4

Time integration

4.1 Introduction

Let $I := [0, t_e]$, $f : I \to \mathbb{R}^N$, $F : \mathbb{R}^N \to \mathbb{R}^N$ and $u_0 \in \mathbb{R}^N$. Consider an initial value problem: determine $u(t) \in \mathbb{R}^N$ such that

$$\frac{du}{dt} + F(u) = f(t) \quad \text{for} \quad t \in I, \quad u(0) = u_0. \tag{4.1}$$

As we will see further on, the Stokes- and Navier-Stokes systems of DAEs in (3.22) and (3.38) take this form if one eliminates the pressure variable by restricting to the subspace of (discrete) divergence free velocities. Related to existence and uniqueness of a solution of (4.1) we give a standard result from the literature (Picard-Lindelöf theorem). For $b > 0$ define $G_b := \left\{ v \in \mathbb{R}^N : \|v - u_0\| \leq b \right\}$ with $\| \cdot \|$ any given norm on $\mathbb{R}^N$. Assume that for $a > 0$ the function $f : [0, a] \to \mathbb{R}^N$ is continuous and that F satisfies the Lipschitz condition:

$$\|F(v) - F(w)\| \leq L\|v - w\| \quad \text{for all} \quad v, w \in G_b. \tag{4.2}$$

Then the initial value problem in (4.1) *has a unique solution* $u(t)$ for $t \in [0, \alpha]$ with $\alpha := \min\{a, bL^{-1}\}$. In the remainder we assume that f and F satisfy these conditions (for suitable a, b, L) and that $t_e \leq \alpha$, i.e., (4.1) has a unique solution.

We discuss a few discretization methods for the general problem (4.1). In our applications the systems are very stiff and thus we need implicit methods. A classical and still very popular method is the θ-scheme:

$$\frac{u^{n+1} - u^n}{\Delta t} + \theta F(u^{n+1}) + (1 - \theta)F(u^n) = \theta f(t_{n+1}) + (1 - \theta)f(t_n), \tag{4.3}$$

with $\theta \in [0,1]$. For $\theta = 1$ this is the *implicit Euler scheme* and for $\theta = \frac{1}{2}$ this method is known as the *Crank-Nicolson method*. Another popular method is the *BDF2 scheme*:

$$\frac{3}{2}u^{n+1} - 2u^n + \frac{1}{2}u^{n-1} + \Delta t F(u^{n+1}) = \Delta t f(t_{n+1}). \tag{4.4}$$

Note that the θ-scheme is a *one*-step method, whereas the BDF2 method is a linear *two*-step scheme. Another method that is used in our applications is the following *fractional-step θ-scheme*. For a given $\theta \in (0, \frac{1}{2})$, the fractional-step θ-scheme is based on a subdivision of each time interval $[n\Delta t, (n+1)\Delta t]$ in three subintervals with endpoints $(n+\theta)\Delta t$, $(n+1-\theta)\Delta t$, $(n+1)\Delta t$. For given u^n the approximations $u^{n+\theta}$, $u^{n+1-\theta}$, u^{n+1} at these endpoints are defined by

$$\frac{u^{n+\theta} - u^n}{\theta \Delta t} + \alpha F(u^{n+\theta}) + (1-\alpha)F(u^n) = f(t_n) \tag{4.5a}$$

$$\frac{u^{n+1-\theta} - u^{n+\theta}}{(1-2\theta)\Delta t} + (1-\alpha)F(u^{n+1-\theta}) + \alpha F(u^{n+\theta}) = f(t_{n+1-\theta}) \tag{4.5b}$$

$$\frac{u^{n+1} - u^{n+1-\theta}}{\theta \Delta t} + \alpha F(u^{n+1}) + (1-\alpha)F(u^{n+1-\theta}) = f(t_{n+1-\theta}). \tag{4.5c}$$

Standard measures for the quality of discretization methods for (stiff) initial value problems are consistency, stability, a smoothing property and the amount of dissipativity. Below we treat these quality measures for the methods that we consider.

Consistency

The implicit Euler method has a consistency order of 1. The Crank-Nicolson, BDF2 and fractional-step θ-scheme, with $\theta = 1 \pm \frac{1}{2}\sqrt{2}$, all have consistency order 2. We derive this consistency result for the fractional-step θ-scheme.

Lemma 4.1.1 *Assume an arbitrary $f \in C^2([0,t_e])$ and $\lambda \in \mathbb{R}$. Let $u(t)$ be the solution of $\frac{du}{dt} - \lambda u = f$, $u(0) = u_0$. Let u^{n+1} be the result of the fractional-step θ-scheme (4.5) applied to this problem with $u^n := u(t_n)$. Then for $\theta = 1 \pm \frac{1}{2}\sqrt{2}$ we have*

$$|u(t_{n+1}) - u^{n+1}| \leq c\,(\Delta t)^3, \tag{4.6}$$

with a constant c independent of Δt and n.

Proof. We take $\theta = 1 \pm \frac{1}{2}\sqrt{2}$. For the solution $u(t)$ we have

$$u(t) = e^{\lambda(t-t_n)}u(t_n) + \int_{t_n}^{t} e^{\lambda(t-\tau)} f(\tau)\,d\tau, \quad t \geq t_n.$$

Hence, with $z := \lambda\Delta t$,

$$u(t_{n+1}) = e^z u(t_n) + \int_{t_n}^{t_{n+1}} e^{\lambda(t_{n+1}-\tau)} f(\tau)\, d\tau \ .$$

A straightforward calculation, in which we use that $2\theta^2 - 4\theta + 1 = 0$ holds, results in

$$\begin{aligned} u^{n+1} &= g(z)u^n + \Delta t\big[\theta(1+z) - (1-\alpha)\theta^2 z + \mathcal{O}(z^2)\big] f(t_n) \\ &+ \Delta t\big[(1-\theta)(1+\theta z) + (1-\alpha)(3\theta^2 - 4\theta + 1)z + \mathcal{O}(z^2)\big] f(t_{n+1-\theta}) \\ &= g(z)u^n + \Delta t\big(\theta(1+z)f(t_n) + (1-\theta)(1+\theta z)f(t_{n+1-\theta})\big) + \mathcal{O}(\Delta t^3), \end{aligned} \tag{4.7}$$

with

$$g(z) := \frac{\big(1 + (1-\alpha)\theta z\big)^2 \big(1 + \alpha(1-2\theta)z\big)}{\big(1 - \alpha\theta z\big)^2 \big(1 - (1-\alpha)(1-2\theta)z\big)}. \tag{4.8}$$

Taylor expansion results in

$$g(z) = 1 + z + \frac{1}{2}z^2\big[1 + (1-2\alpha)(2\theta^2 - 4\theta + 1)\big] + \mathcal{O}(z^3) \quad (z \to 0).$$

