}_\beta + D_\beta \tilde{A}_\alpha) - 2i(C\gamma^\mu)_{\alpha\beta} A_\mu = 0 \tag{24}$$

have been used in (22). Identifying the interaction term $\mathscr{L}^I = \mathscr{A}$ in (14) with the EM one-form superpotential, the two-superform field strength $\mathscr{F} = d\mathscr{A}$ is,

$$\mathscr{F} = d\mathscr{A} = \frac{1}{2} f_{\mu\nu} L^\mu L^\nu + i\lambda_\alpha (\gamma_\mu L)^\alpha L^\mu + \cdots \tag{25}$$

where the $\cdots$ terms are linear in the one forms $L_B, L_\Sigma^\alpha, L_Z^{\mu\nu}$ which vanish on shell. The field strength components are the ones given in (23)- (24).

Our model describes the coupling to a particular choice of $U(1)$ gauge superfield strength $W_\alpha(x,\theta)$

$$W_\alpha(\theta) = i\tilde{\lambda}_\alpha - \frac{i}{2} f_{\mu\nu}(\bar{\theta}\gamma^{\mu\nu})_\alpha - i\mathrm{D}(\bar{\theta}\gamma_5)_\alpha. \tag{26}$$

which satisfies the standard superspace constraints for the SUSY gauge theories,

$$\begin{gathered} F_{\alpha\beta} = 0, \quad F_{\mu\alpha} = W_\beta (\gamma_\mu)^\beta{}_\alpha, \\ D_\alpha W_\beta = -\tfrac{i}{2}(C\gamma^{\mu\nu})_{\alpha\beta} F_{\mu\nu}, \quad \partial_\mu W_\beta (\gamma^\mu)^\beta{}_\alpha = 0. \end{gathered} \tag{27}$$

It is known (see e.g. [10]) that the coupling of the N=1 superparticle to the gauge superfield strength $W_\alpha(x,\theta)$ satisfying the constraints (27) leads to a κ-invariant interaction. Actually our system is not only invariant under the global superMaxwell symmetries but also invariant under τ reparametrization and the κ symmetries.

5 Conclusions

In this article we considered non-central extensions of the (super) Poincaré group and their relation with (Susy) electromagnetic backgrounds. We have introduced a massive particle invariant under Maxwell symmetries which on shell describes the motion of a particle in a constant, fixed EM field.

We have also considered supersymmetric extensions of the Maxwell algebra and proposed a κ invariant massless superparticle model (14) with the superMaxwell symmetries. It couples minimally to a constant $U(1)$ gauge superfield strength satisfying the superspace constraints (see (27)).

Acknowledgements We acknowledge Sotirios Bonanos, Gary Gibbons, Kiyoshi Kamimura, Jerzy Lukierski and Chris Pope for their collaboration in the papers on which this work is based. We also acknowledge financial support from projects FPA2007-66665-C02-01, 2009SGR502, Polish Ministry of Science and High Education grant NN202 318534 and Consolider CPANCSD2007-00042.

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Multiple Scattering: Dispersion, Temperature Dependence, and Annular Pistons

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Abstract We review various applications of the multiple scattering approach to the calculation of Casimir forces between separate bodies, including dispersion, wedge geometries, annular pistons, and temperature dependence. Exact results are obtained in many cases.

1 Quantum vacuum energy

Quantum vacuum energies, or Casimir energies, are important at all energy scales, from subnuclear to cosmological. Applications are starting to appear in nanotechnology. Furthermore it is most likely that the source of dark energy that makes up some 70% of the energy budget of the universe is quantum vacuum fluctuations. In particular, the 7-year WMAP data is completely consistent with the existence of a cosmological constant [1],

$$w \equiv \frac{p}{\rho} = -1.10 \pm 0.14 (68\%\,\mathrm{CL}), \tag{1}$$

which is precisely what would be expected if dark energy arose from this source [2]. Finally, zero-point fluctuations may be the most fundamental aspect of quantum field theory.

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_9,

2 Multiple-scattering formulation

The multiple scattering formulation is easiest stated for a scalar field, which is rather ‘easily’ generalized to electromagnetism. For example, see [3]. Vacuum energy is given by the famous trace-log formula,

$$E = \frac{i}{2}\mathrm{Tr}\,\ln G \to \frac{i}{2}\mathrm{Tr}\,\ln G G_0^{-1}, \tag{2}$$

where in terms of the background potential V,

$$(-\partial^2 + V)G = 1, \quad -\partial^2 G_0 = 1. \tag{3}$$

Now we define the T-matrix,

$$T = S - 1 = V(1 + G_0 V)^{-1}, \tag{4}$$

and if the potential has two disjoint parts, $V = V_1 + V_2$, it is easy to derive the interaction between the two bodies (potentials):

$$E_{12} = -\frac{i}{2}\mathrm{Tr}\ln(1 - G_0 T_1 G_0 T_2) \tag{5a}$$

$$= -\frac{i}{2}\mathrm{Tr}\ln(1 - V_1 G_1 V_2 G_2), \tag{5b}$$

where $G_i = (1 + G_0 V_i)^{-1} G_0$, $i = 1, 2$, and likewise T_i refers to V_i.

3 Quantum vacuum energy—dispersion

Perhaps not surprisingly in retrospect, we find that the usual dispersive form of the electromagnetic energy [4]

$$U = \frac{1}{2}\int (d\mathbf{r}) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\left[\frac{d(\omega\varepsilon)}{d\omega}E^2(\mathbf{r}) + H^2(\mathbf{r})\right] \tag{6}$$

must be used, which, quantum mechanically, corresponds to the vacuum energy form

$$\mathscr{E} = -\frac{i}{2}\int (d\mathbf{r}) \int \frac{d\omega}{2\pi}\left[2\varepsilon\,\mathrm{tr}\,\Gamma + \omega\frac{d\varepsilon}{d\omega}\mathrm{tr}\,\Gamma\right], \tag{7}$$

in terms of the Green’s dyadic Γ. This result follows directly from the trace-log formula for the vacuum energy

$$\mathscr{E} = \frac{i}{2}\int \frac{d\omega}{2\pi}\mathrm{Tr}\,\ln\Gamma, \tag{8}$$

and is equivalent to the variational statement [5]

$$\delta\mathscr{E} = \frac{i}{2}\int \frac{d\omega}{2\pi}\operatorname{Tr}\delta\varepsilon\Gamma. \tag{9}$$

From the energy, precisely because the dispersive derivative terms are present, we recover the Lifshitz formula for the energy per area between parallel dielectric slabs, with permittivity $\varepsilon_{1,2}$, separated by a medium of permittivity ε_3 of thickness a,

$$\frac{\mathscr{E}}{A} = \frac{1}{4\pi^2}\int_0^\infty d\zeta \int_0^\infty dk\,k\left[\ln\left(1 - r_{\text{TE}} r'_{\text{TE}} e^{-2\kappa_3 a}\right) + \ln\left(1 - r_{\text{TM}} r'_{\text{TM}} e^{-2\kappa_3 a}\right)\right], \tag{10}$$

with $\kappa_i = \sqrt{k_\perp^2 + \zeta^2\varepsilon_i}$, $\zeta = -i\omega$ being the imaginary frequency. The TE reflection coefficients are given by

$$r_{\text{TE}} = \frac{\kappa_3 - \kappa_1}{\kappa_3 + \kappa_1}, \qquad r'_{\text{TE}} = \frac{\kappa_3 - \kappa_2}{\kappa_3 + \kappa_2}, \tag{11}$$

while the TM coefficients are obtained from these by the substitution $\kappa_a \to \bar{\kappa}_a = \kappa_a/\varepsilon_a$. For further details of this calculation, see [6].

4 Noncontact gears

The program of calculating the quantum vacuum lateral force between corrugated surfaces and gears has been under active development. The electromagnetic situation of currugated dielectric slabs is illustrated in Fig. 1. For details see [7].

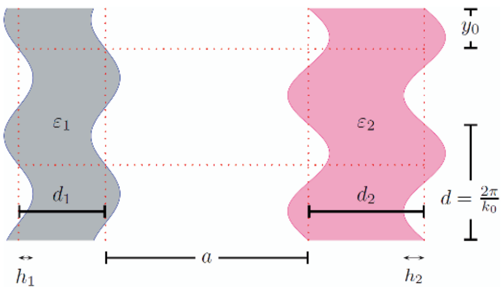


Fig. 1 Parallel dielectric slabs with sinusoidal corrugations.

In the conductor limit ($\varepsilon_i \to \infty$) and for the case of sinusoidal corrugations described by $h_1(y) = h_1\sin[k_0(y+y_0)]$ and $h_2(y) = h_2\sin[k_0 y]$ the lateral force can be evaluated to be in first order in h_1/a and h_2/a

$$F^{(2)}_{\varepsilon\to\infty} = 2k_0 a \sin(k_0 y_0) \left| F^{(0)}_{\text{Cas}} \right| \frac{h_1}{a} \frac{h_2}{a} A^{(1,1)}_{\varepsilon\to\infty}(k_0 a), \tag{12}$$

where

$$A^{(1,1)}_{\varepsilon\to\infty}(t_0) = \frac{15}{\pi^4} \int_{-\infty}^{\infty} dt \int_0^{\infty} \bar{s} d\bar{s} \frac{s}{\sinh s} \frac{s_+}{\sinh s_+} \left[\frac{1}{2} + \frac{(s^2+s_+^2-t_0^2)^2}{8 s^2 s_+^2} \right], \tag{13}$$

where $s^2 = \bar{s}^2 + t^2$ and $s_+^2 = \bar{s}^2 + (t+t_0)^2$. The first term in (13) corresponds to the Dirichlet scalar case [8], which here corresponds to the E mode (referred to in [9] as the TM mode). We note that $A^{(1,1)}_{\varepsilon\to\infty}(0) = 1$. See Fig. 2 for the plot of $A^{(1,1)}_{\varepsilon\to\infty}(k_0 a)$ versus $k_0 a$. We observe that only in the proximity force approximation limit $k_0 a = 0$ is the electromagnetic contribution twice that of the Dirichlet case, and in general the electromagnetic case is less than twice that of the Dirichlet case. This result can be

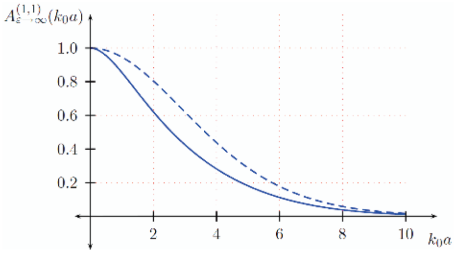


Fig. 2 Plot of $A^{(1,1)}_{\varepsilon\to\infty}(k_0 a)$ versus $k_0 a$. The dotted curve represents 2 times the Dirichlet case.

shown to coincide with the expression found in Emig *et al.* [9] apart from an overall factor of 2, which presumably is a transcription error. The double integral representation in (13) is more useful for numerical evaluation than the single-integral form given in [9] because of the oscillatory nature of the function $\sin x/x$ in the latter. Generalization of these results are forthcoming.

5 Wedge as generalization of cylinder

In a series of papers, we have considered variations on the wedge geometry, such as a wedge defined by perfectly reflecting walls, intersected with a concentric circular cylinder, the arc being either a perfect reflector itself, or the boundary between two dielectric-diamagnetic regions. Most interesting is the case when the wedge itself is constructed as the interface between two such media. See Fig. 3. In order to have a tractable situation, we have considered the diaphanous or isorefractive condition

$$\varepsilon_1 \mu_1 = \varepsilon_2 \mu_2, \tag{14}$$

that is, the speed of light is the same in the two media. (If that is not done for the wedge, the differential equations are no longer separable.) See Refs [10, 11, 12] for more detail.

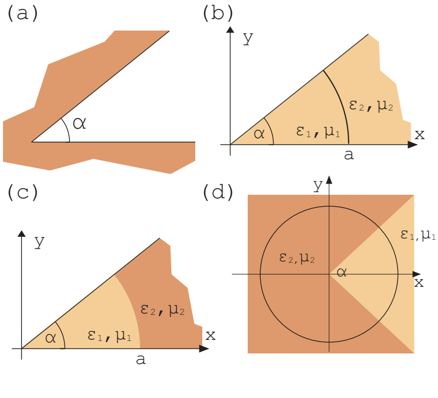


Fig. 3 Wedge geometries. (a) The perfectly conducting wedge geometry. (b) The geometry of a wedge intercut by a perfectly conducting cylindrical arc. (c) Wedge with magnetodielectric arc. (d) Diaphanous wedge in a perfectly conducting cylindrical shell.

Consider now case (d). Using multiple scattering, or the Kontorovich-Lebedev transformation, we obtain the following implicit formula for the eigenvalues for the order ν of the contributing cylindrical partial waves, $D(\nu,\omega)=0$, where ($r=$ reflection coefficient on wedge)

$$\begin{aligned} D(\nu,\omega) &= (1-e^{2\pi i\nu})^2 - r^2(e^{i\nu(2\pi-\alpha)} - e^{i\nu\alpha})^2 \\ &= -4e^{2\pi i\nu}[\sin^2(\nu\pi) - r^2\sin^2(\nu(\pi-\alpha))], \end{aligned} \tag{15}$$

which are selected by the "argument principle," which is just the Cauchy theorem applied to the contour γ shown in Fig. 4.

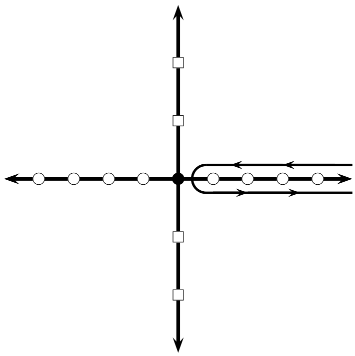


Fig. 4 Contour of integration γ for the argument principle. Shown also are singularities of the integrand along the real and imaginary ν axes.

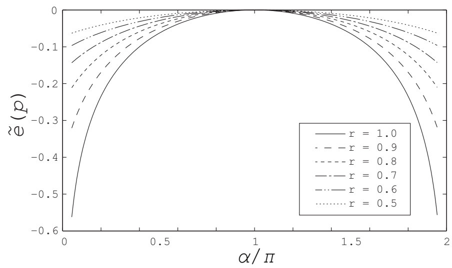


Fig. 5 The function $\tilde{e}(p)$ plotted as a function of opening angle α.

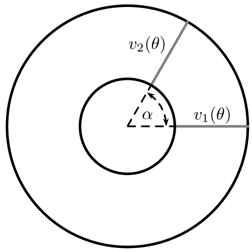


Fig. 6 Two semitransparent plates in an annulus.

In this way, we find the energy per length given by $\tilde{\mathscr{E}} = \frac{1}{8\pi n a^2}\tilde{e}(p)$, $p = \pi/\alpha$ as shown in Fig. 5. Note that only for perfect reflectors does the energy diverge as the opening angle approaches zero.

6 Annular piston—semitransparent plates

The wedge geometry may be generalized by considering two semitransparent plates in a Dirichlet annulus, as shown in Fig. 6. We use multiple scattering in the angular coordinates, and an eigenvalue condition in the radial coordinates; this problem is equally well solvable with radial Green's functions, but this approach may be more generalizable. This section is based on [13].

The Green's function $\mathscr{G}(\mathbf{r},\mathbf{r}')$ will satisfy the equation

$$\left[-\nabla^2 - \omega^2 + V(\mathbf{r})\right]\mathscr{G}(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}'), \tag{16}$$

while $\mathscr{G}^{(0)}$ has $V(\mathbf{r}) = 0$. For the cylindrical geometry of an annulus, the boundary conditions are $\mathscr{G} = 0$ at $\rho = a$ and $\rho = b$, where a and b are the inner and outer radii, respectively. We take the potential to be $V(\mathbf{r}) = v(\theta)/\rho^2$. The corresponding Green's function is

$$\mathscr{G}(\mathbf{r},\mathbf{r}';\omega) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ik(z-z')}\sum_\eta R_\eta(\rho;\omega,k)R_\eta(\rho';\omega,k)g_\eta(\theta,\theta'), \tag{17}$$

in terms of the separation constant η. The normalized radial eigenfunctions appearing here are

$$\left[-\rho\frac{d}{d\rho}\rho\frac{d}{d\rho}-(\omega^2-k^2)\rho^2\right]R_\eta(\rho;\omega,k)=\eta^2R_\eta(\rho;\omega,k), \tag{18}$$

with the boundary conditions $R_\eta(a;\omega,k)=R_\eta(b;\omega,k)=0$. The reduced Green's function satisfies

$$\left[-\frac{d^2}{d\theta^2}+\eta^2+v(\theta)\right]g_\eta(\theta,\theta')=\delta(\theta-\theta'), \tag{19}$$

with periodic boundary conditions.

To obtain the radial functions, we need the solution of the modified Bessel differential equation, of imaginary order, which is zero for $\rho=a$ for all values of η and κ. An obvious solution is

$$\tilde{R}_\eta(\rho;\kappa)=K_{i\eta}(\kappa a)\tilde{I}_{i\eta}(\kappa\rho)-\tilde{I}_{i\eta}(\kappa a)K_{i\eta}(\kappa\rho)=\tilde{R}_{-\eta}(\rho,\kappa), \tag{20}$$

where

$$\tilde{I}_\nu=\frac{1}{2}(I_\nu+I_{-\nu}). \tag{21}$$

The eigenvalues are given by the zeros of $D(\eta)=\tilde{R}_\eta(b;\kappa)$. We don't need the explicit eigenfunctions here.

6.1 Reduced Green's function

The free angular reduced Green's function is given by

$$g_\eta^{(0)}(\theta,\theta')=\frac{1}{2\eta}\left(-\sinh\eta|\theta-\theta'|+\frac{\cosh\eta\pi}{\sinh\eta\pi}\cosh\eta|\theta-\theta'|\right). \tag{22}$$

For a single potential $v(\theta)=\lambda\delta(\theta-\alpha)$ for $\theta,\theta'\in[\alpha,2\pi+\alpha]$, the reduced Green's function is

$$\begin{aligned}g_\eta(\theta,\theta') \;=\; &\frac{1}{2\eta}\left(-\sinh\eta|\theta-\theta'|+\frac{2\eta\cosh\eta\pi\cosh\eta|\theta-\theta'|}{2\eta\sinh\eta\pi+\lambda\cosh\eta\pi}\right.\\ &\left.-\lambda\frac{\cosh\eta(2\pi+2\alpha-\theta-\theta')-\cosh 2\eta\pi\cosh\eta|\theta-\theta'|}{[2\eta\sinh\eta\pi+\lambda\cosh\eta\pi]2\sinh\eta\pi}\right).\end{aligned} \tag{23}$$

6.2 Two semitransparent planes

Now we look at the interaction energy between two semitransparent planes, as illustrated in Fig. 6. Since it is nontrivial to work out the Green's function for two potentials, it is easiest to use the multiple-scattering formalism (5b)

$$E = \frac{1}{2i}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\mathrm{Tr}\ln(1-\mathscr{G}^{(1)}V_1\mathscr{G}^{(2)}V_2). \tag{24}$$

The subscripts on the Vs represent the potentials $V_1(\mathbf{r}) = \lambda_1\delta(\theta)/\rho^2$, and $V_2(\mathbf{r}) = \lambda_2\delta(\theta-\alpha)/\rho^2$. The Green's functions with superscript (i) represent the interaction with only a single potential V_i. From this we obtain a simplified form of the interaction energy:

$$\mathscr{E} = \frac{1}{4\pi}\int_0^{\infty}\kappa\, d\kappa \sum_\eta \ln\left(1-\mathrm{tr}\, g_\eta^{(1)} v_1 g_\eta^{(2)} v_2\right), \tag{25}$$

where $g_\eta^{(i)}$ are given in (23). Then

$$\mathrm{tr}\, g_\eta^{(1)} v_1 g_\eta^{(2)} v_2 = \frac{\lambda_1\lambda_2\cosh^2\eta(\pi-\alpha)}{(2\eta\sinh\eta\pi+\lambda_1\cosh\eta\pi)(2\eta\sinh\eta\pi+\lambda_2\cosh\eta\pi)}. \tag{26}$$

Using the argument principle to determine the angular eigenvalues, we get the following expression for the energy for an annular Casimir piston,

$$\begin{aligned}\mathscr{E} &= \frac{1}{8\pi^2 i}\int_0^{\infty}\kappa\, d\kappa \int_\gamma d\eta \frac{\partial}{\partial\eta}\ln\left[K_{i\eta}(\kappa a)\tilde{I}_{i\eta}(\kappa b)-\tilde{I}_{i\eta}(\kappa a)K_{i\eta}(\kappa b)\right] \\ &\quad\times\ln\left(1-\frac{\lambda_1\lambda_2\cosh^2\eta(\pi-\alpha)/\cosh^2\eta\pi}{(2\eta\tanh\eta\pi+\lambda_1)(2\eta\tanh\eta\pi+\lambda_2)}\right).\end{aligned} \tag{27}$$

The contour of integration for the argument principle is again given in Fig. 4.

This formula can actually be used to evaluate the energy of interaction between the two planes of the piston, by distorting the η contour to lines making angles of $\pm\pi/4$ with respect to the real axis. The results are shown in Fig. 7. In Fig. 7 we define $d = \frac{b+a}{2}\sin\frac{\alpha}{2}$, and the plateaus seen in the second figure may be understood from the proximity force approximation,

$$\frac{\mathscr{E}_{\mathrm{PFA}}}{\mathscr{E}_{\|}} = \frac{1}{16}\frac{b^2}{a^2}\left(1+\frac{a}{b}\right)^4, \tag{28}$$

in comparison to the interaction between infinite parallel plates.

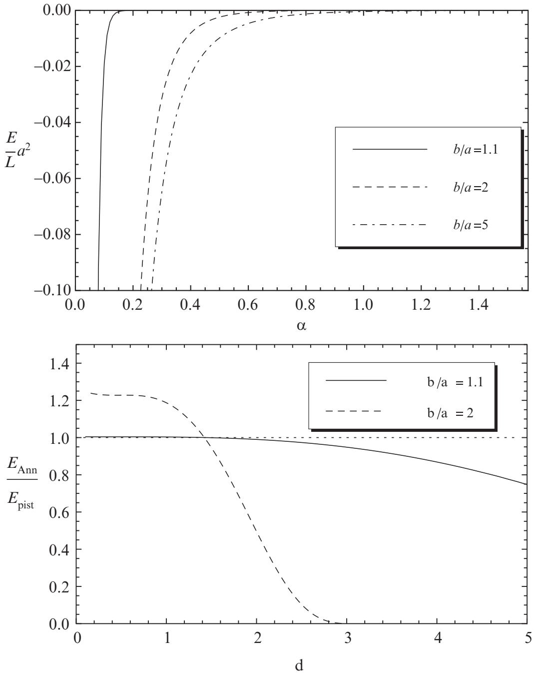
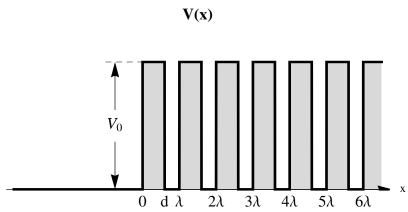


Fig. 7 Energy/length for an annular piston as function of angle (top), and compared to the energy/length for a rectangular piston (bottom).

7 Applications of multiple scattering

As an illustration of practical calculations using the multiple scattering machinery, we illustrate in Fig. 8 a semi-infinite array of periodic potentials, such as a array of dielectric slabs, for which the exact Casimir-Polder force with an atom to the left may be calculated [14].

Fig. 8 A semi-infinite array of periodic potentials. The exact CP force between an atom and this array may be calculated.



7.1 Casimir-Polder force

Consider an atom, of polarizability $\alpha(\omega)$, a distance Z to the left of the array. The Casimir-Polder energy is

$$E = -\int_{-\infty}^{\infty} d\zeta \int \frac{d^2k}{(2\pi)^2} \alpha(i\zeta) \operatorname{tr} \mathbf{g}(Z,Z), \tag{29}$$

where apart from an irrelevant constant the trace of the Green's function is

$$\operatorname{tr} \mathbf{g}(Z,Z) \to \frac{1}{2\kappa} \left[-\zeta^2 \mathscr{R}^{\mathrm{TE}} + (\zeta^2 + 2k^2) \mathscr{R}^{\mathrm{TM}} \right] e^{-2\kappa|Z|}. \tag{30}$$

Here the reflection coefficients are those for the entire array (a is the distance between the potential slabs),

$$\mathscr{R} = \frac{1}{2R} \left[e^{2\kappa a} + R^2 - T^2 - \sqrt{\left(e^{2\kappa a} - R^2 - T^2\right)^2 - 4R^2T^2} \right]. \tag{31}$$

If the potentials consist of dielectric slabs, with dielectric constant ε and thickness d, the TE reflection and transmission coefficients for a single slab are ($\kappa' = \sqrt{\varepsilon\zeta^2 + k^2}$)

$$R^{\mathrm{TE}} = \frac{e^{2\kappa' d} - 1}{\left(\frac{1+\kappa'/\kappa}{1-\kappa'/\kappa}\right) e^{2\kappa' d} - \left(\frac{1-\kappa'/\kappa}{1+\kappa'/\kappa}\right)}, \tag{32a}$$

$$T^{\mathrm{TE}} = \frac{4(\kappa'/\kappa) e^{\kappa' d}}{(1+\kappa'/\kappa)^2 e^{2\kappa' d} - (1-\kappa'/\kappa)^2}. \tag{32b}$$

The TM reflection and transmission coefficients are obtained by replacing, except in the exponents, $\kappa' \to \kappa'/\varepsilon$. (Multilayer potentials have been discussed extensively in the past, see, for example, [15, 16, 17, 18].)

For example, in the static limit, where we disregard the frequency dependence of the polarizability,

$$E = -\frac{\alpha(0)}{2\pi} \frac{1}{Z^4} F(a/Z, d/Z). \tag{33}$$

This is compared with the single slab result in Fig. 9. It is interesting to consider the $Z \to \infty$ limit, which is shown in Fig. 10. When $a/d \to 0$ we recover the bulk limit. Such results apparently will have applications to experiment rather soon [19].

8 Exact temperature results

The scalar Casimir energy between two weak nonoverlapping potentials $V_1(\mathbf{r})$ and $V_2(\mathbf{r})$ at temperature T is [20]

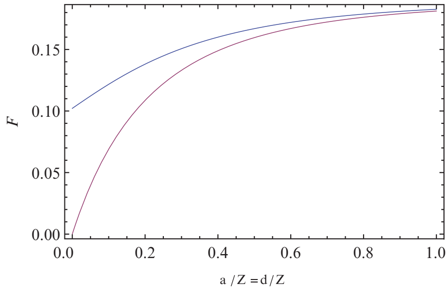


Fig. 9 Casimir-Polder energy between a semi-infinite array of dielectric slabs with $\varepsilon = 2$, compared to the energy (lower curve) if only one slab were present. Here we have assumed that the spacing between the slabs and the widths of the slabs are equal.

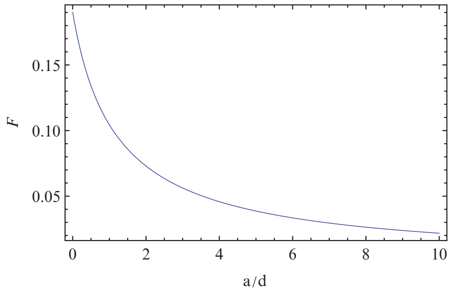


Fig. 10 Casimir-Polder energy for large distances from the array, as a function of the ratio a/d, where a is the distance between the dielectric slabs in the array, and d is the thickness of each slab. Here $\varepsilon = 2$.

$$E_T = -\frac{T}{32\pi^2}\int (d\mathbf{r})(d\mathbf{r}')V_1(\mathbf{r})V_2(\mathbf{r}')\frac{\coth 2\pi T|\mathbf{r}-\mathbf{r}'|}{|\mathbf{r}-\mathbf{r}'|^2}. \tag{34}$$

8.1 Exact proximity force approximation

From (34) we find that the energy between a semitransparent plane and an arbitrarily curved nonintersecting semitransparent surface is for weak coupling

$$E_T = -\frac{\lambda_1\lambda_2 T}{16\pi}\int dS \int_{2\pi T z(S)} dx \frac{\coth x}{x}, \tag{35}$$

where the area integral is over the curved surface. Here $z(S)$ is the distance between the plates at a given point on the surface S. Equation (35) is precisely what one means by the proximity force approximation:

$$E_{\text{PFA}} = \int dS \mathscr{E}_{\parallel}(z(S)), \tag{36}$$

as noted by Decca et al. [21]. See also [22].

8.2 Interaction between semitransparent spheres

We can, for weak scalar coupling, compute the energy between two spheres of radius a and b, whose centers are separated by a distance R:

$$E_T = -\frac{\lambda_1\lambda_2}{16\pi}\frac{ab}{R}\left\{\ln\frac{1-(a-b)^2/R^2}{1-(a+b)^2/R^2}+f(2\pi T(R+a+b))+f(2\pi T(R-a-b))\right.$$
$$\left.-f(2\pi T(R-a+b))-f(2\pi T(R+a-b))\right\}, \tag{37}$$

where $f(y)$ for $y<\pi$ is given by the power series,

$$f(y)=\sum_{n=1}^{\infty}\frac{2^{2n}B_{2n}}{2n(2n-1)(2n)!}y^{2n}, \tag{38}$$

which is obtained from the differential equation

$$y\frac{d^2}{dy^2}f(y)=\coth y-\frac{1}{y},\quad f(0)=f'(0)=0. \tag{39}$$

Results for the energy obtained by solving this differential equation are shown in Fig. 11. For further details see [20].

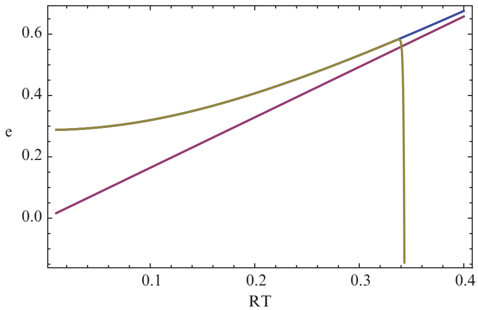


Fig. 11 Comparison between the general and high temperature forms of the energy, as a function of RT. Energies are shown for $a=b=R/4$. The high temperature result is linear in T. Also shown is the power series expansion truncated at 200 terms. which diverges in this case at $RT=1/3$. Plotted is $e=-16\pi RE/(\lambda_1\lambda_2a^2)$.

8.3 Mean distances between spheres

Encountered in the above calculation are mean powers of distances between spheres as defined by

$$\int d\Omega\, d\Omega'|\mathbf{r}-\mathbf{r}'|^p=(4\pi)^2R^pP_p(\hat{a},\hat{b}), \tag{40}$$

for spheres, of radii a and b, respectively, separated by a center-to-center distance R. Here $\hat{a} = a/R$ and $\hat{b} = b/R$, and $P_p(\hat{a},\hat{b})$ can in general be represented by the infinite series

$$P_p(\hat{a},\hat{b}) = \sum_{n=0}^{\infty} \frac{2}{(2n+2)!} \frac{\Gamma(2n-p-1)}{\Gamma(-p-1)} Q_n(\hat{a},\hat{b}). \tag{41}$$

Here the homogeneous polynomials Q_n are

$$Q_0 = 1, \tag{42a}$$
$$Q_1 = 2(\hat{a}^2+\hat{b}^2), \tag{42b}$$
$$Q_2 = 3\hat{a}^4+10\hat{a}^2\hat{b}^2+3\hat{b}^4, \tag{42c}$$
$$Q_3 = 4\hat{a}^6+28\hat{a}^4b^2+28\hat{a}^2\hat{b}^4+4\hat{b}^6. \tag{42d}$$

Here in general,

$$Q_n = \frac{1}{2}\sum_{m=0}^{n} \binom{2n+2}{2m+1} \hat{a}^{2(n-m)}\hat{b}^{2m}. \tag{43}$$

There is also a recursion relation,

$$P_{p-1}(\hat{a},\hat{b}) = \frac{R^{-p}}{1+p}\frac{\partial}{\partial R}R^{1+p}P_p(\hat{a},\hat{b}), \tag{44}$$

since Q_n is homogeneous in R of degree $-2n$.

For integer $p > -2$, P_p is a polynomial of degree $2\lceil p/2\rceil$, and we can immediately find

$$P_p(\hat{a},\hat{b}) = \frac{1}{4\hat{a}\hat{b}}\frac{1}{(p+2)(p+3)}\big[(1+\hat{a}+\hat{b})^{p+3} + (1-\hat{a}-\hat{b})^{p+3} - (1-\hat{a}+\hat{b})^{p+3} - (1+\hat{a}-\hat{b})^{p+3}\big], \tag{45}$$

Although this was derived for integer p it actually holds for all values of p.

For example, when p is a negative integer, we have the explicit forms, which are obtained from (45) by taking the appropriate limit:

$$P_{-1} = 1, \quad \text{Newton's theorem,} \tag{46a}$$
$$P_{-2} = \frac{1}{4\hat{a}\hat{b}}\left[\ln\frac{1-(\hat{a}+\hat{b})^2}{1-(\hat{a}-\hat{b})^2} + \hat{a}\ln\frac{(1+\hat{b})^2-\hat{a}^2}{(1-\hat{b})^2-\hat{a}^2} + \hat{b}\ln\frac{(1+\hat{a})^2-\hat{b}^2}{(1-\hat{a})^2-\hat{b}^2}\right], \tag{46b}$$
$$P_{-3} = -\frac{1}{4\hat{a}\hat{b}}\ln\frac{1-(\hat{a}+\hat{b})^2}{1-(\hat{a}-\hat{b})^2}, \tag{46c}$$
$$P_{-4} = \frac{1}{[1-(\hat{a}+\hat{b})^2][1-(\hat{a}-\hat{b})^2]}. \tag{46d}$$

and further expressions, which can be obtained by use of (44), may be readily verified.

9 Conclusions

The multiple scattering formalism can be used to find numerical results effectively in many situations, as we have seen in this outline. Weak coupling results are exact and often given in closed form. The method can also be used to extract not only interaction energies but self energies, as described in [23].

Acknowledgements We thank the US National Science Foundation and the US Department of Energy for partial support of this research. We thank Elom Abalo, Nima Pourtolami, and K. V. Shajesh for collaborative assistance. We dedicate this paper to Emilio Elizalde.

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PART II
Gravity and Cosmology

Brane Cosmology with an $f(R)$ contribution

Mariam Bouhmadi-López

Abstract A generalised induced gravity brane-world model is proposed. The brane action contains an arbitrary $f(R)$ term, R being the scalar curvature of the brane while the brane is 5-dimensional and is described by a Hilbert-Einstein action. It can be shown that the effect of the $f(R)$ term on the dynamics of a homogeneous and isotropic brane is twofold: (i) an evolving induced gravity parameter and (ii) a shift on the energy density of the brane. This new shift term, which is absent on the Dvali, Gabadadze and Porrati (DGP) model, plays a crucial role to self-accelerate the generalised normal DGP branch of our model. The stability of de Sitter solutions is analysed under homogeneous perturbations. These results are compared with the standard 4-dimensional one.

1 Introduction

Understanding the recent acceleration of the universe is one of the most challenging task nowadays in physics. The first evidence for the acceleration of the universe was provided by the analysis of the Hubble diagram of SNe Ia more than a decade ago [1]. This amusing discovery, together with (i) measurements of the fluctuations in the cosmic microwave background radiation (CMB) which implied that the universe is (quasi) spatially flat and (ii) that the amount of matter which clusters gravitationally is much less than the critical energy density, implied the existence of some stuff usually dubbed the *dark energy component* that drives the late-time acceleration of the universe. Afterwards, more precised measurements of the CMB anisotropy by WMAP [2] and the power spectrum of galaxy clustering by the 2dFGRS and SDSS surveys [3, 4] have confirmed this discovery.

A possible approach to describing the late-time acceleration of the universe is to consider a modified theory of gravity, such that a weakening of this interaction on

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_10,

the appropriate scales induces the recent speed up of the universe (cf. [5, 6, 7, 8]). In other words, the weakening of gravity on large scales would provide an *effective negative pressure* that would induce the late-time acceleration of the universe.

A possible approach is the Dvali, Gabadadze and Porrati (DGP) scenario [9, 10, 11], which corresponds to a five-dimensional (5D) brane-world model. In this model, our universe is a brane; i.e. a 4D hyper-surface, embedded in a Minkowski space-time. Matter is trapped on the brane and only gravity experiences the full bulk. The DGP model has two sets of solutions: the self-accelerating branch and the normal one. The self-accelerating brane, as its name suggests, speeds up at late-time without invoking any unknown dark energy component. On the other hand, the normal branch requires a dark energy component to accommodate the current observations [12, 13]. From a geometrical point of view, the two branches are embedded in a completely different way in the bulk [10]. Despite the very nice features of the self-accelerating DGP branch, it suffers from serious theoretical problems like the ghost issue [14]. The main aim of this paper is to consider a mech anism to self-accelerate the normal branch which is known to be free from the ghost issue [14].

This mechanism will be based on a modified Hilbert-Einstein action on the brane and the simplest gravitational option is to invoke an $f(R)$ term. Extended theories of gravity based on 4D $f(R)$ scenarios have gathered a lot of attention in the last years (cf. for example and reference cited there [5, 6, 7]). It has been shown that these 4D models should follow more or less the expansion of a LCDM universe [15, 16, 17] and could have distinctive signatures on the large scale structure of the universe [18, 19]. On the other hand, several methods have been invoked to reconstruct the shape of $f(R)$ from observations [20, 21, 22], for example, by using the dependence of the Hubble parameter with redshift which can be retrieved from astrophysical observations or a cosmographyc approach. We will show that an $f(R)$ term on the brane action can induce naturally self-accel eration on the normal DGP branch.

2 Induced gravity with an f(R) contribution on the brane action

We start considering a brane, described by a 4D hyper-surface (h with metric g), embedded in a 5D bulk space-time ($\mathscr{B}$, metric $g^{(5)}$), whose action is described by

$$\mathscr{S} = \int_{\mathscr{B}} d^5X \sqrt{-g^{(5)}} \left\{ \frac{1}{2\kappa_5^2} R[g^{(5)}] + \mathscr{L}_5 \right\} + \int_h d^4X \sqrt{-g} \left\{ \frac{1}{\kappa_5^2} K + \mathscr{L}_4 \right\}, \quad (1)$$

where κ_5^2 is the 5D gravitational constant, $R[g^{(5)}]$ is the scalar curvature in the bulk and K the extrinsic curvature of the brane in the higher dimensional bulk, corresponding to the surface boundary term [23]. We will assume that the bulk contains only a cosmological constant; i.e. $\mathscr{L}_5 = -U$. Consequently, the bulk space-time geometry is described by an Einstein space-time

$$G_{AB}[g^{(5)}] = -\kappa_5^2 U g^{(5)}_{AB}. \tag{2}$$

The 4D Lagrangian corresponds to

$$\mathscr{L}_4 = \alpha f(R) + \mathscr{L}_m, \tag{3}$$

where R is the scalar curvature of the induced metric on the brane, g, and α is a constant that measures the strength of the generalised induced gravity term $f(R)$ and has mass square units. Therefore, the function $f(R)$ has mass square units. On the other hand, $\mathscr{L}_m$ corresponds to the matter Lagrangian of the brane which in particular may include a brane tension. The previous action, includes as a particular case the DGP scenario [9, 10] when the bulk is flat, $f(R) = R$ and $\alpha = 1/2\kappa_4^2$ where κ_4^2 is proportional to the 4D gravitational constant.

We will be mainly interested in the cosmology of a homogeneous and isotropic brane, therefore, it is quite useful to follow the approach introduced by Shiromizu, Maeda and Sasaki in[1] [24]. Then, the projected Einstein equation on the brane reads, where we have assumed a mirror symmetry across the brane,

$$G_{\mu\nu}[g] = -\frac{1}{2}\kappa_5^2 U g_{\mu\nu} + \kappa_5^4 \Pi_{\mu\nu} - E_{\mu\nu}. \tag{4}$$

Here, $\Pi_{\mu\nu}$ corresponds to the quadratic energy momentum tensor [24]

$$\Pi_{\mu\nu} = -\frac{1}{4}\tau_{\mu\sigma}\tau_\nu{}^\sigma + \frac{1}{12}\tau\tau_{\mu\nu} + \frac{1}{8}g_{\mu\nu}(\tau_{\rho\sigma}\tau^{\rho\sigma} - \frac{1}{3}\tau^2), \tag{5}$$

and $E_{\mu\nu}$ is the (trace-free) projected Weyl tensor on the brane.

The total energy momentum on the brane can be defined as

$$\tau_{\mu\nu} \equiv -2\frac{\delta\mathscr{L}_4}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathscr{L}_4. \tag{6}$$

It is useful to split the previous energy momentum tensor into two terms

$$\tau_{\mu\nu} = \tau^{(m)}_{\mu\nu} + \tau^{(f)}_{\mu\nu}. \tag{7}$$

The first term $\tau^{(m)}_{\mu\nu}$ corresponds to the energy momentum tensor of matter (which include in particular the brane tension) on the brane. The second term

$$\tau^{(f)}_{\mu\nu} = -2\alpha\left\{\frac{df}{dR}R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu}\left[g_\mu{}^\alpha g_\nu{}^\beta - g_{\mu\nu}g^{\alpha\beta}\right]\nabla_\alpha\nabla_\beta\left(\frac{df}{dR}\right)\right\}. \tag{8}$$

[1] For a different approach to deduce the equations of evolution of a DGP brane with curvature modifications on the brane action see the references[25, 26]. See as well [27] for a brane-world model with an $f(R)$ term.

corresponds to the energy momentum tensor due to the generalised induced gravity term, $f(R)$, on the brane. Now, if f is proportional to the scalar curvature of the brane, then $\tau_{\mu\nu}^{(f)}$ is proportional to the Einstein tensor of the brane; i.e. the standard induced gravity brane-world scenario is recovered:

$$\tau_{\mu\nu}^{(f)} = -2\alpha G_{\mu\nu}. \tag{9}$$

Using the 5D Codacci equation, the bulk Einstein equation, and the junction condition at the brane, it turns out that the total energy momentum tensor of the brane is conserved $\tau_{\mu\nu}$ [24], i.e.

$$\nabla^\nu \tau_{\mu\nu} = 0. \tag{10}$$

On the other hand, because[2]

$$\nabla^\nu \tau_{\mu\nu}^{(f)} = 0, \tag{11}$$

we can conclude that the energy momentum tensor of matter on the brane is conserved

$$\nabla^\nu \tau_{\mu\nu}^{(m)} = 0. \tag{12}$$

3 Dynamics of a homogeneous and isotropic brane

In what follows, we consider a homogeneous and isotropic brane. The matter sector on the brane can be described by a perfect fluid with energy density $\rho^{(m)}$ and pressure $p^{(m)}$, where $\rho^{(m)}$ is conserved as we have pointed above. On the other hand, an effective energy density and an effective pressure associated to the energy momentum tensor coming from the $f(R)$ term on action can be defined as follows [29]

$$\rho^{(f)} = -2\alpha\left[3\left(H^2+\frac{k}{a^2}\right)f' - \frac{1}{2}(Rf'-f) + 3H\dot{R}f''\right], \tag{13}$$

$$p^{(f)} = 2\alpha\left\{\left(2\dot{H}+3H^2+\frac{k}{a^2}\right)f' - \frac{1}{2}(Rf'-f)\left[\ddot{R}f''+(\dot{R})^2 f'''+2H\dot{R}f''\right]\right\} \tag{14}$$

Notice that the definition of $\rho^{(f)}$ and $p^{(f)}$ is different from the standard 4D definition in $f(R)$ models [29]. On the other hand, the energy density is conserved on the brane.

The modified Friedmann equation on the brane can be written as

[2] We have proved this equation using the 4D Bianchi identity on the brane; i.e. $\nabla^\nu G_{\mu\nu} = 0$, and the relation between the non commutative character of two covariant derivatives and its relation to the Riemann curvature tensor (again on the brane), see for example equation 3.2.12 of [28]. Therefore, the conservation relation (11) can be proven in analogy to how it is done in the standard 4D $f(R)$ scenario.

$$3H^2 = \frac{\kappa_5^4}{12}\rho^2. \tag{15}$$

While, the spatial component of Einstein equation can be expressed as

$$2\dot{H} + 3H^2 = -\frac{\kappa_5^4}{12}\rho(\rho + 2p), \tag{16}$$

where the energy density ρ and the pressure p are defined as

$$\rho = \rho^{(m)} + \rho^{(f)}, \quad p = p^{(m)} + p^{(f)}. \tag{17}$$

For simplicity, on equations (15) and (16) we have used the spatially flat chart of the brane. We have also assumed no dark radiation on the brane; i.e. the bulk corresponds to a 5D maximally symmetric space-time. Notice that even in more general cases the dark radiation term will have no influence on the late-time dynamics of the brane as this term is constrained to be already subdominant by the time of nucleosynthesis [30].

4 de Sitter branes

A de Sitter space-time is the simplest cosmological solution that exhibits acceleration and therefore it is worthwhile to prove the existence of this solution in our model and study its stability. This would be a first step towards describing in a realistic way the late-time acceleration of the universe in an $f(R)$ brane-world model. This approach will also enable us to look for self-accelerating solutions on the modified normal DGP branch. So, in this section, we first obtain the fixed points of the model corresponding to a de Sitter space-time and then we study their stability under homogeneous perturbations.

4.1 Background solutions

In our model, the Hubble parameter for de Sitter solutions can be expressed as[3]

$$2\kappa_5^4\alpha^2F_0^2H_0^2 = 1 + \frac{1}{3}\kappa_5^4\alpha^2F_0(R_0F_0 - f_0) + \varepsilon\sqrt{1 + \frac{2}{3}\kappa_5^4\alpha F_0\left[\alpha(R_0F_0 - f_0)\right]} \tag{18}$$

here $\varepsilon = \pm 1$, the subscript 0 stands for quantities evaluated at the de Sitter space-time, $R_0 = 12H_0^2$ and $F = df/dR$. We recover the DGP model for $f(R) = R$. In fact,

[3] For a maximally symmetric brane in our model, the matter content of the brane behaves like a cosmological constant. As such a term can always be reabsorbed in the $f(R)$ term we will disregard the matter content in our analysis of de Sitter branes.

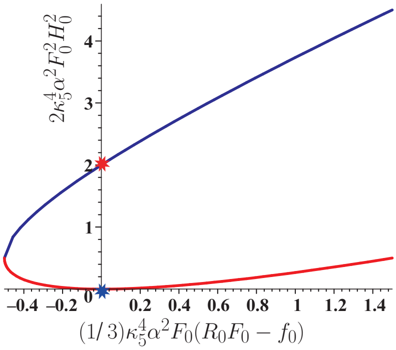

Fig. 1 The figure shows the behaviour of the rescaled squared Hubble rate $2\kappa_5^4\alpha^2F_0^2H_0^2$ for the two branches that generalise the DGP solution versus the rescaled energy density $\rho^{(c)}$ defined as $\frac{1}{3}\kappa_5^4\alpha^2F_0(R_0F_0-f_0)$. The blue star corresponds to the normal DGP branch which is flat. The red star corresponds to the self-accelerating DGP branch. On the other hand, the blue curve corresponds to the generalised (by the inclusion of the $f(R)$ term) self-accelerating branch, while the red curve corresponds to the generalised (by the inclusion of the $f(R)$ term) normal branch.

in that case, the de Sitter self-accelerating DGP branch is obtained for $\varepsilon = 1$ and the normal DGP branch or the non-self-accelerating solution for $\varepsilon = -1$. When the brane action contains curvature corrections to the Hilbert-Einstein action given by the brane scalar curvature, the branch with $\varepsilon = -1$ is no longer flat and accelerates (cf. Figs. 1, 2). Consequently, an $f(R)$ term on the brane action induce in a natural way self-acceleration on the normal branch. Most importantly, it is known that such a branch is free from the ghost problem (see [14] and references therein). The reason behind the self-acceleration of the generalised normal brane is the presence of the effective energy density

$$\rho_0^{(c)} = \alpha(F_0R_0 - f_0) \tag{19}$$

on the modified Friedmann equation on the brane. This can be easily shown by comparing the Friedmann equation (18) with that of modified gravity on brane world-models [31].

4.2 Stability of the self-accelerating solutions

We next analyse the stability of de Sitter solutions under homogeneous perturbations up to first order on $\delta H = H(t) - H_0$. We will follow the method used in [32].

The perturbed Friedmann equation (15) implies an evolution equation for δH:

$$\delta\ddot{H} + 3H_0\delta\dot{H} + m_{\text{eff}}^2\delta H = 0, \tag{20}$$

where m_{eff}^2 is defined as

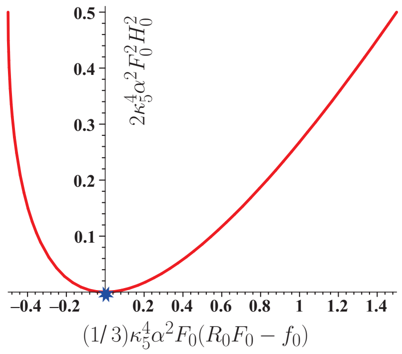

Fig. 2 The figure corresponds to a zoom of the normal branch as it appears on the figure 4.1.

$$m_{\rm eff}^2 = m_{(4)}^2 + m_{\rm shift}^2 + m_{\rm pert}^2. \tag{21}$$

where

$$\begin{aligned} m_{(4)}^2 &= \frac{1}{3}\left(\frac{F_0}{f_{RR}} - 2\frac{f_0}{F_0}\right), \\ m_{\rm back}^2 &= -\frac{2}{\alpha^2\kappa_5^4F_0^2}\left[1-\sqrt{1+\frac{2}{3}\alpha^2\kappa_5^4F_0(f_0-\kappa_5^2UF_0)}\right], \\ m_{\rm pert}^2 &= \frac{F_0}{3f_{RR}}\left[1-\sqrt{1+\frac{2}{3}\alpha^2\kappa_5^4F_0(f_0-\kappa_5^2UF_0)}\right]^{-1}. \end{aligned} \tag{22}$$

and $f_{RR} = d^2f/dR^2$. All this quantities are evaluated at the de Sitter background solution. Any de Sitter solution is stable as long as $m_{\rm eff}^2$ is positive.

The terms defined in (22) have the following physical meaning: (i) $m_{(4)}^2$ is the analogous quantity to $m_{\rm eff}^2$ in a 4D f(R) model [32], (ii) $m_{\rm back}^2$ is a purely background effect due to the shift on the Hubble parameter respect to the standard 4D case and (iii) $m_{\rm pert}^2$ is a purely perturbative extra-dimensional effect.

If we assume that we are close to the 4D regime; i.e. the Hubble rate of the brane is close to its analogous quantity in a 4D $f(R)$ model, then $m_{\rm back}^2 > 0$ and $m_{\rm pert}^2 < 0$. Consequently, $m_{\rm back}^2$ tends to make the perturbation heavier. However, the perturbative effect encoded on $m_{\rm pert}^2$ would make the perturbation lighter. It can be shown that the extra-dimension has a *benigner* effect in the 4D f(R) model; i.e. $m_{\rm eff}^2 > m_{(4)}^2$, as long as[4]

$$F_0^2 < 4f_0f_{RR}. \tag{23}$$

[4] We have assumed the natural condition $F_0 > 0$; i.e. the effective gravitational constant of the brane is positive. On the other hand, we have also assumed that we are slightly perturbing the Hilbert-Einstein action of the brane, i.e. $f_0 \sim R_0$. Therefore, f_0 is positive because $R_0 = 12H_0^2$.

5 Conclusions

A mechanism to self-accelerate the normal DGP branch has been presented which unlike the original self-accelerating DGP branch is known to be free of the ghost problem. The mechanism is based in including curvature modifications on the brane action. For simplicity, we choose those terms to correspond to an $f(R)$ contribution, which in addition is known to be the only higher order gravity theories that avoid the so called Ostrogradski instability in 4D models [7]. Notice as well that by embedding the DGP model in a higher dimensional space-time, the ghost issue present in the original DGP model may be cured [33] while preserving the existence of a self-accelerating solution [34]. See also [35, 36].

It is know that 4D $f(R)$ models are not free from theoretical problems, so in constructing an $f(R)$ brane-world model, we should of course try to avoid these theoretical troubles. We have just undertaken a first step towards constructing realistic self-accelerating solutions in the normal DGP branch. There are still many issues to be addressed, for example which $f(R)$ should we pick up to be in agreement with the cosmological observations and the solar system tests? We leave these interesting issues for future works.

Acknowledgements It is a pleasure to thank the conference organisers for their invitation. M.B.L. is supported by the Portuguese Agency Fundação para a Ciência e Tecnologia through the fellowship SFRH/BPD/26542/2006.

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$f(R)$-Gravity Matched With Large Scale Structure and Cosmological Observations

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Abstract The so called $f(R)$-gravity could be, in principle, able to explain the accelerated expansion of the Universe without adding unknown forms of dark energy/dark matter but, more simply, extending the General Relativity by generic functions of the Ricci scalar. However, a part several phenomenological models, there is no final $f(R)$-theory capable of fitting all the observations and addressing all the issues related to the presence of dark energy and dark matter. Astrophysical observations are pointing out huge amounts of "dark matter" and "dark energy" needed to explain the observed large scale structures and cosmic accelerating expansion. Up to now, no experimental evidence has been found, at fundamental level, to explain such mysterious components. The problem could be completely reversed considering dark matter and dark energy as "shortcomings" of General Relativity.

1 Introduction

Although being the best fit to a wide range of data, the ΛCDM model is affected by strong theoretical shortcomings that have motivated the search for alternative models [1]. Dark Energy (DE) models mainly rely on the implicit assumption that Einstein's General Relativity (GR) is the correct theory of gravity. Nevertheless, its validity on the larger astrophysical and cosmological scales has never been tested, and it is therefore conceivable that both cosmic speed up and Dark Matter (DM) represent signals of a breakdown of GR. Following this line of thinking, the choice of a generic function $f(R)$ as the gravitational Lagrangian, where R is the Ricci scalar, can be derived by matching the data and by the "economic" requirement that no exotic ingredients have to be added. This is the underlying philosophy of what are referred to as $f(R)$ gravity [2]. It is worth noticing that Solar System experiments show the validity of GR at these scales so that $f(R)$ theories should not differ too

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_11,

much from GR at this level [3]. In other words, the PPN limit of such models must not violate the experimental constraints on Eddington parameters. A positive answer to this request has been recently achieved for several $f(R)$ theories [4], nevertheless it has to be remarked that this debate is far to be definitively concluded. Although higher order gravity theories have received much attention in cosmology, since they are naturally able to give rise to the accelerating expansion (both in the late and in the early universe [5]), it is possible to demonstrate that $f(R)$ theories can also play a major role at astrophysical scales [6, 7]. In fact, modifying the gravity action can affect the gravitational potential in the low energy limit.

Provided that the modified potential reduces to the Newtonian one on the Solar System scale, this implication could represent an intriguing opportunity rather than a shortcoming for $f(R)$ theories. In fact, a corrected gravitational potential could offer the possibility to fit galaxy rotation curves without the need of Dark Matter. In addition, one could work out a formal analogy between the corrections to the Newtonian potential and the usually adopted Dark Matter models. In order to investigate the consequences of $f(R)$ theories on both cosmological and astrophysical scales, let us first remind the basics of this approach and then discuss dark energy and dark matter issues as curvature effects.

2 Dark energy as a curvature effect

From a mathematical viewpoint, $f(R)$ theories generalize the Hilbert - Einstein Lagrangian as $\mathscr{L} = \sqrt{-g}f(R)$ without assuming *a priori* the functional form of Lagrangian density in the Ricci scalar. The field equations are obtained by varying with respect to the metric components to get [8] :

$$f'(R)R_{\alpha\beta} - \frac{1}{2}f(R)g_{\alpha\beta} = f'(R)^{;\mu\nu}\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu}\right) + T^{M}_{\alpha\beta} \qquad (1)$$

where the prime denotes derivative with respect to the argument and $T^{M}_{\alpha\beta}$ is the standard matter stress - energy tensor. Defining the *curvature stress - energy tensor* as

$$T^{curv}_{\alpha\beta} = \frac{1}{f'(R)}\left\{\frac{1}{6}g_{\alpha\beta}\left[f(R) - Rf'(R)\right] + f'(R)^{;\mu\nu}(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\beta}g_{\mu\nu})\right\}. \qquad (2)$$

Eqs.(1) may be recast in the Einstein - like form as :

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = T^{curv}_{\alpha\beta} + T^{M}_{\alpha\beta}/f'(R) \qquad (3)$$

where matter non - minimally couples to geometry through the term $1/f'(R)$. The presence of term $f'(R)_{;\mu\nu}$ renders the equations of fourth order, while, for $f(R) = R$, the curvature stress - energy tensor $T^{curv}_{\alpha\beta}$ identically vanishes and Eqs.(3) reduce to the standard second - order Einstein field equations. As it is clear, from (3), the curvature stress - energy tensor formally plays the role of a further source term in the field equations so that its effect is the same as that of an effective fluid of purely geometrical origin.

However the metric variation is just one of the approaches towards $f(R)$ gravity: in fact, one can face the problem also considering the so called Palatini approach (e.g. see [9, 10]) where the metric and connection fields are considered independent. Apart from some differences in the interpretation, one can deal with a fluid of geometric origin in this case as well. The scheme outlined above provides all the ingredients we need to tackle with the dark side of the Universe. Depending on the scales, such a curvature fluid can play the role of DM and DE. From the cosmological point of view, in the standard framework of a spatially flat homogenous and isotropic Universe, the cosmological dynamics is determined by its energy budget through the Friedmann equations. The cosmic acceleration is achieved when the r.h.s. of the acceleration equation remains positive (in physical units with $8\pi G = c = 1$):

$$\frac{\ddot{a}}{a} = -\frac{1}{6}\left(\rho_{tot} + 3p_{tot}\right) , \tag{4}$$

where a is the scale factor, $H = \dot{a}/a$ the Hubble parameter, the dot denotes derivative with respect to cosmic time, and the subscript tot denotes the sum of the curvature fluid and the matter contribution to the energy density and pressure. From the above relation, the acceleration condition, for a dust dominated model, leads to:

$$\rho_{curv} + \rho_M + 3p_{curv} < 0 \rightarrow w_{curv} < -\frac{\rho_{tot}}{3\rho_{curv}} \tag{5}$$

so that a key role is played by the effective quantities:

$$\rho_{curv} = \frac{8}{f'(R)}\left\{\frac{1}{2}\left[f(R) - Rf'(R)\right] - 3H\dot{R}f''(R)\right\} , \tag{6}$$

and

$$w_{curv} = -1 + \frac{\ddot{R}f''(R) + \dot{R}\left[\dot{R}f'''(R) - Hf''(R)\right]}{\left[f(R) - Rf'(R)\right]/2 - 3H\dot{R}f''(R)} . \tag{7}$$

As a first simple choice, one may neglect ordinary matter and assume a power - law form $f(R) = f_0 R^n$, with n a real number, which represents a straightforward generalization of the Einstein GR in the limit $n = 1$. One can find power - law solutions for $a(t)$ providing a satisfactory fit to the SNeIa data and a good agreement with the estimated age of the Universe in the range $1.366 < n < 1.376$ [5]. The data fit turns out to be significant (see Fig. 1) improving the χ^2 value and, it fixes the best fit value at $n = 3.46$ when it is accounted only the baryon contribute $\Omega_b \approx 0.04$ (according with BBN prescriptions). It has to be remarked that considering DM does not modify the result of the fit, supporting the assumption of no need for DM in this model. From the evolution of the Hubble parameter in term of redshift one can even calculate the Age of Universe. The best fit value $n = 3.46$ provides $t_{univ} \approx 12.41$ Gyr. It is worth noting that considering $f(R) = f_0 R^n$ gravity represents only the

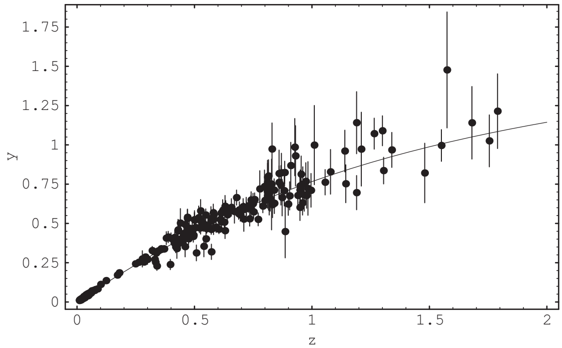

Fig. 1 The Hubble diagram of 20 radio galaxies together with the "gold" sample of SNeIa, in term of the redshift as suggested in [11]. The best fit curve refers to the $f(R)$ - gravity model without Dark Matter.

simplest generalization of Einstein theory. In other words, it has to be considered that R^n - gravity represents just a working hypothesis as there is no overconfidence that such a model is the correct final gravity theory. In a sense, we want only to suggest that several cosmological and astrophysical results can be well interpreted in the realm of a power law extended gravity model. This approach gives no rigidity about the value of the power n, although it would be preferable to determine a model capable of working at different scales. Furthermore, we do not expect to be able to reproduce the whole cosmological phenomenology by means of a simple power law model, which has been demonstrated to be not sufficiently versatile.

For example, we can demonstrate that this model fails when it is analyzed with respect to its capability of providing the correct evolutionary conditions for the perturbation spectra of matter overdensity [12]. This point is typically addressed as one of the most important issues which suggest the need for Dark Matter. In fact, if one wants to discard this component, it is crucial to match the observational results related to the Large Scale Structure of the Universe and the Cosmic Microwave Background which show, respectively at late time and at early time, the signature of the initial matter spectrum. As important remark, we note that the quantum spectrum of primordial perturbations, which provides the seeds of matter perturbations, can be positively recovered in the framework of R^n - gravity. In fact, $f(R) \propto R^2$ can represent a viable model with respect to CMBR data and it is a good candidate for cosmological Inflation. To develop the matter power spectrum suggested by this model, we resort to the equation for the matter contrast obtained in [12] in the case of fourth order gravity. This equation can be deduced considering the conformal Newtonian gauge for the perturbed metric [12] :

$$ds^2 = (1+2\psi)dt^2 - a^2(1+2\phi)\Sigma_{i=1}^{3}(dx^i) . \tag{8}$$

In GR, it is $\phi = -\psi$, since there is no anisotropic stress; in extended gravity, this relation breaks, in general, and the $i \neq j$ components of field equations give new relations between ϕ and ψ. In particular, for $f(R)$ gravity, due to nonvanishing $f_{R;i;j}$ (with $i \neq j$), the $\phi - \psi$ relation becomes scale dependent. Instead of the perturbation equation for the matter contrast δ, we provide here its evolution in term of the growth index $f = d\ln\delta/d\ln a$, that is the directly measured quantity at $z \sim 0.15$:

$$f'(a) - \frac{f(a)^2}{a} + \left[\frac{2}{a} + \frac{1}{a}E'(a)\right] f(a) - \frac{1-2Q}{2-3Q} \cdot \frac{3\Omega_m a^{-4}}{nE(a)^2\tilde{R}^{n-1}} = 0, \quad (9)$$

$E(a) = H(a)/H_0$, $\tilde{R}$ is the dimensionless Ricci scalar, and

$$Q = -\frac{2f_{RR}c^2k^2}{f_R a^2}. \quad (10)$$

For $n = 1$ the previous expression gives the ordinary growth index relation for the Cosmological Standard Model. It is clear, from (9), that such a model suggests a scale dependence of the growth index which is contained into the corrective term Q so that, when $Q \to 0$, this dependence can be reasonably neglected. In the most general case, one can resort to the limit $aH < k < 10^{-3}hMpc^{-1}$, where (9) is a good approximation, and non-linear effects on the matter power spectrum can be neglected.
Studying numerically (9), one obtains the growth index evolution in term of the scale factor; for the sake of simplicity, we assume the initial condition $f(a_{ls}) = 1$ at the last scattering surface as in the case of matter-like domination. The results are summarized in Fig.(2), where we show, in parallel, the growth index evolution in R^n - gravity and in the ΛCDM model.

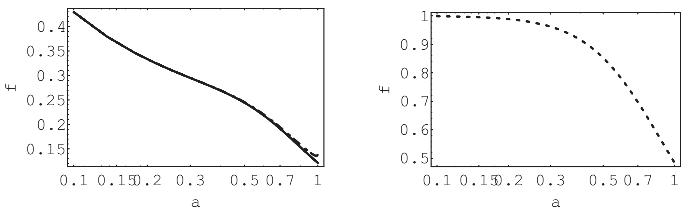

Fig. 2 Scale factor evolution of the growth index f: (*left*) modified gravity, in the case $\Omega_m = \Omega_{bar} \sim 0.04$, for the SNeIa best fit model with $n = 3.46$, (*right*) the same evolution in the case of a ΛCDM model. In the case of R^n - gravity it is shown also the dependence on the scale k. The three cases $k = 0.01,\ 0.001,\ 0.0002$ have been checked. Only the latter case shows a very small deviation from the leading behavior.

In the case of $\Omega_m = \Omega_{bar} \sim 0.04$, one can observe a strong disagreement between the expected rate of the growth index and the behavior induced by power law fourth order gravity models. These results seem to suggest that an extended gravity model which considers a simple power law of Ricci scalar, although cosmologically relevant at late times, is not viable to describe the evolution of Universe at all scales. In other words, such a scheme seems too simple to give account for the whole cosmological phenomenology. In fact, in [12] a gravity Lagrangian considering an exponential correction to the Ricci scalar $f(R) = R + A\exp(-BR)$ (with A, B two constants), gives more interesting results and displays a grow factor rate which is in agreement with the observational results at least in the Dark Matter case. To corroborate this point of view, one has to consider that when the choice of $f(R)$ is performed starting from observational data (pursuing an inverse approach) as in [14], the reconstructed Lagrangian is a non - trivial polynomial in term of the Ricci scalar. A result which directly suggests that the whole cosmological phenomenology can be accounted only with a suitable non - trivial function of the Ricci scalar rather than a simple power law function. As matter of fact, the results obtained with respect to the study of the matter power spectra in the case of R^n - gravity do not invalidate the whole approach, since they can be referred to the too simple form of the model.

3 Dark matter as a curvature effect

The results obtained at cosmological scales motivates further analysis of $f(R)$ theories. In a sense, one is wondering whether the curvature fluid, which works as DE, can also play the role of effective DM thus yielding the possibility of recovering the observed astrophysical phenomenology by the only visible matter. It is well known that, in the low energy limit, higher order gravity implies a modified gravitational potential. Therefore, in our discussion, a fundamental role is played by the new gravitational potential descending from the given fourth order gravity theories we are referring to. By considering the case of a pointlike mass m and solving the vacuum field equations for a Schwarzschild - like metric, one gets from a theory $f(R) = f_0 R^n$, the modified gravitational potential [6] :

$$\Phi(r) = -\frac{Gm}{2r}\left[1 + \left(\frac{r}{r_c}\right)^{\beta}\right] \tag{11}$$

where

$$\beta = \frac{12n^2 - 7n - 1 - \sqrt{36n^4 + 12n^3 - 83n^2 + 50n + 1}}{6n^2 - 4n + 2} \tag{12}$$

which corrects the ordinary Newtonian potential by a power - law term. In particular, this correction sets in on scales larger than r_c which value depends essentially on the mass of the system. The corrected potential (11) reduces to the standard $\Phi \propto 1/r$ for $n = 1$ as it can be seen from the relation (12).

The result (11) deserves some comments. As discussed in detail in [6], we have assumed the spherically symmetric metric and imposed it into the field equations (1) considered in the weak field limit approximation. As a result, we obtain a corrected Newtonian potential which accounts for the strong non-linearity of gravity related to the higher-order theory. However, we have to notice that Birkhoff's theorem does not hold, in general, for $f(R)$ gravity but other spherically symmetric solutions than the Schwarzschild one can be found in these extended theories of gravity [15]. The generalization of (11) to extended systems is achieved by dividing the system in infinitesimal mass elements and summing up the potentials generated by each single element. In the continuum limit, we replace the sum with an integral over the mass density of system taking care of eventual symmetries of the mass distribution (see [6] for details). Once the gravitational potential has been computed, one may evaluate the rotation curve $v_c^2(r)$ and compare it with the data. For extended systems, one has typically to resort to numerical techniques, but the main effect may be illustrated by the rotation curve for the pointlike case, that is:

$$v_c^2(r) = \frac{Gm}{2r}\left[1+(1-\beta)\left(\frac{r}{r_c}\right)^{\beta}\right] . \tag{13}$$

Compared with the Newtonian result $v_c^2 = Gm/r$, the corrected rotation curve is modified by the addition of the second term in the r.h.s. of (13). For $0 < \beta < 1$, the corrected rotation curve is higher than the Newtonian one. Since measurements of spiral galaxies rotation curves signals a circular velocity higher than those which are predicted on the basis of the observed luminous mass and the Newtonian potential, the above result suggests the possibility that our modified gravitational potential may fill the gap between theory and observations without the need of additional DM.

It is worth noting that the corrected rotation curve is asymptotically vanishing as in the Newtonian case, while it is usually claimed that observed rotation curves are flat (i.e., asymptotically constant). Actually, observations do not probe v_c up to infinity, but only show that the rotation curve is flat within the measurement uncertainties up to the last measured point. This fact by no way excludes the possibility that v_c goes to zero at infinity. In order to observationally check the above result, we have considered a sample of LSB galaxies with well measured HI + Hα rotation curves extending far beyond the visible edge of the system. LSB galaxies are known to be ideal candidates to test Dark Matter models since, because of their high gas content, the rotation curves can be well measured and corrected for possible systematic errors by comparing 21 - cm HI line emission with optical Hα and [NII] data. Moreover, they are supposed to be Dark Matter dominated so that fitting their rotation curves without this elusive component is a strong evidence in favor of any successful alternative theory of gravity.

Our sample contains 15 LSB galaxies with data on both the rotation curve, the surface mass density of the gas component and R - band disk photometry extracted from a larger sample selected by de Blok & Bosma [16]. We assume the stars are distributed in an infinitely thin and circularly symmetric disk with surface density $\Sigma(r) = \Upsilon_\star I_0 exp(-r/r_d)$ where the central surface luminosity I_0 and the disk scale-length r_d are obtained from fitting to the stellar photometry. The gas surface density has been obtained by interpolating the data over the range probed by HI measurements and extrapolated outside this range. When fitting to the theoretical rotation curve, there are three quantities to be determined, namely the stellar mass - to - light (M/L) ratio, $\Upsilon_\star$ and the theory parameters (β, r_c). It is worth stressing that, while fit results for different galaxies should give the same β, r_c is related to one of the integration constants of the field equations. As such, it is not a universal quantity and its value must be set on a galaxy - by - galaxy basis. However, it is expected that galaxies having similar properties in terms of mass distribution have similar values of r_c so that the scatter in r_c must reflect somewhat the scatter in the circular velocities. In order to match the model with the data, we perform a likelihood analysis determining for each galaxy, using, as fitting parameters β, $\log r_c$ (with r_c in kpc) and the gas mass fraction[1] f_g. As it is evident considering the results from the different fits, the experimental data are successfully fitted by the model (see [6] for details). In particular, for the best fit range of β ($\beta = 0.80 \pm 0.08$), one can conclude that R^n gravity with $2.3 < n < 5.3$ (best fit value $n = 3.2$ which well overlaps the above mentioned range of n fitting SNeIa Hubble diagram) can be a good candidate to solve the missing matter problem in LSB galaxies without any Dark Matter.

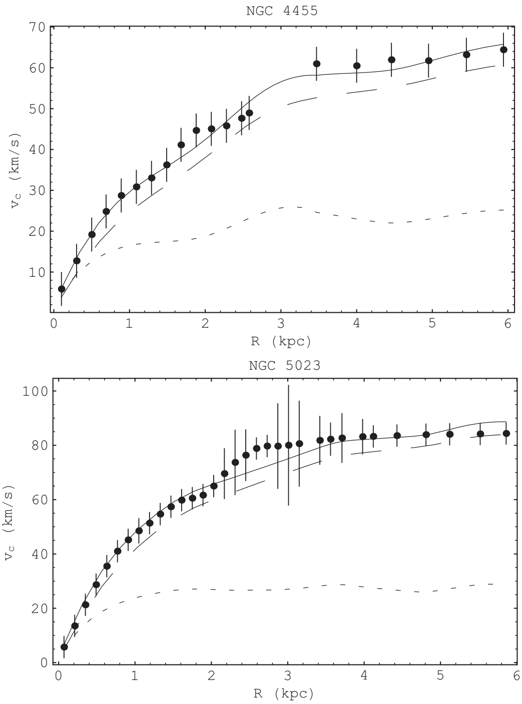

Fig. 3 Best fit theoretical rotation curve superimposed to the data for the LSB galaxy NGC 4455 (left) and NGC 5023 (right). To better show the effect of the correction to the Newtonian gravitational potential, we report the total rotation curve $v_c(r)$ (solid line), the Newtonian one (short dashed) and the corrected term (long dashed).

[1] This is related to the M/L ratio as $\Upsilon_\star = [(1-f_g)M_g]/(f_g L_d)$ with $M_g = 1.4 M_{HI}$ the gas (HI + He) mass, $M_d = \Upsilon_\star L_d$ and $L_d = 2\pi I_0 r_d^2$ the disk total mass and luminosity.

At this point, it is worth wondering whether a link may be found between R^n gravity and the standard approach based on Dark Matter haloes since both theories fit equally well the same data. As a matter of fact, it is possible to define an *effective Dark Matter halo* by imposing that its rotation curve equals the correction term to the Newtonian curve induced by R^n gravity. Mathematically, one can split the total rotation curve derived from R^n gravity as $v_c^2(r) = v_{c,N}^2(r) + v_{c,corr}^2(r)$ where the second term is the correction. Considering, for simplicity a spherical halo embedding a thin exponential disk, we may also write the total rotation curve as $v_c^2(r) = v_{c,disk}^2(r) + v_{c,DM}^2(r)$ with $v_{c,disk}^2(r)$ the Newtonian disk rotation curve and $v_{c,DM}^2(r) = GM_{DM}(r)/r$ the Dark Matter one, $M_{DM}(r)$ being its mass distribution. Equating the two expressions, we get:

$$M_{DM}(\eta) = M_{vir}\left(\frac{\eta}{\eta_{vir}}\right)\frac{2^{\beta-5}\eta_c^{-\beta}(1-\beta)\eta^{\frac{\beta-5}{2}}\mathscr{I}_0(\eta) - \mathscr{V}_d(\eta)}{2^{\beta-5}\eta_c^{-\beta}(1-\beta)\eta^{\frac{\beta-5}{2}}\mathscr{I}_0(\eta_{vir}) - \mathscr{V}_d(\eta_{vir})}\,. \tag{14}$$

with $\eta = r/r_d$, $\Sigma_0 = \Upsilon_\star i_0$, $\mathscr{V}_d(\eta) = I_0(\eta/2)K_0(\eta/2) \times I_1(\eta/2)K_1(\eta/2)^2$ and:

$$\mathscr{I}_0(\eta,\beta) = \int_0^\infty \mathscr{F}_0(\eta,\eta',\beta)k^{3-\beta}\eta'^{\frac{\beta-1}{2}}\mathrm{e}^{-\eta'}d\eta' \tag{15}$$

with $\mathscr{F}_0$ only depending on the geometry of the system and "*vir*" indicating virial quantities. (14) defines the mass profile of an effective spherically symmetric Dark Matter halo whose ordinary rotation curve provides the part of the corrected disk rotation curve due to the addition of the curvature corrective term to the gravitational potential. It is evident that, from an observational viewpoint, there is no way to discriminate between this dark halo model and R^n gravity.

Having assumed spherical symmetry for the mass distribution, it is immediate to compute the mass density for the effective dark halo as $\rho_{DM}(r)=(1/4\pi r^2)dM_{DM}/dr$. The most interesting features of the density profile are its asymptotic behaviors that may be quantified by the logarithmic slope $\alpha_{DM} = d\ln\rho_{DM}/d\ln r$ which can be computed only numerically as function of η for fixed values of β (or n). As expected, α_{DM} depends explicitly on β, while (r_c, Σ_0, r_d) enter indirectly through η_{vir}. The asymptotic values at the center and at infinity denoted as α_0 and α_∞ result particularly interesting. It turns out that α_0 almost vanishes so that in the innermost regions the density is approximately constant. Indeed, $\alpha_0 = 0$ is the value corresponding to models having an inner core such as the cored isothermal sphere and the Burkert model [17]. Moreover, it is well known that galactic rotation curves are typically best fitted by cored dark halo models. On the other hand, the outer asymptotic slope is between -3 and -2, that are values typical of most dark halo models in literature. In particular, for $\beta = 0.80$ one finds $(\alpha_0, \alpha_\infty) = (-0.002, -2.41)$, which are quite similar to the value for the Burkert model $(0, -3)$. It is worth noting that the Burkert model has been empirically proposed to provide a good fit to the LSB and dwarf galaxies rotation curves. The values of $(\alpha_0, \alpha_\infty)$ we find for the best fit effec-

[2] Here I_l and K_l, with $l = 1,2$ are the Bessel functions of first and second type.

tive dark halo therefore suggest a possible theoretical motivation for the Burkert-like models. Due to the construction, the properties of the effective Dark Matter halo are closely related to the disk one. As such, we do expect some correlation between the dark halo and the disk parameters. To this aim, exploiting the relation between the virial mass and the disk parameters , one can obtain a relation for the Newtonian virial velocity $V_{vir} = GM_{vir}/r_{vir}$:

$$M_d \propto \frac{(\frac{3}{4\pi}\delta_{th}\Omega_m\rho_{crit})^{\frac{1-\beta}{4}} r_d^{\frac{1+\beta}{2}} \eta_c^{\beta}}{2^{\beta-6}(1-\beta)G^{\frac{5-\beta}{4}}} \frac{V_{vir}^{\frac{5-\beta}{2}}}{\mathscr{I}_0(V_{vir},\beta)} \,. \tag{16}$$

We have numerically checked that (16) may be well approximated as $M_d \propto V_{vir}^a$ which has the same formal structure as the baryonic Tully - Fisher (BTF) relation $M_b \propto V_{flat}^a$ with M_b the total (gas + stars) baryonic mass and V_{flat} the circular velocity on the flat part of the observed rotation curve. In order to test whether the BTF can be explained thanks to the effective Dark Matter halo we are proposing, we should look for a relation between V_{vir} and V_{flat}. This is not analytically possible since the estimate of V_{flat} depends on the peculiarities of the observed rotation curve such as how far it extends and the uncertainties on the outermost points. For given values of the disk parameters, we therefore simulate theoretical rotation curves for some values of r_c and measure V_{flat} finally choosing the fiducial value for r_c that gives a value of V_{flat} as similar as possible to the measured one. Inserting the relation thus found between V_{flat} and V_{vir} into (16) and averaging over different simulations, we finally get :

$$\log M_b = (2.88 \pm 0.04) \log V_{flat} + (4.14 \pm 0.09) \tag{17}$$

while a direct fit to the observed data gives [18] :

$$\log M_b = (2.98 \pm 0.29) \log V_{flat} + (3.37 \pm 0.13) \,. \tag{18}$$

The slope of the predicted and observed BTF are in good agreement thus leading further support to our approach. The zeropoint is markedly different with the predicted one being significantly larger than the observed one. However, it is worth stressing that both relations fit the data with similar scatter. A discrepancy in the zeropoint can be due to our approximate treatment of the effective halo which does not take into account the gas component. Neglecting this term, we should increase the effective halo mass and hence V_{vir} which affects the relation with V_{flat} leading to a higher than observed zeropoint. Indeed, the larger is M_g/M_d, the more the points deviate from our predicted BTF thus confirming our hypothesis. Given this caveat, we can conclude, with confidence, that R^n gravity offers a theoretical foundation even for the empirically found BTF relation.

