10

Mathematical model

10.1 Introduction

We consider the model for transport of a dissolved species as given in (1.24). In (1.24) the unknown quantity (concentration) is denoted by $c = c(x,t)$, the velocity field by $\mathbf{u}$ and the diffusion coefficient by D. In this and the next chapter we use a different notation for these quantities: the unknown function (concentration) is denoted by $u(x,t)$ (instead of c), the velocity field by $\mathbf{w}$ (instead of $\mathbf{u}$) and the diffusion coefficient by α. In this notation the mass transport equation, in strong formulation, is as follows:

$$\frac{\partial u}{\partial t} + \mathbf{w} \cdot \nabla u - \operatorname{div}(\alpha \nabla u) = f \quad \text{in} \quad \Omega_i(t), \; i = 1,2, \; t \in [0,T], \tag{10.1a}$$

$$[\alpha \nabla u \cdot \mathbf{n}]_\Gamma = 0, \tag{10.1b}$$

$$[\beta u]_\Gamma = 0, \tag{10.1c}$$

$$u(\cdot,0) = u_0 \quad \text{in} \quad \Omega_i(t), \; i = 1,2, \tag{10.1d}$$

$$u(\cdot,t) = 0 \quad \text{on} \quad \partial\Omega, \; t \in [0,T]. \tag{10.1e}$$

We assume that $\Omega_1 = \Omega_1(t)$, $t \in [0,T]$, is *given*, with $\partial\Omega_1$ sufficiently smooth and $\partial\Omega_1 \cap \partial\Omega = \emptyset$, i.e. one phase is completely surrounded by the other one. A typical example is a droplet surrounded by another fluid. $\mathbf{n}$ denotes the unit normal at Γ pointing from Ω_1 into Ω_2. In (10.1a) we have standard parabolic convection-diffusion equations, which are coupled by the interface conditions in (10.1b) and (10.1c). The diffusion coefficient $\alpha = \alpha(x,t)$ is assumed to be piecewise constant:

$$\alpha = \alpha_i > 0 \quad \text{in} \quad \Omega_i(t).$$

In general we have $\alpha_1 \neq \alpha_2$. The interface condition in (10.1b) results from the conservation of mass principle. The condition in (10.1c) is the so-called *Henry condition*. In this condition the coefficient $\beta = \beta(x,t)$ is strictly positive and piecewise constant:

$$\beta = \beta_i > 0 \quad \text{in} \quad \Omega_i(t).$$

In general we have $\beta_1 \neq \beta_2$, in which case the solution u is *discontinuous across the interface*. We assume that for the function u_0 in the initial condition (10.1d) the conditions in (10.1b), (10.1c) are satisfied. For simplicity we only consider homogeneous Dirichlet boundary conditions in (10.1e).

As noted above, the interface $\Gamma = \Gamma(t)$ is assumed to be given and to be sufficiently smooth. Also $\mathbf{w} = \mathbf{w}(x, t)$ is assumed to be a given sufficiently smooth velocity field. Clearly, in the setting of a two-phase flow problem the interface and the velocity field result from the Navier-Stokes equations which model the fluid dynamics. If in the two-phase flow system there is no significant influence of the concentration u on fluid dynamics, it is reasonable to assume that in the transport problem (10.1) the interface Γ and velocity field $\mathbf{w}$ are given quantities. In certain other cases, for example if there is a strong dependence of the surface tension coefficient τ on the concentration u, this assumption can be unrealistic.

In this chapter we present suitable weak formulations of the mass transport model (10.1). These formulations are used in the derivation of Galerkin finite element discretizations in Chap. 11. Although the mass transport model (10.1) consists of (relatively simple) convection-diffusion equations in the subdomains, its numerical treatment requires special finite element techniques, since the diffusion coefficient is discontinuous across the interface (which is not aligned with the triangulation) and the solution u has to satisfy a jump condition across the interface.

In this chapter we distinguish the following two cases:

- Firstly, in Sect. 10.2 we treat the special case in which both the interface and the velocity field are assumed to be *stationary*, i.e., independent of t. A model example is a droplet at a stationary position is a stationary flow field. For this case a weak formulation easily follows from results known in the literature.
- The second, more general, and from a practical point point of view more interesting, case of a *non-stationary* interface and a time-dependent velocity field is treated in Sect. 10.3. This general case requires a more elaborate analysis.

The distinction of these two cases is also useful for the analysis of the finite element methods treated in Chap. 11.

Remark 10.1.1 The discontinuity of u across the interface can be avoided by introducing transformed quantities $\tilde{u} := \beta u$, $\tilde{\alpha} := \alpha/\beta$, $\tilde{\mathbf{w}} := \mathbf{w}/\beta$. Then (10.1a)-(10.1c) can be reformulated as

$$\beta^{-1}\frac{\partial \tilde{u}}{\partial t} + \tilde{\mathbf{w}} \cdot \nabla \tilde{u} - \operatorname{div}(\tilde{\alpha}\nabla\tilde{u}) = f \quad \text{in} \quad \Omega_i, \ i = 1, 2, \ t \in [0, T], \tag{10.2a}$$

$$[\tilde{\alpha}\nabla\tilde{u} \cdot \mathbf{n}]_\Gamma = 0, \tag{10.2b}$$

$$[\tilde{u}]_\Gamma = 0. \tag{10.2c}$$

In this formulation we have continuity of $\tilde{u}$ across Γ but, compared to (10.1a), a discontinuous subdomain dependent scaling factor β^{-1} in front of the time derivative, which causes difficulties.