For $\theta = 1 \pm \frac{1}{2}\sqrt{2}$ we have $2\theta^2 - 4\theta + 1 = 0$ and thus

$$g(z) = e^z + \mathcal{O}(z^3) \tag{4.9}$$

holds. The quadrature rule $\int_0^1 v(t)\,dt \approx \xi v(0) + (1-\xi)v(1-\xi)$ is exact for all linear functions v iff $\xi = 1 \pm \frac{1}{2}\sqrt{2}$. Thus for $\theta = 1 \pm \frac{1}{2}\sqrt{2}$ we have

$$\begin{aligned} \int_{t_n}^{t_{n+1}} e^{\lambda(t_{n+1}-\tau)} f(\tau)\, d\tau &= \Delta t\big(\theta e^z f(t_n) + (1-\theta)e^{\theta z} f(t_{n+1-\theta})\big) + \mathcal{O}(\Delta t^3) \\ &= \Delta t\big(\theta(1+z)f(t_n) + (1-\theta)(1+\theta z)f(t_{n+1-\theta})\big) + \mathcal{O}(\Delta t^3). \end{aligned}$$

Using this in combination with (4.7), (4.9) we get

$$\begin{aligned} u^{n+1} &= e^z u(t_n) + \Delta t\big(\theta(1+z)f(t_n) + (1-\theta)(1+\theta z)f(t_{n+1-\theta})\big) + \mathcal{O}(\Delta t^3) \\ &= e^z u(t_n) + \int_{t_n}^{t_{n+1}} e^{\lambda(t_{n+1}-\tau)} f(\tau)\, d\tau + \mathcal{O}(\Delta t^3) \\ &= u(t_{n+1}) + \mathcal{O}(\Delta t^3), \end{aligned}$$

and thus the result is proved. □

A similar bound as in (4.6) can be derived for the case that F is a nonlinear function which satisfies the Lipschitz condition in (4.2). Thus for $\theta = 1 \pm \frac{1}{2}\sqrt{2}$ the fractional-step θ-scheme has consistency order 2.

Remark 4.1.2 For the fractional-step θ-scheme to have consistency order 2 it is *necessary* to take the value $\theta = 1 \pm \frac{1}{2}\sqrt{2}$ in the following sense. Consider the special case $\lambda = 0$, $u_0 = 0$, $f(t) = t$, $n = 0$. Then we have $u(t_1) = u(\Delta t) = \frac{1}{2}(\Delta t)^2$ and a simple computation yields $u^1 = (1-\theta)^2(\Delta t)^2$. Thus for (4.6) to hold *with a θ-value independent of Δt* we need $(1-\theta)^2 = \frac{1}{2}$, i.e., $\theta = 1 \pm \frac{1}{2}\sqrt{2}$.

Remark 4.1.3 Consider the following variant of the fractional-step θ-scheme, with $G(u,t) := F(u) - f(t)$:

$$\frac{u^{n+\theta} - u^n}{\theta \Delta t} + \alpha G(u^{n+\theta}, t_{n+\theta}) + (1-\alpha) G(u^n, t_n) = 0$$

$$\frac{u^{n+1-\theta} - u^{n+\theta}}{(1-2\theta)\Delta t} + (1-\alpha) G(u^{n+1-\theta}, t_{n+1-\theta}) + \alpha G(u^{n+\theta}, t_{n+\theta}) = 0$$

$$\frac{u^{n+1} - u^{n+1-\theta}}{\theta \Delta t} + \alpha G(u^{n+1}, t_{n+1}) + (1-\alpha) G(u^{n+1-\theta}, t_{n+1-\theta}) = 0.$$

This scheme is equal to three steps of the θ-scheme (4.3), where α or $1-\alpha$ takes the role of θ in (4.3), and for the three substeps we use time steps $\theta\Delta t$, $(1-2\theta)\Delta t$ and $\theta\Delta t$, respectively. For an accuracy analysis we consider the same test problem as in Lemma 4.1.1 and take $u^n := u(t_n)$, $\theta = 1 \pm \frac{1}{2}\sqrt{2}$. Along the same lines as in the proof of Lemma 4.1.1 one can derive the following, with $z := \lambda\Delta t$:

$$\begin{aligned} u^{n+1} &= g(z)u^n + (1-\alpha)\Delta t\big[\theta(1+z)f(t_n) + (1-\theta)(1+\theta z)f(t_{n+1-\theta})\big] \\ &\quad + \alpha\Delta t\big[(1-\theta)\big(1+(1-\theta)z\big)f(t_{n+\theta}) + \theta f(t_{n+1})\big] + \mathcal{O}(\Delta t^3). \end{aligned}$$

For $\int_0^1 v(t)\,dt$ the quadrature rules $\theta v(0) + (1-\theta)v(1-\theta)$ and $(1-\theta)v(\theta) + \theta v(1)$ are exact for all linear functions. Hence, using the Taylor expansion $e^z = 1 + z + \mathcal{O}(z^2)$ we get

$$\begin{aligned} &\int_{t_n}^{t_{n+1}} e^{\lambda(t_{n+1}-\tau)} f(\tau)\,d\tau \\ &= (1-\alpha)\Delta t\big[\theta(1+z)f(t_n) + (1-\theta)(1+\theta z)f(t_{n+1-\theta})\big] \\ &+ \alpha\Delta t\big[(1-\theta)\big(1+(1-\theta)z\big)f(t_{n+\theta}) + \theta f(t_{n+1})\big] + \mathcal{O}(\Delta t^3). \end{aligned}$$

Thus as in the proof of Lemma 4.1.1 we obtain

$$u^{n+1} = e^z u(t_n) + \int_{t_n}^{t_{n+1}} e^{\lambda(t_{n+1}-\tau)} f(\tau)\,d\tau + \mathcal{O}(\Delta t^3) = u(t_{n+1}) + \mathcal{O}(\Delta t^3).$$

Hence, this variant has consistency order 2, too.

Stability

For an error analysis of time discretization methods for stiff problems *stability* properties have to be considered. For a stability analysis, these methods are applied to the test problem

$$\frac{du}{dt} = \lambda u, \qquad \lambda \in \mathbb{C},\ \mathrm{Re}(\lambda) \le 0. \tag{4.10}$$

A solution of this test problem satisfies the growth relation

$$|u(t_{n+1})| = |e^{\lambda \Delta t}||u(t_n)| = e^{\mathrm{Re}(\lambda)\Delta t}|u(t_n)|. \tag{4.11}$$

Due to $\mathrm{Re}(\lambda) \le 0$ the growth factor satisfies $0 \le e^{\mathrm{Re}(\lambda)\Delta t} \le 1$. For one-step methods applied to this test problem one obtains $|u^{n+1}| = g(\lambda\Delta t)|u^n|$, with a so-called *stability function* $g(z)$ which is an approximation of the growth factor $|e^z|$ in (4.11). For the implicit Euler, Crank-Nicolson and fractional-step θ-scheme (cf. (4.8)) the stability function is given by:

$$\begin{aligned} g_{EB}(z) &:= \left|\frac{1}{1-z}\right| \\ g_{CN}(z) &:= \left|\frac{1+\frac{1}{2}z}{1-\frac{1}{2}z}\right| \\ g_{FS}(z) &:= \left|\frac{\big(1+(1-\alpha)\theta z\big)^2\big(1+\alpha(1-2\theta)z\big)}{\big(1-\alpha\theta z\big)^2\big(1-(1-\alpha)(1-2\theta)z\big)}\right|. \end{aligned}$$