Although the results outlined along this paper are referred to a simple choice of fourth order gravity models ($f(R) = f_0 R^n$) they could represent an interesting paradigm. In fact, even if such a model is not suitable to provide the correct form of the matter power spectra, and this suggests that a more complicated Lagrangian is needed to reproduce the whole dark sector phenomenology at all scales, we have shown that considering extensions of GR can allow to explain some important issues of cosmological and astrophysical phenomenology. We have seen that extended gravity models can reproduce SNeIa Hubble diagram without Dark Matter, giving significant predictions even with regard to the age of Universe. In addition, the modification of the gravitational potential which arises as a natural effect in the framework of higher order gravity can represent a fundamental tool to interpret the flatness of rotation curves of LSB galaxies. Furthermore, if one considers the model parameters settled by the fit over the observational data on LSB rotation curves, it is possible to construct a phenomenological analogous of Dark Matter halo whose shape is similar to the one of the Burkert model. Since the Burkert model has been empirically introduced to give account of the Dark Matter distribution in the case of LSB and dwarf galaxies, this result could represent an interesting achievement since it gives a theoretical foundation to such a model. By investigating the relation among dark halo and the disk parameters, we have deduced a relation between M_d and V_{flat} which reproduces the baryonic Tully - Fisher. In fact, exploiting the relation between the virial mass and the disk parameters, one can obtain a relation for the virial velocity which can be satisfactory approximated as $M_d \propto V_{vir}^a$. Even such a result seems very intriguing since it gives again a theoretical interpretation for a phenomenological relation. As a matter of fact, although not definitive, these results on $f(R)$ can represent a viable approach for future investigations and in particular support the quest for a unified view of the Dark Side of the Universe that could be interpreted as gravitational effects indeed.

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An analysis of the phase space of Hořava-Lifshitz cosmologies

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Abstract Using the dynamical system approach, properties of cosmological models based on the Hořava-Lifshitz gravity are systematically studied. A result of this investigation is that in the detailed balance case one of the attractors in the theory corresponds to the oscillatory behavior described by Brandenberger. Instead the cosmological models generated by Hořava-Lifshitz gravity without the detailed balance assumption have indeed the potential to describe the transition between the Friedmann and the dark energy eras. The whole analysis leads to the plausible conclusion that a cosmology compatible with the present observations of the universe can be achieved only if the detailed balance condition is broken.

1 Introduction

Recently, Hořava made a proposal for an ultraviolet completion of general relativity (GR) [1] which seems to be renormalizable (at least at the level of power counting) by introducing irrelevant operators that explicitly break Lorentz invariance. Lorentz invariance is expected to be recovered at low energies, as an accidental symmetry of the theory.

Originally, this Hořava-Lifshitz (HL) theorywas formulated imposing the so-called *projectability condition* and the *detailed balance condition*. The first condition is related to the space-time dependence of the lapse function, N, which characterizes a canonical $3+1$ decomposition of the metric field g, while the second is

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_12,

a restriction on the form of the potential terms which may appear in the Lagrangian that leads to simplifications since it reduces the final number of couplings.

Presently—as this article is being written—the consistency status of the theory is not completely clear, nor its low energy limit [2, 3, 4, 5, 6, 7, 8, 9]. In the projectable version seems to be less problematic, since the above listed problems can in principle be evaded by the non-local form of the Hamiltonian constraint [6]. Also, imposing detailed balance leads to a cosmological constant with the *wrong sign*, which is in contrast with cosmological observations [1, 10, 11]. If detail balance is not imposed a richer phenomenology seems to appear, where cosmological applications may lead to new results in inflation, bouncing cosmology, dark matter, and dark energy (see, for example, [10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24]).

At this point it is important to investigate the key aspects of the theory, which may help in clarifying the status of the different HL proposals as plausible candidates of a quantum theory of gravity. In the present work we present the main results of [25] which investigate the *cosmological phase space* of the HL model. To address this involved issue, we will use the dynamical system approach to cosmology as formulated by Collins and applied by Wainwright and Ellis [26]. Such approach has the advantage of relying on dynamical system variables which are directly related to cosmological observables—like the matter density parameter—and of being relatively easy to apply to very complicated cosmological models. During the past years this method has been able to unfold some very interesting properties of Bianchi universes [26], as well as of scalar-tensor and higher-order gravity cosmological models (for some detailed examples, see [27, 28, **?**, 30]).

2 Hořava-Lifshitz gravity

In the HL theory, the dynamical variables are defined to be the laps N, the shift N_i and the space metric g_{ij}, Latin indices running from 0 to 3. The space-time metric is defined using the ADM construction, as

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \,, \tag{1}$$

where $N^i = g^{ij} N_j$ as usual. The action S is written in terms of geometric objects, characteristics of the ADM slicing of space-time, like the 3d-covariant derivative ∇_i, the spatial curvature tensor is R_{ijkl}, and the extrinsic curvature K_{ij} is defined as

$$K_{ij} = \frac{1}{2N}\left(-\dot{g}_{ij} + \nabla_i N_j + \nabla_j N_i\right), \tag{2}$$

where the dot stands for time derivative.

In terms of the above tensor fields, the HL action can be written as

$$S = \int dt\, dx^3 N \sqrt{g}\left(\mathscr{L}_{kinetic} - \mathscr{L}_{potential} + \mathscr{L}_{matter}\right) \,, \tag{3}$$

being the kinetic term universally given by

$$\mathscr{L}_{kinetic} = \alpha(K_{ij}K^{ij} - \lambda K^2)\,, \tag{4}$$

with α and λ playing the role of coupling constants. The potential term is, in principle, a generic function of R_{ijkl} and ∇_i. Here we will work with two different types of potentials: the detailed balance potential [10]

$$\mathscr{L}_{potential-detail} = \beta C_{ij}C^{ij} + \gamma\varepsilon^{ijk}R_{il}\nabla_j R^l_k + \zeta R_{ij}R^{ij} + \eta R^2 + \delta R + \sigma\,, \tag{5}$$

where $\sqrt{g}C^{ij} = \varepsilon^{ikl}\nabla_k(R^j_l - 1/4Rg^j_l)$, and the potential defined in [12] (the SVW case),

$$\begin{aligned}\mathscr{L}_{potential-SVW} = {} & g_8\nabla_i R_{jk}\nabla^i R^{jk} + g_7 R\nabla^2 R + g_6 R^i_j R^j_k R^k_i + g_5 R(R_{jk}R^{jk}) + g_4 R^3 \\ & + g_3 R_{jk}R^{jk} + g_2 R^2 + g_1 R + g_0\,,\end{aligned} \tag{6}$$

where the coupling constants are not all dimensionless. Finally, the matter term corresponds to the coupling of the matter fields to gravity.

Apart the assumption $\lambda \neq 1/3$ we will constraint as little as possible the different ranges of values on each coupling constant to see how much information comes out of the dynamical system approach. Then, we will add this information to the constraints arising from other considerations [33], to finally obtain the most promising form of the potential.

If one considers a FLRW *ansatz* on the 4D metric, only a subset of the coupling constants plays a role in the dynamics and, being N a function of time only by definition, all issues related to the projectability condition turn out to be irrelevant. Then, we set $N \longrightarrow N(t)$, $N_i \longrightarrow 0$, $g_{ij} \longrightarrow a(t)\gamma_{ij}$ where γ_{ij} is a maximally symmetric 3D metric, of constant curvature $R = 6k$, with $k = (-1, 0, 1)$.

The inclusion of matter content in the theory in its general form has not been worked out yet [10, 4, 34, 35]. Here, we will add to the gravity field equations a cosmological stress-energy tensor, such that in the low-energy limit we recover the usual GR formulation. Since one of our goals is to investigate the relation between HL and dark energy (cosmic acceleration), we will only consider here $p = w\rho$ with $w > 0$, so that dark energy cannot be introduced by hand in the model.

3 Hořava-Lifshitz cosmology: detailed balance case

Let us write the relevant field equations on a FLRW ansatz. We found that in the detailed balance case, the system can be written as follows

$$\alpha(3\lambda-1)\left[\dot{H}+H^2\right] = -\frac{1}{6}(1+3w)\rho + \mathbb{A}\alpha(1-3\lambda)\frac{k^2}{a^4} + \mathbb{A}\alpha(3\lambda-1)\Lambda^2\,, \tag{7}$$

$$\alpha(3\lambda-1)\left[3H^2+6\mathbb{A}\Lambda\frac{k}{a^2}\right]=\rho+3\mathbb{A}\alpha(3\lambda-1)\frac{k^2}{a^4}+3\mathbb{A}\alpha(3\lambda-1)\Lambda^2\,,\quad (8)$$

$$\dot{\rho}+3(w+1)H\rho=0\,,\quad (9)$$

where we have chosen to leave the parameters α,λ,Λ explicit because of their cosmological relevance, and we have defined the variable $\mathbb{A}=\frac{-\zeta}{\alpha(1-3\lambda)^2}$ which is always positive, since $\zeta\leq 0$ owing to the detailed balance constraint. Notice that, already at this level, only a subset of all the initially defined couplings $(\alpha,\lambda,\Lambda,\mathbb{A})$, plays a role in the cosmology.

In order to analyze the phase space of this cosmological model, let us define the variables

$$\Omega=\frac{\rho}{3\alpha H^2}\,,\qquad z=\frac{\mathbb{A}\Lambda^2}{H^2}\,,\qquad K=2\Lambda\frac{k\mathbb{A}}{a^2H^2}\,,\qquad C=\frac{k^2\mathbb{A}}{a^4H^2}\,,\quad (10)$$

with the cosmic time $\mathcal{N}=\ln[a(t)]$.

The cosmological equations (7)-(9) are then equivalent to the system [25]

$$z'=z[3+C+K-3z-3w(C-K+z-1)]\,,\quad (11)$$

$$C'=C\,[C+K-3z-3w(C-K+z-1)-1]\,,\quad (12)$$

$$K'=K\,[1+C+K-3z-3w(C-K+z-1)]\quad (13)$$

$$1+K-z-C+\frac{\Omega}{1-3\lambda}=0.\quad (14)$$

This system presents three invariant submanifolds $z=0$, $K=0$, $C=0$ which, by definition, cannot be crossed by any orbit. This implies that no global attractor can exist in this type of HL cosmology [25]. Also, the structure of (14) reveals that the system is non-compact and asymptotic analysis would be required in order to complete the study of the phase space. However here we will limit ourselves to the finite analysis. We refer the reader to [25] for a complete asymptotic analysis.

The finite fixed points can be found setting z',C',K' in (11)-(13) to be zero and solving the corresponding algebraic equations. The results are shown in Table 1. The solutions associated to the fixed point can be derived from the Raychaudhuri equation

$$\dot{H}=\frac{1}{2}\,[3z-C-K-3+3w(C-K+z-1)]\,H^2\,,\quad (15)$$

and the results are shown in Table 1, too. As one can see there, we have a general Friedmann solution which depends on the barotropic factor w of standard matter, a pure radiation like solution, a Milne universe, and an exponential solution. Note that the Friedmann solution is generated when the term related to the spatial curvature k is dominant. Instead, when the effective Hořava radiation is dominant, the corresponding cosmology presents a "radiation-dominated like" solution, as expected from the form of the corresponding terms in (7)-(9). Substitution into Eqs. (7)-(9) reveals that this is actually a vacuum solution and that the exponential solution corresponds to a an oscillating evolution with period $T=2\pi\frac{3\lambda-1}{\Lambda}\sqrt{\frac{\alpha}{|\zeta|}}$. Such solution can be connected to the scenario proposed in [15]. The Milne solution, instead, is not an actual solution of the system[1].

The stability of the fixed points can be determined by evaluating the eigenvalues of the Jacobian matrix associated with the system (11)-(13), as prescribed by the Hartman-Grobman Theorem [31]. The results can be found in Table 1. We can observe that the thermodynamical properties of matter influence the stability of the Friedmann point $\mathscr{A}$ and of the point $\mathscr{B}$. This means that, if $0 < w < 1/3$, the typical completely finite orbit implies an initial radiation-like behavior that evolves towards a Friedmann or a Milne behavior (or both), to eventually approach an oscillating state. Instead, if $1/3 < w < 1$, $\mathscr{A}$ is a source so that the typical orbit will start with a Friedmann evolution and evolve towards a radiation-like or a Milne evolution before converging to an oscillatory behavior. In both these scenarios we *do not* find any transition to a dark energy era.

Table 1 Finite fixed points of the system (11)-(13) and their associated solutions.

Point	Coordinates $[\Omega, z, C, K]$	Solution	Energy density	Stability
$\mathscr{A}$	$[3\lambda-1,0,0,0]$	$a=a_0(t-t_0)^{\frac{2}{3(1+w)}}$	$\rho=\rho_0(t-t_0)^{-2}$	$\begin{cases}\text{saddle} & 0\le w\le 1/3\\ \text{repeller} & 1/3\le w\le 1\end{cases}$
$\mathscr{B}$	$[0,0,1,0]$	$a=a_0(t-t_0)^{\frac{1}{2}}$	$\rho=0$	$\begin{cases}\text{repeller} & 0\le w\le 1/3\\ \text{saddle} & 1/3\le w\le 1\end{cases}$
$\mathscr{C}$	$[0,0,0,-1]$	$a=a_0(t-t_0)$	$\rho=0$	saddle
$\mathscr{D}$	$[0,1,0,0]$	$a=a_0e^{\tau(t-t_0)}$	$\rho=0$	attractor

$$\tau = i\sqrt{\frac{|\zeta|}{\alpha}\frac{\Lambda}{3\lambda-1}}$$

4 Hořava-Lifshitz cosmology: no detailed balance case (SVW potential)

If we do not impose detailed balance, the cosmological equations in presence of matter can be written as (we closely follow the notation of [12])

$$\left(1-\frac{3\xi}{2}\right)\left(\dot{H}+H^2\right)+\frac{1}{2}\kappa^2\rho(1+3w)-\frac{\chi_1}{6}+\frac{\chi_3 k^2}{6a^4}+\frac{\chi_4 k}{3a^6}=0\,, \tag{16}$$

[1] However this does not constitute a real problem because, as we will see, this point is always unstable and there is no orbit which can reach it.

$$\left(1-\tfrac{3\xi}{2}\right)H^2-\tfrac{\chi_2 k}{6a^2}-\kappa^2\rho-\tfrac{\chi_1}{6}-\tfrac{\chi_3 k^2}{6a^4}-\tfrac{\chi_4 k}{6a^6}=0\,, \tag{17}$$

$$\dot{\rho}+3(w+1)H\rho=0\,, \tag{18}$$

where

$$\kappa^2=\tfrac{1}{6\alpha}\,, \tag{19}$$

$$\chi_1=\tfrac{g_0\alpha^3}{6}\,, \tag{20}$$

$$\chi_2=-6g_1\alpha^2>0\,, \tag{21}$$

$$\chi_3=12\alpha\,(3g_2+g_3)\,, \tag{22}$$

$$\chi_4=24\,(9g_4+3g_5+g_6)\,, \tag{23}$$

and we have defined $\xi=1-\lambda$ assuming also that $\xi\neq 2/3$. Notice that taking $\xi>2/3$ would imply that the energy density in the Friedmann equation has a negative sign. In fact, this range of the parameter corresponds to spin zero modes of the theory, which can be excluded at the cosmological level and are related to unwanted ghost modes. Also the sign of the term χ_1 determines the sign of an (effective) cosmological constant in the model and χ_2 can be always taken to be negative [12]. Comparing the system above with the one in (7)-(9) one can note that, as pointed out in [12], there is not much difference between the two cases. For example, apart from the values of the constants, (8) and (17) differ only by the term associated with χ_4. However, as we are going to show, the associated cosmological dynamics will be non-trivially changed.

Let us define the variables

$$\Omega=\frac{\kappa^2\rho}{H^2},\qquad x=\frac{k^2\chi_3}{6a^4H^2},\qquad y=\frac{k\chi_4}{6a^6H^2},\qquad z=\frac{\chi_1}{6H^2},\qquad K=\frac{k\chi_2}{6a^2H^2}\,, \tag{24}$$

and the cosmic time $\mathcal{N}=\ln[a(t)]$. The cosmological equations (16)-(18) are then equivalent to the system [25]

$$x'=\frac{2(1-3w)x^2}{2-3\xi}+\frac{x}{2-3\xi}\left[2K(3w+1)-6(w-1)y-6(w+1)z-(1-3w)(2-3\xi)\right]\,, \tag{25}$$

$$y'=-\frac{6(w-1)y^2}{2-3\xi}+\frac{y}{2-3\xi}\left[2K(3w+1)+2(1-3w)x-6(w+1)z+3(w-1)(2-3\xi)\right]\,, \tag{26}$$

$$z'=-\frac{6(w+1)z^2}{2-3\xi}+\frac{z}{2-3\xi}\left[2K(3w+1)+2(1-3w)x-6(w-1)y+3(w+1)(2-3\xi)\right]\,, \tag{27}$$

$$K' = \frac{2K^2(3w+1)}{2-3\xi} + \frac{K}{2-3\xi}[2(1-3w)x - 6(w-1)y - 6(w+1)z + (3w+1)(2-3\xi)], \quad (28)$$

$$K + x + y + z + \Omega + 1 - \frac{3}{2}\xi = 0 \quad (29)$$

As expected, the new degrees of freedom, associated with additional terms present in this case, result in an additional dimension for the phase space. The system above possesses four invariant submanifolds, namely $x = 0$, $y = 0$, $z = 0$, $K = 0$. This implies that, also in this case, no global attractor can exist. As before, the structure of (29) is such that the system is non-compact but we will limit ourselves only to the finite analysis.

The finite fixed points are found by setting $x', y, z', K' = 0$ in (25)-(28) and solving the resulting system of algebraic equations. The results are shown in Table 2. Note that the presence of a fixed point depends on the exact sign of the constant χ_i, as well as on the value of ξ. For example, given the fact that the variable Ω is defined positive, the fixed point $\mathscr{A}$, can exist only if its coordinates are non negative, and this happens for $\xi < \frac{2}{3}$ only.

The solutions associated to the fixed point can be derived from the Raychaudhuri equation

$$\dot{H} = -\frac{H^2}{4-6\xi}\left[2K + 2x + 6y - 6z - 6w(x+y+z-1-K) - 9(w+1)\xi + 6\right]. \quad (30)$$

The corresponding results are shown in Table (2). The new terms in the system (16)-(18) induce a new fixed point characterized by a behavior $t^{1/3}$ which corresponds to the domination of a new cosmic component which goes like a^{-6}. Substitution into Eqs. (7)-(9) reveals that, in this case, only the Friedmann solution and the exponential solution yield identities. Specifically, the exponential solution represents a de Sitter solution if $\frac{\chi_1}{2-3\xi} > 0$, otherwise it is associated with oscillations. In other words, if one wants standard matter to interact with HL gravity in the standard way (gravity makes matter to attract itself), then the de Sitter solution is only present if $\chi_1 > 0$, i.e., if the cosmological constant has the right sign, as expected.

In the same way as in the previous case, the stability of the fixed points can be determined by evaluating the eigenvalues of the Jacobian matrix associated with the system (11), as prescribed by the Hartman-Grobman theorem [31]. The results can be also found in Table 2. The stabilities of the fixed points in this model are different from the corresponding ones in the previous case although, again, the only stable finite fixed point continues to be the de Sitter one.

Unfortunately, due to the higher dimensionality of the phase space, the dynamics of this model are not as easy to extract as the ones in the previous paragraph. However in the general structure of the equations there is no feature which prevents the existence of an orbit connecting the unstable Friedmann phase with the de Sitter attractor. This means that HL cosmologies without detailed balance can actually admit a transition between Friedmann evolution and a dark energy era.

Table 2 Finite fixed points of the system (25)-(28), their associated solutions and stability.

Point	Coordinates $[\Omega, x, y, z, K]$	Solution	Energy density	Stability
$\mathscr{A}$	$\left[\frac{1}{2}(2-3\xi), 0, 0, 0, 0\right]$	$a = a_0(t-t_0)^{\frac{2}{3(1+w)}}$	$\rho = \rho_0(t-t_0)^{-2}$	saddle
$\mathscr{B}$	$\left[0, \frac{1}{2}(2-3\xi), 0, 0, 0\right]$	$a = a_0(t-t_0)^{\frac{1}{2}}$	$\rho = 0$	saddle
$\mathscr{C}$	$\left[0, 0, 0, 0, \frac{1}{2}(3\xi-2)\right]$	$a = a_0(t-t_0)$	$\rho = 0$	saddle
$\mathscr{D}$	$\left[0, 0, 0, \frac{1}{2}(2-3\xi), 0\right]$	$a = a_0 e^{\tau(t-t_0)}$	$\rho = 0$	attractor
$\mathscr{E}$	$\left[0, 0, \frac{1}{2}(2-3\xi), 0, 0\right]$	$a = a_0(t-t_0)^{\frac{1}{3}}$	$\rho = 0$	repeller

$$\tau = \sqrt{\frac{\alpha_1}{3(2-3\xi)}}$$

5 Discussion and conclusions

In this paper we have used dynamical system techniques to analyze the non vacuum cosmology of Horâva-Lifshitz gravity both in the presence and in the absence of detailed balance. Our analysis has allowed to both gain an understanding of the qualitative behavior of the cosmology and to obtain interesting new constraints on the parameters of the model.

In the first case the phase space exhibits four finite fixed points, three of which represent physical solutions of the system. Although one of these points is associated to an unstable classical Friedmann solution that could be certainly useful to model the nucleosynthesis and the structure formation periods, our analysis does not reveal any useful fixed point which could model an inflationary or dark energy phase. It has been proposed that, because of the changes in the value of the speed of light contained in the theory, the absence of an explicit inflationary phase might not be such a serious shortcoming of the theory [10], although at first glance it is difficult in this setting to produce a dark energy era. A more conclusive analysis of this issue, however, will require a complete numerical study, which is left to future work. An interesting result of our investigation is that one of the attractors in the theory corresponds to an oscillatory behavior. Such oscillations can be associated with a bouncing universe, which can be connected to the analysis in [15].

Maybe the most important result in this paper is related to the HL cosmology without detailed balance. In this case, in fact, the additional freedom in the values of the parameters allows the existence of cosmic histories which contain a Friedmann era and evolve towards a dark energy one. This follows because the phase space contains a fixed point associated to the standard Friedmann solution which is unstable and another one which can be associated to a de Sitter solution which is an attractor. The last point can then model a dark energy era. However, the existence of these fixed points is only a necessary condition: because of the presence of invariant manifolds and the constraints on the parameters only a subset of the phase space and the parameter space will realize this scenario.

On the other hand the fact that the fixed points all lie on invariant submanifolds guarantee that such orbits can exist. Unfortunately the high dimension of the phase space makes it quite hard to perform any qualitative analysis. Therefore only numerical methods will allow the investigation of the details of these orbits. Notwithstanding these problems, we feel that it is safe to conclude that a cosmology compatible with the present observations can be obtained, in the HL framework, *only* if the detailed balance is broken. Such result makes this type of HL gravity a very promising phenomenological model for both the study of dark energy and quantum gravity.

Acknowledgements This work was partially funded by Ministerio de Educación y Ciencia, Spain, projects CICYT-FEDER-FPA2005-02211 and FIS2006-02842, CSIC under the I3P program, and AGAUR, contract 2005SGR-00790 and grant 2009BE-100132. SC was funded by Generalitat de Catalunya through the Beatriu de Pinós contract 2007BP-B1 00136. EE's research was performed while on leave at Department of Physics and Astronomy, Dartmouth College, 6127 Wilder Laboratory, Hanover, NH 03755, USA.

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Gravitational Waves Astronomy: a cornerstone for gravitational theories

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Abstract Realizing a gravitational wave (GW) astronomy in next years is a great challenge for the scientific community. By giving a significant amount of new information, GWs will be a cornerstone for a better understanding of gravitational physics. In this paper we re-discuss that the GW astronomy will permit to solve a captivating issue of gravitation. In fact, it will be the definitive test for Einstein's general relativity (GR), or, alternatively, a strong endorsement for extended theories of gravity (ETG).

1 Introduction

The scientific community hopes in a first direct detection of GWs in next years [1]. The realization of a GW astronomy, by giving a significant amount of new information, will be a cornerstone for a better understanding of gravitational physics. In fact, the discovery of GW emission by the compact binary system PSR1913+16, composed by two Neutron Stars [2], has been, for physicists working in this field, the ultimate thrust allowing to reach the extremely sophisticated technology needed for investigating in this field of research. In this paper we re-discuss that the GW astronomy will permit to solve a captivating issue of gravitation. In fact, it will be the definitive test for Einstein's GR, or, alternatively, a strong endorsement for ETG [3].

Although Einstein's GR [4] achieved great success (see for example the opinion of Landau who says that GR is, together with quantum field theory, the best scientific theory of all [5]) and withstood many experimental tests, it also displayed many shortcomings and flaws which today make theoreticians question whether it is the definitive theory of gravity, see [6]-[19] and references within. As distinct from other field theories, like the electromagnetic theory, GR is very difficult to quantize. This fact rules out the possibility of treating gravitation like other quantum theories,

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_13,

and precludes the unification of gravity with other interactions. At the present time, it is not possible to realize a consistent quantum gravity theory which leads to the unification of gravitation with the other forces. From an historical point of view, Einstein believed that, in the path to unification of theories, quantum mechanics had to be subjected to a more general deterministic theory, which he called *generalized theory of gravitation*, but he did not obtain the final equations of such a theory (see for example the biography of Einstein which has been written by Pais [20]). At present, this point of view is partially retrieved by some theorists, starting from the Nobel Laureate G. 't Hooft [21].

One can define *ETG* those semi-classical theories where the Lagrangian is modified, in respect of the standard Einstein-Hilbert gravitational Lagrangian [5], adding high-order terms in the curvature invariants (terms like R^2, $R^{\alpha\beta}R_{\alpha\beta}$, $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$, $R\Box R$, $R\Box^k R$) or terms with scalar fields non-minimally coupled to geometry (terms like $\phi^2 R$) [6]-[19]. In general, one has to emphasize that terms like those are present in all the approaches to the problem of unification between gravity and other interactions. Additionally, from a cosmological point of view, such modifications of GR generate inflationary frameworks which are very important as they solve many problems of the standard universe model (see [22] for a review).

In the general context of cosmological evidence, there are also other considerations which suggest an extension of GR. As a matter of fact, the accelerated expansion of the universe, which is observed today, implies that cosmological dynamics is dominated by the so called Dark Energy, which gives a large negative pressure. This is the standard picture, in which this new ingredient is considered as a source on the right-hand side of the field equations. It should be some form of un-clustered, non-zero vacuum energy which, together with the clustered Dark Matter, drives the global dynamics. This is the so called "concordance model" (ΛCDM) which gives, in agreement with the CMBR, LSS and SNeIa data, a good picture of the observed Universe today, but presents several shortcomings such as the well known "coincidence" and "cosmological constant" problems [23]. An alternative approach is changing the left-hand side of the field equations, to see if the observed cosmic dynamics can be achieved by extending General Relativity [6]-[19]. In this different context, it is not required to find candidates for Dark Energy and Dark Matter, that, till now, have not been found; only the "observed" ingredients, which are curvature and baryonic matter, have to be taken into account. Considering this point of view, one can think that gravity is different at various scales and there is room for alternative theories [6]-[19]. In principle, the most popular Dark Energy and Dark Matter models can be achieved considering $f(R)$ theories of gravity, where R is the Ricci curvature scalar, and/or scalar-sensor gravity (STG) [6]-[19]. In this picture, even the sensitive detectors for gravitational waves (GWs), like bars and interferometers, whose data analysis recently started [1], could, in principle, be important to confirm or rule out the physical consistency of GR or of any other theory of gravitation. This is because, in the context of ETG, some differences between GR and the others theories can be pointed out starting from the linearized theory of gravity, see [3] and [24]-[28].

Now, let us consider this issue in more detail.

2 Using gravitational waves to discriminate

GWs are a consequence of Einstein's GR [4], which presuppose GWs to be ripples in the space-time curvature travelling at light speed [29, 30]. Only asymmetric astrophysics sources can emit GWs. The most efficient are coalescing binaries systems, while a single rotating pulsar can rely only on spherical asymmetries, usually very small. Supernovae could have relevant asymmetries, being potential sources[1].

The most important cosmological source of GWs is, in principle, the so called stochastic background of GWs which, together with the Cosmic Background Radiation (CBR), would carry, if detected, a huge amount of information on the early stages of the Universe evolution [25], [31]-[35]. The existence of a relic stochastic background of GWs is a consequence of generals assumptions. Essentially it derives from a mixing between basic principles of classical theories of gravity and of quantum field theory [31]-[34]. The model derives from the inflationary scenario for the early universe [22], which is tuned in a good way with the WMAP data on the CBR (in particular exponential inflation and spectral index ≈ 1 [36]. The GWs perturbations arise from the uncertainty principle and the spectrum of relic GWs is generated from the adiabatically-amplified zero-point fluctuations [31]-[34]. The analysis has been recently generalized to ETG in [25] and [35].

In 1957, F.A.E. Pirani, who was a member of the Bondi's research group, proposed the geodesic deviation equation as a tool for designing a practical GW detector [37]. Pirani showed that if a GW propagates in a spatial region where two test masses are present, the effect is to drive the masses to have oscillations.

In 1959, Joseph Weber studied a detector that, in principle, might be able to measure displacements smaller than the size of the nucleus [38]. He developed an experiment using a large suspended bar of aluminium, with a high resonant Q at a frequency of about 1 kHz. Then, in 1960, he tried to test the general relativistic prediction of GWs from strong gravity collisions [39] and, in 1969, he claimed evidence for observation of gravitational waves (based on coincident signals) from two bars separated by 1000 km [40]. He also proposed the idea of doing an experiment to detect gravitational waves using laser interferometers [40]. In fact, all the modern detectors can be considered like being originated from early Weber's ideas [1].

In recent papers [26, 27] it has been shown that GWs from ETG generate different oscillations of test masses, with respect to GWs from standard GR. Thus, an accurate analysis of such a motion can be used in order to discriminate among various theories.

In general, GWs manifest them-self by exerting tidal forces on the test-masses which are the mirror and the beam-splitter in the case of an interferometer [1].

Working with $G = 1$, $c = 1$ and $\hbar = 1$ (natural units), the line element for a GW arising from standard GR and propagating in the z direction is [3, 28, 41]

$$ds^2 = dt^2 - dz^2 - (1+h_+)dx^2 - (1-h_+)dy^2 - 2h_\times dxdy, \tag{1}$$

where $h_+(t-z)$ and $h_\times(t-z)$ are the weak perturbations due to the $+$ and the $\times$ polarizations which are expressed in terms of synchronous coordinates in the transverse-traceless (TT) gauge [3, 28, 41].

In the case of standard GR the motion of test masses, due to GWs and analysed in the gauge of the local observer, is well known [41]. By putting the beam-splitter in the origin of the coordinate system, the components of the separation vector are the mirror's coordinates. At first order in h_+, the displacements of the mirror due by the $+$ polarization of a GW propagating in the z direction are given by [41]:

$$\delta x_M(t) = \frac{1}{2} x_{M0} h_+(t) \tag{2}$$

and

$$\delta y_M(t) = -\frac{1}{2} y_{M0} h_+(t), \tag{3}$$

where x_{M0} and y_{M0} are the initial (unperturbed) coordinates of the mirror. The $\times$ polarization generates an analogous motion for test masses which are rotated of 45-degree with respect the z axis [41].

In all ETG a third *massive* polarization of GWs is present [3], [24]-[27], which is usually labelled with Φ, and the line element for such a third polarization can be always put in a conformally flat form in both of the cases of STG and $f(R)$ theories [24]-[27]:

$$ds^2 = [1+\Phi(t-v_G z)](-dt^2+dz^2+dx^2+dy^2). \tag{4}$$

v_G in (4) is the particle's velocity (the group velocity in terms of a wave-packet [3], [24]-[27]). In the case of STG the third mode can be massless. In that case $v_G = 1$ and, at first order in Φ, the displacements of the mirror due to these massless scalar GWs are given by [26]

$$\delta x_M(t) = \frac{1}{2} x_{M0} \Phi(t) \tag{5}$$

and

$$\delta y_M(t) = \frac{1}{2} y_{M0} \Phi(t). \tag{6}$$

In the case of massive scalar GWs and of $f(R)$ theories it is [26, 27]

$$\begin{aligned} \delta x_M(t) &= \tfrac{1}{2} x_{M0} \Phi(t) \\ \delta y_M(t) &= \tfrac{1}{2} y_{M0} \Phi(t) \\ \delta z_M(t) &= -\tfrac{1}{2} m^2 z_{M0} \psi(t), \end{aligned} \tag{7}$$

where [26, 27]

$$\ddot{\psi}(t) \equiv \Phi(t). \tag{8}$$

Note: the most general definition is $\psi(t-v_G z) + a(t-v_G z) + b$, but one assumes only small variations of the positions of the test masses, thus $a = b = 0$ [26, 27].

Then, in the case of massive GWs a longitudinal component is present because of the presence of a small mass m [26, 27]. As the interpretation of Φ is in terms of a wave-packet, solution of the Klein-Gordon equation [26, 27]

$$\Box\Phi = m^2\Phi, \tag{9}$$

it is also

$$\psi(t - v_G z) = -\frac{1}{\omega^2}\Phi(t - v_G z). \tag{10}$$

Thus, if advanced projects on the detection of GWs will improve their sensitivity allowing to perform a GWs astronomy (this is due because signals from GWs are quite weak) [1], one will only have to look which is the motion of the mirror in respect to the beam splitter of an interferometer in the locally inertial coordinate system in order to understand which is the correct theory of gravity. If such a motion will be governed only by Eqs. (2) and (3) we will conclude that GR is the ultimate theory of gravity. If the motion of the mirror is governed also by Eqs. (5) and (6), in addition to the motion arising from Eqs. (2) and (3), we will conclude that massless STG is the correct theory of gravitation. Finally, if the motion of the mirror is governed also by Eqs. (7) in addition to the ordinary motion of Eqs. (2) and (3), we will conclude that the correct theory of gravity will be massive STG which is equivalent to $f(R)$ theories. Even if such signals will be quite weak, a consistent GWs astronomy will permit to understand which is the direction of the propagating GW by using coincidences between various detectors and to compute a hypothetical group velocity v_G by using delay times, thus, all the quantities of the above equations could be, in principle, determined.

3 Conclusion remarks

We re-discussed that the GW astronomy will permit to solve a captivating issue of gravitation. If advanced projects on the detection of GWs will improve their sensitivity allowing to perform a GWs astronomy, such a GWs astronomy will be the definitive test for Einstein's GR, or, alternatively, a strong endorsment for ETG. In fact, a careful analysis of the motion of the mirror of the interferometer with respect to the beam splitter will permit to discriminate among GR and ETG.

Acknowledgements I thank the Associazione Scientifica Galileo Galilei, for supporting this paper. I strongly thank Emilio Elizalde and the other organizers for inviting me at the Workshop "Cosmology, the Quantum Vacuum and Zeta Functions" for the celebration of Emilio Elizalde's sixtieth birthday.

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Hamilton–Jacobi Method and Gravitation

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Abstract Studying the behaviour of a quantum field in a classical, curved, spacetime is an extraordinary task which nobody is able to take on at present time. Independently by the fact that such problem is not likely to be solved soon, still we possess the instruments to perform exact predictions in special, highly symmetric, conditions. Aim of the present contribution is to show how it is possible to extract quantitative information about a variety of physical phenomena in very general situations by virtue of the so-called Hamilton–Jacobi method. In particular, we shall prove the agreement of such semi-classical method with exact results of quantum field theoretic calculations.

1 Introduction

Suppose we are interested in studying the behaviour of a field $\Phi(x)$ (scalar, for sake of simplicity) in a curved spacetime endowed with a (trapping) horizon (e.g. in the vicinity of a black hole). Based on physical intuition, we expect that the interaction from the quantum field and the classical background gives rise to different phenomena, as: Hawking radiation through the horizon; decay of unstable particles scattering off the gravitational field; vacuum particle creation in regions of strong gravity; radiation from (possibly, naked) singularities, etc. The aformentioned topics would pertain the investigation of a quantum theory of gravity, the lack of which obliges us to work with standard techniques.

The field is governed by the Klein–Gordon equation,

$$\left(\Box_x - \frac{m^2}{\hbar^2}\right)\Phi(x) = 0\,, \tag{1}$$

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_14,

where the parameter m^2 is interpreted as the field mass and for convenience we have inserted $\hbar$ explicitly. Field quantization can be performed in either canonical or path integral ways:

$$\Phi(\mathbf{x}) = \int_{\mathscr{A}} \mathrm{D}p \cdot \mathrm{D}\tilde{\mathbf{x}} \cdot \mathrm{D}N \times (\text{Gauge-fixing conditions}) \times \exp(iI[p, \tilde{\mathbf{x}}, N]) \quad (2)$$

where $\mathscr{A}$ represents appropriate boundary conditions and $I[p,x,N]$ is the Hamiltonian formulation of the action. The fact that $\Phi(N, \mathbf{x})$ has to satisfy the equation of motion (1) imposes constraints on the allowed boundary conditions, $\mathscr{A}$.
Solution to (2) is largely unknown due to the difficulty of computing path integrals in curved spacetimes. However, some information is accessible in the WKB regime of approximation. In this case, one generally finds appropriate to look for a solution in the form,

$$\Phi(x) = D(x)e^{-\frac{I(x)}{\hbar}} + O(\hbar) \quad (3)$$

where the small parameter $\hbar$ is used to govern the WKB expansion. Inserting (3) into (1) and equating powers of $\hbar$, we obtain to the lowest orders

$$\begin{aligned} \hbar^{-2}: &\qquad -\nabla^a I \nabla_a I + m^2 = 0 \\ \hbar^{-1}: &\qquad 2\nabla D \cdot \nabla I + D\Box I = 0 \\ \hbar^{0}: &\qquad \Box D = 0 \\ \cdots: &\qquad \cdots \end{aligned} \quad (4)$$

Exact computation of the pre-factor $D(x)$ is complicated even in very addomesticated situations and therefore it is beyond our present goal.
Let us split I into a real and a purely imaginary part: $I(x) := I_R(x) - iS(x)$; then (4) becomes:

$$-(\nabla I_R)^2 + (\nabla S)^2 + m^2 = 0. \quad (5)$$

If the imaginary part of I varies with x much more rapidly than the real part, that is, if $|\nabla S| \gg |\nabla I_R|$, it follows from (5) that S will be an approximate solution to the (Lorentzian) Hamilton-Jacobi equation

$$g^{ab}\partial S_a \partial S_b + m^2 = 0. \quad (6)$$

Furthmore, the wave function (3) will then be predominantly of the form e^{iS}. Of course, going from the exact path integral form (2) to the approximate regime (3) with I solution to (4), we loose specification of the boundary conditions $\mathscr{A}$. It will be evident later that this lost is only apparent.

The basic idea proposed some time ago by Parick & Wilzcek [1] is to interpret the spacetime horizon – say, for example, of a black hole – as a sort of barrier and to study the tunnelling of field quanta through it. Certainly, the horizon behavies in quite a different way with respect to usual quantum mechanical potential barriers. In ordinary quantum mechanics, the barrier is represented by the region between the turning points of the classical trajectories. Here instead, the horizon is just a point on the classical characteristic curves. Evaluating the tunnelling in quantum mechanics means computing the ratio between particle wavefunction on the two sides of the barrier. In the black hole case, instead, the required ratio is generated by a discontinuity in the wavefunction. Moreover, in the familiar context of tunneling through a barrier, an imaginary part comes from a negative eigenvalue of the small disturbance operator around the classical bounce, while the Euclidean action is real. In the black hole case, instead, as we shall see later, it is the action itself that is complex.

Given the whole sort of specifications above, we can conclude the reasoning and invoke the well known result according to which, the creation probability per unit time of quanta of mass m is given – to leading order in $\hbar$ – by the WKB formula

$$\Gamma \propto \exp\left(-\frac{2}{\hbar}S\right) \tag{7}$$

with S solution to (6). Remarkably, as it has been shown in [2, 3, 4, 5, 6], the procedure outlined so far reproduces the infamous Planckian spectrum of Hawking radiation in the case of black hole spacetimes: $\Gamma \propto \exp(-\beta\omega_H)$, with ω_H the energy of tunnelling particles through the black hole horizon and β interpreted as the inverse temperature of the thermalized field quanta. The Hamilton–Jacobi method of tunnelling has therefore proved an elegant way to interpret Hawking radiation as a tunnelling process and to derive in relatively simple way the associated temperature ($T = \beta^{-1}$). As we shall try to show in the following, the method does not exhaust its power in the computation of black hole Hawking temperature, generalizing indeed to a wider class of spacetime horizons (e.g. cosmological horizons) and to other kinds of semi-classical phenomena.

We use the conventions according to which the metric signature is $(-,+,+,+)$; first latin indices as a,b run over $0,\ldots,3$, mid-latin indices as i,j only over $0,1$. From nown on, we implement natural units, so that $c = \hbar = G = k_B = 1$.

2 The Kodama–Hayward formalism for spherically symmetric spacetimes

In the following, we shall limit ourselves to focus only on spherically symmetric spacetimes where no gravitational waves production is involved. The line element can be locally written as [7]

$$ds^2 = \gamma_{ij}(x)dx^i dx^j + R^2(x)d\Omega^2 \,, \tag{8}$$

where the two-dimensional metric $\gamma_{ij}(x)$ is referred to as the normal metric, $\{x^i\}$ are associated coordinates and $R(x^i)$ is the areal radius, considered as a scalar field in the two-dimensional normal space. We recall that to have a truly dynamical solution, i.e. to avoid Birkhoff's theorem, the spacetime must be filled with matter everywhere. Examples are the Vaidya solution, which contains a flux of radiation at

infinity, and FRW solutions which contain a perfect fluid.
A dynamical trapping horizon, if it exists, is located at $0 = \chi(x)|_H$, with $\chi(x) := \gamma^{ij}(x)\partial_i R(x)\partial_j R(x)$, provided that $\partial_i \chi(x) = 0$. The dynamical surface gravity associated with the horizon is given by the normal space scalar $\kappa_H = \frac{1}{2}\Box_\gamma R(x)|_H$ as proved in [7].
In the spherical symmetric dynamical case, it is possible to introduce the so-called Kodama vector field K, with $(K^a G_{ab})^{;b} = 0$, that can be taken as its defining property, [8]. It follows that K is a natural generalization of the Killing vector of stationary spacetimes. Given the metric (8), the non-vanishing Kodama vector components are $K^i = \varepsilon^{ij}\partial_j R(x)/\sqrt{-\gamma}$ ($\varepsilon^{01} = +1$). The Kodama vector gives a preferred flow of time and in this sense it generalizes the flow of time given by the Killing vector in the static case. As a consequence, we may introduce the invariant energy associated with a particle of mass m by means of the scalar quantity on the normal space

$$\omega = -K \cdot dS \tag{9}$$

where S is the particle action which we assume to satisfy the reduced Hamilton–Jacobi equation

$$\gamma^{ij}\partial_i S \partial_j S + m^2 = 0\,. \tag{10}$$

Remarkably, the probability rate (7) does not depend by the choice of coordinates, since the horizon location, the horizon surface gravity, the Kodama energy are all invariantly defined in the space normal to the spheres of symmetry [9].

The basic idea can now be roughly described as follows: the reduced Hamilton–Jacobi equation (10) supplemented by the Kodama energy formula (9) constrains particle's momenta, e.g. $\partial_+ S = \partial_+ S(x^\pm, m, \omega)$; thus, the mass parameter m gives two complementary energy scales so that, according to the physical phenomenon involved, the two scales exchange the leading role in the analysis.
More in detail, suppose we are interested in the physics of the horizon: tunnelling through the horizon – typically related to Hawking/Unruh effects – corresponds to the existence of a simple pole in particle's momenta. In this case, it turns out that the mass parameter can be neglected so that, to all the extents, particles move along null trajectories. On the other hand, if we are now interested in bulk effects, away from any horizon, then the mass parameter plays a crucial role being possibly responsable for a branch point singularity in tunnelling particle's momenta.
Let us make an example in order to make clearer what we mean. As fully described in [10], the FRW spacetime with spatial curvature $\hat{k} = \frac{k}{l^2}$ ($k = 0, \pm 1$ and l an opportune length scale) represents a dynamical, spherically symmetric spacetime exhibiting a cosmological horizon in correspondence of what we shall call the Hubble radius, namely $R_H(t) := (H^2 + \hat{k}/a^2)^{-1/2}$ and $R(t,r) := a(t)r$. The Kodama energy is $\omega = \sqrt{1-\hat{k}r^2}(-\partial_t I + rH\partial_r I) \equiv \sqrt{1-\hat{k}r^2}\,\tilde{\omega}$. The Hamilton–Jacobi equation reads $-(\partial_t S)^2 + \frac{(1-\hat{k}r^2)}{a^2(t)}(\partial_r S)^2 + m^2 = 0$, so that the radial particle's momentum is

$$\partial_r S = -\frac{aH\tilde{\omega}(ar) \pm a\sqrt{\omega^2 - m^2(1-(ar/R_H)^2)}}{1-(ar/R_H)^2}\,. \tag{11}$$

Near the horizon, the mass coefficient vanishes so that we can set $m = 0$. Thus, making a null-horizon radial expansion, the action for particles coming out of the horizon towards the inner (untrapped) region is $S = 2\int dr\partial_r I$, with $\partial_r S$ exhibiting a simple pole at the horizon. To deal with the simple pole in the integrand, we implement Feynman's $i\varepsilon$–prescription, something which resambles the recovering of the boundary conditions encoded in the path integral approach mentioned above. Because of (7), $\Gamma \sim \exp(-\omega_H/T)$, $\omega_H > 0$ for physical particles and $T = -\kappa_H/2\pi$ ($\kappa_H < 0$ for trapping horizons of the inner type such as the Hubble radius, cf. [11]) the dynamical temperature associated to FRW horizon.

To treat instead the decay of unstable composite particles inside the Hubble horizon (i.e., in the untrapped region), we need to identify the energy of the particle before the decay as the Kodama energy, ω; then we denote by m the effective mass parameter of one of the decay products, after the decay. With these understandings, we find out that for the unstable particle sitting at rest at the origin of the comoving coordinates, there is an imaginary part of the action as the decay product tunnels into the region $0 < r < r_0$ to escape beyond r_0, with r_0 implicitly defined through $[a(t)r_0]^2 = R_0^2 := \left(1 - \frac{\omega^2}{m^2}\right) R_H^2$. Assuming a two-particle decay, the rate is

$$\Gamma = \Gamma_0 \, e^{-2\pi R_H\,(m-\omega)} \tag{12}$$

and Γ_0 depending on the interaction coupling (e.g. $\Gamma_0 \sim \lambda^2$ for a $\lambda\phi^3$ interaction). Equation (12) agrees with Volovik result for de Sitter space [12] and with asymptotic quantum field theory calculation by Bros *et al*, [13].

3 Vacuum particle creation and emission from naked singularities

A perfectly legitimate question we can ask ourselves is weather the method is extendable to the case of static black holes as well. With regard to this, we consider the exterior region of a spherically symmetric, static, black hole spacetime and repeat the same argument. Quite generally, we can write the line element as

$$ds^2 = -e^{2\psi(r)}C(r)dt^2 + \frac{dr^2}{C(r)} + r^2 d\Omega^2 \,. \tag{13}$$

The analysis of the radial momentum is made easier by setting the Kodama energy $\omega = 0$: in the intention, this would correspond to particle creation from vacuum,

$$\int dr\,\partial_r S = m\int_{r_1}^{r_2} dr \frac{1}{\sqrt{-C(r)}} \,. \tag{14}$$

The integration is taken over any interval (r_1, r_2) where $C(r) > 0$. Equation (14) shows that, under very general conditions, in static black hole spacetimes there could be a decay rate whenever a region where $C(r)$ is positive exists. However, it is an

easy task to show that the spacelike singularity of the Schwarzschild black hole does not emit particles in the semi-classical regime: in the interior, the Kodama vector is spacelike, thus no energy can be introduced there.
The situation is very different when a naked singularity is present. Considering a neutral particle in the Reissner-Nordström solution with mass M and charge $Q > 0$ (for definiteness), the line element is

$$ds^2 = -\frac{(r-r_-)(r-r_+)}{r^2}dt^2 + \frac{r^2}{(r-r_-)(r-r_+)}dr^2 + r^2 d\Omega^2 \,, \tag{15}$$

with $r_\pm$ functions of (M,Q) denoting the inner and outer horizons. The function $C(r) = (r-r_-)(r-r_+)/r^2$ is negative in between the two horizons, where the Kodama vector is spacelike, so there the action is real. On the other hand, it is positive within the outer communication domain, $r > r_+$, but also within the region contained by the inner Cauchy horizon, that is $0 < r < r_-$. Thus, because of (14) and assuming the particles come created in pairs, we obtain that, modulo the pre-factor over which we have nothing to say, there is a creation probability per unit time and unit volume (equation (16) not depending upon the creation event) of neutral particles of mass m by the strong gravitational field near the Reissner-Nordström naked singularity (M,Q) which goes as

$$\Gamma \sim \left(\frac{M-Q}{M+Q}\right)^{mM} e^{-2Qm}. \tag{16}$$

At a first look, the process of particle production in the region close to the singularity raises the issue of the stability of the solution. However, this does not seem to be a problem. In fact, radiation created by the bulk close to the singularity comes into the singularity with infinite red-shift and approaches the future inner, classically unstable, horizon with infinite blue-shift. Thus, the contribution of the radiation coming in the singularity to the back-reaction is negligible and the causal structure of the singularity safe; while the blue-shifted radiation approaching the future sheet of the inner horizon will contribute to its quantum instability (Cf. [14] for further investigation).

A complementary and potentially interesting effect is the emission from the naked singularity itself. We investigate this problem for the case of two-dimensional dilaton gravity, and will come back to Reissner-Nordström solution afterward.

Consider the two-dimensional metric

$$ds^2 = \sigma^{-1}dx^+dx^- \,, \qquad \sigma := \lambda^2 x^+x^- - a(x^+ - x_0^+)\theta(x^+ - x_0^+) \tag{17}$$

where λ is related to the cosmological constant by $\Lambda = -4\lambda^2$ and a represents the wave amplitude. This metric arises as a solution of $2D$ dilaton gravity coupled to a bosonic field with stress tensor $T_{++} = 2a\,\delta(x^+ - x_0^+)$, describing a shock wave. $\sigma = 0$ is a naked singularity partly to the future of a flat space region (linear dilaton vacuum). The heavy arrow in the figure represents the history of the shock wave responsible for the existence of the timelike singularity.

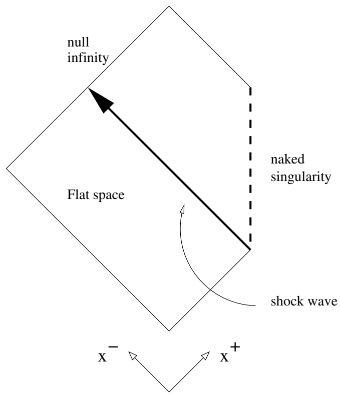

Fig. 1 The naked singularity formed by the shock wave.

The goal is to compute the outgoing flux which is given by: $2T_{++} - 2T_{--}$. In order to do this, we notice that the Hamilton–Jacobi equation implies either $\partial_+ S = 0$ or $\partial_- S = 0$, S being the action. To find the ingoing flux we integrate along x^+ till we encounter the naked singularity, using $\partial_- S = 0$, so that $S = \int dx^+ \partial_+ I = \int dx^+ \frac{\omega}{\sigma}$. The absorption probability is $\Gamma(\omega) = \Gamma_0 \exp[-2\pi\omega/(\lambda^2 x^- - a)]$. The flux is computed by integrating the probability over the coordinate frequency $\tilde{\omega} = \frac{\omega}{2\sigma}$ (that is, the variable conjugate to the time coordinate), with the density of states measure $\frac{d\tilde{\omega}}{2\pi}$: $T_{++} = \Gamma_0 \frac{(\lambda^2 x^- - a)^2}{2\pi^3\sigma^2}$.
To find T_{--}, we integrate now along x^- starting from the naked singularity, this time using $\partial_+ S = 0$. A similar calculation gives $T_{--} = \Gamma_0 \frac{\lambda^4 (x^+)^2}{2\pi^3\sigma^2}$.

T_{+-} is given by the conformal anomaly, $T = 4\sigma T_{+-} = R/24\pi$ (for one bosonic degree of freedom). Matching to the anomaly gives $\Gamma_0 = \pi^2/24 \sim O(1)$.
Note that the stress tensor diverges approaching the singularity, indicating that its resolution will not be possible within classical gravity but requires instead quantum gravity [15].
Indeed, all these results agree with the one-loop calculation to be found in [16].

Returning now to the Reissner-Nordström solution, could it be that the naked singularity emits particles? In the four-dimensional case one easily sees that the action has no imaginary part along null trajectories either ending or beginning at the singularity. Formally this is because the Kodama energy coincides with the Killing energy in such a static manifold and there is no infinite red-shift from the singularity to infinity. Even considering the metric as a genuinely two-dimensional, however, it is possible to show that the action does not exhibit any imaginary part [10].
It is fair to say that the Reissner-Nordström naked singularity will not emit particles in this approximation something which seems to be coherent with quantum field theory results, [10, 17].

4 Conclusions

We have shown that semi-classical tunnelling method can handle several quantum effects: radiation from dynamical horizons (both cosmological and collapsing); gravitational enhancement of particle decay otherwise forbidden by conservation laws; radiation from two-dimensional naked singularities. Normally, great efforts are needed to analyze quantum effects in gravity, while instead the tunneling picture promptly gives strong indications of what's going on. The obtained agreement between both the particle decay rates and the radiation from naked singularities in the tunnelling picture and the (asymptotic of the) exact results – when they exist in particular conditions – gives confidence, in our opinion, of the validity of the method even in more general situations.

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A characteristic signature of fourth order gravity

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Abstract We present for the first time the complete matter power spectrum for R^n gravity which has been derived from the *fourth order* scalar perturbation equations. This leads to the discovery of a characteristic signature of fourth order gravity in the matter power spectrum, the details of which have not seen before in other studies in this area and therefore provides a crucial test for fourth order gravity on cosmological scales.

1 Introduction

Ever since the *Concordance model* [1] was proposed as the best fit to all available cosmological data sets, there have been many attempts to understand the nature of Dark Energy. However, despite enormous effort over the past few years, this problem remains one of the greatest puzzles in contemporary physics. One of the theoretical proposals that has received a considerable amount of attention recently, is that Dark Energy has a geometrical origin. This idea has been driven by the fact that modifications to General Relativity appear in the low energy limit of many fundamental schemes [2, 3] and that these modifications lead naturally to cosmologies which admit a Dark Energy like era [4] without the introduction of any additional cosmological fields. Most of the work on this idea has focused on *fourth order gravity*, in which the standard Hilbert-Einstein action is modified with terms that are at most of order four in the metric tensor. The features of fo urth order gravity have been analyzed with different techniques [5] and all these studies suggest that these

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_15,

cosmologies can give rise to a phase of accelerated expansion, which is considered to be an important footprint of Dark Energy.

The work described above has largely focused on the dynamics of homogeneous cosmologies which have the standard Robertson-Walker geometry and are therefore also isotropic. Although these results have many of the desirable features that we are looking for, such as a matter dominated epoch and late-time acceleration, there are still some key issues that need to be addressed before one could claim to have a cosmological description which is able to compete with the standard ΛCDM cosmology. The calculation and analysis of the evolution of linear perturbations and their comparison with observations is clearly among the most important of these open problems. Over the past year this problem has been studied by several authors using the metric approach to perturbations, by either considering different ways of parameterizing the non-Einstein modifications of gravity or by simplifying the underlying fourth-order perturbation equations using a quasi-static approximation [6].

In what follows, we demonstrate that considerable progress can be made to this problem by using the *1+3 covariant approach* to cosmological perturbations [7]. Using a specific recasting of the field equations (based on the Ricci and Bianchi identities), the development of equations describing cosmological perturbations in theories of gravity characterized by an action which is a general analytic function of the Ricci scalar $f(R)$, becomes both transparent and straightforward, allowing for the exact integration of the perturbation equations without making any approximations. In order to easily discuss the key features of the perturbation dynamics and the associated power spectrum, we focus on R^n-gravity. This theory is characterized by the action $L=\sqrt{-g}\,[\chi R^n+\mathscr{L}_M]$ and is the simplest possible example of fourth order gravity.

2 Perturbation dynamics and the power spectrum

Before we can discuss the evolution of density perturbations, a suitable background cosmology must first be found. In [8, 9], the complete dynamics of homogeneous and isotropic cosmologies were studied in detail using the dynamical system approach (see [10] and references therein). It was found that in R^n gravity, it is possible to have a transient matter-dominated decelerated expansion phase, followed by a smooth transition to a Dark Energy like era which drives the cosmological acceleration. The first phase, characterized by the baratropic equation of state parameter w, has an expansion history determined by a scale-factor $a=t^{2n/3(1+w)}$ where we restrict ourselves to $n>0$ for this background as negative values of n would represent a contracting model. This solution provides exactly the setting during which structure formation can take place and is therefore an ideal background solution for our study of density perturbations.

Scalar perturbations, which describe density perturbations may be extracted from any first order tensor T_{ab} orthogonal to u^a by using a *local* decomposition [11], so that repeated application of the operator $\tilde{\nabla}_a \equiv h_a^b\nabla_b$ on T_{ab} extracts the scalar part of the perturbation variables. In this way we can define the following scalar quantities

$$\Delta_m = \frac{S^2}{\mu_m}\tilde{\nabla}^2\mu_m\,,\quad Z = S^2\tilde{\nabla}^2\Theta\,,\quad C = S^4\tilde{\nabla}^2\tilde{R}\,,\quad \mathscr{R} = S^2\tilde{\nabla}^2 R\,,\quad \mathfrak{R} = S^2\tilde{\nabla}^2\dot{R}\,. \quad (1)$$

where Δ^m_a, Z respectively represent the fluctuations in the matter energy density μ_m and expansion Θ, and $\mathscr{R}$, $\mathfrak{R}$ determine the fluctuations in the Ricci scalar R and its momentum $\dot{R}$. This set of variables completely characterizes the evolution of density perturbations. Then, using eigenfunctions of the spatial Laplace-Beltrami operator defined in [7]: $\tilde{\nabla}^2 Q = -\frac{k^2}{S^2}Q$, where $k = 2\pi S/\lambda$ is the wave number and $\dot{Q} = 0$, we can expand every first order quantity in the above equations:

$$X(t,\mathbf{x}) = \sum X^{(k)}(t)\, Q^{(k)}(\mathbf{x})\,, \quad (2)$$

where $\sum$ stands for both a summation over a discrete index or an integration over a continuous one. In this way, it is straightforward, although lengthy, to derive a pair of second order equations describing the k^{th} mode for density perturbations in $f(R)$ gravity. They are:

$$\ddot{\Delta}^{(k)}_m + \left[\left(\frac{2}{3} - w\right)\Theta - \frac{\dot{R}f''}{f'}\right]\dot{\Delta}^{(k)}_m - \left[w\frac{k^2}{S^2} - w(3p^R + \mu^R) - \frac{2w\dot{R}\Theta f''}{f'} - \frac{\left(3w^2-1\right)\mu^m}{f'}\right]\Delta^{(k)}_m$$
$$= \frac{1}{2}(w+1)\left[-2\frac{k^2}{S^2}\frac{f''}{f'} - 1 + \left(f - 2\mu^m + 2\dot{R}\Theta f''\right)\frac{f''}{f'^2} - 2\dot{R}\Theta\frac{f^{(3)}}{f'}\right]\mathscr{R}^{(k)} - \frac{(w+1)\Theta f''}{f'}\dot{\mathscr{R}}^{(k)}\,,$$
$$f''\ddot{\mathscr{R}}^{(k)} + \left(\Theta f'' + 2\dot{R}f^{(3)}\right)\dot{\mathscr{R}}^{(k)} - \left[\frac{k^2}{S^2}f'' + 2\frac{K}{S^2}f'' + \frac{2}{9}\Theta^2 f'' - (w+1)\frac{\mu^m}{2f'}f'' - \frac{1}{6}(\mu^R + 3p^R)f''\right.$$
$$\left.-\frac{f'}{3} + \frac{f}{6f'}f'' + \dot{R}\Theta\frac{f''^2}{f'} - \ddot{R}f^{(3)} - \Theta f^{(3)}\dot{R} - f^{(4)}\dot{R}^2\right]\mathscr{R}^{(k)} = -\left[\frac{1}{3}(3w-1)\mu^m\right.$$
$$\left.+\frac{w}{1+w}\left(f^{(3)}\dot{R}^2 + (p^R + \mu^R)f' + \frac{7}{3}\dot{R}\Theta f'' + \ddot{R}f''\right)\right]\Delta^{(k)}_m - \frac{(w-1)\dot{R}f''}{w+1}\dot{\Delta}^{(k)}_m\,. \quad (3)$$

where $f' = \partial f(R)/\partial R$, the quantities μ_R, p_R are the energy density and pressure of the *curvature fluid* defined in [12] and $K = 0, +1, -1$ is the usual spatial curvature scalar. It is easy to see that for the $f(R) = R$ case, these equations reduce to the standard equations for the evolution of the scalar perturbations in General Relativity.

Already on super-Hubble scales, $k/aH \ll 1$, a number of important features are found which allows one to differentiate (3) from their General Relativity counterparts [12]. Firstly, it is clear that the evolution of density perturbations is determined by a *fourth order* differential equation rather than a second order one. This implies that the evolution of the density fluctuations contains, in general, four modes rather that two and can give rise to a more complex evolution than the one of General Relativity (GR). Secondly, the perturbations are found to depend on the scale for any equation of state for standard matter (while in General Relativity the evolution of the dust perturbations are scale-invariant). This means that even for dust, the evolution of super-horizon and sub-horizon perturbations are different. Thirdly, it is found that the growth of large density fluctuations can occur also in backgrounds in which the expansion rate is increasing in time (see figure 1). This is in striking contrast with what one finds in General Relativity and would lead to a time-varying gravitational potential, putting tight constraints on the Integrated Sachs-Wolfe effect for these models.

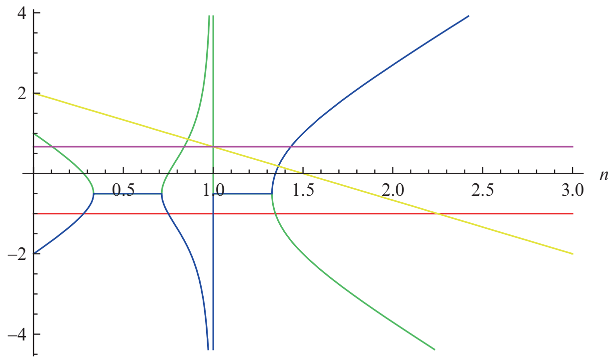

Fig. 1 Plot against n of the real part of the long wavelength modes for R^n-gravity in the dust case (blue, red green and yellow lines) together with the GR modes (red and purple line). Note that there is at least one growing mode for any value of n. This means that even in cases where the expansion rate is accelerating, i.e., $n > \frac{3}{2(1+3)}$, large-scale density perturbations grow.

Let us now turn to the case of a general wave mode k. One of the most instructive way of understanding the details of the evolution of density perturbations for a general k is to compute the matter power spectrum $P(k)$, defined by the relation [13] $\langle \Delta_m(\mathbf{k}_1)\Delta_m(\mathbf{k}_2)\rangle = P(k_1)\delta(\mathbf{k}_1+\mathbf{k}_2)$, where $\mathbf{k}_i$ are two wavevectors characterizing two Fourier components of the solutions of (3) and $P(\mathbf{k}_1) = P(k_1)$ because of isotropy in the distribution of the perturbations. This quantity tells us how the fluctuations of matter depend on the wavenumber at a specific time and carries information about the amplitude of the perturbations (but not on their spatial structure). In General Relativity, the power spectrum on large scales is constant, while on small scales it is suppressed in comparison with the large scales (i.e., modes which entered the horizon during the radiation era) [14]. In the case of pure dust in General Relativity the power spectrum is scale invariant. Substituting the details of the background, the values of the parameter n, the barotropic factor w, the spatial curvature index K and the wavenumber k into (3) one is able to obtain $P(k)$ numerically.

The k-structure of equations (3) suggest that in fourth order gravity there exist at least three different growth regimes of the perturbations. This is confirmed by our results (see figure 2). In particular, in the case of dust we have three regimes for any values of the remaining parameters: (i) on very large scales the spectrum it is like what one finds for General Relativity, i.e., scale invariant; (ii) as k becomes bigger the scale invariance is broken and oscillations in the spectrum appear; (iii) for even larger k the spectrum becomes again scale invariant. However, on these scales the spectrum can contain either an excess or deficit of power depending on the value of

n. In particular for $n \approx 1^+$ small scales have more power than large scales, but, as one moves towards larger values of n, the small scale modes are suppressed.

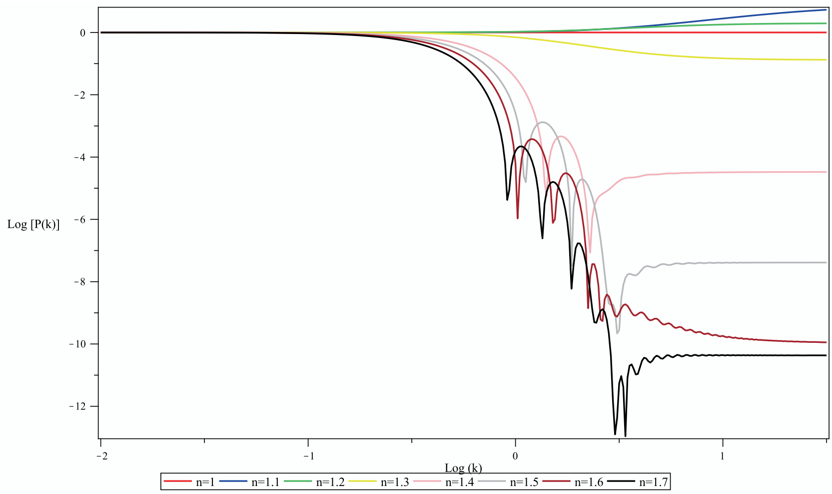

Fig. 2 Plot of the power spectrum at $\tau = 1$ for R^n-gravity and $n > 1$. Note that the spectrum is composed of three parts corresponding to three different evolution regimes for the perturbations.

3 Results and discussion

The features of the spectrum that we have presented above can be best interpreted by comparing the system (3) which produced it with the equations for the evolution of scalar perturbations for two interacting fluids in General Relativity [15]. Immediately one notices that they have the same structure, i.e., there are friction and source terms due to the interaction and the gravitation of the two effective fluids. It is then natural to ask ourselves if this analogy can be useful to better understand the physics of these models. The answer is affirmative. First of all a more correct way to draw this analogy would be to write the system of equations for Δ_m and fluctuations in the energy density of the curvature fluid $\Delta_R = S^2\tilde{\nabla}^2\mu_R/\mu_R$ and analyze their structure rather than using the ones above. On very large and on very small scales, the coefficients of the (Δ_m, Δ_R) system become independent of k, so that the evolution of the perturbations does not change as a function of scale and the power spectrum is consequently scale invariant. On intermediate scales the interaction between the two fluids is maximized and the curvature fluid acts as a relativistic component whose pressure is responsible for the oscillations and the dissipation of the small scale per-

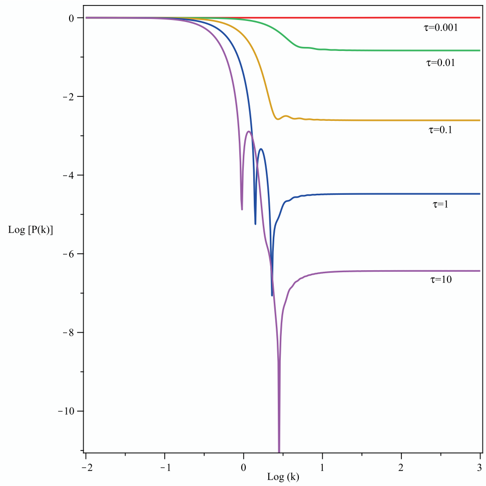

Fig. 3 Evolution the Power spectrum for R^n-gravity for $n = 1.4$. The spectrum has been normalized in such a way that the curves coincide at large scales. Note how, as time passes, small scale perturbations are dissipated and oscillation appear.

turbations in the same way in which the photons operate in a baryon-photon system. This suggests the following interesting interpretation for the perturbation variables $\mathscr{R}$ and $\mathfrak{R}$. These quantities can be interpreted as representing the modes associated with the contribution of the additional scalar degree of freedom typical of $f(R)$-gravity. In this sense the spectrum can be explained physically as a consequence of the interaction between these scalar modes and standard matter. The result is a considerable loss of power for a relatively small variation of the parameter n. For example, in the case $n = 1.4$ the difference in power between the two scale invariant parts of the spectru m for $n = 1.1$ is of one order of magnitude while for $n = 1.6$ is about ten orders of magnitude. It should be noted that existing analysis[16] of this model require $n = 3.5$ in order for predictions to be consistent with measurements of rotation curves of low surface brightness galaxies and SNe Ia. Given the huge drop in power at small scales in the power spectrum for $n = 3.5$ one expects that this model could be easily ruled out.

Further information on the dynamics of the matter perturbations can be obtained examining the time evolution of the power spectrum. In figure 3 we give the power spectrum for $n = 1.4$ at different times. One can see that, as the universe expands, the small scale part of the spectrum is more and more suppressed and oscillations start to form, suggesting that in this model small scale perturbations tend to be dissipated in time. On the other hand on large scales they do not evolve, which might appear in contrast with what mentioned above. However this is a byproduct of the normalization: for clarity we have normalized the spectrum in such a way that every curve has the same power in long wavelength limit. A more in depth discussion of the features presented above can be found in [17].

Probably the most important consequence of the form of the spectrum presented above is the fact that the effect of these type of fourth order corrections is evident only for a special range of scales, while the rest of the spectrum has the same k dependence of GR (but different amplitude). This implies that we have a spectrum that both satisfies the requirement for scale invariance and has distinct features that one could in principle detect, by combining future Cosmic Microwave Background (CMB) and large scale surveys (LSS) [18, 19].

Acknowledgements The authors wish to thank Dr J. Larena for useful discussion and suggestions. KNA and SC are supported by Claude Leon Foundation fellowships. This work was supported by the National Research Foundation (South Africa) and the *Ministrero degli Affari Esteri - DIG per la Promozione e Cooperazione Culturale* (Italy) under the joint Italy/South Africa science and technology agreement.

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Horizons and singularity in Clifton's spherical solution of $f(R)$ vacuum

Valerio Faraoni

Abstract Due to the failure of Birkhoff's theorem, black holes in $f(R)$ gravity theories in which an effective time-varying cosmological "constant" is present are, in general, dynamical. Clifton's exact spherical solution of $R^{1+\delta}$ gravity, which is dynamical and describes a central object embedded in a spatially flat universe, is studied. It is shown that apparent black hole horizons disappear and a naked singularity emerges at late times.

1 Introduction

The study of Type Ia supernovae [43, 41, 44, 42, 45, 61, 30, 46, 3] revealed that the universe is currently accelerating its expansion. This discovery has generated an enormous amount of activity and theoretical models in order to find an explanation of this phenomenon. The most common models are based on General Relativity (GR) and invoke mysterious forms of dark energy (see [32] for a list of references). However, dark energy, possibly even phantom energy, is too exotic and *ad hoc* and attempts have been made to model the cosmic acceleration without dark energy. $f(R)$ theories of gravity reminiscent of the quadratic corrections to the Einstein-Hilbert action introduced by renormalization have been re-introduced in the metric [5, 10], Palatini [63], and metric-affine [51, 52, 53, 56, 57] formulations and have received much attention in recent years (see [55, 16] for reviews and [58, 54, 6, 20, 7, 39, 48] for introductions). Emilio Elizalde has given many contributions to the development of $f(R)$ gravity and cosmology.

While many cosmological and other aspects of $f(R)$ gravity (stability, weak-field limit, ghost content) have been discussed in recent years, it is important to understand spherical solutions (both vacuum and interior) in these theories [28, 1, 50, 49, 31, 2, 15, 37, 62].

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_16,

Metric $f(R)$ gravity is described by the action

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\, f(R) + S^{(matter)}\,, \tag{1}$$

where $f(R)$ is a non-linear function of its argument and $S^{(matter)}$ is the matter part of the action. R denotes the Ricci scalar of the metric g_{ab} with determinant g, $\kappa \equiv 8\pi G$ where G is Newton's constant, and we adopt the notations of [64].

The Jebsen-Birkhoff theorem of GR fails in these theories, adding to the variety of spherical solutions [23]. Of particular interest are black holes in generalized gravity, which have been studied especially in relation to their thermodynamics ([24] and references therein). Since $f(R)$ theories are designed to produce an effective dynamical cosmological constant, physically relevant spherically symmetric and black hole solutions are likely to describe central objects embedded in cosmological backgrounds. This kind of solution is poorly understood even in GR, although a few examples are available there [59, 21, 9, 8, 35, 36, 25, 47, 29, 40, 34, 26]. Even less is known about $f(R)$ black holes, which are certainly worth exploring. Here we consider a specific solution of vacuum $f(R) = R^{1+\delta}$ gravity discovered in [11]. Solar System experiments set the limits[1] $\delta = (-1.1 \pm 1.2)\cdot 10^{-5}$ on the parameter δ [11, 4, 12, 13, 65], while local stability requires $f''(R) \geq 0$ [17, 18, 19, 38], hence we restrict to the range $0 < \delta < 10^{-5}$.

The solution of [11] is dynamical and describes a time-varying central object embedded in a spatially flat universe in vacuum $R^{1+\delta}$ gravity. This solution is made possible by the fact that the fourth order field equations of metric $f(R)$ gravity

$$f'(R)R_{ab} - \frac{f(R)}{2}\, g_{ab} = \nabla_a\nabla_b f'(R) - g_{ab}\Box f'(R) \tag{2}$$

in vacuo can be rewritten as effective Einstein equations

$$R_{ab} - \frac{1}{2}\, g_{ab}R = \frac{1}{f'(R)}\left[\nabla_a\nabla_b f' - g_{ab}\Box f' + g_{ab}\frac{(f - Rf')}{2}\right] \tag{3}$$

with geometric terms acting as effective matter on the right hand side. This time-varying effective matter invalidates the Jebsen-Birkhoff theorem and can propel the acceleration of the universe. An equivalent representation of metric $f(R)$ gravity as an $\omega = 0$ Brans-Dicke theory with a special scalar field potential reveals explicitly the presence of a massive scalar degree of freedom $f'(R)$ responsible for these effects [55]. Since analytical spherical and dynamical solutions of $f(R)$ gravity in asymptotically Friedmann-Lemaitre-Robertson-Walker (FLRW) backgrounds are harder to find than in GR (where only a few are known anyway), Clifton's solution is particularly valuable.

[1] See also [14] for this specific form of the function $f(R)$.

2 Clifton's solution and its horizons

In this section we describe the Clifton solution [11] and the work [22] locating the horizons of this solution.

The spherically symmetric and time-dependent solution of vacuum $R^{1+\delta}$ gravity of [11] is given by

$$ds^2 = -A_2(r)dt^2 + a^2(t)B_2(r)\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right], \tag{4}$$

and (using the isotropic radius and the notations of [11])

$$A_2(r) = \left(\frac{1 - C_2/r}{1 + C_2/r}\right)^{2/q}, \tag{5}$$

$$B_2(r) = \left(1 + \frac{C_2}{r}\right)^4 A_2(r)^{q+2\delta-1}, \tag{6}$$

$$a(t) = t^{\frac{\delta(1+2\delta)}{1-\delta}}, \tag{7}$$

$$q^2 = 1 - 2\delta + 4\delta^2. \tag{8}$$

Once δ is fixed, two classes of solutions exist, corresponding to the sign of C_2qr. The line element (4) reduces to the FLRW one if $C_2 \to 0$. In the limit $\delta \to 0$ in which the theory reduces to GR, (4) reduces to the Schwarzschild metric in isotropic coordinates provided that $C_2qr > 0$, hence positive and negative values of r are possible according to the sign of C_2, but we assume $r > 0, C_2 > 0$ and take the positive root in the expression $q = \pm\sqrt{1 - 2\delta + 4\delta^2}$, so that $q \simeq 1 - \delta$ as $\delta \to 0$. The solution (4)-(8) is conformal to the Fonarev solution [27] which is conformally static [33], and therefore is also conformally static, similar to the Sultana-Dyer [59, 21, 9, 8] and certain generalized McVittie solutions [26] of GR.