We will consider the model in the formulation (10.1a)-(10.1e), which compared to (10.2) is closer to physics.

10.2 Weak formulation: stationary interface

In this section, based on results known in the literature on parabolic equations, we derive a weak formulation for the transport problem in (10.1). For this known theory to be applicable we have to assume that the interface and the velocity field are stationary, i.e., Γ and $\mathbf{w}$ do not depend on t, cf. also Remark 10.2.5. Due to the fact that the underlying problem is a two-phase flow with two incompressible immiscible phases it is reasonable to make the following assumptions about the velocity field $\mathbf{w} = \mathbf{w}(x)$:

$$\operatorname{div}\mathbf{w} = 0 \quad \text{in } \ \Omega_i, \ i = 1,2, \quad \mathbf{w}\cdot\mathbf{n} = 0 \quad \text{on } \ \Gamma, \quad \|\mathbf{w}\|_{L^\infty(\Omega)} \le c < \infty. \tag{10.3}$$

In the remainder of this section we assume that (10.3) holds.

For a weak formulation we introduce suitable Hilbert spaces. First we define the space of functions for which all weak first derivatives exist on both Ω_1 and Ω_2 and which in addition are zero (in trace sense) on $\partial\Omega$. In the literature this space is usually denoted by $H_0^1(\Omega_1 \cup \Omega_2)$:

$$H_0^1(\Omega_1 \cup \Omega_2) := \left\{ v \in L^2(\Omega) : \ v_{|\Omega_i} \in H^1(\Omega_i), \ i = 1,2, \ v_{|\partial\Omega} = 0 \right\}.$$

For $v \in H_0^1(\Omega_1 \cup \Omega_2)$ we write $v_i := v_{|\Omega_i}, \ i = 1,2$. Furthermore

$$H := L^2(\Omega), \quad V := \left\{ v \in H_0^1(\Omega_1 \cup \Omega_2) : \ [\beta v]_\Gamma = 0 \right\}. \tag{10.4}$$

Note:

$$v \in V \ \Leftrightarrow \beta v \in H_0^1(\Omega). \tag{10.5}$$

On H we use the scalar product

$$(u,v)_0 := (\beta u, v)_{L^2} = \int_\Omega \beta u v \, dx,$$

which clearly is equivalent to the standard scalar product on $L^2(\Omega)$. The corresponding norm is denoted by $\|\cdot\|_0$. For $u, v \in H^1(\Omega_i)$ we define $(u,v)_{1,\Omega_i} := \beta_i \int_{\Omega_i} \nabla u_i \cdot \nabla v_i \, dx$ and furthermore

$$(u,v)_{1,\Omega_1\cup\Omega_2} := (u,v)_{1,\Omega_1} + (u,v)_{1,\Omega_2}, \quad u, v \in V.$$

The corresponding norm is denoted by $|\cdot|_{1,\Omega_1\cup\Omega_2}$. This norm is equivalent to

$$\left(\|\cdot\|_0^2 + |\cdot|_{1,\Omega_1\cup\Omega_2}^2\right)^{\frac{1}{2}} =: \|\cdot\|_{1,\Omega_1\cup\Omega_2}.$$

We emphasize that the norms $\|\cdot\|_0$ and $\|\cdot\|_{1,\Omega_1\cup\Omega_2}$ depend on β. The space $\big(V,(\cdot,\cdot)_{1,\Omega_1\cup\Omega_2}\big)$ is a Hilbert space. We obtain a Gelfand triple $V \hookrightarrow H \equiv H' \hookrightarrow V'$, with dense and continuous embeddings $\hookrightarrow$. In the following the same spaces $L^2(0,T;V)$, $C([0,T];H)$ as in Sect. 2.2.3 are used, cf. Sect. 2.2.1 for a definition.

The bilinear form

$$a(u,v) := (\alpha u, v)_{1,\Omega_1\cup\Omega_2} + (\mathbf{w}\cdot\nabla u, v)_0, \quad u,v \in V, \tag{10.6}$$

is continuous on V and using (10.3) we get, for $u \in V$,

$$\begin{aligned}
&(\mathbf{w}\cdot\nabla u, u)_0 \\
&= \sum_{i=1,2} \beta_i \int_{\Omega_i} \mathbf{w}\cdot\nabla u_i\, u_i \, dx \\
&= \int_\Gamma \mathbf{w}\cdot\mathbf{n}[\beta u^2]_\Gamma \, ds - \sum_{i=1,2} \beta_i \int_{\Omega_i} \operatorname{div}\mathbf{w}\, u_i^2 \, dx - (\mathbf{w}\cdot\nabla u, u)_0 \\
&= -(\mathbf{w}\cdot\nabla u, u)_0.
\end{aligned} \tag{10.7}$$

Hence, $(\mathbf{w}\cdot\nabla u, u)_0 = 0$ holds. This yields ellipticity of $a(\cdot,\cdot)$:

$$a(u,u) \geq \min_{i=1,2} \alpha_i \, |u|^2_{1,\Omega_1\cup\Omega_2} \quad \text{for all} \;\; u \in V. \tag{10.8}$$

Now consider the following weak formulation of (10.1a)-(10.1e). Given $f \in V'$, $u_0 \in H$, determine $u \in W^1(0,T;V) := \big\{\, v \in L^2(0,T;V) : \, v' \in L^2(0,T;V' \,\big\}$ such that

$$u(0) = u_0, \quad \frac{d}{dt}(u(t),v)_0 + a(u,v) = f(v) \quad \text{for all} \;\; v \in V. \tag{10.9}$$