The variant of the fractional-step θ-scheme discussed in Remark 4.1.3 also has the stability function g_{FS}. For the BDF2 method one obtains $u^n = c_0\left(\frac{2+\sqrt{1+2z}}{3-2z}\right)^n + c_1\left(\frac{2-\sqrt{1+2z}}{3-2z}\right)^n$, with $z := \lambda\Delta t$ and constants c_0, c_1 that depend on the starting values u^0, u^1. The stability function of the BDF2 method is given by

$$g_{BDF}(z) := \max\left\{\left|\frac{2+\sqrt{1+2z}}{3-2z}\right|, \left|\frac{2-\sqrt{1+2z}}{3-2z}\right|\right\}.$$

For a given method with stability function g the so-called *stability region* is defined by

$$S := \{\, z \in \mathbb{C} :\ g(z) \le 1 \,\}.$$

The method is said to be A-stable if

$$\mathbb{C}^- := \{\, z \in \mathbb{C} :\ \mathrm{Re}(z) \le 0 \,\} \subset S$$

holds. From standard literature on time discretization methods for (stiff) initial value problems, cf. [134], it is known that the backward -Euler, Crank-Nicolson method and BDF2 method are A-stable.

We consider the fractional-step θ-scheme with $\theta = 1 \pm \frac{1}{2}\sqrt{2}$. Due to the structure of the fractional-step θ-scheme it is natural to restrict to $\alpha \in [0,1]$. First the case $\theta = 1 - \frac{1}{2}\sqrt{2}$ is treated.

Lemma 4.1.4 *Take* $\alpha \in [0,1]$, $\theta = 1 - \frac{1}{2}\sqrt{2}$. *The fractional-step* θ*-scheme is* A*-stable iff* $\alpha \in [\frac{1}{2}, 1]$.

Proof. For $\alpha > 0$ we have

$$\lim_{z \to -\infty} g_{FS}(z) = \left| \frac{1-\alpha}{\alpha} \right| = \frac{1-\alpha}{\alpha}.$$

Since $\frac{1-\alpha}{\alpha} > 1$ for $\alpha < \frac{1}{2}$ the method is not A-stable for $\alpha < \frac{1}{2}$. We consider $\alpha \geq \frac{1}{2}$. The denominator in the function g_{FS} has no zero in $\mathbb{C}^-$ and thus g_{FS} is the norm of a function that is analytic on $\mathbb{C}^-$. From the maximum principle for analytic functions it follows that

$$\max_{z \in \mathbb{C}^-} g_{FS}(z) = \max_{y \in \mathbb{R}} g_{FS}(iy).$$

Due to $g_{FS}(iy) = g_{FS}(-iy)$ we can restrict to $y \in [0, \infty)$. Note that $g_{FS}(0) = 1$ and $\lim_{y \to \infty} g_{FS}(iy) = \frac{1-\alpha}{\alpha} \leq 1$. A straightforward computation yields that on $[0, \infty)$ the derivative of the function $y \to g_{FS}(iy)$ is less than or equal to 0. Hence $\max_{y \in \mathbb{R}} g_{FS}(iy) \leq g_{FS}(0) = 1$. □

We now consider $\theta = 1 + \frac{1}{2}\sqrt{2}$, $\alpha \in [0, 1]$. For $\alpha < 1$ the denominator has a zero at $z_0 = (1-\alpha)^{-1}(1-2\theta)^{-1} < 0$. For the value $z = z_0$ the nominator is not equal to zero. Hence $\lim_{z \to z_0} g_{FS}(z) = \infty$ and thus the method is not A-stable. For $\alpha = 1$ it can be shown with the same arguments as in the proof of Lemma 4.1.4 that the method is A-stable. *Below, for the fractional-step θ-scheme we restrict to* $\theta = 1 - \frac{1}{2}\sqrt{2}$, $\alpha \in [\frac{1}{2}, 1]$, or $\theta = 1 + \frac{1}{2}\sqrt{2}$, $\alpha = 1$. For these parameter values the method has consistency order 2 and is A-stable.

Smoothing property

A further criterion which is relevant for comparing these methods is the notion of smoothing, which quantifies the amount of damping of the numerical solutions of (4.10) with λ such that $\mathrm{Re}(\lambda) \to -\infty$ (i.e. of "high" frequencies). For $\mathrm{Re}(\lambda) \to -\infty$ the growth factor $e^{\mathrm{Re}(\lambda)\Delta t}$, cf. (4.11), tends to zero. The smoothing property measures how well this strong damping behavior for $\mathrm{Re}(\lambda) \to -\infty$ is reflected in the numerical scheme. The method has a *smoothing property* if there exists a constant $\delta < 1$ such that for the corresponding stability function g we have

$$\lim_{\mathrm{Re}(z) \to -\infty} g(z) \leq \delta. \tag{4.12}$$

The size of δ is a measure for the strength of the smoothing: a small δ value corresponds to a strong smoothing. A strong smoothing is a desirable property of a numerical scheme. One easily verifies that for the backward Euler and the BDF2 methods we have a maximal smoothing effect, namely with $\delta = 0$. For the Crank-Nicolson method there is no smoothing at all: $\delta = 1$. For the fractional-step θ-method we have a smoothing effect with $\delta = \frac{1}{\alpha} - 1$, and thus the smoothing effect increases for larger α.

Dissipativity

The last property that we consider is the *amount of dissipativity* of a method. This is a measure for the quality of the numerical method when applied to (4.10) with a *periodic* solution of the form $u(t) = e^{ixt}$, $x \in \mathbb{R}$, i.e. with $\lambda = ix$. Hence, in (4.11) we then have a growth factor $e^{\mathrm{Re}(\lambda)\Delta t} = 1$. In this case we have to consider the corresponding stability functions $g(z)$ with $z = ix$, $x \in \mathbb{R}$. The amount of dissipativity is measured by the *deviation* of

$$d(x) := g(ix), \quad x \in \mathbb{R}, \tag{4.13}$$

from the optimal value 1. For the implicit Euler method we have

$$d_{EB}(x) = \frac{1}{\sqrt{1+x^2}},$$

and thus an increasing amount of dissipativity for larger x values. For the Crank-Nicolson we have

$$d_{CN}(x) = 1,$$

and thus *no dissipativity*. For the fractional-step θ-scheme the following holds. For $\theta = 1 + \frac{1}{2}\sqrt{2}$, $\alpha = 1$ we have $d_{FS}(x) = g_{FS}(ix) = \big(1 + (1-2\theta)x^2\big)^{\frac{1}{2}}\big(1 + \theta^2 x^2\big)^{-1}$, which is monotonically decreasing with value 0 for $x \to \infty$, thus in this case there is a large amount of dissipativity for large x values. For the case $\theta = 1 - \frac{1}{2}\sqrt{2}$, $\alpha \in [\frac{1}{2}, 1]$ the dissipativity function depends on α: $d_{FS,\alpha}(x) := g_{FS}(ix)$. We have $\lim_{x\to\infty} d_{FS,\alpha}(x) = \frac{1}{\alpha} - 1$. Inspection of the function $d_{FS,\alpha}$ yields that it is constant for $\alpha = \frac{1}{2}$ and strictly decreasing for $\alpha \in (\frac{1}{2}, 1]$. Furthermore, we have $d_{FS,\alpha'}(x) < d_{FS,\alpha}(x)$ if $\frac{1}{2} \le \alpha < \alpha' \le 1$ and $x > 0$. Thus we have more dissipativity for larger values of α. For a few cases the dissipativity function $d_{FS,\alpha}$ is illustrated in Fig. 4.1. Due to $d_{FS}(x) = d_{FS}(-x)$ it suffices to show results for $x \ge 0$.