In order to identify possible apparent horizons, it is convenient to cast the metric (4) in the Nolan gauge. Using first the Schwarzschild-like radius

$$\tilde{r} \equiv r\left(1 + \frac{C_2}{r}\right)^2, \tag{9}$$

giving $dr = \left(1 - \frac{C_2^2}{r^2}\right)^{-1} d\tilde{r}$ and then the areal radius

$$\rho \equiv \frac{a(t)\sqrt{B_2(r)}\,\tilde{r}}{\left(1 + \frac{C_2}{r}\right)^2} = a(t)\,\tilde{r}A_2(r)^{\frac{q+2\delta-1}{2}}, \tag{10}$$

the line element (4) takes the form

$$ds^2 = -A_2 dt^2 + a^2 A_2^{2\delta-1} d\tilde{r}^2 + \rho^2 d\Omega^2 \,. \tag{11}$$

Denoting the differentiation with respect to time with an overdot and using the identities

$$d\tilde{r} = \frac{d\rho - A_2^{\frac{q+2\delta-1}{2}} \dot{a}\,\tilde{r}\,dt}{a\left[A_2^{\frac{q+2\delta-1}{2}} + \frac{2(q+2\delta-1)}{q}\frac{C_2}{\tilde{r}} A_2^{\frac{2\delta-1-q}{2}}\right]} \equiv \frac{d\rho - A_2^{\frac{q+2\delta-1}{2}} \dot{a}\,\tilde{r}\,dt}{a A_2^{\frac{q+2\delta-1}{2}} C(r)} \,, \tag{12}$$

one obtains

$$C(r) = 1 + \frac{2(q+2\delta-1)}{q}\frac{C_2}{\tilde{r}} A_2^{-q} = 1 + \frac{2(q+2\delta-1)}{q}\frac{C_2 a}{\rho} A_2^{\frac{2\delta-1-q}{2}} \,, \tag{13}$$

which turns the metric into the Painlevé-Gullstrand-like form

$$ds^2 = -A_2\left[1 - \frac{A_2^{2(\delta-1)}}{C^2}\dot{a}^2\tilde{r}^2\right] dt^2 - \frac{2A_2^{\frac{-q+2\delta-1}{2}}}{C^2}\dot{a}\,\tilde{r}\,dt\,d\rho$$
$$+ \frac{d\rho^2}{A_2^q C^2} + \rho^2 d\Omega^2 \,. \tag{14}$$

Now we introduce a new time coordinate $\bar{t}$ defined by

$$d\bar{t} = \frac{1}{F(t,\rho)}\left[dt + \beta(t,\rho) d\rho\right] \tag{15}$$

in order to eliminate the cross-term $dt d\rho$. Here $F(t,\rho)$ is an integrating factor which guarantees that $d\bar{t}$ is an exact differential and is determined by

$$\frac{\partial}{\partial\rho}\left(\frac{1}{F}\right) = \frac{\partial}{\partial t}\left(\frac{\beta}{F}\right) . \tag{16}$$

The line element becomes

$$ds^2 = -A_2\left[1 - \frac{A_2^{2(\delta-1)}}{C^2}\dot{a}^2\tilde{r}^2\right] F^2 d\bar{t}^2$$
$$+ 2F\left\{A_2\beta\left[1 - \frac{A_2^{2(\delta-1)}}{C^2}\dot{a}^2\tilde{r}^2\right] - \frac{A_2^{\frac{-q+2\delta-1}{2}}}{C^2}\dot{a}\tilde{r}\right\} d\bar{t}\,d\rho$$
$$+ \left\{-A_2\left[1 - \frac{A_2^{2(\delta-1)}}{C^2}\dot{a}^2\tilde{r}^2\right]\beta^2 + \frac{2A_2^{\frac{-q+2\delta-1}{2}}}{C^2}\dot{a}\tilde{r}\beta + \frac{1}{A_2^q C^2}\right\} d\rho^2$$
$$+ \rho^2 d\Omega^2 \,. \tag{17}$$

The choice

$$\beta = \frac{A_2^{\frac{-q+2\delta-3}{2}}}{C^2} \frac{\dot{a}\tilde{r}}{1 - \frac{A_2^{2(\delta-1)}}{C^2} \dot{a}^2\tilde{r}^2} \tag{18}$$

eliminates the $dtd\rho$ term and casts the metric in the Nolan gauge

$$ds^2 = -A_2 DF^2 d\bar{t}^2 + \frac{1}{A_2^q C^2} \left[1 + \frac{A_2^{-q-1} H^2 \rho^2}{C^2 D} \right] d\rho^2 + \rho^2 d\Omega^2 , \tag{19}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter of the background universe and

$$D \equiv 1 - \frac{A_2^{2(\delta-1)}}{C^2} \dot{a}^2 \tilde{r}^2 = 1 - \frac{A_2^{-q-1}}{C^2} H^2 \rho^2 . \tag{20}$$

Using the second of these equations, the line element (19) assumes the simple form

$$ds^2 = -A_2 DF^2 d\bar{t}^2 + \frac{d\rho^2}{A_2^q C^2 D} + \rho^2 d\Omega^2 . \tag{21}$$

The apparent horizons, if they exist, are located at $g^{\rho\rho} = 0$, which yields $A_2^q C^2 D = 0$ and $A_2^q \left(C^2 - H^2 R^2 A_2^{-q-1} \right) = 0$. Therefore, $g^{\rho\rho}$ vanishes if $A_2 = 0$ or $H^2 R^2 = C^2 A_2^{q+1}$. A_2 vanishes at $r = C_2$, which describes the Schwarzschild horizon when $\delta \to 0$ (the GR limit). This locus corresponds to a spacetime singularity because the Ricci scalar $R = \frac{6\left(\dot{H}+2H^2\right)}{A_2(r)}$ diverges as $r \to C_2$ (it reduces to the usual FLRW value $6\left(\dot{H}+2H^2\right)$ as $C_2 \to 0$). This singularity is strong according to Tipler's classification [60] because the areal radius $\rho = a\tilde{r}A_2^{\frac{q+2\delta-1}{2}}$ vanishes when $r = C_2$ for $\delta > 0$, in contrast with the Schwarzschild metric corresponding to $\delta = 0$ in which $\rho = \tilde{r} = 4C_2$ at $r = C_2$.

The second possibility $H^2\rho^2 = C^2 A_2^{q+1}$ yields

$$H\rho = \pm \left[1 + \frac{2(q+2\delta-1)}{q} \frac{C_2 a}{\rho} A_2^{\frac{2\delta-1-q}{2}} \right] A_2^{\frac{q+1}{2}} , \tag{22}$$

with the positive sign corresponding to an expanding universe. When $\delta \to 0$, this equation reduces to $H\rho = \left[1 + \frac{2\delta C_2 a}{\rho} A_2^{-\left(1-\frac{3\delta}{2}\right)} \right] A_2^{1-\delta}$.

To gain some insight, consider the following two limits. As $C_2 \to 0$ (the central object disappears and the solution is FLRW space), $r = \tilde{r}$ and ρ become a comoving and a proper radius, respectively, while (22) reduces to $H\rho = 1$ with solution $\rho_c = 1/H$, the radius of the cosmological horizon. In the limit $\delta \to 0$ in which the theory reduces to GR, (22) reduces to $A_2 = 0$ or $r = C_2$ with $H \equiv 0$.

Using (7) and (10), the left hand side of (22) is expressed as

$$HR = \frac{\delta(1+2\delta)}{1-\delta} t^{\frac{2\delta^2+2\delta-1}{1-\delta}} \frac{C_2}{x} \frac{(1-x)^{\frac{q+2\delta-1}{q}}}{(1+x)^{\frac{-q+2\delta-1}{q}}}, \tag{23}$$

where $x \equiv C_2/r$, while the right hand side of (22) is

$$\left(\frac{1-x}{1+x}\right)^{\frac{q+1}{q}} \left[1 + \frac{2(q+2\delta-1)}{q} \frac{x}{(1-x)^2}\right]. \tag{24}$$

Eq. (22) then becomes

$$\frac{1}{t^{\frac{1-2\delta-2\delta^2}{1-\delta}}} = \frac{(1-\delta)}{\delta(1+2\delta)C_2} \frac{x(1+x)^{\frac{-2q+2\delta-2}{q}}}{(1-x)^{\frac{2(\delta-1)}{q}}} \cdot \left[1 + \frac{2(q+2\delta-1)}{q} \frac{x}{(1-x)^2}\right] \tag{25}$$

(note that $\frac{1-2\delta-2\delta^2}{1-\delta}$ is positive for $0 < \delta < \frac{\sqrt{3}-1}{2} \simeq 0.366$).

At late times t, the left hand side of (25) vanishes, $x \simeq 0$, and there exists a unique root of the equation locating the apparent horizons, which corresponds to a cosmological horizon, consistently with the fact that $r \to \infty$ as $x = C_2/r \to 0$. The limit $x \to 0$ can also be obtained when the parameter $C_2 \to 0$, in which case $H\rho \to 1$

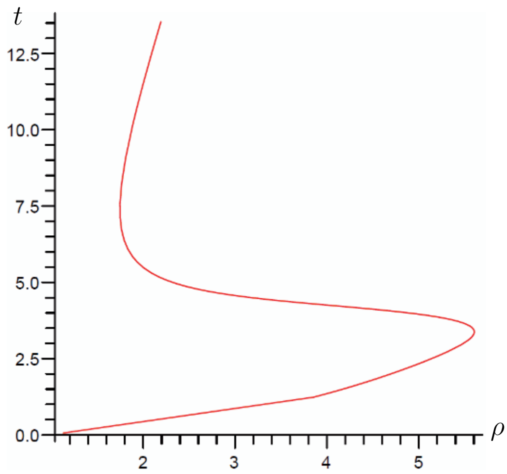

Fig. 1 Radii of the apparent horizons of Clifton's solution (vertical axis) versus time (horizontal axis) for the parameter values $C_2 = 1$ and $\delta = 0.13$.

and $r \simeq \rho \simeq H^{-1} = \frac{1-\delta}{\delta(1+2\delta)}\, t$ is the radius of the cosmological horizon of the FLRW space without a central object. Hence, *there is only a cosmological apparent horizon and no black hole apparent horizons at late times*: the central singularity at $\rho = 0$ becomes naked.

The radii ρ of the apparent horizons and the time t can be expressed in the parametric form

$$\rho(x) = t(x)^{\frac{\delta(1+2\delta)}{1-\delta}} \frac{C_2}{x} (1-x)^{\frac{q+2\delta-1}{q}} (1+x)^{\frac{q-2\delta+1}{q}} , \tag{26}$$

$$t(x) = \left\{ \frac{(1-\delta)}{\delta(1+2\delta)C_2} \frac{x(1+x)^{\frac{2(-q+\delta-1)}{q}}}{(1-x)^{\frac{2(\delta-1)}{q}}} \left[1 + \frac{2(q+2\delta-1)x}{q(1-x)^2}\right] \right\}^{\frac{1-\delta}{2\delta^2+2\delta-1}} \tag{27}$$

using x as a parameter. Fig. 1 reports ρ versus t for the parameter values $C_2 = 1$ and $\delta = 0.13$, showing that two inner horizons develop after the Big Bang covering the central singularity $\rho = 0$, then they approach each other, merge, and disappear, while a third, cosmological horizon keeps expanding. The $\rho = 0$ singularity becomes naked after this merging event.

3 Discussion and conclusions

Cosmologists may be detecting deviations from GR and therefore it is necessary to understand spherical solutions of $f(R)$ gravity, which has been proposed as a simple alternative to the mysterious dark energy. Since the Jebsen-Birkhoff theorem fails in these theories, spherical solutions do not have to be static. $f(R)$ theories are designed with a built-in dynamical cosmological constant to model the present acceleration of the universe, hence analytical spherical solutions describing a central object embedded in a FLRW background are the relevant ones. Unfortunately, such solutions are poorly understood even in GR [59, 21, 9, 8, 35, 36, 25, 47, 29, 40, 34, 26]. It seems difficult to find *generic* solutions describing black holes embedded in FLRW backgrounds. Finding numerically spherical interior solutions of $f(R)$ gravity is also an active area of research [28, 1, 50, 49, 31, 2, 15, 37, 62]. All these issues deserve further attention in the future.

Acknowledgements It is a pleasure to thank Rituparno Goswami for a discussion, the conference organizers for their invitation, and the Natural Sciences and Engineering Research Council of Canada (NSERC) for financial support.

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Gravitational Zero Point Energy and the Induced Cosmological Constant

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Abstract We discuss how to extract information about the cosmological constant from the Wheeler-DeWitt equation, considered as an eigenvalue of a Sturm-Liouville problem in a generic spherically symmetric background. The equation is approximated to one loop with the help of a variational approach with Gaussian trial wave functionals. A canonical decomposition of modes is used to separate transverse-traceless tensors (graviton) from ghosts and scalar. We show that no ghosts appear in the final evaluation of the cosmological constant. A zeta function regularization and a ultra violet cutoff are used to handle with divergences. A renormalization procedure is introduced to remove the infinities. We compare the result with the one obtained in the context of noncommutative geometries

1 Introduction

One of the biggest challenges of our century is the explanation of why the observed cosmological constant is so small when compared to the one estimated by Zero Point Energy (ZPE) computations in Quantum Field Theory. Indeed there exists a difference of 120 orders of magnitude between them. However, it appears that a definitive answer is still lacking. One possible approach to this problem comes from the Wheeler-DeWitt equation (WDW)[1], which is described by

$$\mathscr{H}\Psi = \left[(2\kappa)\, G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{g}}{2\kappa}\left({}^{3}R - 2\Lambda\right)\right]\Psi = 0, \tag{1}$$

where $\kappa = 8\pi G$, G_{ijkl} is the super-metric and ${}^{3}R$ is the scalar curvature in three dimensions. The main reason to use such an equation is that its most general formu-

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_17,

lation intrinsically includes a cosmological term. Moreover, if we formally re-write the WDW equation as[1] [2]

$$\frac{1}{V}\frac{\int\mathscr{D}\left[g_{ij}\right]\Psi^{*}\left[g_{ij}\right]\int_{\Sigma}d^{3}x\hat{\Lambda}_{\Sigma}\Psi\left[g_{ij}\right]}{\int\mathscr{D}\left[g_{ij}\right]\Psi^{*}\left[g_{ij}\right]\Psi\left[g_{ij}\right]}=\frac{1}{V}\frac{\left\langle\Psi\left|\int_{\Sigma}d^{3}x\hat{\Lambda}_{\Sigma}\right|\Psi\right\rangle}{\langle\Psi|\Psi\rangle}=-\frac{\Lambda}{\kappa},\tag{2}$$

where

$$V=\int_{\Sigma}d^{3}x\sqrt{g}\tag{3}$$

is the volume of the hypersurface Σ and

$$\hat{\Lambda}_{\Sigma}=(2\kappa)\,G_{ijkl}\pi^{ij}\pi^{kl}-\sqrt{g}^{3}R/(2\kappa)\,,\tag{4}$$

we recognize that the WDW equation can be represented by an expectation value. In particular, (2) represents the Sturm-Liouville problem associated with the cosmological constant. In this form the ratio Λ_c/κ represents the expectation value of $\hat{\Lambda}_{\Sigma}$ without matter fields. The related boundary conditions are dictated by the choice of the trial wave functionals which, in our case are of the Gaussian type. Different types of wave functionals correspond to different boundary conditions. The choice of a Gaussian wave functional is justified by the fact that we would like to explain the cosmological constant (Λ_c/κ) as a ZPE effect. To fix ideas, we will work with the following form of the metric

$$ds^{2}=-N^{2}(r)\,dt^{2}+\frac{dr^{2}}{1-\frac{b(r)}{r}}+r^{2}\left(d\theta^{2}+\sin^{2}\theta d\phi^{2}\right),\tag{5}$$

where $b(r)$ is subject to the only condition $b(r_t)=r_t$. As a first step, we begin to decompose the gravitational perturbation in such a way to obtain the graviton contribution enclosed in (2).

2 Extracting the graviton contribution

We can gain more information if we consider $g_{ij}=\bar{g}_{ij}+h_{ij}$,where $\bar{g}_{ij}$ is the background metric and h_{ij} is a quantum fluctuation around the background. Thus (2) can be expanded in terms of h_{ij}. Since the kinetic part of $\hat{\Lambda}_{\Sigma}$ is quadratic in the momenta, we only need to expand the three-scalar curvature $\int d^3x\sqrt{g}^3R$ up to the quadratic order. However, to proceed with the computation, we also need an orthogonal decomposition on the tangent space of 3-metric deformations[4, 5]:

$$h_{ij}=\frac{1}{3}(\sigma+2\nabla\cdot\xi)\,g_{ij}+(L\xi)_{ij}+h_{ij}^{\perp}.\tag{6}$$

The operator L maps ξ_i into symmetric tracefree tensors

[1] See also Ref.[3] for an application of the method to a $f(R)$ theory.

$$(L\xi)_{ij} = \nabla_i \xi_j + \nabla_j \xi_i - \frac{2}{3} g_{ij} (\nabla \cdot \xi), \tag{7}$$

$h_{ij}^{\perp}$ is the traceless-transverse component of the perturbation (TT), namely $g^{ij} h_{ij}^{\perp} = 0$, $\nabla^i h_{ij}^{\perp} = 0$ and h is the trace of h_{ij}. It is immediate to recognize that the trace element $\sigma = h - 2(\nabla \cdot \xi)$ is gauge invariant. If we perform the same decomposition also on the momentum π^{ij}, up to second order (2) becomes

$$\frac{1}{V} \frac{\left\langle \Psi \left| \int_\Sigma d^3x \left[\hat{\Lambda}_\Sigma^{\perp} + \hat{\Lambda}_\Sigma^{\xi} + \hat{\Lambda}_\Sigma^{\sigma} \right]^{(2)} \right| \Psi \right\rangle}{\langle \Psi | \Psi \rangle} = -\frac{\Lambda}{\kappa}. \tag{8}$$

Concerning the measure appearing in (2), we have to note that the decomposition (6) induces the following transformation on the functional measure $\mathscr{D}h_{ij} \to \mathscr{D}h_{ij}^{\perp} \mathscr{D}\xi_i \mathscr{D}\sigma J_1$, where the Jacobian related to the gauge vector variable ξ_i is

$$J_1 = \left[\det \left(\triangle g^{ij} + \frac{1}{3} \nabla^i \nabla^j - R^{ij} \right) \right]^{\frac{1}{2}}. \tag{9}$$

This is nothing but the famous Faddev-Popov determinant. It becomes more transparent if ξ_a is further decomposed into a transverse part ξ_a^T with $\nabla^a \xi_a^T = 0$ and a longitudinal part $\xi_a^{\parallel}$ with $\xi_a^{\parallel} = \nabla_a \psi$, then J_1 can be expressed by an upper triangular matrix for certain backgrounds (e.g. Schwarzschild in three dimensions). It is immediate to recognize that for an Einstein space in any dimension, cross terms vanish and J_1 can be expressed by a block diagonal matrix. Since $\det AB = \det A \det B$, the functional measure $\mathscr{D}h_{ij}$ factorizes into

$$\mathscr{D}h_{ij} = \left(\det \triangle_V^T \right)^{\frac{1}{2}} \left(\det \left[\frac{2}{3} \triangle^2 + \nabla_i R^{ij} \nabla_j \right] \right)^{\frac{1}{2}} \mathscr{D}h_{ij}^{\perp} \mathscr{D}\xi^T \mathscr{D}\psi \tag{10}$$

with $\left(\triangle_V^{ij} \right)^T = \triangle g^{ij} - R^{ij}$ acting on transverse vectors, which is the Faddeev-Popov determinant. In writing the functional measure $\mathscr{D}h_{ij}$, we have here ignored the appearance of a multiplicative anomaly[6]. Thus the inner product can be written as

$$\int \mathscr{D}\rho \Psi^* \left[h_{ij}^{\perp} \right] \Psi^* \left[\xi^T \right] \Psi^* [\sigma] \Psi \left[h_{ij}^{\perp} \right] \Psi \left[\xi^T \right] \Psi [\sigma], \tag{11}$$

where

$$\mathscr{D}\rho = \mathscr{D}h_{ij}^{\perp} \mathscr{D}\xi^T \mathscr{D}\sigma \left(\det \triangle_V^T \right)^{\frac{1}{2}} \left(\det \left[\frac{2}{3} \triangle^2 + \nabla_i R^{ij} \nabla_j \right] \right)^{\frac{1}{2}}. \tag{12}$$

Nevertheless, since there is no interaction between ghost fields and the other components of the perturbation at this level of approximation, the Jacobian appearing in the numerator and in the denominator simplify. The reason can be found in terms of connected and disconnected terms. The disconnected terms appear in the Faddeev-Popov determinant and these ones are not linked by the Gaussian integration. This means that disconnected terms in the numerator and the same ones appearing in the denominator cancel out. Therefore, (8) factorizes into three pieces. The piece containing $\hat{\Lambda}_{\Sigma}^{\perp}$ is the contribution of the transverse-traceless tensors (TT): essentially is the graviton contribution representing true physical degrees of freedom. Regarding the vector term $\hat{\Lambda}_{\Sigma}^{T}$, we observe that under the action of infinitesimal diffeomorphism generated by a vector field ε_i, the components of (6) transform as follows[4]

$$\xi_j \longrightarrow \xi_j + \varepsilon_j, \qquad h \longrightarrow h + 2\nabla \cdot \xi, \qquad h_{ij}^{\perp} \longrightarrow h_{ij}^{\perp}. \tag{13}$$

The Killing vectors satisfying the condition $\nabla_i \xi_j + \nabla_j \xi_i = 0$, do not change h_{ij}, and thus should be excluded from the gauge group. All other diffeomorphisms act on h_{ij} nontrivially. We need to fix the residual gauge freedom on the vector ξ_i. The simplest choice is $\xi_i = 0$. This new gauge fixing produces the same Faddeev-Popov determinant connected to the Jacobian J_1 and therefore will not contribute to the final value. We are left with

$$\frac{1}{V}\frac{\left\langle \Psi^{\perp}\left|\int_{\Sigma} d^3x \left[\hat{\Lambda}_{\Sigma}^{\perp}\right]^{(2)}\right|\Psi^{\perp}\right\rangle}{\left\langle \Psi^{\perp}|\Psi^{\perp}\right\rangle} + \frac{1}{V}\frac{\left\langle \Psi^{\sigma}\left|\int_{\Sigma} d^3x \left[\hat{\Lambda}_{\Sigma}^{\sigma}\right]^{(2)}\right|\Psi^{\sigma}\right\rangle}{\left\langle \Psi^{\sigma}|\Psi^{\sigma}\right\rangle} = -\frac{\Lambda^{\perp}}{\kappa} - \frac{\Lambda^{\sigma}}{\kappa}. \tag{14}$$

Note that in the expansion of $\int_{\Sigma} d^3x\sqrt{g}R$ to second order, a coupling term between the TT component and scalar one remains. However, the Gaussian integration does not allow such a mixing which has to be introduced with an appropriate wave functional. Extracting the TT tensor contribution from (2) approximated to second order in perturbation of the spatial part of the metric into a background term $\bar{g}_{ij}$, and a perturbation h_{ij}, we get

$$\hat{\Lambda}_{\Sigma}^{\perp} = \frac{1}{4V}\int_{\Sigma} d^3x\sqrt{\bar{g}}G^{ijkl}\left[(2\kappa)K^{-1\perp}(x,x)_{ijkl} + \frac{1}{(2\kappa)}\left(\tilde{\triangle}_L\right)_j^a K^{\perp}(x,x)_{iakl}\right], \tag{15}$$

where

$$\left(\tilde{\triangle}_L h^{\perp}\right)_{ij} = \left(\triangle_L h^{\perp}\right)_{ij} - 4R_i^k h_{kj}^{\perp} + {}^3R h_{ij}^{\perp} \tag{16}$$

is the modified Lichnerowicz operator and $\triangle_L$ is the Lichnerowicz operator defined by

$$(\triangle_L h)_{ij} = \triangle h_{ij} - 2R_{ikjl}h^{kl} + R_{ik}h_j^k + R_{jk}h_i^k \qquad \triangle = -\nabla^a\nabla_a. \tag{17}$$

G^{ijkl} represents the inverse DeWitt metric and all indices run from one to three. Note that the term $-4R_i^k h_{kj}^{\perp} + {}^3R h_{ij}^{\perp}$ disappears in four dimensions. The propagator $K^{\perp}(x,x)_{iakl}$ can be represented as

$$K^{\perp}(\vec{x},\vec{y})_{iakl} = \sum_{\tau}\frac{h_{ia}^{(\tau)\perp}(\vec{x})\, h_{kl}^{(\tau)\perp}(\vec{y})}{2\lambda(\tau)}, \tag{18}$$

where $h_{ia}^{(\tau)\perp}(\vec{x})$ are the eigenfunctions of $\tilde{\triangle}_L$. τ denotes a complete set of indices and $\lambda(\tau)$ are a set of variational parameters to be determined by the minimization of (15). The expectation value of $\hat{\Lambda}_{\Sigma}^{\perp}$ is easily obtained by inserting the form of the propagator into (15) and minimizing with respect to the variational function $\lambda(\tau)$. Thus the total one loop energy density for TT tensors becomes

$$\frac{\Lambda}{8\pi G} = -\frac{1}{2}\sum_{\tau}\left[\sqrt{\omega_1^2(\tau)} + \sqrt{\omega_2^2(\tau)}\right]. \tag{19}$$

The above expression makes sense only for $\omega_i^2(\tau) > 0$, where ω_i are the eigenvalues of $\tilde{\triangle}_L$. In the next section, we will explicitly evaluate (19) for a background of spherically symmetric type.

3 One loop energy density

3.1 Conventional regularization and renormalization

The reference metric (5) can be cast into the following form

$$ds^2 = -N^2(r(x))\,dt^2 + dx^2 + r^2(x)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right), \tag{20}$$

where

$$dx = \pm\frac{dr}{\sqrt{1-\frac{b(r)}{r}}} \tag{21}$$

and $b(r)$ a generic shape function. Specific examples are

$$b(r) = \frac{\Lambda_{dS}}{3}r^3; \qquad b(r) = -\frac{\Lambda_{AdS}}{3}r^3 \qquad \text{and} \qquad b(r) = 2MG. \tag{22}$$

However, we would like to maintain the form of the line element (20) as general as possible. With the help of Regge and Wheeler representation[7], the Lichnerowicz operator $\left(\tilde{\triangle}_L h^{\perp}\right)_{ij}$ can be reduced to

$$\left[-\frac{d^2}{dx^2} + \frac{l(l+1)}{r^2} + m_i^2(r)\right] f_i(x) = \omega_{i,l}^2 f_i(x) \quad i = 1,2 \quad , \tag{23}$$

where we have used reduced fields of the form $f_i(x) = F_i(x)/r$ and where we have defined two r-dependent effective masses $m_1^2(r)$ and $m_2^2(r)$

$$\begin{cases} m_1^2(r) = \frac{6}{r^2}\left(1-\frac{b(r)}{r}\right) + \frac{3}{2r^2}b'(r) - \frac{3}{2r^3}b(r) \\ \\ m_2^2(r) = \frac{6}{r^2}\left(1-\frac{b(r)}{r}\right) + \frac{1}{2r^2}b'(r) + \frac{3}{2r^3}b(r) \end{cases} \quad (r \equiv r(x)). \tag{24}$$

In order to use the W.K.B. method considered by 't Hooft in the brick wall problem[8], from (23) we can extract two r-dependent radial wave numbers

$$k_i^2\left(r,l,\omega_{i,nl}\right)=\omega_{i,nl}^2-\frac{l\left(l+1\right)}{r^2}-m_i^2\left(r\right)\quad i=1,2\quad . \tag{25}$$

Then the counting of the number of modes with frequency less than ω_i is given approximately by

$$\tilde{g}\left(\omega_i\right)=\int_0^{l_{\max}}\nu_i\left(l,\omega_i\right)\left(2l+1\right)dl. \tag{26}$$

$\nu_i\left(l,\omega_i\right)$ is the number of nodes in the mode with $\left(l,\omega_i\right)$, such that $\left(r\equiv r\left(x\right)\right)$

$$\nu_i\left(l,\omega_i\right)=\frac{1}{\pi}\int_{-\infty}^{+\infty}dx\sqrt{k_i^2\left(r,l,\omega_i\right)}. \tag{27}$$

Here it is understood that the integration with respect to x and $l_{\max}$ is taken over those values which satisfy $k_i^2\left(r,l,\omega_i\right)\geq 0$. With the help of $(26,27)$, (19) becomes

$$\frac{\Lambda}{8\pi G}=-\frac{1}{\pi}\sum_{i=1}^{2}\int_0^{+\infty}\omega_i\frac{d\tilde{g}\left(\omega_i\right)}{d\omega_i}d\omega_i. \tag{28}$$

This is the one loop graviton contribution to the induced cosmological constant. The explicit evaluation of (28) gives

$$\frac{\Lambda}{8\pi G}=\rho_1+\rho_2=-\frac{1}{4\pi^2}\sum_{i=1}^{2}\int_{\sqrt{m_i^2(r)}}^{+\infty}\omega_i^2\sqrt{\omega_i^2-m_i^2\left(r\right)}d\omega_i, \tag{29}$$

where we have included an additional 4π coming from the angular integration. The use of the zeta function regularization method to compute the energy densities ρ_1 and ρ_2 leads to

$$\rho_i\left(\varepsilon\right)=\frac{m_i^4\left(r\right)}{64\pi^2}\left[\frac{1}{\varepsilon}+\ln\left(\frac{4\mu^2}{m_i^2\left(r\right)\sqrt{e}}\right)\right]\quad i=1,2\quad , \tag{30}$$

where we have introduced the additional mass parameter μ in order to restore the correct dimension for the regularized quantities. Such an arbitrary mass scale emerges unavoidably in any regularization scheme. The renormalization is performed via the absorption of the divergent part into the re-definition of a bare classical quantity. Here we have two possible choices: the induced cosmological constant Λ or the gravitational Newton constant G. In any case a certain degree of arbitrariness is present because of the scale parameter μ. However, it is instructive a comparison of the result in (30) with the one which can be obtained by imposing a UV cutoff. A direct calculation leads to $(i=1,2)$

$$\int_{\sqrt{m_i^2(r)}}^{+\infty}\omega_i^2\sqrt{\omega_i^2-m_i^2\left(r\right)}d\omega_i$$

$$\underset{x_i=\omega_i/\sqrt{m_i^2(r)}}{=} \frac{m_i^4(r)}{4}\left[x_i^3\sqrt{x_i^2-1}-\frac{x_i}{2}\sqrt{x_i^2-1}-\frac{1}{2}\ln\left(x_i+\sqrt{x_i^2-1}\right)\right]_1^{\omega_{UV}/\sqrt{m_i^2(r)}}$$

$$\simeq \frac{m_i^4(r)}{4}\left[\frac{\omega_{UV}^4}{m_i^4(r)}-\frac{\omega_{UV}^2}{2m_i^2(r)}-\frac{1}{2}\ln\left(\frac{2\omega_{UV}}{\sqrt{m_i^2(r)}}\right)\right], \tag{31}$$

where $\omega_{UV} \gg \sqrt{m_i^2(r)}$. Nevertheless, for some backgrounds in some ranges,

$$m_0^2(r) = m_1^2(r) = -m_2^2(r). \tag{32}$$

Thus, in these cases

$$\frac{\Lambda}{8\pi G} = \rho_1+\rho_2 = -\frac{1}{4\pi^2}\left[\int_{\sqrt{m_0^2(r)}}^{+\infty}\omega^2\sqrt{\omega^2-m_0^2(r)}d\omega+\int_0^{+\infty}\omega^2\sqrt{\omega^2+m_0^2(r)}d\omega\right]$$

$$\simeq -\frac{1}{4\pi^2}\left[\frac{\omega_{UV}^4}{2}+\frac{m_0^4(r)}{8}\ln\left(\frac{m_0^2(r)\sqrt{e}}{4\omega_{UV}^2}\right)\right], \tag{33}$$

where we have used

$$\int_0^{+\infty}\omega^2\sqrt{\omega^2+m_0^2(r)}d\omega$$

$$\underset{x=\omega/\sqrt{m_0^2(r)}}{=} \frac{m_0^4(r)}{4}\left[x^3\sqrt{x^2+1}+\frac{x}{2}\sqrt{x^2+1}-\frac{1}{2}\ln\left(x+\sqrt{x^2+1}\right)\right]_0^{\omega_{UV}/\sqrt{m_0^2(r)}}. \tag{34}$$

The Schwarzschild Schwarzschild-de Sitter (SdS) and Schwarzschild-Anti de Sitter (SAdS) backgrounds satisfy relation (32) in a region close to the throat. Indeed, by expanding $b(r)$ close to the throat, one gets $(r \equiv r(x))$

$$\begin{cases} m_1^2(r) = \frac{6}{r^2}-\frac{15r_t}{2r^3}-\frac{6b'(r_t)}{r^2}+\frac{15b'(r_t)r_t}{2r^3} \\ m_2^2(r) = \frac{6}{r^2}-\frac{9r_t}{2r^3}-\frac{4b'(r_t)}{r^2}+\frac{9b'(r_t)r_t}{2r^3} \end{cases} \tag{35}$$

and for example, for the Schwarzschild case where $b(r) = r_t = 2MG$, we get

$$\begin{cases} m_1^2(r) = -\frac{3r_t}{2r^3} \\ m_2^2(r) = +\frac{3r_t}{2r^3} \end{cases}. \tag{36}$$

Note that (39) works when the effective masses satisfy relation (32), otherwise the zeta function and the cutoff regularizations produce different results as shown by (31). The divergence can be eliminated by separating the cosmological constant Λ, into a bare cosmological constant Λ_0 and a divergent quantity Λ^{div}, where

$$\Lambda^{div} = \frac{Gm_0^4(r)}{\varepsilon 32\pi^2}, \tag{37}$$

or

$$\Lambda_{UV}^{div} = -\frac{G}{4\pi^2}\left[\frac{\omega_{UV}^4}{2} + \frac{m_0^4(r)}{8}\ln\left(\frac{\mu^2\sqrt{e}}{4\omega_{UV}^2}\right)\right]. \tag{38}$$

In both cases, the remaining finite value for the cosmological constant reads

$$\frac{\Lambda_0}{8\pi G} = (\rho_1(\mu) + \rho_2(\mu)) = \rho_{eff}^{TT}(\mu, r) = \frac{m_0^4(r)}{32\pi^2}\ln\left(\frac{4\mu^2}{m_0^2(r)\sqrt{e}}\right). \tag{39}$$

3.2 The example of non-commutative theories

Non Commutative theories provide a powerful method to naturally regularize divergent integrals appearing in (29). Basically, the number of states is modified in the following way[11]

$$dn = \frac{d^3x d^3k}{(2\pi)^3} \Longrightarrow dn_i = \frac{d^3x d^3k}{(2\pi)^3}\exp\left(-\frac{\theta}{4}k_i^2\right), \tag{40}$$

with

$$k_i^2 = \omega_{i,nl}^2 - m_i^2(r) \quad i = 1,2. \tag{41}$$

This deformation corresponds to an effective cut off on the background geometry (20). The UV cut off is triggered only by higher momenta modes $\gtrsim 1/\sqrt{\theta}$ which propagate over the background geometry. The virtue of this kind of deformation is its exponential damping profile, which encodes an intrinsic nonlocal character into fields $f_i(x)$. Plugging (27) into (26) and taking account of (40), the number of modes with frequency less than ω_i, $i = 1,2$ is given by

$$\tilde{g}(\omega_i) = \frac{1}{\pi}\int_{-\infty}^{+\infty} dx \int_0^{l_{\max}} (2l+1)\sqrt{\omega_{i,nl}^2 - \frac{l(l+1)}{r^2} - m_i^2(r)}\exp\left(-\frac{\theta}{4}k_i^2\right) dl \tag{42}$$

and the induced cosmological constant becomes

$$\frac{\Lambda}{8\pi G} = \frac{1}{6\pi^2}\left[\int_{\sqrt{m_0^2(r)}}^{+\infty}\sqrt{\left(\omega^2 - m_0^2(r)\right)^3}e^{-\frac{\theta}{4}\left(\omega^2 - m_0^2(r)\right)} + \int_0^{+\infty}\sqrt{\left(\omega^2 + m_0^2(r)\right)^3}e^{-\frac{\theta}{4}\left(\omega^2 + m_0^2(r)\right)}\right], \tag{43}$$

which integrated leads to

$$\frac{\Lambda}{8\pi G}=\frac{1}{12\pi^2}\left(\frac{4}{\theta}\right)^2\left(y\cosh\left(\frac{y}{2}\right)-y^2\sinh\left(\frac{y}{2}\right)\right)\,K_1\left(\frac{y}{2}\right)+y^2\cosh\left(\frac{y}{2}\right)K_0\left(\frac{y}{2}\right),\tag{44}$$

where $K_0(y)$ and $K_1(y)$ are the modified Bessel function and

$$y=\frac{m_0^2(r)\,\theta}{4}.\tag{45}$$

The asymptotic properties of (44) show that the one loop contribution is everywhere regular. Indeed, we find that when $y\to+\infty$,

$$\frac{\Lambda}{8\pi G}\simeq\frac{1}{6\pi^2\theta^2}\sqrt{\frac{\pi}{y}}\left[3+\left(8y^2+6y+3\right)\exp(-y)\right]\to 0.\tag{46}$$

Conversely, when $y\to 0$, we obtain

$$\frac{\Lambda}{8\pi G}\simeq\frac{4}{3\pi^2\theta^2}\left[2-\left(\frac{7}{8}+\frac{3}{4}\ln\left(\frac{y}{4}\right)+\frac{3}{4}\gamma\right)y^2\right]\to\frac{8}{3\pi^2\theta^2}\tag{47}$$

a finite value for Λ. Note that expression (44) can be used when the background satisfies the relation (32). For the other cases, we find that the effective masses contribute in the same way at one loop. Thus (43) becomes

$$\begin{aligned}\frac{\Lambda}{8\pi G}=\frac{1}{6\pi^2}&\left[\int_{\sqrt{m_1^2(r)}}^{+\infty}\sqrt{\left(\omega^2-m_1^2(r)\right)^3}e^{-\frac{\theta}{4}\left(\omega^2-m_1^2(r)\right)}\right.\\&\left.+\int_{\sqrt{m_2^2(r)}}^{+\infty}\sqrt{\left(\omega^2-m_2^2(r)\right)^3}e^{-\frac{\theta}{4}\left(\omega^2-m_2^2(r)\right)}\right].\end{aligned}\tag{48}$$

For example, when

$$m_1^2(r)=m_2^2(r),\tag{49}$$

(48) reduces to

$$\frac{\Lambda}{8\pi G}=\frac{1}{6\pi^2}\left(\frac{4}{\theta}\right)^2\left(\frac{1}{2}y(1-y)K_1\left(\frac{y}{2}\right)+\frac{1}{2}y^2K_0\left(\frac{y}{2}\right)\right)\exp\left(\frac{y}{2}\right).\tag{50}$$

The asymptotic expansion of (50) leads to

$$\frac{\Lambda}{8\pi G}\simeq\frac{1}{6\pi^2}\left(\frac{4}{\theta}\right)^2\frac{3}{8}\sqrt{\frac{\pi}{y}}\to 0,\tag{51}$$

when $y\to\infty$. On the other hand, when $z\to 0$, one gets

$$\frac{\Lambda}{8\pi G}\simeq\frac{1}{6\pi^2}\left(\frac{4}{\theta}\right)^2\left[1-\frac{z}{2}+\left(-\frac{7}{16}-\frac{3}{8}\ln\left(\frac{z}{4}\right)-\frac{3}{8}\gamma\right)z^2\right]\to\frac{8}{3\pi^2\theta^2},\tag{52}$$

i.e. a finite value of the cosmological term.

4 Summary and conclusions

In this contribution, the effect of a ZPE on the cosmological constant has been investigated using two specific geometries such as dS and AdS metrics. The computation has been done by means of a variational procedure with a Gaussian Wave Functional which should be a good candidate for a ZPE calculation. We have found that only the graviton is relevant[9]. Actually, the appearance of a ghost contribution is connected with perturbations of the shift vectors[4]. In this work we have excluded such perturbations. As usual, in ZPE calculation we meet the problem of divergences which are regularized with zeta function techniquesor by introducting a UV cutoff. After regularization , we have adopted to remove divergences by absorbing them into the induced cosmological constant Λ. Another possibility of keeping under control divergences comes from a NCG induced minimal length. As a result we get a modified counting of graviton modes. This let us obtain everywhere regular values for the cosmological constant, independently of the chosen background, which nevertheless is of a spherically symmetric type. Although the result seems to be promising, we have to note that the evaluation is at the Planck scale.

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Lensing effects in ringholes and the multiverse black holes

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Abstract Ringholes are space-time tunnelings connecting two asymptotically flat regions by means of a throat with the topology of a torus. This report considers the processes of semiclassical thermal emission from ringholes. It is shown that at or near the throat the ringholes can be characterized as a mixture of two thermal sources, one at positive temperature and the other at negative temperature which, respectively, emit usual black body radiation and phantom-like radiation, leading to two possible limiting situations, one similar to a wormhole in that it behaves just like a diverging lens, and the other similar to a black hole in that it behaves only as a converging lens.

1 On wormholes, rinholes and the multiverse

The so called ringholes [1] may now become a space-time construct pertaining to general relativity with remarkable pedagogical and even observational interests. A set of potentially observable effects from space-time tunnelings was first considered in a couple of papers at the end of the twenty century [2,3]. More recently, Shatskiy [4] has studied in greater detail the lensing effect produced by a wormhole in a beam of light rays. He obtained that the image that could be obtained from a single luminous source would be that of a bright circle (See Fig. 1). In the light of this result it was speculated [5] that, since wormholes may connect our universe with a parallel universe, the circular signature which was induced by the lensing effect from a wormholes might be uncovering the very existence of a parallel universe. However, it is worth noticing that there are many astronomical objects, including black holes, galaxies and negative-energy stars, which may leave a bright ring as a result from the induced lensing. Owing to their optical properties, which simultaneously corre-

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_18,

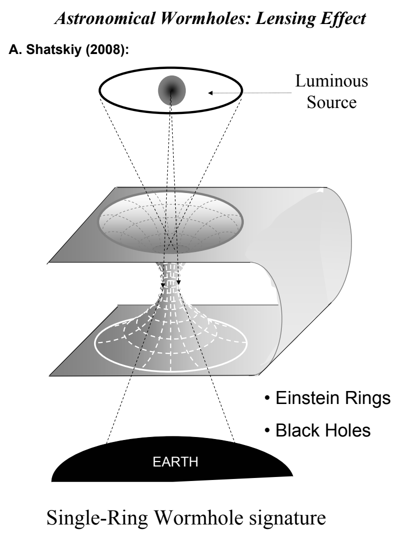

Fig. 1 Gravitational lensing effect induced by a wormhole into the light rays from a luminous source. The figure shows the originally calculated ring which could be indistinguishable from the rings induced by black holes, negative-energy stars or galaxies. When the continuous values of the azimuthal and polar angles θ and ϕ of the throat sphere are also taken into account, then the inner region of the circle is filled with the luminosity yield from the deflected rays as well, producing a bright disc, such as it was pointed out by Shatskiy himself [10].

sponded to those of a divergent and a convergent lens [1], ringholes actually might be expected to ultimately be some kind of mixture of black- and worm-holes, at least from a thermodynamical point of view (Fig. 2). It is still unclear however whether ringholes really exist in nature though the possible imprint that they would leave in the universe appears as particularly distinctive in the form of two concentric bright rings (see Fig. 3(b)) [6], a signature which might have been already identified in a glowing double ring recently detected by the Hubble telescope [7] and may be pointing to a direct detection of some features of a universe other than ours own. If so, this would be the first observational direct evidence of the existence of a real multiverse. Indeed a ringhole is nothing but a wormhole where the spherical symmetry on the throat has been replaced for that of a torus [1] (See Fig. 2 for a pictorial representation of a ringhole). That more complicated topology allows any light ray or particle to follow an itinerary through the ringhole interior along which they may not find any exotic or phantom matter but just ordinary matter, actually a safer way than that which can be followed through wormholes where the traveler inexorably becomes involved with matter having exotic properties, potentially incompatible with the very concept of life [1]. I would like to emphasize now that there exists at least

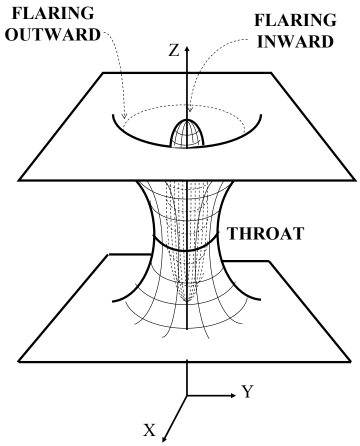

Fig. 2 Pictorial representation of the space-time of a ringhole connecting two asymptotically flat regions belonging to the same universe, or to two different universes, as placed on a rectangular coordinate system. It satisfies the topology of a torus and, therefore, it can be embedded in a surface which partly flares outward, like a wormhole with spherical symmetry, and partly flares inward like a compact object made up of ordinary energy, such as a black hole or a neutron star.

one highly nontrivial standpoint from whose perspective one might reach the rather tentative conclusion that, whereas the laws of quantum mechanics may prohibit the existence of isolated wormholes, they in any event are perfectly compatible with the presence of ringholes, no matter whether they are isolated or not. This apparently unexpected result came about from the consideration of the so called quantum interest conjecture of Ford and Roman [8], which follows from the quantum inequalities and according to which a positive energy pulse must always overcompensate any negative energy pulse by an amount which is a monotonically increasing function of the pulse separation. An isolated wormhole thermally radiating phantom energy at a negative temperature [9] seems to violate such a conjecture. However, in principle, at least one might always accommodate the relative proportion of phantom to ordinary matter in a ringhole in such a way that, when thermally radiating, the intensity of the positive-energy emitted pulses always overcompensated those containing phantom energy. It is in this sense that ringholes are more likely to exist than wormholes.

In fact, it is a result of the present paper that the very structure of ringholes makes us to expect that the semiclassical thermal radiation processes spontaneously taking place in them actually be a mixture from those thermal radiations being emitted from black bodies simultaneously radiating at positive and negative temperatures, a process which, as I have said above, at least could always be made to satisfy the Ford and Roman conjecture [8] by adjusting the geometrical and topological parameters that define a ringhole (Fig. 3 (a)), and that we show in this paper to unavoidably lead to a remanent final construct representable in the form of some sort of either wormhole or black hole that would be unable to radiate any more, provided that no accretion of dark or phantom energy took place.

After reviewing the geometry of ringholes and their occurrence in the universe and multiverse, by using a heuristic procedure based on the Hayward formulation [9], this report will deal with the semiclassical thermal properties of ringholes and their connections with the allowed geometrical structure of these tunnels, always respecting the quantum interest conjecture (or its complementary version) in terms of a vacuum made up of positive (negative) internal energy, placed in a given multiverse where these toroidal tunnelings can connect two universes to each other.

2 The ringhole space-time

In this section we shall very briely review the characteristics of the space-time of a ringhole, particularly emphasizing its properties as a double (diverging and converging) optical lens and the plausible imprint that ringholes may leave from luminous objects placed behind them in our own universe or in other universe if we consider a multiverse context. Figs. 2 and 3 (a) show the set of geometrical parameters [11], we can finally obtain for the three-dimensional space-time metric that corresponds to a torus [1]

$$ds^2 = -C_2 r^2 dt^2 + b^2 \left[1 + \frac{C_1 a^2 \sin^2\varphi_2}{r^6\left(1-\frac{A^2}{r^4}\right)}\right] d\varphi_2^2 + m^2 d\varphi_1^2 \tag{1}$$

where

$$A = a^2 - b^2,\ \ m = a - b\cos\varphi_2,\ \ r = \sqrt{a^2+b^2-2ab\cos\varphi_2}, \tag{2}$$

with C_1 and C_2 arbitrary integration constants, and a and b are the radius of the circumference generated by the circular axis of the torus and that of a torus section, respectively, both radii being constant in metric (1), with $a > b$. Metric (1) is defined for $0 \leq t \leq \infty$, $a-b \leq r \leq a+b$ and the angles (see Fig. 2 (a)) $0 \leq \varphi_1,\ \varphi_2 \leq 2\pi$.

In order to check the properties of a ringhole as an optical lens, we now write the static spacetime metric of a single, traversable ringhole in the form [1,11]

$$ds^2 = -dt^2 + \left(\frac{n_\ell}{r_\ell}\right)^2 d\ell^2 + m_\ell^2 d\varphi_1^2 + \left(\ell^2 + b_0^2\right) d\varphi_2^2, \tag{3}$$

where $-\infty < t < +\infty$, with $-\infty < \ell < +\infty$ the proper radial distance of each transversal section of the torus, and

$$m_\ell = a - \left(\ell^2 + b_0^2\right)^{1/2} \cos\varphi_2, \;\; n_\ell = \left(\ell^2 + b_0^2\right)^{1/2} - a\cos\varphi_2, \tag{4}$$

$$r_\ell = \sqrt{a^2 + \ell^2 + b_0^2 - 2\left(\ell^2 + b_0^2\right)^{1/2} a\cos\varphi_2}, \tag{5}$$

in which b_0 is the throat radius. As ℓ increases from $-\infty$ to 0, b decreases monotonously from $+\infty$ to its minimum value b_0 at the throat radius, and as ℓ increases onward to $+\infty$, b increases monotonously to $+\infty$ again. Now, for metric (3) to describe a traversable ringhole we must embed it in a three-dimensional Euclidean space at fixed time t [1]. We should consider a three-geometry that would respect the topology of a torus and satisfy $a \geq b \leq \ell$, so it will suffice to confining attention to the maximum- and minimum-circumference slices, that is $\varphi_2 = \pi, 0$, through it. In the first case, $r = m = n = a + b$, and

$$ds^2 = \frac{dr^2}{1 - \frac{b_0^2}{b^2}} + r^2 d\varphi_1^2.$$

We visualize then this slice as removed from space-time (3) and embedded in the three-dimensional Euclidean space which is taken to be given in terms of cylindrical coordinates such as

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2 = \left[1 + \left(\frac{dz}{dr}\right)^2\right] dr^2 + r^2 d\phi^2, \tag{6}$$

with $dz/dr = \left(b^2/b_0^2 - 1\right)^{-1/2}$ in order for metric (6) to be the same as the previous one. This condition displays the way in which, provided we have fixed $a \geq b$, the function $b \equiv b(\ell)$ shapes the ringhole spatial geometry. In case that we consider the minimum-circumference slice, $\varphi_2 = 0$, through the geometry of three-space at a fixed time, then $r = m = -n = a - b$, and the previous metric is obtained again, so that we always achieve the latter condition, no matter the choice of slice.

The requirement that ringholes be connectible to asymptotically flat space-time entails at the throat that the embedding surface flares outward for $2\pi - \varphi_2^c > \varphi_2 > \varphi_2^c$, and flares inward for $-\varphi_2^c < \varphi_2 < \varphi_2^c$, with $\varphi_2^c = \arccos(b/a)$, which respectively satisfy the condition $d^2r/dz^2 > 0$ and $d^2r/dz^2 < 0$, at or near the throat. It follows [1] that one had to expect lensing effects to occur at or near the ringhole throat, that is to say, the mouths would act like a diverging lens for world lines along $2\pi - \varphi_2^c > \varphi_2 > \varphi_2^c$, and like a converging lens for world lines along $-\varphi_2^c < \varphi_2 < \varphi_2^c$. No lensing actions would therefore take place at the angular horizons placed at $\varphi_2 = \varphi_2^c$ and $\varphi_2 = 2\pi - \varphi_2^c$.

In fact, in the case of ringholes, instead of producing just a single flaring outward for light rays passing through the wormhole throat [11], this multiply connected topology, in addition to that flaring outward (diverging) effect, also produces a flaring inward (converging) effect [1] on the light rays that pass through its throat, in such a way that an observer on Earth would interpret light passing through the ringhole throat from a single luminous source as coming from two bright, glowing concentric rings, which forms up the distinctive peculiar pattern from ringholes (See

Fig. 3 (b) [6]). That pattern cannot be generated by any other possible disturbing astronomical object other than the very implausible set of three luminous massive objects (let us say galaxies) which must be so perfectly aligned along the sigh line that its occurrence becomes extremely unlikely [7]. The angular radii of the two concentric bright rings shown in Fig. 3(b) are given by the approximate expressions

$$\Theta_W \sim \sqrt{\frac{4GMd_{LS}}{c^2 d_L d_S}\left(1-\frac{\varphi_2^c}{2\pi}\right)}$$

$$\Theta_B \sim \sqrt{\frac{4GMd_{LS}\varphi_2^c}{c^2 \pi d_L d_S}},$$

where d_{LS} is the distance between the luminous source and the ringhole's mouth closest to Earth (named lens), d_S is the distance between the source and Earth, and d_L is the distance between the ringhole's mouth closest to Earth (lens) and Earth. We note that if $a >> b$, then $\Theta_B > \Theta_W$, and if $b \sim a$, then $\Theta_B < \Theta_W$. the extreme case for which we have a single ring with nearly double brightness ($\Theta_B = \Theta_W$) would take place when $b = a/2$.

3 Heuristic approach to thermal emission from ringholes

The ultimate physical reason why a wormhole radiates phantom energy at a negative temperature lies [9] on the feature that the surface embedding the wormhole space-time flares outward at or near the throat, so that, unlike most gravitational systems made out of ordinary matter, such as stars or black holes, wormhole throat really always behaves like an optical diverging lens under any circumstances. That is no longer the case with a ringhole where the optical properties that can be ascribed to its throat stem now from the feature that, depending on the value taken on the angle, φ_2, formed up by the radius of the torus section, b, with that of the circumference generated by the circular axis of the torus, a, with $a > b$ (see Fig. 2 (a)), the embedding surface flares either outward if $2\pi - \varphi_2^c > \varphi_2 > \varphi_2^c$, or inward when $-\varphi_2^c < \varphi_2 < \varphi_2^c$, with $\varphi_2^c = \arccos(b/a)$.

Neither flaring nor lensing actions would therefore take place at the two existing angular horizons at $\varphi_2 = \varphi_2^c$ and $\varphi_2 = 2\pi - \varphi_2^c$. It appears then most natural to expect that from the embedding surface sector defined by $2\pi - \varphi_2^c > \varphi_2 > \varphi_2^c$ ringholes would radiate phantom or exotic radiation at a negative temperature such as wormholes do [9], and from the embedding surface sector $-\varphi_2^c < \varphi_2 < \varphi_2^c$, separated from the previous one by means of the above two angular horizons, the same ringhole simultaneously radiated ordinary positive-energy particles at a positive temperature such as black holes do [12], both along thermally chaotic processes. We can arrange the parameters in such a way that the positive-energy radiation always overcompensated the negative-energy radiation in case that the vacuum surrounding the ringhole be characterized by a positive-energy fluid, like quintessential dark

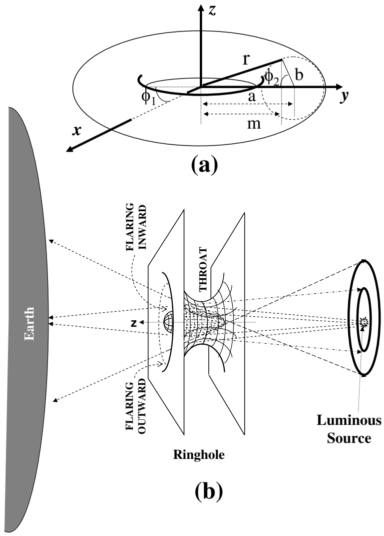

Fig. 3 Gravitational lensing effect produced by a ringhole from a single luminous source. (a) Parameters defining the toroidal ringhole throat in terms of which metric (1) is defined. (b) Rays passing near the outer and inner surfaces respectively flare outward and inward, leading to a image from a single luminous source placed behind the ringhole which is made of two concentric bright rings. The relative mutual positions of these rings would depend on the distance between the ringhole and the luminous source. If that distance is small enough then the larger outer ring comes from the flaring inward surface, and conversely, if the distance source-ringhole is increased then the outer ring comes from the outward surface, the larger that distance the greater the difference between the radii of the two bright rings.

energy [13], or vice versa, the negative-energy radiation always overcompensated the positive-energy radiation when the surrounding vacuum would be a fluid with negative internal energy, like phantom energy [14].

We would now estimate the expressions for these temperatures and their dependence on the geometrical parameters defining the ringhole surface. The generalized surface gravity for the case of a ringhole can be written as an extension to toroidal symmetry from the surface gravity in the spherically symmetric case that corresponds to a wormhole. It gets the form

$$\kappa = \frac{b_0^2}{2b\left(a^2+b^2-2ab\cos\varphi_2\right)} -2\pi\left(a^2+b^2-2ab\cos\varphi_2\right)^{1/2}(\varepsilon - p_r), \tag{7}$$

with ε the energy density and p_r the radial pressure which depends on the angle φ_2, too. In the interval $2\pi - \varphi_2^c > \varphi_2 > \varphi_2^c$ the combination $\varepsilon - p_r$ becomes negative [1] and hence κ is definite positive. In case that $-\varphi_2^c < \varphi_2 < \varphi_2^c$ then [1] $\varepsilon - p_r > 0$ and therefore the generalized surface gravity κ becomes definite negative.

Now, from the general expression for gravitational temperature T_G (see for example [9])

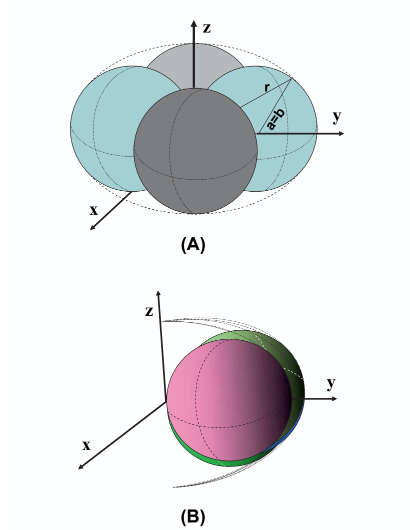

Fig. 4 Limiting ringholes in which $a^2 = b^2$. We have then two possible final situations. (A) if $a = b$, then the process of thermal emission of positive- and negative radiation tends to that limiting geometry where all ordinary energy contributing the ringhole with $a > b$ is exhausted, leaving a geometry which is equivalent to that of a wormhole in that the surface embedding it flares outward but not inward, and is therefore made out of exotic, phantomlike energy characterized by a negative temperature; and (B) if $a = -b$ then the process of thermal emission tends to that limiting geometry where now all exotic energy contributing the ringhole with $|a| > |b|$ is exhausted, leaving a geometry which is now equivalent to that of a black hole, instead of a wormhole, in that the surface embedding the ringhole spacetime metric (1) always flares inward, being therefore made out of ordinary matter with positive internal energy, which is characterized by a positive temperature. Whether or not the inherent violation of the quantum interest conjecture prevented the onset into a new period of thermal emission starting with the limiting ringhole in case (A), or into the emission process itself from the very beginning in case (B), is a matter to be further investigated.

$$T_G = \frac{-\left.\kappa\right|_{b=b_0}}{2\pi}, \tag{8}$$

it follows then that, as it was to be expected for an absolute value of the radius a sufficiently larger than that for radius b, (i) whereas the temperature on the embedding surface that flares outward is always negative, that on the flaring inward embedding surface is always positive and its absolute value is larger or smaller than that for the negative temperature, depending on the nature of the surrounding vacuum, and (ii) in spite of that, because the involved negative-temperature system and the positive-temperature system can never come in contact in the present case (because the temperature vanishes at the two angular horizons), and hence the former system is not definite hotter or coulder than the latter one, at least at the first stages of a large-a ringhole evaporation, the intensity and energy of the radiation pulses emitted at the positive temperature may well overcompensate those generated at a negative temperature or vice versa, so satisfying the generalized quantum interest conjecture (see next section, Fig. 5). It also follows that as the above overall thermal process

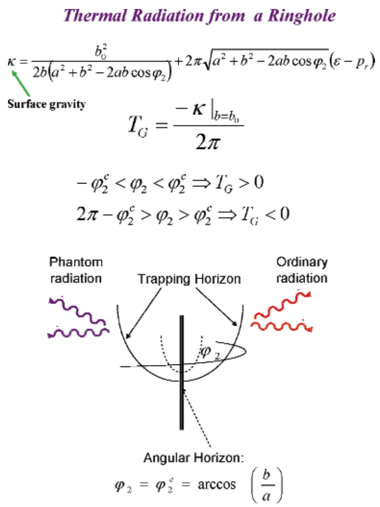

Fig. 5 The interplay between the angular horizons and the trapping horizons during the thermal emission of ringholes.

of combined emission progresses, the ringhole may be converted into either some sort of wormhole in that the surface embedding it would tend to only flare outward; that precisely happening when a becomes exactly equal to b (see Fig. 4(A)) or some sort of black hole in that the embedding surface tended to just flare inward, and this takes place when $a = -b$ (see Fig. 4(B)). These limiting geometric ringhole configurations either can only continue emitting phantom radiation at a negative temperature as far as the restricted quantum interest conjecture is violated, such as it also happens with spherically symmetric wormholes, or can emit ordinary radiation at a positive temperature such as it happens with spherically symmetric black holes. It is in this sense that ringholes with these limiting geometries are equivalent either to a wormhole or to a black hole. There is still another aspect in which wormholes or black holes and the above limiting ringhole configurations are again physically though not geometrically equivalent. It is in that both types of tunneling (wormholes and limiting ringholes with $a = b$) and both types of objects (black holes and limiting ringholes with $a = -b$) would all leave the same gravitational signature on sky when light coming to us from a luminous object is placed behind them, along the line of sight, i.e. a single glowing ring of the kind already considered by Shatskiy for single wormholes [4]. Unfortunately, such bright rings are not at all distinguishable from e.g. Einstein rings, or that is produced by stars with negative energy. It is worth remarking, moreover, that the thermal emission process of ringholes can take

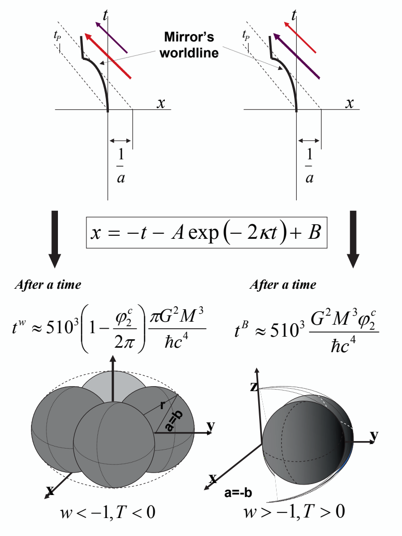

Fig. 6 Thermal radiation from ringholes envisaged as the radiation from a receding mirror, analogously to how it happens in black holes [12]. Also in the present case the mode solutions for the receding mirror are the same as the late-time asymptotic modes for a torus of a mixture phantom/dark energy going to form the ringhole.

only place because one can always define a trapping horizon (see Fig. 5), similarly to how we did for the case of wormholes [4]. Again, such as it was envisaged for the case of black holes [15], also in the case of ringholes the thermal radiation can be regarded to be the particle production originating from moving mirrors out of a cosmic vacuum made up of a mixture of dark and phantom energies. Taking the usual mirror trajectory, $x = -t - A\exp(-2\kappa t) + B$, with A and B given constants, we gain get the two limiting situations depicted in Fig. 4 (see Fig. 6). Thus, whereas the limiting space-time 4(A) with $a = b$ is obtained by only emitting phantom radiation at $T < 0$, after a time

$$t^W \sim 5\times 10^3\frac{\pi G^2M^3}{\hbar c^4}\left(1-\frac{\varphi_2^c}{2\pi}\right),$$

the limiting space-time 4(B) with $a = -b$, can finally be reached by usual thermal radiation at $T > 0$, after a time

$$t^B \sim 5\times 10^3\frac{G^2M^3\varphi_2^c}{\hbar c^4}.$$

We finally note, moreover, that, whereas in a vacuum with positive energy the Ford-Roman quantum interest conjecture [8] holds, if the vacuum is made out of phantom energy, or any fluid having negative internal energy, then what could be called a *quantum altruism conjecture* starts holding, implying that a pulse with positive energy must always be overcompensated by one with negative energy, the difference between the two pulses being in this case proportional to the time elapsed from the emission of the positive-energy pulse and the emission of the negative-energy pulse (see Fig. 7). In fact, since the two subsystems respectively having positive and neg-

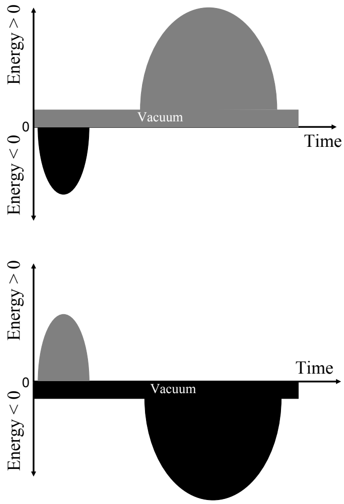

Fig. 7 Upper part: The quantum interest conjecture. Pulses of negative energy are allowed by quantum theory in a vacuum with positive energy provided the three conditions first introduced by Ford and Roman are fulfilled [8]. Lower part: The quantum altruism conjecture. Pulses with positive energy would be permitted by quantum theory in a vacuum with negative internal energy under the following three conditions. 1) The longer the positive pulse lasts, the weaker it must be. 2) A pulse with negative energy must follow whose magnitude exceeds that of the initial positive-energy pulse. 3) The longer the time interval between the two pulses, the larger the positive-energy pulse must be.

ative temperature in a single ringhole can never be in contact, in spite that $|T > 0|$ is always smaller than $|T < 0|$ by virtue of (7) and (8), it can be clearly deduced that, on the one hand, if the universe in which we immerse the rihghole is dominated by a positive internal energy fluid (i.e. $\rho + p > 0$), such as dark energy quintessence, the whole ringhole system will be progressively enriched in its positive temperature component over that with negative temperature as time is going on, in such a way that any emitted pulse with negative internal energy can always be overcompensated by a pulse with positive internal energy, and conversely, if on the other hand the universe is dominated by a negative internal energy fluid (i.e. $\rho + p < 0$), such as the phantom energy, then the whole ringhole system will be progressively enriched in its negative-temperature component over that with positive temperature as time is going on, and therefore any emitted pulse with positive internal energy can always be overcompensated by a pulse with negative internal energy, in both cases the difference between the two pulses increasing as the time elapsed between them becomes longer. The main conclusion of the present report ought to be that ringholes, if they at all exist, could offer us a particularly useful way to eventually observe luminous phenomena taking place in other universes, so confirming the very existence of parallel universes, and ultimately of the multiverse itself. Before closing up this report, I would like to add a brief comment on the very debated issue of the loss of quantum coherence, when applied to the case of the thermal emission from ringholes. Since the thermal evaporation of ringholes would always lead to final situations where some structures with nonzero energy are left, one can ensure that the information that went to enter the ringhole throat manifesting in the form of phantom and dark energy inhomogeneities, would always be preserved inside the final structures, so avoiding the emergence of any information paradox in ringhole thermal emission.

Acknowledgements This work was conceived and partly developed in the Estación Ecológica de Biocosmología of Medellín, Spain. The author is indebted to C. Sigüenza for useful discussions and S. Robles-Pérez for technical help. This work was supported by the Spanish MICINN under Research Project No. FIS2008-06332/FIS.

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Hamiltonian ADM Gravity in Non-Harmonic Gauges with Well Defined Non-Euclidean 3-Spaces: How Much Darkness can be Explained as a Relativistic Inertial Effect?

Luca Lusanna

Abstract In special and general relativity the synchronization convention of distant clocks may be simulated with a mathematical definition of global non-inertial frames (the only ones existing in general relativity due to the equivalence principle) with well-defined instantaneous 3-spaces. For asymptotically Minkowskian Einstein space-times this procedure can be used at the Hamiltonian level in the York canonical basis, where it is possible for the first time to disentangle tidal gravitational degrees of freedom from gauge inertial ones. The most important inertial effect connected with clock synchronization is the York time ${}^3K(\tau, \sigma^r)$, not existing in Newton gravity. This fact opens the possibility to describe some aspects of *darkness* as a relativistic inertial effect in Einstein gravity by means of a Post-Minkowskian reformulation of the Celestial Reference System ICRS.

In classical and quantum physics predictability is possible only if the relevant partial differential equations have a well-posed Cauchy problem, whose pre-requisite is the existence of a well defined 3-space (i.e. a clock synchronization convention) supporting the Cauchy data.

In Galilei space-time there is no problem: time and Euclidean 3-space are absolute.

Instead there is no intrinsic notion of 3-space, simultaneity, 1-way velocity of light (two distant clocks are involved) in the absolute Minkowski space-time: only the light-cone is intrinsically given as the locus of incoming and outgoing radiation. The light postulate says that the 2-way (only one clock is involved) velocity of light c is isotropic and constant. Its codified value replaces the rods (i.e. the standard of length) in modern metrology, where an atomic clock gives the standard of time. Einstein's $1/2$ synchronization convention [1] selects the Euclidean 3-spaces $x^o =$

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[1] An inertial observer A send a ray of light at x^o_i towards the (in general accelerated) observer B; the ray is reflected towards A at a point P of B world-line and then reabsorbed by A at x^o_f; by

S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_19,

$ct = const.$ of the inertial frames centered on inertial observers: only in this case the 2-way and 1-way light velocities coincide. However, with realistic accelerated observers the convention breaks down and till recently there was no definition of global non-inertial 3-spaces due to the coordinate singularities present in the *1+3 point of view* (only the world-line of a time-like observer is given) both with Fermi coordinates (crossing of the 3-spaces) and rotating frames (the horizon problem of the rotating disk).

In [1] the theory of global non-inertial frames is fully developed in the *3+1 point of view*: besides the observer world-line one gives an admissible 3+1 splitting of Minkowski space-time, i.e. a nice foliation whose leaves are instantaneous 3-spaces. Lorentz-scalar observer-dependent radar 4-coordinates $\sigma^A = (\tau; \sigma^r)$ are used: τ is an arbitrary increasing function of the observer proper time and σ^r are curvilinear 3-coordinates on the 3-spaces Σ_τ with the observer as origin. Each 3-space is asymptotically Euclidean with asymptotic inertial observers at spatial infinity. The inverse transformation $\sigma^A \mapsto x^\mu = z^\mu(\tau, \sigma^r)$ defines the embeddings of the 3-spaces Σ_τ into Minkowski space-time and the induced 4-metric is $g_{AB}[z(\tau, \sigma^r)] = [z^\mu_A\, \eta_{\mu\nu}\, z^\nu_B](\tau, \sigma^r)$, where $z^\mu_A = \partial z^\mu / \partial \sigma^A$ and ${}^4\eta_{\mu\nu} = \varepsilon\,(+---)$ is the flat metric ($\varepsilon = \pm 1$ according to either the particle physics $\varepsilon = 1$ or the general relativity $\varepsilon = -1$ convention). While the 4-vectors $z^\mu_r(\tau, \sigma^u)$ are tangent to Σ_τ, so that the unit normal $l^\mu(\tau, \sigma^u)$ is proportional to $\varepsilon^\mu{}_{\alpha\beta\gamma}\,[z^\alpha_1\, z^\beta_2\, z^\gamma_3](\tau, \sigma^u)$, we have $z^\mu_\tau(\tau, \sigma^r) = [N\, l^\mu + N^r\, z^\mu_r](\tau, \sigma^r)$ ($N(\tau, \sigma^r) = \varepsilon\,[z^\mu_\tau\, l_\mu](\tau, \sigma^r)$ and $N_r(\tau, \sigma^r) = -\varepsilon\, g_{\tau r}(\tau, \sigma^r)$ are the lapse and shift functions).

The foliation is nice and admissible if it satisfies the conditions:
1) $N(\tau, \sigma^r) > 0$ in every point of Σ_τ (the 3-spaces never intersect, avoiding the coordinate singularity of Fermi coordinates);
2) $\varepsilon\, {}^4g_{\tau\tau}(\tau, \sigma^r) > 0$, so to avoid the coordinate singularity of the rotating disk, and with the positive-definite 3-metric ${}^3g_{rs}(\tau, \sigma^u) = -\varepsilon\, {}^4g_{rs}(\tau, \sigma^u)$ having three positive eigenvalues (these are the Møller conditions [1]);
3) all the 3-spaces Σ_τ must tend to the same space-like hyper-plane at spatial infinity (so that there are always asymptotic inertial observers to be identified with the fixed stars).

These conditions imply that global *rigid* rotations are forbidden in relativistic theories. In [1] there is the expression of the admissible embedding corresponding to a 3+1 splitting of Minkowski space-time with parallel space-like hyper-planes (not equally spaced due to a linear acceleration) carrying differentially rotating 3-coordinates without the coordinate singularity of the rotating disk. It is the first consistent global non-inertial frame of this type.

As shown in [1, 2] every isolated system (particles, strings, fluids, fields) admitting a Lagrangian $\mathcal{L}(matter)$ can be reformulated as a *parametrized Minkowski theory*, in which the new embedding-dependent Lagrangian is $\mathcal{L}(matter, g_{AB}[z])$.

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convention P is synchronous with the mid-point between emission and absorption on A's world-line, i.e. $x^o_P = x^o_i + \frac{1}{2}\,(x^o_f - x^o_i)$.

This action is invariant under frame-preserving 4-diffeomorphisms, so that the embeddings are *gauge variables* and the ten components of $g_{AB}[z]$ are the special-relativistic *inertial potentials* [2]. A change of clock synchronization (of the shape of Σ_τ) and/or of the 3-coordinates into the 3-spaces is a gauge transformation: physics does not change, only the appearances of phenomena change.

In this formulation the description of matter has to be done with quantities which know the instantaneous 3-spaces Σ_τ. For instance a Klein-Gordon field $\tilde{\phi}(x)$ will be replaced with $\phi(\tau,\sigma^r) = \tilde{\phi}(z(\tau,\sigma^r))$; the same for every other field. Instead for a relativistic particle with world-line $x^\mu(\tau)$ we must make a choice of its energy sign: then it will be described by 3-coordinates $\eta^r(\tau)$ defined by the intersection of the world-line with Σ_τ: $x^\mu(\tau) = z^\mu(\tau,\eta^r(\tau))$. Differently from all the previous approaches to relativistic mechanics, the dynamical configuration variables are the 3-coordinates $\eta^r_i(\tau)$ and not the world-lines $x^\mu_i(\tau)$ (to rebuild them in an arbitrary frame we need the embedding defining that frame!).

The *inertial rest-frame instant form* of the isolated system [1, 2] is obtained by restricting the embedding to the inertial rest-frame centered on the Fokker-Pryce center of inertia: its Euclidean Wigner-covariant 3-spaces are orthogonal to the conserved 4-momentum of the isolated system. Every isolated system can be described as a decoupled non-local (and therefore *un-observable*) canonical non-covariant Newton-Wigner external center of mass [3], with an associated external realization of the Poincare' algebra, carrying a pole-dipole structure: the invariant mass M and the rest spin $\bar{S}$ of the isolated system. By construction, they depend upon Wigner-covariant relative variables describing the internal dynamics of the isolated system [4].

The world-lines $x^\mu_i(\tau)$ of the particles are derived (interaction-dependent) quantities and in general they do not satisfy vanishing Poisson brackets: already at the classical level a non-commutative structure emerges!

The definition of relativistic atomic physics (scalar positive-energy charged particles plus the electro-magnetic field in the radiation gauge with Grassmann-valued electric charges to regularize self-energies) and of its Poincare' generators becomes possible [3, 4, 5] in this framework. The identification of the Darwin potential, to be

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[2] They generate the relativistic apparent forces in the non-inertial frame and in the non-relativistic limit they reduce to the Newtonian inertial potentials. The extrinsic curvature ${}^3K_{rs}(\tau,\sigma^u) = [\frac{1}{2N}(N_{r|s} + N_{s|r} - \partial_\tau {}^3g_{rs})](\tau,\sigma^u)$, describing the *shape* of the instantaneous 3-spaces of the non-inertial frame as embedded 3-manifolds of Minkowski space-time, is a functional of the independent inertial potentials ${}^4g_{AB}$.

[3] It is convenient to replace it with its initial value, namely with the Jacobi data of the Hamilton-Jacobi formulation.

[4] Inside the Wigner 3-spaces there is an unfaithful internal realization of the Poincare' algebra, determined by the energy-momentum tensor, whose energy is the invariant mass and whose angular momentum is the rest spin. The internal 3-momentum vanishes being the rest-frame condition. The internal center of mass inside the Wigner 3-spaces is eliminated by the vanishing of the internal (interaction-dependent) Lorentz boosts, avoiding a double counting of this collective variable.

added to the Coulomb one, in this classical setting establishes a contact with the theory of relativistic bound states, whose constituents must be synchronized (absence of relative times).