Here $\frac{d}{dt}(u(t),v)_0 = u'(v)$ corresponds to a weak derivative $u' \in L^2(0,T;V')$ as explained in Lemma 2.2.6. Hence, due to (2.28) $u \in C([0,T];H)$ holds and thus the initial condition $u = u_0$ is well-defined. From Theorem 2.2.7 it follows that the weak formulation (10.9) *has a unique solution.*

Remark 10.2.1 This existence and uniqueness result still holds (cf. [238, 106] and Remark 2.2.8) if instead of ellipticity of the bilinear form $a(\cdot,\cdot)$, cf. (10.8), one has the weaker property

$$a(u,u) \geq c_0 |u|^2_{1,\Omega_1\cup\Omega_2} - c_1\|u\|_0^2 \quad \text{for all} \;\; u \in V,$$

with constants $c_0 > 0$ and c_1 independent of u. Using $|(\mathbf{w}\cdot\nabla u, u)_0| \leq c|u|_{1,\Omega_1\cup\Omega_2}\|u\|_0$ it easily follows that this property holds *without* using the first two assumptions in (10.3). We introduce these assumptions because they simplify the presentation of the analysis for the continuous problem and we need them in our analysis of the Nitsche-XFEM method in Sect. 11.2.

The weak derivative $u' \in L^2(0,T;V')$ in (10.9), which satisfies $u'(v) = \frac{d}{dt}(u(t),v)_0$ for $v \in V$, can be replaced by a more regular one if we assume some regularity of the data f and u_0. Related to this regularity issue we first consider the *stationary* problem: for $f \in H$,

$$\text{find} \;\; u \in V \;\; \text{such that} \;\; a(u,v) = (f,v)_0 \quad \text{for all} \;\; v \in V. \tag{10.10}$$

We assume that the unique solution u of this problem satisfies $u_i \in H^2(\Omega_i)$, $i = 1,2$ and

$$\|u\|_{2,\Omega_1\cup\Omega_2} := \left(\|u\|^2_{1,\Omega_1\cup\Omega_2} + |u|^2_{2,\Omega_1\cup\Omega_2}\right)^{\frac{1}{2}} \leq c\,\|f\|_0 \tag{10.11}$$

holds, with a constant c independent of f. For the stationary problem it is no restriction to assume $\beta_1 = \beta_2$, since the general case can be reduced to that by a transformation as in (10.2). For the symmetric case $\mathbf{w} = 0$, this regularity result is given in [65]. For the general case such regularity results are derived in Chap. 3 of [162] (cf. also [161]). Using this regularity assumption it follows from Theorem II.3.2 in [236] that the following holds:

Theorem 10.2.2 *Assume that* (10.11) *is satisfied. Take*

$$f \in H, \quad u_0 \in V_{\text{reg}} := \left\{\, v \in V \,:\, v_i \in H^2(\Omega_i), \quad i = 1,2 \,\right\}. \tag{10.12}$$

Then the unique solution $u \in W^1(0,T;V)$ *of* (10.9) *satisfies* $u \in C([0,T];V_{\text{reg}})$ *and its weak derivative* $u' =: \frac{du}{dt}$ *has the regularity property*

$$\frac{du}{dt} \in L^2(0,T;V) \cap C([0,T];H). \tag{10.13}$$

Hence u *satisfies, for almost all* $t \in (0,T)$*:*

$$(\frac{du}{dt}, v)_0 + a(u,v) = (f,v)_0 \quad \textit{for all} \;\; v \in V. \tag{10.14}$$

We now show that the variational problem (10.14) is indeed a correct weak formulation of the problem (10.1a)-(10.1e).

Lemma 10.2.3 *Take* $f \in H$, $u_0 \in V_{\text{reg}}$. *Assume that* (10.1a)-(10.1e) *has a solution* $u(x,t)$ *which is sufficiently smooth such that for* $u : t \to u(\cdot,t)$ *we have* $u \in C([0,T];V_{\text{reg}})$ *and* $\frac{du}{dt} \in L^2(0,T;H)$. *This* u *solves the variational problem* (10.14).
Conversely, if $u \in C([0,T];V_{\text{reg}})$ *with* $u(0) = u_0$ *solves the variational problem* (10.14) *then* u *satisfies* (10.1a) *in a weak* $L^2(\Omega_i)$ *sense and* (10.1b), (10.1c), (10.1e) *in trace sense.*

Proof. Take $u \in C([0,T];V_{\text{reg}})$ with $\frac{du}{dt} \in L^2(0,T;H)$, and $v \in V$. Using $[\beta v]_\Gamma = 0$ and the notation $\{w\}_\Gamma := \frac{1}{2}\big((w_1)_{|\Gamma} + (w_2)_{|\Gamma}\big)$ for the average of a function $w \in H^1(\Omega_1 \cup \Omega_2)$ we get

$$[\alpha\nabla u\cdot\mathbf{n}\beta v]_\Gamma=[\alpha\nabla u\cdot\mathbf{n}]_\Gamma\,\{\beta v\}_\Gamma+\{\alpha\nabla u\cdot\mathbf{n}\}_\Gamma\,[\beta v]_\Gamma=[\alpha\nabla u\cdot\mathbf{n}]_\Gamma\,\{\beta v\}_\Gamma.$$