Due to the fact that the stability function g_{FS} is the norm of a rational function we have

$$\lim_{\mathrm{Re}(z)\to-\infty} g_{FS}(z) = \lim_{x\to\infty} g_{FS}(ix).$$

This property also holds for the stability functions of the other three methods. Thus there is a conflict between good smoothing ($\lim_{\mathrm{Re}(z)\to-\infty} g_{FS}(z)$ close to zero, cf. (4.12)) and low dissipativity ($g_{FS}(ix) \approx 1$ for a large range of x values). From the analysis above and Fig. 4.1 we see that for $\theta = 1 - \frac{1}{2}\sqrt{2}$ and $\alpha = \frac{1}{2}$ the fractional-step θ-scheme has the same properties as the Crank-Nicolson method, namely no smoothing ($\delta = 1$) and no dissipativity. For $\theta = 1 + \frac{1}{2}\sqrt{2}$, $\alpha = 1$, the fractional-step θ-scheme has properties similar to those of the implicit Euler method: optimal smoothing ($\delta = 0$) and strong dissipativity. A good compromise is found by taking $\theta = 1 - \frac{1}{2}\sqrt{2}$ and $\alpha \in (0, \frac{1}{2})$. A popular parameter choice, cf. [207, 244], is

$$\theta := 1 - \frac{1}{2}\sqrt{2}, \quad \alpha := \frac{1-2\theta}{1-\theta} = 2 - \sqrt{2}. \tag{4.14}$$

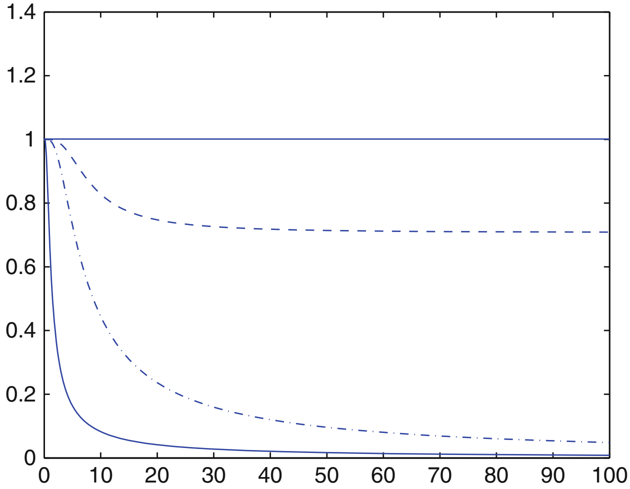

Fig. 4.1. Dissipativity functions $d_{FS,\alpha}$ for $\theta = 1 - \frac{1}{2}\sqrt{2}$, $\alpha \in \{\frac{1}{2}, 2 - \sqrt{2}, 1\}$, and $\theta = 1 + \frac{1}{2}\sqrt{2}$, $\alpha = 1$ (top to bottom).

For these values (cf. Fig. 4.1) the method has "modest" dissipativity and it has a "reasonable" smoothing property with $\delta = \frac{1}{\alpha} - 1 = \frac{1}{2}\sqrt{2}$. Furthermore, due to $\theta\alpha = (1 - 2\theta)(1 - \alpha)$ the systems in (4.5) for the unknowns $u^{n+\theta}$, $u^{n+1-\theta}$, u^{n+1}, respectively, have the same form. *In the remainder we only consider the fractional-step θ-scheme with the parameter values as in* (4.14).

The dissipativity functions $d(x)$ for the implicit Euler, Crank-Nicolson, BDF2 and fractional-step θ-scheme ($\theta = 1 - \frac{1}{2}\sqrt{2}$, $\alpha = 2 - \sqrt{2}$) are illustrated in Fig. 4.2. In all four cases we have $d(-x) = d(x)$ and therefore we show the functions only for $x \geq 0$.

In practice often the Crank-Nicolson method is used. A disadvantage of this method, however, is that it has no smoothing property. The fractional-step θ-schema is a method which has both a good smoothing property and modest dissipativity.

In our applications we will use the implicit Euler method (a simple method with a strong smoothing property), the Crank-Nicolson method and the fractional-step θ-scheme. Note that the implicit Euler and the Crank-Nicolson method are special cases of the θ-scheme.

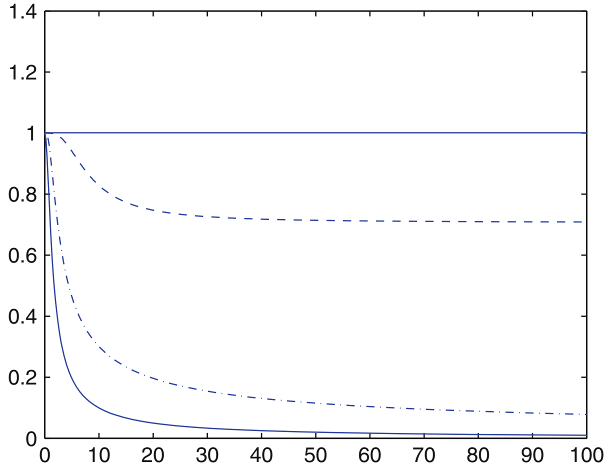

Fig. 4.2. Dissipativity functions d_{CN}, d_{FS}, d_{BDF} and d_{EB} (top to bottom).