Also a new formulation of *relativistic quantum mechanics and entanglement* was given [6]. The use of the static Jacobi data for the external center of mass avoids the causality problems connected with the instantaneous spreading of wave packets. Due to the need of clock synchronization for the definition of the instantaneous 3-spaces, the Hilbert space $H = H_{com,HJ} \otimes H_{rel}$ ($H_{com,HJ}$ is the Hilbert space of the external center of mass in the Hamilton-Jacobi formulation, while H_{rel} is the Hilbert space of the internal relative variables) is not unitarily equivalent to $H_1 \otimes H_2 \otimes ...$, where H_i are the Hilbert spaces of the individual particles. As a consequence, at the relativistic level the zeroth postulate of non-relativistic quantum mechanics does not hold: the Hilbert space of composite systems is not the tensor product of the Hilbert spaces of the sub-systems. The non validity of the zeroth postulate and the *non-locality* of Poincare' generators imply a *kinematical non-locality* and a *kinematical spatial non-separability* introduced by special relativity, which reduce the relevance of *quantum non-locality* in the study of the foundational problems of quantum mechanics which have to be rephrased in terms of relative variables.

The replacement of clock synchronization with an admissible 3+1 splitting can be used also in general relativity (GR), where also the space-time becomes dynamical [7], being determined by Einstein equations modulo 4-coordinate transformations (the gauge group of GR). We will define global non-inertial frames (the only ones existing in the large in GR due to the equivalence principle) with admissible 3+1 splittings and radar 4-coordinates in globally hyperbolic, asymptotically Minkowskian space-times in the framework of ADM canonical gravity. With suitable boundary conditions, eliminating super-translations [8], the asymptotic symmetries reduce to the ADM Poincare' group [5] and the non-Euclidean 3-spaces are orthogonal to the conserved ADM 4-momentum at spatial infinity [9]: this is a *non-inertial rest frame* of the 3-universe (see [1] for the non-inertial rest-frame instant form in special relativity). There are asymptotic inertial observers with spatial axes identified by means of the fixed stars of star catalogues.

As a consequence, the 3-universe (the isolated system "gravitational field plus matter") can be described as a decoupled non-covariant non-observable external pseudo-particle carrying a pole-dipole structure, whose mass and spin are identified by the ADM weak energy and by the ADM angular momentum. Instead the ADM 3-momentum vanishes, since this determines the rest-frame condition. The vanishing of the ADM Lorentz boosts eliminate the internal center of mass of the 3-universe.

In absence of matter Christodoulou - Klainermann space-times [10] are compatible with this description.

[5] For $G = 0$ it reduces to the Poincare' group of the matter in Minkowski non-inertial frames. In this way, after a restriction to inertial frames we can recover all the results of the standard model of elementary particles, which are connected with properties of the representations of the Poincare' group in inertial frames of Minkowski space-time.

Now the dynamical variable is not the embedding but the 4-metric, which determines the dynamical chrono-geometrical structure of space-time by means of the line element: it teaches to massless particles which are the allowed trajectories in each point [6]. Since tetrad gravity is more natural for the coupling of gravity to the fermions, the 4-metric is decomposed in terms of cotetrads, ${}^4g_{AB} = E^{(\alpha)}_A\, {}^4\eta_{(\alpha)(\beta)}\, E^{(\beta)}_B$ [7], and the ADM action, now a functional of the 16 fields $E^{(\alpha)}_A(\tau,\sigma^r)$, is taken as the action for ADM tetrad gravity [9]. This leads to an interpretation of gravity based on a congruence of time-like observers endowed with orthonormal tetrads: in each point of space-time the time-like axis is the unit 4-velocity of the observer, while the spatial axes are a (gauge) convention for observer's gyroscopes.

In canonical ADM tetrad gravity there are 16 fields, 16 conjugate momenta, 14 first-class constraints, generators of Hamiltonian gauge transformations, 14 gauge variables, the *GR inertial effects* and 2+2 physical variables, the *tidal effects* (the gravitational waves after linearization). As shown in [9, 12], in our family of space-times the Dirac Hamiltonian turns out to be the *weak* ADM energy [8] plus constraints. Therefore in this family of space-times there is *not a frozen picture*, like in the family of spatially compact without boundary space-times considered in loop quantum gravity, where the Dirac Hamiltonian is a combination of constraints.

In [9] a York canonical basis, adapted to ten first-class constraints, was identified: this allows for the first time to get the explicit identification of the inertial and tidal variables. It implementes the York map of [13] and diagonalizes the York-Lichnerowicz approach [14]. Its final form is ($\alpha_{(a)}(\tau,\sigma^r)$ are angles, ${}^3e_{(a)r}(\tau,\sigma^r)$ are cotriads on the 3-space, $1+n(\tau,\sigma^r)$ and $\bar{n}_{(a)}(\tau,\sigma^r)$ are the lapse and shift functions respectively)

$\varphi_{(a)}$	$\alpha_{(a)}$	n	$\bar{n}_{(a)}$	θ^r	$\tilde{\phi}$	$R_{\bar{a}}$
$\pi_{\varphi_{(a)}} \approx 0$	$\pi^{(\alpha)}_{(a)} \approx 0$	$\pi_n \approx 0$	$\pi_{\bar{n}_{(a)}} \approx 0$	$\pi^{(\theta)}_r$	$\pi_{\tilde{\phi}} = \frac{c^3}{12\pi G}\, {}^3K$	$\Pi_{\bar{a}}$

$$
{}^3e_{(a)r} = R_{(a)(b)}(\alpha_{(c)})\, {}^3\bar{e}_{(b)r} = R_{(a)(b)}(\alpha_{(c)})\, V_{rb}(\theta^i)\, \tilde{\phi}^{1/3}\, e^{\sum_{\bar{a}}^{1,2} \gamma_{\bar{a}a} R_{\bar{a}}},
$$

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[6] In 2013 the ESA-ACES mission [11] on the synchronization of atomic clocks between Earth and the Space Station will make the first precision measurement of the gravitational redshift created by the geo-potential, i.e. of the $1/c^2$ modifications of the Minkowski light-cone. Every approach to quantum gravity will have to reproduce these data. A varying light-cone is a non-perturbative effect in every quantum field theory, string included, because to define the Fock space one needs the Fourier decomposition of fields on a fixed background space-time with a fixed light-cone. On the other hand in loop quantum gravity one has still to find a well defined coarse graining identifying Minkowski space-time and perturbations around it.

[7] (α) are flat indices; the cotetrads $E^{(\alpha)}_A$ are the inverse of the tetrads $E^A_{(\alpha)}$ connected to the world tetrads by $E^\mu_{(\alpha)}(x) = z^\mu_A(\tau,\sigma^r)\, E^A_{(\alpha)}(z(\tau,\sigma^r))$.

[8] It is a volume integral over 3-space of a coordinate-dependent energy density. It is weakly equal to the *strong* ADM energy, which is a flux through a 2-surface at spatial infinity.

$$
\begin{aligned}
{}^4g_{\tau\tau} &= \varepsilon\,[(1+n)^2 - \sum_a \bar{n}^2_{(a)}], \qquad {}^4g_{\tau r} = -\varepsilon\,\bar{n}_{(a)}\,{}^3\bar{e}_{(a)r},\\
{}^4g_{rs} &= -\varepsilon\,{}^3g_{rs} = -\varepsilon\,\tilde{\phi}^{2/3}\sum_a V_{ra}(\theta^i)\,V_{sa}(\theta^i)\,e^{2\sum_{\bar{a}}^{1,2}\gamma_{\bar{a}a}R_{\bar{a}}},
\end{aligned}
\tag{1}
$$

In this York canonical basis the *inertial effects* are described by the arbitrary gauge variables $\alpha_{(a)}$, $\varphi_{(a)}$, $1+n$, $\bar{n}_{(a)}$, θ^i, ${}^3K$, while the *tidal effects*, i.e. the physical degrees of freedom of the gravitational field, by the two canonical pairs $R_{\bar{a}}$, $\Pi_{\bar{a}}$, $\bar{a} = 1,2$. The momenta $\pi_r^{(\theta)}$ and the 3-volume element $\tilde{\phi} = \sqrt{det\,{}^3g_{rs}}$ have to be found as solutions of the super-momentum and super-hamiltonian (i.e. the Lichnerowicz equation) constraints, respectively.

The gauge variables $\alpha_{(a)}$, $\varphi_{(a)}$ parametrize the extra O(3,1) gauge freedom of the tetrads (the gauge freedom for each observer to choose three gyroscopes as spatial axes and to choose the law for their transport along the world-line). We have studied in detail the Schwinger time gauges where we impose the gauge fixings $\varphi_{(a)}(\tau,\sigma^r) \approx 0$, $\alpha_{(a)}(\tau,\sigma^r) \approx 0$ so that the tetrads become adapted to the 3+1 splitting (the time-like tetrad coincides with the unit normal to the 3-space).

The gauge angles θ^i (i.e. the director cosines of the tangents to the three coordinate lines in each point of Σ_τ) describe the freedom in the choice of the 3-coordinates σ^r on each 3-space: their fixation implies the determination of the shift gauge variables $\bar{n}_{(a)}$, namely the appearances of gravito-magnetism in the chosen 3-coordinate system.

One momentum is a gauge variable (a reflex of the Lorentz signature): the *York time*, i.e. the trace ${}^3K(\tau,\sigma^r)$ of the *extrinsic curvature* of the non-Euclidean 3-spaces as 3-sub-manifolds of space-time. This inertial effect (absent in Newtonian gravity with its absolute Euclidean 3-space) describes the GR remnant of the special-relativistic gauge freedom in clock synchronization. Its fixation determines the lapse function.

In the York canonical basis the Hamilton equations generated by the Dirac Hamiltonian $H_D = \hat{E}_{ADM} + (constraints)$ are divided in four groups: A) four contracted Bianchi identities, namely the evolution equations for $\tilde{\phi}$ and $\pi_i^{(\theta)}$ (they say that given a solution of the constraints on a Cauchy surface, it remains a solution also at later times); B) four evolution equation for the four basic gauge variables θ^i and ${}^3K$: these equations determine the lapse and the shift functions once four gauge fixings for the basic gauge variables are added; C) four evolution equations for the tidal variables $R_{\bar{a}}$, $\Pi_{\bar{a}}$; D) the Hamilton equations for matter, when present.

Once a gauge is completely fixed, the Hamilton equations become deterministic. Given a solution of the super-momentum and super-Hamiltonian constraints and the Cauchy data for the tidal variables on an initial 3-space, we can find a solution of Einstein's equations in radar 4-coordinates adapted to a time-like observer. To it there is associated a special 3+1 splitting of space-time with dynamically selected instantaneous 3-spaces in accord with [7]. Then we can get pass to adapted world

4-coordinates ($x^\mu = z^\mu(\tau,\sigma^r) = x_o^\mu + \varepsilon_A^\mu\,\sigma^A$) and we can describe the solution in every 4-coordinate system by means of 4-diffeomorphisms.

In [15] we study the coupling of N charged scalar particles plus the electro-magnetic field to ADM tetrad gravity in this class of asymptotically Minkowskian space-times without super-translations. To regularize the self-energies both the electric charge and the sign of the energy of the particles are Grassmann-valued. The introduction of the non-covariant radiation gauge allows to reformulate the theory in terms of transverse electro-magnetic fields and to extract the generalization of the Coulomb interaction among the particles in the Riemannian instantaneous 3-spaces of global non-inertial frames.

From the Hamilton equations in the York canonical basis [15], followed by a Post-Minkowskian linearization with the asymptotic flat Minkowski 4-metric at spatial infinity as background, it has been possible to develop a theory of gravitational waves with asymptotic background propagating in the non-Euclidean 3-spaces Σ_τ of a family of *non-harmonic 3-orthogonal* gauges [9] parametrized by the values of the York time ${}^3K(\tau,\sigma^r)$ (the left gauge freedom in the shape of Σ_τ).

The conceptual problem of the GR gauge freedom in the choice of the 4-coordinates is *solved at the experimental level inside the Solar system by the choice of a convention for the description of matter*: a) for satellites near the Earth (like the GPS ones) one uses NASA 4-coordinates compatible with the terrestrial ITFR2003 and geocentric GCRS IAU2000 [16] frames; b) for planets in the Solar System one uses the barycentric BCRS-IAU2000 [16] frame. These frames are compatible with "quasi-inertial frames" in Minkowski space-time. These are metrological choices like the choice of a certain atomic clock as standard of time.

In astronomy the positions of stars and galaxies are determined from the data (luminosity, light spectrum, angles) on the sky as living in a 4-dimensional nearly-Galilei space-time with the celestial ICRS [16] frame considered as a "quasi-inertial frame" (all galactic dynamics is Newtonian gravity), in accord with the standard FRW ΛCDM cosmological model when the constant intrinsic 3-curvature of 3-spaces is zero (as implied by the CMB data[17]). To reconcile all the data with this 4-dimensional reconstruction one must postulate the existence of dark matter and dark energy as the dominant components of the classical universe after the recombination 3-surface!

Our proposal is to define a *Post-Minkowskian ICRS* with non-Euclidean 3-spaces, whose intrinsic 3-curvature (due essentially to gravitational waves) is small, in such a way that the York time be (at least partially) fitted to the observational data implying the presence of dark matter. As shown in [15] the Post-Newtonian limit of the Post-Minkowskian Hamilton equations of particles in this family of gauges reproduces Kepler equations plus a v/c term depending on the York time (the arbitrary gauge function). Therefore there is the concrete possibility (under investigation) to explain the rotation curves of galaxies [18] as a *relativistic inertial effect inside Einstein GR* (choice of a York time compatible with observations [19]) without modi-

[9] The 3-metric in Σ_τ is diagonal like in astronomical frames GCRS and BCRS.

fications: a) of Newton gravity like in MOND [20]; b) of GR like in $f(R)$ theories [21]; c) of particle physics with the introduction of WIMPS [22]. Then, the next step will be to study the dependence on the York time of quantities like redshift, luminosity distance, gravitational lensing.... and to see which information on the York time can be extracted from the data supporting dark energy.

In conclusion the reformulation of clock synchronization as the existence of well-defined non-Euclidean 3-spaces with the gauge freedom of the York time plus the proposed way out from the GR gauge problem using the observational metrological conventions may help in reducing the dark side of the universe to a relativistic inertial effect inside Einstein GR by means of a Post-Minkowskian definition of ICRS, which will be also useful for the ESA-GAIA mission [23] (cartography of the Milky Way) and for the possible anomalies inside the Solar System [24].

Finally the transition to cosmology should be done with approaches of the type of backreaction [25].

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Dark energy and cosmic magnetic fields: electromagnetic relics from inflation

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Abstract We consider an extended electromagnetic theory in which the scalar state which is usually eliminated be means of the Lorenz condition is allowed to propagate. On super-Hubble scales, such a state is given by the temporal component of the electromagnetic potential and contributes as an effective cosmological constant to the energy-momentum tensor. Its initial amplitude is set by quantum fluctuations generated during inflation and it is shown that the predicted value for the cosmological constant agrees with observations provided inflation took place at the electroweak scale. We also consider more general theories including non-minimal couplings to the space-time curvature in the presence of the temporal electromagnetic background. We show that both in the minimal and non-minimal cases, the modified Maxwell's equations include new effective current terms which can generate magnetic fields from sub-galactic scales up to the present Hubble horizon. The corresponding amplitudes could be enough to seed a galactic dynamo or even to account for observations just by collapse and differential rotation in the protogalactic cloud.

1 Introduction

Despite the fact that a cosmological constant provides a simple and accurate description of cosmic acceleration, from a theoretical point of view it would be even more satisfactory to have a fundamental explanation of the tiny value of such a constant. In this sense, several modifications of the gravitational interaction on cosmological scales have been proposed in the literature.

However, apart from gravity there is another long-range interaction in nature which is nothing but electromagnetism. In this work we will consider the potential role of modified electromagnetic theories in the dark energy problem [1, 2, 3]. This

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_20,

is a quite natural approach to the problem since the behaviour of electromagnetic interaction on astrophysical and cosmological scales is still far from clear, the most evident examples being the unknown origin of the μG magnetic fields observed in galaxies and clusters [4] and, more remarkably, the very recent claim of detection of extra galactic magnetic fields [5]. Thus, the interesting possibility of finding a link between dark energy and the origin of cosmic magnetic field will be also explored.

2 Covariant quantization in flat space-time

Let us start by briefly reviewing the standard covariant quantization method in Minkowski space-time [6] since this will be useful in the rest of the work. The starting point is the modified electromagnetic action:

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\xi}{2} (\partial_\mu A^\mu)^2 + A_\mu J^\mu \right). \tag{1}$$

Because of the presence of the gauge breaking ξ-term, this action is no longer invariant under arbitrary gauge transformations, but only under residual ones given by: $A_\mu \rightarrow A_\mu + \partial_\mu \theta$, provided $\Box\theta = 0$. The equations of motion obtained from this action now read:

$$\partial_\nu F^{\mu\nu} + \xi \partial^\mu (\partial_\nu A^\nu) = J^\mu. \tag{2}$$

In order to recover ordinary Maxwell's equation, the Lorenz condition $\partial_\mu A^\mu = 0$ must be imposed so that the ξ term disappears. At the classical level this can be achieved by means of appropriate boundary conditions on the field. Indeed, taking the four-divergence of the above equation, we find:

$$\Box(\partial_\nu A^\nu) = 0 \tag{3}$$

where we have made use of current conservation. This means that the field $\partial_\nu A^\nu$ evolves as a free scalar field, so that if it vanishes for large $|t|$, it will vanish at all times. At the quantum level, the Lorenz condition cannot be imposed as an operator identity, but only in the weak sense $\partial_\nu A^{\nu(+)}|\phi\rangle = 0$, where $(+)$ denotes the positive frequency part of the operator and $|\phi\rangle$ is a physical state. This condition is equivalent to imposing $[\mathbf{a}_0(k) + \mathbf{a}_{\|}(k)]|\phi\rangle = 0$, with $\mathbf{a}_0$ and $\mathbf{a}_{\|}$ the annihilation operators corresponding to temporal and longitudinal electromagnetic states. Thus, in the covariant formalism, the physical states contain the same number of temporal and longitudinal photons, so that their energy densities, having opposite signs, cancel each other. Therefore, only the transverse photons contribute to the energy density.

3 Covariant quantization in an expanding universe

So far we have only considered Maxwell's theory in flat space-time, however when we move to a curved background, and in particular to an expanding universe, then consistently imposing the Lorenz condition in the covariant formalism turns out to be difficult to realize. Indeed, let us consider the curved space-time version of action (1):

$$S = \int d^4x\sqrt{g}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\xi}{2}(\nabla_\mu A^\mu)^2 + A_\mu J^\mu\right] \tag{4}$$

Now the modified Maxwell's equations read:

$$\nabla_\nu F^{\mu\nu} + \xi\nabla^\mu(\nabla_\nu A^\nu) = J^\mu \tag{5}$$

and taking again the four divergence, we get:

$$\Box(\nabla_\nu A^\nu) = 0 \tag{6}$$

We see that once again $\nabla_\nu A^\nu$ behaves as a scalar field which is decoupled from the conserved electromagnetic currents, but it is non-conformally coupled to gravity. This means that, unlike the flat space-time case, this field can be excited from quantum vacuum fluctuations by the expanding background in a completely analogous way to the inflaton fluctuations during inflation. This poses the question of the validity of the Lorenz condition at all times. Thus, for example, let us consider quantization in an expanding background interpolating between two asymptotically flat regions and prepare our system in an initial state $|\phi\rangle$ belonging to the physical Hilbert space, i.e. satisfying $\partial_\nu \mathscr{A}_{in}^{\nu\,(+)}|\phi\rangle = 0$ in the initial flat region. Because of the expansion of the universe, the positive frequency modes in the *in* region with a given temporal or longitudinal polarization will become a linear superposition of positive and negative frequency modes in the *out* region and with different polarizations [2, 7]. Thus, the system will end up in a final state which no longer satisfies the weak Lorenz condition i.e. in the *out* region $\partial_\nu \mathscr{A}_{out}^{\nu\,(+)}|\phi\rangle \neq 0$.

Although there are alternative quantization procedures which avoid this problem, in this work we will explore the possibility of quantization in an expanding universe without imposing the Lorenz condition.

4 Extended electromagnetism without the Lorenz condition

Let us then explore the possibility that the fundamental theory of electromagnetism is given by the modified action (4) where we allow the $\nabla_\mu A^\mu$ field to propagate. Since we are not imposing the Lorenz condition, in principle, important viability problems for the theory could arise, namely:

- Modification of classical Maxwell's equations
- Electric charge non-conservation
- New unobserved photon polarizations
- Negative norm (energy) states
- Conflicts with QED phenomenology

However, as we will show in the following, none of these problems is actually present. Having removed one constraint, the theory contains one additional degree of freedom. Thus, the general solution for the modified equations (5) can be written as:

$$\mathscr{A}_\mu = \mathscr{A}_\mu^{(1)} + \mathscr{A}_\mu^{(2)} + \mathscr{A}_\mu^{(s)} + \partial_\mu \theta \tag{7}$$

where $\mathscr{A}_\mu^{(i)}$ with $i = 1,2$ are the two transverse modes of the massless photon, $\mathscr{A}_\mu^{(s)}$ is the new scalar state, which is the mode that would have been eliminated if we had imposed the Lorenz condition and, finally, $\partial_\mu \theta$ is a purely residual gauge mode, which can be eliminated by means of a residual gauge transformation in the asymptotically free regions, in a completely analogous way to the elimination of the A_0 component in the Coulomb quantization. The fact that Maxwell's electromagnetism could contain an additional scalar mode decoupled from electromagnetic currents, but with non-vanishing gravitational interactions, was already noticed in a different context in [8].

In order to quantize the free theory, we perform the mode expansion of the field with the corresponding creation and annihilation operators for the *three* physical states:

$$\mathscr{A}_\mu = \int d^3k \sum_{\lambda=1,2,s} \left[\mathbf{a}_\lambda(k) \mathscr{A}_{\mu k}^{(\lambda)} + \mathbf{a}_\lambda^\dagger(k) \overline{\mathscr{A}_{\mu k}^{(\lambda)}} \right] \tag{8}$$

where the modes are required to be orthonormal with respect to the appropriate scalar product (see for instance [9]). Notice that the three modes can be chosen to have positive normalization, and therefore:

$$\left[\mathbf{a}_\lambda(k), \mathbf{a}_{\lambda'}^\dagger(k') \right] = \delta_{\lambda\lambda'} \delta^{(3)}(k-k'), \quad \lambda, \lambda' = 1,2,s \tag{9}$$

We see that the sign of the commutators is positive for the three physical states, i.e. the negative norm state can be eliminated in the free theory.

The evolution of the new mode is given by (6), so that on super-Hubble scales, $|\nabla_\mu \mathscr{A}_k^{(s)\mu}| = const.$ which, as shown in [1], implies that the field contributes as a cosmological constant in (4). Indeed, the energy-momentum tensor derived from (4) reads in that limit:

$$T_{\mu\nu} = \frac{\xi}{2} g_{\mu\nu} (\nabla_\alpha A^\alpha)^2 \tag{10}$$

which is the energy-momentum tensor of a cosmological constant. Notice that, as seen in (6), the new scalar mode is a massless free field and it is possible to calculate the corresponding power spectrum generated during inflation, $P_{\nabla A}(k) = 4\pi k^3|\nabla_\mu \mathscr{A}_k^{(s)\mu}|^2$. In the super-Hubble limit, we get in a quasi-de Sitter inflationary phase characterized by a slow-roll parameter ε:

$$P_{\nabla A}(k) = \frac{9H_{k_0}^4}{16\pi^2}\left(\frac{k}{k_0}\right)^{-4\varepsilon} \tag{11}$$

where H_{k_0} is the Hubble parameter when the k_0 mode left the horizon [1]. Notice that this result implies that $\rho_A \sim (H_{k_0})^4$ (see [10] and references therein for problems with infrared divergences during inflation). The measured value of the cosmological constant then requires $H_{k_0} \sim 10^{-3}$ eV, which corresponds to an inflationary scale $M_I \sim 1$ TeV. Thus we see that the cosmological constant scale can be naturally explained in terms of physics at the electroweak scale. This is one of the most relevant aspects of the present model in which, unlike existing dark energy theories based on scalar fields, dark energy can be generated without including any potential term or dimensional constant.

As shown above, the field amplitude remains frozen on super-Hubble scales, so that no modification of Maxwell's equation is generated on those scales. However as the amplitude starts decaying once the mode enters the horizon in the radiation or matter eras, the ξ term in (5) generates an effective current which can produce magnetic fields on cosmological scales, as we will show below.

Notice that in Minkowski space-time, the theory (4) is completely equivalent to standard QED. This is so because, although non-gauge invariant, the corresponding effective action is equivalent to the standard BRS invariant effective action of QED [2].

To summarize, none of the above mentioned consistency problems for the theory in (4) arise, thus:

- Electric charge is conserved since only the gauge electromagnetic sector is modified, but not the sector of charged particles which preserves its gauge symmetry.
- The new state only couples gravitationally and evades laboratory detection.
- The new state has positive norm (energy).
- The effective action is completely equivalent to standard QED in the flat space-time limit.
- Although ordinary Maxwell's equations are modified on small scales, the only effect of the new term is the generation of cosmic magnetic fields.

On the other hand, despite the fact that the background evolution in the present case is the same as in ΛCDM, the evolution of metric perturbations could be different. We have calculated the evolution of metric, matter density and electromagnetic perturbations [3]. The propagation speeds of scalar, vector and tensor perturbations are found to be real and equal to the speed of light, so that the theory is classically stable. On the other hand, it is possible to see that all the parametrized post-Newtonian (PPN) parameters [11] agree with those of General Relativity, i.e. the theory is compatible with all the local gravity constraints for any value of the homogeneous background electromagnetic field [1, 12].

Concerning the evolution of scalar perturbations, we find that the only relevant deviations with respect to ΛCDM appear on large scales $k \sim H_0$ and that they depend on the primordial spectrum of electromagnetic fluctuations. However, the effects on the CMB temperature and matter power spectra are compatible with observations except for very large primordial fluctuations [3].

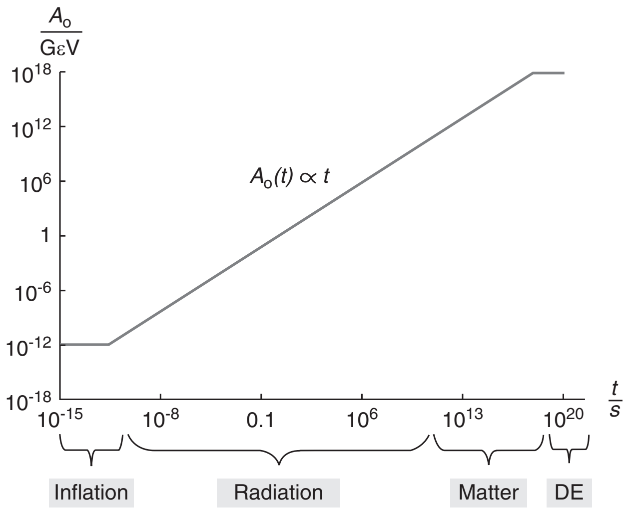

Fig. 1 Cosmological evolution of the temporal component from electroweak-scale inflation until present

5 Cosmological evolution

Let us now consider the cosmological evolution of this new electromagnetic mode. For that purpose, we will consider a homogeneous electromagnetic field with $A_\mu = (A_0(t), A(t))$. The corresponding equations motion in a flat Robertson-Walker background in cosmological time t read:

$$\begin{aligned} \ddot{A}_0 + 3H\dot{A}_0 + 3\dot{H}A_0 = 0 \\ \ddot{A} + H\dot{A} = 0 \end{aligned} \tag{12}$$

In the case in which the scale factor behaves as a simple power law with $H = p/t$, the solutions for the above equations grow as: $A_0(t) \propto t$ and $A(t) \propto t^{1-p}$. Thus we see that the temporal component grows faster than the spatial one and therefore, at late times, on cosmological scales we can ignore the spatial contribution and the new scalar state is essentially given by the electric potential. We can also compute the contributions from the temporal and spatial components to the energy density, thus we get:

$$\rho_{A_0} = \frac{\lambda}{2}\left(\dot{A}_0 + 3HA_0\right)^2 = const. \qquad \rho_A = \frac{1}{2a^2}\left(\dot{A}\right)^2 \propto a^{-4} \tag{13}$$

We see that also, as commented before, the temporal component behaves as a cosmological constant, whereas the contribution from the spatial part decays as radiation and therefore does not affect the universe isotropy on large scales.

In Fig.1 we show the cosmological evolution of the electric potential A_0. We see that the field is constant during inflation, it grows linearly in time in the matter and radiation eras and becomes also constant when the electromagnetic dark energy starts dominating. Notice that the present value of $A_0 \simeq 0.3M_P$ is determined by the initial value of the field generated during inflation from quantum fluctuations.

6 Generation of cosmic magnetic fields

It is interesting to note that the ξ-term can be seen, at the equations of motion level, as a conserved current acting as a source of the usual Maxwell field. To see this, we can write $-\xi\nabla^\mu(\nabla_\nu A^\nu) \equiv J^\mu_{\nabla\cdot A}$ which, according to (6), satisfies the conservation equation $\nabla_\mu J^\mu_{\nabla\cdot A} = 0$ and we can express (5) as:

$$\nabla_\nu F^{\mu\nu} = J^\mu_T \tag{14}$$

with $J^\mu_T = J^\mu + J^\mu_{\nabla\cdot A}$ and $\nabla_\mu J^\mu_T = 0$. Physically, this means that, while the new scalar mode can only be excited gravitationally, once it is produced it will generally behave as a source of electromagnetic fields. Therefore, the modified theory is described by ordinary Maxwell equations with an additional ”external” current. For an observer with four-velocity u^μ moving with the cosmic plasma, it is possible to decompose the Faraday tensor in its electric and magnetic parts as: $F_{\mu\nu} = 2E_{[\mu}u_{\nu]} + \frac{\varepsilon_{\mu\nu\rho\sigma}}{\sqrt{g}}B^\rho u^\sigma$, where $E^\mu = F^{\mu\nu}u_\nu$ and $B^\mu = \varepsilon^{\mu\nu\rho\sigma}/(2\sqrt{g})F_{\rho\sigma}u_\nu$. Due to the infinite conductivity of the plasma, Ohm's law $J^\mu - u^\mu u_\nu J^\nu = \sigma F^{\mu\nu}u_\nu$ implies $E^\mu = 0$. Therefore, in that case the only contribution would come from the magnetic part. Thus, from Maxwell's equations, we get:

$$F^{\mu\nu}{}_{;\nu}u_\mu = \frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{g}}B_\rho u_{\sigma;\nu}u_\mu = J^\mu_{\nabla\cdot A}u_\mu \tag{15}$$

that for comoving observers in a FLRW metric imply (see also [13]):

$$\omega\cdot B = \rho_g^0 \tag{16}$$

where $v = dx/d\eta$ is the conformal time fluid velocity, $\omega = \nabla \times v$ is the fluid vorticity, ρ_g^0 is the effective charge density today whose power spectrum can be obtained from (11), and the B components scale as $B_i \propto 1/a$ as can be easily obtained from $\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma;\nu} = 0$ to the lowest order in v. Thus, the presence of the non-vanishing cosmic effective charge density necessarily creates both magnetic field and vorticity. Due to the presence of the effective current, we find that vorticity grows as $|\omega| \propto a$, from radiation era until present.

Using (16), it is possible to translate the existing upper limits on vorticity coming from CMB anisotropies [13] into *lower* limits on the amplitude of the magnetic fields generated by this mechanism. Thus we find that for a nearly scale invariant vorticity power spectrum, magnetic fields $B_\lambda > 10^{-12}$ G are typically generated with coherence lengths ranging from sub-galactic scales up to the present Hubble radius. Those fields could act as seeds for a galactic dynamo or even account for observations just by collapse and differential rotation of the protogalactic cloud [14]

7 Non-minimal couplings

Let us now generalize the previous results by considering the most general expression for the electromagnetic action in the presence of gravity [15], including all the possible terms leading to linear second order differential equations:

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\xi}{2}(\nabla_\mu A^\mu)^2 + \sigma R_{\mu\nu}A^\mu A^\nu + \omega R A_\mu A^\mu\right]. \quad (17)$$

Notice that this expression does not contain any dimensional parameter or potential term and σ and ω are arbitrary dimensionless constants. In order to fix them, we will consider the weak-field limit of the theory. Thus, the space-time metric can be written as a small perturbation around Minkowski space-time, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ (in this section we will ignore the universe expansion) and the electromagnetic potential reads $A_\mu = \bar{A}_\mu + a_\mu$ with $\bar{A}_\mu = \bar{A}_0\delta_\mu^0$ and $\bar{A}_0 \simeq 0.3M_P$ as shown before. The corresponding Maxwell's equations obtained from (17) read to first order:

$$\partial_\nu F^{\mu\nu} + \xi\partial^\mu(\nabla_\nu A^\nu)_{(1)} = J_g^\mu. \quad (18)$$

where $F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, $(\nabla_\nu A^\nu)_{(1)}$ denotes the contribution to first order and the non-minimal terms give rise to an effective current given also to first order by: $J_g^\mu = 2(\sigma R_{(1)}^{\mu\nu} + \omega R_{(1)}\eta^{\mu\nu})\bar{A}_\nu$. Imposing this effective current to be conserved, i.e. $\partial_\mu J_g^\mu = 0$, we obtain $\sigma = -2\omega$, i.e. the non-minimal coupling must involve the conserved Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$. Notice also that conservation implies that taking the divergence of (18) we get $\Box(\nabla_\mu A^\mu)_{(1)} = 0$, i.e. to first order it is possible to impose the Lorenz condition at the classical level as in ordinary electromagnetism. Thus, for weak gravitational fields we recover ordinary Maxwell electromagnetism, the only difference is the appearance of a gravitationally-generated electromagnetic current, which is only present provided the background electromagnetic potential is non-vanishing and in the presence of space-time curvature.

This theory is a particular case of the more general class of vector-tensor theories [11]. For this particular case, the PPN parameters are in agreement with observations provided: $|\sigma| \lesssim 10^{-5}$.

Concerning the stability of the theory, we have analyzed the behavior of the inhomogeneous perturbations around the Minkowski background. The corresponding propagation speeds for the scalar, vector and tensor perturbations are: $c_s^2 = 1$, $c_v^2 \simeq 1 + 16\pi G\sigma^2 \bar{A}_0^2$ and $c_t^2 \simeq 1 + 16\pi G\sigma \bar{A}_0^2$, where we have expanded for $|\sigma| \ll 1$. Notice that the scalar modes propagate at the speed of light irrespective of the value of the parameter σ. However, the speed of photons c_v would be larger than the "speed of light" $c = 1$ which determines the null cones of the Minkowski geometry. Although, in principle, this could give rise to inconsistencies with causality, this is not the case since *stable causality* [16] is ensured in this model for small σ.

In order to study the presence of quantum instabilities (ghosts), we analyze the positiveness of the energy density of the three types of perturbations considered before. Thus, following [17, 12], we find that for scalar modes the energy density vanishes identically if we impose the Lorenz condition, as in ordinary electromagnetism. For vector and tensor modes, the energy densities are positive for small $|\sigma|$.

Also due to the smallness of the parameter σ, the cosmological evolution of the homogeneous mode becomes modified in a negligible way by the presence of the coupling to the Einstein tensor. This ensures that the inflationary generation and evolution discussed in previous sections for the minimal theory is also a good description in the non-minimal case.

8 Cosmic magnetic fields from non-minimal couplings

Let us now consider the possible effects of the new effective electromagnetic current $J_g^\mu = 2\sigma G^{\mu 0}\bar{A}_0$ in (18). Using Einstein equations to relate $G^{\mu\nu}$ to the matter content, we obtain:

$$J_g^\mu = 16\pi G\sigma T^{\mu 0}\bar{A}_0 \tag{19}$$

so that the effective electromagnetic current is essentially determined by the four-momentum density. Moreover, if we assume $T^{\mu\nu} = (\rho + p)u^\mu u^\nu - p\eta^{\mu\nu}$ at first order, we can see that the energy density of any perfect fluid has an associated electric charge density given, for small velocities, by:

$$\rho_g = J_g^0 = 16\pi G\sigma\rho\bar{A}_0 \tag{20}$$

and the three-momentum density generates an electric current density given by

$$J_g = 16\pi G\sigma(\rho + p)v\bar{A}_0 \tag{21}$$

This theory effectively realizes the old conjecture by Schuster, Einstein and Blackett [18] of gravitational magnetism, i.e. neutral mass currents generating electromagnetic fields.

In the case of a particle of mass m at rest, (20) introduces a small contribution to the *active* electric charge (the source of the electromagnetic field), given by $\Delta q = 16\pi G\sigma m\bar{A}_0 \simeq 15\sigma(m/M_P)$, but does not modify the *passive* electric charge (that determining the coupling to the electromagnetic field). In fact, this would give different active charges to electrons and protons due to their mass difference and, in addition, would provide the neutron with a non-vanishing active electric charge. However, the effect is very small in both cases $\Delta q \simeq 4\sigma 10^{-18} e$ where $e = 0.303$ is the electron charge in Heaviside-Lorentz units. Present limits on the electron-proton charge asymmetry and neutron charge are both of the order $10^{-21}e$ [19], implying $|\sigma| \lesssim 10^{-3}$ which is less stringent than the PPN limit discussed before.

On the other hand, for any compact object, even in the case it is neutral, the effective electric current will generate an intrinsic magnetic moment given by:

$$m = \beta \frac{\sqrt{G}}{2} L \tag{22}$$

with L the corresponding angular momentum and β a constant parameter whose value is:

$$\beta = 16\pi\sqrt{G}\sigma\bar{A}_0 \tag{23}$$

Notice that relation (22) resembles the Schuster-Blackett law, which is an empirical relation between the magnetic moments and the angular momenta found in a wide range of astrophysical objects from planets, to galaxies, including those related to the presence of rotating neutron stars such as GRB or magnetars [20]. However, let us mention that the observational evidence on this relation is still not conclusive. From observations, the β parameter is found to be in the range 0.001 to 0.1.

Imposing the PPN limits on the σ parameter, we find $\beta \lesssim 10^{-4}$, which is just below the observed range. Thus for a typical spiral galaxy, a direct calculation provides: $B \sim \sigma 10^{-4}$ G, i.e. according to the PPN limits, the field strength could reach 10^{-9} G without amplification.

9 Conclusions

In this work we have studied extended (minimal and non-minimal) electromagnetic theories with an additional scalar state. The energy density of the new scalar mode on cosmological scales is shown to behave as an effective cosmological constant, whose value is determined by the amplitude of quantum fluctuations generated during inflation. As a matter of fact, the measured value of the cosmological constant is naturally explained provided inflation took place at the electroweak scale. The model is free from classical or quantum instabilities and is consistent with all the local gravity constraints. On the other hand, it is also compatible with observations from CMB and large scale structure and contains the same number of free parameters as ΛCDM. Unlike dark energy models based on scalar fields, acceleration in this model arises from the kinetic term of the new electromagnetic mode, without the introduction of unnatural dimensional parameters or potential terms. The results presented in this work show that, unlike previous proposals, the nature of dark energy can be established without resorting to new physics. On the other hand, the modified Maxwell's equations contain additional current terms which can generate cosmic magnetic fields with large coherence lengths. Those fields could act as seed for standard amplification mechanisms, thus establishing an interesting link between dark energy and the origin of cosmic magnetic fields.

Acknowledgments: This work has been supported by MICINN (Spain) project numbers FIS 2008-01323 and FPA 2008-00592, CAM/UCM 910309, MEC grant BES-2006-12059 and MICINN Consolider-Ingenio MULTIDARK CSD2009-00064.

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On the Viability of a Non-Analytical $f(R)$-Theory

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Abstract In this paper, we show how a *power-law* correction to the Einstein-Hilbert action provides a viable modified theory of gravity, passing the Solar-System tests, when the exponent is between the values 2 and 3. Then, we implement this paradigm on a cosmological setting outlining how the main phases of the Universe thermal history are properly reproduced.
As a result, we find two distinct constraints on the characteristic length scale of the model, *i.e.*, a lower bound from the Solar-System test and an upper one by guaranteeing the matter dominated Universe evolution.

1 Basic statements

From the very beginning, the possibility to riformulate General Relativity by using a generic function of the Ricci scalar (see, for example, [1] for a recent review and references therein) appeared as a natural issue offered by the fundamental principles established by Einstein. However, it is important to remark that any modification of the Einstein-Hilbert (EH) Lagrangian is reflected onto a deformed gravitational-field dynamics at any length scale investigated or observed. Thus, the success of such $f(R)$ gravity in the solution of a specific problem has to match consistency with observation in different length scales [2, 3, 4]. A viable self-consistent model can be often obtained at the price to consider a generalized gravitational lagrangian containing a large number of free parameters. Nevertheless, the wide spectrum of

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_21,

possible choices for $f(R)$ can appear as a weakness point in view of the predictivity of the theory, because a significant degree of degeneracy is expected in the m odel.

Here, we consider an opposite point of view, by studying the viability of a power-law correction to the EH action having a single free parameter (a length scale) once the power-law exponent is fixed. We investigate the implementation of the Solar-System test to our model [5] and then we pursue a cosmological study of the resulting modified Friedmann-Lemaître-Robertson-Walker (FLRW) dynamics. As expected, this scenario gives us a rather stringent range of variation for the free length scale where searching for new gravitational physics.

2 Non-analytical power-law $f(R)$ model

In this paper, we consider the following modified gravitational action in the so-called *Jordan frame*

$$S = -\frac{1}{2\chi} \int d^4x \sqrt{-g}\, f(R) \,, \qquad f(R) = R + qR^n \,, \tag{1}$$

where n is a non-integer dimensionless parameter and $q < 0$ has dimensions of $[L]^{2n-2}$ (in the equation above $\chi = 8\pi G$, using $c = 1$ and G being the Newton constant, moreover, the signature is set as $[+,-,-,-]$). Such a form of $f(R)$ gives the following constraints for n: if $R > 0$, all n-values are allowed; if $R < 0$, the condition $n = \ell/(2m+1)$ must hold (where, here and in the following, m and ℓ denote positive integer). It is straightforward to verify that S in (1) is non-analytical in $R = 0$ for non-integer, rational n, *i.e.*, it does not admit Taylor expansion near $R = 0$.

Let us now define the *characteristic length scale* of our model as

$$L_q(n) \equiv |q|^{1/(2n-2)} \,, \tag{2}$$

while variations of the total action $S_{tot} = S + S_M$ (where S_M denotes the matter term) with respect to the metric give, after manipulations and modulo surface terms:

$$f' R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f - \nabla_\mu \nabla_\nu f' + g_{\mu\nu} \Box f' = \chi\, T_{\mu\nu} \,, \tag{3}$$

where $T_{\mu\nu}$ is the Energy-Momentum Tensor (EMT). Here and in the following $(...)'$ indicates the derivative with respect to R, $\Box \equiv g^{\rho\sigma}\nabla_\rho\nabla_\sigma$ and ∇_μ or $(...)_;$ denotes the covariant derivative (Greek indices run form 0 to 3).

We can gain further information on the value of n by analyzing the conditions that allow for a consistent weak-field stationary limit. Having in mind to investigate the weak field limit of our theory to obtain predictions at Solar-System scales, we can decompose the corresponding metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is a small (for our case, static) perturbation of the Minkowskian metric $\eta_{\mu\nu}$. In this limit, the vacuum Einstein equations read

$$R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R - nq(R^{n-1})_{;\mu;\nu} + nq\eta_{\mu\nu}\Box R^{n-1} = 0 \,, \qquad R = 3nq\Box R^{n-1} \,. \tag{5}$$

The structure of such field equations leads us to focus our attention on the restricted region of the parameter space $2 < n < 3$. This choice is enforced by the fulfillment of the conditions by which all other terms are negligible with respect to the linear and the lowest-order non-Einsteinian ones.

3 Viability of the theory: the Solar-System test

From the analysis of the weak-field limit in the Jordan frame, *i.e.*, (5), we learn the possibility to find a post-Newtonian solution by solving (5) up to the next-to-leading order in h, *i.e.*, up to $\mathscr{O}(h^{n-1})$, and neglecting the $\mathscr{O}(h^2)$ contribution only for the cases $2 < n < 3$. These considerations motivate the choice we claimed above concerning the restriction of the parameter n.

The most general spherically-symmetric line element in the weak-field limit is

$$ds^2 = (1+\Phi)dt^2 - (1-\Psi)dr^2 - r^2 d\Omega^2 \,, \tag{6}$$

where Φ and Ψ are the two generalized gravitational potentials and $d\Omega^2$ is the solid-angle element. Within this framework, the modified Einstein equations (5) rewrite

$$\begin{aligned}
R_{tt} - \tfrac{1}{2}R - nq\nabla^2 R^{n-1} &= 0\,, & R &= \nabla^2\Phi + \tfrac{2}{r^2}(r\Psi)_{,r}\,,\\
R_{rr} + \tfrac{1}{2}R - nq(R^{n-1})_{,r,r} + nq\nabla^2 R^{n-1} &= 0\,, & R_{tt} &= \tfrac{1}{2}\nabla^2\Phi\,,\\
R_{\theta\theta} + \tfrac{1}{2}r^2 R - nqr(R^{n-1})_{,r} + nqr^2\nabla^2 R^{n-1} &= 0\,, & R_{rr} &= -\tfrac{1}{2}\Phi_{,r,r} - \tfrac{1}{r}\Psi_{,r}\,,\\
R + 3nq\nabla^2 R^{n-1} &= 0\,, & R_{\theta\theta} &= -\Psi - \tfrac{r}{2}\Phi_{,r} - \tfrac{r}{2}\Psi_{,r}\,,\\
\left[\nabla^2 \equiv \tfrac{d^2}{dr^2} + \tfrac{2}{r}\tfrac{d}{dr}\right]\,, & & R_{\phi\phi} &= \sin^2\theta R_{\theta\theta}\,,
\end{aligned}$$

where $(\ldots)_{,}$ denotes ordinary differentiation. The system above is solved by

$$R = Ar^{\frac{2}{n-2}}\,, \qquad A = \left[-\frac{6nq(3n-4)(n-1)}{(n-2)^2}\right]^{\frac{1}{2-n}}\,, \tag{8a}$$

$$\Phi = \sigma + \frac{\delta}{r} + \Phi_n\left(\frac{r}{L_q}\right)^{2\,\frac{n-1}{n-2}}\,, \qquad \Phi_n \equiv \left[-\frac{6n(3n-4)(n-1)}{(n-2)^2}\right]^{\frac{1}{2-n}}\frac{(n-2)^2}{6(3n-4)(n-1)}\,, \tag{8b}$$

$$\Psi = \frac{\delta}{r} + \Psi_n\left(\frac{r}{L_q}\right)^{2\,\frac{n-1}{n-2}}\,, \qquad \Psi_n \equiv \left[-\frac{6n(3n-4)(n-1)}{(n-2)^2}\right]^{\frac{1}{2-n}}\frac{(n-2)}{3(3n-4)}\,, \tag{8c}$$

where the integration constant δ has the dimensions of $[L]$ and the dimensionless integration constant σ can be set equal to zero without loss of generality. The integration constant A has the dimensions of $[L]^{(2n-2)/(2-n)}$, and Φ_n and Ψ_n are dimensionless, accordingly. Moreover, one can check that Φ_n and Ψ_n are well-defined only in the case $n = (2m+1)/\ell$ while we get $A > 0$ since we assume $q < 0$. In agreement to the geodesic motion as expanded in the weak field limit, the integration constant δ results equal to $\delta = -r^S$, where $r^S = 2GM$ is the *Schwarzschild radius* of a central object of mass M.

The most suitable arena to evaluate the reliability and the validity range of the weak-field solution (8) is the Solar System [2, 4]. To this end, we can specify (8b)-(8c) for the typical length scales involved in the problem and we split Φ and Ψ into two terms, the Newtonian part and a modification, *i.e.*,

$$\Phi \equiv \Phi_N + \Phi_M \equiv -r^S_\odot/r + \Phi_n (r/L_q)^{2(n-1)/(n-2)} \,, \tag{9a}$$

$$\Psi \equiv \Psi_N + \Psi_M \equiv -r^S_\odot/r + \Psi_n (r/L_q)^{2(n-1)/(n-2)} \,, \tag{9b}$$

here, the integration constant δ of (8b)-(8c) is $\delta = -r^S_\odot \equiv 2GM_\odot$ ($M_\odot$ being the Solar mass). While the weak-field approximation of the Schwarzschild metric is valid within the range $r^S_\odot \ll r < \infty$ because it is asymptotically flat, the modification terms have the peculiar feature to diverge for $r \to \infty$. It is therefore necessary to establish a validity range $r_{Min} \ll r \ll r_{Max}$, related to n and L_q, where this solution is physically predictive [6].

Since we aim to provide a physical picture at least of the planetary region of the Solar System, we are led to require that Φ_M and Ψ_M remain small perturbations with respect to Φ_N and Ψ_N, so that it is easy to recognize the absence of a minimal radius except for the condition $r \gg r^S_\odot$. The typical distance $L^*_\odot$ corresponds to the request

$$|\Phi_N(L^*_\odot)| \sim |\Phi_M(L^*_\odot)| \,, \qquad |\Psi_N(L^*_\odot)| \sim |\Psi_M(L^*_\odot)| \,. \tag{10}$$

For $r^S_\odot \ll r \ll L^*_\odot$, the system obeys thus Newtonian physics and experiences the post-Newtonian term as a correction. Another maximum distance $L^{**}_\odot$ can be defined, according to the request that the weak-field expansion (6) should hold, regardless to the ratios Φ_M/Φ_N and Ψ_M/Ψ_N. $L^{**}_\odot$ results to be defined by

$$|\Phi_N(L^{**}_\odot)| \ll |\Phi_M(L^{**}_\odot)| \sim 1 \,, \qquad |\Psi_N(L^{**}_\odot)| \ll |\Psi_M(L^{**}_\odot)| \sim 1 \,. \tag{11}$$

We remark that $L^*_\odot$ and $L^{**}_\odot$ are defined as functions of n and L_q, *i.e.*,

$$L^*_\odot \sim |r^S_\odot/\Phi_n|^{\frac{n-2}{3n-4}} L_q^{\frac{2n-2}{3n-4}} \,, \qquad L^{**}_\odot \sim L_q \big/ |\Phi_n|^{\frac{n-2}{2n-2}} \,, \tag{12}$$

and it is important to underline that, for the validity of our scheme, the condition $L^*_\odot \gg r^S_\odot$ must hold, *i.e.*, $L_q \gg r^S_\odot |\Phi_n|^{(n-2)/(2n-2)}$.

Neglecting the lower-order effects concerning the eccentricity of the planetary orbit, we can deal with the simple model of a planet moving on circular orbit around the Sun with an orbital period T given by $T = 2\pi(r/a)^{1/2}$ ($a = (d\Phi/dr)/2$ being the centripetal acceleration). For our model, from (8b), we get

$$T_n = \frac{2\pi r^{3/2}}{(GM_\odot)^{1/2}} \left[1 + 2\Phi_n \, \frac{n-1}{n-2} \left(r^{\frac{3n-4}{n-2}}\right) \Big/ \left(r^S_\odot L_q^{\frac{2n-2}{n-2}}\right)\right]^{-1/2} \,. \tag{13}$$

We now can compare the correction to the Keplerian period $T_K = 2\pi r^{3/2}(GM_\odot)^{-1/2}$, with the experimental data of the period T_{exp} and its uncertainty δT_{exp}. We then impose the correction to be smaller than the experimental uncertainty, *i.e.*,

$$\frac{\delta T_{exp}}{T_{exp}} \geqslant \frac{|T_K - T_n|}{T_K} \sim |\Phi_n| \, \frac{n-1}{n-2} \left(r_P^{\frac{3n-4}{n-2}} \right) / \left(r_\odot^S L_q^{\frac{2n-2}{n-2}} \right) , \tag{14}$$

where r_P is the mean orbital distance of a given planet from the Sun.

Let us now specify our analysis for the example of the Earth [2]. In this case, $T_{exp} \simeq 365.2563$ days and $\delta T_{exp} \simeq 5.0 \cdot 10^{-10}$ days (with $r_P \simeq 4.8482 \times 10^{-6}$ pc). This way, for the Earth, we can get a lower bound $L_q > L_{q\oplus}^{Min}$ for the characteristic length scale of our model, as function of n, *i.e.*,

$$L_{q\oplus}^{Min}(n) = \left[1.3689 \times 10^{12} \, \frac{|\Phi_n|}{r_\odot^S} \, \frac{n-1}{n-2} \, r_P^{\frac{3n-4}{n-2}} \right]^{\frac{n-2}{2n-2}} , \tag{15}$$

where Φ_n is defined in (8b) and $L_{q\oplus}^{Min} \sim 4 \times 10^{-3}$ pc, for a typical value $n \simeq 2.66$. We remark that $L_{q\oplus}^{Min}$, by virtue of (8b), is defined only for $n = (2m+1)/\ell$.

Our analysis clarifies how the predictions of the corresponding equations for the weak-field limit appear viable in view of the constraints arising from the Solar-System physics. Indeed, the lower bound for L_q does not represent a serious shortcoming of the model, as we are going to discuss in Sec.6, where a plot of $L_{q\oplus}^{Min}(n)$ and of $L_\odot^*(L_q)$ and $L_\odot^{**}(L_q)$ will be also addressed.

4 Cosmological implementation of the non-analytical $f(R)$ model

In order to study how our *f(R)* model affects the cosmological evolution, we start from the modified gravitational action (1) and we assume the standard Robertson-Walker (RW) line element in the synchronous reference system, *i.e.*,

$$ds^2 = dt^2 - a(t)^2 \left[dr^2/(1-Kr^2) + r^2 d\Omega^2 \right] , \tag{16}$$

where $a(t)$ is the scale factor and K the spatial curvature constant. Using such expression, the 00-component of (3) results, for symmetry using the Bianchi identity, the only independent one and it writes as

$$f' R_{00} - \tfrac{1}{2} f + 3(\dot{a}/a) f'' \dot{R} = \chi \, T_{00} \, . \tag{17}$$

where the dot indicates the time derivative. We assume as matter source a perfect-fluid EMT, *i.e.*, $T_{\mu\nu} = (p+\rho) u_\mu u_\nu - p g_{\mu\nu}$, in a comoving reference system (thus $T_{00} = \rho$), where p is the thermostatic pressure, ρ the energy density and u_μ denotes the 4-velocity. The 0-component of the conservation law, *i.e.*, $T^\nu_{\mu;\nu} = 0$ with $\nu = 0$, assuming the *equation of state* (EoS) $p = w\rho$, gives the following expression for the energy density: $\rho = \rho_0 \left[a/a_0 \right]^{-3(1+w)}$.

Using now $f = R + qR^n$ with $q < 0$, we are able to explicitly write (17):

$$
\begin{aligned}
&2\tilde{\chi}\, a^{1-3w} + 6^n n q a^{5-2n}\, \ddot{a}\, (-K - \dot{a}^2 - a\ddot{a})^{n-1} + \\
&+ a^2 \left[-6K - 6\dot{a}^2 + 6^n q a^{2(1-n)} (-K - \dot{a}^2 - a\ddot{a})^n \right] + \\
&+ 6^n (n-1) n q \dot{a} a^{2(2-n)} (-K - \dot{a}^2 - a\ddot{a})^{n-2} \left[-2\dot{a}^3 - 2K\dot{a} + a\dot{a}\ddot{a} + a^2 \dddot{a} \right] = 0\,,
\end{aligned}
\tag{18}
$$

where $\tilde{\chi} = \chi \rho_0 a_0^{3(1+w)}$. Let us now assume a power-law $a = a_0\,[t/t_0]^x$ for the scale factor and, for the sake of simplicity, we set $\bar{a} = a_0 t_0^{-x}$ (clearly, $[\bar{a}] = [L^{1-x}]$). Here and in the following, we use the subscript $(...)_0$ to denote quantities measured today. In this case, (18) can be recast in the form

$$
\begin{aligned}
&-6\bar{a}^2 K t^{2x} - 6\bar{a}^4 x^2 t^{4x-2} + q\bar{a}^4 t^{4x} \left(C_1 t^{-2} - 6\bar{a}^{-2} K t^{-2x} \right)^n + 2\tilde{\chi}\bar{a}^{1-3w} t^{x(1-3w)} = \\
&= nqx\bar{a}^{6-2n} t^{6x} \left(C_1 \bar{a}^2 t^{-2} - 6K t^{-2x} \right)^n \frac{(C_2 K t^2 + x C_3 t^{2x})}{(K t^2 + C_4 t^{2x})^2}\,,
\end{aligned}
\tag{19}
$$

where $C_1 = 6x(1-2x)$, $C_2 = (x(2n-1)-1)$, $C_3 = x\bar{a}^2(x+2n-3)(2x-1)$, $C_4 = x\bar{a}^2(2x-1)$.

4.1 Radiation-dominated Universe

Here, we assume the radiation-dominated Universe EoS $w = 1/3$ ($\rho \sim a^{-4}$). In the following, we will discuss the three distinct regimes, in the asymptotic limit as $t \to 0$, for $x < 1$, $x > 1$ and $x = 1$, separately.

In the case $x < 1$, all terms containing explicitly the curvature K of (19) results to be negligible for $t \to 0$ and asymptotic solutions are allowed if and only if $x \leqslant n/2$ which, in the case $2 < n < 3$ we are considering, is always satisfied. The leading-order term of (19) writes as

$$
q\bar{a}^4 C_1^n \left[1 - (C_3/C_4^2) n x^2 \bar{a}^2 \right] t^{4x-2n} = 0\,,
\tag{20}
$$

and $x = 1/2$ and $x = [2 + 3n - 2n^2 \pm (4 + 8n + n^2 - 12n^3 + 4n^4)^{1/2}]/2n$ are the solutions. Such second expression results to be negative or imaginary for $2 < n < 3$ and must be excluded. Thus, the only solution for $x < 1$, in the asymptotic limit for $t \to 0$, is the well-known radiation dominated behavior $a \sim t^{1/2}$. In the other two cases, *i.e.*, for $x \geqslant 1$, it is easy to recognize that no asymptotic solutions are allowed. Therefore, the approach to the initial singularity is not characterized by power-law inflation behavior when spatial curvature is non-vanishing.

Let us now assume a vanishing spatial curvature in (19). In can be show how, for $K = 0$, the radiation-dominated solution with $w = 1/3$ and $x = 1/2$ is an *exact* solution (non-asymptotic and allowed for all n-values) giving $\rho_0 = 3/(4\chi t_0^2)$, matching the standard FLRW case. In the case $x > 1$, the leading-order terms of (19) read, for $t \to 0$ and $K = 0$,

$$q\bar{a}^4 C_1^n \left[1 - (C_3/C_4^2) n x^2 \bar{a}^2\right] t^{4x-2n} + 2\tilde{\chi} = 0\,. \tag{21}$$

Three distinct regimes have to be now separately discussed. For $x > n/2$, the leading order of the equation above does not admit solutions since it writes simply $2\tilde{\chi} = 0$ and, for $x < n/2$, the solutions of (21) are those obtained in the case for $x < 1$. Instead, for $x = n/2$, and defining $H_0 = (n/2)/t_0$, one gets

$$\rho_0 = \frac{\tilde{\rho}_0(n)\, q_0}{4\chi t_0^2}\,, \quad \tilde{\rho}_0(n) = \tfrac{3^n}{2}\,(1-n)^{(n-1)} n^n (n(4+(6-5n)n)-4)(n/2)^{2-2n}, \tag{22}$$

where we have introduced the dimensionless parameter $q_0 = H_0^{2n-2} q$. We remark that the constraint $n = (2m+1)/\ell$ (which is in agreement with respect to the one obtained from Solar-System test) must hold in order to have $\rho_0 > 0$ since we have assumed $q < 0$ and therefore $q_0 < 0$. The function $\tilde{\rho}_0$ results to increase as n goes from 2 to 3 and, in particular, one can get $216 < \tilde{\rho}_0 < 21\,024$. Finally, for $x = 1$ and $K = 0$, (19) reads $[1 - n(2n-2)]\, t^{4-2n} = 0$, giving $n = [1 \pm \sqrt{3}]/2$. As the previous case, the regime $x = 1$ does not admit solutions in the region $2 < n < 3$.

4.2 Matter-dominated Universe

Let us now study the matter-dominated Universe EoS $w = 0$ ($\rho \sim a^{-3}$). As previously done, we analyze the three distinct regimes for $x < 1$, $x > 1$ and $x = 1$, and, in the limit for $t \to \infty$, it is easy to recognize that there are no power-law solutions in all these cases for $K \neq 0$. Setting $K = 0$, the $x \geqslant 1$ regimes do not provide any power-law form for cosmological dynamics either. On the other hand, for $x < 1$ and assuming zero spatial curvature in (19), we get the following equation:

$$[-6x^2 \bar{a}^4]\, t^{4x-2} + 2\tilde{\chi}\bar{a}\, t^x = \bar{a}^4 q C_1^n [-1 + C_3 \bar{a}^2 n x^2 / C_4^2]\, t^{4x-2n}\,. \tag{23}$$

Since $4x - 2 > 4x - 2n$, the term on the right hand side can be neglected in the limit of large t and the equation above admits three distinct situations: $x < 2/3$, $x > 2/3$ and $x = 2/3$. Both cases with $x \neq 2/3$ do not admit solution. The case $x = 2/3$ admits instead an asymptotic solution for $t \to \infty$. In fact, (23) reduces to $8\bar{a}^3 = 6\tilde{\chi}$ and the FLWR matter-dominated power-law solution $a = \bar{a} t^{2/3}$ is reached setting $\rho_0 = 4/(3\chi t_0^2)$.

In conclusion, we can infer that, for $f(R) = R + qR^n$, the standard matter-dominated FLRW behavior of the scale factor $a \sim t^{2/3}$ is the only asymptotic (as $t \to \infty$) power-law solution.

4.2.1 Range of t-values:

As shown above, the matter dominated solution $a \sim t^{2/3}$ is obtained for $K = 0$ and asimptotically as $t \to \infty$. In order to neglect all the K-terms in our $f(R)$ model, we start directly from the expression of the Ricci scalar [7]. Using a power-law scale factor, we get the t-range (if $x \neq 1/2$ and $x < 1$)

$$t \ll \big|\, [x(2x-1)]/[K/\bar{a}^2]\, \big|^{1/(2-2x)} \,. \tag{24}$$

For the matter-dominated era and using standard cosmological parameters [8], one can get the upper limit $K/\bar{a}^2 \lesssim 0.006\,(H_0)^{2/3}$, to estimate the value of $K/\bar{a}^2$. Thus, setting $x = 2/3$, we get the bound $t \ll\, 235/H_0$, independently of the form of $f(R)$.

At the same time, if we set $x = 2/3$, the asymptotic solution $\rho_0 = 4/(3\chi t_0^2)$ is reached neglecting the right hand side ($\ll 1$) of (23), *i.e*, if t is constrained by the lower limit: $t \gg\, \mu(n,q_0)/H_0$, where (we remind that $q_0 = H_0^{2n-2}\,q$)

$$\mu(n,q_0) = \big|\, q_0\big[-(4/3)^n + 2^{(2n+1)}3^{-n}\,n(2n-7/3)\big]\,\big|^{1/2(n-1)} \,. \tag{25}$$

Let us now recall that the matter-dominated era began, assuming $H_0^{-1} \simeq 4.3\times 10^{17}$s, at $t_{Eq} \simeq 5.1\times 10^{-6}/H_0$. In this sense, we can safely assume $\mu(n,q_0) \leqslant 5.1\times 10^{-8}$, which implies an upper limit for $|q_0|$, *i.e.*, $|q_0| \leqslant |q_0|^{Max}$, where

$$|q_0|^{Max}(n) = \big[5.1\times 10^{-8}\big]^{2(1-n)}\,\big|-(4/3)^n + 2^{(2n+1)}3^{-n}\,n(2n-7/3)\big|^{-1} \,. \tag{26}$$

It is easy to check that the function $|q_0|^{Max}(n)$ is decreasing as n goes from 2 to 3, in particular, one gets: $10^{-16} \lesssim |q_0|^{Max} \lesssim 10^{-31}$.

5 The inflationary paradigm

After discussing the power-law evolution of the Universe proper of the radiation- and matter-dominated eras, we now analyze the inflationary behavior characterizing the very early dynamics (for an interesting approach to the inflationary scenario within the modified gravity scheme, see [9, 10]). In this respect, we hypothesize an exponential behavior for the scale factor of the Universe $a = a_0\, e^{s(t-t_0)} = \bar{a}\, e^{st}$, where $s > 0$ and $\bar{a} = a_0 e^{-st_0}$. In the following, we concentrate the attention on the solution for vanishing spatial curvature $K = 0$ and, in this case, (18) rewrites as

$$\bar{a}^4\, e^{4st}\big[q(-12)^n\, s^{2n}(1-n/2) - 6s^2\big] + 2\tilde{\chi}\,(\bar{a}\, e^{st})^{1-3w} = 0 \,. \tag{27}$$

Let us now assume $w = -1$ (*i.e.*, $\rho = \rho_I = const.$) during inflation. Using the definition $q_0 = H_0^{2n-2}\,q$, the equation above reduces to

$$\big[(-1)^n 12^n\, q_0(1-n/2)\big]\, s_0^{2n} - 6s_0^2 + \kappa = 0 \,, \tag{28}$$

where $\kappa = 2\chi\rho_I H_0^{-2}$ and s_0 is a dimensionless parameter defined as $s_0 = s/H_0$. Since H_0 denotes the Hubble parameter measured today and estimating $H_I = \sqrt{\chi\rho_I/3}$ (*i.e.*, accordingly to its Friedmannian value) during inflation [7], one can obtain $\kappa \sim H_I^2/H_0^2 \sim 10^{100}$. For such values, it is easy to realize that considering the case

$n = 2\ell/(2m+1)$, the equation above does not admit real solution, thus we now discuss, consistently with the previous analyses, only $n = (2m+1)/(2\ell+1)$.

In order to integrate (28), we focus on a particular value of the power-law $f(R)$ exponent, *e.g*, $n = 29/13 \sim 2.23$. Using (15), for this value of n one can safely consider $L_{q\oplus}^{Min} \sim 1.44 \times 10^{-5}$ pc and, having in mind that $L_q = |q_0|^{1/(2n-2)}/H_0$ with $H_0 \simeq 4.2 \times 10^9$ pc, we get $|q_0| > |q_0|^{Min} \sim 2.56 \times 10^{-36}$.

Let us now fix the parameter q_0 to a reasonable value like $q_0^* \sim -10^3|q_0|^{Min}$ (such assumption will be physically motivated in the next Section). In this case, the solution of (28) is $s_0 \sim 2.45 \times 10^{29}$. This analysis demonstrates that an exponential early expansion of the Universe is still associated to a vacuum constant energy, even for the modified Friedmann dynamics. However, we see that the rate of expansion is significantly lower than the Friedmann-like one of about a factor in s_0 of 10^{20}. Although our estimation relies on the Friedmannian relation between H_I and ρ_I (the latter is taken of the order of the Grand Unification energy-scale), nevertheless the values of s_0 remains many order of magnitude below the standard value $\sim 10^{50}$ even if we change H_I for several order of magnitude. Despite this difference, it is still possible to arrange the cosmological parameter in order to have a satisfactory inflationary scenario, as far as we require a longer duration of the de Sitter phase.

6 Physical remarks

As already discussed in Sec.2, the parameter q has dimension $[L]^{2n-2}$. We have therefore defined a characteristic length scale of the model as $L_q(n) = |q|^{1/(2n-2)}$. Assuming $f(R)$ corrections to be smaller than the experimental uncertainty of the orbital period of the Earth around the Sun, the lower bound (15) for $L_q(n)$ was found. In order to identify the allowed scales for our model and in view of the upper constraint on the parameter $q_0 = H_0^{2n-2}q$ derived in the cosmological framework, we can now define the upper limit for $L_q(n)$ as

$$L_q^{Max}(n) = [|q_0|^{Max}]^{1/(2n-2)}/H_0 \,, \tag{29}$$

which, considering (26), yields to the constraints 65.59 pc $< L_q^{Max} <$ 78.37 pc, for $2 < n < 3$. Assuming $H_0 \simeq 4.2 \times 10^9$ pc, the two bounds for the characteristic length scales here discussed, *i.e*, (15) and (29), are plotted in Fig.1(A). At the same time two other typical lengths have been outlined in (12) for the Solar System. $L_\odot^*$ represents the minimum distance to have post-Newtonian and Newtonian terms of the same order. While $L_\odot^{**}$ was defined according to the request that the weak-field expansion holds. Setting now $n = 23/9 \simeq 2.55$, one can show from (15) and (29) that the allowed scales are 0.0013 pc $\lesssim L_q \lesssim$ 71.72 pc. In this range, $L_\odot^*$ and $L_\odot^{**}$ can be plotted as in Fig.1(B).

Summarizing, our analysis states a precise range of validity for the power-law $f(R)$ model we consider. Indeed, for a generic value of n (*i.e.*, not close to 2 or 3) the fundamental length of the model is constrained to range from the super Solar-

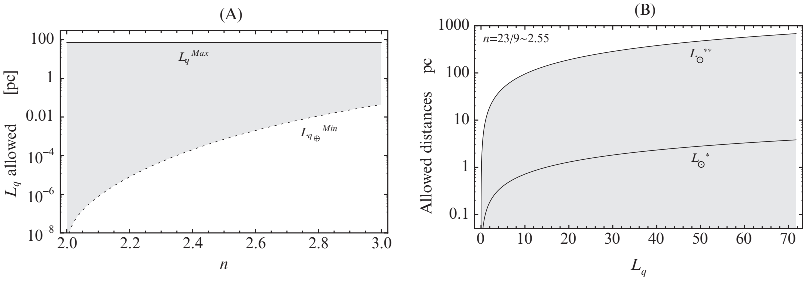

Fig. 1 **Panel A:** $L_{q\oplus}^{Min}$ of (15) and L_q^{Max} of (29). The gray zone represents the allowed characteristic-length scales of the model. We stress that $L_{q\odot}^{Min}$ is defined only if $n = (2m+1)/\ell$, as represented by the dotted line. **Panel B:** $L_\odot^*$ and $L_\odot^{**}$ of (12). The gray zone represents here the allowed distances for the model.

System scale up to a sub-galactic one. Therefore, in agreement to (9a), we have to search significant modification for the Newton law in gravitational system lying in this interval of length scales, like for instance, stellar clusters.

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Towards the unification of late-time acceleration and inflation by k-essence model

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Abstract Based on the formulation of the reconstruction for the k-essence model, which was recently proposed in [1], we explicitly construct cosmological model to unifying the late-time acceleration and the inflation in the early universe.

1 Introduction

By several observations of the universe, it is widely believed that the expansion of the present universe is accelerating [2, 3, 4]. In order to explain the acceleration, many kinds of models have been proposed. In this report, we focus on so-called k-essence model [5] in these models. The k-essence model is derived from k-inflation model [6] and we may regard the tachyon dark energy model [7], ghost condensation model [8], and scalar field quintessence model [9] with variations of the k-essence model.

In this report, based on the formulation of the reconstruction [1] for the k-essence model, we explicitly construct cosmological model to unifying the late-time acceleration and the inflation in the early universe (For consideration of unifying the inflation with the late-time acceleration in modified gravity, see [10].). In [1], the k-essence models which reproduce the arbitrary FRW cosmology, that is, the arbitrary time-development of the scale factor or the Hubble rate, has been explicitly constructed. For general reconstruction, see [13, 14]. In [1], two cases have been considered: One is the case that the action only contains the kinetic term and another is more general case including potential etc. In the former case, it was found that the exact ΛCDM model cannot be constructed although there is a model infinitely closing to ΛCDM model. In the model, however, the solution corresponding to ΛCDM model is unfortunately not stable. In the latter case, it has been found

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_22,

that there appear infinite number of redundant functions of the scalar fields, which are not directly related with the time development of the scale factor. By adjusting one of them, however, we can always obtain the model where the solution we need could be stable.

2 Formulation of reconstruction

In this section, we review on the formulation of the reconstruction proposed in [1] and give the k-essence models which reproduce the arbitrary FRW cosmology, that is, the arbitrary time-development of the scale factor or the Hubble rate. We now consider a rather general model, whose action is given by

$$S=\int d^4x\sqrt{-g}\left(\frac{R}{2\kappa^2}-K(\phi,X)+L_{\mathrm{matter}}\right), \quad X\equiv\partial^\mu\phi\partial_\mu\phi\,. \tag{1}$$

Here ϕ is a scalar field.

Now the Einstein equation has the following form:

$$\frac{1}{\kappa^2}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right)=-K(\phi,X)\,g_{\mu\nu}+2K_X(\phi,X)\,\partial_\mu\phi\partial_\nu\phi+T_{\mu\nu}\,. \tag{2}$$

Here $K_X(\phi,X)\equiv\partial K(\phi,X)/\partial X$ and $T_{\mu\nu}$ is the energy-momentum tensor of the matter. On the other hand, the variation of ϕ gives

$$0=-K_\phi(\phi,X)+2\nabla^\mu\left(K(\phi,X)\,\partial_\mu\phi\right)\,. \tag{3}$$

Here $K_\phi(\phi,X)\equiv\partial K(\phi,X)/\partial\phi$ and we have assumed that the scalar field ϕ does not directly couple with the matter.

In this section, we assume the FRW universe whose spacial part is flat: $ds^2=-dt^2+a(t)^2\sum_{i=1,2,3}(dx^i)^2$, and the scalar field ϕ only depends on time. Then the FRW equations are given by

$$\begin{aligned}\frac{3}{\kappa^2}H^2&=2X\frac{\partial K(\phi,X)}{\partial X}-K(\phi,X)+\rho_{\mathrm{matter}}\,,\\ -\frac{1}{\kappa^2}\left(2\dot{H}+3H^2\right)&=K(\phi,X)+p_{\mathrm{matter}}(t)\,.\end{aligned} \tag{4}$$

We included the matters with constant EoS parameters w_i. Then the energy density of the matters is given by $\sum_i\rho_{0i}a^{-3(1+w_i)}$ with constants ρ_{0i} and the pressure is given by $\sum_i w_i\rho_{0i}a^{-3(1+w_i)}$. Since the redefinition of ϕ can be absorbed into the redefinition of $K(\phi,X)$, we may identify the scalar field ϕ with the time coordinate t, $\phi=t$. Then we can rewrite the equations in (4) in the following form

$$K(t,-1)=-\frac{1}{\kappa^2}\left(2\dot{H}+3H^2\right)-\sum_i w_i\rho_{0i}a^{-3(1+w_i)}\,,$$

$$\left.\frac{\partial K(\phi,X)}{\partial X}\right|_{X=-1} = \frac{1}{\kappa^2}\dot{H} + \frac{1}{2}\sum_i (1+w_i)\,\rho_{0i}a^{-3(1+w_i)}\,. \tag{5}$$

By using the appropriate function $g(\phi)$, if we choose

$$\begin{aligned}
K(\phi,X) &= \sum_{n=0}^{\infty}(X+1)^n K^{(n)}(\phi)\,,\\
K^{(0)}(\phi) &\equiv -\frac{1}{\kappa^2}\left(2g''(\phi)+3g'(\phi)^2\right) - \sum_i w_i\rho_{0i}a_0^{-3(1+w_i)}\mathrm{e}^{-3(1+w_i)g(\phi)}\\
K^{(1)}(\phi) &= \frac{1}{\kappa^2}g''(\phi) + \frac{1}{2}\sum_i (1+w_i)\,\rho_{0i}a_0^{-3(1+w_i)}\mathrm{e}^{-3(1+w_i)g(\phi)}\,,
\end{aligned} \tag{6}$$

we find the following solution for the FRW equations (4),

$$H = g'(t) \quad \left(a = a_0\mathrm{e}^{g(t)}\right)\,. \tag{7}$$

Here $K^{(n)}(\phi)$ with $n = 2,3,\cdots$ can be arbitrary functions. The case that $K^{(n)}(\phi)$'s with $n = 2,3,\cdots$ vanish was studied in [11, 12] and the instability was investigated.

It is often convenient to use redshift z instead of cosmological time t since the redshift has direct relation with observations. The redshift is defined by $a(t) = a(t_0)/(1+z) = \mathrm{e}^{N-N_0}$. Here t_0 is the cosmological time of the present universe, N_0 could be an arbitrary constant, and N is called as e-folding and directly related with the redshift z. We now consider the reconstruction by using N instead of the cosmological time t and identify the scalar field ϕ with N Then since $d/dt = Hd/dN$, in terms of N, the equations in (5) have the following expressions

$$\begin{aligned}
K\left(t,-H^2\right) &= -\frac{1}{\kappa^2}\left(2H\frac{dH}{dN}+3H^2\right) - \sum_i w_i\rho_{0i}a^{-3(1+w_i)}\,,\\
H^2\left.\frac{\partial K(\phi,X)}{\partial X}\right|_{X=-H^2} &= \frac{1}{\kappa^2}H\frac{dH}{dN} + \frac{1}{2}\sum_i (1+w_i)\,\rho_{0i}a^{-3(1+w_i)}\,.
\end{aligned} \tag{8}$$

By using the appropriate function $f(\phi)$, if we choose

$$\begin{aligned}
K(\phi,X) &= \sum_{n=0}^{\infty}\left(\frac{X}{f(\phi)^2}+1\right)^n \tilde{K}^{(n)}(\phi)\,,\\
\tilde{K}^{(0)}(\phi) &\equiv -\frac{1}{\kappa^2}\left(2f(\phi)f'(\phi)+3f(\phi)^2\right) - \sum_i w_i\rho_{0i}a_0^{-3(1+w_i)}\mathrm{e}^{-3(1+w_i)(N-N_0)}\\
\tilde{K}^{(1)}(\phi) &= \frac{1}{\kappa^2}f(\phi)f'(\phi) + \frac{1}{2}\sum_i (1+w_i)\,\rho_{0i}a_0^{-3(1+w_i)}\mathrm{e}^{-3(1+w_i)(N-N_0)}\,,
\end{aligned} \tag{9}$$

we find the following solution for the FRW equations (4),

$$H = f(N)\,, \quad \phi = N\,. \tag{10}$$

Note that if we define a new field φ by $\varphi = \int \frac{d\phi}{f(\phi)}$, which can be solved as ϕ as a function of φ as $\phi(\varphi)$ and we find $\varphi = t$ up to an additive constant corresponding to the constant of the integration. We can also identify the expansion in (9) with that in (6): $\sum_{n=0}^{\infty}\left(\frac{X}{f(\phi)^2}+1\right)^n \tilde{K}^{(n)}(\phi) = \sum_{n=0}^{\infty}\left(\tilde{X}+1\right)^n K^{(n)}(\varphi)$. Here $\tilde{X} \equiv \partial^\mu\varphi\partial_\mu\varphi$, $K^{(n)}(\varphi) \equiv \tilde{K}^{(n)}(\phi(\varphi))$. Then even for the expansion in (9), we can use the arguments about the stability and the existence of the Schwarzschild solution, which will be given in the following sections.