Using this we obtain

$$\begin{aligned}(\frac{du}{dt},v)_0+a(u,v)&=(\frac{du}{dt},v)_0+(\mathbf{w}\cdot\nabla u,v)_0\\ &-\sum_{i=1,2}\int_{\Omega_i}\mathrm{div}(\alpha_i\nabla u_i)\beta_i v_i\,dx+\int_\Gamma[\alpha\nabla u\cdot\mathbf{n}\beta v]_\Gamma\,ds\\ &=\sum_{i=1,2}\int_{\Omega_i}\big(\frac{du_i}{dt}+\mathbf{w}\cdot\nabla u_i-\mathrm{div}(\alpha_i\nabla u_i)\big)\beta_i v_i\,dx\\ &+\int_\Gamma[\alpha\nabla u\cdot\mathbf{n}]_\Gamma\,\{\beta v\}_\Gamma\,ds.\end{aligned}\tag{10.15}$$

If u satisfies (10.1a), (10.1b) we thus obtain

$$(\frac{du}{dt},v)_0+a(u,v)=(f,v)_0\quad\text{for all}\quad v\in V,$$

i.e., (10.14) holds. Conversely, if $u\in C([0,T];V_{\mathrm{reg}})$ with $u(0)=u_0$ solves the variational problem (10.14) we obtain for $u_i(t,x):=u(t)(x)_{|\Omega_i}$

$$\begin{aligned}&\sum_{i=1,2}\int_{\Omega_i}\big(\frac{\partial u_i}{\partial t}+\mathbf{w}\cdot\nabla u_i-\mathrm{div}(\alpha_i\nabla u_i)-f\big)\beta_i v_i\,dx\\ &\quad+\int_\Gamma[\alpha\nabla u\cdot\mathbf{n}]_\Gamma\,\{\beta v\}_\Gamma\,ds=0\end{aligned}$$

for all $v\in V$. This implies that $\frac{\partial u_i}{\partial t}+\mathbf{w}\cdot\nabla u_i-\mathrm{div}(\alpha_i\nabla u_i)=f$ in $L^2(\Omega_i)$ sense and $[\alpha\nabla u\cdot\mathbf{n}]_\Gamma=0$ in trace sense. The properties in (10.1c) and (10.1e) hold due to $u\in V$. □

For the result in (10.15) it is essential that we multiply the equation (10.1a) by βv and not by v. This explains why in the scalar products $(\cdot,\cdot)_0$ and $(\cdot,\cdot)_{1,\Omega_1\cup\Omega_2}$ we use the weighting with the (piecewise constant) function β.

Remark 10.2.4 Using (10.5) the weighting with β in the scalar products in (10.14) can be eliminated, resulting in the following equivalent variational equation:

$$(\frac{du}{dt},v)_{L^2}+(\alpha\nabla u,\nabla v)_{L^2}+(\mathbf{w}\cdot\nabla u,v)_{L^2}=(f,v)_{L^2}\quad\forall\,v\in H_0^1(\Omega).\tag{10.16}$$

In the finite element discretization in Chap. 11 we prefer the formulation in (10.14), since it uses the same space V both as solution space and as test space. In (10.16) we have V and $H_0^1(\Omega)$ as solution and test space, respectively.

Remark 10.2.5 With additional technical manipulations, the analysis presented in this section can be generalized such that it also covers the case of a time dependent (sufficiently smooth) velocity field $\mathbf{w} = \mathbf{w}(x,t)$. Then the bilinear form $a(\cdot,\cdot)$ in (10.6) depends on t, i.e., we have $a(t;u,v)$. This bilinear form is continuous and elliptic uniformly in $t \in [0,T]$ and an analysis as in e.g. [256], Sect. 26, can be applied. This analysis does *not* apply to the case of a non-stationary interface.

10.3 Weak formulation: non-stationary interface

In this section we derive a weak formulation of the transport problem (10.1) for the general case that $\Gamma = \Gamma(t)$ is time-dependent and the velocity field $\mathbf{w} = \mathbf{w}(x,t)$ may depend on t. We assume that $\Gamma(t)$ is sufficiently smooth for all $t \in [0,T]$. In the analysis presented in the previous section it is essential for the formulation of the parabolic mass transport problem in the space $L^2(0,T;V)$ that the space V *does not depend on* t. If, however, the interface is non-stationary, then $H_0^1(\Omega_1 \cup \Omega_2)$, and thus also V, is time dependent. Due to this, the analysis of the previous section is not applicable in case $\Gamma = \Gamma(t)$. Instead, a *space-time* variational formulation should be used. Although the transport problem (10.1) is relatively simple, since it consists of two coupled parabolic problems, we are not aware of any literature in which a rigorous analysis of an appropriate weak formulation of this problem for the case of a non-stationary interface is given. Below we present such an analysis.

For the velocity field we assume that for almost all $t \in [0,T]$:

$$\mathbf{w}(\cdot,t) \in H^1(\Omega)^3, \quad \|\mathbf{w}(\cdot,t)\|_{L^\infty(\Omega)} \le c < \infty, \quad \operatorname{div}\mathbf{w} = 0 \text{ in } \Omega.$$

Furthermore, we assume that the interface $\Gamma(t)$ is *transported by the velocity field* $\mathbf{w}$.