4.2 The θ-scheme for the Navier-Stokes problem

The DAE system (3.38) is rewritten in the form

$$\begin{aligned} \frac{d\vec{\mathbf{u}}}{dt}(t) + \mathbf{M}^{-1}\mathbf{B}^T\vec{\mathbf{p}}(t) &= \mathbf{M}^{-1}\, g(\vec{\mathbf{u}}, t), \\ \mathbf{B}\,\vec{\mathbf{u}}(t) &= 0, \end{aligned} \tag{4.15}$$

where

$$g(\vec{\mathbf{u}}, t) := \vec{\mathbf{f}} - \mathbf{A}\vec{\mathbf{u}}(t) - \mathbf{N}(\vec{\mathbf{u}}(t))\vec{\mathbf{u}}(t).$$

The Stokes DAE system (3.22) has a similar form, with $g(\vec{\mathbf{u}}, t) := \vec{\mathbf{f}} - \mathbf{A}\vec{\mathbf{u}}(t)$. We eliminate the incompressibility constraint $\mathbf{B}\vec{\mathbf{u}}(t) = 0$ and the corresponding Lagrange multiplier $\vec{\mathbf{p}}(t)$ to replace the DAE system by an equivalent ODE system. This can be achieved by applying the $\mathbf{M}$-orthogonal projection $\mathbf{P}$ on $\ker \mathbf{B}$:

$$\mathbf{P} = \mathbf{I} - \mathbf{M}^{-1}\mathbf{B}^T(\mathbf{B}\mathbf{M}^{-1}\mathbf{B}^T)^{-1}\mathbf{B}.$$

The projection $\mathbf{P}$ is orthogonal w.r.t. the scalar product $\langle \cdot\,, \cdot \rangle_{\mathbf{M}} := \langle \mathbf{M}\,\cdot\,, \cdot \rangle$, and $\mathbf{P}\,\vec{\mathbf{v}} = \vec{\mathbf{v}}$ for all $\vec{\mathbf{v}} \in \ker \mathbf{B}$, furthermore $\mathbf{P}\mathbf{M}^{-1}\mathbf{B}^T = 0$. Hence, instead of a DAE system we obtain a system of ordinary differential equations:

A solution $\vec{\mathbf{u}}(t)$ of (4.15) satisfies

$$\frac{d\vec{\mathbf{u}}}{dt}(t) = \mathbf{P}\mathbf{M}^{-1}\, g(\vec{\mathbf{u}}, t). \tag{4.16}$$

If for a given initial condition $\vec{\mathbf{u}}(0) = \vec{\mathbf{u}}_0$, with $\mathbf{B}\mathbf{u}_0 = 0$, and $t \in [0, t_e]$ (with t_e sufficiently small) the problem in (4.16) has a unique solution, then this $\vec{\mathbf{u}}$ is also a solution of (4.15). For a given velocity $\vec{\mathbf{u}}(t)$ the corresponding pressure $\vec{\mathbf{p}}$ is defined by the equation

$$\mathbf{B}\mathbf{M}^{-1}\mathbf{B}^T\vec{\mathbf{p}}(t) = \mathbf{B}\mathbf{M}^{-1}\,g(\vec{\mathbf{u}},t). \tag{4.17}$$

The matrix $\mathbf{B}\mathbf{M}^{-1}\mathbf{B}^T$ is nonsingular (on the subspace of FE pressure functions with $(p_h,1)_{L^2}=0$) due to the LBB stability of the pair of finite element spaces used.

Remark 4.2.1 For the Stokes case we have that $\vec{\mathbf{u}} \to g(\vec{\mathbf{u}},t)$ is affine and thus $\vec{\mathbf{u}} \to \mathbf{B}\mathbf{M}^{-1}\,g(\vec{\mathbf{u}},t)$ is affine, too. Hence a Lipschitz condition as in (4.2) is satisfied with a constant L independent of the radius b of the ball G_b. From the Picard-Lindelöf theorem it then follows that for given $\vec{\mathbf{u}}(0)=\vec{\mathbf{u}}_0$ the ODE system (4.16) corresponding to the Stokes problem has a unique solution for $t \in [0,t_e]$ and t_e arbitrary. For the Navier-Stokes case a Lipschitz condition as in (4.2) can be shown to hold only if t_e is sufficiently small. Hence in that case existence and uniqueness of a solution is guaranteed only for a sufficiently short time interval.

The θ-scheme (4.3) can be applied to the ODE system (4.16), which results in

$$\frac{\vec{\mathbf{u}}^{n+1}-\vec{\mathbf{u}}^n}{\Delta t} = \theta\,\mathbf{P}\mathbf{M}^{-1}\,g(\vec{\mathbf{u}}^{n+1},t_{n+1}) + (1-\theta)\mathbf{P}\mathbf{M}^{-1}\,g(\vec{\mathbf{u}}^n,t_n). \tag{4.18}$$

We assume that, for a given $\vec{\mathbf{u}}^0$, this recursion has a unique solution (which holds for Δt sufficiently small). In addition we assume that $\mathbf{B}\vec{\mathbf{u}}^0=0$ holds. From (4.18) and $\mathbf{B}\mathbf{P}=0$ it then follows that $\mathbf{B}\vec{\mathbf{u}}^n=0$ holds for all n. Based on (4.17) we introduce a pressure variable $\vec{\mathbf{p}}^k$ such that the corresponding finite element pressure function p_h satisfies $(p_h,1)_{L^2}=0$ and such that $\mathbf{B}\mathbf{M}^{-1}\mathbf{B}^T\vec{\mathbf{p}}^k = \mathbf{B}\mathbf{M}^{-1}\,g(\vec{\mathbf{u}}^k,t_k)$ holds. Using the definition of the projection $\mathbf{P}$ the recurrence relation in (4.18) can be rewritten as

$$\begin{aligned}\frac{\vec{\mathbf{u}}^{n+1}-\vec{\mathbf{u}}^n}{\Delta t} &+ \mathbf{M}^{-1}\mathbf{B}^T\left(\theta\vec{\mathbf{p}}^{n+1}+(1-\theta)\vec{\mathbf{p}}^n\right)\\ &= \theta\mathbf{M}^{-1}\,g(\vec{\mathbf{u}}^{n+1},t_{n+1}) + (1-\theta)\mathbf{M}^{-1}\,g(\vec{\mathbf{u}}^n,t_n).\end{aligned}$$

Thus for given $\vec{\mathbf{u}}^n$ the pair $\vec{\mathbf{u}}^{n+1}$, $\vec{\tilde{\mathbf{p}}} := \theta\vec{\mathbf{p}}^{n+1}+(1-\theta)\vec{\mathbf{p}}^n$ is a solution of

$$\tfrac{\vec{\mathbf{u}}^{n+1}-\vec{\mathbf{u}}^n}{\Delta t} + \mathbf{M}^{-1}\mathbf{B}^T\vec{\tilde{\mathbf{p}}} = \theta\mathbf{M}^{-1}\,g(\vec{\mathbf{u}}^{n+1},t_{n+1}) + (1-\theta)\mathbf{M}^{-1}\,g(\vec{\mathbf{u}}^n,t_n), \tag{4.19}$$

$$\mathbf{B}\,\vec{\mathbf{u}}^{n+1} = 0. \tag{4.20}$$

For given $\vec{\mathbf{u}}^n$ this saddle point problem has (for Δt sufficiently small) a unique solution pair $(\vec{\mathbf{u}}^{n+1},\vec{\tilde{\mathbf{p}}})$ (on the subspace of pressure functions that satisfy $(p_h,1)_{L^2}=0$). Thus instead of (4.18) for computing $\vec{\mathbf{u}}^{n+1}$ we can use the equivalent formulation in (4.19)-(4.20) for computing $\vec{\mathbf{u}}^{n+1}$, $\vec{\tilde{\mathbf{p}}}$. An important advantage of the latter formulation is that the projection $\mathbf{P}$ has been eliminated. In the derivation it is essential that the mass matrix $\mathbf{M}$ does not depend on t. Summarizing, the θ-method for the Navier-Stokes DAE system takes the form