3 Stability of solution

We now investigate the stability of the solution (7) or (10). First we consider the case without matter. From (4), we can derive the following equation which does not contain the variable g'',

$$3\frac{1-y^2}{1+X}X = -\frac{\dot{H}}{H^2} + \frac{\kappa^2}{H^2}\sum_{n=2}^{\infty}\left((n-1)X-n-1\right)X(X+1)^{n-2}K^{(n)}(\phi)\,, \tag{11}$$

where $y = \frac{g'}{H}$. Using (11), we can rewrite $dy/dN = (1/H)\,dy/dt$ in the form which does not contain g:

$$\begin{aligned}\frac{dy}{dN} &= 3X\frac{1-y^2}{1+X}\left(\frac{\dot{\phi}}{X}+y\right)\\ &-\frac{\kappa^2}{H^2}\sum_{n=2}^{\infty}\left[(\dot{\phi}+yX)\left((n-1)X-n-1\right)+\dot{\phi}n(X+1)\right](X+1)^{n-2}K^{(n)}(\phi)\,.\end{aligned} \tag{12}$$

When we consider the perturbation from a solution $\phi = t$ by putting $\phi = t + \delta\phi$ in (12), we obtain

$$\frac{d\delta\dot{\phi}}{dN} = \left[-3 - \frac{g''}{g'^2} - \frac{d}{dN}\left\{\frac{\kappa^2}{6g'^2}(8K^{(2)} - \frac{2}{\kappa^2}g'')\right\}\right]\delta\dot{\phi}\,. \tag{13}$$

If the quantity inside [ ] is negative, the fluctuation $\delta\dot{\phi}$ becomes exponentially smaller with time and therefore the solution becomes stable. Note that the stability is determined only in terms of $K^{(2)}$ and does not depend on other $K^{(n)}$ $(n \neq 2)$. Then if we choose $K^{(2)}$ properly, the solution corresponding to arbitrary development of the universe becomes stable.

We now investigate the stability when we include the matter. Then the equation corresponding to (11) has the following form:

$$3\frac{1-y^2}{1+X}X = -\frac{\dot{H}}{H^2} + \frac{\kappa^2}{H^2}\sum_{n=2}^{\infty}\Big((n-1)X-n-1\Big)X(X+1)^{n-2}K^{(n)}(\phi) \tag{14}$$

$$+\frac{\kappa^2}{H^2}\frac{X-1}{2(X+1)}\rho_{\rm matter}-\frac{\kappa^2}{2H^2}p_{\rm matter}-\frac{\kappa^2}{H^2}\frac{X}{X+1}\sum_i\rho_{0i}a_0^{-3(1+w_i)}\mathrm{e}^{-3(1+w_i)g(\phi)}\,.$$

Then we find

$$\begin{aligned}\frac{dy}{dN}&=3X\frac{1-y^2}{1+X}\left(\frac{\dot\phi}{X}+y\right)\\&\quad-\frac{\kappa^2}{H^2}\sum_{n=2}^{\infty}\left[(\dot\phi+yX)\left((n-1)X-n-1\right)+\dot\phi n(X+1)\right](X+1)^{n-2}K^{(n)}(\phi)\\&\quad+\frac{\kappa^2}{2H^2X}\left(-\frac{X-1}{X+1}\left(\dot\phi+yX\right)-\dot\phi\right)\rho_{\rm matter}+\frac{\kappa^2}{2H^2}yp_{\rm matter}\\&\quad+\frac{\kappa^2}{2H^2}\sum_i\left(\left(\dot\phi+yX\right)\frac{2}{X+1}-\dot\phi(1+w_i)\right)\rho_{0i}a_0^{-3(1+w_i)}\mathrm{e}^{-3(1+w_i)g(\phi)}\,.\end{aligned}\tag{15}$$

When we include the matter, the situation becomes a little bit complicated since not only H but the scale factor a appears in the equation. Then we need equation to describe the time development of a. By defining $\delta\lambda$ for convenience as

$$\delta\lambda\equiv3\sum_i(1+w_i)\rho_{0i}a(t)^{-3(1+w_i)}\frac{\kappa^2}{6g'^2}\left(\frac{\delta a}{a}-g'\delta\phi\right)\,.\tag{16}$$

we obtain the following equations,

$$\frac{d}{dN}\begin{pmatrix}\delta\dot\phi\\ \delta\lambda\end{pmatrix}\bigg|_{\phi=t,H=g'(t)}=\begin{pmatrix}A&B\\C&D\end{pmatrix}\begin{pmatrix}\delta\dot\phi\\ \delta\lambda\end{pmatrix}\tag{17}$$

$$\begin{aligned}A&\equiv-3+\frac{g''}{g'^2}+\frac{\kappa^2}{2g'^2}\sum_i(1+w_i)\rho_{0i}a(t)^{-3(1+w_i)}\\&\qquad-\frac{d}{dN}\ln\left\{8K^{(2)}-\frac{2}{\kappa^2}g''-\sum_i(1+w_i)\rho_{0i}a(t)^{-3(1+w_i)}\right\}\,,\\B&\equiv3-\frac{24K^{(2)}}{8K^{(2)}-\frac{2}{\kappa^2}g''-\sum_i(1+w_i)\rho_{0i}a(t)^{-3(1+w_i)}}\,,\\C&\equiv3\sum_i(1+w_i)\rho_{0i}a(t)^{-3(1+w_i)}\left(\frac{\kappa^2}{6g'^2}\right)^2\\&\qquad\times\left\{8K^{(2)}-\frac{6g'^2}{\kappa^2}-\frac{2}{\kappa^2}g''-\sum_i(1+w_i)\rho_{0i}a(t)^{-3(1+w_i)}\right\}\,,\\D&\equiv\frac{d}{dN}\ln\left\{\frac{\kappa^2}{2g'^2}\sum_i(1+w_i)\rho_{0i}a(t)^{-3(1+w_i)}\right\}-\frac{\kappa^2}{2g'^2}\sum_i(1+w_i)\rho_{0i}a(t)^{-3(1+w_i)}\,.\end{aligned}$$

Generally, 2×2 matrix must have negative trace and positive determinant in order that two eigenvalues could be negative since the two eigenvalues are given by

$\frac{1}{2}\{\mathrm{tr}M \pm \sqrt{(\mathrm{tr}M)^2 - 4(\det M)}\}$. So we just need to calculate the determinant and the trace of the matrix for investigating the stability of the fixed point $\phi = t$, $H = g'(t)$. Since the expressions in (17) are very complicated, we consider the case that the solution is given by the behavior mimicing ΛCDM solution in the Einstein gravity and the matter contents are given in the present universe. Then we find

$$a(t) \sim A \sinh^{\frac{2}{3}}[\alpha t], \quad g'(t) \sim \frac{2}{3}\alpha \coth[\alpha t], \tag{18}$$

$$\frac{\kappa^2}{\alpha^2}\sum_i (1+w_i)\rho_{0i} a(t)^{-3(1+w_i)} \sim \frac{2}{3}\frac{1}{\sinh^2[\alpha t]}, \tag{19}$$

$$\frac{\kappa^2}{\alpha^2}\sum_i w_i(1+w_i)\rho_{0i} a(t)^{-3(1+w_i)} \sim \frac{4}{9}\times 1.86\times 10^{-4}\frac{1}{\sinh^{\frac{8}{3}}[\alpha t]}, \tag{20}$$

where $A \equiv (\rho_{m0}/\rho_\Lambda)^{\frac{1}{3}}$ and $\alpha \equiv \kappa\sqrt{3\rho_\Lambda}/2$. Note that in (18) and (19), we neglect the contribution from radiation, and in (20), there only appears the contribution from radiation. Therefore the expressions in (18) - (20) could be valid at least when $t \geq 10^9$ years. By using (18) - (20), we find the following expressions of the determinant and trace of the matrix in (17):

$$\mathrm{tr}\begin{pmatrix} A & B \\ C & D \end{pmatrix} \sim -6 + \frac{3}{2}\frac{1}{\cosh^2[\alpha t]} - \frac{8\frac{\kappa^2}{\alpha^3}K^{(2)\prime} - \frac{4}{3}\frac{\cosh[\alpha t]}{\sinh^3[\alpha t]}}{\frac{2}{3}\coth[\alpha t]\left(8\frac{\kappa^2}{\alpha^2}K^{(2)} + \frac{2}{3}\frac{1}{\sinh^2[\alpha t]}\right)}, \tag{21}$$

$$\begin{aligned}
\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} &\sim \left\{8\frac{\kappa^2}{\alpha^2}K^{(2)} + \frac{2}{3}\frac{1}{\sinh^2[\alpha t]}\right\}^{-1} \\
&\times \left[\left(-27\frac{\sinh[\alpha t]}{\cosh^3[\alpha t]} + 36\tanh[\alpha t]\right)\frac{\kappa^2}{\alpha^3}K^{(2)\prime}\right. \\
&+ \left(72 - 36\frac{1}{\cosh^2[\alpha t]} - 18\frac{1}{\cosh^4[\alpha t]} - 12\times 1.86\times 10^{-4}\frac{1}{\sinh^{\frac{8}{3}}[\alpha t]}\right)\frac{\kappa^2}{\alpha^2}K^{(2)} \\
&-6\times 1.86\times 10^{-4}\frac{1}{\cosh^2[\alpha t]\sinh^{\frac{8}{3}}[\alpha t]} - \frac{3}{2}\frac{1}{\cosh^4[\alpha t]\sinh^2[\alpha t]} \\
&\left.+3\frac{1}{\sinh^2[\alpha t]\cosh^2[\alpha t]} + 4\times 1.86\times 10^{-4}\frac{1}{\sinh^{\frac{8}{3}}[\alpha t]}\right].
\end{aligned} \tag{22}$$

Note that $1 < \cosh[\alpha t] \leq 2$ and $0 < \sinh[\alpha t] \leq 1.7$ in evolution of the universe, For example, if we consider the case that $K^{(2)}$ is constant, then the trace of the matrix is always negative and the determinant is positive when $K^{(2)} \geq 0$ since $72\frac{\kappa^2}{\alpha^2}K^{(2)}$ and $3/(\sinh^2[\alpha t]\cosh^2[\alpha t])$ are dominant terms in (22). Therefore even if $K^{(2)}$ is constant, the fixed point solution mimicking ΛCDM solution in the Einstein gravity becomes stable as long as $K^{(2)} \geq 0$.

4 Schwarzschild solution

We now consider the condition that there could be the Schwarzschild or Schwarzschild-(A)dS solution for the action (1).

We now assume ϕ is a constant: $\phi = \phi_0$, or the change of the value of ϕ is very small. Then the Einstein equation (2) and the equation (3) given by the variation of ϕ reduce to

$$\frac{1}{\kappa^2}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = -K\left(\phi_0,0\right)g_{\mu\nu} + T_{\mu\nu}\,, \tag{23}$$

$$0 = K_\phi\left(\phi_0,0\right)\,. \tag{24}$$

When $T_{\mu\nu} = 0$, if $K\left(\phi_0,0\right)$ does not vanish, a solution of (23) is given by Schwarzschild-(A)dS space-time. On the other hand, if $K\left(\phi,0\right)$ vanishes, the Schwarzschild solution, which is asymptotically the Minkowski space-time, is a solution. The equation (24) requires that $K_\phi\left(\phi,0\right)$ has an extremum in general or $K\left(\phi,0\right)$ can be a constant independent of the value of ϕ. Especially if $K\left(\phi,0\right)$ vanishes identically, (23) gives the Schwarzschild solution.

We now write $K\left(\phi,X\right)$ as in (6), $K\left(\phi,X\right) = \sum_{n=0}\left(1+X\right)^n K_n(\phi)$. Then if $K\left(\phi,0\right) = 0$, that is, if $K\left(\phi,0\right)$ vanishes independent of the value of ϕ, we find

$$\sum_{n=0} K_n(\phi) = 0\,. \tag{25}$$

Especially we may choose

$$K_3(\phi) = -K_0(\phi) - K_1(\phi) - K_2(\phi)\,. \tag{26}$$

Then if the condition (25) or (26) is satisfied, the Schwalzschild space-time is always a solution independent of the value of ϕ as long as ϕ is a constant. Then any point source of matter makes the Schwarzschild space-time which generates the Newton potential. Then the correction to the Newton law could not appear. Note that the value of the scalar field ϕ changes by the evolution of the universe but as long as the condition (25) or (26) is satisfied, in a local region where ϕ is almost constant, the correction to the Newton law could be negligible.

5 Construction of a model unifying late-time acceleration and inflation

In this section, by using the formulation of the reconstruction, we try to construct a model describing the late-time acceleration and the inflation in the early universe.

In the Einstein gravity, we have the following FRW equation

$$\frac{3}{\kappa^2}H^2 = \rho_{\rm total}\,, \quad -\frac{1}{\kappa^2}\left(2\dot H + 3H^2\right) = p_{\rm total}\,. \tag{27}$$

Here $\rho_{\rm total}$ and $p_{\rm total}$ are total energy density and the total pressure. Then we may define total equation of state parameter $w_{\rm total}$ by

$$w_{\rm total} = \frac{p_{\rm total}}{\rho_{\rm total}} = -1 - \frac{2\dot H}{3H^2} = -1 - \frac{2}{3H}\frac{dH}{dN}\,. \tag{28}$$

As matters, we include the radiation and dust, which may corresponds to the baryon and dark matter. Then as in the energy density of matter behaves as in (9),

$$\rho = \rho_{\rm radiation} {a_0}^{-4} \mathrm{e}^{-4(N-N_0)} + \rho_{\rm dust} {a_0}^{-3} \mathrm{e}^{-3(N-N_0)}\,. \tag{29}$$

We may consider the following Hubble rate

$$H^2 = H_0^2 \left(N_I^2 + N^2\right)^{-\gamma} + \frac{\kappa^2}{3}\left(\rho_{\rm radiation} a_0^{-4} \mathrm{e}^{-4(N-N_0)} + \rho_{\rm dust} a_0^{-3} \mathrm{e}^{-3(N-N_0)}\right)\,. \tag{30}$$

Here H_0, N_I, and γ are parameters assumed to be positive. Then by using (8), we find

$$\begin{aligned} K^{(0)}(\phi) &= \frac{H_0^2}{\kappa^2}\left\{2\gamma\phi\left(N_I^2+\phi^2\right)^{-\gamma-1} + 3\left(N_I^2+\phi^2\right)^{-\gamma}\right\}\,, \\ K^{(1)}(\phi) &= -\frac{H_0^2}{\kappa^2}\gamma\phi\left(N_I^2+\phi^2\right)^{-\gamma-1}\,. \end{aligned} \tag{31}$$

Note that in the expression of $K^{(0)}(\phi)$ and $K^{(1)}(\phi)$, the parameters describing the matters like $\rho_{\rm radiation}$ and $\rho_{\rm dust}$ are not included.

We now investigate the cosmology described by the Hubble rate in (30). If H_0 is large enough, in the early universe, where we assume $N \to 0$, the first term in (30) could dominate $H^2 \sim H_0^2 \left(N_I^2 + N^2\right)^{-\gamma}$. Then by using (28), we find

$$w_{\rm total} \sim -1 + \frac{2\gamma N}{3\left(N_I^2 + N^2\right)}\,. \tag{32}$$

Then $N \to 0$, we find $w_{\rm total} \to -1$, which corresponding to the effective cosmological constant with $w = -1$. Therefore the inflation in the early universe could be generated. When N becomes larger, the first term could become smaller and the second term in (30) corresponding to the radiation could dominate. Maybe more exactly, the second and third terms could be generated by the reheating after the inflation, after that there could be complicated process for the matters like pair annihilation (creation), baryogenesis (or leptogenesis). In this present report, we do not discuss about the detailed process. After the radiation, the third term in (30) corresponding to the dust could dominate as in the usual scenario. The contributions from the radiation and matter decrease exponentially as a function of N, the first term in in (30) dominate in the late universe again. Then $w_{\rm total}$ is given by (32) again. For large N, corresponding to the late universe, we find $w_{\rm total} \to -1$, again and therefore the expansion of the universe accelerates again. Note that, however, different from the case of the small cosmological constant, H is not a constant but decreasing function of N as $H \sim H_0 N^{-\gamma}$. Then the smallness of the scale of the acceleration in the present universe might be naturally explained.

We now discuss more quantitively. Since $H \to H_0 N_I^{-\gamma}$ when $N \to 0$, if we assume $N_I \sim \mathscr{O}(1)$, the scale of H_0 corresponds to the weak scale. In the action (1) with (31), the dimensional parameter appears only in the combination of

$$M^4 = \frac{H_0^2}{\kappa^2} = M_{\rm Planck}^2 H_0^2 \,. \tag{33}$$

Here $M_{\rm Planck} \sim 10^{19}\,{\rm GeV}$ is the Planck scale. Then if the scale of the inflation is the Planck scale, $H_0 \sim M_{\rm Planck}$, M is also the Planck scale, $M \sim M_{\rm Planck}$. If H_0 is a GUT scale, $H_0 \sim 10^{16}\,{\rm GeV}$, the scale of M is $10^{17\sim 18}\,{\rm GeV}$. We should note that M is only one parameter with the dimension of mass in the action. We now consider how very small scale corresponding to the late acceleration can appear from only one dimensional parameter. In the present universe, if we denote the present value of H by $H_{\rm present} \sim 10^{-33}\,{\rm eV}$, under the assumption $N_I \sim \mathscr{O}(1)$, we find $C \equiv \frac{\text{Plank scale}}{H_0} \sim 10^{61} \sim {\rm e}^{140}$. Then when we assume $H_0 \sim M_{\rm Planck}$, we have $\frac{H_0}{H_{\rm present}} \sim C$. If we naively assume $N \sim \ln C \gg N_I$, we find $\gamma \sim 28$. Then we find that the very small scale corresponding to the late acceleration can appear from only one dimensional parameter M. Then the fine tuning problem might be relaxed. If the equation of state parameter $w_{\rm darkenergy}$ of the dark energy could be given by (32) with $N \gg N_I$ as $w_{\rm darkenergy} \sim w_{\rm total}$, we obtain $w_{\rm darkenergy} \sim -0.87$, which is little bit greater than the value obtained from the observation $-0.14 < 1 + w < 0.12$[3].

6 Summary

In this report, we explicitly construct cosmological model of k-essence to unifying the late-time acceleration and the inflation in the early universe (Note that such reconstruction scheme for k-essence models may be extended also for presence of Lagrange multiplier [15].). The construction is based on the formulation of the reconstruction for the k-essence model, which was recently proposed in [1]. The action (1) can be expanded as a power series of $1 + X = 1 + \partial^\mu \phi \partial_\mu \phi$ as in (6). For the cosmological evolution of the Hubble rate H, only the first two terms in the series are relevant. The third term is relevant for the stability. The fourth or higher terms are relevant for the existence of the Schwarzschild solution, which may reproduce the Newton law if the scalar field is not directly coupled with the matter.

In the constructed model (31), only one dimensional parameter, whose scale could be equal to or a little bit smaller than the Planck scale but the small scale of the present Hubble scale can be produced. We have not discussed about the reheating and structure formation in the universe etc., which will be discussed in the forthcoming paper.

Acknowledgements We are grateful to J. Matsumoto and S. D. Odintsov for very helpful discussions. This work has been supported by Global COE Program of Nagoya University (G07) provided by the Ministry of Education,Culture, Sports, Science & Technology of Japan, and by the JSPS Grant-in-Aid for Scientific Research (S) # 22224003.

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Conformal Equivalence in Classical Gravity: the Example of “Veiled” General Relativity

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Abstract In the theory of General Relativity, gravity is described by a metric which couples minimally to the fields representing matter. We consider here its “veiled” versions where the metric is conformally related to the original one and hence is no longer minimally coupled to the matter variables. We show on simple examples that observational predictions are nonetheless exactly the same as in General Relativity, with the interpretation of this “Weyl” rescaling “à la Dicke”, that is, as a spacetime dependence of the inertial mass of the matter constituents.

1 Introduction

Many extensions of General Relativity which are under current investigation (for example $f(R)$ gravity, see e.g. [1], or quintessence models, see e.g. [2]) fall in the class of scalar-tensor theories (see e.g. [3]) where gravity is represented by a scalar field $\tilde{\phi}$ together with a metric $\tilde{\boldsymbol{g}}$ which minimally couples to the matter variables. Now, as is well-known (see [4] where references to the earlier literature can also be found), the “Jordan frame” variables $\tilde{\phi}$ and $\tilde{\boldsymbol{g}}$ can be traded for the “Einstein frame” variables $(\phi_*, \boldsymbol{g}_*)$ with $\tilde{\boldsymbol{g}} = e^{2\Omega}\boldsymbol{g}_*$, the conformal factor Ω being chosen so that the action for gravity becomes Einstein-Hilbert’s, the “price to pay” being that the matter fields no longer minimally couple to the metric $\boldsymbol{g}_*$.

Although there seems to be an agreement in the recent literature about the mathematical equivalence of these two “frames” (as long as Ω does not blow up) there is still some debate about their “physical” equivalence, the present trend (see e.g. [1] and references therein) being that calculations may be performed in the Einstein frame but interpretation should be done in the Jordan frame (for the opposite

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_23,

view see e.g. [5], where a comprehensive review of the earlier literature can also be found).

It should be clear however, see [6], that, just as one can formulate and interpret a theory using any coordinate system (proper account being taken of inertial accelerations if need be), one should be able to formulate and interpret (classical) gravity using any conformally related metric, proper account being taken of non-minimal coupling if need be. (For recent papers supporting this view, see e.g. [7, 8, 9].)

In this paper we shall try to make this equivalence "crystal clear" by showing that some familiar predictions of General Relativity can equivalently be made in its "veiled" versions where the metric is conformally related to the original one and hence is no longer minimally coupled to the matter variables.

2 Conformal transformations and "veiled" General Relativity

In the theory of General Relativity:

- Events are represented by the points P of a 4-dimensional manifold $\mathscr{M}$ equipped with a Riemannian metric $\boldsymbol{g}$, with components $g_{\mu\nu}(x^\alpha)$ in the (arbitrary) coordinate system x^α labelling the points P.
- Matter is represented by a collection of tensorial fields on $\mathscr{M}$, denoted $\psi_{(a)}(P)$.
- Gravity is encoded in the metric $\boldsymbol{g}$ which couples minimally to the fields $\psi_{(a)}$. This means that the action for matter is obtained from the form it takes in flat spacetime in Minkowskian coordinates by replacing $\eta_{\mu\nu}$ by $g_{\mu\nu}$.
- Finally the action for gravity is postulated to be Einstein-Hilbert's.

Hence the familiar total action:

$$S[g_{\mu\nu},\psi_{(a)}]=\frac{1}{16\pi}\int d^4x\sqrt{-g}R+S_m[g_{\mu\nu},\psi_{(a)}]\,, \tag{1}$$

where g is the determinant of the metric components $g_{\mu\nu}$ and R the scalar curvature. Our conventions are: signature $(-+++)$, $R=g^{\mu\nu}R_{\mu\nu}$, $R_{\mu\nu}=R^\sigma{}_{\mu\sigma\nu}$, $R^\mu{}_{\nu\rho\sigma}=\partial_\rho\Gamma^\mu{}_{\nu\sigma}+\cdots$. We use Planck units where $c=\hbar=G=1$.

The field equations are obtained by extremising S with respect to the metric $g_{\mu\nu}$ and the matter fields $\psi_{(a)}$, which yields the equally familiar Einstein equations,

$$G_{\mu\nu}=8\pi T_{\mu\nu}\,,\qquad \frac{\delta S_m}{\delta\psi_{(a)}}=0\,, \tag{2}$$

where $G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor and where $T_{\mu\nu}=-\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}$ is the total stress-energy tensor. As is well-known $T_{\mu\nu}$ is constrained by the Bianchi identity to be divergence-less,

$$D_\nu T^{\mu\nu}=0\,, \tag{3}$$

D being the covariant derivative associated with $\boldsymbol{g}$. Recall that this conservation law implies that the worldline of uncharged test particles are represented by geodesics of the metric $\boldsymbol{g}$.

Let us now equip our manifold $\mathscr{M}$ with another metric $\bar{\boldsymbol{g}}$, with components $\bar{g}_{\mu\nu}$ in the same coordinate system x^α, which is conformally related to the original one:

$$g_{\mu\nu} = \Phi\, \bar{g}_{\mu\nu}\,, \tag{4}$$

$\Phi(x^\alpha)$ being an arbitrary function of the coordinates, that we shall restrict to be everywhere positive.[1]

Using the fact that $\sqrt{-g} = \Phi^2\sqrt{-\bar{g}}$ and that the Ricci tensors and scalar curvatures are related as

$$R_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{\bar{D}_{\mu\nu}\Phi}{\Phi} - \frac{\bar{g}_{\mu\nu}}{2}\frac{\bar{\Box}\Phi}{\Phi} + \frac{3}{2}\frac{\partial_\mu\Phi\partial_\nu\Phi}{\Phi^2}\,, \quad R = \frac{1}{\Phi}\left(\bar{R} - 3\frac{\bar{\Box}\Phi}{\Phi} + \frac{3}{2}\frac{(\bar{\partial}\Phi)^2}{\Phi^2}\right), \tag{5}$$

($\bar{R}_{\mu\nu}$, $\bar{R}$ and $\bar{D}$ being the Ricci tensor, the scalar curvature and the covariant derivative associated with the metric $\bar{\boldsymbol{g}}$), it is easy to find the "veiled" version of Einstein's equations (2),

$$\Phi\,\bar{G}_{\mu\nu} - \bar{D}_{\mu\nu}\Phi + \bar{g}_{\mu\nu}\bar{\Box}\Phi + \frac{3}{2\Phi}\left(\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}\bar{g}_{\mu\nu}(\bar{\partial}\Phi)^2\right) = 8\pi\bar{T}_{\mu\nu}\,, \quad \frac{\delta S_m}{\delta\psi_{(a)}} = 0\,, \tag{6}$$

where S_m is now expressed in terms of $\bar{g}_{\mu\nu}$, $S_m[g_{\mu\nu},\psi_{(a)}] = S_m[\Phi\,\bar{g}_{\mu\nu},\psi_{(a)}]$, and where $\bar{T}_{\mu\nu} = -\frac{2}{\sqrt{-\bar{g}}}\frac{\delta S_m}{\delta\bar{g}^{\mu\nu}}$ so that $\bar{T}_{\mu\nu} = \Phi\,T_{\mu\nu}$, with $g_{\alpha\beta}$ replaced by $\Phi\,\bar{g}_{\alpha\beta}$ in $T_{\mu\nu}$. As for the Bianchi identity (3), it translates into

$$\bar{D}_\nu\bar{T}^{\mu\nu} = \frac{\bar{\partial}^\mu\Phi}{2\Phi}\,\bar{T}\,. \tag{7}$$

The total stress-energy tensor is no longer conserved.

Equations (6), (7) can also be straightforwardly obtained from the Einstein-Hilbert action (1). Indeed it reads, using (4) and up to a boundary term,

$$S[\bar{g}_{\mu\nu},\Phi,\psi_{(a)}] = \frac{1}{16\pi}\int d^4x\sqrt{-\bar{g}}\left(\Phi\,\bar{R} + \frac{3}{2}\frac{(\bar{\partial}\Phi)^2}{\Phi}\right) + S_m[\Phi\,\bar{g}_{\mu\nu},\psi_{(a)}]\,. \tag{8}$$

Extremisation with respect to $\bar{g}_{\mu\nu}$ and $\psi_{(a)}$ yields (6). As for the extremisation with respect to Φ it is redundant since it turns out to be equivalent to the trace of equation (6). This reflects the fact that, $\bar{g}_{\mu\nu}$ remaining unconstrained, Φ is an arbitrary

[1] $(\mathscr{M},\mathrm{g})$ or $(\mathscr{M},\bar{\mathrm{g}})$ are often called, rather improperly, "frames", when a more accurate wording would be "representations" of space and time, see [6].

function and not a dynamical field.[2]

Let us now be more specific about the matter action S_m.

As an example (others are considered in the Appendix), take matter to be an electron characterized by its inertial mass m and charge q interacting with the electromagnetic field A_μ created by an infinitely massive proton, so that S_m is the Lorentz action where $\eta_{\mu\nu} \to g_{\mu\nu}$:[3]

$$S_m[g_{\mu\nu}, L] = -m \int_L \sqrt{-g_{\mu\nu} dx^\mu dx^\nu} + q \int_L A_\mu dx^\mu \,, \tag{9}$$

where L is a path determined by $x^\mu = x^\mu(\lambda)$. The equation of motion of the electron, $\frac{\delta S_m}{\delta L} = 0$, is the familiar Lorentz equation,

$$m u^\nu D_\nu u^\mu = q F^\mu{}_\nu u^\nu \,, \tag{10}$$

where $u^\mu = dx^\mu / d\tau$ with $g_{\mu\nu} u^\mu u^\nu = -1$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

Equivalently, S_m reads, in terms of the metric $\bar{g}_{\mu\nu}$,

$$S_m[\bar{g}_{\mu\nu}, \Phi, L] = -\int_L \bar{m} \sqrt{-\bar{g}_{\mu\nu} dx^\mu dx^\nu} + q \int_L \bar{A}_\mu dx^\mu \,, \tag{11}$$

where $\bar{A}_\mu = A_\mu$ and

$$\bar{m} = \sqrt{\Phi} m \,. \tag{12}$$

As for the Lorentz equation (10), it becomes

$$\bar{m} \left[\bar{u}^\nu \bar{D}_\nu \bar{u}^\mu + \frac{1}{2\Phi} \partial_\nu \Phi \left(\bar{g}^{\mu\nu} + \bar{u}^\mu \bar{u}^\nu \right) \right] = q \bar{F}^\mu{}_\nu \bar{u}^\nu \,, \tag{13}$$

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[2] One notes the resemblance of the action (8) and the field equations (6) with the Brans-Dicke action and field equations [10] when their parameter ω is $\omega = -3/2$, see e.g. [11]. The difference (which makes $\omega = -3/2$ Brans-Dicke theory different from General Relativity) is that, in Brans-Dicke theory, matter is minimally coupled to the metric $\bar{\boldsymbol{g}}$ (not $\boldsymbol{g}$),

$$S_{\rm BD}^{\omega=-3/2} = \frac{1}{16\pi} \int d^4x \sqrt{-\bar{g}} \left(\Phi \bar{R} + \frac{3}{2} \frac{(\bar{\partial}\Phi)^2}{\Phi} \right) + S_m[\bar{g}_{\mu\nu}, \Psi_{(a)}] \,.$$

In the spirit of [12], one could therefore introduce a "detuned" version of General Relativity based on the action,

$$S_{\rm detunedGR} = \frac{1}{16\pi} \int d^4x \sqrt{-\bar{g}} \left(\Phi \bar{R} + \frac{3}{2} \frac{(\bar{\partial}\Phi)^2}{\Phi} \right) + S_m[\Phi F(\Phi) \bar{g}_{\mu\nu}, \Psi_{(a)}] \,,$$

which reduces to "veiled" General Relativity if $F(\Phi) = 1$ and to $\omega = -3/2$ Brans-Dicke theory if $F(\Phi) = \Phi^{-1}$. We shall not pursue this idea any further here.

[3] In Planck units m and q are two dimensionless numbers which are determined in a local inertial frame where gravity is "effaced"[13] and where the laws of Special Relativity hold.

with $\bar{u}^\mu = dx^\mu/d\bar{\tau}$ and $\bar{g}_{\mu\nu}\bar{u}^\mu\bar{u}^\nu = -1$.

In locally Minkowskian coordinates X^μ in the neighbourhood of some point P where $\bar{g}_{\mu\nu} \approx \eta_{\mu\nu}$ and if Φ is approximately constant, this equation takes the form,

$$\bar{m}\frac{dU^\mu}{d\tau_M} \approx qF^\mu{}_\nu U^\nu\,, \tag{14}$$

with $U^\mu = dX^\mu/d\tau_M$ and $\eta_{\mu\nu}U^\mu U^\nu = -1$. This equation is the same as the one governing the motion of the electron in Special Relativity apart from the fact that its mass is rescaled by the factor $\sqrt{\Phi(P)}$, see [6].[4]

As an illustration of the consequences of the rescaling of the mass in veiled General Relativity, consider for example a transition between the levels n and n' of, say, the hydrogen atom. Its frequency is given by Bohr's formula,

$$\bar{\nu}(P) = \left(\frac{1}{n'^2} - \frac{1}{n^2}\right)\frac{\bar{m}(P)q^4}{2} \qquad \text{with} \qquad \bar{m}(P) = \sqrt{\Phi(P)}\,m\,. \tag{15}$$

It depends on P, that is, on when and where it is measured. Hence the frequency $\bar{\nu}(P) \equiv \bar{\nu}$ of the transition measured at point P ("there and then") and the frequency $\bar{\nu}(P_0) \equiv \bar{\nu}_0$ of the same transition measured at P_0 ("here and now") are related by:[5]

$$\bar{\nu} = \sqrt{\frac{\Phi(P)}{\Phi(P_0)}}\,\bar{\nu}_0\,. \tag{16}$$

3 Conformal equivalence in cosmology

Let us show here on a few examples that the standard cosmological models of General Relativity or its conformally related sister theories all lead to the same physical predictions and hence are observationally indistinguishable.

The field equations to solve are the veiled Einstein equations (6)-(7) for $\bar{g}_{\mu\nu}$ and Φ.

We look for simplicity for spatially flat Robertson-Walker metrics,

$$d\bar{s}^2 = \bar{a}^2(t)(-dt^2 + dr^2)\,, \tag{17}$$

[4] This space-time dependence of the (inertial) mass can be interpreted as a local rescaling of the unit of mass, see [6]. It can also be interpreted as the result of the "interaction" of the "scalar field" Φ with matter. It must be remembered however that this "interaction" is an artefact of the introduction of the metric $\bar{\boldsymbol{g}}$, and that the "scalar force" which appears in (13) or (7) can be globally effaced by returning to the original metric $\boldsymbol{g}$, just like an inertial force can be effaced by going to an inertial frame.

[5] This difference between the two numbers $\bar{\nu}$ and $\bar{\nu}_0$ can be interpreted as simply due to the fact that they are expressed using a different unit of time at P and P_0, see [6].

where the scale factor $\bar{a}$ and the scalar field Φ depend on t only. By construction equations (6)-(7) are undetermined and we shall choose here, to make our point more strikingly, Φ to be the dynamical field describing gravity by imposing

$$\bar{a}(t) = 1\,. \tag{18}$$

Therefore the metric $\bar{\boldsymbol{g}}$ is flat.[6]

Matter is represented by the stress-energy tensor of a perfect fluid (see Appendix): $\bar{T}_{\mu\nu} = (\bar{\rho} + \bar{p})\bar{u}_\mu \bar{u}_\nu + \bar{p}\,\bar{g}_{\mu\nu}$ that we choose to be at rest with respect to the Minkowskian coordinate grid (t,r):[7] $\bar{u}^\mu = (1,0)$; as for the (veiled) density and pressure $\bar{\rho}$ and $\bar{p}$ they depend on t.

The equations of motion (6)-(7) for Φ then reduce to, a prime denoting derivation with respect to t,

$$\frac{3}{4\Phi}\Phi'^2 = 8\pi\bar{\rho}\,, \qquad \bar{\rho}' = \frac{\Phi'}{2\Phi}(\bar{\rho} - 3\bar{p})\,, \tag{19}$$

which can be solved once an equation of state is given. For $\bar{p} = w\bar{\rho}$ for example,

$$\Phi = \left(\frac{t}{t_0}\right)^{4/(1+3w)}\,, \quad \bar{\rho} = \frac{3}{2\pi(1+3w)t_0^2}\left(\frac{t}{t_0}\right)^{2(1-3w)/(1+3w)}\,. \tag{20}$$

Let us now turn to the relation between the luminosity distance D and redshift z that the model predicts.

As usual, we focus on a given atomic transition line in the spectrum of a distant galaxy at point $P = (t,r)$. The observer is at point $P_0 = (t_0, 0)$, and the atomic line emitted by this galaxy is observed at frequency ν_0. As given in (16), if $\bar{\nu}$ is the frequency of this transition measured at point P, the frequency of the same transition measured at point P_0 will be $\bar{\nu}_0 = \sqrt{\Phi(P_0)/\Phi(P)}\,\bar{\nu}$. Therefore the observed redshift is given by

$$1 + z = \frac{\bar{\nu}_0}{\nu_0} = \sqrt{\frac{\Phi(t_0)}{\Phi(t)}}\frac{\bar{\nu}}{\nu_0}\,. \tag{21}$$

The luminosity distance is given, by definition, as

$$D = \sqrt{\frac{L}{4\pi\ell}}\,, \tag{22}$$

where L is the absolute luminosity of the galaxy and ℓ is the apparent luminosity per unit area observed at point P_0. Since the mass of the electron in veiled General Relativity varies according to $\bar{m} = \sqrt{\Phi}\,m$, it is crucial here to recall that the absolute

[6] This does not mean that t and r represent time and position in an inertial frame since the word-lines of free particles are not straight lines. They rather solve, see (13): $\bar{u}^\nu \bar{D}_\nu \bar{u}^\mu = -\frac{1}{2\Phi}(\bar{\partial}^\mu \Phi + \bar{u}^\mu \bar{u}^\nu \partial_\nu \Phi)$, whose solution is, C being three constants: $\bar{V} \equiv \bar{u}/\bar{u}^0 = C/\sqrt{C^2 + \Phi(t)} \neq$ const..

[7] This is the familiar "Weyl postulate".

luminosity is *not* equal to the luminosity measured at the point of emission P (where the frequency of the transition is $\bar{\nu}$) but is defined as if the galaxy were at the point of reception P_0 (where the frequency of the transition is $\bar{\nu}_0$) so that we have [8]

$$L = N\frac{\bar{\nu}_0}{\Delta t} = N\bar{\nu}_0^2\,, \tag{23}$$

where N is the number of photons emitted by this transition during a period $\Delta t = 1/\bar{\nu}_0$. The apparent luminosity is given by

$$\ell = N\frac{\nu_0^2}{S} = N\frac{\nu_0^2}{4\pi r^2}\,, \tag{24}$$

where $S = 4\pi r^2$ is the surface area of a sphere of radius r since the metric $\bar{\boldsymbol{g}}$ is flat. Inserting (23) and (24) into (22), we find, using (21),

$$D = \frac{\bar{\nu}_0}{\nu_0}\,r = \sqrt{\frac{\Phi(t_0)}{\Phi(t)}}\,\frac{\bar{\nu}}{\nu_0}\,r\,. \tag{25}$$

In order finally to relate $\bar{\nu}$ to ν_0 and r to T we must study the propagation of light from P to P_0. Light follows the null cones of $(\mathscr{M}, \bar{g}_{\mu\nu} = \eta_{\mu\nu})$ so that r is the time light takes to propagate from P to P_0, and the frequency $\bar{\nu}$ measured at P is the same as the frequency ν_0 observed at P_0:

$$(\bar{\nu} = \nu_0\,, \quad r = t_0 - t) \qquad \Longrightarrow \qquad z = \sqrt{\frac{\Phi(t_0)}{\Phi(t)}} - 1\,, \quad D = \sqrt{\frac{\Phi(t_0)}{\Phi(t)}}\,(t_0 - t)\,. \tag{26}$$

Let us, for cosmetics, trade an integration on t by an integration on z:

$$t_0 - t = \int_t^{t_0} dt = -\int_0^z \frac{dz}{dz/dt} = \int_0^z \frac{2\Phi^{3/2}}{\Phi'}dz\,. \tag{27}$$

This leads us to the relationship between the luminosity-distance and redshift that our cosmological model in veiled General Relativity predicts:

$$D = (1+z)\int_0^z \frac{dz}{H}\,, \tag{28}$$

where $H \equiv \Phi'/(2\Phi^{3/2})$ must be expressed in terms of $z = \sqrt{\frac{\Phi(t_0)}{\Phi(t)}} - 1$ after integration of the equations of motion (19) for Φ.

Now, in General Relativity, that is, in the "unveiled frame", $ds^2 = a^2\,d\bar{s}^2$ with $a = \sqrt{\Phi}$, where matter is minimally coupled to the metric $g_{\mu\nu} = a^2\eta_{\mu\nu}$, H is nothing but the "Hubble parameter":

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[8] This (crucial) coupling of the inertial masses to the scalar field Φ is forgotten in some papers, see e.g. [14], which hence (wrongly) conclude to the inequivalence of the Jordan and Einstein frames.

$$H \equiv \frac{\Phi'}{2\Phi^{3/2}} = \frac{a'}{a^2} = \frac{1}{a}\frac{da}{d\tau}\,, \tag{29}$$

with $d\tau \equiv a\,dt$. Moreover the equations of motion (19) for Φ are identical to the standard Friedmann-Lemaître equations,

$$3H^2 = 8\pi\rho\,, \quad \dot{\rho} + 3H(\rho + p) = 0\,, \tag{30}$$

with $\rho \equiv \bar{\rho}/\Phi^2$ and $p \equiv \bar{p}/\Phi^2$ (see Appendix). Finally, the text-book derivation of the relation luminosity-distance versus redshift yields (28). Therefore the predicted relationship between the observables z and D is the same, whether we represent gravity by a curved Robertson-Walker metric $g_{\mu\nu} = a^2\eta_{\mu\nu}$ minimally coupled to matter as in General relativity, or by a flat metric $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ together with a scalar field Φ coupled to matter, in its "veiled" version.

The physical *interpretation* of (28) is however different. Indeed, in the particular version of veiled General Relativity that we considered here:

- The evolution of the universe is not interpreted by cosmic expansion. Since we chose $\Phi = a^2$ there is in fact no cosmic expansion at all: $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$; but we defined on this flat manifold a scalar field Φ which evolves in time and describes the interaction of gravity and matter.
- There is no redshifting of photons, since the frequency of an atomic transition *measured* at P is equal to the frequency of that same transition as *observed* at P_0 ($\bar{\nu} = \nu_0$).
- However the interaction of Φ with matter implies that the mass $\bar{m}$ of the electron varies in time ($\bar{m} = \sqrt{\Phi}\,m = am$). Therefore the frequency of an atomic transition as measured in a lab there and then at P is not the same as the frequency measured here and now at P_0: $\bar{\nu} = \sqrt{\Phi(P)/\Phi(P_0)}\,\bar{\nu}_0 = (a/a_0)\bar{\nu}_0$. This redshifting due to a varying mass is exactly the same as the one due to a cosmological redshift in General Relativity.

Pursuing the above interpretation, the temperature of the cosmic microwave background can be considered constant, since photons are not redshifted, and chosen to be the present temperature $T_0 = 2.725$K, throughout the whole history of the universe (that is, during the whole time-evolution of the gravitational field Φ). The universe was in thermal equilibrium when the electron mass was smaller by a factor of more than 10^3 compared to the mass today, that is when the ground state binding energy of the hydrogen was less than 0.0136 eV. The "Big-Bang" is flat space at time $t = 0$ when the masses of the matter constituents are zero.

In conclusion, the above considerations show that the physical interpretation of the equations can be very different in General Relativity or its veiled versions, but the resulting relations between observables are completely independent of the conformal representation (or "frame") one chooses.

4 Conformal equivalence in local gravity

We shall see here that the tests of General Relativity in the Solar System (gravitational redshift, bending of light, perihelion advance, Shapiro effect...) can just as well be constructed using veiled General Relativity.

For definiteness let us describe the gravitational field of the Sun by the Schwarzschild solution of the vacuum Einstein equations written in Droste coordinates $x^\mu = (t, r, \theta, \phi)$,

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = -(1-2M/r)\, dt^2 + \frac{dr^2}{1-2M/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \quad (31)$$

where M is the (active) gravitational mass of the Sun. The propagation of light and the motion of planets in the Solar System are represented by (null) geodesics of this Schwarzschild spacetime. Proper time as measured in, say, Planck units, by a clock travelling in the Solar System is represented by the length of its timelike worldline $x^\mu(\lambda)$, that is, by the number $\tau = \int \sqrt{-g_{\mu\nu} dx^\mu dx^\nu}$.

Let us now introduce the following "veiled" Schwarzschild line element $ds^2 = \Phi\, d\bar{s}^2$ with $\Phi = 1 - 2M/r$ so that

$$d\bar{s}^2 \equiv \bar{g}_{\mu\nu} dx^\mu dx^\nu = -dt^2 + \frac{dr^2}{(1-2M/r)^2} + \frac{r^2}{1-2M/r}(d\theta^2 + \sin^2\theta d\phi^2)\,, \quad (32)$$

which solves the "veiled" vacuum Einstein equations (6) (we shall restrict our attention to the region outside the horizon, $r > 2M$).[9]

Light follows the null geodesics of $\bar{g}_{\mu\nu}$ which are the same as those of $g_{\mu\nu}$. Therefore the prediction for the bending of light is the same as in General Relativity.

Test particles do not follow geodesics of $\bar{g}_{\mu\nu}$ and their equation of motion is given by (13) (with $q = 0$). However this equation is just a rewriting of the geodesic equation in the metric $g_{\mu\nu}$. Therefore the trajectories $r = r(\phi)$ in the equatorial plane $\theta = \pi/2$ are the same in both General Relativity and its veiled version. The prediction for, say, the perihelion advance of Mercury is hence the same.

Consider now an atom at rest at r and an observer at rest at r_0. Since t is proper time, the frequency ν_0 of an atomic transition, as observed at r_0, will be the same as the frequency $\bar{\nu}$ measured at r: $\bar{\nu} = \nu_0$. However, in close analogy with the cosmological case, since the mass of the electron undergoing this transition depends on r as (12), $\bar{m} = \sqrt{\Phi(r)}\, m$, $\bar{\nu}$ is related to the frequency $\bar{\nu}_0$ of the transition as measured at r_0 by (16): $\bar{\nu} = \bar{\nu}_0 \sqrt{\Phi(r)/\Phi(r_0)}$. Hence the gravitational redshift is predicted to be

[9] One may wonder if the line elements $d\bar{s}^2 = ds^2/\Phi$, ds^2 being the Schwarzschild solution, are the only solutions of the "veiled" vacuum Einstein equations (which, beware, are not $\bar{G}_{\mu\nu} = 0$!). The answer is yes since, by construction, these equations are undetermined and Φ can be chosen at will to solve them. One then chooses $\Phi = 1$ and invokes the uniqueness of the Schwarzschild solution.

$$z \equiv \frac{\bar{\nu}_0}{\nu_0} - 1 = \sqrt{\frac{\Phi(r_0)}{\Phi(r)}} - 1 = \sqrt{\frac{1-2M/r_0}{1-2M/r}} - 1\,, \tag{33}$$

which is exactly the same as the prediction of General Relativity.

Finally let us consider predictions for the tests of General Relativity relying on time measurements (such as the Shapiro effect, GPS,...). In veiled General Relativity, the proper time interval $d\bar{\tau} = \sqrt{-\bar{g}_{\mu\nu}\,dx^\mu dx^\nu}$ between two adjacent events $x^\mu = (t,r,\theta,\phi)$ and $x^\mu + dx^\mu$ differs from that of General Relativity $d\tau$: $d\bar{\tau} = d\tau/\sqrt{\Phi}$. However, if we recall that time measurements are based on atomic clocks, that is, time intervals are counted in units of a frequency of an atomic transition, we readily find that the observed number of 'ticks' will be the same,

$$N_{\text{ticks}} = \bar{\nu}\,d\bar{\tau} = \sqrt{\Phi}\,\nu\,\frac{d\tau}{\sqrt{\Phi}} = \nu\,d\tau\,, \tag{34}$$

where $\bar{\nu}$ and ν are the frequencies of an atomic transition defined in veiled and unveiled General Relativity, respectively. Thus predictions for all the time measurements in veiled General Relativity again exactly agree with those in General Relativity.

5 Conclusion

In 1912 Nordström proposed a theory where gravity was represented by a scalar field Φ on Minkowski spacetime with metric $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$. Of course, matter was non-minimally coupled to that field, so that its interaction to gravity be described (see e.g. [16]). In 1914 Einstein and Fokker introduced a conformally flat metric $g_{\mu\nu} = \Phi\,\eta_{\mu\nu}$ which turned Nordström's equation of motion of test particles into the geodesic equation of the metric $\boldsymbol{g}$. Hence matter was minimally coupled to $\boldsymbol{g}$. As for the Klein-Gordon field equation for Φ it became an equation relating the scalar curvature of $\boldsymbol{g}$ to the trace of the stress-energy tensor of matter. It was clear (at least to Einstein and Fokker !) that the two versions of the theory were strictly equivalent, Nordström's formulation being the "veiled" one. And if Nordström's theory was soon abandoned it was not because it had been formulated first in flat spacetime but because its predictions (deduced either from its "veiled" or "unveiled" formulations) were in contradiction with observations.

In this paper we did nothing more than what Einstein and Fokker did in 1914 but applied the idea to General Relativity itself, in order to show, in a hopefully clear way, that, even if the description of phenomena could be different in General Relativity and in its conformally related sister theories, the predictions for the relationships between (classical) observables were strictly the same.

It should then become obvious that the same conclusion holds too when dealing with extensions of General Relativity such as $f(R)$ theories, coupled quintessence or, more generally, scalar-tensor theories (even if the scalar field Φ is then truly dynamical): the Jordan frame, where matter is minimally coupled to the metric, and the Einstein frame, where the action for gravity is Hilbert's, are equivalent, mathematically and physically, at least when dealing with classical phenomena and the motion of objects which are weakly gravitationally bound. Preferring to interpret the phenomena in the Jordan frame is somewhat similar to preferring to work in an inertial frame in Special Relativity: this allows to forget about the spacetime dependence of the inertial mass of the matter constituents just like one can forget about inertial forces in an inertial frame.

This analogy between inertial forces and non-minimal couplings points to quantum phenomena where the equivalence between the Jordan and Einstein frames may not hold.

Another point which deserves further investigation is the equivalence of conformally related frames when it comes to the motion of compact bodies whose gravitational binding energy is significant. It is known for example that a small black hole follows a geodesic in General Relativity [17]. In scalar tensor theories weakly gravitating bodies follow geodesics of the Jordan frame metric (to which matter is minimally coupled) but small black holes follow geodesics of the Einstein metric, see [18] and e.g. [19]. How this result, which is interpreted as a violation of the Strong Equivalence Principle, can be obtained using the Jordan frame exclusively remains to be elucidated. (see [20] for a recent study on the motion of small bodies in covariant classical field theories).

Acknowledgements N.D. thanks the Yukawa Institute for its hospitality and financial support during the YKIS2010 Symposium and Gravity and Cosmology 2010 Workshop. The work of M.S. is supported in part by Korea Institute for Advanced Study under the KIAS Scholar program, by the Grant-in-Aid for the Global COE Program "The Next Generation of Physics, Spun from Universality and Emergence" from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan, by JSPS Grant-in-Aid for Scientific Research (A) No. 21244033, and by JSPS Grant-in-Aid for Creative Scientific Research No. 19GS0219.

Appendix

We considered in the main text the example of matter being an electron in the field of an infinitely massive proton.

As a second example, consider matter to be a massive scalar field ψ with action,

$$S_m^{[\xi]} = -\frac{1}{2}\int d^4x\sqrt{-g}\left[(\partial\psi)^2 + \left(m^2 + \frac{\xi}{6}R\right)\psi^2\right]. \tag{35}$$

When $\xi = 0$ and $\xi = 1$, its extremisation with respect to ψ yields the familiar Klein-Gordon equations which read, respectively,

$$\Box\psi - m^2\psi = 0\,, \qquad \Box\psi - \left(m^2 + \frac{R}{6}\right)\psi = 0\,. \tag{36}$$

As for the veiled versions of (35), they are, respectively,

$$S_m^{[0]} = -\frac{1}{2}\int d^4x\sqrt{-\bar{g}}\,\Phi\left[(\bar{\partial}\psi)^2 + \bar{m}^2\psi^2\right],$$
$$S_m^{[1]} = -\frac{1}{2}\int d^4x\sqrt{-\bar{g}}\left[(\bar{\partial}\bar{\psi})^2 + \left(\bar{m}^2 + \frac{\bar{R}}{6}\right)\bar{\psi}^2\right] \quad (37)$$

where $g_{\mu\nu} = \Phi\bar{g}_{\mu\nu}$, $\bar{m} = \sqrt{\Phi}m$, and where $\bar{\psi} = \sqrt{\Phi}\,\psi$ in $S_m^{[1]}$. The extremisations of $S_m^{[0]}$ with respect to ψ and of $S_m^{[1]}$ with respect to $\bar{\psi}$ yield, respectively,

$$\bar{\Box}\psi - \bar{m}^2\psi = -\bar{\partial}\psi\cdot\frac{\bar{\partial}\Phi}{\Phi}, \qquad \bar{\Box}\bar{\psi} - \left(\bar{m}^2 + \frac{\bar{R}}{6}\right)\bar{\psi} = 0 \quad (38)$$

which are nothing but a rewriting of equations (36). In locally Minkowskian coordinates X^μ in the neighbourhood of some point P where $\bar{g}_{\mu\nu} \to \eta_{\mu\nu}$ and where Φ is approximately constant they reduce to their Special Relativistic forms, where the mass of the field has to be rescaled: $m \to \bar{m} = m\sqrt{\Phi(P)}$.[10]

In the case $\xi = 0$ the coupling of ψ to Φ can be globally effaced, by returning to the metric $\boldsymbol{g}$. In the case $\xi = 1$ the Klein-Gordon equation is, as is well-known, conformally invariant.

In the conformal invariant case, one might be confused by the fact that the stability of the field ψ depends on the sign of $(m^2 + R/6)$, while one can easily change its sign by a conformal transformation. This seemingly paradoxical situation is resolved by investigating more carefully the relation between the field in two different conformal frames.

As an example, let us consider the case when $g_{\mu\nu} = \eta_{\mu\nu}/(H\eta)^2$ is the (expanding part of) de Sitter metric (with $-\infty < \eta < 0$), and $m^2 < 0$ but $m^2 + R/6 = m^2 + 2H^2 > 0$, so that the field ψ is stable. Now consider the conformal transformation to the frame $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$. Then we have $\bar{m}^2 = m^2/(H\eta)^2 < 0$. Thus the field is badly unstable because the mass-squared is not only negative but diverges at $\eta = 0$. However if we recall that $\bar{\psi} = (-H\eta)^{-1}\psi$, this instability is solely due to the ill behaviour of the conformal factor as $\eta \to -0$.

Now let us consider a converse case when $g_{\mu\nu} = \eta_{\mu\nu}$ and $m^2 < 0$, so that the field ψ is unstable: $\psi \propto e^{|m|\eta}$ diverges exponentially. Turn now to the expanding de Sitter frame $\bar{g}_{\mu\nu} = \eta_{\mu\nu}/(H\eta)^2$, with $-\infty < \eta < 0$. Then the effective mass-squared $\bar{m}^2 + \bar{R}/6 = m^2/(H\eta)^2 + 2H^2$ will eventually become positive as $\eta \to -0$, hence the field must be stable in the expanding de Sitter frame. This seeming paradox can be resolved by noting the fact that $\bar{\psi} = (-H\eta)\psi \propto H\eta e^{|m|\eta}$. Thus time is bounded from above at $\eta = 0$, and hence there is literally 'no time' for the instability to develop.[11]

[10] Note that the same rescaling of mass occurs in a conformal transformation of the Dirac equation, see e.g. [15].

[11] We thank Andrei Linde for raising this issue.

As a last example consider matter to be a perfect fluid. Its stress-energy tensor and equations of motion, deduced from their special relativistic expressions by the replacement $\eta_{\mu\nu} \to g_{\mu\nu}$ are

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}\,, \qquad D_j T^{\mu\nu} = 0 \tag{39}$$

where ρ and p are the energy density and pressure of the fluid measured in a local inertial frame and where u^i is its 4-velocity normalised as $g_{\mu\nu}u^i u^j = -1$. Now, since $T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}$ (where we need not specify S_m), we have

$$\bar{T}_{\mu\nu} \equiv -\frac{2}{\sqrt{-\bar{g}}}\frac{\delta S_m}{\delta \bar{g}^{\mu\nu}} = \Phi T_{\mu\nu}\,, \tag{40}$$

so that the “veiled” version of (39) is, cf (7),

$$\bar{T}_{\mu\nu} = (\bar{\rho} + \bar{p})\bar{u}_\mu \bar{u}_\nu + \bar{p}\,\bar{g}_{\mu\nu}\,, \qquad \bar{D}_\nu \bar{T}^{\mu\nu} = \frac{\bar{\partial}^\mu \Phi}{2\Phi}\,\bar{T}\,, \tag{41}$$

where $\bar{g}_{\mu\nu}\bar{u}^i\bar{u}^j = -1$, and with $\bar{\rho} = \Phi^2\rho$ and $\bar{p} = \Phi^2 p$.[12]

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Finite-Time Singularities in Modified $\mathscr{F}(R,G)$-Gravity and Singularity Avoidance

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Abstract We study finite-time future singularities in $\mathscr{F}(R,G)$-gravity, where R and G are the Ricci scalar and the Gauss-Bonnet invariant, respectively. In particular, we reconstruct the $F(G)$-gravity and $\mathscr{F}(R,G)$-gravity models realizing the finite-time future singularities. We discuss a possible way to cure the finite-time future singularities in $\mathscr{F}(R,G)$-gravity by taking into account higher-order curvature corrections or effects of viscous fluids.

1 Introduction

Among the possible alternatives in order to explain the Dark Energy Issue, are the so called Modified Theories of Gravity[1, 2].

We would like to consider modified $\mathscr{F}(R,G)$-gravity, where the action is described by a function of the Ricci scalar R and the Gauss-Bonnet invariant $G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\xi\sigma}R^{\mu\nu\xi\sigma}$.

Many of modified gravity models bring the future universe evolution to finite-time singularities. Some of these singularities are softer than other and not all physical quantities (scale factor, effective energy density and pressure) necessarily diverge at this finite future time. Note that singular solutions correspond to accelerated universe, and often appear as the final evolution of unstable de Sitter space.

The presence of finite-time singularities may cause serious problems in the black holes or stellar astrophysics[3]. Thus, it is of some interest to explore the $\mathscr{F}(R,G)$-gravity models realizing singularities and if any natural scenario to cure such singularities exists.

We use units of $k_{\mathrm{B}} = c = \hbar = 1$ and denote the gravitational constant $8\pi G_N$ by $\kappa^2 \equiv 8\pi/{M_{\mathrm{Pl}}}^2$ with the Planck mass $M_{\mathrm{Pl}} = G_N^{-1/2} = 1.2 \times 10^{19}\mathrm{GeV}$.

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_24,

2 The model

The action of $\mathscr{F}(R,G)$-gravity is given by

$$S=\int d^4x\sqrt{-g}\left[\frac{\mathscr{F}(R,G)}{2\kappa^2}+\mathscr{L}_{\text{matter}}\right], \tag{1}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$ and $\mathscr{L}_{\text{matter}}$ is the matter Lagrangian. The spatially-flat FRW space-time is described by the metric

$$ds^2=-dt^2+a^2(t)d\mathbf{x}^2\,, \tag{2}$$

where $a(t)$ is the scale factor of the universe.

From the action in (1), the FRW-equations of motion (EOM) are derived as

$$\rho_{\text{eff}}=\frac{3}{\kappa^2}H^2\,,\quad p_{\text{eff}}=-\frac{1}{\kappa^2}\left(2\dot{H}+3H^2\right), \tag{3}$$

where ρ_{eff} and p_{eff} are the effective energy density and pressure of the universe, respectively, and these are defined as

$$\rho_{\text{eff}}=\frac{1}{\mathscr{F}'_R}\left\{\rho+\frac{1}{2\kappa^2}\left[\left(\mathscr{F}'_R R-\mathscr{F}\right)-6H\dot{\mathscr{F}}'_R+G\mathscr{F}'_G-24H^3\dot{\mathscr{F}}'_G\right]\right\}, \tag{4}$$

$$p_{\text{eff}}=\frac{1}{\mathscr{F}'_R}\left\{p+\frac{1}{2\kappa^2}\left[-\left(\mathscr{F}'_R R-\mathscr{F}\right)+4H\dot{\mathscr{F}}'_R+2\ddot{\mathscr{F}}'_R-G\mathscr{F}'_G\right.\right.$$
$$\left.\left.+16H\left(\dot{H}+H^2\right)\dot{\mathscr{F}}'_G+8H^2\ddot{\mathscr{F}}'_G\right]\right\}. \tag{5}$$

Here, $H=\dot{a}(t)/a(t)$ is the Hubble parameter and the dot denotes the time derivative. ρ and p are the energy density and pressure of matter, whereas $\mathscr{F}'_R=\partial_R\mathscr{F}(R,G)$ and $\mathscr{F}'_G=\partial_G\mathscr{F}(R,G)$. For general relativity with $\mathscr{F}(R,G)=R$, $\rho_{\text{eff}}=\rho$ and $p_{\text{eff}}=p$ and therefore (4) and (5) are the Friedmann equations. Consequently, (4) and (5) imply that the contribution of modified gravity can formally be included in the effective energy density and pressure of the universe.

3 Finite-time future singularities

We consider the case in which the Hubble parameter is expressed as

$$H=\frac{h}{(t_0-t)^\beta}+H_0\,, \tag{6}$$

where h, t_0 and H_0 are positive constants and $t<t_0$ because it should be for expanding universe. β is a positive constant or a negative non-integer number, so that,

when t is close to t_0, H or some derivative of H and therefore the curvature become singular.

Such choice of Hubble parameter corresponds to accelerated universe, becouse if (6) is a solution of the EOM (3), it is easy to see that the strong energy condition ($\rho_{\text{eff}} + 3p_{\text{eff}} \geq 0$) is always violated when $\beta > 0$, or is violated for small value of t when $\beta < 0$. It means that in any case the singularity could emerge as final evolution of accelerated universe.

The finite-time future singularities can be classified in the following way[4]:

- Type I and Big Rip. It corresponds to $\beta > 1$ and $\beta = 1$. H and R ($\sim H^2$) diverge.
- Type II (sudden). It corresponds to $-1 < \beta < 0$. R ($\sim \dot{H}$) diverges.
- Type III. It corresponds to $0 < \beta < 1$. H and R ($\sim \dot{H}$) diverge.
- Type IV. It corresponds to $\beta < -1$ but β is not any integer number. Some derivative of H and therefore the curvature becomes singular.

We note that in the present paper, we call singularities for $\beta = 1$ and those for $\beta > 1$ as the "Big Rip" singularities and the "Type I" singularities, respectively.

4 Reconstruction method

In order to study the finite-time singularities in $\mathscr{F}(R,G)$-gravity, we use the reconstruction method.

We assume that the contribute of ordinary matter and radiation in expanding singular universe is too small with respect to the modified gravity, and we study the pure gravitational action of $\mathscr{F}(R,G)$-gravity, i.e., the action in (1) without $\mathscr{L}_{\text{matter}}$. In this case, it follows from (4) and (5) that the EOM of $\mathscr{F}(R,G)$-gravity are given by[6]:

$$24H^3\dot{\mathscr{F}}'_G + 6H^2\mathscr{F}'_R + 6H\dot{\mathscr{F}}'_R + (\mathscr{F} - R\mathscr{F}'_R - G\mathscr{F}'_G) = 0\,, \tag{7}$$

$$\begin{aligned} &8H^2\ddot{\mathscr{F}}'_G + 2\ddot{\mathscr{F}}'_R + 4H\dot{\mathscr{F}}'_R + 16H\dot{\mathscr{F}}'_G(\dot{H} + H^2) \\ &\qquad + \mathscr{F}'_R(4\dot{H} + 6H^2) + \mathscr{F} - R\mathscr{F}'_R - G\mathscr{F}'_G = 0\,. \end{aligned} \tag{8}$$

In the case of pure gravity, these two equations are linearly dependents.

Moreover, we have

$$R = 6\left(2H^2 + \dot{H}\right)\,, \quad G = 24H^2\left(H^2 + \dot{H}\right)\,. \tag{9}$$

It is easy to see that, in the case of Type I, II and III singularities, G and R tend to infinitive when $t \to t_0$ in Equation (6), and in the case of Type IV singularities tend to zero.

By using proper functions $Z(t)$, $P(t)$ and $Q(t)$ of a scalar field which is identified with the cosmic time t, we can rewrite the action in (1) without $\mathscr{L}_{\text{matter}}$ to

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}\,(Z(t)R + P(t)G + Q(t))\,. \tag{10}$$

By the variation with respect to t, we obtain

$$Z'(t)R+P'(t)G+Q'(t)=0\,, \tag{11}$$

from which in principle it is possible to find $t=t(R,G)$. Here, the prime denotes differentiation with respect to t. By substituting $t=t(R,G)$ into (10), we find the action in terms of $\mathscr{F}(R,G)$

$$\mathscr{F}(R,G)=Z(R,G)R+P(R,G)G+Q(R,G)\,. \tag{12}$$

We describe the scale factor as

$$a(t)=a_0\exp(g(t))\,, \tag{13}$$

where a_0 is a constant and $g(t)$ is a function of time. By using the (7), (8) and (13), the matter conservation law $\dot{\rho}+3(\rho+p)$ and then neglecting the contribution from matter, we get the differential equation

$$Z''(t)+4\dot{g}^2(t)P''(t)-\dot{g}(t)Z'(t)+(8\dot{g}\ddot{g}-4\dot{g}^3(t))P'(t)+2\ddot{g}(t)Z(t)=0\,, \tag{14}$$

where the Hubble parameter is $H(t)=\dot{g}(t)$. By using (7), $Q(t)$ becomes

$$Q(t)=-24\dot{g}^3(t)P'(t)-6\dot{g}^2(t)Z(t)-6\dot{g}(t)Z'(t)\,. \tag{15}$$

It means that in principle, by solving (14) on the singular solution (6), it is possible to reconstruct $\mathscr{F}(R,G)$ producing finite-time future singularities.

In general, if $Z(t)\neq 0$, $\mathscr{F}(R,G)$ can be written in the following form:

$$\mathscr{F}(R,G)=Rg(R,G)+f(R,G)\,, \tag{16}$$

where $g(R,G)\neq 0$ and $f(R,G)$ are generic functions of R and G. From (7) and (8), we obtain

$$\begin{aligned}\rho_{\text{eff}} = -\frac{1}{2\kappa^2 g(R,G)}\Big[&24H^3\dot{\mathscr{F}}'_G+6H^2\left(R\frac{dg(R,G)}{dR}+\frac{df(R,G)}{dR}\right)+6H\dot{\mathscr{F}}'_R\\ &+(\mathscr{F}-R\mathscr{F}'_R-G\mathscr{F}'_G)\Big]\,,\end{aligned} \tag{17}$$

$$\begin{aligned}p_{\text{eff}} = \frac{1}{2\kappa^2 g(R,G)}\Big[&8H^2\ddot{\mathscr{F}}'_G+2\ddot{\mathscr{F}}'_R+4H\dot{\mathscr{F}}'_R+16H\dot{\mathscr{F}}'_G(\dot{H}+H^2)\\ &+\left(R\frac{dg(R,G)}{dR}+\frac{df(R,G)}{dR}\right)(4\dot{H}+6H^2)+\mathscr{F}-R\mathscr{F}'_R-G\mathscr{F}'_G\Big]\,,\end{aligned} \tag{18}$$

where ρ_{eff} and p_{eff} are given by the expressions in (3).

The modification of gravity could be included into the Equation of State (EoS) of an inhomogeneus dark fluid with energy density ρ_{eff} and pressure p_{eff}

$$p_{eff} = \omega\rho_{eff} + \mathscr{G}(H,\dot{H}...)\,, \tag{19}$$

where ω is the constant EoS parameter of matter and $\mathscr{G}(H,\dot{H}...)$ is a viscosity term given by

$$\begin{aligned}\mathscr{G}(H,\dot{H}...) = \frac{1}{2\kappa^2 g(R,G)}\Big\{&(1+\omega)(\mathscr{F} - R\mathscr{F}'_R - G\mathscr{F}'_G)\\ &+\left(R\frac{dg(R,G)}{dR} + \frac{df(R,G)}{dR}\right)\left[6H^2(1+\omega)+4\dot{H}\right]\\ &+H\dot{\mathscr{F}}'_R(4+6\omega) + 8H\dot{\mathscr{F}}'_G\left[2\dot{H}+H^2(2+3\omega)\right] + 2\ddot{\mathscr{F}}'_R + 8H^2\ddot{\mathscr{F}}'_G\Big\}\end{aligned} \tag{20}$$

The use of this equation requires that $g(R,G) \neq 0$ on the solution.

By combining the two equations in (3), we obtain

$$\mathscr{G}(H,\dot{H}...) = -\frac{1}{\kappa^2}\left[2\dot{H} + 3(1+\omega)H^2\right]\,, \tag{21}$$

where (19) has been used.

5 Singularities in $\mathscr{F}(R,G)$-gravity

Big Rip singularity in $F(G)$-gravity

As a simple example of reconstruction method, we examine the Big Rip singularity in Gauss-Bonnet $F(G)$-gravity, where $\mathscr{F}(R,G) = R + F(G)$ and $F(G)$ is a function of Gauss-Bonnet invariant only. In this case, by putting $Z(t) = 1$, the action of (10) can be written in terms of two proper functions $P(t)$ and $Q(t)$ and the variation with respect to t yields

$$P'(t)G + Q'(t) = 0\,, \tag{22}$$

from which we can find $t = t(G)$ and the action in terms of R and $F(G)$

$$F(G) = P(G)G + Q(G)\,. \tag{23}$$

Equations (14) and (15) read

$$2\frac{d}{dt}\left(\dot{g}^2(t)\frac{dP(t)}{dt}\right) - 2\dot{g}^3(t)\frac{dP(t)}{dt} + \ddot{g}(t) = 0\,, \tag{24}$$

$$Q(t) = -24\dot{g}^3(t)\frac{dP(t)}{dt} - 6\dot{g}^2(t)\,. \tag{25}$$

For the Big Rip singularity, $\beta = 1$ in (6). If we assume $H_0 = 0$ (the constant is negligible in the asyptotic singular limit $t \to t_0$), $\dot{g}(t) = h/(t_0 - t)$ and the most general solution of (24) is given by

$$P(t)=\frac{1}{4h(h-1)}(2t_0-t)t+c_1\frac{(t_0-t)^{3-h}}{3-h}+c_2\,, \tag{26}$$

where c_1 and c_2 are generic constants. From (25), we get

$$Q(t)=-\frac{6h^2}{(t_0-t)^2}-\frac{24h^3\left[\frac{(t_0-t)}{2h(h-1)}-c_1(t_0-t)^{2-h}\right]}{(t_0-t)^3}\,. \tag{27}$$

Furthermore, from (22) we obtain t in terms of G and, by solving (23), we find the most general form of $F(G)$ which realizes the Big Rip singularity

$$F(G)=\frac{\sqrt{6h^3(1+h)}}{h(1-h)}\sqrt{G}+c_1G^{\frac{1+h}{4}}+c_2G\,. \tag{28}$$

This is an exact solution of the EOM in the case of Big Rip. The term c_2G is a topological invariant. In general, if for large values of G, $F(G)\sim\alpha G^{1/2}$, where $\alpha(\neq 0)$ is a constant, the Big Rip singularity could appear for any value of $h\neq 1$. Note that $c_2G^{(1+h)/4}$ is an invariant with respect to the Big Rip solution.

Other types of singularities and more general $\mathscr{F}(R,G)$-gravity case

In a similar way, it is possible to reconstruct $F(G)$-gravity models in wich the other types of singularities could appear, when $\beta\neq 1$ in (6) and the scale factor, when $H_0=0$, behaves as

$$a(t)=\exp\left[\frac{h(t_0-t)^{1-\beta}}{\beta-1}\right]\,. \tag{29}$$

We give some results.

The asymptotic solution (in the limit $t\to t_0$) of $F(G)$ when $\beta>1$ is expressed as

$$F(G)=-12\sqrt{\frac{G}{24}}\,. \tag{30}$$

Hence, if for large values of G, $F(G)\sim-\alpha\sqrt{G}$ with $\alpha>0$, a Type I singularity could appear.

When $\beta<1$, the asymptotic solution of $F(G)$ becomes

$$F(G)\sim\alpha|G|^{\gamma}\,,\quad \gamma=\frac{2\beta}{3\beta+1}\,, \tag{31}$$

where α is a constant. If for large values of G, $F(G)$ has this form with $0<\gamma<1/2$, we find $0<\beta<1$ and a Type III singularity could emerge. If for $G\to-\infty$, $F(G)$ has the form in (31) with $-\infty<\gamma<0$, we find $-1/3<\beta<0$ and a Type II (sudden) singularity could appear. Moreover, if for $G \to 0^-$, $F(G)$ has the form in (31) with $1 < \gamma < \infty$, we obtain $-1 < \beta < -1/3$ and a Type II singularity could occur. If for $G \to 0^-$, $F(G)$ has the form in (31) with $2/3 < \gamma < 1$, we obtain $-\infty < \beta < -1$ and a Type IV singularity could appear. We also require that $\gamma \neq 2n/(3n-1)$, where n is a natural number.

As a consequence, a large class of realistic models of $F(G)$-gravity, which reproduce the current acceleration and the early-time inflation, could generate future time-singularities, as for example[7]:

$$F_1(G) = \frac{a_1 G^n + b_1}{a_2 G^n + b_2}\,, \quad F_2(G) = \frac{a_1 G^{n+N} + b_1}{a_2 G^n + b_2}\,, \quad F_3(G) = a_3 G^n (1 + b_3 G^m)\,. \tag{32}$$

All this models contain power functions of G and for some choices of parameters could produce singularities.

With reconstruction method it is possible to derive also more general $\mathscr{F}(R,G)$-models producing finite-time singularities. For example, in the model $\mathscr{F}(R,G) = R - \alpha G/R$, where α is a positive constant, could appear the Type I singularity, whereas in the model $\mathscr{F}(R) = R + \alpha R^\gamma$, where α and γ are constants, could appear Types II, III or IV singularities (for a review, see [5]).

6 Curing the finite-time future singularities

We discuss a possible way to cure the finite-time future singularities in $F(G)$-gravity and $\mathscr{F}(R,G)$-gravity. In the case of large curvature, the quantum effects become important and lead to higher-order curvature corrections. It is therefore interesting to resolve the finite-time future singularities with some power function of G or R.

We consider the description of modified gravity as inhomogeneous fluid. If some singularities occur, (21) behaves as

$$\mathscr{G}(H,\dot{H}...) \simeq -\frac{3(1+\omega)h^2}{\kappa^2}(t_0-t)^{-2\beta} + \frac{2\beta h}{\kappa^2}(t_0-t)^{-\beta-1}\,. \tag{33}$$

One way to prevent a singularity appearing could be that the function $\mathscr{G}(H,\dot{H}...)$ becomes inconsistent with the behavior of (33) in the singular limit ($t \to t_0$).

Let us consider a simple example in order to cure Big Rip singularity in $F(G)$-gravity. Suppose that for large values of G,

$$R + F(G \to \infty) \longrightarrow R + \gamma G^m\,, \quad m \neq 1\,, \tag{34}$$

with $\gamma \neq 0$. For $H = h/(t_0 - t)$, namely the Big Rip case, we have

$$\mathscr{G}(H,\dot{H}...) \simeq \frac{\alpha}{(t_0-t)^{4m}}\,. \tag{35}$$

Hence, if $m > 1/2$, $\mathscr{G}(H, \dot{H}...)$ tends to infinity faster than (33) and we avoid this kind of singularity.

As general results, we find that the term γG^m with $m > 1/2$ and $m \neq 1$ cure the singularities occurring when $G \to \pm\infty$ (Type I, II and III). Moreover, the term γG^m with $m \leq 0$ cure the singularities occurring when $G \to 0^-$ (Type II, IV).

In $f(R)$-gravity (namely, R plus a function of R), by using the term γR^m, the same consequences are found. The term γR^m with $m > 1$ cures the Type I, II and III singularities. The term γR^m with $m < 2$ cures the Type IV singularity.

Within the framework of $\mathscr{F}(R,G)$-gravity, we can use the terms such as G^m/R^n to cure the singularities. For example, we can avoid the Type I singularities if the asymptotic behavior of the model is given by $\gamma G^m R^n$, with $m, n > 0$.

7 Effects of viscous fluid in singular universe

As the last point, we explore the role of perfect/viscous fluids within singular modified gravity, investigating how the singularities may change or disappear, due to the contribution of quintessence or phantom fluids.

We consider the class of modified gravity $\mathscr{F}(R,G) = R + f(R,G)$, where $f(R,G)$ is a function of the Ricci scalar R and the Gauss-Bonnet invariant G, and we suppose the presence in the universe of cosmic viscous fluid, whose EoS is given by

$$p = \omega\rho - 3H\zeta\,, \tag{36}$$

where p and ρ are the pressure and energy density of fluid, respectively, and ω is the EoS parameter. ζ is the bulk viscosity and in general it could depend on ρ, but we will consider the simplest case of constant viscosity only (for more general cases, see [8]). On thermodynamical grounds, in order to have the positive sign of the entropy change in an irreversible process, ζ has to be a positive quantity.

The FRW-equations of motion are:

$$\rho_G + \rho = \frac{3}{8\pi G_N} H^2\,, \quad p_G + p = -\frac{1}{8\pi G_N}\left(2\dot{H} + 3H^2\right)\,. \tag{37}$$

The modified gravity is formally included into the modified energy density ρ_G and the modified pressure p_G, which correspond to (17)-(18) for $g(R,G) = 1$.