We consider the 3D case, i.e. $\Omega \subset \mathbb{R}^3$. The analysis applies, with only minor modifications to the general case $\Omega \subset \mathbb{R}^d$. For simplicity we assume that $\Omega_1(t)$ is connected and completely contained in Ω. i.e., $\partial\Omega_1(t) \cap \partial\Omega = \emptyset$, $\Gamma(t) = \partial\Omega_1(t)$. In the subsections below we first introduce suitable space-time Sobolev spaces and derive some properties of these spaces (Sect. 10.3.1), then we introduce a space-time weak formulation (Sect. 10.3.2) and finally prove well-posedness of this weak formulation (Sect. 10.3.3).

10.3.1 Preliminaries

The spaces and techniques treated in this section can be found in papers on parabolic problems in so-called noncylindrical domains, which means that the spatial domain in which the problem is formulated depends on t. The first extensive treatment of this topic is given in [167]. Below we use some results from this paper.

The remainder of this section is somewhat technical. For the readers convenience we outline the main results. We first introduce spaces on the space-time domain $Q_T := \Omega \times (0,T)$. The spaces V_β and W_β introduced in (10.18), (10.22) are generalizations of the spaces $L^2(0,T;V)$ and $W^1(0,T;V)$ used for the stationary interface case in the previous section. These spaces V_β, W_β are used in the space-time weak formulation presented in Sect. 10.3.2. Dense subspaces of piecewise smooth functions $\mathcal{V}_\beta \subset V_\beta$, cf. (10.20), and $\mathcal{W}_\beta \subset W_\beta$, cf. (10.21), are introduced. An important difference between these two subspaces is that functions from $\mathcal{V}_\beta$ are zero on the whole boundary of Q_T, whereas functions from $\mathcal{W}_\beta$ are in general zero only on $\partial\Omega \times [0,T]$. An alternative definition of the space W_β, which is useful for the analysis of well-posedness in Sect. 10.3.3, is derived in Proposition 10.3.2. For initial conditions to be well-defined we can use the embedding property $W_\beta \hookrightarrow C([0,T];L^2(\Omega))$ proved in Lemma 10.3.4. In the analysis of well-posedness we need two partial integration identities that hold in the space W_β, namely the ones derived in Corollary 10.3.5 and Lemma 10.3.6.

The (open) space-time cylinder is denoted by $Q_T := \Omega \times (0,T) \subset \mathbb{R}^4$, and the space-time interface is given by $\Gamma_* := \{\,(x,t) \in Q_T : x \in \Gamma(t),\ t \in (0,T)\,\}$. The space-time cylinder is split in subdomains

$$Q_i := \{\,(x,t) \in Q_T : x \in \Omega_i(t),\ t \in (0,T)\,\}, \quad i = 1,2.$$

For $v \in L^2(Q_T)$ we define $v_i := v_{|Q_i}$. We introduce a scalar product in which, on Q_i, we take first derivatives with respect to all *spatial* variables, but *no derivative* with respect to t:

$$\|u\|^2_{H^{1,0}(Q_i)} := \sum_{j=1}^{3} \Big\|\frac{\partial u}{\partial x_j}\Big\|^2_{L^2(Q_i)} + \|u\|^2_{L^2(Q_i)}, \quad u \in C^1(\overline{Q}_i). \tag{10.17}$$

The induced Hilbert space is given by

$$H^{1,0}(Q_i) := \overline{C^1(\overline{Q}_i)}^{\|\cdot\|_{H^{1,0}(Q_i)}} = \left\{ u \in L^2(Q_i) : \frac{\partial u}{\partial x_j} \in L^2(Q_i), \quad j = 1,2,3 \right\}.$$

From [167] it follows that under mild assumptions on Γ_* there exist bounded linear trace operators

$$\begin{aligned} \gamma_i &: H^{1,0}(Q_i) \to L^2(\Gamma_*), \quad i = 1,2, \\ \gamma_\Omega &: H^{1,0}(Q_2) \to L^2(\partial\Omega \times (0,T)). \end{aligned}$$

In the remainder we assume that such bounded linear trace operators exist. Then for $u \in L^2(Q_T)$ with $u_i \in H^{1,0}(Q_i)$, $i = 1,2$, the operators $u_{|\partial\Omega} = \gamma_\Omega u$ and $[u] = [u]_{\Gamma_*} = \gamma_1 u - \gamma_2 u$ are well-defined. The first space that plays an important role in the space-time weak formulation of the mass transport equation is the following analogon of the space V in (10.4):

$$V_\beta := \{\, u \in L^2(Q_T) : u_i \in H^{1,0}(Q_i),\ i = 1,2,\ u_{|\partial\Omega} = 0,\ [\beta u]_{\Gamma_*} = 0 \,\}. \tag{10.18}$$

This space is equipped with the norm

$$\|u\|_V^2 := \sum_{i=1}^{2} \|u_i\|_{H^{1,0}(Q_i)}^2, \quad u \in V_\beta.$$

This space can also be characterized as follows. Let

$$H_0^{1,0}(Q_T) := \overline{C_0^1(Q_T)}^{\|\cdot\|_{H^{1,0}(Q_T)}},$$

with $\|\cdot\|_{H^{1,0}(Q_T)}$ as in (10.17) but with Q_i replaced by Q_T, be the space-time analogon of $H_0^1(\Omega)$ (i.e., no derivatives w.r.t. t). Then

$$v \in V_\beta \;\Leftrightarrow\; \beta v \in H_0^{1,0}(Q_T) \tag{10.19}$$

holds. Using this we obtain as alternative characterization of V_β:

$$V_\beta = \overline{\mathcal{V}_\beta}^{\|\cdot\|_V}, \quad \text{with } \mathcal{V}_\beta := \left\{ \beta^{-1}\phi : \phi \in C_0^1(Q_T) \right\}. \tag{10.20}$$