$$\begin{aligned} \mathbf{M}\frac{\vec{\mathbf{u}}^{n+1} - \vec{\mathbf{u}}^{n}}{\Delta t} &+ \theta[\mathbf{A}\vec{\mathbf{u}}^{n+1} + \mathbf{N}(\vec{\mathbf{u}}^{n+1})\vec{\mathbf{u}}^{n+1}] + \mathbf{B}^T\vec{\mathbf{p}} \\ &= \theta\vec{\mathbf{f}}^{n+1} - (1-\theta)[\mathbf{A}\vec{\mathbf{u}}^{n} + \mathbf{N}(\vec{\mathbf{u}}^{n})\vec{\mathbf{u}}^{n} - \vec{\mathbf{f}}^{n}] \\ \mathbf{B}\vec{\mathbf{u}}^{n+1} &= 0. \end{aligned} \tag{4.21}$$

The θ-schema applied to the Stokes problem results in a system as in (4.21), with the two terms $\mathbf{N}(\cdot)$ replaced by 0. In each time step a system of equations for the unknowns $\vec{\mathbf{u}}^{n+1}, \vec{\mathbf{p}}$ has to be solved. For the (Navier-)Stokes problem this saddle point system is (non)linear. Iterative methods for solving this system are treated in Chap. 5.

Remark 4.2.2 In the derivation above, we applied the *method of lines* approach, in which we first discretize the space variable and then the time variable. In view of the time discretization for two-phase flow problems, treated in Chap. 8, we comment on an alternative approach, often called *Rothe's method*, in which first the time variable and then the space variable is discretized. We explain this for the Stokes case. Starting point is the time dependent Stokes problem in which the pressure has been eliminated, i.e. a formulation as in (2.33). This is a variational formulation of an ODE in the function space $\mathbf{V}_{\mathrm{div}}$. To this problem one can apply the θ-scheme for discretization of the time variable, resulting in the following problem: given $\mathbf{u}^0 \in \mathbf{V}_{\mathrm{div}}$, for $n \geq 0$, determine $\mathbf{u}^{n+1} \in \mathbf{V}_{\mathrm{div}}$ such that

$$\begin{aligned} &\frac{1}{\Delta t}(\mathbf{u}^{n+1} - \mathbf{u}^{n}, \mathbf{v})_{L^2} + \theta a(\mathbf{u}^{n+1}, \mathbf{v}) \\ &= \theta \mathbf{g}^{n+1} - (1-\theta)\big[a(\mathbf{u}^{n}, \mathbf{v}) - \mathbf{g}^{n}\big] \quad \text{for all} \;\; \mathbf{v} \in \mathbf{V}_{\mathrm{div}}. \end{aligned} \tag{4.22}$$

This is a "projected" (due to $\mathbf{V}_{\mathrm{div}}$) *stationary* Stokes problem for the unknown function $\mathbf{u}^{n+1}$. Since finite element subspaces of $\mathbf{V}_{\mathrm{div}}$ are in general difficult to construct, we reformulate this problem as a saddle point problem in $\mathbf{V} \times Q = H_0^1(\Omega)^3 \times L_0^2(\Omega)$. Define the bilinear form

$$\hat{a}(\mathbf{u}, \mathbf{v}) = \frac{1}{\Delta t}(\mathbf{u}, \mathbf{v})_{L^2} + \theta a(\mathbf{u}, \mathbf{v}), \quad \mathbf{u}, \mathbf{v} \in \mathbf{V}, \;\theta \in (0, 1].$$

This bilinear form is elliptic and continuous on $\mathbf{V}$. We can apply the abstract theory in Sect. 15.3, Theorems 15.3.1 and 15.3.4, from which it follows that the problem (4.22) has a unique solution $\mathbf{u}^{n+1}$ which can also be characterized by the following Oseen problem: determine $\mathbf{u}^{n+1} \in \mathbf{V}$ and $p \in Q$ such that

$$\begin{aligned} &\frac{1}{\Delta t}(\mathbf{u}^{n+1} - \mathbf{u}^{n}, \mathbf{v})_{L^2} + \theta a(\mathbf{u}^{n+1}, \mathbf{v}) + b(\mathbf{v}, p) \\ &= \theta \mathbf{g}^{n+1} - (1-\theta)\big[a(\mathbf{u}^{n}, \mathbf{v}) - \mathbf{g}^{n}\big] \quad \text{for all} \;\; \mathbf{v} \in \mathbf{V}, \\ &b(\mathbf{u}, q) = 0 \quad \text{for all} \;\; q \in Q. \end{aligned} \tag{4.23}$$

To this problem we can apply a Galerkin discretization with spaces $\mathbf{V}_h \subset \mathbf{V}$, $Q_h \subset Q$. Using standard nodal bases, we then obtain a fully discrete problem as in (4.21), with $N(\cdot) = 0$. The mass and stiffness matrices $\mathbf{M}$ and $\mathbf{A}$ and the right-hand sides $\vec{\mathbf{f}}^n$ are the same in the two approaches. Hence, the two methods yield the same results.

Although these two approaches turn out to be equivalent in case of an non-stationary Stokes problem with Hood-Taylor finite element spaces for spatial discretization and the θ-scheme for time discretization we comment on a subtle difference between the methods that will become important if we treat two-phase flow problems. For the method of lines the approach is as follows: we start with a saddle point problem for $(\mathbf{u}, p)$, apply spatial Galerkin discretization, eliminate p_h, apply time discretization, introduce p_h again. For Rothe's method: start with a saddle point problem for $(\mathbf{u}, p)$, eliminate p, apply time discretization, introduce p again, apply spatial Galerkin discretization. We see that in the former method we eliminate and re-introduce the spatially discrete pressure variable p_h, whereas in the latter this is done for the spatially continuous variable p. If in a time step $t_n \to t_{n+1}$ one wants to use *different* pressure finite element spaces, then this pressure elimination and re-introduction can be problematic for the method of lines approach, whereas this is not the case for Rothe's method.