The fluid energy conservation law is a consequence of the EOM (37):

$$\dot{\rho} + 3H\rho(1+\omega) = 9H^2\zeta\,. \tag{38}$$

The presence of fluid could influence the behaviour of singular $f(R,G)$-models (i.e. models that in absence of fluids produce some singularities). We will check the solutions of the fluid energy density when H is singular.

Non viscous case

In the non-viscous case $\zeta = 0$ (perfect fluid), the solution of (38) assumes the classical form:

$$\rho = \rho_0 a(t)^{-3(1+\omega)}\,, \tag{39}$$

where ρ_0 is a positive constant and $a(t)$ is the scale factor of the universe. By combining (39) with (29), it is easy to see that for $\beta > 1$ (Type I singularity), ρ grows up and diverges exponentially if $\omega < -1$. In the presence of phantom fluid, the EOM (37) become inconsistent with respect to the singular form of Hubble parameter in (6), and the Type I singularity is not realized in $f(R,G)$-gravity.

When $0 < \beta < 1$, the fluid energy density ρ is avoidable on the Type III singular solution, whereas for Type II and IV singular models ($\beta < 0$), the presence of quintessence or phantom fluids makes the singularities worse. In particular, in the case of $\beta < -1$, the dynamical behaviour of (37) could become inconsistent, because ρ behaves as $(t_0 - t)$ and it is larger than the time-dependent part of H^2 ($\sim (t_0 - t)^{-\beta}$) when $t \to t_0$.

Constant viscosity

Suppose to have the bulk viscosity equal to a constant, $\zeta = \zeta_0$, and the Hubble parameter in the general form of (6). The asymptotic solutions of (38) in the singular limit $t \to t_0$ are:

$$\rho \simeq \frac{3h\zeta_0}{(1+\omega)(t_0 - t)^{\beta}}\,, \qquad \beta > 1\,, \tag{40}$$

$$\rho \simeq \frac{9\zeta_0 h^2}{(2\beta - 1)(t_0 - t)^{2\beta - 1}}\,, \qquad 1 > \beta > 0\,, \tag{41}$$

$$\rho \simeq \frac{9hH_0\zeta_0}{(\beta - 1)(t_0 - t)^{\beta - 1}} + \frac{3H_0\zeta_0}{1+\omega}\,, \quad 0 > \beta\,, H_0 \neq 0\,. \tag{42}$$

In the first and second cases ($\beta > 0$), it is possible to see that ρ diverges more slowly than H^2, so that viscous fluid does not influence the asymptotically behaviour of Types I and III singular models in (37), due to the constant viscosity.

In the third case, we consider fluids that tend to a non-negligible energy density when $\beta < 0$. It automatically leads to $H_0 \neq 0$ in (6) and ρ behaves as in (42). Large bulk viscosity ζ_0 becomes relevant in the EOM. Moreover, if $\omega < -1$, the effective energy density (namely, $\rho_G + \rho$) could be negative and avoid the Type II and IV singularities for expanding universe (where $H_0 > 0$).

8 Conclusion

We have investigated the finite-time future singularities in $F(G)$-gravity and $\mathscr{F}(R,G)$-gravity. We can reconstruct the $F(G)$-gravity and $\mathscr{F}(R,G)$-gravity models in which the singularities could appear. Note that all types of future-time singularities could appear in $\mathscr{F}(R,G)$-modified gravity. In addition, we have discussed a possible way to resolve the finite-time future singularities in $F(G)$-gravity and $\mathscr{F}(R,G)$-gravity under quantum effects of higher-order curvature corrections or the presence of perfect/viscous fluid in the universe.

Acknowledgements I wish to thank K. Bamba, S. D. Odintsov and S. Zerbini, whom the paper 'Finite-time future singularities in modified Gauss-Bonnet and $\mathscr{F}(R,G)$ gravity and singularity avoidance' published in Eur. Phys. J. C67 has been written with.

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Asymptotic Darkness in the Hořava-Lifshitz Gravity

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Abstract In this talk, We review the fundamental problem encountered to quantize General Relativity, in terms of quantum field theory. We describe the renormalization of Newton constant as an irrelevant coupling, the idea of Asymptotic Safety and its inviability due to the ultraviolet behavior of the theory, also known as "Asymptotic Darkness". Then, We introduce the Hořava-Lifshitz models of gravity, and argue how this new framework could bypass the above reasoning, due to the explicit breaking of Lorentz invariance that seems to implied the non-existence of BH horizons.

1 Introduction

General relativity (GR) describes successfully the gravitation of mater and energy in a wide range of scales from milliliters to cosmological distances. Since gravity affects all types of known particles, is the most universal interaction we have. Unfortunately, it has proven to be difficult to reconcile GR with theories of particle physics, that are naturally described in terms of quantum field theories (QFT). Basically all attempts to understand GR from a QFT perspective have failed, as we write this article.

There are many programs that try to quantize gravity, some have a long run, other failed soon, but at some point all these attempts have to face the fact that GR simply does not fulfil the minimal requirements to describe its ultraviolet-completion in terms of a standard QFT.

A standard QFT should be a local, Lorentz-invariant theory that follows the principles of quantum mechanics. These theories, are well defined as longs as they are

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_25,

in the UV regime, Gaussian theories that admit at most, perturbations by relevant operators. Any other situation, will take us outside the well understood grounds of QFT and therefore will require additional structures to define somehow another consistent scenarios.

That GR is not a the standard QFT is a well known fact, that nonetheless has not receive sufficient attention. The gravitational interaction is written in terms of Newton constant G_N and the metric g,

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} R\,, \tag{1}$$

Then, in four dimensions G_N has scaling dimension (with energy units) of -2. Based on standard arguments of renormalization group flow, applied to an expansion of the metric around flat space $g_{\mu\nu} = \eta_{\mu\nu} + 8\pi G_N h_{\mu\nu}$, is possible to show that we are in the presence of an irrelevant coupling. Therefore, it makes no sense to assign a finite value to the Newton coupling constant in the infrared regime (IR), to them try to extend such interaction to the ultraviolet regime (UV). In other words, perturvative quantization is not possible.

The above result, does not fully eliminates the existence of a QFT completion of GR. Provided we renounce to perturbative renormalizablility, we still could achieve this target. In other words, we still could have a quantum field theory that has a UV attractive fixed point, but now our UV theory will be non-free i.e. a non-Gaussian interacting theory. This UV-description would correspond to a conformal field theory (CFT) that among other things, has a entropy that scales with the energy as

$$S \sim E^{3/4} \tag{2}$$

Nevertheless, even if this would be the case of GR, we confront another serious problems related to the UV/IR mixing in GR that is normally described as "Asymptotic Darkness". To illustrate this idea, consider a physical process at very high center-of-mass energies. As soon as the energy is localized in a region smaller than its Schwarzschild radius, a Black Hole (BH) is produced. In fact, more generally, the collapse of any other sufficiently massive and small configuration to a black hole is inevitable due to the hoop theorem. Therefore, the spectrum of a quantum completion of GR should be dominated by BHs. If we increase the energy, instead of probing smaller distances, we increase the size of the resulting BHs. This striking connection between UV/IR physics predicts a spectrum dominated by dark objects in the UV.

A key observation on the nature of BHs, is that for this objects, the entropy only grows with the area. Therefore the dominant contribution to the entropy in the UV regime is given by BH states, and since the entropy is proportional to the area and not the volume, is simply not possible to have a UV description in terms of a four-dimensional CFT. In terms of the energy, for asymptotically flat space we have

$$S \sim E^2 \tag{3}$$

Therefore, based on the above scaling of the entropy in GR, we are ready to exclude the possibility that GR can be understood as the IR flow of a standard QFT.

We can only concluded then, that in order to quantize gravity some (if not all) of the basic axioms used in the above discussion has to be given up. Recalled that we have used up to now: locality, lorentz invariance, quantum mechanics and of course, our believe that GR is relevant theory of gravity in the IR regime.

2 Other avenues

That we have to abandon some of the above listed axioms should come as no surprise. The actual implementation of this change, is on the other hand, a much more adventurous enterprize.

An interesting possibility is that GR is nothing more than an effective field theory, that due to coarse graining effects has forgotten about its relation to the original UV degrees of freedom. This approach would put GR at the same level of thermodynamics or hydrodynamics. In this case the metric is a sort of composite operator that emerges only at low energies and therefore is not usefull to investigate about its UV constituents. In the same form that Pions are not useful to describe quark physics in QCD. In fact, is tempting to see the universality of the GR coupling to mater as a natural consequence of a sort of ”thermodynamic limit”. More explicitly, there are many results in the literature that support this point of view, like the calculation of Jacobson, where the field equations of GR can be recover from a ”thermodynamics of space-time”, based on generalizations of the more canonical BH thermodynamics. That a hydrodynamic approach in the AdS/CFT correspondence has successfully been used to describe GR perturbations around BH solutions. That in some mini-super-space models in AdS, the corresponding GR field equations have been reproduced by the thermodynamical limit of the dual CFT configurations, etc. A more extremist point of view has been consider by E. Verlinde, where n ot only gravity but our whole universe should be though as a thermodynamical limit of unknown underlying UV theory. Here, we will leave these ideas untouched, to focuss on other possibilities, since we believe that somehow, they just postpone the question of what are the relevant UV degrees of freedom completing GR.

Once we agree that the metric is the relevant field that describes gravitational interactions, we have no other option than abandon some of the axioms that characterize a QFT i.e. we have to give up locality and/or quantum mechanics and/or Lorentz invariance. In fact, one of the most promising theories to quantize gravity, string theory, is built-in with non-local interactions. within this framework, a good example of how string theory bypasses our previous conclusion is nicely seen in the case of the AdS/CFT duality. Here closed string theory in five dimensional AdS is holographically equivalent to a four dimensional CFT in flat space. since closed strings contain gravity among its fields, we are defining quantum gravity via duality to a CFT. In this case, is easy to see that the entropy of a AdS Bh matches the entropy of the dual CFT since in five dimensional AdS the BH entropy scales as $S \sim E^{3/4}$ matching the four dimensional CFT scaling.

Inthe above case, the holographic nature of the duality is ultimately what saves the day, and allows a UV description of our gravitational theory. Unfortunately, the above scenario gives only partial answers to the general question of quantizing GR, since we are tied to negative cosmological constant with important technical problems that have produce only limited results base on highly symmetric scenarios.

Instead of passing through a catalog of results and short comings of string theory, let us focus on a different possibility, corresponding to abandon Lorentz invariance once for all. In condensed mater field theory, it is not strange to find examples, realized in nature, where a system is described in the UV by a non relativistic field theory that nevertheless runs to the IR into a relativistic field theory. In these cases, Lorentz invariance is obtained only as an accidental symmetry and has no intrinsic axiomatic character. The above phenomena was studied by Lifshitz in connection to solid state physics, and has lately been implemented in a gravitational framework by Hořava. These class of gravitational theories that explicitly break Lorentz invariance are named Hořava-Lifshitz theories of Gravity (HL) and although its theoretical consistency is not clear as we write this talk, these theories open up the possibility of circumvent old problems with these new twist. Doe to space limitations, here we will concentrate only on the main characteristics of these theories that have implications to Asymptotic Darkness and Black Hole thermodynamics.

3 Hořava-Lifshitz gravity

Recently, Hořava made a proposal for an ultraviolet completion of GR normally referred to as the Hořava-Lifshitz (HL) theory [1]. The salient characteristic of the HL proposal is that it seems to be renormalizable, at least at the level of power counting. This ultraviolet behavior is obtained by introducing irrelevant operators that explicitly break Lorentz invariance but ameliorate the ultraviolet divergences. On the other hand, Lorentz invariance is expected to be recovered at low energies, as an accidental symmetry of the theory.

Originally, the HL proposal came with the possibility of imposing or not the so-called *projectability condition* and the *detailed balance condition*. The first condition is related to the space-time dependence of the lapse function, N, which characterizes a canonical $3+1$ decomposition of the metric field g, while the second is a restriction on the form of the potential terms which may appear in the Lagrangian that leads to simplifications since it reduces the final number of couplings. Notice therefore that we have, in principle, four different incarnations of this proposal.

Since its publication, the HL theory has been the object of an exhaustive research regarding its different properties and implications to space-time physics. In particular, a lot of attention has been paid to its internal consistency, to how to define the infrared limit, its compatibility with GR, and the potential application of the results obtained to cosmology.

Presently—as this talk is prepared—the consistency status of the theory is not completely clear, nor its low energy limit and, hence, how GR is recovered at the different regimes. In fact, for the non-projectable version, although the low energy limit has been found [2], there seem to remain important problems, regarding strongly coupled features that may preclude any type of perturbation approach [3, 4] that on the other hand, may be eliminated by the inclusion of a whole family of new operators that where overseen in the original proposal [5]. Also, imposing detailed balance leads to a cosmological constant with the *wrong sign*, from what we should expect to be at odds with cosmological observations [1, 9]. In comparison, the projectable version seems to be less problematic, since the above listed problems can in principle be evaded by the non-local form of the Hamiltonian constraint [8]. Also, if detail balance is not imposed a richer phenomenology seems to appear, where cosmological applications may lead to new results in inflation, bouncing cosmology, dark matter, and dark energy (see, for example, [9]-[20]).

At this point it is important to investigate the key aspects of the theory, which may help in clarifying the status of the different HL proposals as plausible candidates of a quantum theory of gravity. In particular is evident from our previous discussion that BH in HL theory are a key ingredient that need to be carefully analyzed.

3.1 The model

In the HL theory, the gravitational dynamical variables are defined to be the laps N, the shift N_i and the space metric g_{ij}, Latin indices running from 1 to 3. The space-time metric is defined using the ADM construction, as

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i)(dx^j + N^j)\,, \tag{4}$$

where $N^i = g^{ij}N_j$ as usual. The action S is written in terms of geometric objects, characteristics of the ADM slicing of space-time, like the 3d-covariant derivative ∇_i, the spatial curvature tensor R_{ijkl}, and the extrinsic curvature K_{ij}. In terms of the above tensor fields, the HL action can be written as

$$S = \int dt\, dx^3\, N\sqrt{g}\left(\mathscr{L}_{kinetic} - \mathscr{L}_{potential} + \mathscr{L}_{matter}\right)\,, \tag{5}$$

being the kinetic term universally given by

$$\mathscr{L}_{kinetic} = \alpha(K_{ij}K^{ij} - \lambda K^2)\,, \tag{6}$$

with α and λ playing the role of coupling constants. Originally, the potential term was a generic function of R_{ijkl} and ∇_i. Then, with the work of **??** it was realized that this generic function should also depend on $a_i = \nabla_i \ln(N)$.

The action generically brakes covariance down to the subgroup of 3-dimensional diffeomorphisms invariance and reparametrization of time i.e. $x \to \tilde{x}(t,x)$ and $t \to$

$\tilde{t}(t)$. Assigning dimension -1 to space and dimension -3 to time, it can be seen that it is enough to restrict the potential to be made out of operators up to dimension 6, to get a power counting renormalizable theory.

In general the potential can be written in a implicit form as

$$\mathscr{L}_{potential} = \sum \beta_n O^n(a_i, \nabla_j, R_{ijkl}) \tag{7}$$

where, O^n is a general operator of maximum dimension 6 and β_n are the corresponding coupling constants. Here, we just show some of the possible couplings to have an idea of the structure of the potential,

$$\begin{aligned} \mathscr{L}_{potential} = & \beta_8 \nabla_i R_{jk} \nabla^i R^{jk} + \beta_7 R \nabla^2 R + \beta_6 R^i_j R^j_k R^k_i + \beta_5 R(R_{jk}R^{jk}) \\ & + \beta_4 R^3 + \beta_3 R_{jk}R^{jk} + \beta_2 R^2 + \beta_1 R + \beta_0 + \beta a^i a_i + \ldots, \end{aligned} \tag{8}$$

So we have a candidate theory of quantum gravity that seems to be well define in perturbation theory since it is power counting renormalizabe. But, what is the status of the Asymptotic Darkness in this theory? Are the BHs as universal as in GR, do we have a well define notion of horizon, and Bh thermodynamics?

All the above questions are fundamental to understand the validity of HL approach to gravity, and ultimately the answers will reshape our understanding of this framework. In the next section we shortly give an overview of our current knowledge of this deep and fundamental questions, that presently, as we write this talk are not even fully explored. Therefore our presentation should be taken as a first step to study these subjects

3.2 BH and asymptotic darkness in HL

It is clear that we can look for spherically symmetric solutions that asymptotically have conserved charges like mass and angular momentum. Also, it has been shown in [22], that in the case we set cosmological constant to zero, HL theories are reducible to Einstein-Aether theory, in the case where the aether vector is hypersurface orthogonal. Therefore we known that in some form, BH should be present in the theory. In fact, spherically symmetric solutions of different kinds that depend on the particular incarnation of the HL theory chosen have been found already in the literature (see for example [23, 24, 25]).

In the simplest cases (like in [23]) with asymptotic flat metric adopts the form

$$\begin{aligned} ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ f(r) = 1 + wr^2 - \sqrt{r(w^2r^3 + 4wM)} \end{aligned} \tag{9}$$

where w depends on different couplings of the HL theory and M corresponds to the mass of the BH. These solutions contain BHs that in general have two different horizons if $wM^2 > 0.5$ and only one in if $wM^2 = 0.5$. There is a singularity in the center of the space at $r = 0$. This solutions approach Schwarzschild BH for $r >> (M/w)^{1/3}$.

Many articles have been written studying the thermodynamics properties of these BH and its related cousins (see for some of them [26]), but in the majority of these studies, it has been assumed that although the gravity sector breaks Lorentz invariance, the mater sector does not i.e. it has been used minimal coupling to investigate the phenomenology of this metrics, in particular Hawking radiation, horizon properties, and particle probing.

Nevertheless, in general we should allow for symmetry breaking of lorentz invariance in the mater sector, that will produce very different outcomes regarding the structure of these named BH. Just to name some consequences, the notion of where the horizon is located, may be particle dependent. This issue can be studied using the appropriate generalization of geodesic equations as in [27]-[30] . This also implies that the notion of Hawking temperature is particle-dependent, if particles have dispersive geodesics. On the top of the above, even the notion of BH thermodynamics becomes ill defined since the analogue of Hawking radiation depends on dispersion, and ceases to be thermal [31], [32].

Also, the actual formation of BH due to the physical process could be not possible. Simple BH in HL-gravity show repulsive forces near the inner singularity and therefore the density of mater is bounded from above. This behavior is related to bouncing solutions in the collapse of mater due to gravity. Nevertheless, to have a clear picture of the above more studies have to be undertaken.

From the above partial results, the emerging picture that seems to be appearing is that BHs are not well define objects in HL-gravity, where horizons are probe dependent and therefore information can scape to infinity. It looks like these BHs are more similar to a low-pass filters, that are black only at low energy but transparent a higher energies.

If this is the case, in this framework defined by all of the HL-gravities, there is no room for Asymptotic Darkness and therefore the main obstruction to understand the UV completion of GR as a QFT disappears. Of course, our QFT of gravity will be a non-relativistic theory of a Lifshitz type.

Acknowledgements We thank the organizers of the Workshop ”Cosmology, the Quantum Vacuum, and Zeta Functions”. This work was partly supported by CICYT-FEDER-FPA2008-01430, the “Universitat Autònoma de Barcelona” PR-404-01-2/08, FPU grant AP2007-00420, CSIC under the program Bolsas de trabajo.

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Testing Modified Gravity with Gravitational Wave Astronomy

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Abstract The emergent area of gravitational wave astronomy promises to provide revolutionary discoveries in the areas of astrophysics, cosmology, and fundamental physics. One of the most exciting possibilities is to use gravitational-wave observations to test alternative theories of gravity. In this contribution we describe how to use observations of extreme-mass-ratio inspirals by the future Laser Interferometer Space Antenna to test a particular class of theories: Chern-Simons modified gravity.

1 Introduction

Gravity, as we see it from our four-dimensional spacetime perspective, appears as the weakest of all physical interactions known to date. Despite this fact, it is the force that governs the large-scale structure of the universe. In the context of Einstein's theory of relativity, gravity also determines the spacetime geometry and hence the relations between the events that take place on it.

As is well known, Newtonian mechanics together with Newton's law of gravitation are sufficient to describe a wide range of phenomena governed by gravity, from the motion of objects near the surface of our planet Earth to the motion in the Solar system, and even at much larger scales. Relativistic effects show up when we make very precise observations of astronomical systems, and also of systems that involve either strong gravitational fields or fast motions. However, due to the weakness of gravity, these effects are difficult to measure with present technology and, as a consequence, only certain regimes of relativistic gravity have been tested so far. These tests confirm, to their level of precision, the validity of general relativity (GR) (see [1] for a review). These experimental tests include observations of the motion of

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_26,

different objects in the solar system and observations of millisecond binary pulsars. In the first case, a dimensionless measure of the Newtonian gravitational potential yields[1]

$$\frac{\Phi_{Newtonian}}{c^2} = \frac{GM_{\odot}}{c^2\, 1\mathrm{AU}} \sim 10^{-8}\,. \qquad (1)$$

In the second case [2], the same dimensionless measure yields

$$\frac{\Phi_{Newtonian}}{c^2} \sim \frac{GM_{\odot}}{c^2\, r^{\mathrm{periastron}}_{\mathrm{Hulse-Taylor}}} \sim 10^{-6}\,. \qquad (2)$$

We can compare these numbers with an estimation for the case of binary black holes (BHs) near the merger phase

$$\frac{\Phi_{Newtonian}}{c^2} \sim \frac{GM_{\mathrm{BH}}}{c^2\,(\mathrm{a\ few}\ r_{\mathrm{Horizon}})} \sim 10^{-1} - 1\,. \qquad (3)$$

This indicates that despite the accurate measurements achieved up to now, both from the solar system and from pulsars, there is still a long way to go until we can have observations of situations with truly strong gravitational fields. This means that there are sectors of the gravitational theory that we have not yet tested. In other words, we know that General Relativity correctly describes Nature in certain regimes, but do not know whether this is also the case when gravitational fields are extreme, in the sense of (3), where alternative theories of gravity might be relevant.

To access the gravitational regime not yet tested one can try to resort to electromagnetic observations (see [4] for a review) as new observatories in the high-energy end of the spectrum have good potential for such a goal. Another possibility is to resort to a different messenger, namely GWs, or a combination of electromagnetic and GW observations through multi-messenger astronomy in the future. Gravitational Wave Astronomy (GWA) is an emergent area that promises to bring revolutionary discoveries to the areas of astrophysics, cosmology, and fundamental physics. There are a number of ground laser interferometric detectors (LIGO [5], VIRGO [6], etc.) that will detect GWs, in the high frequency range ($f \sim 10 - 10^3$ Hz) during the next decade. There are also ongoing developments for future detectors in space, like the Laser Interferometer Space Antenna (LISA) [7] or DECIGO [8]. In particular, LISA will operate in the low frequency band ($f \sim 10^{-4} - 1$ Hz), a band not accessible from the ground due to seismic noise, and probably the richest band in terms of interesting astrophysical and cosmological sources.

Apart from these detectors, there is work in progress to use networks of millisecond pulsars to detect GWs in the ultra-low frequency range ($f \sim 10^{-9} - 10^{-8}$ Hz). Millisecond pulsars have already being used to test alternative theories of grav-

[1] Here, G denotes the gravitational Newton constant, c the speed of light, $M_{\odot}$ the mass of the Sun, and r different distance measures.

[2] We here choose data from the well-known Hulse and Taylor binary pulsar (PRS B1913+16) [2], the one that provided the first strong evidence for the existence of gravitational waves (GWs) [3].

ity, like scalar-tensor theories (see, e.g. [9]). These pulsars are remarkable stable rotators, and as such, they require only simple models to describe their spin-down and times-of-arrival (TOAs) with a precision of $< 1\ \mu$s over many years of observations. GWs are not included in the analysis, so their existence will induce differences between the measured and theoretical TOAs, the so-called *timing residuals*. To determine the exact origin of the timing residuals, that is, effects different from GWs (calibration effects, errors in planetary ephemeris, irregularities in the pulsar spin-down, etc.) it is necessary to correlate the timing residuals of multiple pulsars. It has been estimated that with a timely progress in technology, a successful detection of GWs should happen within a decade [10] (or alternatively the experiments will rule out current predictions for GW sources in this frequency band).

GWs are a double-edge tool: On the one hand, their detection is quite a difficult problem that requires very advanced technology. On the other hand, they are an ideal tool to test strong gravity (in the sense of 3), since GWs carry almost uncorrupted information from their sources. In the next sections we discuss the following points: (i) The basics of the planned LISA mission; (ii) The main properties of EMRIs and their GW emission; (iii) How to use EMRIs to test alternative theories of gravity with a focus on Dynamical Chern-Simons Modified Gravity (DCSMG). This discussion is based on work described in [11].

2 LISA: The laser interferometer space antenna

LISA [7] is a joint NASA-ESA mission designed to detect and analyze the gravitational radiation coming from astrophysical and cosmological sources in the low-frequency band (corresponding to oscillation periods in the range 10 s - 10 hours). LISA consists of three identical spacecrafts flying in a triangular constellation, with equal arms of $5 \cdot 10^6$ km each. As GWs from distant sources reach LISA, they warp space-time (locally generating curvature), stretching and compressing the triangle. Thus, by precisely monitoring the separation between the spacecrafts, we can measure the GWs, and their shape and timing teach us about the nature and evolution of the systems that emitted them.

The LISA constelation is in orbit around the Sun, at a plane inclined by 60 degrees to the ecliptic. The triangle appears to rotate once around its center in the course of a year's revolution around the Sun (see Fig. 1). The center of the LISA triangle traces an Earth-like orbit in the ecliptic plane, trailing Earth by 20 degrees. The free-fall orbits of the three spacecraft around the Sun maintain this triangular formation, with the triangles appearing to rotate about its center once per year.

The sensitivity of LISA as a GW observatory is described by the strength of its response to impinging GWs as a function of frequency. At low frequencies it is limited by acceleration noise; at mid frequencies by laser shot noise and optical-path measurement errors; and at high frequencies by the fact that the GW wavelength becomes shorter than the LISA arm length, reducing the efficiency of the interferometric measurement (see Fig. 2).

Whereas ground-based detectors will make the first detections in the high-frequency band and also inaugurate the field of GWA, space-based detectors like LISA, operating in the low-frequency band, will enable us to explore this new field in detail. In this band, there are several important sources of GWs (see Fig. 2), such as massive BH mergers, i.e., mergers of BHs that grow in galactic centers and become a binary after their host galaxies collide. Another LISA source of GWs are millions of galactic binaries, which form a GW foreground (except for the more bright ones, which can be separately identified as foreground). Yet another important source is the capture and inspiral of stellar-mass compact objects (white dwarfs, or neutron stars, or stellar BHs) by massive BHs at galactic centers. This source, commonly referred to as Extreme-Mass-Ratio Inspirals (EMRIs), is the one we will focus on in this contribution as a high-precission tool for fundamental physics tests. Apart from these three sources, there are also prospects of detecting eventual stochastic GW backgrounds from the early universe (inflation,superstrings,topological defects, etc.).

3 Extreme-mass-ratio inspirals: EMRIs

EMRIs are composed of a stellar-mass compact object (SCO) spiraling into a massive BH (MBH) located in a galactic center. Since the masses of interest for the SCO are around $m_\star = 1 - 10^2\,M_\odot$, and for the MBH are in the range $M_\bullet = 10^5 - 10^7\,M_\odot$, the mass-ratio for these systems is $\mu = m_\star/M_\bullet \sim 10^{-7} - 10^{-3}$. During the inspiral phase, an EMRI losses energy and angular momentum via the emission of GWs, producing a shrinking of the orbit, which means that the period decreases and, as a consequence, the GW frequency increases.

There are several astrophysical mechanisms that have the potential to produce EMRIs (see [13] for a review on EMRI astrophysics and other aspects). The most studied mechanism is based on the properties of the SCO's dynamics in stellar cups

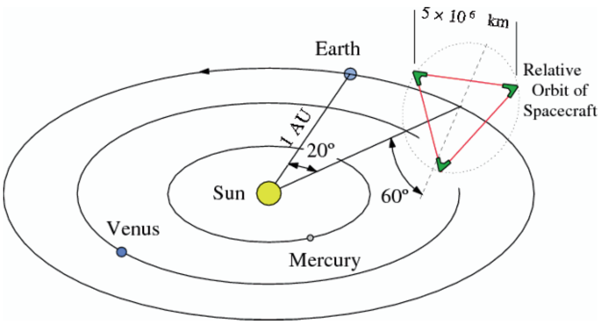

Fig. 1 Figure reproduced from [12] with permission. It illustrates the configuration of the motion of the LISA constellation.

around MBHs at galactic centers. There is a small but non-negligible probability that one of these objects may *fall* into a bounded trajectory with respect to the MBH, due to gravitational interactions with other bodies. If so, GW emission would then force the system to decay and plunge into the MBH in a period significantly smaller than the Hubble time scale. Initially, the orbit can be quite eccentric, with eccentricities in the range $1-e \sim 10^{-6}-10^{-3}$, but by the time they enter into the LISA band the eccentricity is expected to be substantially reduced (due to GW emission), although it will probably still be significant (in the range $e \sim 0.5-0.9$). A remarkable fact about EMRIs is that during the last year before plunge they emit on the order of 10^5 or more GW cycles, carrying a lot of information about the MBH strong field region.

Moreover, it has been estimated that LISA will be able to detect GW signals of around $10-10^3$ EMRIs per year up to distances of $z \lesssim 1$ [14, 15]. These signals will be hidden in the LISA instrumental noise and in the GW foreground produced mainly by compact binaries in the LISA band. Thus, in order to extract the EMRI signals we need a very accurate theoretical description of the gravitational waveforms. The main difficulty in producing these is in the description of the gravitational effects of the SCO on its own trajectory. These effects produce deviations in the SCO's motion away from a geodesic around the MBH. One can think of

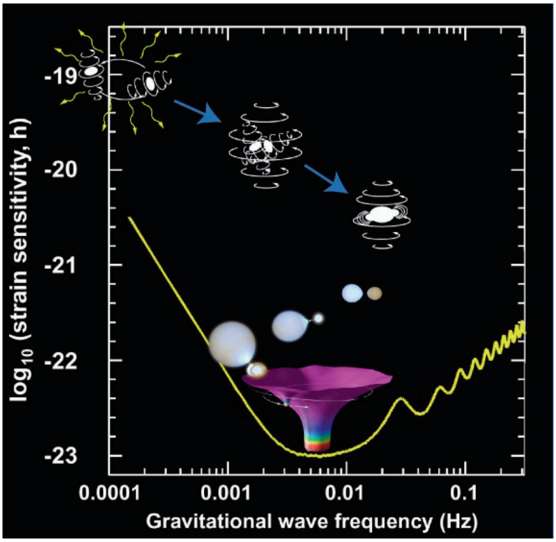

Fig. 2 This figure shows the LISA sensitivity response function in terms of the frequency. It allows shows the main sources of GW for LISA: (i) Massive BH mergers, entering the band from the upper left corner (inspiral phase) and evolving in frequency until they merger and the final BH rings down. (ii) Galactic binaries. They are almost monochromatic sources and there many of them, forming a foreground from which only a fraction can be individually distinguished. (iii) EMRIs. The capture and subsequent inspiral of a stellar-mass object into a Massive BH (see Sect. 3).

such a deviation as induced by the action of a local force, the so-called *self-force*. Analogously, one can think of the SCO as moving on a geodesic of the spacetime generated both by the MBH and the SCO itself. There is an entire research program devoted to the computation of the self-force carried out by a community that annually gathers at the CAPRA Ranch Meetings on Radiation Reaction. In the last few years, there has been tremendous progress in this direction, among other things leading to the first computations of the gravitational self-force for generic orbits around a non-spinning MBH. Details on the self-force research program and recent advances can be found in the reviews [16, 17] and references therein.

LISA observations of EMRIs have the potential to make revolutionary discoveries in Astrophysics, Cosmology, and Fundamental Physics. With regard to the first, Astrophysics, we expect the following discoveries: to better understand the dynamics in galactic centers (mass segregation, resonant relaxation, massive perturbers, etc); to obtain information about the mass spectrum of stellar BHs in galactic nuclei in order to understand the formation of stellar BHs and their relation to their progenitors; to obtain information on the distribution of the MBH spins for masses up to a few million solar masses, which has implications for galaxy formation models; perhaps to detect the inspiral of an Intermediate-Mass BH (IMBHs) into a MBH, which will give direct evidence for the existence of IMBHs; etc. Regarding Cosmology, it has been proposed [18] that precise measurements of the Hubble parameter are possible by correlating LISA EMRI observations (which act as standard "sirens" and provide precise measurements of luminosity distances up to $z \sim 1$) with galaxy redshift surveys, which would provide statistical redshift information of the EMRI events (which cannot be inferred from the GW observations). Applying this idea to a simplified cosmological model, it has been estimated that using 20 or more EMRI events to $z \sim 0.5$, one could measure the Hubble constant to better than one percent precision.

Finally, and of most relevance for this contribution, EMRIs have also a great potential for Fundamental Physics, based on the fact that EMRI GW are long and carry a detailed *map* of the MBH spacetime, i.e., of the MBH multipole moments. It has been estimated that LISA can measure the main parameters of an EMRI system with high precision [19]:

$$\Delta(\ln M_\bullet)\,,\; \Delta\left(\ln\frac{m_\star}{M_\bullet}\right),\; \Delta\left(\frac{S_\bullet}{M_\bullet^2}\right) \;\sim\; 10^{-4}\,,\;\; \Delta\Omega \;\sim\; 10^{-3}\,, \tag{4}$$

where $S_\bullet$ denotes the MBH spin and Ω the solid angle (related to sky localization of the source). It is also expected (see [20, 21]) that LISA will be able to measure $3-5$ MBH multipole moments with good accuracy. Therefore, EMRI LISA observations provide a unique opportunity to test the no-hair theorem and also to perform tests of alternative theories of gravity, which is the main subject of the remainder of this contribution.

4 Testing theories of gravity with EMRIs: EMRIs in DCSMG

There are many modifications of General Relativity available in the literature, and hence in principle it seems difficult to justify a particular choice. However, not all theories are created equal. In particular, not all theories available are consistent in all regimes; in many of them the status of BH solutions is unclear, in the sense that either the solutions are not known or, if they are known, it is unclear that they are unique or that they represent the final state of gravitational collapse. On the other hand, in order to gain some insight on the effects modifications of gravity may have on GW observations it is good to explore several different particular theories. In this sense, one can also propose certain criteria for a theory to be a reasonable candidate to test GR with LISA [22].

An example of such a theory is DCSMG (see [23] for a recent review). This modification to General Relativity was introduced by Jackiw and Pi [24] and it consists of the addition of a new term to the Einstein-Hilbert Lagrangian that generalizes the standard 3-D Chern-Simons term. This new term is a parity violating interaction that is motivated by several quantum gravity approaches, like string theory and loop quantum gravity. This modification is also motivated from an effective field theory standpoint, through the inclusion of high-curvature terms to the action (see [25] for an application of this approach to inflationary cosmology).

In this 4D theory, the action is given by:

$$\mathscr{S} = \mathscr{S}_{\rm EH} + \mathscr{S}_{\rm CS} + \mathscr{S}_{\phi} + \mathscr{S}_{\rm matter}\,, \tag{5}$$

where $\mathscr{S}_{\rm EH}$ is the Einstein-Hilbert action

$$S_{\rm EH} = \kappa \int d^4x\, \sqrt{-{\rm g}}\, R\,, \qquad \kappa = \frac{1}{16\pi G}\,, \tag{6}$$

which is modified by the addition of a term containing the Pontryagin density $({}^*RR = R_{\alpha\beta\gamma\delta}\,{}^*R^{\alpha\beta\gamma\delta} = \frac{1}{2}\varepsilon^{\alpha\beta\mu\nu}R_{\alpha\beta\gamma\delta}R^{\gamma\delta}{}_{\mu\nu})$

$$\mathscr{S}_{\rm CS} = \frac{\alpha}{4}\int d^4x\, \sqrt{-{\rm g}}\; \phi\; {}^*RR\,. \tag{7}$$

Here, the Pontryagin density, a topological invariant in 4D, is multiplied by a scalar field, ϕ, to produce a modification of the GR field equations, which is proportional to the coupling constant α. In addition we have the action term of the scalar field:

$$\mathscr{S}_{\phi} = -\beta \int d^4x\, \sqrt{-{\rm g}}\, \left[\frac{1}{2}{\rm g}^{\mu\nu}\left(\nabla_\mu\phi\right)\left(\nabla_\nu\phi\right) + V(\phi)\right], \tag{8}$$

where β is another coupling constant. In the original version of the theory, the scalar field ϕ was forced to be a fixed function, devoid of dynamics and a contribution to the action. This leads to an additional constraint, the vanishing of the Pontryagin density, which is too restrictive. In particular, it disallows spinning BH solutions for scalar fields whose gradient is time-like [26] and forbids perturbations of non-spinning BHs [27]. Finally, $\mathscr{S}_{\rm matter}$ is the action of any additional matter fields.

We now summarize the main results on the study of EMRIs in DCSMG [11]. The first point is that spinning MBHs are no longer described by the Kerr metric (although non-spinnning ones are described by the Schwarzschild metric). Using the small-coupling and slow-rotation approximations, the exterior, stationary and axisymmetric gravitational field of a rotating BH in dynamical DCSMG modified gravity, in Boyer-Lindquist type coordinates, is given by [28]:

$$ds^2 = ds^2_{\rm Kerr} + \frac{5\xi a}{4r^4}\left[1 + \frac{12M}{7r} + \frac{27M^2}{10r^2}\right]\sin^2\theta dt d\varphi\,, \tag{9}$$

where $ds^2_{\rm Kerr}$ is the line element for the Kerr metric, M and a are the MBH mass and spin parameter, and $\xi = \alpha^2/(\kappa\beta)$ [see (7) and (8)]. The multipolar structure of the modified metric remains completely determined by only two moments (no-hair or two-hair theorem): the mass monopole and the current dipole. The relation, however, between these two moments and higher-order ones is modified from the GR expectation at multipole $\ell \geq 4$. On the other hand, the solution for the DCSMG scalar field ϕ is:

$$\phi = \frac{5}{8}\frac{\alpha}{\beta}\frac{a}{M}\frac{\cos(\theta)}{r^2}\left(1 + \frac{2M}{r} + \frac{18M^2}{5r^2}\right)\,, \tag{10}$$

which is axisymmetric and fully determined by the MBH geometry [28]. Hence, the no-hair theorem still holds in this theory.

Regarding the equations of motion for the SCO, it has been shown [11] that point-particles follow geodesics in this theory, as in GR. Moreover, it turns out that the metric given in (9) has the same symmetries as the Kerr metric (stationary and axisymmetric), including the existence of a 2-rank Killing tensor. As a consequence, the geodesics equations are also fully integrable, and the difference with respect to Kerr can be encoded in a single function [11]. One can see that the innermost-stable circular orbit (ISCO) location is DCSMG shifted by [28]:

$$R_{\rm ISCO} = \underbrace{6M \mp \frac{4\sqrt{6}a}{3} - \frac{7a^2}{18M}}_{\text{GR Piece}} \pm \underbrace{\frac{77\sqrt{6}a}{5184}\frac{\alpha^2}{\beta\kappa M^4}}_{\text{CS Modification}} \tag{11}$$

where the upper (lower) signs correspond to co- and counter-rotating geodesics. Notice that the DCSMG correction acts *against* the spin effects. One can also check that the three fundamental frequencies of motion [29] change with respect to the GR values.

The next important question to address is how GW emission and propagation is affected in DCSMG. First of all, it has been shown [11] that observers far away from the sources can only observe the same polarizations as in GR, although there is an additional mode, a breathing mode, that has an impact in the strong-field dynamics but decays too fast with distance to be observable in the GW emission. The DCSMG EMRI analysis of [11] was carried out in the so-called *semi-relativistic* approximation [30], where the motion is assumed geodesic and GWs are assumed to propagate in a flat spacetime. Neglecting *radiation reaction* effects, the dephasing between DCSMG and GR GWs is only due to modifications in the MBH geometry. This dephasing will not prevent in principle detection of GWs from EMRIs with LISA (from *short* periods of data ~ 3 weeks, where radiation reaction effects can be neglected), but instead it will bias the estimation of parameters, leading to an uncontrolled systematic error.

The study of radiation reaction effects in DCSMG [11] was carried out using the short-wave approximation (see, e.g. [31]). It was found that to leading order the GW emission formulae are unchanged with respect to GR. That is, we can introduce an effective GW energy-momentum tensor that has exactly the same form as in GR (the Isaacson tensor [32]). There are subdominant contributions to the radiation reaction mechanism due to the presence of the DCSMG scalar field ϕ.

By comparing waveforms computed in GR with waveforms computed in DCSMG (assuming the same orbital parameters: eccentricity, pericenter, and inclination), a a rough estimate [11] of the accuracy to which DCSMG gravity could be constrained via a LISA observation was given. This estimate can be expressed as:

$$\xi^{1/4} \lesssim 10^5\,\mathrm{km} \left(\frac{\delta}{10^{-6}}\right)^{1/4} \left(\frac{M_\bullet}{M_{\mathrm{MW}}}\right), \tag{12}$$

where δ is the accuracy to which ξ can be measured, which depends on the integration time, the signal-to-noise ratio, the type of orbit considered and how much radiation-reaction affects the orbit. Moreover, M_{MW} is the mass of the presumable BH at Sgr A* in our Milky Way galaxy, with a canonical value of $\sim 4.5 \cdot 10^6 M_\odot$. Notice that IMRIs (with total masses in the range $10^3 - 10^4 M_\odot$) are favored over EMRIs. This result is to be compared with the binary pulsar constrained $\xi^{1/4} \lesssim 10^4$ km [28]. These results imply that it may be possible to place strong constraints (up to two orders of magnitude more stringent than binary pulsar ones) with IMRI GW observations. Moreover, a GW test can constrain the dynamical behavior of the theory in the neighbourhood of BHs, which is simply not possible with neutron star observations.

At present, there is work in progress that focuses on the inclusion of radiation reaction effects and the use of better statistical tools to estimate the ability of LISA to constraint DCSMG. A key point in this regard is that, to leading order, the GW emission in DCSMG is unchanged with respect to GR, which can be used to simplify the analysis, allowing for GR-like expressions for the rate of change of constants of motion due to the GW emission.

Acknowledgements CFS acknowledges support from the Ramón y Cajal Programme of the Spanish Ministry of Education and Science (MEC) and by a Marie Curie International Reintegration Grant (MIRG-CT-2007-205005/PHY) within the 7th European Community Framework Programme. Financial support from the contracts ESP2007-61712 (MEC), FIS2008-06078-C03-

01 and FIS2008-06078-C03-03 (MICINN) is gratefully acknowledged. NY acknowledges support from NSF grant PHY 07-45779.

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Gravitons in Flatland

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Abstract We review some features of three-dimensional (3D) massive gravity theories. In particular, we stress the role of the Schouten tensor, explore an analogy with Lovelock gravity and discuss renormalizabilty.

Edwin Abbot's 1884 novella *Flatland* [1] was the first of many explorations of a hypothetical world in which there are only two space dimensions (although Charles Hinton's *A Plane World*, also published in 1884, may deserve equal credit[1]). In the context of special relativity, Flatland may be considered to be a three-dimensional (3D) Minkowski spacetime. Of course, the term "Flatland" might be considered inappropriate in the context of General Relativity, but the 3D Einstein equations imply that the spacetime curvature is entirely determined by the matter content, so spacetime is still locally flat outside sources [2, 3, 4]; this means that 3D GR does not admit gravitational waves, and there are consequently no massless gravitons. However, it is possible to add higher-order terms (in a derivative expansion) to the Einstein-Hilbert (EH) action such that when the action is expanded about the Minkowski vacuum (in powers of the metric perturbation) the quadratic approximation to it is a Minkowski space field theory propagating *massive* gravitons. This contribution reviews the current status of these 3D "massive gravity" theories, focusing on the small fluctuations about Minkowski space: i.e. on gravitons in Flatland.

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[1] We thank Gary Gibbons for bringing Hinton's work to our attention.

S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_27,

We start from the 3D Fierz-Pauli (FP) equations for a totally symmetric rank-s tensor $\varphi^{(s)}$, which is also traceless for $s \geq 2$. These equations consist of the dynamical equation

$$\left(\Box - m^2\right) \varphi^{(s)} = 0 \tag{1}$$

together with the subsidiary condition

$$\partial^\nu \varphi^{(s)}_{\nu\mu_2\cdots\mu_s} = 0 \,. \tag{2}$$

They are equivalent to the single equation

$$[\mathbb{H} - sm]\,[\mathbb{H} + sm]\,\varphi^{(s)} = 0\,, \tag{3}$$

where m is a mass parameter and $\mathbb{H}$ is the spin-s unit-mass "helicity operator" :

$$\mathbb{H}_{\mu_1\ldots\mu_s}{}^{\nu_1\ldots\nu_s} = s\varepsilon_{(\mu_1}{}^{\tau(\nu_1}\,\delta^{\nu_2}_{\mu_2}\cdots\delta^{\nu_s)}_{\mu_s)}\,\partial_\tau \,. \tag{4}$$

It is evident from (3) that two spin-s modes are propagated, of helicities $\pm s$. The generalized equation

$$[\mathbb{H} - sm_+]\,[\mathbb{H} + sm_-]\,\varphi^{(s)} = 0 \tag{5}$$

also propagates two modes of helicities s and $-s$ but with, respectively, masses m_+ and m_- (both of which may assumed to be positive). Since parity flips the sign of helicity, it follows that parity is violated whenever $m_+ \neq m_-$. In particular, we may take $m_- \to \infty$ for fixed $m_+ = \mu$, in which case the second-order equation (5) degenerates to the first-order "self-dual" equation [5, 6, 7]

$$[\mathbb{H} - s\mu]\,\varphi^{(s)} = 0 \,. \tag{6}$$

Note that this still implies the subsidiary condition (2).

For any of the above massive spin-s field equations, there is a systematic procedure for finding an equivalent set of equations for a spin-s *gauge theory* [8]. We first relax the tracelessness condition on φ (in the case that $s \geq 2$) and then solve the subsidiary condition by writing

$$\varphi^{(s)}_{\mu_1\cdots\mu_s} = -\frac{1}{s!}\varepsilon_{\mu_1}{}^{\tau_1\nu_1}\cdots\varepsilon_{\mu_s}{}^{\tau_s\nu_s}\partial_{\tau_1}\cdots\partial_{\tau_s}h_{\nu_1\cdots\nu_s} \equiv G_{\mu_1\cdots\mu_s} \tag{7}$$

for a rank-s gauge potential h, with rank-s field-strength G invariant under the gauge transformation[2]

$$h_{\mu_1\cdots\mu_s} \to h_{\mu_1\cdots\mu_s} + \partial_{(\mu_1}\xi_{\mu_2\cdots\mu_s)} \tag{8}$$

for arbitrary rank-$(s-1)$ symmetric tensor parameter ξ. The subsidiary condition is then replaced by the Bianchi-type identity

$$\partial^\nu G_{\nu\mu_1\cdots\mu_{s-1}} \equiv 0\,, \tag{9}$$

[2] For spin $s \geq 3$ fields in 4D it is usual to assume a weaker gauge invariance in which the parameter is traceless, in which case the field strength is always second-order in derivatives.

but the tracelessness condition on φ, which we initially relaxed, must now be re-imposed as a condition on G.

Applied to the "self-dual" equation (6) for $s=1$, this procedure yields the field equations of "topologically massive electrodynamics", as has been known for some time [9]. The application to the same self-dual equation for $s=2$ was presented in [10]; in this case we have the equations

$$[\mathbb{H}-2\mu]\,G=0\,,\qquad \eta^{\mu\nu}G_{\mu\nu}=0\,, \tag{10}$$

for symmetric tensor field strength G expressed in terms of a symmetric tensor potential, which we now write as $h_{\mu\nu}$, and view as a metric perturbation: $h=g-\eta$. Then G has the interpretation as the linearized Einstein tensor:

$$G_{\mu\nu}=G^{(lin)}_{\mu\nu}\equiv R^{(lin)}_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}R^{(lin)}\,, \tag{11}$$

where $R^{(lin)}_{\mu\nu}$ is the linearized Ricci tensor and $R^{(lin)}$ its Minkowski trace. The equations (10) can now be shown to be equivalent to the linearized version of the one equation

$$G_{\mu\nu}+\frac{1}{\mu}C_{\mu\nu}=0\,, \tag{12}$$

where $C_{\mu\nu}$ is the Cotton tensor (the 3D analog of the Weyl tensor):

$$\sqrt{-\det g}\,C_{\mu\nu}\equiv\varepsilon_\mu{}^{\tau\rho}D_\tau S_{\rho\nu}\,,\qquad S_{\mu\nu}\equiv R_{\mu\nu}-\frac{1}{4}g_{\mu\nu}R\,, \tag{13}$$

where D is the usual covariant derivative constructed from the Levi-Civita connection. The tensor $S_{\mu\nu}$ is the 3D Schouten tensor, about which we shall have more to say later. Equation (12) can be derived from the Lagrangian density

$$\mathscr{L}=-\sqrt{-\det g}\,R+\frac{1}{\mu}\mathscr{L}_{LCS}\,, \tag{14}$$

where $\mathscr{L}_{LCS}$ is the Lorentz Chern-Simons (LCS) term. This is the action of 'topologically massive gravity" (TMG) [11]. Note the unconventional sign of the Einstein-Hilbert (EH) term; it is needed for positive energy of the one massive spin 2 mode that is propagated in the Minkowski vacuum. Whether the energy of an asymptotically Minkowski solution of the non-linear theory is necessarily positive is still a matter of debate.

Applying the same procedure to the second-order FP equations (3) for spin 2, we arrive at the Lagrangian density for "new massive gravity" (NMG) [12], which we may write in the form[3]

[3] The sign of the EH term depends on the metric signature convention; as we use here the "mostly plus" convention, the sign is opposite to that of [12] where the "mostly minus" convention was used.

$$\mathscr{L} = \sqrt{-\det g}\left[-R + \frac{1}{m^2}K\right], \qquad K \equiv G^{\mu\nu}S_{\mu\nu}\,. \tag{15}$$

Note the occurrence, once again, of the Schouten tensor. This derivation of NMG may be run in reverse to prove that it propagates two massive spin-2 modes in a Minkowski vacuum, with helicities ± 2, but this fact does not guarantee that neither mode is a ghost (negative kinetic energy). This is not an issue for TMG because one may always adjust the overall sign of the action to ensure that the one propagating mode is physical, but this may not be sufficient when there are two propagating modes. In fact, if the same method is applied to the Proca equations then one arrives at an equivalent set of "extended Proca" equations [13], but the "extended Proca" action propagates one of the two spin-1 modes as a ghost, and the same phenomenon occurs for spin 3 [8]. So the absence of ghosts in NMG is far from obvious. Nevertheless, there is an alternative form of the action involving an auxiliary tensor field that allows a simple proof of the equivalence of the linearized *action* to the standard FP action, which is known to be ghost-free. This was reviewed in [14]. The absence of ghosts in linearized NMG may also be verified directly by a canonical analysis [15].

The scalar $K = G^{\mu\nu}S_{\mu\nu}$ has the "conformal covariance" property [12]

$$g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}\int d^3x\sqrt{|g|}K \quad \propto \quad K\,. \tag{16}$$

This implies that the quadratic approximation to K is invariant under linearized Weyl transformations, as stressed in [15]. In fact, the scalar $G^{\mu\nu}S_{\mu\nu}$ has this property in any dimension. To see this, one should first appreciate that the definition of the Schouten tensor is dimension dependent; for spacetime dimension $D > 2$,

$$S_{\mu\nu} = \frac{1}{D-2}\left[R_{\mu\nu} - \frac{1}{2(D-1)}Rg_{\mu\nu}\right]. \tag{17}$$

This tensor first arose in the decomposition of the Riemann tensor into the traceless Weyl conformal tensor W and a remainder:

$$R_{\mu\nu\rho\sigma} = W_{\mu\nu\rho\sigma} + (g\circ S)_{\mu\nu\rho\sigma}\,, \tag{18}$$

where $\circ$ indicates the Kulkarni-Nomizu product of two second-rank tensors (for symmetric tensors this product has the symmetries of the Riemann tensor). The Schouten tensor also has the interpretation as a (dependent) gauge potential for conformal boosts[4] and this explains why, in 3D, the Cotton tensor may be expressed in terms of it. Next, we note the *identity*[5]

[4] This gauge potential is set equal to the Schouten tensor on imposing a constraint on conformal curvatures [16] in close analogy to the way that the affine connection becomes a function of the metric and its derivatives when the torsion is constrained to vanish.

[5] An equivalent identity has been noted independently in [17].

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2 \equiv W^{\mu\nu\rho\sigma}W_{\mu\nu\rho\sigma} - 4(D-3)G^{\mu\nu}S_{\mu\nu}\,, \tag{19}$$

which is valid for any $D \geq 3$, although both sides vanish identically for $D = 3$. For $D = 4$ the left hand side is the integrand of the Gauss-Bonnet invariant, and hence is a total derivative. This shows that the 4D scalar $G^{\mu\nu}S_{\mu\nu}$ equals the square of the Weyl tensor, up to a total derivative. For $D > 4$ the left hand side is the Lovelock term [18]; it is not a total derivative but it has the feature that it does not contribute to the quadratic action in an expansion about Minkowski space[6]. Thus, even for $D > 4$ it remains true that the quadratic approximation to $G^{\mu\nu}S_{\mu\nu}$ equals the square of the linearized Weyl tensor, up to a total derivative, and hence that it is invariant under linearized Weyl transformations.

Let us now turn to the generalized, parity-violating, FP equations (5). Applying the same procedure reviewed above for the other cases, we arrive at the Lagrangian density of "general massive gravity" (GMG) [12]

$$\mathscr{L} = \sqrt{-\det g}\left[-R + \frac{1}{m^2}G^{\mu\nu}S_{\mu\nu}\right] + \frac{1}{\mu}\mathscr{L}_{LCS} \tag{20}$$

where

$$m^2 = m_+m_-\,, \qquad \mu = \frac{m_+m_-}{m_- - m_+}\,. \tag{21}$$

Again, the derivation does not guarantee the absence of ghosts. However, the canonical analysis that shows NMG to be ghost free can be easily generalized to GMG [10], as we now review. One begins by making a time-space split and imposing a convenient gauge condition

$$\partial_i h_{i\mu} = 0; \qquad \mu = (0,i) \quad i = 1,2\,. \tag{22}$$

The 3-metric perturbation h can now be expressed in terms of three independent functions as follows:

$$h_{\mu\nu} = \begin{pmatrix} n & m\hat{\partial}_i\phi_2 \\ m\hat{\partial}_j\phi_2 & \hat{\partial}_i\hat{\partial}_j\phi_1 \end{pmatrix}, \qquad \hat{\partial}_i \equiv \varepsilon^{ij}\partial_j\,. \tag{23}$$

Substitution into the linearized GMG action yields an action involving (n,ϕ_1,ϕ_2) that is fourth order in derivatives but only second-order in time derivatives. Introducing new independent functions (N,Φ_1,Φ_2) by the *space* non-local field redefinitions[7]

$$n + \Box\phi_1 - 2m^2(\phi_1 - \phi_2) = \frac{1}{\nabla^2}N\,, \qquad \phi_a = \frac{1}{\nabla^2}\Phi_a \quad (a = 1,2), \tag{24}$$

we then arrive at an equivalent action with Lagrangian density

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6 A number of other similarities between NMG and Lovelock gravity have been noted in [19, 20, 21].

7 Space non-local field redefinitions are allowed since they cannot change the canonical structure, which depends on the time derivatives.

$$\mathscr{L} = \frac{1}{2}\Phi_a\left[\delta_{ab}\Box - M^2_{ab}\right]\Phi_b + 2m^2N^2\,, \tag{25}$$

where the 2×2 matrix M^2_{ab} is real symmetric with eigenvalues (m^2_+, m^2_-). This confirms that there are two propagating modes with masses $m_\pm$. Crucially, both modes have positive energy, so GMG is ghost-free.

It is remarkable that the final result (25) is Lorentz invariant, despite the fact that we arrived at it via manipulations that explicitly violate Lorentz invariance, but the Lorentz transformations that leave invariant (25) are *not the same* as the Lorentz transformations that leave invariant the linearized GMG action. This is simply a reflection of the fact that a free field theory has an infinite-dimensional invariance group. In the present case, this infinite dimensional group has at least two, mutually space non-local and probably non-commuting, Lorentz subgroups. The introduction of interactions that preserve one of these Lorentz subgroups will break the other one. Precisely because we view linearized NMG as the quadratic approximation to NMG, it is the manifest Lorentz transformations of this action, in its original form, that are relevant to the determination of the spin of propagated modes. For this reason, one cannot read off the spins from the Lagrangian (25); however we already know that the two modes have spin 2 from the equivalence of the field equations to those of the generalized FP equations.

A feature of curvature-squared terms is that they contribute to the quadratic kinetic terms in an expansion about Minkowski space, and hence to the propagator as well as to the vertices of Feynmann diagrams. Specifically, they introduce $1/p^4$ type terms in the propagator, where p is the 3-momentum, and this makes the generic curvature squared gravity theory power-counting renormalizable in 4D [22]. Unfortunately, this comes at the cost of unitarity. There is one exceptional case in which unitarity is not violated, although renormalizability is lost. This is the model obtained by adding to the EH term the square of the scalar curvature (R^2); this is equivalent, for an appropriate choice of signs, to a scalar field coupled to gravity (see [23] for a review). There is another exceptional case in which renormalizability is lost (without a gain in unitarity). This is the model obtained by addition of the square of the Weyl curvature tensor (W^2), but without an R^2 term. Exceptional cases can arise because the metric perturbation $h_{\mu\nu}$ is not an irreducible representation of the Lorentz group but includes the scalar trace h. There is a term in the propagator that projects onto the pure spin 2 part of $h_{\mu\nu}$ and a term that projects onto the trace, irrespective of whether h propagates any physical spin-0 mode. Both terms in the propagator go like $1/p^2$ in the context of the EH term alone. Addition of the square of the Weyl conformal curvature tensor (W^2) causes the spin 2 projector part of the propagator to go like $1/p^4$ whereas addition of the square of the curvature scalar (R^2) causes the scalar projector part of the propagator to go like $1/p^4$, and *both* are needed for power-counting renormalizability. Thus, omitting either the R^2 term or the W^2 term implies a loss of renormalizability.

The situation in 3D, where the generic curvature-squared gravity theory is power-counting super-renormalizable, is potentially better since at least the $K = G^{\mu\nu}S_{\mu\nu}$ term may be included without violating unitarity, as we have just seen. However, the linearized Weyl invariance of K implies that the $1/p^2$ behaviour of the scalar projection term in the propagator is not affected by the curvature-squared term of NMG, so that NMG is not power-counting renormalizable [15]. Here it should be said that it has been claimed that NMG is super-renormalizable [24], but we have not understood the argument[8]. In any case, it is certainly true that NMG is exceptional within the class of curvature-squared theories in 3D in much the same way that the '$R+W^2$' theory is exceptional in 4D. This can be seen more explicitly using the results of [25] for the propagator of the general 3D gravity model with curvature squared terms. Consider the Lagrangian density

$$\mathscr{L} = \sqrt{|g|}\left[\sigma R + \frac{a}{m^2}K + \frac{b}{m^2}R^2\right] \tag{26}$$

for constants (σ, a, b); the choice $(\sigma, a, b) = (-1, 1, 0)$ yields NMG. If we expand about the Minkowski vacuum we find that

$$\mathscr{L} = \frac{1}{2}h^{\mu\nu}\mathscr{O}_{\mu\nu,\rho\sigma}h^{\rho\sigma} + \dots \tag{27}$$

where $\mathscr{O}$ is a fourth-order linear differential tensor operator and the dots indicate interaction terms. The operator $\mathscr{O}$ may be expressed in terms of two orthogonal projection operators, for spin 2 and spin 0 [26]; in momentum space, these are

$$P^{(2)}_{\mu\nu,\rho\sigma} = \frac{1}{2}\left(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho} - \theta_{\mu\nu}\theta_{\rho\sigma}\right), \qquad P^{(0,s)}_{\mu\nu,\rho\sigma} = \frac{1}{2}\theta_{\mu\nu}\theta_{\rho\sigma}, \tag{28}$$

where

$$\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}. \tag{29}$$

Specifically, one finds that

$$\mathscr{O}_{\mu\nu,\rho\sigma} = \left[-\frac{1}{2}\sigma p^2 + \frac{ap^4}{2m^2}\right]P^{(2)}_{\mu\nu,\rho\sigma} + \left[\frac{1}{2}\sigma p^2 + \frac{bp^4}{2m^2}\right]P^{(0,s)}_{\mu\nu,\rho\sigma}. \tag{30}$$

This operator is not invertible, but we may invert within each of the subspaces defined by the two projectors. The result is the propagator

$$2m^2\left\{\frac{P^{(2)}_{\mu\nu,\rho\sigma}}{p^2\left(ap^2 - m^2\sigma\right)} + \frac{P^{(0,s)}_{\mu\nu,\rho\sigma}}{p^2\left(bp^2 + m^2\sigma\right)}\right\}. \tag{31}$$

As long as $ab \neq 0$ this behaves like $1/p^4$ as $p^2 \to \infty$, but the spin 2 part goes like $1/p^2$ when $a = 0$ and the spin 0 part goes like $1/p^2$ when $b = 0$.

Returning to 4D, there is one curvature-squared model that is potentially renormalizable, and that is conformal gravity. The action for conformal gravity is just the integral of W^2, without the EH term. The propagator is now purely $1/p^4$ because

[8] In particular, it appears that the means used to arrive at this conclusion are not specific to 3D and could be used to obtain the same result in 4D.

the trace of the metric perturbation is a gauge degree of freedom. It is often said that this model has ghosts since any perturbation away from conformality leads to ghosts, but consistency requires that the conformal invariance be preserved, even by quantum corrections. The one-loop conformal anomalies cancel for some conformal supergravity models (see [27] for a review) so these may be viable theories, although they have not yet found any compelling application.

The action for 3D conformal gravity is just the LCS term [28]. In other words, one omits the EH term from TMG, but this propagates no modes. One may also omit the EH term from NMG, in which case one gets a model that propagates a single massless mode [15], of no definite spin because spin is not defined for massless particles. If a LCS term is added (equivalently, if the EH term is omitted from the GMG action) then one gets a 4th order "New Topologically Massive Gravity" (NTMG) model that propagates a single massive spin-2 mode [10, 29]. In any of these models without an EH term, the trace of the metric perturbation is a gauge degree of freedom in the quadratic approximation, so the propagator is either pure $1/p^4$ or behaves this way in the short distance limit. This fact was claimed in [15] to imply renormalizabilty. However, the trace of the metric perturbation is not a gauge degree of freedom of the interacting theory. Its equation of motion is $K = 0$, which is identically satisfied in the linearized limit (since K has no term linear in fields) but not otherwise. The usual power-counting arguments apply to a perturbation theory in which all non-gauge degrees of freedom are represented in the propagator, but this condition is not satisfied here. It remains to be seen what effect this has.

If massive gravity theories are not power-counting renormalizable then they are still no worse than general relativity in 4D. In the latter case, we know that supersymmetry can soften the ultra-violet divergences, and there are some hints that the maximally supersymmetric $\mathscr{N} = 8$ supergravity may be finite (see e.g. [30]). In view of this, it is of interest to consider the massive 3D supergravity theories. The representation theory for massive particle supermultiplets in 3D is formally the same as that for massless supermultiplets in 4D so $\mathscr{N} = 8$ is maximal for massive 3D supergravity too, although the total number of supersymmetry charges is 16 rather than 32. So far only the $\mathscr{N} = 1$ theory has been constructed in detail [10, 31] although the $\mathscr{N} = 2$ massive 3D supergravity has been constructed as a linear theory in Minkowski space [10] and the $\mathscr{N} = 8$ theory can be constructed in the same approximation [32]. Assuming that there is an $\mathscr{N} = 8$ massive 3D supergravity, representation theory implies that it must preserve parity since the state of helicity $+2$ is in the same supermultiplet as the state of helicity -2. In other words, we expect NMG to have an $\mathscr{N} = 8$ supersymmretric extension but not TMG. The representation theory would allow $\mathscr{N} = 7$ as maximal for super-TMG but the details suggest that $\mathscr{N} = 6$ is actually maximal for a parity violating massive 3D gravity. Thus, $\mathscr{N} = 8$ new massive supergravity is the most promising candidate for a 3D massive gravity theory with 'improved' ultraviolet behaviour, but it remains to be seen how significant any improvement will be.

Acknowledgements PKT thanks the conference organizers for the invitation to deliver this talk on the occasion of Emilio Elizalde's 60th birthday. The authors are grateful to Roel Andringa, Mees de Roo, Jan Rosseel and Ergin Sezgin for discussions in the course of collaborations on supersymmetric extensions of massive 3D gravity models. The work of OH is supported by the DFG – The German Science Foundation and in part by funds provided by the U.S. Department of Energy (DoE) under the cooperative research agreement DE-FG02-05ER41360. PKT thanks the EPSRC for financial support.

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Very Special Relativity and Noncommutative Space-Time

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Abstract The Very Special Relativity (VSR) introduced by Cohen and Glashow [16] has a robust mathematical realization on noncommutative space-time, in particular on noncommutative Moyal plane, with *light-like noncommutativity* [35]. The realization is essentially connected to the twisted Poincaré algebra and its role as symmetry of noncommutative space-time and the corresponding quantum field theories [11, 12]. In our setting the VSR invariant theories are specified with a single deformation parameter, the noncommutativity scale Λ_{NC}. Preliminary analysis with the available data leads to $\Lambda_{NC} \geq 1-10$ TeV.

1 Introduction

General arguments based on quantum mechanics and gravitational theory in the process of measurement [19, 20], as well as on open string theory in the presence of an antisymmetric background field [3, 33], indicate that the concept of space-time as a continuous manifold indeed breaks down at very short distances into a quantum object and a possible description of physics at such scales is in terms of noncommuting coordinate operators. The ultimate question whether space-time is a quantum object and what should replace the well-established relativistic invariance is one of the fundamental issues in high energy physics. Although currently we do not have any observation or experiment signaling departure from Lorentz symmetry, with the advance of technologies we will be able to trace such deviations with ever increasing precision. With the prospect of upcoming experiments various possible deviations from Lorentz invariance at high energies have been studied, both theoretically and phenomenologically (for an incomplete list, see, e.g., [17, 18, 26, 28, 29, 37]).

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_28,

The Cohen-Glashow Very Special Relativity (VSR) [16] is defined as symmetry under certain subgroups of Poincaré group, containing space-time translations and a proper subgroup of Lorentz group $SO(3,1)$ with the property that when supplemented with parity, CP or time-reversal T it enlarges to the full Lorentz group. In other words, a theory with VSR symmetry is not strictly Lorentz invariant and also not parity or time-reversal invariant. Thus, the Lorentz violation and CP violation are linked together. The problem open in [16] is whether Lorentz and nearly CP invariant theories, like the Standard Model, could emerge as effective theories from a more fundamental scheme, perhaps operative at the Planck scale.

The Lorentz symmetry has a rich representation spectrum, which turned out to be essential in building physical models based on Lorentz invariant quantum field theories: the states of the Fock spaces of quantum field theories are particle state, labeled by their mass and spin and obeying the spin-statistics relation. The VSR subgroups of the Lorentz group turn out to have a trivial spectrum, admitting only one-dimensional representations. Being proper subgroups, the representations of the VSR subgroups of Lorentz are automatically representations of the Lorentz group, but the reciprocal is not true. As a result, if we construct a VSR invariant quantum field theory based on the one-dimensional representations of the VSR subgroups, when requiring also P, T or CP invariance, although the theory becomes invariant under the whole Lorentz group, the fact about the one-dimensional representations of VSR does not change and hence the effective theory would be doomed by its very poor representation content. This is what we call "representation problem" in the VSR invariant theories. VSR invariant theories can in principle be constructed by adding Lorentz violating terms (the terms which reduce the symmetry of the Lagrangian to VSR) as perturbations to ordinary Lorentz invariant Lagrangians, such that the theories have the usual matter content allowed by Lorentz invariance. However, such a realization of VSR may not be thought of as a "fundamental" or "master" theory which leads to Lorentz invariant theories at low energies; this approach does not provide a firm theoretical setting for building the VSR invariant theories.

An alternative way for resolving the "representation problem" [35] could be achieved noting that one can include in the picture of symmetries not only the commutation relations defining an algebra, but also the action of the generators of the symmetry on the tensor product of their representation spaces (the so-called coproduct). In more mathematical terms, the reasoning in terms of Lie groups/algebras can be extended to considering (deformed) Hopf algebras, in particular the *twisted* Poincaré algebra, which provides the symmetry of noncommutative space-time with canonical commutation relations [11, 12]. The twists [21] are deformations which leave the commutation relations and structure constants of the algebra untouched, but affect other properties of the Hopf algebra, i.e. the co-algebra structure [6, 15, 31]. Since the commutation relations of generators are not deformed, it follows automatically that the Casimir operators are the same and the representation content of a twisted Hopf algebra is identical to the one of the undeformed algebra. On the other hand, the deformation of the co-algebra structure reduces the symmetry of the scheme. This latter feature enables us to use the concept of twist to reduce the Lorentz symmetry to its VSR subgroups.

2 The Cohen-Glashow Very Special Relativity

Energy-momentum conservation, and hence invariance under rigid space-time translations, should be preserved in VSR invariant theories. The minimal version of the VSR algebra contains, besides the generators of translations P_μ, the subgroup $T(2)$ of the Lorentz group, which is generated by

$$T_1 = K_x + J_y \quad \text{and} \quad T_2 = K_y - J_x, \tag{1}$$

where J_i and K_i $i = x, y, z$ are respectively generators of rotations and boosts. It is then immediate to check that $[T_1, T_2] = 0$ and hence $T(2)$ is an Abelian subalgebra of Lorentz algebra $so(1,3)$. Moreover, upon action of parity P,

$$T_1 \longrightarrow T_1^P = -K_x + J_y\ , \qquad T_2 \longrightarrow T_2^P = -K_y - J_x\ , \tag{2}$$

and similarly under T. It is straightforward to see that the algebra obtained from T_1, T_2, T_1^P and T_2^P closes on the whole Lorentz group and therefore, T_1, T_2, P_μ form (the smallest possible) VSR algebra.

The group $T(2)$ can be identified with the translation group on a two dimensional plane. The other larger versions of VSR are obtained by adding one or two Lorentz generators to $T(2)$, which have geometric realizations on the two dimensional plane: (i) $E(2)$, the 3-parametric group of two dimensional Euclidean motion, generated by T_1, T_2 and J_z; (ii) $HOM(2)$, the group of orientation-preserving similarity transformations, or homotheties, generated by T_1, T_2 and K_z; (iii) $SIM(2)$, the group isomorphic to the four-parametric similitude group, generated by T_1, T_2, J_z and K_z.

All the above-mentioned groups share the property of $T(2)$ that, by adding the parity or time-reversal conjugates of the generators, they enhance to the full Lorentz algebra. The special feature of $T(2)$ VSR is that, besides having an invariant vector $n_\mu = (1,0,0,1)$, it has as well an invariant two form [16], and this will provide its connection to the Moyal plane. $E(2)$ VSR [16], although it has an invariant vector $n_\mu = (1,0,0,1)$, it does not have any invariant two form. The $HOM(2)$ and $SIM(2)$ do not admit any invariant vector or tensors. We wish to emphasize that VSR subgroups only admit one dimensional representations. While all the representations of VSR are also representations of the Lorentz group, the converse is not true.

One way to realize $T(2)$ or $E(2)$ VSR is to make use of the fact that they admit invariant vector or tensor and use the idea of *inverse* spontaneous Lorentz symmetry breaking and give VEVs to a vector or a tensor in such a way that in low energies the VEV goes away, or become negligible and we recover the full Lorentz symmetry. This was indeed the idea put forward by Cohen and Glashow and some other authors [16, 17, 18, 26, 37]. However, this may spoil the desirable features of Lorentz invariant theories and in principle is a phenomenological approach which introduces many parameters in the theory. $HOM(2)$ and $SIM(2)$ do not admit invariant tensors and their formulation should be done in some other ways. The $SIM(2)$ case as the largest VSR has been studied more (see e.g. [4, 5, 22, 23]). An interesting connection between VSR and Finslerian geometry has also been proposed [24, 25].

3 Noncommutative spaces and twisted Poincaré symmetry

The Poincaré algebra is the isometry of the Minkowski space. There are $3+1$ dimensional space-times whose "isometry" group is either of the VSR subgroups. Noncommutative spaces which are defined through the commutation relations

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}(x) \tag{3}$$

among their coordinates, where $\theta^{\mu\nu}$ is in general a function of coordinates (Jacobi identity being satisfied), provide a setup to address this issue.

The commutation relations (3) usually spoil the Lorentz invariance and, if θ has space-time dependence, also the translational invariance. Nonetheless, depending on θ, specific subgroups of the Poincaré group under which the commutation relation (3) is preserved still provide a symmetry (or "isometry") of the noncommutative space-time.

Due to the lack of translational invariance in the cases when θ depends on coordinates, it is clear that a connection between VSR and noncommutative space-time could in principle be found only for the coordinate-independent noncommutativity parameter.

Depending on the structure of the r.h.s. of (3), there exist three types of noncommutative deformations of the space-time which can be realized through twists of the Poincaré algebra [11, 12, 30]:

- *Constant* $\theta^{\mu\nu}$: the Heisenberg-type commutation relations, defining the *Moyal space*:
$$[x^\mu, x^\nu] = i\theta^{\mu\nu}\,, \tag{4}$$
where $\theta^{\mu\nu}$ is a constant antisymmetric matrix. This is the most studied case and various aspects of QFTs on the Moyal space have been analyzed.
- *Linear* $\theta^{\mu\nu}$, with the Lie-algebra type commutators:
$$[x^\mu, x^\nu] = iC^{\mu\nu}_\rho x^\rho\,. \tag{5}$$
This case describes an (associative but) noncommutative space if $C^{\mu\nu}_\rho$ are structure constants of an associative Lie algebra.
- *Quadratic noncommutativity*, with the quantum group type of commutation relations:
$$[x^\mu, x^\nu] = \frac{1}{q}R^{\mu\nu}_{\rho\sigma}x^\rho x^\sigma\,. \tag{6}$$

All the above-mentioned cases of noncommutative space-time have originally been studied in [8] with respect to the formulation of NC QFTs on those spaces.

The essential element for our discussion is that for specific choices of θ the commutation relations can be obtained from the associative star-products coming from introducing twisted co-product for the Poincaré algebra [11, 12] (see also [30, 40]). The advantage of using the twisted Poincaré language for constructing physical theories is that, in spite of the lack of full Lorentz symmetry, the fields carry representations of the full Lorentz group [13, 14] and the spin-statistics theorem is still valid.

One would be tempted to say that the construction of a NC quantum field theory through the Weyl-Moyal correspondence is equivalent to the procedure of redefining the multiplication of functions, so that it is consistent with the twisted coproduct of the Poincaré generators [11].

However, the definition of noncommutative fields and the action of the twisted Poincaré transformations on them is not a trivial one. Ordinary relativistic fields are defined by the method of induced representations. In the commutative setting, Minkowski space is realized as the quotient of the Poincaré group by the Lorentz group, and a classical field is a section of a vector bundle induced by some representation of the Lorentz group. This construction does not generalize to the noncommutative case, because the universal enveloping algebra of the Lorentz Lie algebra is not a Hopf subalgebra of the twisted Poincaré algebra. As a result, Minkowski space $R^{1,3}$, which in the commutative setting is realized as the quotient of the Poincaré group G by the Lorentz group L, G/L, has no noncommutative analogue.

One proposal for bypassing this predicament is to consider V a Poincaré-module, with trivial action of the momentum generators [13]. Another proposal – which we shall adopt in this paper – is to maintain V as a Lorentz module, but to forbid the transformations which cannot go through [14]. In this way, we induce only the Lorentz transformations corresponding to the stability group of θ, but the fields will carry representations of the full Lorentz group; consequently, the particle spectrum of the noncommutative quantum field theory with twisted Poincaré symmetry will have the richness of the relativistic quantum field theory. We emphasize that the fact that only certain Lorentz transformations are allowed on the noncommutative fields is a strong indication of the Lorentz symmetry violation.

4 Noncommutative space-times invariant under VSR subgroups

Among the three cases of NC space-time discussed in the previous section only the constant $\theta^{\mu\nu}$ case preserves the space-time translational invariance in all directions. In the cases of linear and quadratic noncommutativity, translational invariance along some or all of the space-time directions is lost. Therefore, the Moyal case is the one relevant to the VSR theory.

4.1 $T(2)$ symmetry implies light-like noncommutativity

Motivated by the above arguments, we set about finding a configuration of the antisymmetric matrix $\theta^{\mu\nu}$. Since $T(2)$ is the only VSR which admits an invariant two tensor [16], we focus on this case. If we denote the elements of the $T(2)$ subgroup by

$$\Lambda_1 = e^{i\alpha T_1} \quad \text{and} \quad \Lambda_2 = e^{i\beta T_2}, \tag{7}$$

the invariance condition for the tensor $\theta^{\mu\nu}$ is written as:

$$\Lambda_i{}^{\mu}{}_{\alpha}\Lambda_i{}^{\nu}{}_{\beta}\theta^{\alpha\beta} = \theta^{\mu\nu}, \quad i = 1,2, \tag{8}$$

and infinitesimally:

$$T_i{}^{\mu}{}_{\alpha}\theta^{\alpha\nu} + T_i{}^{\nu}{}_{\beta}\theta^{\mu\beta} = 0, \quad i = 1,2. \tag{9}$$

The matrix realizations of the generators T_1 and T_2 are (see, e.g., [7]):

$$T_1 = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}. \tag{10}$$

Plugging these values into (9) we find the solution

$$\theta^{0i} = -\theta^{3i}, \quad i = 1,2, \tag{11}$$

all the other components of the antisymmetric matrix $\theta^{\mu\nu}$ being zero. To obtain the above result we did not assume any special form for the x-dependence of $\theta^{\mu\nu}$. With the above condition on $\theta^{\mu\nu}$, we see that $\theta^{\mu\nu}\theta_{\mu\nu} = 0$, that is

Regardless of its space-time dependence, a light-like $\theta^{\mu\nu}$ is invariant under $T(2)$.