We introduce a subspace of V_β of functions for which a suitable weak time derivative is well-defined. This space is an analogon for the noncylindrical case of the space $W^1(0,T;V)$ introduced in Sect. 2.2.3. We need the dual space of $H_0^{1,0}(Q_T)$, denoted by

$$H^{-1,0}(Q_T) := H_0^{1,0}(Q_T)'.$$

For $u \in V_\beta$ the distributional time derivative $\frac{\partial u}{\partial t}$ is the linear functional given by

$$\frac{\partial u}{\partial t}(\phi) := -\int_{Q_T} u \frac{\partial \phi}{\partial t}\, dx\, dt, \quad \phi \in C_0^1(Q_T).$$

Define $\mathcal{W}_\beta \subset V_\beta$ by

$$\mathcal{W}_\beta := \left\{ \beta^{-1}\psi : \psi \in C^1(\overline{Q}_T), \;\; \psi_{|\partial\Omega} = 0 \right\}. \tag{10.21}$$

Note that opposite to $\mathcal{V}_\beta$ in (10.20), a function from $\mathcal{W}_\beta$ is not necessarily equal to zero on all of ∂Q_T.

Lemma 10.3.1 *For* $\psi \in \mathcal{W}_\beta$ *we have* $\frac{\partial \psi}{\partial t} \in H^{-1,0}(Q_T)$.

Proof. Take $\psi \in \mathcal{W}_\beta$, $\phi \in C_0^1(Q_T)$. Then $\psi_i \in C^1(\overline{Q}_i)$, $i = 1, 2$, and

$$\begin{aligned} \frac{\partial \psi}{\partial t}(\phi) &= -\int_{Q_T} \psi \frac{\partial \phi}{\partial t}\, dx\, dt \\ &= \sum_{i=1}^{2} \left(-\int_{\Gamma_*} \gamma_i \psi_i\, \phi n_4\, ds\, dt + \int_{Q_i} \frac{\partial \psi_i}{\partial t} \phi\, dx\, dt \right), \end{aligned}$$

where $n_4 = \hat{\mathbf{n}}_4$ is the fourth component (i. e., the temporal direction) of the unit normal $\hat{\mathbf{n}}$ at Γ_*. Using the bounds for the trace operators γ_i and Cauchy-Schwarz inequalities we obtain

$$\Big|\frac{\partial \psi}{\partial t}(\phi)\Big| \leq c\|\phi\|_{H^{1,0}(Q_T)}.$$

Using a density argument it follows that $\frac{\partial \psi}{\partial t} \in H_0^{1,0}(Q_T)' = H^{-1,0}(Q_T)$ holds. □

We introduce the closure of $\mathcal{W}_\beta$ in V_β w.r.t. the topology induced by $\|\cdot\|_V^2 + \|\frac{\partial}{\partial t}\cdot\|^2_{H^{-1,0}(Q_T)}$:

$$W_\beta := \overline{\mathcal{W}_\beta}^{\|\cdot\|_W}, \quad \text{with } \|v\|_W^2 := \|v\|_V^2 + \|\frac{\partial v}{\partial t}\|^2_{H^{-1,0}(Q_T)}. \tag{10.22}$$

The space W_β is contained in $\tilde{W}_\beta := \{\, v \in V_\beta : \frac{\partial v}{\partial t} \in H^{-1,0}(Q_T)\,\}$. We claim that $W_\beta = \tilde{W}_\beta$ holds. This claim is formulated in the following proposition, for which we only give a sketch of a proof, based on [7]. In the remainder we use this proposition only in Lemma 10.3.10 and Theorem 10.3.11.

Proposition 10.3.2 *Assume that for $i = 1, 2$, there are bounded C^1 bijections $\Phi_i : \Omega_i(0) \times (0,T) \to Q_i$ such that $\Omega_i(t) = \{\, \Phi_i(\tilde{x}, t) : \tilde{x} \in \Omega_i(0)\,\}$, $0 < t < T$. For W_β as in* (10.22) *the following holds:*

$$W_\beta = \left\{ v \in V_\beta : \frac{\partial v}{\partial t} \in H^{-1,0}(Q_T) \right\}. \tag{10.23}$$

Proof. First we consider the case of a stationary interface, i.e., $\Gamma(t)$ does not depend on t and thus $\beta(x,t) = \beta(x)$ (the subdomains Q_i are cylindrical). We write $\mathcal{H} = H_0^1(\Omega)$. The spaces $L^2(0,T;\mathcal{H})$ and $L^2(0,T;\mathcal{H}') = L^2(0,T;\mathcal{H})'$ can be identified with $H_0^{1,0}(Q_T)$ and $H^{-1,0}(Q_T)$, respectively. Define

$$X := \left\{ u \in L^2(0,T;\mathcal{H}) : \frac{d(\beta^{-1}u)}{dt} \in L^2(0,T;\mathcal{H}') \right\}, \tag{10.24}$$

with $\frac{dw}{dt} = w'$ the weak time derivative as in Sect. 2.2.3. On X we use the norm

$$\|u\|_X^2 = \|u\|^2_{L^2(0,T;\mathcal{H})} + \|\frac{d(\beta^{-1}u)}{dt}\|^2_{L^2(0,T;\mathcal{H}')}.$$