4.3 Fractional-step θ-scheme for the Navier-Stokes problem

Applying the fractional-step θ-scheme to the Navier-Stokes problem in ODE form (4.16) and transforming it back to its original DAE form along the same lines as in Sect. 4.2 results in

$$\begin{cases} \mathbf{M}\frac{\vec{\mathbf{u}}^{n+\theta}-\vec{\mathbf{u}}^{n}}{\theta\Delta t} + \alpha[\mathbf{A}\vec{\mathbf{u}}^{n+\theta} + \mathbf{N}(\vec{\mathbf{u}}^{n+\theta})\vec{\mathbf{u}}^{n+\theta}] + \mathbf{B}^T\vec{\tilde{\mathbf{p}}}^1 \\ \qquad = \vec{\mathbf{f}}^n - (1-\alpha)[\mathbf{A}\vec{\mathbf{u}}^n + \mathbf{N}(\vec{\mathbf{u}}^n)\vec{\mathbf{u}}^n] \\ \mathbf{B}\vec{\mathbf{u}}^{n+\theta} = 0 \end{cases} \tag{4.24}$$

$$\begin{cases} \mathbf{M}\frac{\vec{\mathbf{u}}^{n+1-\theta}-\vec{\mathbf{u}}^{n+\theta}}{(1-2\theta)\Delta t} + (1-\alpha)[\mathbf{A}\vec{\mathbf{u}}^{n+1-\theta} + \mathbf{N}(\vec{\mathbf{u}}^{n+1-\theta})\vec{\mathbf{u}}^{n+1-\theta}] + \mathbf{B}^T\vec{\tilde{\mathbf{p}}}^2 \\ \qquad = \vec{\mathbf{f}}^{n+1-\theta} - \alpha[\mathbf{A}\vec{\mathbf{u}}^{n+\theta} + \mathbf{N}(\vec{\mathbf{u}}^{n+\theta})\vec{\mathbf{u}}^{n+\theta}] \\ \mathbf{B}\vec{\mathbf{u}}^{n+1-\theta} = 0 \end{cases} \tag{4.25}$$

$$\begin{cases} \mathbf{M}\frac{\vec{\mathbf{u}}^{n+1}-\vec{\mathbf{u}}^{n+1-\theta}}{\theta\Delta t} + \alpha[\mathbf{A}\vec{\mathbf{u}}^{n+1} + \mathbf{N}(\vec{\mathbf{u}}^{n+1})\vec{\mathbf{u}}^{n+1}] + \mathbf{B}^T\vec{\tilde{\mathbf{p}}}^3 \\ \qquad = \vec{\mathbf{f}}^{n+1-\theta} - (1-\alpha)[\mathbf{A}\vec{\mathbf{u}}^{n+1-\theta} + \mathbf{N}(\vec{\mathbf{u}}^{n+1-\theta})\vec{\mathbf{u}}^{n+1-\theta}] \\ \mathbf{B}\vec{\mathbf{u}}^{n+1} = 0. \end{cases} \tag{4.26}$$

If we take parameter values as in (4.14) then the *non*linear problems for the pairs $(\vec{\mathbf{u}}^{n+\theta}, \vec{\mathbf{p}}^1)$, $(\vec{\mathbf{u}}^{n+1-\theta}, \vec{\mathbf{p}}^2)$, $(\vec{\mathbf{u}}^{n+1}, \vec{\mathbf{p}}^3)$ in these three substeps have a similar form. We obtain the fractional-step θ-scheme for the Stokes by replacing all terms $\mathbf{N}(\cdot)$ by 0.

Remark 4.3.1 If one uses the variant of the fractional-step θ-scheme as described in Remark 4.1.3 then in each time interval $[n\Delta t, (n+1)\Delta t]$ three successive substeps of the θ-scheme (4.21) (with different values for θ) are applied.

4.4 Numerical experiments

To analyze the time discretization error for different time integration schemes, we reconsider the test case of a rectangular tube described in Sect. 3.3.1. Instead of a stationary Stokes problem we now consider the non-stationary Stokes problem (2.34) on $\Omega \times [0,T]$ with $T = 2$ for different time step sizes Δt. To obtain a time-dependent velocity and pressure field, we prescribe an oscillating boundary condition $\mathbf{u}(0, x_2, x_3) = s(x_2, x_3)\big(1 + 0.25\sin(2\pi t)\big)$ at the inflow boundary $x_1 = 0$, with s defined in (3.39), an outflow boundary condition $\boldsymbol{\sigma}\mathbf{n} = 0$ for $x_1 = L$ and $\mathbf{u} = 0$ on the remaining boundaries.

For spatial discretization we use the Hood-Taylor finite element pair for $k = 2$ on a triangulation $\mathcal{T}$ which is constructed by subdividing Ω into $16\times4\times4$ sub-cubes each consisting of 6 tetrahedra. For this fixed spatial discretization different time integration schemes are analyzed for different time step sizes $\Delta t = T/n_t$ where $n_t = 25, 50, 100, 200, 400, 800$ denotes the number of time steps applied to obtain the approximations $\vec{\mathbf{u}}^{n_t}$, $\vec{p}^{\,n_t}$ to $\vec{\mathbf{u}}(T)$, $\vec{p}(T)$, respectively. As the exact solutions $\vec{\mathbf{u}}(T), \vec{p}(T)$ of the DAE system (3.22) are not available we instead use reference solutions $\vec{\mathbf{u}}^{\mathrm{ref}}$, $\vec{p}^{\,\mathrm{ref}}$ obtained by applying 2000 steps of the fractional-step θ-scheme with step size $\Delta t = 10^{-3}$.

n_t	$\Vert\vec{\mathbf{u}}^{\mathrm{ref}} - \vec{\mathbf{u}}^{n_t}\Vert_{L^2}$	order	$\Vert\vec{\mathbf{u}}^{\mathrm{ref}} - \vec{\mathbf{u}}^{n_t}\Vert_1$	order	$\Vert\vec{p}^{\,\mathrm{ref}} - \vec{p}^{\,n_t}\Vert_{L^2}$	order
25	3.69 E-5	—	3.54 E-4	—	7.71 E-3	—
50	1.38 E-5	1.42	1.32 E-4	1.43	2.17 E-3	1.83
100	5.69 E-6	1.29	5.40 E-5	1.29	6.51 E-4	1.73
200	2.53 E-6	1.17	2.40 E-5	1.17	2.17 E-4	1.59
400	1.19 E-6	1.09	1.12 E-5	1.09	8.13 E-5	1.42
800	5.75 E-7	1.05	5.43 E-6	1.05	3.38 E-5	1.26

Table 4.1. Convergence behavior of the implicit Euler scheme w.r.t. time step size.