One may use the light-cone frame coordinates

$$x^{\pm} = (t \pm x^3)/2, \qquad x^i,\ i = 1,2. \tag{12}$$

In the above coordinate system the only non-zero components of the light-like noncommutativity (11) is $\theta^{-i} = \theta^{0i} = -\theta^{3i}$ (and $\theta^{+-} = \theta^{+i} = \theta^{ij} = 0$). In the light-cone coordinates (or light-cone gauge) one can take x^+ to be the light-cone time and x^- the light-cone space direction. In this frame, (light-cone) time commutes with the space coordinates, which leads to the possibility of a consistent quantization for this type of field theories.

4.2 $E(2)$ and $SIM(2)$ invariant NC spaces

A constant θ^{-i} breaks rotational invariance in the (x^1, x^2)-plane and hence larger VSR subgroups are not possible in the Moyal NC space case. The $E(2)$ invariant case can be realized in the linear, Lie-algebra type noncommutative spaces and $SIM(2)$ can be realized by quadratic noncommutativity.

The $E(2)$ case

$E(2)$ is made up of T_1, T_2, J_z. $x^\pm$ are invariant under J_z. δ_{ij} and ε_{ij} are the (only) two invariant tensors under J_z while x^i transform as vector under J_z. Therefore, $\theta^{-i} = \ell\varepsilon_{ij}x^j$ and $\theta^{-i} = \ell x^i$ lead to $E(2)$ invariant spaces, namely

$$[x^-, x^i] = i\ell\varepsilon^{\ ij} x^j, \tag{13}$$

or

$$[x^-, x^i] \quad = i\ell x^i. \tag{14}$$

With the above choices, it is evident that the translational symmetry along $x^\pm$ is preserved while along x^i it is lost.

There is a twisted Poincaré algebra which provides the symmetry for the case of (13) while the other case cannot be generated by a twist [36]. In the above, ℓ and λ are deformation parameters of dimension length.

The $SIM(2)$ case

Since K_z acts on $x^\pm$ as scaling (scaling x^- by, say, κ and x^+ by κ^{-1}) while keeping x^i intact, it is readily seen that it is impossible to find θ^{-i} linear in the coordinates which is invariant under $HOM(2)$. It is, however, possible to realize $SIM(2)$ (and hence $HOM(2)$, too) with quadratic θ^{-i}. To have both the K_z and J_z invariant noncommutative structures, from the above discussions we deduce that we should take θ^{-i} which is linear in both x^- and x^i, therefore the two possibilities are

$$[x^-, x^i] = i\frac{\xi-1}{\xi+1}\,\varepsilon^{\ ij}\{x^-, x^j\}, \quad \xi \text{ real} \tag{15}$$

or

$$[x^-, x^i] = i\tan\chi\;\{x^-, x^i\},$$

preserving translational symmetry only along x^+ (where ξ and χ are dimensionless deformation parameters). For neither of the above cases there is any twisted Poincaré of the form discussed in [30] to provide these commutators [36]. The case (15b) in the above mentioned cylindrical coordinates x^-, ρ, ϕ takes the familiar form of a quantum (Manin) plane [32] with x^- and ρ being the coordinates on the Manin plane.

As mentioned above, the Cohen-Glashow VSR requires translational invariance, which is only realized in the constant $\theta^{\mu\nu}$ case, therefore we continue with the discussion of QFTs on the light-like Moyal plane, as the VSR-invariant theories. Further analysis of the linear and quadratic noncommutativity cases will be discussed in a future work [36].

5 NC QFT on light-like Moyal plane as VSR invariant Theory

A Moyal plane with light-like noncommutativity is invariant under the $T(2)$ VSR. A prescription for writing VSR invariant QFT for any given ordinary Lorentz invariant QFT was given in [35]: For any given QFT on commutative Minkowski space its VSR invariant counterpart is a noncommutative QFT, NCQFT, which is obtained by replacing the usual product of functions (fields) with the nonlocal Moyal $*$-product (for a review on NC QFTs, see [38])

$$(\phi * \psi)(x) = \phi(x)\, e^{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial_\mu}\overrightarrow{\partial_\nu}}\, \psi(x)\,, \tag{16}$$

where $\theta^{\mu\nu}$ is the constant light-like noncommutativity matrix given as in (11). Without loss of generality one may use the freedom in choosing the direction of the axes in the (x^1, x^2)-plane such that $\theta' = 0$ and our VSR invariant theory is specified with a single deformation parameter θ.

Due to twisted Poincaré symmetry, the fields carry representations of the full Lorentz group, but the theory is only invariant under transformations of the stability group of $\theta^{\mu\nu}$, $T(2)$ [13, 14]. Consequently, the NC QFT constructed on this space possess also the same symmetry [2], as well as twisted Poincaré symmetry [11, 12].

6 Outlook and conclusions

We have presented a framework for constructing VSR invariant quantum field theories. In analogy with the Poincaré algebra which has the geometric interpretation of the isometry group of the Minkowski space, our realization of the VSR subgroups, among other things, provides a geometric interpretation for these groups, as the isometry groups of specific "noncommutative" space-times, with *light-like noncommutativity*. In particular, demanding invariance under space-time translations restricts us to light-like noncommutative Moyal plane which is specified by a single deformation (noncommutativity) parameter. This case realizes the $T(2)$ invariant Cohen-Glashow VSR. Our realization of VSR theory naturally resolves the "representation problem" that, in spite of the lack of full Lorentz symmetry, one can still label fields by the Lorentz representations in a consistent manner. Essentially, due to the locality in light-cone time, light-like noncommutative field theories can be quantized [36]. Perturbative unitarity was proven to be satisfied [1], unlike the case of space-time noncommutative QFTs [27] and we can rely on the basic notions of fermions and bosons, spin-statistics relation and CPT theorem [10, 34, 39]. However, as shown in [34] for NC QED, C, P and T symmetries are not individually preserved and these symmetries, along with the full Poincaré symmetry may be recovered only in the $\theta^{\mu\nu} \to 0$ limit (or at energies much below the noncommutativity scale).

Through the parameter θ of the NC QFT realization of $T(2)$ VSR which has dimension length-square we define the noncommutativity scale $\Lambda_{NC} = 1/\sqrt{\theta}$. To find bounds on Λ_{NC} we need to compare results based on the NC models to the existing observations and data. These data can range from atomic spectroscopy and Lamb-shift (see, e.g., [9]) to particle physics bounds on the electric-dipole moments of elementary particles. The preliminary analysis leads to $\Lambda \geq 1 - 10$ TeV. A thorough analysis of obtaining bounds on Λ_{NC} will be performed in a future work [36].

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PART III
Zeta Functions in Physics and Mathematics

Pistons Modelled by Potentials

Guglielmo Fucci, Klaus Kirsten and Pedro Morales

Abstract In this article we consider a piston modelled by a potential in the presence of extra dimensions. We analyze the functional determinant and the Casimir effect for this configuration. In order to compute the determinant and Casimir force we employ the zeta function scheme. Essentially, the computation reduces to the analysis of the zeta function associated with a scalar field living on an interval $[0,L]$ in a background potential. Although, as a model for a piston, it seems reasonable to assume a potential having compact support within $[0,L]$, we provide a formalism that can be applied to any sufficiently smooth potential.

1 Introduction

In recent years piston configurations have received a surging interest in the Casimir effect community. The main reason for this fact is that pistons allow for an unambiguous prediction of forces which turn out to be divergence free [2]. The piston is usually represented by an infinitely thin movable plate at which the field has to satisfy some ideal boundary conditions. Different boundary conditions and various shapes of cross-sections have been analyzed and, as expected, the force heavily depends on the different possible choices, see, e.g., [6, 7, 9, 12, 14, 15]. It is the aim of this article to represent pistons of finite thickness by compactly supported potentials. Physical properties of the pistons are encoded in the spectrum of the ordinary differential operator

$$P := -\frac{d^2}{dx^2} + V(x), \tag{1}$$

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_29,

where $x \in I = [0,L]$ and $V(x)$ models the piston. The points $x = 0$ and $x = L$ represent the positions of the fixed plates with suitable boundary conditions chosen. The piston at position a is represented by the potential $V(x)$ which has a support strongly concentrated around a.

It is this problem to which an analysis in the space $I \times \mathscr{N}$ reduces after separation of variables, where $\mathscr{N}$ describes the cross section of the piston and additional dimensions, represented by a smooth Riemannian manifold possibly with boundary.

Although ultimately, in the context of pistons, our interest is in potentials with compact support within I, the formalism is developed for general potential $V(x)$. We will use the zeta function scheme to evaluate various quantities of interest; for introductions to spectral zeta functions and its applications in physics see [4, 5, 8]. The relevant zeta function is represented as a contour integral where the boundary value of the unique solution to a given initial value problem enters. This representation is briefly described in Section 2 and subsequently used to evaluate the functional determinant, the Casimir energy and force. In order to analyze these quantities, the analytic continuation of the zeta function needs to be constructed. This will entail the knowledge of a certain asymptotic behavior of solutions to initial value problems, which is obtained in Section 3 using standard WKB techniques [1, 13]. The representation obtained can be used to find the functional determinant, Section 4, and the Casimir energy and force, Section 5. In Section 6 we restrict to compactly supported potentials and describe the resulting simplifications. In the Conclusions we summarize the most important findings and outline possible further applications of our approach.

2 Contour representation of the piston zeta function

Let $M = [0,L] \times \mathscr{N}$, where $\mathscr{N}$ represents the cross section of the piston and the additional Kaluza-Klein dimensions. For simplicity we assume Dirichlet plates at $x = 0$ and $x = L$ and we assume a sufficiently smooth potential $V(x)$ depending only on $x \in [0,L]$. With $y \in \mathscr{N}$, the relevant energy eigenvalues for a scalar field are determined by the second order differential operator

$$L = -\frac{\partial^2}{\partial x^2} - \Delta_{\mathscr{N}} + V(x), \tag{2}$$

together with Dirichlet boundary conditions at $x = 0$ and $x = L$ and unspecified boundary conditions at the boundary of $\mathscr{N}$. Using separation of variables we write eigenfunctions in the form

$$\phi(x,y) = X(x)\varphi(y),$$

where the $\varphi(y)$ are assumed as eigenfunctions of the Laplacian on $\mathscr{N}$,

$$-\Delta_{\mathscr{N}}\varphi_\ell(y) = \eta_\ell^2\varphi_\ell(y). \tag{3}$$

This implies that the eigenvalues λ of L are given as

$$\lambda = \nu^2 + \eta_\ell^2,$$

where ν^2 is determined by

$$\left(-\frac{\partial^2}{\partial x^2} + V(x)\right) X_\nu(x) = \nu^2 X_\nu(x), \qquad X_\nu(0) = X_\nu(L) = 0. \tag{4}$$

Formally, the zeta function of L can therefore be written as

$$\zeta(s) = \sum_{\ell,\nu} (\nu^2 + \eta_\ell^2)^{-s} \qquad \text{for } \Re s > \frac{D}{2}, \tag{5}$$

with $D = \dim(M)$, the dimension of M. Note that without specifying $\mathcal{N}$ the spectrum η_ℓ is not known, and also ν^2 will not be known unless $V(x)$ is one of the very few potentials allowing for a closed solution of eq. (4).

Despite this lack of knowledge an analytical continuation of $\zeta(s)$ in eq. (5) can be constructed and properties of $\zeta(s)$ on M can be given in terms of the zeta function of $\mathcal{N}$ defined by

$$\zeta_{\mathcal{N}}(s) = \sum_\ell \eta_\ell^{-2s} \qquad \text{for } \Re s > \frac{D-1}{2}.$$

The strategy for the analysis of $\zeta(s)$ in eq. (5) is to rewrite the series as a contour integral using the argument principle or Cauchy's residue theorem [3]. Instead of considering the eigenvalue problem in eq. (4) we consider the *initial value problem* [10, 11]

$$\left(-\frac{\partial^2}{\partial x^2} + V(x)\right) u_\mu(x) = \mu^2 u_\mu(x), \qquad u_\mu(0) = 0, \quad u'_\mu(0) = 1,$$

where $\mu \in \mathbb{C}$. The eigenvalues ν^2 of the original problem are recovered as solutions to the secular equation

$$u_\mu(L) = 0; \tag{6}$$

note, that $u_\mu(L)$ is an analytic function of μ. Eq. (5), for $\Re s > D/2$, can then be rewritten as

$$\zeta(s) = \frac{1}{2\pi i} \sum_\ell \int_\gamma d\mu (\mu^2 + \eta_\ell^2)^{-s} \frac{d}{d\mu} \ln u_\mu(L),$$

where γ encloses all solutions to eq. (6), which are assumed to be on the positive real axis; the changes necessary when zero modes or finitely many negative eigenvalues are present are given in [11].

Let us next consider the contributions from each ℓ by analyzing

$$\zeta_\ell(s) = \frac{1}{2\pi i} \int_\gamma d\mu (\mu^2 + \eta_\ell^2)^{-s} \frac{d}{d\mu} \ln u_\mu(L).$$

Deforming the contour, as usual, to the imaginary axis we find

$$\zeta_\ell(s) = \frac{\sin \pi s}{\pi} \int_{\eta_\ell}^{\infty} dk\, (k^2 - \eta_\ell^2)^{-s} \frac{d}{dk} \ln u_{ik}(L), \tag{7}$$

valid for $1/2 < \Re s < 1$. In order to construct a representation of $\zeta_\ell(s)$ that is valid in a region $\Re s < 1/2$, as is needed for the functional determinant and the Casimir energy, we add and subtract the large-k asymptotics of $u_{ik}(L)$.

3 Asymptotic behavior of boundary values for differential equations

The next mathematical task therefore is to determine the large-k asymptotics of the unique solution to the initial value problem

$$\left(-\frac{d^2}{dx^2} + V(x) + k^2\right) u_{ik}(x) = 0, \qquad u_{ik}(0) = 0, \quad u'_{ik}(0) = 1. \tag{8}$$

We note that the differential equation in (8) has exponentially growing and exponentially decaying terms [1, 13]. Although, ultimately, for (7) we will only need the exponentially growing part, we first have to consider linear combinations of the two in order to be able to impose the initial conditions in eq. (8).

It is convenient and standard to introduce

$$S(x,k) = \partial_x \ln \psi_k(x),$$

where $\psi_k(x)$ satisfies

$$\left(-\frac{d^2}{dx^2} + V(x) + k^2\right) \psi_k(x) = 0. \tag{9}$$

The differential equation satisfied by $S(x,k)$ turns out to be

$$S'(x,k) = k^2 + V(x) - S^2(x,k), \tag{10}$$

where the prime indicates differentiation with respect to x. As $k \to \infty$, the function $S(x,k)$ can be seen to have the asymptotic form

$$S(x,k) = \sum_{i=-1}^{\infty} k^{-i} S_i(x),$$

where the asymptotic orders $S_i(x)$ are given by

$$S_{-1}(x) = \pm 1, \quad S_0(x) = 0, \quad S_1(x) = \pm\frac{V(x)}{2}, \tag{11}$$
$$S_{i+1}(x) = \mp\frac{1}{2}\left(S_i'(x) + \sum_{j=0}^{i} S_j(x)S_{i-j}(x)\right).$$

It is clear that an arbitrary number of asymptotic orders can be easily evaluated using an algebraic computer program. The two different signs in (11) produce the indicated exponentially growing and decaying solutions $\psi_k(x)$ of (9). We denote solutions of (10) corresponding to these two signs by $S^+(x,k)$ and $S^-(x,k)$. The associated solutions of (9) then have the form

$$\psi_k^\pm(x) = A^\pm \exp\left\{\int_0^x dt\, S^\pm(t,k)\right\}.$$

The large-k behavior for $u_{ik}(x)$ is obtained by considering the linear combination

$$u_{ik}(x) = A^+ \exp\left\{\int_0^x dt\, S^+(t,k)\right\} + A^- \exp\left\{\int_0^x dt\, S^-(t,k)\right\},$$

together with the initial conditions in (8) still to be imposed. These initial conditions imply

$$A^+ = -A^-, \qquad A^+ = \frac{1}{S^+(0,k) - S^-(0,k)};$$

note, that without including the $S^-(x,k)$ part the initial conditions could not be satisfied.

We are now in the position to write out the large-k behavior for $u_{ik}(L)$. Let $E(k)$ denote exponentially damped terms as $k \to \infty$. First, we see that, as $k \to \infty$,

$$u_{ik}(L) = \frac{1}{S^+(0,k) - S^-(0,k)} \exp\left\{\int_0^L dt\, S^+(t,k)\right\} + E(k),$$

and therefore

$$\begin{aligned}\ln u_{ik}(L) &= -\ln\left(S^+(0,k) - S^-(0,k)\right) + \int_0^L dt\, S^+(t,k) + E(k)\\ &= -\ln(2k) + k + \sum_{j=0}^{\infty} d_j k^{-j} + E(k),\end{aligned}$$

where the d_j's are easily determined from eq. (11). For instance, the first six are given explicitly by

$$d_0 = 0, \qquad d_1 = \frac{1}{2}\int_0^L dt\, V(t), \qquad d_2 = -\frac{1}{4}[V(L) - V(0)],$$

$$d_3 = \frac{1}{8}[V'(L) - V'(0)] - \frac{1}{8}\int_0^L dt\, V^2(t),$$

$$d_4 = -\frac{1}{16}[V''(L) - V''(0)] + \frac{1}{8}[V^2(L) - V^2(0)],$$

$$d_5 = \frac{1}{32}[V^{(3)}(L) - V^{(3)}(0)] - \frac{5}{32}[V(L)V'(L) - V(0)V'(0)] + \frac{1}{16}\int_0^L dt\, V^3(t) - \frac{1}{32}\int_0^L dt\, V(t)V''(t). \tag{12}$$

In what follows, the potential is always assumed to be as smooth as necessary for the asymptotic orders given by these formulas, and higher ones if needed, to be well defined.

Subtracting and adding the asymptotic behavior up to the order k^{-N}, the zeta function naturally splits into two parts,

$$\zeta_\ell(s) = \zeta_\ell^{(f)}(s) + \zeta_\ell^{(as)}(s),$$

where

$$\zeta_\ell^{(f)}(s) = \frac{\sin \pi s}{\pi}\int_{\eta_\ell}^{\infty} dk\, (k^2 - \eta_\ell^2)^{-s}\frac{d}{dk}\left\{\ln u_{ik}(L) - k + \ln(2k) - \sum_{j=0}^{N} d_j k^{-j}\right\}, \tag{13}$$

$$\zeta_\ell^{(as)}(s) = \frac{\sin \pi s}{\pi}\int_{\eta_\ell}^{\infty} dk\, (k^2 - \eta_\ell^2)^{-s}\frac{d}{dk}\left\{k - \ln(2k) + \sum_{j=0}^{N} d_j k^{-j}\right\}. \tag{14}$$

The k-integrals in $\zeta_\ell^{(as)}(s)$ are easily done, yielding

$$\zeta_\ell^{(as)}(s) = \frac{1}{2\Gamma(s)}\left\{\frac{\Gamma\left(s - \frac{1}{2}\right)}{\sqrt{\pi}}\eta_\ell^{1-2s} - \Gamma(s)\eta_\ell^{-2s} - \sum_{j=1}^{N} j d_j \frac{\Gamma\left(s + \frac{j}{2}\right)}{\Gamma\left(1 + \frac{j}{2}\right)}\eta_\ell^{-j-2s}\right\}.$$

After summing over ℓ, the representation obtained is then valid for $1 > \Re s > (D - N - 2)/2$ and it reads

$$\zeta^{(f)}(s) = \frac{\sin \pi s}{\pi}\sum_\ell \int_{\eta_\ell}^{\infty} dk\, (k^2 - \eta_\ell^2)^{-s}\frac{d}{dk}\left\{\ln u_{ik}(L) - k + \ln(2k) - \sum_{j=0}^{N}\frac{d_j}{k^j}\right\}, \tag{15}$$

$$\zeta^{(as)}(s) = \frac{1}{2\Gamma(s)}\left\{\frac{\Gamma\left(s-\frac{1}{2}\right)}{\sqrt{\pi}}\zeta_{\mathscr{N}}\left(s-\frac{1}{2}\right) - \Gamma(s)\zeta_{\mathscr{N}}(s) - \sum_{j=1}^{N} jd_j \frac{\Gamma\left(s+\frac{j}{2}\right)}{\Gamma\left(1+\frac{j}{2}\right)}\zeta_{\mathscr{N}}\left(s+\frac{j}{2}\right)\right\}. \tag{16}$$

In particular, choosing $N = D-1$, respectively $N = D$, the representation can be used to compute the determinant, respectively the Casimir energy. This will be done in the next sections.

4 Functional determinants

In this section we will evaluate the functional determinant, or, equivalently, $\zeta'(0)$, using the representation of $\zeta(s)$ given by eqs. (15) and (16). The contribution from $\zeta^{(f)}(s)$ is trivially obtained and it reads

$$\zeta^{(f)\prime}(0) = -\sum_{\ell}\left[\ln u_{i\eta_\ell}(L) - \eta_\ell + \ln(2\eta_\ell) - \sum_{j=1}^{D-1} d_j \eta_\ell^{-j}\right],$$

as the sum over ℓ converges by construction. For the evaluation of the contribution from $\zeta^{(as)}(s)$, let us note that in the general situation considered, namely $\mathscr{N}$ can be a manifold of any dimension with or without boundary, for $j = 1, \ldots, (D-1)/2$, we have the following Laurent expansion

$$\zeta_{\mathscr{N}}\left(\frac{j}{2}+\varepsilon\right) = \frac{1}{\varepsilon}\text{Res }\zeta_{\mathscr{N}}(j/2) + \text{FP }\zeta_{\mathscr{N}}(j/2) + \mathscr{O}(\varepsilon). \tag{17}$$

Using standard properties of Γ-functions, this immediately gives

$$\zeta^{(as)\prime}(0) = -\text{FP }\zeta_{\mathscr{N}}\left(-\frac{1}{2}\right) + \text{Res }\zeta_{\mathscr{N}}\left(-\frac{1}{2}\right)(-2+\ln 4) - \frac{1}{2}\zeta'_{\mathscr{N}}(0) - \sum_{j=1}^{D-1} d_j\left(\text{FP }\zeta_{\mathscr{N}}\left(\frac{j}{2}\right) + \text{Res }\zeta_{\mathscr{N}}\left(\frac{j}{2}\right)\left(\gamma + \psi\left(\frac{j}{2}\right)\right)\right).$$

Adding up these two pieces gives the answer for the functional determinant on M in terms of the spectral zeta function on $\mathscr{N}$. Once the 'base' manifold $\mathscr{N}$ is specified, more explicit results can be given.

5 Casimir energy

For the Casimir energy we set $N = D$ and use again the Laurent series (17) for $\zeta_{\mathcal{N}}(j/2+\varepsilon)$. In this case, we find

$$\zeta^{(f)}(-1/2) = -\frac{1}{\pi}\sum_{\ell}\int_{\eta_\ell}^{\infty} dk\,(k^2-\eta_\ell^2)^{1/2}\frac{d}{dk}\left\{\ln u_{ik}(L) - k + \ln(2k) - \sum_{j=1}^{D}\frac{d_j}{k^j}\right\},$$

$$\begin{aligned}\zeta^{(as)}(-1/2+\varepsilon) &= \frac{1}{\varepsilon}\left\{\frac{1}{4\pi}\zeta_{\mathcal{N}}(-1) - \frac{1}{2}\text{Res}\;\zeta_{\mathcal{N}}(-1/2) + \frac{d_1\zeta_{\mathcal{N}}(0)}{2\pi}\right.\\ &\left.\qquad + \sum_{j=2}^{D}\frac{d_j}{2\sqrt{\pi}}\frac{\Gamma\left(\frac{j-1}{2}\right)}{\Gamma\left(\frac{j}{2}\right)}\text{Res}\;\zeta_{\mathcal{N}}\left(\frac{j-1}{2}\right)\right\}\\ &+\frac{1}{4\pi}\left(\zeta'_{\mathcal{N}}(-1)+\zeta_{\mathcal{N}}(-1)(-1+\ln 4)\right) - \frac{1}{2}\text{FP}\;\zeta_{\mathcal{N}}(-1/2)\\ &+\frac{d_1}{2\pi}\left(\zeta'_{\mathcal{N}}(0)+\zeta_{\mathcal{N}}(0)(-2+\ln 4)\right)\\ &+\sum_{j=2}^{D}\frac{d_j}{2\sqrt{\pi}}\frac{\Gamma\left(\frac{j-1}{2}\right)}{\Gamma\left(\frac{j}{2}\right)}\left(\text{FP}\;\zeta_{\mathcal{N}}\left(\frac{j-1}{2}\right) + \text{Res}\;\zeta_{\mathcal{N}}\left(\frac{j-1}{2}\right)\left[H_{(k-3)/2}-2+\ln 4\right]\right),\end{aligned}$$

with the harmonic numbers

$$H_n = \sum_{i=1}^{n}\frac{1}{i}.$$

Multiplying by $1/2$ and adding up the two pieces, the Casimir energy follows. Again, when $\mathcal{N}$ is specified more explicit results can be given. In general, the Casimir energy is divergent, but as seen below, the resulting forces on pistons are finite.

6 Compactly supported potentials

In order to reasonably talk about the force on a piston modeled by a potential we now assume the potential $V(x)$ to have compact support within the interval $[0,L]$; namely, we assume that it does not vanish for $x \in [a-\varepsilon, a+\varepsilon] \subset [0,L]$. This can be considered a model for a piston of thickness 2ε. In this case the asymptotic behavior of $u_{ik}(L)$ will be independent of a as the integrals over $V(x)$ and its powers and derivatives are independent of a (as long as the support is within the interval $[0,L]$). This can be seen explicitly in (12). It is here that sufficiently smooth potentials become a necessary assumption in order for these formulas, and the corresponding ones for higher orders, to be well defined. These formulas also simplify further because $V(0) = V(L) = 0$. The corresponding result can be used to write down the Casimir energy. Because of the independence of the asymptotic behavior of $u_{ik}(L)$ from a, in these circumstances we immediately obtain

$$F_{Cas} = -\frac{1}{2}\frac{\partial}{\partial a}\zeta\left(-\frac{1}{2}\right) = \frac{1}{2\pi}\sum_{\ell}\int_{\eta_\ell}^{\infty} dk\,(k^2-\eta_\ell^2)^{1/2}\frac{\partial}{\partial a}\frac{\partial}{\partial k}\ln u_{ik}(L).$$

The force, in particular its sign, is encoded in boundary values of an initial value problem to an ordinary differential equation.

Results for the determinant are also easily written down from Section 4 with the simpler d_j's used.

7 Conclusions and outlook

In this article we have provided a formalism that allows for the evaluation of functional determinants and Casimir energies and forces for the configuration of a generalized piston. Results are as explicit as they can be without specifying the cross section and the additional Kaluza-Klein dimensions that might be present.

With these results available, one can obtain very explicit answers for a given cross section of the piston and specified Kaluza-Klein dimensions. Furthermore one can use the potential to mimic material properties of the piston. For all cases, with very few exceptions for particular potentials, a numerical determination of the boundary value $u_{ik}(L)$ will be necessary.

Along the same lines different boundary conditions at $x = 0$ and $x = L$ can be considered.

Work along these lines is currently in progress.

Acknowledgements KK is supported by National Science Foundation grant PHY–0554849.

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Local ζ-functions, stress-energy tensor, field fluctuations, and all that, in curved static spacetime

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Abstract This is a quick review on some technology concerning the local *zeta function* applied to Quantum Field Theory in curved static (thermal) spacetime to regularize the stress energy tensor and the field fluctuations. Dedicated to Prof. Emilio Elizalde on the occasion of his 60th birthday.

1 Quasifree QFT in curved static manifolds, Euclidean approach ζ-function technique.

1.1 *The ζ-function determinant.* Suppose we are given a $n \times n$ positive-definite Hermitian matrix A with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. One can define the complex-valued function

$$\zeta(s|A) = \sum_{j=1}^{n} \lambda_j^{-s}, \tag{1}$$

where $s \in \mathbb{C}$. (Notice that λ_j^{-s} is well-defined since $\lambda_j > 0$.) By direct inspection one proves that:

$$\det A = e^{-\frac{d\zeta(s|A)}{ds}|_{s=0}}. \tag{2}$$

This trivial result can be generalized to provide a useful definition of the determinant of an operator working in an infinite-dimensional Hilbert space. To this end, focus on a non-negative self-adjoint operator A whose spectrum is discrete and each eigenspace has a finite dimension, and consider the series with $s \in \mathbb{C}$ (the prime on the sum henceforth means that any possible null eigenvalues is omitted)

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,

DOI 10.1007/978-3-642-19760-4_30,

$$\zeta(s|A) := \sum_j{}' \lambda^{-s} . \tag{3}$$

Looking at (2), the idea [Ha77] is to define, once again.

$$\det A = e^{-\frac{d\zeta(s|A)}{ds}|_{s=0}} ,$$

where now the function ζ on the right-hand side is, in the general case, the *analytic continuation of the function defined by the series* (3) in its convergence domain, since the series may diverge at $s = 0$ – and this is the standard situation in the infinite-dimensional case! – provided that the analytic extension really reaches a neignorhood of the point $s = 0$. The interesting fact is that this procedure truly works in physically relevant cases, related to QFT in curved spacetime, producing meaningful results as we go to discuss in the following section.

1.2 *Thermal QFT in static spacetimes*. A smooth globally hyperbolic spacetime (M,g) is said to be *static* if it admits a (local) time-like Killing vector field ∂_t normal to a smooth spacelike Cauchy surface Σ. Consequently, there are (local) coordinate frames $(x^0,x^1,x^2,x^3) \equiv (t,x)$ where $g_{0i} = 0$ $(i = 1,2,3)$ and $\partial_t g_{\mu\nu} = 0$ and x are local coordinates on Σ. Though the results we are going to present may be generalize to higher spin fields, we henceforth stick to the case of a real scalar field ϕ propagating in M and satisfying an equation of motion of the form

$$P\phi = 0 , \tag{4}$$

where $P := -\nabla_\mu \nabla^\mu + V$, V being a smooth scalar field like

$$V(x) := \xi R + m^2 + V'(x) . \tag{5}$$

We also assume that V' satisfies $\partial_t V' = 0$ so that the space of solutions of (4) is invariant under t-displacements. Furthermore $\xi \in \mathbb{R}$, is a constant, R is the scalar curvature and m^2 the squared mass of the particles associated to the field. The domain of P is the space of real-valued C^∞ functions compactly supported Cauchy data on Σ. In the quasifree case, a straightforward way to define a QFT consistes of the assignment of a suitable Green function of the operator P [FR87], in particular the Feynman propagator $G_F(x,x')$ or, equivalently, the Wightman functions $W_\pm(x,x')$. Then the GNS theorem (e.g. see [KW91]) allows one to construct a corresponding Fock realization of the theory. In a globally hyperbolic static spacetime it is possible to chose t-invariant Green functions. In that case, in static coordinates, one performs the Wick rotation obtaining the Euclidean formulation of the same QFT. This means that (locally) one can pass from the Lorentzian manifold (M,g) to a Riemannian manifold $(M_E, g^{(E)})$ by the analytic continuation $t \to i\tau$ where $t,\tau \in \mathbb{R}$. This defines a (local) Killing vector ∂_τ in the Riemannian manifold M_E and a corresponding (local) "static" coordinate frame (τ,x) therein. As is well-known [FR87], when the orbits of the Euclidean time τ are closed with period β, $T = 1/\beta$ has to be interpreted as the temperature of the quantum state because the Wightman two-point function of the associated quasifree state satisfy the KMS condition at the inverse temperature β. In this approach, the Feynman propagator $G_F(t-t',x,x')$ determines – and (generally speaking [FR87]) it is completely determined by – a proper Green function (in the spectral theory sense) $S_\beta(\tau-\tau',x,x')$ of a corresponding self-adjoint extension A of the operator

$$A' := -\nabla_a^{(E)}\nabla^{(E)a} + V(x) \, : \, C_0^\infty(M_E) \to L^2(M_E, d\mu_{g^{(E)}})\,. \tag{6}$$

$S_\beta(\tau-\tau',x,x')$ is periodic with period β in the $\tau-\tau'$ entry and it is said the Schwinger function. As a matter of fact, S_β turns out to be the integral kernel of A^{-1} when $A>0$.

The *partition function* of the quantum state associated to S_β is the functional integral evaluated over the field configurations periodic with period β in the Euclidean time

$$Z_\beta = \int \mathscr{D}\phi \, e^{-S_E[\phi]}\,, \tag{7}$$

the Euclidean action S_E being ($d\mu_{g^{(E)}} := \sqrt{g^{(E)}}d^4x$)

$$S_E[\phi] = \frac{1}{2}\int_M d\mu_{g^{(E)}}(x)\,\phi(x)(A\phi)(x)\,. \tag{8}$$

Thus, extending the analogous result for finite dimensional Gaussian integral, one has

$$Z_\beta = \left\{\det\left(\frac{A}{\mu^2}\right)\right\}^{-1/2}\,, \tag{9}$$

where μ is a mass scale which is necessary for dimensional reasons. To give a sensitive interpretation of that determinant, the idea [Ha77] is to try to exploit (2). If M_E is a D-dimensional Riemannian compact manifold and A' is bounded below by some constant $b \geq 0$, A' admits the *Friedrichs* self-adjoint extension A which is also bounded below by the same bound of A', moreover the spectrum of A is discrete and each eigenspace has a finite dimension. Then, as we said, one can consider the series

$$\zeta(s|A/\mu^2) := \sum_j{}' \left(\frac{\lambda_j}{\mu^2}\right)^{-s}\,. \tag{10}$$

Remarkably [Ha77, BCEMZ03], in the given hypotheses, the series above converges for $Re\; s > D/2$ and it is possible to continue the right-hand side above into a meromorphic function of s which is regular at $s=0$. Following (2) and taking the presence of μ into account, we define:

$$Z_\beta := e^{\frac{1}{2}\frac{d}{ds}|_{s=0}\zeta(s|A/\mu^2)}\,, \tag{11}$$

where the function ζ on the right-hand side is the analytic continuation of that defined in (10). It is possible the define the ζ function in terms of the heat kernel of the operator A, $K(t,x,y|A)$ [BCEMZ03]. This is the smooth integral kernel of the (Hilbert-Schmidt, trace-class) operators e^{-tA}, $t>0$. One has, for $Re\ s > D/2$,

$$\zeta(s|A/\mu^2) = \int_M d\mu_{g^{(E)}}(x) \int_0^{+\infty} dt\, \frac{\mu^{2s} t^{s-1}}{\Gamma(s)} \left[K(t,x,x|A) - P_0(x,x|A)\right] , \qquad (12)$$

$P(x,y|A)$ is the integral kernel of the projector on the null-eigenvalues eigenspace of A.
When M_E is not compact, the spectrum of A may included a continuous-spectrum part, however, it is still possible to generalize the definitions and the results above considering suitable integrals on the spectrum of A, provided A is strictly positive (e.g, see [Wa79]).
Another very useful tool is the *local* ζ function that can be defined in two differen,t however equivalent, ways [Wa79, Mo98, BCEMZ03]:

$$\zeta(s,x|A/\mu^2) = \int_0^{+\infty} dt\, \frac{\mu^{2s} t^{s-1}}{\Gamma(s)} \left[K(t,x,x|A) - P_0(x,x|A)\right] , \qquad (13)$$

and, ϕ_j being the smooth eigenvector of the eigenvalue λ_j,

$$\zeta(s,x|A/\mu^2) = {\sum_j}' \left(\frac{\lambda_j}{\mu^2}\right)^{-s} \phi_j(x)\phi_j^*(x) . \qquad (14)$$

Both the integral and the series converges for $Re\ s > D/2$ and the local zeta function enjoys the same analyticity properties of the integrated ζ function. For future convenience it is also useful to define, in the sense of the analytic continuation,

$$\zeta(s,x,y|A/\mu^2) = \int_0^{+\infty} dt\, \frac{\mu^{2s} t^{s-1}}{\Gamma(s)} \left[K(t,x,y|A) - P_0(x,y|A)\right] \qquad (15)$$

(see [Mo98, Mo99] for the properties of this off-diagonal ζ-function). In the framework of the ζ-function regularization framework, the *effective Lagrangian* is defined as

$$\mathscr{L}(x|A)_{\mu^2} := \frac{1}{2}\frac{d}{ds}|_{s=0}\zeta(s,x|A/\mu^2) , \qquad (16)$$

and thus, in a thermal theory, $Z_\beta = e^{-S_\beta}$ where $S_\beta = \int d\mu_g \mathscr{L}_{\beta\mu^2}$. A result which generalizes to any dimension an earlier results by Wald [Wa79] is the following [Mo98]. The above-defined effective Lagrangian can be computed by a *point-splitting procedure*: For D even

$$\mathscr{L}(y|A)_{\mu^2} = \lim_{x\to y} \left\{ -\int_0^{+\infty} \frac{dt}{2t} K(t,x,y|A) - \frac{a_{D/2}(x,y)}{2(4\pi)^{D/2}} \ln \frac{\mu^2 \sigma(x,y)}{2} \right.$$

$$+\sum_{j=0}^{D/2-1}\left(\frac{D}{2}-j-1\right)!\frac{a_j(x,y|A)}{2(4\pi)^{D/2}}\left(\frac{2}{\sigma(x,y)}\right)^{D/2-j}\Bigg\}-2\gamma\frac{a_{D/2}(y,y)}{2(4\pi)^{D/2}} \tag{17}$$

for D odd (notice that μ disappears)

$$\mathscr{L}(y|A)_{\mu^2}=\lim_{x\to y}\left\{-\int_0^{+\infty}\frac{dt}{2t}K(t,x,y|A)-\sqrt{\frac{2}{\sigma(x,y)}}\frac{a_{(D-1)/2}(x,y)}{2(4\pi)^{D/2}}\right.$$
$$\left.+\sum_{j=0}^{(D-3)/2}\frac{(D-2j-2)!!}{2^{(D+1)/2-j}}\frac{a_j(x,y|A)}{2(4\pi)^{D/2}}\left(\frac{2}{\sigma(x,y)}\right)^{D/2-j}\right\}. \tag{18}$$

Above, $\sigma(x,y)$ is one half the square of the geodesical distance of x from y and the coefficients a_j are the well-known off-diagonal coefficients of the small-t expansion of the heat-kernel. These coefficients, in spite of their non symmetric definition, turns out to by invariant when interchanging x and y [Mo99, Mo00].

2 Stress-energy tensor and field fluctuations

2.1 *Generalizations of the local ζ function technique.* Physically relevant quantities are the *(quantum) field fluctuation* and the *averaged (quantum) stress tensor*, respectively:

$$<\phi^2(x)>=\frac{\delta}{\delta J(x)}|_{J\equiv 0}\ln\int\mathscr{D}\phi\, e^{-S_E+\int d\mu_{g^{(E)}}\phi^2 J}, \tag{19}$$

$$<T_{ab}(x)>=\frac{2}{\sqrt{g^{(E)}(x)}}\frac{\delta}{\delta g^{(E)ab}(x)}\ln\int\mathscr{D}\phi\, e^{-S_E[g^{(E)}]}. \tag{20}$$

A standard method to compute them is the so-called *point-splitting procedure* [BD82, Fu91, Wa94, Mo00, Mo03]. It is however possible to extend the ζ-function technique [Mo97, IM98, Mo98, Mo99, Mo00] to define suitable ζ functions regularizing those quantities directly, similarly to what done for the effective Lagrangian. Consider the stress tensor. The idea relies upon the following chain of formal identities [Mo97]

$$\sqrt{g^{(E)}(x)}<T_{ab}(x)>\text{``}=\text{''}2\frac{\delta}{\delta g^{(E)ab}(x)}\ln Z_\beta\text{``}=\text{''}\frac{\delta}{\delta g^{(E)ab}(x)}\frac{d}{ds}|_{s=0}\zeta(s|A/\mu^2)$$
$$\text{``}=\text{''}\frac{\delta}{\delta g^{(E)ab}(x)}\frac{d}{ds}|_{s=0}\sum_j{}'\left(\frac{\lambda_j}{\mu^2}\right)^{-s}\text{``}=\text{''}\frac{d}{ds}|_{s=0}\mu^{-2s}\sum_j{}'\frac{\delta\lambda_j^{-s}}{\delta g^{(E)ab}(x)}. \tag{21}$$

Following this route, one *define* the ζ-regularized (or renormalized) stress tensor as

$$<T_{ab}(x|A)>_{\mu^2}:=\frac{1}{2}\frac{d}{ds}|_{s=0}Z_{ab}(s,x|A/\mu^2), \tag{22}$$

where, *in the sense of the analytic continuation of the left-hand side*

$$Z_{ab}(s,x|A/\mu^2) := 2\sum_j{}' \mu^{-2s} \frac{\delta\lambda_j^{-s}}{\delta g^{ab}(x)}\,. \tag{23}$$

The difficult problem is now twofold: how to compute the functional derivative in the right-hand side of (23) and whether or not the series in the right-hand side of (23) defines an analytic function of s in a neighborhood of $s = 0$. We have the result [Mo97, Mo99]:

Theorem 1. *If M_E is compact, $A \geq 0$ and $\mu^2 > 0$, then $Z_{ab}(s,x|A/\mu^2)$ is well-defined and is a C^∞ function of x which is also meromorphic in $s \in \mathbb{C}$. In particular, it is analytic in a neighborhood of $s = 0$.*

The result above has been checked even in several noncompact manifolds (containing singularities) [Mo97, BCEMZ03]. In that case, the summation in the right-hand side of (23) has to be replaced by a suitable spectral integration. The series in the right-hand side of (23) can be explicitly computed as [Mo97, Mo99]:

$$s\sum_j{}' \left\{ \frac{2}{\mu^2}\left(\frac{\lambda_j}{\mu^2}\right)^{-s-1} T_{ab}[\phi_j,\phi_j^*](x) + g_{ab}(x)\left(\frac{\lambda_j}{\mu^2}\right)^{-s} \right\},$$

$T_{ab}[\phi_j,\phi_j^*](x)$ being the classical stress tensor evaluated on the modes of A (see [Mo97, Mo99, BCEMZ03] for details). The series converges for $Re\ s > 3D/2 + 2$. It is similarly possible to define a ζ-function regularizing the field fluctuation [IM98, Mo98]:

$$<\phi^2(x|A)>_{\mu^2} := \frac{d}{ds}|_{s=0}\Phi(s,x|A/\mu^2)\,,$$

where

$$\Phi(s,x|A/\mu^2) := \frac{s}{\mu^2}\zeta(s+1,x|A/\mu^2)\,. \tag{24}$$

The properties of these functions have been studied in [IM98, Mo98] and several applications on concrete cases are considered (*e.g.* cosmic-string spacetime and homogeneous spacetimes). In particular, in [Mo98], the problem of the change of the parameter m^2 in the field fluctuations has been studied.

2.2 *Physically meaningfulness of the procedures.* We are now interested in the physical meaningfulness of the presented regularization techniques. The following general results strongly suggest that it is the case [Mo97, Mo99, Mo03].

Theorem 2. *If M_E is compact, $A \geq 0$ and $\mu^2 > 0$, and the averaged quantities above are those defined above in terms of local ζ-function regularization, then*

(a) $< T_{ab}(x|A) >_{\mu^2}$ *defines a* C^∞ *symmetric tensorial field.*
(b) *Similarly to the classical result,*

$$\nabla^b < T_{bc}(x|A) >_{\mu^2} = -\frac{1}{2} < \phi^2(x|A) >_{\mu^2} \nabla_c V'(x)\,. \tag{25}$$

(c) *Concerning the trace of the stress tensor, it is naturally decomposed in the classical and the known quantum anomalous part*

$$g^{ab}(x) < T_{ab}(x|A) >_{\mu^2} = \left(\frac{\xi_D - \xi}{4\xi_D - 1}\Delta - m^2 - V'(x)\right) < \phi^2(x|A) >_{\mu^2} + \delta_D \frac{a_{D/2}(x,x|A)}{(4\pi)^{D/2}} - P_0(x,x|A)\,, \tag{26}$$

where $\delta_D = 0$ *if* D *is odd and* $\delta_D = 1$ *if* D *is even,* $\xi_D = (D-2)/[4(D-1)]$.
(d) *for any* $\alpha > 0$

$$< T_{ab}(x|A) >_{\alpha\mu^2} = < T_{ab}(x|A) >_{\mu^2} + t_{ab}(x)\ln\alpha\,, \tag{27}$$

where, the form of $t_{ab}(x)$ *which depends on the geometry only and is in agreement with Wald's axioms* [Wa94], *has been given in* [Mo99, Mo03].
(e) *In the case* $\partial_0 = \partial_\tau$ *is a global Killing vector, the manifold admits periodicity* β *along the lines tangent to* ∂_0 *and* M *remains smooth (near any fixed points of the Killing orbits) fixing arbitrarily* β *in a neighborhood and, finally,* Σ *is a global surface everywhere normal to* ∂_0*, then*

$$\frac{\partial}{\partial\beta}\ln Z(\beta)_{\mu^2} = \int_\Sigma dx\sqrt{g(x)} < T^0_0(x,\beta|A) >_{\mu^2}\ . \tag{28}$$

Another general achievement regards the possibility to recover the Lorentzian theory from the Euclidean one [Mo99]:

Theorem 3. *Let* M_E *be compact,* $A \geq 0$, $\mu^2 > 0$. *Also assume that* M_E *is static with global Killing time* ∂_τ *and (orthogonal) global spatial section* Σ *and finally,* $\partial_\tau V' \equiv 0$. *Then*
(a) $\partial_\tau < \phi^2(x|A) >_{\mu^2} \equiv 0$;
(b) $\partial_\tau \mathscr{L}(x|A)_{\mu^2} \equiv 0$;
(c) $\partial_\tau < T_{ab}(x|A) >_{\mu^2} \equiv 0$;
(d) $< T_{0i}(x|A) >_{\mu^2} \equiv 0$ *for* $i = 1,2,3,...,D-1$
where the averaged quantities above are those defined above in terms of local ζ*-function regularization and coordinates* $\tau \equiv x^0, x \in \Sigma$ *are employed.*

These properties allow one to continue the Euclidean considered quantities into imaginary values of the coordinate $\tau \mapsto it$ obtaining *real* functions of the Lorentzian time t.
Some of the properties above (regarding Thm.1, Thm. 2, Thm.3) have been found

to be valid in some noncompact manifolds too (Rindler spacetime, cosmic string spacetime, Einstein's open spacetime, H^N spaces, Gödel spacetime, BTZ spacetime) [Mo97, IM98, Ca98, Ra98, Ra98b, Ra99, Ra05, BMVZ98, RF02, SS04, AMR05]. In particular, the presented theory has been successfully exploited to compute the quantum back reaction on the three-dimensional BTZ metric [BMVZ98] in the case of the singular ground state containing a naked singularity. A semiclassical implementation of the cosmic censorship conjecture has been found in that case.

2.3. *Interplay of zeta-function approach and point-splitting technique.* The procedure of the point-splitting to renormalize the field fluctuation as well as the stress tensor [BD82, Wa94, Mo00, Mo03], when the two-point functions are referred to *quasifree Hadamard-states*, can be summarized as

$$< \phi^2(y) >_{\rm ps} = \lim_{x\to y} \{G(x,y) - H(x,y)\} \,, \tag{29}$$

$$< T_{ab}(y) >_{\rm ps} = \lim_{x\to y} \mathscr{D}_{ab}(x,y) \{G(x,y) - H(x,y)\} + g_{ab}(y)Q(y) \,, \tag{30}$$

where $G(x,y)$ is the symmtric part the two-point Wightman function of the considered quantum state or, in Euclidean approach, the corresponding Schwinger function. $H(x,y)$ is the *Hadamard parametrix* which depends on the *local* geometry only and takes the short-distance singularity into account. $H(x,y)$ is represented in terms of a truncated series of functions of $\sigma(x,y)$. The operator $\mathscr{D}_{ab}(x,y)$ is a bi-tensorial operator obtained by "splitting" the argument of the classical expression of the stress tensor (see [Mo99, Mo03]). Finally $Q(y)$ is a scalar obtained by imposing several physical conditions (essentially, the appearance of the conformal anomaly, the conservation of the stress tensor and the triviality of the Minkowskian limit) [Wa94] in the left-hand side of (30) (see [BD82, Fu91, Wa94, Mo99] for details). More recently, in the framework of Lorentzian generally locally covariant algebraic quantum field theory in curved spacetime, it has established [Mo03] that Q can be omitted, redefining the classical stress-energy tensor, and thus $\mathscr{D}_{ab}(x,y)$, into a way that it does not affect the classical expression of $T_{\mu\nu}$ when computed on solutions of the equations of motion, improving the point-splitting procedure. See [Hac10] where that point-splitting procedure is discussed and applied especially to cosmology. In geodesically convex neighborhoods:

$$\begin{aligned} H_\mu(x,y) = {} & \frac{\sum_{j=0}^{L} u_j(x,y)\sigma(x,y)^j}{(4\pi)^{D/2}(\sigma(x,y)/2)^{D/2-1}} + \delta_D \left[\sum_{j=0}^{M} v_j(x,y)\sigma(x,y)^j \ln\left(\frac{\mu^2\sigma(x,y)}{2}\right)\right] \\ & + \delta_D \sum_{j=0}^{N} w_j(x,y)\sigma(x,y)^j \,. \end{aligned} \tag{31}$$

L,M,N are fixed integers (see [Mo99, Mo03] for details), $\delta_D = 0$ if D is odd and $\delta_D = 1$ otherwise. The coefficients u_j and v_j are smooth functions of (x,y) which are completely determined by the local geometry. The coefficients w_j are determined once one has fixed w_0, and they are omitted [Mo03] when dropping Q. Dealing with Euclidean approaches, it is possible to explicitly compute u_j and v_j in terms of heat-kernel coefficients [Mo98, Mo99]. One has the following result [Mo98, Mo99].

Theorem 4. *If M_E is compact, $A \geq 0$ and $\mu^2 > 0$, and the averaged quantities in the left-hand side below are those defined above in terms of local ζ-function regularization, then*

$$< \phi^2(y|A) >_{\mu^2} = \lim_{x \to y} \left\{ G(x,y) - H_{\mu'}(x,y) \right\} , \tag{32}$$

$$< T_{ab}(y|A) >_{\mu^2} = \lim_{x \to y} \mathscr{D}_{ab}(x,y) \left\{ G(x,y) - H_{\mu'}(x,y) \right\} + g_{ab}(y) Q(y) , \tag{33}$$

where $G(x,y) = \zeta(1,x,y|A/\mu^2)$ given in (15), *$H_{\mu'}$ is completely determined by (31) with the requirement*

$$w_0(x,y) := -\frac{a_{D/2-1}(x,y|A)}{(4\pi)^{D/2}} [2\gamma + \ln \mu'^2] , \tag{34}$$

and the term Q is found to be

$$Q(y) = \frac{1}{D} \left(-P_0(y,y|A) + \delta_D \frac{a_{D/2}(y,y|A)}{(4\pi)^{D/2}} \right) . \tag{35}$$

If $\mathscr{D}_{ab}$ is defined in order to drop Q in the right-hand side of (33), H_{μ^2} is determined by fixing $w_0(x,y) = 0$. The scales μ and μ' satisfies $\mu = c\mu'$ for some constant $c > 0$.

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Ergodic Solenoidal Geometry

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Abstract We present a survey of recent results in the geometry of ergodic solenoids. We discuss the ideas behind the theory and its perspectives. Dedicated to Emilio Elizalde in his 60th anniversary.

1 Introduction.

Geometry is at the origin of numerous applications of Mathematics to other fields, and to Mathematics itself. Classical Differential Geometry is nowadays a fundamental tool in Theoretical Physics. Needless to say that it is one of the most successful modern interaction and has marked the development of both fields. The objects of classical Differential Geometry are manifolds. One can envision many other geometric theories whose fundamental objects can be very different from classical manifolds. This idea is already present in Riemann's fundamental Memoir on the Foundations of Geometry.

We present in this article recent results on *Ergodic Solenoidal Geometry*. We expose an informal presentation of the theory and we refer to our recent articles [2][3][4][5][6][7] for precise definitions, theorems in full generality, and complete proofs.

Ergodic Solenoidal Geometry is a generalization of Differential Geometry where the central objects that extend manifolds are *uniquely ergodic solenoids*. Roughly speaking a *solenoid* is an abstract foliated space by finite dimensional leafs with transverse structure embedding into a finite dimensional space. Thus, contrary to other theories whose objects are foliated spaces, here we request some sort of finite

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Partially supported through grant MEC (Spain) MTM2007-63582 and CNRS (UMR 7539).

S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_31,

dimensional transverse structure. This hypothesis is natural considering that our aim is to generalize finite dimensional Differential Geometry and not, for example, Banachian Differential Geometry. Example of solenoids are manifold with a foliation, but we have many more, as the dyadic solenoid $\hat{\mathbb{T}} = \varprojlim\{\mathbb{T} \to \mathbb{T};\ x \mapsto 2x\}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

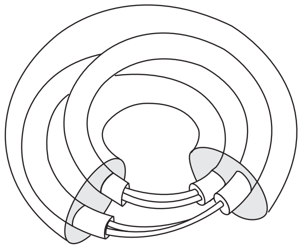

Fig. 1 The dyadic solenoid

The objects corresponding to compact and connected manifolds (we shall restrict to these from now on) are compact connected solenoids, but we need to restrict furthermore the structure in order to be able to generalize basic properties. Note that a compact manifold is a solenoid with trivial atomic transverse structure and a unique leaf. We consider solenoids that are topologically minimal, that is, all leaves are dense. We also consider solenoids possessing *daval* measures (the name comes from "measures that *decompose as volume along leaves*"). These are probability measures that locally disintegrate along the leaves as a product of a measure on a local transversal and a riemannian measure on the local leafs. It is easy to see that the existence of daval measure is equivalent to the existence of *transverse measure* in the sense of the theory of foliations. We recall that such a transverse measure is a collection of measures supported in each local transversal that are transported into each other by holonomy maps. Transverse measures are considered up to equivalence by multiplication by a positive scalar. Obviously any riemannian metric defines a daval measure on a manifold, the transverse measures being trivial atomic masses. But, in general, as is well known from foliation theory, transverse measures do not need to exist.

Thus, for the moment, our generalized objects are compact minimal solenoids admitting a transverse measure. A transverse measure is *ergodic* if for any local transversal, any Borel set that is invariant by the holonomy pseudogroup has zero or full measure. This is obviously the case for the trivial atomic measure associated to connected manifolds, since the holonomy is transitive on the points of the transversal. Thus we realize that it is natural to request to the transverse measure to be ergodic. And this is not enough to have a proper generalization. For manifolds the transverse measure is unique by the precedent argument. Thus we require *unique ergodicity* of the transverse measure: The transverse measure is unique (up to multiplication by a positive scalar as usual). It turns out that unique ergodicity implies topological minimality. Also unique ergodicity is determined by the geometry, but ergodicity is not.

We then arrive to the basic object of our geometry.

The objects in Ergodic Solenoidal Geometry that generalize compact connected manifolds in classical Differential Geometry are compact uniquely ergodic solenoids.

It is important to understand why we do not require just ergodicity. We could do this, but then only the underlying topological space is not sufficient to fully determine the object, more precisely its measurable transverse structure as happens for classical manifolds. Indeed the correct intuition is that the same solenoid endowed with distinct transverse ergodic measures should be considered as two objects of the geometry. By classical ergodic theory, ergodic measures generate all transversal measures. Note that two distinct ergodic measures are mutually singular.

In the following we explain how uniquely ergodic solenoids arise naturally from classical differential topology when we desire to represent geometrically real homology classes by a geometric class of currents on a manifold à la Ruelle-Sullivan, or à la Schwarzman, both being equivalent by Birkhoff ergodic theorem ([3]). Some important problems in geometry are about representing geometrically homology classes, as the famous Hodge conjecture. Indeed solenoidal geometry allows to put in a new context the Hodge conjeture, and allows to isolate de geometric aspects from the algebraic ones. We formulate a natural solenoidal Hodge conjecture in this context (see [4] and section 2). We present then homological and cohomological theories for solenoids (section 3), and how Hodge theory extends to ergodic solenoidal geometry (section 4).

2 Geometric realization of the real homology.

We describe in this section our original motivation for introducing uniquely ergodic solenoids.

We consider a smooth compact connected oriented manifold M of dimension $n \geq 1$. Any closed oriented submanifold $N \subset M$ of dimension $0 \leq k \leq n$ determines a homology class in $H_k(M,\mathbb{Z})$. This homology class in $H_k(M,\mathbb{R})$, as dual of De Rham cohomology, is explicitly given by integration of the restriction to N of differential k-forms on M. Also, any immersion $f: N \to M$ defines an integer homology class in a similar way by integration of pull-backs of k-forms. Unfortunately, because of topological reasons dating back to Thom [11] [12], not all integer homology classes in $H_k(M,\mathbb{Z})$ can be realized in such a way. Geometrically, we can realize any class in $H_k(M,\mathbb{Z})$ by topological k-chains. The real homology $H_k(M,\mathbb{R})$ classes are only realized by formal combinations with real coefficients of k-cells. This is not satisfactory for various reasons. In particular, for diverse purposes it is important to have an explicit realization, as geometric as possible, of real homology classes.

The first contribution in this direction came in 1957 from the work of S. Schwartzman [9]. Schwartzman showed how, by a limiting procedure, one-dimensional curves embedded in M can define a real homology class in $H_1(M,\mathbb{R})$. More precisely, he proved that this happens for almost all curves solutions to a differential equation admitting an invariant ergodic probability measure. Schwartzman's idea is very natural. It consists on integrating 1-forms over large pieces of the parametrized curve and normalizing this integral by the length of the parametrization. Under suitable conditions, the limit exists and defines an element of the dual of $H^1(M,\mathbb{R})$, i.e. an element of $H_1(M,\mathbb{R})$. This procedure is equivalent to the more geometric one of closing large pieces of the curve by relatively short closing paths. The closed curve obtained defines an integer homology class. The normalization by the length of the parameter range provides a class in $H_k(M,\mathbb{R})$. Under suitable hypothesis, there exists a unique limit in real homology when the pieces exhaust the parametrized curve, and this limit is independent of the closing procedure. In the article [3], we study the different aspects of the Schwartzman procedure, that we extend to higher dimension.

Later in 1975, D. Ruelle and D. Sullivan [8] defined, for arbitrary dimension $0 \leq k \leq n$, geometric currents by using oriented k-laminations embedded in M and endowed with a transversal measure. They applied their results to stable and unstable laminations of Axiom A diffeomorphisms. In a later article Sullivan [10] extended further these results and their applications. The point of view of Ruelle and Sullivan is also based on duality. The observation is that k-forms can be integrated on each leaf of the lamination and then all over the lamination using the transversal measure. This makes sense locally in each flow-box, and then it can be extended globally by using a partition of unity. The result only depends on the cohomology class of the k-form. In [2] we review and extend Ruelle-Sullivan theory.

It is natural to ask whether it is possible to realize every real homology class using a topologically minimal Ruelle-Sullivan current. In order to achieve this goal we must enlarge the class of Ruelle-Sullivan currents by considering immersions of abstract oriented solenoids. For these oriented solenoids we can consider k-forms that we can integrate provided that we are given a transversal measure invariant by the holonomy group. We define an immersion of a solenoid S into M to be a regular map $f : S \to M$ that is an immersion in each leaf. If the solenoid S is endowed with a transversal measure μ, then any smooth k-form in M can be pulled back to S by f and integrated. The resulting numerical value only depends on the cohomology class of the k-form. Therefore we have defined a closed current that we denote by (f,S_μ) and that we call a *generalized current*. This gives a homology class $[f,S_\mu] \in H_k(M,\mathbb{R})$. The main result from [4] is the following:

Theorem 1. (Realization Theorem) *Every real homology class in $H_k(M,\mathbb{R})$ can be realized by a generalized current (f,S_μ) where S_μ is an oriented, minimal, uniquely ergodic solenoid.*

This result strengthens De Rham's realization theorem of homology classes by abstract currents, i.e. forms with coefficients distributions. It is a geometric De Rham's Theorem where the abstract currents are replaced by generalized currents that are geometric objects. Moreover, we prove in [6] that such geometric currents are dense in the space of currents.

We can ask why we do need to enlarge the class of Ruelle-Sullivan currents. The result does not hold for minimal Ruelle-Sullivan currents due to the observation that homology classes with non-zero self-intersection cannot be represented by Ruelle-Sullivan currents with no compact leaves ([5]). Therefore it is not possible to represent a real homology class in $H_k(M,\mathbb{R})$ with non-zero self-intersection by a minimal Ruelle-Sullivan current that is not a submanifold. Note that this obstruction only exists when $n-k$ is even. This may be the historical reason behind the lack of results on the representation of an arbitrary homology class by minimal Ruelle-Sullivan currents.

The space of solenoids is large, and we would like to realize the real homology classes by a minimal class of solenoids enjoying good properties. We are first naturally led to topological minimality. As we prove in [2], the spaces of k-solenoids is inductive and therefore there are always minimal k-solenoids. However, the transversal structure and the holonomy group of minimal solenoids can have a rich structure. In particular, such a solenoid may have many distinct transversal measures, each one yielding a different generalized current for the same immersion f. Also when we push Schwartzman ideas beyond 1-homology for some nice classes of solenoids, we see that in general, even when the immersion is an embedding, the generalized current does not necessarily coincide with the Schwartzman homology class of the immersion of each leaf (actually not even this Schwartzman class needs to be well defined). Indeed the classical literature lacks of information about the precise relation between Ruelle-Sullivan and Schwartzman currents, in particular in higher dimension. One would naturally expect that there is some relation between the generalized currents and the Schwartzman current (if defined) of the leaves of the lamination. We study this problem in [3].

The main result in [3] is that there is such relation for the class of minimal, ergodic solenoids with a trapping region. A solenoid with a trapping region has holonomy group generated by a single map. Then the bridge between generalized currents and Schwartzman currents of the leaves is provided by Birkhoff's ergodic theorem. The main result of [3] is the following:

Theorem 2. *Let S_μ be a minimal solenoid endowed with an ergodic transversal measure μ and possessing a trapping region W. Let $f : S_\mu \to M$ be an immersion of S_μ into M such that $f(W)$ is contained in a ball of M. Then for μ-almost all leaves $l \subset S_\mu$, the Schwartzman homology class of $f(l) \subset M$ is well defined and coincides with the homology class $[f, S_\mu]$.*

If moreover S is uniquely ergodic, then this happens for all leaves.

The solenoids constructed for the proof of the Realization Theorem in [4] do satisfy the hypothesis of this theorem and the transversal measure is unique, that is, the solenoids are uniquely ergodic.

Solenoidal Hodge Conjecture.

The Hodge Conjecture is an statement about the geometric realization of an integral class of pure type (p,p) in a complex (projective) manifold. If we drop the condition of the class being integral, then theorem 1 suggests a natural conjecture for *real* homology classes of pure type as follows.

For a compact Kähler manifold M of complex dimension n, a complex immersed solenoid $f : S_\mu \to M$ (that is, a solenoid where the images $f(l)$ of the leaves $l \subset S_\mu$ are complex immersed submanifolds), of dimension $k = 2(n-p)$, defines a class in $H_{n-p,n-p}(M) = H^{p,p}(M)^* \subset H_k(M,\mathbb{R})$. It is natural to formulate the following conjecture (see [4]):

Conjecture 1. ***(Solenoidal Hodge Conjecture)*** Let M be a compact Kähler manifold. Then any class in $H^{p,p}(M)$ is represented by a complex immersed solenoid of dimension $k = 2(n-p)$.

Note that the standard Hodge Conjecture is stated for projective complex manifolds, since it fails for Kähler manifolds [14]. The counterexamples of [14] are non-algebraic complex tori. It is easy to see that conjecture 1 holds for complex tori (using non-minimal complex solenoids).

3 Differential geometry of solenoids.

We describe in this section how theories and tools of differential topology do extend to Ergodic Solenoidal Geometry. The following discussion is based on [7].

3.1 De Rham cohomology

Let S be a solenoid. The space of p-forms $\Omega^p(S)$ consist of p-forms on leaves with function coefficients that are smooth on leaves and partial derivatives of all orders continuous transversally. Using the differential d in the leaf-wise directions, we obtain the De Rham differential complex $(\Omega^*(S),d)$. The De Rham cohomology groups of the solenoid are defined as the quotients

$$H^p_{DR}(S) := \frac{\ker(d : \Omega^p(S) \to \Omega^{p+1}(S))}{\operatorname{im}(d : \Omega^{p-1}(S) \to \Omega^p(S))}\,. \tag{1}$$

We can also consider the spaces $\Omega^p_m(S)$ differential forms with function coefficients that are smooth on leaves and measurable transversally. Then define in the same way the De Rham measurable cohomology groups $H^p_{DRm}(S)$ using the complex $(\Omega^*(S),d)$. Note the natural map $H^p_{DR}(S) \to H^p_{DRm}(S)$.

Proposition 1. *Let $\underline{\mathbb{R}}_c$ and $\underline{\mathbb{R}}_m$ be respectively the sheaf of functions which are locally constant on leaves and transversally continuous, resp. measurable. Then we have isomorphisms*

$$H^p_{DR}(S) \cong H^p(S, \underline{\mathbb{R}}_c)\,,$$

and

$$H^p_{DRm}(S) \cong H^p(S, \underline{\mathbb{R}}_m)\,.$$

Remark 1. The spaces $\Omega^p(S)$ are topological vector spaces. Therefore the De Rham cohomology (1) inherits a natural topology. In general, these spaces are infinite dimensional (even for compact solenoids). In some references, it is customary to take the closure of the spaces $\operatorname{im} d$ in definition (1), obtaining the *reduced De Rham cohomology groups*

$$\bar{H}^p_{DR}(S) = \frac{\ker d|_{\Omega^p}}{\overline{\operatorname{im} d|_{\Omega^{p1}}}}\,.$$

This is equivalent to quotienting $H^p_{DR}(S)$ by $\overline{\{0\}}$, obtaining thus Hausdorff vector spaces.

We shall list some basic properties of the De Rham cohomology:

1. **Functoriality**. Let S_1, S_2 be two solenoids. A smooth map $f : S_1 \to S_2$ is a map sending leaves to leaves and transversally continuous. f defines a map on De Rham cohomologies, $f^* : H^p_{DR}(S_2) \to H^p_{DR}(S_1)$, by $f^*[\omega] = [f^*\omega]$. This applies in particular to an immersion of a solenoid into a smooth manifold $f : S \to M$, or to the inclusion of a leaf $i : l \to S$.
2. **Mayer-Vietoris sequence**. Let U, V be two open subsets of a solenoid S. There is a short exact sequence of complexes: $\Omega^\bullet(U \cup V) \to \Omega^\bullet(U) \oplus \Omega^\bullet(V) \to \Omega^\bullet(U \cap V)$.
3. **Homotopy**. A homotopy between two maps $f_0, f_1 : S_1 \to S_2$ is a map $F : S_1 \times [0,1] \to S_2$ (where $S_1 \times [0,1]$ is given the solenoid structure with leaves $l \times [0,1]$, for $l \subset S_1$ a leaf of S_1) such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. We say that the maps f_0, f_1 are homotopic, written $f_0 \sim f_1$. In this case $f_0^* = f_1^* : H^p_{DR}(S_2) \to H^p_{DR}(S_1)$.
4. **Homotopy type**. We say that two solenoids S_1, S_2 are of the same homotopy type if there are maps $f : S_1 \to S_2$, $g : S_2 \to S_1$, such that $f \circ g \sim Id_{S_2}$, $g \circ f \sim Id_{S_1}$. Then the cohomology groups of S_1 and S_2 are isomorphic.

3.2 Fundamental classes

Let S be an *oriented* compact k-solenoid (i.e., the dimension of the leaves is k). The De Rham cohomology groups do not depend on any measure of S. If $\mu = (\mu_T)$ is a transversal measure, then the integral $\int_{S_\mu}$ descends to cohomology giving a map

$$\int_{S_\mu} : H^k_{DR}(S) \to \mathbb{R}\,. \tag{2}$$

We define the solenoidal homology as

$$H_p(S,\underline{\mathbb{R}}_c) := H^p(S,\underline{\mathbb{R}}_c)^* = H^p_{DR}(S)^*.$$

Then the map (2) defines a homology class $[S_\mu] \in H^k(S,\underline{\mathbb{R}}_c)^* = H_k(S,\underline{\mathbb{R}}_c)$. We shall call this element the *fundamental class* of S_μ.

Any map $f : S_1 \to S_2$ defines a map $f^* : H^p_{DR}(S_2) \to H^p_{DR}(S_1)$ and hence, by dualizing, a map $f_* : H_p(S_1,\underline{\mathbb{R}}_c) \to H_p(S_2,\underline{\mathbb{R}}_c)$. Applying this to an immersion $f : S_\mu \to M$ of an oriented, measured, compact solenoid into a smooth manifold, then we have the equality

$$f_*[S_\mu] = [f,S_\mu] \ ,$$

with the generalized Ruelle-Sullivan class defined in the previous section.

Note that if S has a dense leaf (in particular when S it is minimal, i.e. all leaves are dense), then $H_0(S,\underline{\mathbb{R}}_c) = \mathbb{R}$. On the other hand, the dimension of the top degree homology counts the number of mutually singular tranverse measures on S.

Theorem 3. *Let S be a compact, oriented k-solenoid. Then* $H_k(S,\underline{\mathbb{R}}_c)$ *is isomorphic to the real vector space generated by all transversal measures.*

This result supports the intuition that the natural objects of Ergodic Solenoidal Geometry are uniquely ergodic solenoids with only one transversal measure.

Remark 2. There is no Poincaré duality for $H^*_{DR}(S)$ in general. Moreover these spaces may be infinite dimensional (even for uniquely ergodic solenoids): if S is a two-torus foliated by irrational lines, then $H^1_{DR}(S)$ can be infinite-dimensional.

3.3 Singular cohomology

We consider the space $\mathrm{Map}(I^n,S)$ of continuous maps $T : I^n \to S$ mapping into a leaf, and endow it with the uniform convergence topology. The degenerate maps (see [1]) form a closed subspace, therefore the quotient, $\mathrm{Map}'(I^n,S)$, has a natural quotient topology. The space of singular chains $C_n(S)$ is the free abelian group generated by $\mathrm{Map}'(I^n,S)$. There is a natural boundary map $\mathbf{d} : C_n(S) \to C_{n-1}(S)$.

Let G be any topological abelian group. Define the cochains

$$C^n(S,G) = \mathrm{Hom}_{cont}(C_n(S),G)$$

as the continuous homomorphisms. That is, $\varphi : C_n(S) \to G$ such that if $T_k : I^n \to S$ are maps which converge to $T_o : I^n \to S$ in the uniform topology, then $\varphi(T_k) \to \varphi(T_o)$. Define the differential $\delta : C^n(S) \to C^{n+1}(S)$ by $\delta\varphi(T) = \varphi(\mathbf{d}T)$. The solenoid singular cohomology of S with coefficients in G is defined as:

$$H^n(S,G) := \frac{\ker(\delta : C^n(S,G) \to C^{n+1}(S,G))}{\mathrm{im}(\delta : C^{n-1}(S,G) \to C^n(S,G))} \,.$$

We have some basic properties:

1. **Functoriality**. Let $f : S_1 \to S_2$ be a solenoid map. Then there is a map $f_* : C_n(S_1) \to C_n(S_2)$, $f_*(T) = f \circ T$, and a map $f^* : C^n(S_2, G) \to C^n(S_1, G)$, $f^*(\varphi) = \varphi \circ f$. Clearly $f^*\delta = \delta f^*$, so the map descends to cohomology: $f^* : H^n(S_2, G) \to H^n(S_1, G)$.
2. **Homotopy**. Suppose that $f, g : S_1 \to S_2$ are two homotopic solenoid maps. The usual construction yields a chain homotopy H between f^* and g^* (one only have to check that this map sends continuous cochains into continuous cochains). Therefore $f^* = g^* : H^n(S_2, G) \to H^n(S_1, G)$.
3. **Homotopy type**. If S_1, S_2 are of the same homotopy type, then $H^n(S_2, G) \cong H^n(S_1, G)$.
4. If $U = D^k \times K(U)$ is a flow-box, then U is of the same homotopy type than $\{*\} \times K(U)$. Therefore $H^n(U) = 0$ for $n > 0$, and $H^0(U) = \text{Map}_{cont}(K(U), G)$. In particular, this implies that

$$\underline{\mathbb{R}}_c \to C^0(-, \mathbb{R}) \stackrel{\delta}{\to} C^1(-, \mathbb{R}) \stackrel{\delta}{\to} \dots$$

   is a resolution. Therefore there is an isomorphism $H^n(S, \mathbb{R}) \cong H^n(S, \underline{\mathbb{R}}_c)$.
5. **Mayer-Vietoris sequence**. For two open sets U, V with $S = U \cup V$, define $C_n(S; U, V)$ as the subcomplex generated by those singular chains completely contained in either U or V. Define accordingly $C^n(S; U, V)$. It is not difficult to see that the restriction $C^n(S, G) \to C^n(S, G; U, V)$ is chain homotopy equivalence. Therefore the exact sequence $0 \to C^n(S, G; U, V) \to C^n(U, G) \oplus C^n(V, G) \to C^n(U \cap V, G) \to 0$ gives rise to a long exact sequence:

$$\dots \to H^p(U \cup V, G) \to H^p(U, G) \oplus H^p(V, G) \to H^p(U \cap V, G) \to H^{p+1}(U \cup V, G) \to \dots$$

3.4 De Rham L^2-cohomology

Now consider a k-solenoid S with a transversal measure μ. There is a notion of cohomology which takes into account the transversal measure structure. For this, we work with forms which are L^2-transversal relative to μ.

Definition 1. A function f is $L^2(\mu)$-*transversally smooth* if in any (good) flow-box $U = D^k \times K(U)$ all partial derivatives on the first variable exist and are in $L^2(\mu_{K(U)})$, i.e. if we write f as $f(x, y)$ then for all $r \geq 0$,

$$\int_{K(U)} ||f(-, y)||^2_{C^r} d\mu_{K(U)}(y) < \infty \, .$$

We consider the space of forms

$$\Omega^p_{L^2(\mu)}(S)$$

which are $L^2(\mu)$-transversally smooth, i.e. locally these are forms $\alpha = \sum f_I(x,y)dx_I$, where f_I are $L^2(\mu)$-transversally smooth functions. There is a well-defined differential along leaves $d: \Omega^p_{L^2(\mu)}(S) \to \Omega^{p+1}_{L^2(\mu)}(S)$ which defines the complex $(\Omega^*_{L^2(\mu)}(S), d)$. We define the *De Rham L^2-cohomology* vector space as the quotients

$$H^p_{DR}(S_\mu) := \frac{\ker(d : \Omega^p_{L^2(\mu)}(S) \to \Omega^{p+1}_{L^2(\mu)}(S))}{\mathrm{im}(d : \Omega^{p-1}_{L^2(\mu)}(S) \to \Omega^p_{L^2(\mu)}(S))}. \tag{3}$$

We also introduce the reduced De Rham L^2-cohomology:

$$\bar{H}^p_{DR}(S_\mu) := \frac{\ker d}{\overline{\mathrm{im}\, d}}. \tag{4}$$

Note that there are natural maps

$$H^p_{DR}(S) \to H^p_{DR}(S_\mu) \to H^p_{DRm}(S),$$

since $C^{\infty,0}$-functions are $L^2(\mu)$-transversally smooth. The integration map $\int_{S_\mu}$ is well-defined for forms in $\Omega^k_{L^2(\mu)}$, since a $L^2(\mu)$-transversally smooth k-form is automatically $L^1(\mu)$-transversal (all measures are finite measures on compact transversals). So we have $\int_{S_\mu} : H^k_{DR}(S_\mu) \to \mathbb{R}$.

Let $\underline{\mathbb{R}}_\mu$ be the sheaf of measurable functions which are locally constant on leaves and $L^2(\mu)$-transversally. A standard Poincaré lemma shows that there is a resolution of sheaves

$$\underline{\mathbb{R}}_\mu \to \Omega^0_{L^2(\mu)} \to \Omega^1_{L^2(\mu)} \to \ldots \to \Omega^k_{L^2(\mu)}.$$

So we get a natural isomorphism

$$H^p_{DR}(S_\mu) \cong H^p(S, \underline{\mathbb{R}}_\mu).$$

We review basic properties of the De Rham L^2-cohomology:

1. There is not cup product, and therefore the $H^*_{DR}(S_\mu)$ are just vector spaces (not rings).
2. Functoriality. If $f : S_1 \to S_2$ is a solenoidal map, then we require that $\mu_2 = f_*\mu_1$. This means that for any local transversal T_1 of S_1, $f(T_1)$ is a local transversal of S_2 and the transported measure $f_*\mu_1$ is a constant multiple of μ_2 on the transversal. Note that this is automatic when the solenoids are uniquely ergodic. Then for any form ω which is $L^2(\mu_2)$-transversally smooth we have that $f^*\omega$ is $L^2(\mu_1)$-transversally smooth.
3. Mayer-Vietoris. It holds exactly as in subsection 3.1.
4. Poincaré duality. We shall see that it holds for the reduced L^2-cohomology for ergodic solenoids (see [7]).