The space on the right-hand side in (10.23) is denoted by $\tilde{W}_\beta$. The equivalence $v \in \tilde{W}_\beta \Leftrightarrow \beta v \in X$ holds and $\|\cdot\|_W$ and $\|\beta\cdot\|_X$ are equivalent norms on $\tilde{W}_\beta$. The scaling with β^{-1} in (10.24) is not essential for properties of the space X, since it can be incorporated as a weighting factor in the $L^2(\Omega)$ scalar product. From the literature, e.g. [256], it follows that $C^\infty(0,T;\mathcal{H})$ is dense in X. Using the density of $C_0^\infty(\Omega)$ in $\mathcal{H}$ it follows that $\mathcal{W} := \{\, \psi \in C^1(Q_T) : \psi_{|\partial\Omega} = 0\,\}$ is dense in X. Take $v \in \tilde{W}_\beta$. Then $\beta v \in X$ and there is a sequence (ψ_m) in

$\mathcal{W}$ with $\lim_{m\to\infty} \|\psi_m - \beta v\|_X = 0$. This implies $\lim_{m\to\infty} \|\beta^{-1}\psi_m - v\|_W = 0$ for the sequence $(\beta^{-1}\psi_m)$ from $\mathcal{W}_\beta$. Thus $\mathcal{W}_\beta$ is dense in $\tilde{W}_\beta$, i.e. $W_\beta = \overline{\mathcal{W}_\beta}^{\|\cdot\|_W} = \tilde{W}_\beta$ holds, which proves the claim for the case of a stationary interface.

We now treat the general case. Define the cylindrical subdomain $\tilde{Q}_i = \Omega_i(0) \times (0,T)$, $i = 1,2$. A function $v \in V_\beta$ can be represented in transformed variables $(\tilde{x}, t)$ by $\tilde{v}(\tilde{x},t) := v(\Phi_i(\tilde{x},t),t)$, $(\tilde{x},t) \in \tilde{Q}_i$. Due to the smoothness assumption on the bijection Φ_i, the Sobolev norm of the transformed function $\|\tilde{v}\|_V^2 := \sum_{i=1}^2 \|\tilde{v}_i\|_{H^{1,0}(\tilde{Q}_i)}$ is equivalent to $\|v\|_V^2$, cf. [8] Sect. 3.34. Furthermore, $\|\frac{\partial \tilde{v}}{\partial t}\|_{H^{-1,0}(\tilde{Q}_T)}$ is equivalent to $\|\frac{\partial v}{\partial t}\|_{H^{-1,0}(Q_T)}$. Using these norm equivalences and the density result for the special case of a stationary interface (cylindrical subdomains) the density result for the general case can be proved. $\square$

In the variational formulation of the transport problem presented in Sect. 10.3.2 below the spaces V_β and W_β play an important role. For the analysis in Sect. 10.3.3 we need some further properties, which are derived in the remainder of this section.

As mentioned above, we assume that the interface $\Gamma(t)$ is transported by the velocity field $\mathbf{w}(x,t)$. From this it follows that

$$\hat{\mathbf{n}} \cdot \begin{pmatrix} \mathbf{w} \\ 1 \end{pmatrix} = 0 \tag{10.25}$$

holds, with $\hat{\mathbf{n}} \in \mathbb{R}^4$ the unit normal at Γ_*, pointing outward from Q_1. For this normal the identity

$$\hat{\mathbf{n}} = \nu \begin{pmatrix} \mathbf{n}_\Gamma \\ -\mathbf{w}\cdot\mathbf{n}_\Gamma \end{pmatrix}, \quad \nu := \frac{1}{\sqrt{1 + (\mathbf{w}\cdot\mathbf{n}_\Gamma)^2}} \tag{10.26}$$

holds, where $\mathbf{n}_\Gamma \in \mathbb{R}^3$ is the unit normal at Γ.

Lemma 10.3.3 *For all $\psi \in \mathcal{W}_\beta$ the identity*

$$\begin{aligned} &\frac{\partial\psi}{\partial t}(\beta\psi) - \int_{Q_T} \psi\mathbf{w}\cdot\nabla(\beta\psi)\,dx\,dt \\ &= \frac{1}{2}\|\beta\psi(\cdot,T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\beta\psi(\cdot,0)\|^2_{L^2(\Omega)} \end{aligned} \tag{10.27}$$

holds.

Proof. Take $\psi \in \mathcal{W}_\beta$, $\phi \in \mathcal{V}_\beta$. Then $\beta\phi \in C_0^1(Q_T)$ holds. From partial integration and using (10.25) we obtain, with the notation $\hat{\mathbf{n}} =: \begin{pmatrix} \hat{\mathbf{n}}_x \\ n_4 \end{pmatrix}$,

$$
\begin{aligned}
&\frac{\partial\psi}{\partial t}(\beta\phi) - \int_{Q_T} \psi\mathbf{w}\cdot\nabla(\beta\phi)\,dx\,dt = -\int_{Q_T} \psi\frac{\partial(\beta\phi)}{\partial t} + \psi\mathbf{w}\cdot\nabla(\beta\phi)\,dx\,dt \\
&= \sum_{i=1}^{2}\int_{Q_i} \big(\frac{\partial\psi_i}{\partial t} + \mathbf{w}\cdot\nabla\psi_i\big)\beta\phi\,dx\,dt - \int_{\Gamma_*} [\psi]\beta\phi(n_4 + \hat{\mathbf{n}}_x\cdot\mathbf{w})\,ds\,dt \qquad (10.28)\\
&= \sum_{i=1}^{2}\int_{Q_i} \big(\frac{\partial\psi_i}{\partial t} + \mathbf{w}\cdot\nabla\psi_i\big)\beta\phi\,dx\,dt.
\end{aligned}
$$