For a fixed spatial coordinate $x = (2, 0.5, 0.5)$ in the center of the domain Ω, the first velocity component $u_1(x,t)$ and pressure $p(x,t)$ are shown as a function of time $t \in [0,2]$ in Fig. 4.3. Also given are the results for the implicit Euler scheme ($\theta = 1$), the Crank-Nicolson scheme ($\theta = 0.5$) and the

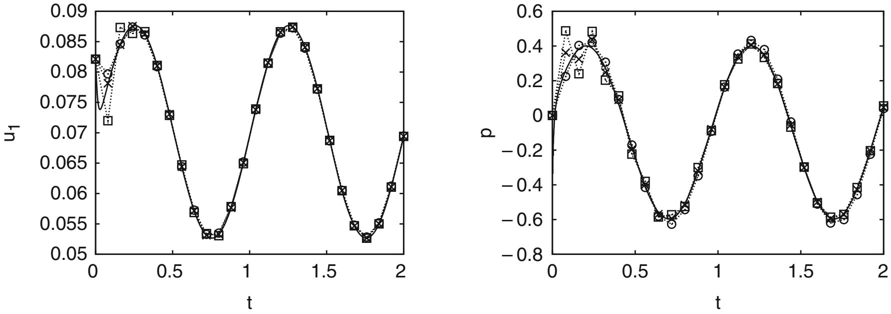

Fig. 4.3. Velocity u_1 (left) and pressure p (right) at point $x = (2, 0.5, 0.5) \in \Omega$ as a function of time. Shown are the reference solution (solid line) and implicit Euler (circles), Crank-Nicolson (squares) and fractional-step (crosses) solutions for 25 time steps, respectively.

n_t	$\|\vec{\mathbf{u}}^{\text{ref}} - \vec{\mathbf{u}}^{n_t}\|_{L^2}$	order	$\|\vec{\mathbf{u}}^{\text{ref}} - \vec{\mathbf{u}}^{n_t}\|_1$	order	$\|\vec{p}^{\text{ ref}} - \vec{p}^{\ n_t}\|_{L^2}$	order
25	2.75 E-5	—	6.87 E-4	—	9.16 E-3	—
50	3.84 E-6	2.84	1.38 E-4	2.31	9.65 E-5	6.57
100	6.56 E-7	2.55	6.66 E-6	4.38	4.21 E-5	1.20
200	1.63 E-7	2.00	1.60 E-6	2.06	1.04 E-5	2.02
400	4.07 E-8	2.01	3.98 E-7	2.01	2.57 E-6	2.01
800	9.99 E-9	2.03	9.78 E-8	2.03	6.56 E-7	1.97

Table 4.2. Convergence behavior of the Crank-Nicolson scheme w.r.t. time step size.

n_t	$\|\vec{\mathbf{u}}^{\text{ref}} - \vec{\mathbf{u}}^{n_t}\|_{L^2}$	order	$\|\vec{\mathbf{u}}^{\text{ref}} - \vec{\mathbf{u}}^{n_t}\|_1$	order	$\|\vec{p}^{\text{ ref}} - \vec{p}^{\ n_t}\|_{L^2}$	order
25	5.85 E-7	—	7.93 E-6	—	2.67 E-4	—
50	3.40 E-7	0.78	3.31 E-6	1.26	5.50 E-5	2.28
100	9.49 E-8	1.84	9.08 E-7	1.87	1.14 E-5	2.27
200	2.37 E-8	2.00	2.30 E-7	1.98	2.56 E-6	2.15
400	5.70 E-9	2.06	5.58 E-8	2.05	6.02 E-7	2.09
800	1.24 E-9	2.20	1.21 E-9	2.20	1.31 E-7	2.20

Table 4.3. Convergence behavior of the fractional-step θ-scheme w.r.t. time step size.

fractional-step θ-scheme applying 25 steps with time step size $\Delta t = 0.08$. We notice an oscillatory behavior of the Crank-Nicolson scheme in the first time steps which is probably due to the fact that this method does not have a good smoothing property, as explained in Sect. 4.1.

Tables 4.1–4.3 show the convergence w.r.t. time step size for the different time discretization schemes. The numerical experiments confirm the first order convergence of the implicit Euler scheme and second order convergence of the Crank-Nicolson and the fractional-step θ-scheme.

Comparing the second order schemes we observe that the errors for the fractional-step scheme are smaller than those of the Crank-Nicolson scheme by about a factor of 10. Note, however, that in the fractional-step scheme three macro-steps are performed per time step, and thus for a fair comparison it should be compared to a Crank-Nicolson scheme with a time step size divided by 3. This would lead to Crank-Nicolson errors which are roughly $3^2 = 9$ times smaller than those in Table 4.2 and thus are of the same order of magnitude as the errors in the fractional-step scheme given in Table 4.3.

4.5 Discussion and additional references

In this chapter we restricted ourselves to basic, but still very popular, time discretization methods for non-stationary (Navier-)Stokes equations. We briefly discuss a few related aspects.

Error analyses of a fully (space and time) discrete problem as in (4.21) are presented in [141, 142, 143]. In the literature there are only very few studies on *adaptive* time stepping for solving non-stationary Navier-Stokes equations; a recent paper is [154]. There are several variants of the fractional-step θ-scheme for the Navier-Stokes equations based on different operator splittings. Some of these are discussed in [122]. A popular variant is based on a semi-implict treatment of the nonlinear term in the Navier-Stokes equations. In such a method one replaces the term $\mathbf{N}(\vec{\mathbf{u}}^{n+\theta})\vec{\mathbf{u}}^{n+\theta}$ in (4.24) by $\mathbf{N}(\vec{\mathbf{u}}^{n})\vec{\mathbf{u}}^{n+\theta}$, and similarly for the nonlinear terms in (4.25), (4.26), cf. [206]. Alternatively, instead of replacing $\vec{\mathbf{u}}^{n+\theta}$ by $\vec{\mathbf{u}}^{n}$ one can also replace it by a more accurate extrapolation of $\vec{\mathbf{u}}^{n}$ and $\vec{\mathbf{u}}^{n-1}$. Other semi-implicit methods are explained in [206].

A class of methods that is particularly popular in the engineering literature are the so-called projection methods. These methods have a predictor-corrector structure, in which in the predictor step, which does not involve pressure, a new velocity field is determined and in the corrector step, which involves a pressure variable, this new velocity field is “projected” onto the subspace of divergence free functions. To explain the main idea we consider a basic variant of this method in semi-discrete form only, i.e. we discretize in time but not in space, and formulate it in strong formulation. Let $\mathbf{u}^n \in \mathbf{V} := H_0^1(\Omega)^3$ be a given approximation of $\mathbf{u}(\cdot, t_n)$. We define $\tilde{\mathbf{u}}^{n+1} \in \mathbf{V}$ as the solution of

$$\frac{1}{\Delta t}\big(\tilde{\mathbf{u}}^{n+1} - \mathbf{u}^n\big) - \frac{1}{Re}\Delta\tilde{\mathbf{u}}^{n+1} + (\mathbf{u}^n \cdot \nabla)\tilde{\mathbf{u}}^{n+1} = \mathbf{g}(t_{n+1}).$$

The approximation $\tilde{\mathbf{u}}^{n+1} \approx \mathbf{u}(\cdot, t_{n+1})$ is projected onto the space of divergence free functions by solving a saddle point problem: determine $\mathbf{u}^{n+1} \in \mathbf{V}$ and $q \in L_0^2(\Omega)$ such that

$$\begin{aligned} \frac{1}{\Delta t}\big(\mathbf{u}^{n+1} - \tilde{\mathbf{u}}^{n+1}\big) + \nabla q = 0 \quad & \text{in} \quad \Omega \\ \operatorname{div} \mathbf{u}^{n+1} = 0 \quad & \text{in} \quad \Omega \\ \mathbf{u}^{n+1} \cdot \mathbf{n} = 0 \quad & \text{on} \quad \partial\Omega. \end{aligned}$$

An detailed study of this projection method and variants of it can be found in [176].