3.5 Bundles over solenoids

Let S be a k-solenoid. A vector bundle of rank n over S consists of a $(k+n)$-solenoid E and a projection map $\pi : E \to S$ satisfying the following condition: there is an open covering U_α for S, and solenoid isomorphisms $\psi_\alpha : E_\alpha = \pi^{-1}(U_\alpha) \stackrel{\cong}{\to} U_\alpha \times \mathbb{R}^n = D^k \times K(U_\alpha) \times \mathbb{R}^n$, such that $\pi = pr_1 \circ \psi_\alpha$, where $pr_1 : U_\alpha \times \mathbb{R}^n \to U_\alpha$ denotes the projection, and the transition functions

$$\psi_\alpha \circ \psi_\beta^{-1} : (U_\beta \cap U_\alpha) \times \mathbb{R}^n \to (U_\beta \cap U_\alpha) \times \mathbb{R}^n$$

are of the form $(x,y,v) \mapsto (x,y,g_{\alpha\beta}(x,y)(v))$, where $g_{\alpha\beta}$ is a $C^{\infty,0}$-smooth function from $U_\alpha \cap U_\beta$ to $\mathrm{GL}(n)$.

Some points are easy to check:

1. The usual constructions of vector bundles remain valid here: direct sums, tensor products, symmetric and anti-symmetric products. Also there are notions of sub-bundle and of quotient bundle.
2. A section of a bundle $\pi : E \to S$ is a map $s : S \to E$ such that $\pi \circ s = Id$. We denote the space of sections as $\Gamma(E)$.
3. If S_μ is a measured solenoid, and $E \to S$ is a vector bundle, then we have the notion of sections which are $L^2(\mu)$-transversally smooth. Locally, in a chart $E_\alpha = D^k \times K(U) \times \mathbb{R}^n \to U_\alpha = D^k \times K(U)$, the section is written $s(x,y) = (x,y,v(x,y))$. We require that v is C^∞ on x and $L^2(\mu)$ on y. This does not depend on the chosen trivialization.
4. If $f : S_1 \to S_2$ is a solenoid map, and $\pi : E \to S_2$ is a vector bundle, then the pull-back $f^*E = \{(p,v) \in S_1 \times E \mid f(p) = \pi(v)\}$ is naturally a vector bundle over S_1.
5. The tangent bundle TS of S is an example of vector bundle. We have bundles of (p,q)-tensors $TS^{\otimes p} \otimes (TS^*)^{\otimes q}$ on any solenoid S. In particular, we have bundles of p-forms (anti-symmetric contravariant tensors) $\bigwedge^p T^*S$. Its sections are the p-forms $\Omega^p(S)$.
6. A metric on a bundle E is a section of $\mathrm{Sym}^2(E^*)$ which is positive definite at every point. A metric on S is a metric on the tangent bundle. An orientation of a bundle E is a continuous choice of orientation for each of the fibers of E. An orientation of S is an orientation of its tangent bundle.

We define $\Omega^p(E) = \Gamma(\bigwedge^p T^*S \otimes E)$. A connection on a vector bundle $E \to S$ is a map

$$\nabla : \Gamma(E) \to \Omega^1(E),$$

such that $\nabla(f \cdot s) = f\nabla s + df \wedge s$. Consider a local trivialization in a flow-box U_α with coordinates (x,y). Then $\nabla|_{U_\alpha} = d + a_\alpha$, where $a_\alpha \in \Omega^1(U_\alpha, \mathrm{End}\, E)$. Under a change of trivialization $g_{\alpha\beta}$, for two trivializing open subsets U_α, U_β, we have the usual formula $a_\beta = g_{\alpha\beta}^{-1} a_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta}$.

A partition of unity argument proves that there are always connections on a vector bundle $E \to S$. The space of connections is an affine space over $\Omega^1(\mathrm{End}\, E)$.

Given a connection ∇ on E, there is a unique map $d_\nabla : \Omega^p(E) \to \Omega^{p+1}(E)$, $p \geq 0$, such that $d_\nabla s = \nabla s$ for $s \in \Gamma(E)$, and $d_\nabla(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d_\nabla \beta$, for $\alpha \in \Omega^p(S)$, $\beta \in \Omega^q(E)$. It is easy to see that $\hat{F}_\nabla : \Gamma(E) \to \Omega^2(E)$, given by $\hat{F}_\nabla(s) = d_\nabla d_\nabla s$, has a tensorial character (i.e., it is linear on functions). Therefore there is a $F_\nabla \in \Omega^2(\mathrm{End}\, E)$, called *curvature* of ∇, such that $\hat{F}_\nabla(s) = F_\nabla \cdot s$. Locally on a trivialization U_α, we have the formula $F_\nabla = da_\alpha + a_\alpha \wedge a_\alpha$.

Given connections on vector bundles, there are induced connections on associated bundles (dual bundle, tensor product, direct sum, symmetric product, pull-back under a solenoid map, etc.). This follows in a straightforward way from the standard theory. In particular, if $l \hookrightarrow S$ is a leaf of a solenoid S, then we can perform the pull-back of the bundle and connection to the leaf, which consists on restricting them to l. This gives a bundle and connection of a complete k-dimensional manifold. Also, if $f : S \to M$ is an immersion of a solenoid in a smooth n-manifold, and $E \to M$ is a bundle with connection, then the pull-back construction produces a bundle with connection on S.

Consider a vector bundle $E \to S$ endowed with a metric. We say that a connection ∇ is compatible with the metric if it satisfies

$$d\langle s,t\rangle = \langle \nabla s, t\rangle + \langle s, \nabla t\rangle \,.$$

In the particular case of the tangent bundle TS of a Riemannian solenoid S, we have the Levi-Civita connection ∇^{LC}, which is the unique connection compatible with the metric and with torsion $T_\nabla(X,Y) = \nabla_X Y - \nabla_Y X = 0$. This is the Levi-Civita connection on each leaf, and the transversal continuity follows easily.

3.6 Chern classes

We can also define a complex vector bundle over a solenoid, by using $\mathbb{C}^n$ as fiber, and taking the transition functions with values in $\mathrm{GL}(n,\mathbb{C})$. An hermitian metric on a complex vector bundle is a positive definite hermitian form in each fiber with smoothness of type $C^{\infty,0}$ on any local trivialization.

Let $E \to S$ be a complex vector bundle over a solenoid of rank n. Put a hermitian structure on E, and consider any hermitian connection ∇ on E. Then the curvature F_∇ is a 2-form with values in $\mathrm{End}\, E$, i.e. $F_\nabla \in \Omega^2(\mathrm{End}\, E)$. The Bianchi identity says

$$d_\nabla F_\nabla = 0 \,.$$

This holds leaf-wise, so it holds on the solenoid.

Consider the elementary functions: $\mathrm{Tr}_i : M_{r\times r} \to \mathbb{C}$, given by $\mathrm{Tr}_i(A) = \mathrm{Tr}(\bigwedge^i A)$. Then the Chern classes are

$$c_i(E) = \left[\mathrm{Tr}_i\left(\frac{\sqrt{-1}}{2\pi} F_\nabla\right)\right] \in H^{2i}_{DR}(S) \,.$$

These classes are well defined (since the forms inside are closed, which again follows by working on leaves) and do not depend on the connection (different connections give forms differing by exact forms), see [13, Chapter III].

We have some facts:

1. If M is a manifold, we recover the usual Chern classes.
2. If $f : S_1 \to S_2$ is a solenoid map, then $f^* c_i(E) = c_i(f^* E)$. In particular,
   - If $f : S \to M$ is an immersion of a solenoid in a manifold and $E|_S = f^* E$, then $c_i(E|_S) = f^* c_i(E)$.
   - If $j : l \to S$ is the inclusion of a leaf, then $c_i(E|_l) = j^* c_i(E)$.

Question. Are the Chern classes defined as elements in $H^{2i}(S, \mathbb{Z})$?
(for line bundles it is true).

4 Hodge Theory of solenoids

4.1 Sobolev norms

Let S_μ be a compact Riemannian k-solenoid which is oriented and endowed with a transversal measure. We denote the associated (finite) daval measure also by μ. Now consider a vector bundle $E \to S$ and endow it with a metric. The space of sections of class $C^{\infty,0}$ is denoted $\Gamma(S,E)$. The space of $L^2(\mu)$-transversally smooth sections (sections of class C^∞ along leaves and L^2 in the transversal directions) is denoted by $\Gamma_{L^2(\mu)}(S,E)$.

Now let us introduce suitable completions of these spaces of sections. Fix a connection ∇ for E and the Levi-Civita connection for TS. There is an L^2-norm on sections of E, given by

$$(s,t)_E = \int_S \langle s,t \rangle \, d\mu \, .$$

We can complete the spaces of sections to obtain spaces of L^2-sections $L^2(S,E)$. We consider also Sobolev norms $W^{l,2}$ as follows. Take s a section of E. Then we set

$$||s||^2_{W^{l,2}_\mu} = \int_S \sum_{i=0}^{l} |\nabla^i s|^2 \, d\mu \, .$$

Completing with respect to this norm gives a Hilbert space consisting of sections with regularity $W^{l,2}$ on leaves and $L^2(\mu)$-transversally, denoted $W^{l,2}_\mu(S,E)$. These spaces do not depend on the choice of metrics and connections.

For future use, we also introduce the norms C^r_μ, which give spaces of sections with C^r-regularity on leaves and $L^2(\mu)$-transversally. Take s a section of E. Assume it has support in a flow box $U = D^k \times K(U)$, and assume that E has been trivialized by an orthonormal frame. Then

$$||s||^2_{C^r_\mu} = \int_{K(U)} ||s(\cdot,y)||^2_{C^r}\, d\mu_{K(U)}(y)\,.$$

These norms are patched (via partitions of unity, in a non-canonical way) to get a norm on the spaces of sections on the whole solenoid. The topology defined by this norm is independent of the partition of unity. The spaces of sections are denoted $C^r_\mu(S,E)$. Note that $\bigcap_{r\geq 0} C^r_\mu(S,E) = \Gamma_{L^2(\mu)}(S,E)$.

We can define the norm $W^{l,2}_\mu$ by using Fourier transforms. For this we have to restrict to a flow-box $U = D^k \times K(U)$. We Fourier-transform the section $s(x,y)$ in the leaf-wise directions, to get $\hat{s}(\xi,y)$, and then take the integral

$$\int_{K(U)} \left(\int (1+|\xi|^2)^l |\hat{s}(\xi,y)|^2 d\xi \right) d\mu_{K(U)}(y).$$

Proposition 2 (Sobolev). *$W^{s,2}_\mu(S,E) \subset C^p_\mu(S,E)$, for $s > [k/2]+p+1$.*

This is similar to Proposition 1.1 in Chapter IV of [13]. The proof carries over to the solenoid situation verbatim. As a consequence,

$$\bigcap_{r\geq 0} W^{r,2}_\mu(S,E) = \Gamma_{L^2(\mu)}(S,E)\,.$$

4.2 Pseudodifferential operators

Let E,F be two vector bundles over S of ranks n,m respectively. A differential operator L of order l is an operator

$$L : \Gamma(S,E) \to \Gamma(S,F)$$

which locally on a flow-box $U = D^k \times K(U)$ is of the form

$$L(s) = \sum_{|\alpha|\leq l} A_\alpha(x,y) D^\alpha s\,,$$

where A_α are $(n\times m)$-matrices of functions (with regularity $C^{\infty,0}$) and $\alpha = (\alpha_1,\ldots,\alpha_k)$ is a multi-index, with $|\alpha| = \sum \alpha_i$, $D^\alpha = \frac{\mathbf{d}^{|\alpha|}}{\mathbf{d}^{\alpha_1}x_1\ldots\mathbf{d}^{\alpha_k}x_k}$. Note that a differential operator gives rise to differential operators on each leaf. Moreover, L extends to

$$L : W^{p,2}_\mu(S,E) \to W^{p-l,2}_\mu(S,F)\,.$$

The usual properties, like the existence of adjoints, extend to this setting.

The symbol of a differential operator on a solenoid is defined in the same fashion as for the case of manifolds, and coincides with the symbol of the differential operator on the leaves. We recall that the symbol $\sigma_l(L) \in \mathrm{Hom}(\pi^*E,\pi^*F)$, $\pi : TS \to S$, has the form

$$\sigma_l(L)(x,y,v) = \sum_{|\alpha|=l} A_\alpha(x,y) v_1^{\alpha_1} \dots v_k^{\alpha_k} .$$

The properties of the symbol map, such as the rule of the symbol of the composition of differential operators, or the symbol of the adjoint, hold here. This is just the fact that they can be done leaf-wise, and the continuous transversality is easy to check.

Differential operators can be generalized to pseudodifferential operators as in the case of manifolds. A pseudodifferential operator of order l on a flow-box $U = D^k \times K(U)$ is an operator

$$L(p) : \Gamma_c(U,E) \to \Gamma(U,F)$$

which sends a (compactly supported) section $s(x,y)$ to

$$L(p)s(x,y) = \int p(x,\xi,y)\hat{s}(\xi,y)e^{i\langle x,\xi\rangle} d\xi ,$$

where $\hat{s}(\xi,y)$ is the (leaf-wise) Fourier transform, and $p(x,\xi,y)$ is a function defined in $D^k \times \mathbb{R}^k \times K(U)$, smooth on x and ξ, continuous on y, and satisfying:

- $|D_x^\beta D_\xi^\alpha p(x,\xi,y)| \leq C_{\alpha\beta l}(1+|\xi|)^{l-|\alpha|}$, for constants $C_{\alpha\beta l}$,
- the limit $\sigma_l(p)(x,\xi,y) = \lim_{\lambda\to\infty} \frac{p(x,\lambda\xi,y)}{\lambda^l}$ exists,
- $p(x,\xi,y) - \sigma_l(p)(x,\xi,y)$ should be of order $\leq l-1$ for $|\xi| \geq 1$.

A pseudodifferential operator of order l on S is an operator $L : \Gamma(S,E) \to \Gamma(S,F)$ which is locally of the form $L(p_U)$ for some p_U as above. The symbol of L is $\sigma_l(L) = \sigma_l(p_U)$ for a local representative $L|_U = L(p_U)$. This symbol is well-defined and independent of choices, which is a delicate point but it is analogous to the case of manifolds (see [13]). The usual properties of the symbol map (composition, adjoint) hold here.

A pseudodifferential operator of order l is an operator of order l, i.e., it extends as a continuous map to

$$L : W_\mu^{p,2}(S,E) \to W_\mu^{p-l,2}(S,E) .$$

This is done as in Theorem 3.4 of [13, Ch. IV], by noting that $||L(p)s(\cdot,y)||_{W^{p-l,2}} \leq C||s(\cdot,y)||_{W^{p,2}}$, where C is a constant depending on $C_{\alpha\beta l}$.

The key of the theory is the fact that we can construct a pseudodifferential operator given a symbol $\sigma_l(L)$.

Proposition 3. *Let S be a compact solenoid. Then there is an exact sequence* $0 \to \mathrm{OP}_{l-1}(E,F) \to \mathrm{PDiff}_l(E,F) \to \mathrm{Symb}_l(E,F) \to 0$, *where* $\mathrm{OP}_{l-1}(E,F)$ *is the space of operators of order* $l-1$, $\mathrm{PDiff}_l(E,F)$ *the space of pseudodifferential operators of order l, and* $\mathrm{Symb}_l(E,F)$ *the space of symbols of order l.*

4.3 Elliptic operator theory for solenoids

We say that a pseudodifferential operator $L:E\to F$ of order l is elliptic if the symbol $\sigma_l(L)$ satisfies that $\sigma_l(L)(x,v):E_x\to F_x$ is an isomorphism for each $x\in S$, $v\in T_xS$, $v\neq 0$.

Theorem 4. *Let L be an elliptic pseudodifferential operator of order l. Then there exists a pseudo-inverse, a pseudodifferential operator $\tilde{L}$ of order $-l$ such that $L\circ\tilde{L}=Id+K_1$ and $\tilde{L}\circ L=Id+K_2$, where K_1,K_2 are operators of order -1.*

This is done as in Theorem 4.4 [13, Ch. IV]. The basic idea is to construct a pseudo-inverse by using Proposition 3. Note that K_1,K_2 are not ussually compact operators (this is due to the failure of the Rellich lemma in our situation), so we will not have finite-dimensionality of the kernel and cokernel of elliptic operators.

Corollary 1. *Let L be an elliptic pseudodifferential operator of order l, and let $\mathscr{K}_{L_s}=\ker(L:W^{s,2}_\mu(S,E)\to W^{s-l,2}_\mu(S,F))$. Then $\mathscr{K}_{L_s}\subset\Gamma_{L^2(\mu)}(S,E)$, and it is independent of s.*

An operator $L:\Gamma(E)\to\Gamma(E)$ is called self-adjoint if $L^*=L$. If L is an elliptic self-adjoint operator, then there is a pseudo-inverse G which is self-adjoint (just take the pseudo-inverse $\tilde{L}$ provided by Theorem 4 and let $G=(\tilde{L}+\tilde{L}^*)/2$). Then we have that $L\circ G=G\circ L$, because

$$\langle (L\circ G-G\circ L)s,s\rangle=\langle Gs,Ls\rangle-\langle Ls,Gs\rangle=0\,.$$

In particular, $K_1=K_2$ in Theorem 4.

For self-adjoint operators, we have the following result

Theorem 5. *Let L be an elliptic self-adjoint operator of order l. Then*

$$W^{s,2}_\mu(S,E)=\ker L\oplus\overline{\operatorname{im} L}\,.$$

and an analogous result for $\Gamma_{L^2(\mu)}(S,E)$.

A complex of differential operators is a sequence

$$\Gamma(E_0)\stackrel{L_0}{\longrightarrow}\Gamma(E_1)\stackrel{L_1}{\longrightarrow}\ldots\stackrel{L_{m-1}}{\longrightarrow}\Gamma(E_m)\,,$$

where E_i are vector bundles, and L_i are differential operators such that $L_i\circ L_{i-1}=0$. The complex is called elliptic if the sequence of symbols

$$\pi^*E_0\stackrel{\sigma(L_0)}{\longrightarrow}\pi^*E_1\stackrel{\sigma(L_1)}{\longrightarrow}\ldots\stackrel{\sigma(L_{m-1})}{\longrightarrow}\pi^*E_m\,,$$

is exact for each $v\neq 0$. We define the cohomology of the complex as

$$H^q(S,E)=\frac{\ker(L_q:\Gamma(E_q)\to\Gamma(E_{q+1}))}{\operatorname{im}(L_{q-1}:\Gamma(E_{q-1})\to\Gamma(E_q))}\,,$$

and the L^2-cohomology by

$$H^q(S_\mu,E)=\frac{\ker(L_q:\Gamma_{L^2(\mu)}(E_q)\to\Gamma_{L^2(\mu)}(E_{q+1}))}{\mathrm{im}(L_{q-1}:\Gamma_{L^2(\mu)}(E_{q-1})\to\Gamma_{L^2(\mu)}(E_q))}\,.$$

The *reduced* L^2-cohomology is

$$\bar{H}^q(S_\mu,E)=\frac{\ker L_q}{\overline{\mathrm{im}\,L_{q-1}}}\,.$$

This is the group $H^q(S_\mu,E)$ quotiented by the closure of $\{0\}$, making it a Hausdorff space.

We construct the Laplacian operators of the elliptic complex as follows:

$$\Delta_j=L_j^*L_j+L_{j-1}L_{j-1}^*:\Gamma_{L^2(\mu)}(E_j)\to\Gamma_{L^2(\mu)}(E_j)\,.$$

These are self-adjoint elliptic operators. There is an associated operator G given by Theorem 5. Denote

$$\mathscr{H}^j(E)=\ker\Delta_j\,.$$

And note that $\Delta_j s=0$ if and only if $L_j s=0$ and $L_j^* s=0$. We remove the subindex j from now on.

Theorem 6. *We have the following:*

1. $\overline{\mathrm{im}\,\Delta}=\overline{\mathrm{im}\,L}\oplus\overline{\mathrm{im}\,L^*}$*, and it is an orthogonal decomposition.*
2. $\Gamma_{L^2(\mu)}(S,E_j)=\mathscr{H}^j(E)\oplus\overline{\mathrm{im}\,L}\oplus\overline{\mathrm{im}\,L^*}$.
3. *There is a canonical isomorphism* $\mathscr{H}^j(E)\cong\bar{H}^j(S_\mu,E)$.

4.4 Harmonic theory

The Riemannian metric and the orientation give rise to a natural volume form along leaves $\mathrm{vol}\in\Omega^k(S)$. The usual Hodge-$*$ operator (see [13]) can be defined for forms on S, actually, it is the $*$ operator on leaves. This operator $*:\Omega^p(S)\to\Omega^{k-p}(S)$ is defined by

$$\alpha\wedge*\beta=(\alpha,\beta)\,\mathrm{vol}\,,$$

for $\alpha,\beta\in\Omega^p(S)$, where $(\cdot,\cdot)$ is the point-wise metric induced on forms. Note that $*$ extends to $*:\Omega^p_{L^2_\mu}(S)\to\Omega^{k-p}_{L^2_\mu}(S)$, since it is leaf-wise isometric. Note that $\mathrm{vol}=*1$.

Lemma 1. $d^*=\pm*d*$. □

The Laplacian is defined as $\Delta=dd^*+d^*d$. Note that if $\Delta s=0$ then $(s,\Delta s)=(s,dd^*s)+(s,d^*ds)=(d^*s,d^*s)+(ds,ds)=||d^*s||^2+||ds||^2$. So $d^*s=0$ and $ds=0$. We define the space of harmonic forms:

$$\mathcal{K}^j(S_\mu) = \mathcal{H}_\Delta(\wedge^j T^* S)\,.$$

Then the theory of elliptic operators says the following

Theorem 7. *We have*

- *The space of harmonic sections* $\mathcal{K}^j(S_\mu) \subset \Omega^j_{L^2(\mu)}(S)$.
- *There is a natural isomorphism* $\bar{H}^j_{DR}(S_\mu) \cong \mathcal{K}^j(S_\mu)$.

Corollary 2. *Poincaré duality:*

$$* : \mathcal{K}^p(S_\mu) \to \mathcal{K}^{k-p}(S_\mu)$$

is an isomorphism.

If S is ergodic, then $H^0(S_\mu) \cong H^k(S_\mu) \cong \mathbb{R}$ (with the isomorphism given by integration $\int_{S_\mu}$. Therefore

$$\int_{S_\mu} : \bar{H}^p_{DR}(S_\mu) \otimes \bar{H}^{k-p}_{DR}(S_\mu) \to \mathbb{R}$$

is a perfect pairing.

In general, the spaces $\mathcal{K}^p(S_\mu)$ are not finite dimensional. For instance, take a solenoid which is a fibration, i.e., S is a compact $(n+k)$-manifold such that there is a submersion $\pi : S \to B$ onto an n-dimensional manifold, and the transversal measure is induced by a measure μ on B. Then we have a fiber bundle $\mathbb{H}^p \to B$ such that $\mathbb{H}^p_y = H^p(F_y)$, $F_y = \pi^{-1}(y)$. Then $\mathcal{K}^p(S_\mu) \cong \Gamma_{L^2(\mu)}(\mathbb{H}^p)$.

Nonetheless, we propose the following conjecture, as it is natural from the standpoint of Ergodic Solemoidal Geometry:

Conjecture 2. If S_μ is a uniquely ergodic solenoid then the spaces $\mathcal{K}^p(S_\mu)$ are of finite dimension.

Acknowledgements As a student, the second author first learned Riemannian geometry from Emilio Elizalde notes in the class "Métodos Matemáticos" at the University of Barcelona. Thus it is a double pleasure to congratulate him for his birthday with this geometric article, and to thank him for his excellent notes.

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Zeta-regularization and exact WKB method for a general 1D Schrödinger equation

André Voros

Abstract We summarize our presentation, based on published material.

We have found an exact analytical solution scheme for the stationary Schrödinger equation of 1D quantum mechanics,

$$(-\mathrm{d}^2/\mathrm{d}q^2 + [V(q)+\lambda])\,\psi(q) = 0, \qquad q \in \mathbb{R}$$

valid for polynomial potentials $V(q)$ of degree N, initially homogeneous (q^N) [1], then arbitrary [2] (general Sturm–Liouville problem). As surveyed in [3]: basically, we reduce the unknowns to $(N+2)$ infinite complex sequences called "(mutually) conjugate spectra"; then, by an *exact WKB* analysis in the complex domain, we find each of these sequences to obey an *exact quantization formula*, of Bohr–Sommerfeld type but *implicit*, as it invokes the two neighboring conjugate spectra.

- For the eigenvalue problem restricted to the half-line $[0,+\infty)$: those unknown sequences are the discrete eigenvalue spectrum itself (with either Dirichlet or Neumann condition at $q=0$) plus its conjugates under a finite cyclic group of complex-analytic dilations. (Solutions for other boundary conditions easily follow.)

- For any fixed-$\tilde{q}$ value $\psi(\tilde{q})$ (or derivative $\psi'(\tilde{q})$): the unknowns are again all those "spectra" but now for a definite $\tilde{q}$-parametric polynomial $V_{\tilde{q}}(q)$.

In either case, the domains of the exact quantization formulae must consist of *semiclassically compliant* sequences (i.e., already asymptotically correct). In the end, the Schrödinger solutions get specified as *fixed points* of certain explicit (countable-dimensional) complex mappings.

Now, in numerical tests for even homogeneous potentials $V(q) = q^N$ (up to degree $N = 400$) [1] and for moderate inhomogeneous perturbations of q^4 and q^6,

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_32,

those mappings appear to be contractive and reconstructing the exact Schrödinger solutions as the unique limits reached under forward iterations.

In particular, the *homogeneous* case reduces in each sector (Dirichlet, resp. Neumann) to a single fixed-point equation for a real mapping, and this case is now *proved globally contractive* [4] (bounds are not yet totally explicit).

As for inhomogeneous potentials, contractivity *has yet to be proved*. In the quartic problem $V(q) = q^4 + vq^2$, a numerical instability even arises for large $v > 0$; still, we now analytically control the singular-perturbation regime $v \to +\infty$, and in fact for all $V(q) = q^N + vq^M$ $(N > M > N/2 - 1)$ [5].

Besides leading to solutions in the form of fixed-point conditions, our approach features the use of *zeta-regularized* spectral determinants [6, 7] and their very general (Wronskian) *functional identities* [8]. The latter in turn imply a mysterious equivalence between our treatment for (Schrödinger) Ordinary Differential Equations (ODE), and the *Bethe Ansatz* for integrable models (IM) in 2D Statistical Mechanics and Conformal Quantum Field Theories [9, 10, 11]. At present this "ODE/IM correspondence" is confined to homogeneous potentials $V(q) = q^N$ (but with non-integer N, plus a centrifugal term L^2/q^2, allowed as well). It is still desirable: a) to find a common framework unifying both sides (ODE and IM); and b) to extend that correspondence to general inhomogeneous $V(q)$ on the ODE side.

We are grateful to the Organizers for their invitation and partial support.

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Generalized Zeta Function Regularization and the Multiplicative Anomaly

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1 Introduction

It is well known that (Euclidean) Partition function is a very important object in Relativistic Quantum Field Theory: the full propagator and all other n-point correlation functions can be computed by it. The formalism can be extended also in curved space-time [1]. The relativistic nature of quantum fields, namely the fact that an infinite number of degrees of freedom is involved, plays a crucial role. As a result, ultraviolet divergences are present, and regularization and renormalization are necessary.

In the one-loop approximation or in the external field approximation, one may describe quantum (scalar) field by means of path (Euclidean) integral and expressing the Euclidean partition function in terms of functional determinants associated with differential operators. Namely, the partition function is proportional to

$$Z_1 = \left(\det L\right)^{-1/2} , \tag{1}$$

with L an elliptic self-adjoint non negative differential, the fluctuation operator. Then, the computation of Euclidean one-loop partition function reduces to the computations of functional determinants. The functional determinants are divergent, ultraviolet divergences are present and may be regularized by making use of suitable regularization.

As a simplest and illustrative example, let us consider $\lambda\phi^4$ self-interacting scalar field. Let us split the quantum field as $\phi = \Phi_0 + \eta$, where Φ_0 is a classical background field. Thus the one-loop fluctuation operator is

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_33,

$$L = -\Delta + m^2 + \frac{\lambda}{2}\Phi_0^2\,. \tag{2}$$

We recall that in gauge theory, A is singular due to the gauge invariance and a gauge fixing+ ghost contributions are necessary. The one-loop quantum partition function $Z[A]$, S_0 being the classical action

$$Z[L] \simeq e^{-S_0}\int d[\eta]\, e^{-\frac{1}{2}\int d^4x \eta L \eta} \tag{3}$$

reduces to a Gaussian functional integral, and as well known, it can be computable in terms of the real eigenvalues λ_n of the fluctuation operator, namely $L\phi_n = \lambda_n\phi_n$. Since $\phi = \sum_n c_n\phi_n$, the formal functional measure $d[\phi]$ may be defined as (μ arbitrary renormalization parameter)

$$d[\phi] = \prod_n \frac{dc_n}{\sqrt{\mu}}\,. \tag{4}$$

As a consequence, the one-loop quantum ”prefactor” is

$$Z_1[L] = \prod_n \frac{1}{\sqrt{\mu}}\int_{-\infty}^{\infty} dc_n e^{-\frac{1}{2}\lambda_n c_n^2} = \left[\det(\mu^{-2}L)\right]^{-1/2} \tag{5}$$

and the one-loop Euclidean Effective Action reads

$$\Gamma_E =: -\log Z = S_0 + \frac{1}{2}\log(\det \mu^{-2}L)\,. \tag{6}$$

What about the evalutation of the above functional determinants? Recall the well known Schwinger argument: one starts from the formal relation

$$\log\det L = \mathrm{Tr}\log L\,. \tag{7}$$

Thus

$$\delta\log\det L = \mathrm{Tr}(L^{-1}\delta L) \tag{8}$$

and consequentely one arrives at the formal expression

$$(\log\det L) = -\left(\int_0^{\infty} dt\, t^{-1}\mathrm{Tr}\, e^{-tL}\right)\,. \tag{9}$$

With regard to this expression, for large t, there are no problems, since L is assumed to be non negative, but for small t, the Heat Kernel expansion in regular smooth and without bondary case and $D = 4$, reads (see for example [2])

$$\mathrm{Tr}\, e^{-tL} \simeq \sum_{r=0}^{\infty} A_r t^{r-2}\,. \tag{10}$$

It follows that the Schwinger representation of functional determinants is divergent a $t = 0$, and one needs for a regularization. One of the simplest and most useful is the dimensional regularization [3], which in our formulation consists in the replacement

$$t^{-1} \to \frac{t^{\varepsilon-1}}{\Gamma(1+\varepsilon)}\,. \tag{11}$$

As a result, the related regularized functional determinant with ε sufficiently large is

$$\log\det L(\varepsilon) = -\int_0^\infty dt\,\frac{t^{\varepsilon-1}}{\Gamma(1+\varepsilon)}\mathrm{Tr}\,e^{-tL} = -\frac{\zeta(\varepsilon,L)}{\varepsilon}\,, \tag{12}$$

where the generalized zeta function associated with L, defined for $Re s > 2$

$$\zeta(s,L) = \frac{1}{\Gamma(s)}\int_0^\infty dt\,t^{s-1}\mathrm{Tr}\,e^{-tL}\,, \tag{13}$$

has been introduced. In order to be able to handle the cutoff, one makes use of the celebrated Seeley Theorem: If L is elliptic and differential operator, defined on a smooth and compact manifold, the analytic continuation of $\zeta(s,L)$ in the whole complex space s is regular at $s = 0$. Making use of this dimensional regularization, and making a Taylor expansion at $\varepsilon = 0$, one arrives at

$$\log\det L(\varepsilon) = -\frac{1}{\varepsilon}\zeta(0,L) - \zeta'(0,L) + O(\varepsilon)\,. \tag{14}$$

Thus, one obtains a justification of the zeta-function regularized functional determinant [4, 5, 6], namely

$$\log\det L = -\zeta'(0,L)\,. \tag{15}$$

In four dimension, the computable Seeley-de Witt coefficient $A_2 = \zeta(0,L)$ controls the ultraviolet divergence, while $\zeta'(0,A)$ gives the finite contribution, and this, in general, is difficult to evalute (see for example [2] and references therein).

2 Multiplicative anomaly in the regular case

In some cases, if, for example, one is dealing with a vector valued fields (charged scalar field), the L becomes a matrix valued differential operator. In the evaluation of the determinant, one first computes the algebraic one. As a consequenge, one is dealing with products of operators. A crucial point arises: the zeta-function regularized determinants do not satisfy the relation $\det(AB) = \det A\det B$, or equivalently

$$\ln\det(AB) = \ln\det A + \ln\det B\,. \tag{16}$$

In fact, in general, there exists the so-called multiplicative anomaly, which may be defined as:

$$a(A,B) = \ln\det(AB) - \ln\det(A) - \ln\det(B)\,. \tag{17}$$

Here it is left understodd that the determinants of the two elliptic operators, A and B are regularized by means of the zeta-function regularization. This multiplicative anomaly has been discovered by Wodzicki (see for example [7, 8, 9] and references therein).

In the simple but important case in which A and B are two commuting invertible self-adjoint elliptic operators of second order, the multiplicative anomaly can be evaluated by the Wodzicki formula (a discussion can be found in [10] and references therein).

$$a(A,B) = \frac{1}{8}\,\mathrm{res}\left[(\ln(AB^{-1}))^2\right]\,, \tag{18}$$

where the non-commutative residue, denoted by res, related to a classical pseudo-differential operator Q of order zero may defined by the logarithmic term in t of the following generalized heat-kernel expansion

$$\mathrm{Tr}(Qe^{-tH}) = \sum_j c_j t^{(j-D)/2} - \frac{\mathrm{res}\,Q}{2}\ln t + O(t\ln t)\,, \tag{19}$$

where H is an elliptic non negative operator of second order, irrelevant for the evaluation of res$Q$.

However, from a pratical point of view, the non-commutative residue can also be evaluated by means of the local formula found by Wodzicki, namely

$$\mathrm{res}\,Q = (2\pi)^{-D}\int_{M_D} dx \int_{|k|=1} Q_{-D}(x,k)dk\,. \tag{20}$$

Here the homogeneity component of order $-D$ of the complete symbol appears. Recall that a classical pseudo-differential operator Q of order zero has a complete symbol $e^{ikx}Qe^{-ikx}$, admitting the following asymptotics expansion, valid for large $|k|$

$$Q(x,k) \simeq \sum_{j=0}^{\infty} Q_{-j}(x,k)\,. \tag{21}$$

In this expansion, the related coefficients satisfy the homogeneity property $Q_j(x,\lambda k) = \lambda^{-j} Q_{-j}(x,k)$.

2.1 Non interacting charged boson field

Let us consider a physical example: a free charged boson field at finite temperature $\beta = 1/T$ and chemical potential μ. The related grand canonical partition function is standard and reads

$$Z_{\beta,\mu} = \int_{\phi(\tau)=\phi(\tau+\beta)} D\phi_i e^{-\frac{1}{2}\int_0^\beta d\tau \int d^3x \phi_i A_{ij}\phi_j}\,, \tag{22}$$

where

$$A_{ij} = \left(L_\tau + L_3 - \mu^2\right)\delta_{ij} + 2\mu\varepsilon_{ij}\sqrt{L_\tau}\,, \quad L_3 = -\Delta_3 + m^2\,, \tag{23}$$

Δ_3 being the Laplace operator on R^3, continuous spectrum k^2) and $L_\tau = -\partial_\tau^2$, discrete spectrum over the Matsubara frequencies $\omega_n^2 = \frac{4\pi^2}{\beta^2}$. Thus, the grand canonical partition function may be written as (see, for example, [11] and references therein)

$$\ln Z_{\beta,\mu} = -\ln\det\|A_{ik}\|\,. \tag{24}$$

Now the algebraic determinant, denoted by $|A|$, can be evaluated and gives

$$|A_{ik}| = (K_+K_-)\,, \tag{25}$$

with

$$K_\pm = L_3 + (\sqrt{L_\tau} \pm i\mu)^2\,. \tag{26}$$

However, it is easy to show that another factorization exists [11], i.e.

$$|A_{ik}| = (L_+L_-)\,, \tag{27}$$

where

$$L_\pm = L_\tau + (\sqrt{L_3} \pm \mu)^2\,. \tag{28}$$

Now a simple calculation gives

$$|A_{ik}| = L_+L_- = K_+K_-\,, \tag{29}$$

and in both cases one is dealing with the product of two pseudo-differential operators (ΨDOs), the couple L_+ and L_- being also formally self-adjoint. Thus, the partition function may be written as

$$\ln Z_{\beta,\mu} = -\ln\det K_+ - \ln\det K_- + a(K_+,K_-)\,, \tag{30}$$

or as

$$\ln Z_{\beta,\mu} = -\ln\det L_+ - \ln\det L_- + a(L_+,L_-)\,. \tag{31}$$

The evaluation of the multiplicative anomalies which appear in the above expressions can be done making use of the Wodzicki formula and a complete agreement is found between the two expressions of the partition function. Thus, if one neglects the multiplicative anomaly, one arrives at a mathematical inconsistency [11].

3 Multiplicative anomaly in the singular case

However there exist cases in which the the analytic continuation of the zeta function is not regular at $z = 0$. Recall that the usual Seeley Theorem, is based on standard heat kernel expansion (here $D = 4$, and boundaryless case)

$$\mathrm{Tr} e^{-tL} \simeq \sum_{j=0}^{\infty} A_j t^{j-2} . \tag{32}$$

As a consequence the standard meromorphic continuation admits only simple poles

$$\zeta(s|L) = \frac{1}{\Gamma(s)} \left[\sum_{j=0}^{\infty} \frac{A_j(L)}{s+j-2} + J(s) \right] , \tag{33}$$

the function $J(s)$ being analytic. It follows that $\zeta(s|L)$ is regular at $s = 0$ and $\zeta(0|L) = A_2(L)$, and $\zeta'(0|L)$ is well-defined and gives the regularized expression for $\det \ln L$.

If we have a non standard Heat-Kernel expansion

$$\mathrm{Tr} e^{-tL} \simeq \sum_{j=0}^{\infty} A_j t^{j-2} + \sum_{j=0}^{\infty} P_j \ln t \, t^{j-2} \tag{34}$$

namely, additional $\ln t$ terms are present, one has a generalization of Seeley result:

$$\zeta(s|L) = \frac{1}{\Gamma(s)} \left[\sum_{j=0}^{\infty} \frac{A_j(L)}{s+j-2} - \sum_{j=0}^{\infty} \frac{P_j(L)}{(s+j-2)^2} + J(s) \right] . \tag{35}$$

As a consequence, double poles are present and, in general, $\zeta(s|L)$ may have a simple pole at $s = 0$. This may happen with pseudodifferential operators or differential operators defined on non compact manifolds.

Within this new contest, two issues have to be discussed. The first one is: how $\ln \det L$ may be defined and the second one: how the Multiplicative Anomaly may be computed in these singular cases?

With regard to the first issue, the starting point is to observe that the functional determinat of self-adjoint operator L is formally a "divergent infinite product"

$$\prod_n \lambda_n , \tag{36}$$

where λ_n are the eingenvalues of L. The divergenge is present because L is a self-adjoint unbouned operator. To deals with it, Mathematicians introduce a canonical regularization by considering the analytic continuation of associated zeta function $\zeta(s|L) = \sum_n \lambda^{-s}$ and then by definition

$$\prod_{k=1}^{\infty} \lambda_k \equiv e^{-\left(\mathrm{Res}(\frac{\zeta(s)}{s^2})\right)_{s=0}} , \tag{37}$$

where Res is the usual Cauchy residue.

In the regular case, one Taylor expands $\zeta(s)$ at $s = 0$ and one obtains

$$\prod_{k=1}^{\infty} \lambda_k \equiv e^{-\zeta'(0)} , \tag{38}$$

in agreement with Ray-Singer-Hawking prescription. However, if one has a simple pole, namely $\zeta(s|L) = \frac{\omega(s)}{s}$, then

$$\prod_{k=1}^{\infty} \lambda_k \equiv e^{-\frac{\omega''(0)}{2}}\,, \tag{39}$$

In agreement with [12, 13]. This prescription is quite general and is valid for generic singular behaviour of ζ at $s = 0$.

What about the second issue? To our knowledge, Wodzicki approach and associated formula are valid only in the regular case. With regard to this issue, we note that, in general, one may proceed defining the regularized functional determinant of the operator L as regularization of a divergent product, namely

$$\ln\det L = -\operatorname{Res}\left(\frac{\zeta(s|L)}{s^2}\right)_{s=0}. \tag{40}$$

3.1 A multiplicative anomaly formula for shift operators

Consider elliptic differential self-adjoint operators: $H = H_0 + V_1$ and $H_V = H + V = H_0 + V_2$ with $V = V_2 - V_1$ constant shifts. The main idea is to express all quantities as a function of $\zeta(s|H)$. Now, the spectral theorem gives

$$\zeta(s|H_V) = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{(-V)^n}{n!} \Gamma(s+n)\zeta(s+n|H)\,. \tag{41}$$

$$\zeta(s|H H_V) = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{(-V)^n}{n!} \Gamma(s+n)\zeta(2s+n|H)\,. \tag{42}$$

Note that here only the meromorphic continuation of $\zeta(H|s)$ appears.

Recalling that the Multiplicative Anomaly may be defined as

$$\mathscr{A} = \ln\det(H(H+V)) - \ln\det(H+V) - \ln\det H\,, \tag{43}$$

one has

$$\mathscr{A} = \mathscr{A}_0 + \mathscr{A}_V\,, \tag{44}$$

with

$$\mathscr{A}_0 = -\operatorname{Res}\left(\frac{\zeta(2s|H) - 2\zeta(s|H)}{s^2}\right)_{s=0}, \tag{45}$$

$$\mathscr{A}_V = -\operatorname{Res}\left(\frac{1}{s^2\Gamma(s)} \sum_{n=1}^{\infty} \frac{(-V)^n}{n!} \Gamma(s+n)\left[\zeta(2s+n|H) - \zeta(s+n|H)\right]\right)_{s=0}, \tag{46}$$

A direct computation shows that the first term does not give any contribution to the Cauchy residue, and in the second term, only a *finite number* of terms survive, the ones corresponding to the poles for $\mathrm{Re}\, s > 0$, and these are, say n_0. Thus, one has

$$\mathscr{A} = -\,\mathrm{Res}\left(\frac{1}{s^2\Gamma(s)}\sum_{n=1}^{n_0}\frac{(-V)^n}{n!}\Gamma(s+n)\left[\zeta(2s+n|H) - \zeta(s+n|H)\right]\right)|_{s=0}\,. \quad (47)$$

This is a new result and gives a general expression for the Multiplicative Anomaly in the case considered. It should be noted that the above expression involves only the meromorphic continuation of $\zeta(s|H)$, and we remind that this follows from the Heat-Kernel expansion of heat trace $\mathrm{Tr}\, e^{-tH}$.

Example: If one has poles of third order,

$$\zeta(s|H) = \frac{1}{\Gamma(s)}\sum_{j=0}^{\infty}\left[\frac{A_j}{s+j-2} - \frac{P_j}{(s+j-2)^2} + \frac{C_j}{(s+j-2)^3} + J(s)\right]. \quad (48)$$

In this case $\zeta(s|H)$ has a pole of second order at $s = 0$, and we get

$$\mathscr{A} = \frac{V^2}{4}\left([A_0 + (1-\gamma)P_0] + \frac{1}{24}[10 - 2\pi^2 - 24\gamma + 12\gamma^2 - G]C_0\right), \quad (49)$$

$\gamma = -\psi(1)$ being the Euler-Mascheroni constant and G is a computable constant. In the standard regular case, all P_j and C_j are vanishing and one gets the Wodzicki formula result [7].

$$\mathscr{A} = \frac{1}{4}V^2 A_0 = \frac{1}{96\pi}\int d^4x(V_1 - V_2)^2\,. \quad (50)$$

Let us conclude this Section with a simple example: a vector valued massive scalar field ϕ defined in $R \times H_3/\Gamma$, ultrastatic space-time with a non compact hyperbolic manifold with finite volume [14].

$$I = \int\left[-\frac{1}{2}\phi\Delta\phi + \frac{m^2\phi^2}{2}\right]\sqrt{g}\,d^4x \quad (51)$$

The Heat-Kernel expansion reads

$$\mathrm{Tr}e^{-tL} \sim \sum_{j=0}^{\infty}\left[A_j(L) + P_j(L)\ln t\right]t^{j-2} \quad (52)$$

$$A_0 = \frac{Vol}{16\pi^2}, \quad P_0(L) = 0\,, \qquad P_1(L) = -\frac{Vol}{16\pi^2}\frac{\pi R}{6v_F}\,, \quad (53)$$

$$P_2(L) = \frac{Vol}{16\pi^2}\frac{\pi R\delta^2}{6v_F}\,. \quad (54)$$

with v_F finite volume of fundamental domain of hyperbolic non compact manifold and $\delta^2 = m^2 + \frac{R}{6}$. Note that $P_0 = 0$, thus the Multiplicative Anomaly is equal to the regular case.

4 Concluding remarks

The Multiplicative Anomaly is present in dealing with functional determinants of products of differential operators. In the regular case, it is a local functional of the fields and can be computed. In the singular case, where the zeta functions are not analytic at $s = 0$, we have shown that it is still a local functional and we have provided a formula fot its evaluation.

Within one-loop physics, apparently no new physics seems to be associated with Multiplicative Anomaly, also in the presence of generalized zeta-function regularization. However its inclusion is necessary for mathematical consistency: charged scalar field at finite temperature is an example.

Furthermore, dealing with spinor fields, one has the Euclidean massive Dirac operator.

$$K = p_\mu \gamma^\mu + iM = A + iM\,, \quad A^+ = A = p_\mu \gamma^\mu\,. \tag{55}$$

Problem: How to evaluate $\ln\det A$? In $D = 4$, with $L = A^2$ being the spinorial Laplace operator in curved space, one has [15]

$$\ln\det K = \frac{1}{2}\ln\det(L + M^2) + i\frac{\pi}{2}\zeta(0|L + M^2) + c_1 M^2 A_1(L) + c_2 M^4 A_0(L)\,. \tag{56}$$

where A_0 and A_1 are the associated Seeley-de Witt coefficients. Two remarks: first the last term contains $A_1(L) \equiv R$, Ricci scalar, this is Sakarov induced gravity idea [8]. Second, this last term may be interpreted as multiplicative anomaly contribution [15]. With regard to this last issue, recently , I. Shapiro and other have reported a non trivial non-local M. A . in Quantum ED in curved space-time [16].

Acknowledgements We would like to thank our friend Emilio Elizalde for fruitful collaboration on this interesting issue in all these years.

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Part IV
Non-standard Approaches

Iso-Minkowskian Geometry For Interior Dynamical Problems

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Abstract By recalling that the exact validity of special relativity in vacuum has been experimentally established beyond doubt, we indicate mathematical, physical, chemical experimental and industrial evidence according to which physical media alter the Minkowskian spacetime; we outline the novel *iso-Minkowskian geometry* specifically built for interior dynamical problems; and we point out its universality for all possible spacetimes characterized by a symmetric metric in (3+1)-dimensions.

1 Apparent lack of exact validity of the Minkowskian and Riemannian geometries for interior problems

Research conducted by numerous scholars during the past fifty years (see general review [1] and specialized treatments [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]) has identified mathematical, physical, chemical, experimental and industrial evidence according to which the Minkowskian geometry, special relativity and the Lorentz-Poincaré (LP) symmetry are not exactly valid for *interior dynamical problems* (e.g., extended-deformable particles and electromagnetic waves propagating within physical media). Needless to say, the *exact validity* of the Minkowskian geometry, special relativity and the LP symmetry for *exterior dynamical problems* (point particles and electromagnetic waves propagating in vacuum) and their *approximate validity* for interior problems remain beyond scientific doubt.

The reasons for said insufficiency are numerous indeed, and include: the impossibility of introducing inertial reference frames within physical media (such as air or water) due to known resistive forces, with consequential inability of formulating the very principle of relativity, let alone testing it experimentally; physical media solely admit the privileged frame at rest with themselves in direct conflict with relativity

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_34,

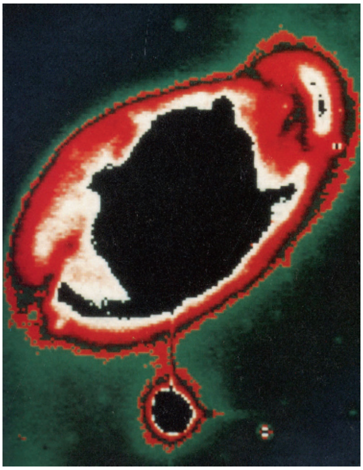

Fig. 1 A view of one of the first experimental deviations of the Minkowskian geometry within physical media identified in the 1970s by H. Arp [39] (see Sect. 3), the galaxy NGC 4319 (top) and the quasar Mark 205 (bottom) that result as being physically connected according to gamma spectroscopy, yet have largely different cosmological redshifts ($z = 0.0056$ and $z = 0.07$, respectively), whose quantitative representation requires different geometric features for the different interior physical media of the galaxy and the quasar

axioms; massive particles, such as electrons, can travel in water faster than the local speed of light, thus forcing the assumption in *water* of the speed of light in *vacuum* as the maximal causal speed, in which case the sum of two local speeds of light does not yield the local speed of light in disagreement with the relativistic sum of speeds.

During the 20th century, all these insufficiency have been generally dismissed via the reduction of light to photons traveling in empty space while experiencing scattering, absorption and remission by the atoms of the medium. However, such a reduction is afflicted by major insufficiencies. As an illustration, for the case of light propagating in water, we have: the impossibility of a numerical representation of the large angle of refraction (since photons must scatter in all directions at the impact with the water surface); the impossibility of a numerical representation of the large reduction of the speed of light by about $1/3$ (since scattering, absorption and re-emission of photons can at best account for a small reduction of speed); the impossibility of reducing to photons electromagnetic waves with a large wavelength, e.g., of one meter, that experience the same phenomenology as that of light; the impossibility of the very existence of light within opaque media with consequential obliteration of the entire conceptual, mathematical and physical framework of special relativity; and numerous other insufficiencies [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

It should be indicated that Einstein introduced the reduction to photons solely for light with such a frequency to admit quantized black-body effects, which reduction is still entirely valid to this day and permits a quantitative representation of the small percentage of the light beam in water lost to dispersion. Definitely, Einstein did not voice the reduction of light to photons to claim the validity of his theory within physical media. The latter claim has been proffered by Einstein's *followers* without any serious scrutiny and despite rather visible inconsistencies. In fact, the reduction to photons of the *entire* beam of light in water against Einstein's teaching implies that an extremely big number of photons must traverse an extremely big number of nuclei without any deflection, as an evident necessary condition to maintain the propagation of a light beam in water along a visible straight line.

It is today known that the sole scientific, that is, quantitative-numerical, representation of *all* experimental; evidence for *all* frequencies is given by a return to the Maxwell conception of light (as well as photons as wave packets) as being *transverse electromagnetic waves* propagating in the universal substratum known as the *ether* with historical expression for the speed within transparent physical media

$$C(x,v,\omega,\delta,\tau,\ldots) = \frac{c}{n(x,v,\omega,\delta,\tau,\ldots)}\,, \tag{1}$$

where c is the speed of light in vacuum and n is the familiar index of refraction with an unrestricted dependence on all needed local variables, such as coordinates x, speed v, frequency ω, density δ, temperature τ, etc.

Evidently, the lack of exact character of the Minkowskian geometry necessarily implies the lack of exact character of the (pseudo-) Riemannian geometry for interior gravitational problems. Independently from physical and geometrical evidence, a most forceful argument is topological. The topology of exterior problems is notoriously local-differential, thus solely capable of representing a finite set of isolated points. By contrast, the correct topology for interior problems, especially interior gravitational problem with very high densities, must be of nonlocal-integral character due to the evident mutual penetration of extended and hyperdense charge distributions. Consequently, the Riemannian geometry cannot possibly be exactly valid for interior gravitational problems of stars, quasars and black holes beginning with its topological foundations.

Almost needless to say, the indicated limitations of the Riemannian geometry have no impact on its historical value because geometries can at best provide an approximation of out rather complex physical reality, while the *approximate* validity of the Riemannian geometry for interior gravitational problems remains beyond scientific doubt.

2 The universal iso-Minkowskian geometry

The return to the Maxwellian conception of light as electromagnetic wave with local speed (1) brings into focus the so-called *Lorentz problem,* referred to the construction of the symmetry leaving invariant a locally varying speed of light. As well known to historians, Lorentz first attempted the achievement of the universal symmetry of local speed $C = c/n(x,v,\omega,\delta,\tau,\ldots)$, but encountered major technical difficulties that forced him to consider the simpler case of constant speed c by setting up in this way the foundation of special relativity.

The author has dedicated most of his research life to the Lorentz problem beginning with his Ph. D. studies in the mid 1960s. The first outcome of these studies is that Lorentz's inability to achieve the invariance of local speed (1) was due to *insufficiency of the basic theory, Lie's theory.* Independently from topological and other insufficiencies, Lie's theory is known as being *linear, local-differential and Hamiltonian.* By contrast, serious studies of interior problems ate large, including Lorentz problem, require a treatment which is *nonlinear, nonlocal-integral and non-Hamiltonian,* the latter characteristics being referred to the *variational nonselfadjointness* of interior problems [2] and the consequential impossibility of their representation via the sole knowledge of a Hamiltonian.

Hence, the solution of the Lorentz problem left no other alternative than that of working out a structural generalization of Lie's theory of nonlinear, nonlocal and non-Hamiltonian character. Along these lines, the author proposed in 1967 [16] a *Lie-admissible generalization of Lie's theory* subsequently specialized for the description of systems than, besides being nonlinear, nonlocal and non-Hamiltonian, are also *irreversible over time* (see memoir [17] of 1978, general presentation [18] of 2006 and review [4, 5]).

The Lie-admissible irreversible treatment of interior problems is excessively complex for the limited length of this note. Consequently, we have to restrict ourselves to a study of the subclass of *nonlinear, nonlocal and non-Hamiltonian interior problems that are reversible over time.* For the case of light propagating within a transparent medium such as water, the above subclass essentially requires ignoring in first approximation the percentage of the light beam lost to dispersion, under which light propagation is indeed reversible over time.

The latter class of systems can be quantitatively treated via the subclass of Lie-admissible theories known as *isotopies* (i.e., axiom-preserving lifting) of Lie's theory [2, 3, 4, 5, 6, 7], today known as the *Lie-Santilli isotheory,* [8, 9, 10, 11, 12, 13, 14, 15] which are based based on the lifting of the trivial unit of Lie's theory into the most general possible, integral-differential, positive-definite unit known as *Santilli isounit*

$$I = \text{Diag.}(1,1,1,\dots) \to \hat{I}(x,v,\omega,\delta,\tau,\dots) = 1/T > 0\ , \tag{2}$$

with the joint lifting of the Lie algebras into the *Lie-Santilli algebras*

$$\left[J_i, J_j\right] = J_i J_j - J_j J_i = C_{ij}^k J_k \to \left[J_i \hat{,} J_j\right] = J_i T J_j - J_j T J_i = \hat{C}_{ij}^k J_k\ , \tag{3}$$

as well as of Lie's transformation groups into the *Lie-Santilli transformation isogroups*

$$A(w) = e^{iJw} A(0) e^{-iwJ} \to A(w) = e^{iJTw} A(0) e^{-iwTJ}\ , \tag{4}$$

with the Lie's parameters w and generators J remaining unchanged under isotopies.

To understand the complexity of the Lorentz problem, and the decades of "out of the mainstream" research required for its solution, let us recall that he invariance of the *constant* speed c is known as being canonical at the classical level and unitary at the operator lever, thus enjoying a majestic axiomatic and physical consistency, including: the same numerical predictions under the same conditions at different time; preservation over time of Hermiticity-observability; verification of causality and conservation laws; etc.

By contrast, the invariance of the locally varying speed of light (1) soon emerged as being *noncanonical* at the classical level and *nonunitary* at the operator level and, thus verifying the so-called *Theorems of Catastrophic Mathematical and Physical Inconsistencies of Noncanonical and Nonunitary Theories* (see original works [18, 19, 20, 21, 22, 23] and review [5]a, including: the prediction of different numerical values under the same conditions at different times (inconsistency that, alone, prevents any possible invariance of $C = c/n$); loss over time of Hermiticity and, therefore, of observability (an occurrence known as the *Lopez Lemma* [20]); violation of causality and conservation laws; and other inconsistencies.

The resolution of these inconsistency problems required decades of solitary studies and was solely achieved in 1996 in mathematical memoir [24] with t the correct formulation of the covering Lie-Santilli isotheory via a new mathematics, today known as *isomathematics* [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] consisting of the isotopies of the entire mathematics used in Lie's theory, including the lifting of numbers, functional analysis, differential calculus, etc. into such a form admitting $\hat{I}(x,v,\omega,\delta,\tau,\ldots)$, rather than I, as the correct left and right unit at all levels. Readers with a vast knowledge of Lie's theory, but not experts in the covering Lie-Santilli isotheory, should be alerted that, in the event only *one* of the isotopies is not used, e.g., the isotheory is treated with the conventional differential calculus, the inconsistency theorems are activated resulting in the absence of mathematical and physical maturity.

Following these decades in the prior construction of the isotopies of Lie's theory, the author constructed the srep-by-step isotopies of all aspects of special relativity, including th isotopies of: the rotational symmetry [25, 26]; the SU(2)-spin symmetry [27, 28]; the Lorentz symmetry at the classical [29] and operator [30] levels; the Poincaré symmetry [31]; the spinorial covering of the Poincaré symmetry [32]; including the isotopies of the axioms and physical laws of special relativity, first formulated in monographs [23] of 1991, and then studies in various works (see monographs [3, 4, 5, 6, 7] and vast literature quoted therein).

The main lines of spacetime isotopies are now elementary. The most fundamental geometric point of this paper is that *the alteration of any characteristics of light cannot occur without a modification of the Minkowskian spacetime.* Consequently, the alteration of the speed of light requires a corresponding lifting of the Minkowskian metric from its historical form with constant c, $\eta = \mathrm{Diag.}(1,1,1,-c^2)$, to a generalized metric characterizing the local speed c^2/n^2 whose most general possible symmetric realization can be expressed with the lifting [29]

$$\begin{aligned}
\eta &= \mathrm{Diag.}\left(1,1,1,-\to c^2\right)\\
\hat{\eta} &= T(x,v,\omega,\delta,\tau,\ldots)\\
\eta &= \mathrm{Diag.}\left(1/n_1^2, 1/n_2^2, 1/n_3^2, -c^2/n_4^2\right),\\
n_\mu &= n_\mu(x,v,\omega,\delta,\tau,\ldots) > 0\,, \qquad (5)
\end{aligned}$$

($\mu = 1,2,3,4$), the corresponding lifting of the spacetime invariant

$$\begin{aligned}
x^2 &= \left(x^\mu \eta_{\mu\nu} x^\nu\right) \times I \\
&= \left(x_1^2 + x_2^2 + x_3^2 - t^2 c^2\right) \times I \in \mathscr{R} \rightarrow \hat{x}^{\hat{2}} \\
&= \left[x^\mu \left(T_\mu^\rho \eta_{\rho\nu}\right) x^\nu\right] \times \hat{I} \\
&= \left(\frac{x_1^2}{n_1^2} + \frac{x_2^2}{n_2^2} + \frac{x_3^2}{n_3^2} - t^2 \frac{c^2}{n_4^2}\right) \times \hat{I} \in \hat{\mathscr{R}}\,,
\end{aligned} \tag{6}$$

and of the lifting of the basic unit

$$I = \text{Diag.}(1,1,1,1) \rightarrow \hat{I} = \text{Diag.}\left(n_1^2, n_2^2, n_3^2, n_4^2\right)\,, \tag{7}$$

where: $n_4 = n$ acquires the meaning of the *time characteristic function* of the medium considered; the n_k, $k = 1,2,3$, are the *space characteristic functions;* $\mathscr{R}$ ($\hat{\mathscr{R}}$) is the field of real numbers (real isonumbers); and the multiplication by $\hat{I}$ is necessary for $\hat{x}^{\hat{2}}$ being an isonumber. Note that the characteristic functions n_μ are far from being the usual "free parameters" since they represent measurable characteristics of the medium, such as size, density, index of refraction, inhomogeneity and anisotropy normalized to the value $n_\mu = 1$, $\mu = 1,2,3,4$ for the vacuum (see [3, 4, 5, 6, 7] for details).

Since the generators are not altered in the transition from the Lie to the Lie-Santilli covering theory, the isotopies of all possible spacetime (as well as internal) symmetries are done via the use of now elementary rules (2)–(7) that lead Santilli [29] in 1983 to the first formulation of the universal symmetry of invariance (6) that we can write here for the simple case in $(3,4)$-dimension (see monograph [4]b for the general case)

$$x'^1 = x^1, \qquad x'^2 = x^2, \tag{8a}$$

$$x'^3 = \hat{\gamma}\left(x^3 - \hat{\beta}\frac{n_4}{n_3}x^4\right), \qquad x'^4 = \hat{\gamma}\left(x^4 - \hat{\beta}\frac{n_3}{n_4}x^3\right), \tag{8b}$$

$$\hat{\gamma} = \frac{1}{\sqrt{1-\hat{\beta}^2}}, \qquad \hat{\beta} = \beta\frac{n_4}{n_3} = \frac{v_3}{c}\frac{n_4}{n_3}. \tag{8c}$$

The corresponding isotopies of the Poincaré symmetry [24, 25, 26, 27, 28, 29, 30, 31, 32] are today known as the *Lorentz-Poincaré-Santilli (LPS) isosymmetry.*

On primitive grounds, the geometry underlying the isotopies of the Minkowskian spacetime is given by a novel geometry first formulated in [29] of 1983, then studied in various works, finalized in paper [33] of 1998, and today known as the *Minkowski-Santilli isogeometry,* or *isogeometry* for short, essentially consists in the reconstruction of the entire formulation of the Minkowskian geometry with respect to isounit (7). The proof of the following property is instructive for the non-initiated reader:

Lemma 1 ([29, 34, 35]). *The Minkowski-Santilli isogeometry is "directly universal" for symmetric* $(3+1)$*-dimensional spacetimes, in the sense of admitting as particular cases all possible spacetime geometries, thus including the Minkowskian,*

Fig. 2 Views taken by the author in Palm Harbor, Florida, of the horizon when the Sun is at the Zenith (left), at Sunset (top right) and Sunrise (bottom right), illustrating the predominant blue color when the Sun is at the Zenith and the predominant red color at both Sunset and Sunrise. As reviewed in Sect. 3, these colors constitute visible evidence of deviations from the Minkowskian geometry in our atmosphere in full agreement with Arp's discovery indicated in Fig. 1

Riemannian, Finslerian and other geometries ("universality") directly in th isometric, thus without transforming the coordinates of the observer ("direct universality").

Some of the features of the isogeometry should not appear unusual due to the novelty of the underlying isomathematics. For instance, the Minkowski-Santilli isogeometry admits the entire machinery of the Riemannian geometry (such as Christoffel's symbols, covariant derivative, etc.), trivially, due to the explicit dependence of the isometric $\hat{\eta}_{\mu\nu} = T^{\rho}_{\mu}\eta_{\rho\nu}$ in the local coordinates. Yet, *the novel isogeometry has null curvature,* trivially, as a central condition for being a correct isotopy of the Minkowski geometry. This occurrence too should not be surprising because, in the final analysis, the center of a massive body (with spherical symmetry) has null gravitational force.

As an illustrative example, one can *identically* reformulate any Riemannian metric, such as the Schwartzschild metric, in terms of the characteristic quantities $1/n^2_\mu$ (see [4]b for details). In this case, the geometry does indeed require curvature when the metric is referred to the trivial unit I. However, the same geometry show no curvature when formulated with respect to the isounit $\hat{I}$, trivially, because the latter has the *inverse* value n^2_μ, resulting in an invariant flatness under lifting (6). Note that the elimination of curvature is *necessary,* to our best understanding, to achieve the universal invariance of all Riemannian line elements, as well as to bypass the activation of the inconsistency theorems caused by the conventional covariance [18, 19, 20, 21, 22, 23]. At any rate,the lack of experimental detection of gravitational waves pointed out by C. Corda [36] appears to confirm all these lines.

In closing, mathematical inclined readers should be aware that all the above results are permitted by a structural generalization of the conventional, 20th century, local-differential topology into its isotopic covering initiated by the mathematicians Gr. T. Tsagas and D. S. Sourlas [37], completed by R. M. Falcon Ganfornina and J. Nunez Valdes [14, 38] and today known as the *Tsagas-Sourlas-Ganformina-Nunez (TSGN) isotopology.*

3 Experimental verifications

The novel isomathematics, related geometries and physical formulations for nonrelativistic and relativistic, classical and operator formulations for interior dynamical problems (reversible over time) have nowadays experimental verifications in all quantitative sciences, including classical physics, particle physics, nuclear physics, superconductivity, chemistry, biology, astrophysics and cosmology (see [2, 5]a for details). Evidently, we cannot possibly review in this short note all these verifications. Hence, we limit ourselves to the review of the verifications of direct geometric nature, those based on deviations from the Minkowskian geometry within physical media in the absence of gravitation.

Remember that Doppler's law is an ultimate manifestation of the Minkowskian geometry uniquely derivable from the LP symmetry. By contrast, the covering isogeometry and related LPS isosymmetry uniquely predict the following generalized law for the frequency of electromagnetic waves propagating within a physical medium, known as *Doppler-Santilli isoshift law*

$$\omega' = \frac{1-\hat{\beta}\cos(\alpha)}{\sqrt{1-\hat{\beta}^2}}\omega\,, \tag{9}$$

The above covering law predicts that the isoshift is not generally null for null speeds (due to the indicated dependence of the characteristic quantities on the speed v) and we write for null angle of aberration

$$\mathrm{Lim}\,\omega'_{v\to 0} \approx \mathrm{Lim}_{v\to 0}\left(1-\frac{v_3}{c}\frac{n_4}{n_3}+\cdots\right)\omega = 1-K(r,v,\omega.\delta,\ldots),\quad K>0 \tag{10}$$

This novel event was predicted in [3]b of 1991, is today known as *Santilli isoredshift* and is referred to *a shift toward the red for light propagating within transparent physical media without any relative motion between the source, the medium and the observer.* We merely have inevitable interactions between light and the medium under which light loses energy $E = h\omega$ with consequential reduction of the frequency ω due to the impossibility of atoms in the medium of losing energy since they are generally in their stable ground state.

The first experimental evidence known to this author on the existence of the isoredshift (although not interpreted as such) has been the discovery by H. Arp [39] of quasars that, according to gamma spectroscopic evidence, are physically connected to an associated galaxy, yet their respective cosmological redshifts are dramatically different (see Fig. 1).

Such a difference clearly indicates a departure from the Minkowskian geometry of the vacuum because, under its validity, said large difference in cosmological redshifts would require that the quasar has at least 100 times the speed of the galaxy, under which conditions the quasar and its associated galaxy would have separated completely billions of years ago. Numerous hypotheses were formulated in order to

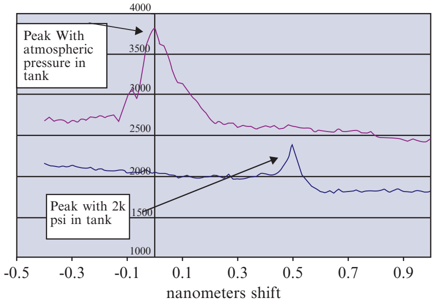

Fig. 3 The first scan confirming Santilli isoredshift obtained at the Isoredshift Testing Station of the Institute for Basic Research in Florida on June 27, 2009 (see [41] for details)

resolve this "anomaly," while maintaining the validity of special relativity, without achieving to date a resolution accepted by the scientific community at large.

Santilli's proposal [3]b of the isotopies of special relativity was based on Arp's discovery that was recommended for further study. In 1992, R. Mignani [40] provided a direct experimental verification of Santilli's isorelativity and the isoshift law by showing that they apply for all known pairs of quasars and associated galaxies.

The isotopies of special relativity [3]b additionally identified the colors of our atmosphere as confirmatory experimental evidence and suggested the conduction of experiments. As established by the conventional quantum scattering theory as well as by evidence in air and water, red light is absorbed by physical media, resulting in the predominance of blue light that originates the color of the sky with the Sun at the Zenith (or the color of water at a sufficient depth).

The predominance of red for the Sun at Sunset and Sunrise was interpreted throughout the 20th century via the abrupt and unexplained assumption of the opposite, namely, that blue light is absorbed by the medium with the increase of the trajectory resulting in the predominant red. Santilli [3]b pointed out that this interpretation is in violation of the quantum scattering theory as well as physical evidence, thus leaving as the sole plausible interpretation the isoredshift of light which is indeed proportional to the trajectory within [physical media.]

Additionally, Santilli pointed out that the predominance of red at Sunset and Sunrise occurs for *direct* sunlight, thus excluding possible interpretation via scattering (that refer to the diffused light); the scattering of photons cannot possibly provide a quantitative representation of the large change of wavelength from blue to red (of about 300 nm); and the presence of the isoredshift is rendered necessary by the fact that the predominance of red is essentially the same at Sunset, where we move *away* from the Sun, as well as Sunrise, where we move *toward* the Sun, thus establishing the isoredshift as dominant over the expected small contributions from the Doppler's law, of course, under a sufficiently long interior trajectory. In view of all the above, Santilli concluded [3]b, suggesting the conduction of experiments on Earth, such as the measurement of a Fraunhofer line of the Sun while moving from the Zenith to the equator, and various other experiments.

Despite the passing of decades, the propagation of the information and the author solicitations for conducting the proposed experiments to various physical and astrophysical laboratories, the above experimental verifications (including Arp's discovery) remained vastly ignored by most physicists and astrophysicists to their evident peril. Consequently, in 2009, Santilli [41] conducted the measurement of the isoredshift in a 20 m long tube containing air at about 140 bars with the resulting measurement of about 0.5 nm isoshift of a blue laser light (see Fig. 3). Reference [40] proved the capability of such an isoredshift to provide a numerical representation of Arp's discovery, the color of our atmosphere, and other interior events.

Reference [41] also pointed out that, while being valid in locally empty spaces, *the Minkowskian geometry is nowhere exactly valid in the universe at large, because at cosmological distances the universe is a medium with high energy density, since it is everywhere filled up with light or stars* (see also [35] for cosmological imp[lications). Consequently, the intergalactic or galactic isoredshift can consequently imply the possible absence of universe expansion, big bang, dark matter and dark energy. Reference [41] concluded with the need to conduct experiments on Earth, a number of them already under way, as the sole grounds for serious science along the teaching of Galileo Galilei.

Acknowledgements The author has no words to thank the organizers and participants of the *2010 Cosmology, the Quantum Vacuum, and Zeta Functions* for penetrating criticisms and invaluable comments.

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Nuclear Fusion Drives Cosmic Expansion

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Abstract From a Twin Universe perspective, it is proposed that stellar nuclear fusion can account for the negative-energy pressure (Dark Energy) that drives our present-day accelerated cosmic expansion. In the mirror twinned universe all processes are duplicated but with reverse negative polarity. Both the Positive and Negative Universes exist on the opposing sides of a topological two-dimensional membrane and therefore shares the same experience of a stretching membrane.

Introduction

Using thermodynamic conservation principles the cosmos existing as a pair of identical anti-parallel universes has been proposed [1]. Parameterization of negative quantities can be formulated in terms of Santilli's isodual theory of antimatter [2]. Sakharov [3] proposed the breaking of charge-parity (CP) symmetry led to the baryonic imbalance that created a surplus of matter in our present-day universe. Petit [4] expanded on this cosmological model whereby a twin universe with reverse arrow of time (T) populated mainly by antimatter would maintain global un-violated CPT-symmetry between the combined universes. The new proposed model postulates that a twin universe exists in an identical state of duality, whereby all contents and processes of each universe have equal magnitude but opposite polarity, including energy-mass that in the anti-parallel universe will have negative quantities producing the necessary gravitational repulsion to drive the cosmic expansion. Since both universes reside on the topological surfaces of a shared membrane, the effective stretching of this common cosmic membrane will be observed as expansion in our side of the universe.

The Wilkinson Microwave Anisotropy Probe [5] has determined the Hubble's constant (H) at 71km.s-1.Mpc-1. Hubble's Law [6] is considered the first obser-

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S.D. Odintsov et al. (eds.), *Cosmology, Quantum Vacuum and Zeta Functions: In Honor of Emilio Elizalde*, Springer Proceedings in Physics 137,
DOI 10.1007/978-3-642-19760-4_35,

vational basis for an expanding universe and supporting evidence of the Big Bang model. Dark Energy is the unknown constituent that propels the present accelerating state of our cosmos, and the equally mysterious Dark Matter conjured up to explain the high-rotational velocities of galaxies. The rest of this article proposes that normal nuclear fusion in stars can account for these unknown cosmic mechanisms without the need for such dark fluids.

Stellar nuclear fusion

Stellar nucleosynthesis is the process of nuclear reactions taking place in stars to build heavier elements. The net mass of fused nuclei is smaller than the sum of the components, with the loss mass released as electromagnetic energy according to Einstein's famous mass-energy equivalence relationship:

$$E = mc^2 \ . \tag{1}$$

Newton's law of universal gravity states that the force between two point masses (m_1, m_2) a distance r apart is given by the following equation:

$$F = G\frac{m_1 m_2}{r^2} \ . \tag{2}$$

If we assume that the masses are of equal magnitude $m = m_1 = m_2$, and the area mass density condensing on the two-demensional membrane $\rho_m = m/\pi r^2$, then the gravitational force of acceleration produced by one point mass on the other is given by:

$$a = \pi G \rho_m \ . \tag{3}$$

Energy production in stars

The observable universe is composed of 70% hydrogen, and the proton-proton (p-p) chain reaction is the predominant thermonuclear fusion process that converts hydrogen nuclei into helium in stars with masses up to that of the Sun.

$$4^1\mathrm{H} \rightarrow ^4\mathrm{He} + 2e^+ + 2\nu_e \ . \tag{4}$$

Along with the formation of a pair of positrons and neutrinos, 26.7MeV of energy is released, equivalent to a mass of $4.8x10^{-29}$kg. For more massive stars, another reaction process is also important that of the carbon-nitrogen-oxygen (CNO) cycle. In the main CNO-I reaction the carbon can be considered a catalyst in converting hydrogen into helium with the carbon being reformed at the completion of the following cycle:

$$^{12}\mathrm{C} \rightarrow ^{13}\mathrm{N} \rightarrow ^{13}\mathrm{C} \rightarrow ^{14}\mathrm{N} \rightarrow ^{15}\mathrm{O} \rightarrow ^{15}\mathrm{N} \rightarrow ^{12}\mathrm{C} \quad (5)$$

As with the p-p cycle, the total released of energy is 26.7MeV due to the mass difference between the fusion of the hydrogen parents to form the helium daughter.

Stretched universal membrane

If we assume that the cosmos is uniformly distributed with point-like stars, the vast empty interstellar space would produce minimal gravitation contraction on the membrane upon which our positive universe resides. If we further assume that an identical negative universe resides on the opposite side of the same membrane, and all quantities and processes are duplicated but in opposite polarity. The stars in the positive universe would undergo the standard nuclear fusion processes that release large amount of electromagnetic waves into the empty space. Consider the same processes on the reverse side of the membrane, whereby the equivalent release of energy condenses out as point masses with negative energy and hence repulsive gravity.

For simplicity of computation, we will assume that the release of solar energy from the various chain reactions condense out as two equal point masses that occupy the volume of the fused helium nuclei $2x10^{-15}m$. With a gravitational constant value of $G = 6.67428x10^{-11}m^3.kg^{-1}s^{-2}$, the gravitation repulsive acceleration is approximately $1.6x10^{-9}m.s^{-2}$. This repulsive force of gravity moves at the speed of light $(c = 299,792,458ms^{-1})$, so the stretched membrane will expand at a rate of:

$$U_r = \frac{\pi G \rho_m}{c} \quad (6)$$

Inputting the model values, the estimated cosmic expansion rate $U_r \sim 5x10^{-18}s^{-1}$. Multiplying this quantum scale of repulsive expansion over an astronomical distance of a Mega-parsec $(Mpc = 3.0857x10^{22}m)$ gives a cosmological expansion rate of $160km.s^{-1}Mpc^{-1}$. Even with this simple model the computed value for the rate of expansion is in reasonable agreement with the present-day measured Hubble constant. Alternatively by interpreting with this simple model, the current Hubble constant of $\sim 70kms^{-1}Mpc^{-1}$ would equate to an average fusion energy release of 11.7MeV.

Summary

The Twin Universe model predicts that the complete cosmos exists as a ten-dimensional entity with two identical but anti-parallel four-dimensional space-time (energy-entropy) universes residing on the opposing surfaces of a two-dimensional

common membrane. Quantities and processes on both sides of the universes are duplicated but of reverse polarity. In the Positive Universe the fusion reactions within stars release vast quantities of energy into the expanse of space as electromagnetic waves. In the reverse Negative Universe the same fusion energy condenses as point masses with negative quantity (repulsive gravity) that stretches out the common membrane producing the observable accelerating expansion of the entire cosmos.

Table 1 Cosmic processes on both sides of the twin universes

Positive Universe	Negative Universe
Fusion generates energy waves	Fusion generates matter particles
Attractive gravity	Repulsive gravity
Measured Hubble's constant $H \sim 71 km.s^{-1}.Mpc^{-1}$	Computed Universal expansion rate $U_r \sim 160 km.s^{-1}.Mpc^{-1}$

The model further predicts that Dark Matter and Dark Energy constitute half of the missing observable energy-mass in the cosmos. Present experimental measurements estimate the percentage of Dark Matter at 23% and Dark Energy at 73%. However, there are proponents that claim both are the same component of Dark Fluid [7], and hence if the differing effects are producing a double-counting of the same unobserved material, then the actual percentage may indeed be 50%.

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