From a continuity argument it follows that

$$
\frac{\partial\psi}{\partial t}(\beta v) - \int_{Q_T} \psi\mathbf{w}\cdot\nabla(\beta v)\,dx\,dt = \sum_{i=1}^{2}\int_{Q_i} \big(\frac{\partial\psi_i}{\partial t} + \mathbf{w}\cdot\nabla\psi_i\big)\beta v\,dx\,dt \qquad (10.29)
$$

holds for all $v \in V_\beta$. In particular it holds for $v = \psi \in \mathcal{W}_\beta \subset V_\beta$. Taking $v = \psi$ in (10.29) and applying partial integration again we get

$$
\begin{aligned}
&\sum_{i=1}^{2}\int_{Q_i} \big(\frac{\partial\psi_i}{\partial t} + \mathbf{w}\cdot\nabla\psi_i\big)\beta\psi\,dx\,dt \\
&= \int_\Omega \beta\psi(\cdot,T)^2\,dx - \int_\Omega \beta\psi(\cdot,0)^2\,dx - \sum_{i=1}^{2}\int_{Q_i} \big(\frac{\partial\psi_i}{\partial t} + \mathbf{w}\cdot\nabla\psi_i\big)\beta\psi\,dx\,dt,
\end{aligned}
$$

since the boundary integral $\int_{\Gamma_*}$ vanishes, cf. (10.28). Using this in (10.29), with $v = \psi$, the result is proved. □

In the next lemma we derive a generalization of the embedding property $W^1(0,T;V) \hookrightarrow C([0,T];H)$ given in (2.28).

Lemma 10.3.4 *There is a continuous embedding $W_\beta \hookrightarrow C([0,T];L^2(\Omega))$.*

Proof. It suffices to prove

$$
\sup_{t\in[0,T]} \|\psi(\cdot,t)\|_{L^2(\Omega)} \le c\|\psi\|_W \quad \text{for all} \;\; \psi \in \mathcal{W}_\beta.
$$

Take $\psi \in \mathcal{W}_\beta$. From Lemma 10.3.3 we have

$$
\begin{aligned}
&\frac{\partial\psi}{\partial t}(\beta\psi) - \int_{Q_T} \psi\mathbf{w}\cdot\nabla(\beta\psi)\,dx\,dt \\
&= \frac{1}{2}\|\beta\psi(\cdot,T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\beta\psi(\cdot,0)\|^2_{L^2(\Omega)}.
\end{aligned} \qquad (10.30)
$$

We take $t_0 \in [\frac{1}{2}T,T]$. Let $\mathcal{W}_\beta(Q_{t_0})$ be as in (10.21), but with T replaced by t_0. Note that $\psi \in \mathcal{W}_\beta = \mathcal{W}_\beta(Q_T)$ implies $\psi \in \mathcal{W}_\beta(Q_{t_0})$. The identity (10.30) holds with Q_T replaced by Q_{t_0}. For the time derivative we then have $\frac{\partial\psi}{\partial t} \in H^{-1,0}(Q_{t_0})$ and

$$\|\frac{\partial\psi}{\partial t}\|_{H^{-1,0}(Q_{t_0})} \leq \|\frac{\partial\psi}{\partial t}\|_{H^{-1,0}(Q_T)} \tag{10.31}$$

holds. Let $\theta = \theta(t)$ be a smooth function with $\theta(t) \in [0,1]$ for all t, $\theta(0) = 0$, $\theta(t) = 1$ for all $t \in [\frac{1}{2}T, T]$. The result (10.30) holds with Q_T replaced by Q_{t_0} and, since $\theta\psi \in \mathcal{W}_\beta$, with ψ replaced by $\theta\psi$. Using this and $\theta(t_0) = 1$, $\theta(0) = 0$ we get, with $c_0 := 2\max\{\beta_1^{-2}, \beta_2^{-2}\}$,

$$\begin{aligned}\|\psi(\cdot,t_0)\|_{L^2(\Omega)}^2 &\leq c_0\frac{1}{2}\|\beta\theta(t_0)\psi(\cdot,t_0)\|_{L^2(\Omega)}^2 - c_0\frac{1}{2}\|\beta\theta(0)\psi(\cdot,0)\|_{L^2(\Omega)}^2 \\ &= c_0\frac{\partial(\theta\psi)}{\partial t}(\beta\theta\psi) - c_0\int_{Q_{t_0}} \theta^2\psi\mathbf{w}\cdot\nabla(\beta\psi)\,dx\,dt.\end{aligned}$$

For the second term on the right-hand side we have

$$\Big|\int_{Q_{t_0}} \theta^2\psi\mathbf{w}\cdot\nabla(\beta\psi)\,dx\,dt\Big| \leq c\|\psi\|_{L^2(Q_{t_0})}\|\nabla(\beta\psi)\|_{L^2(Q_{t_0})} \leq c\|\psi\|_V^2.$$

For the first term we get

$$\begin{aligned}\Big|\frac{\partial(\theta\psi)}{\partial t}(\beta\theta\psi)\Big| &\leq \|\frac{\partial(\theta\psi)}{\partial t}\|_{H^{-1,0}(Q_{t_0})}\|\theta\beta\psi\|_{H^{1,0}(Q_{t_0})} \\ &\leq c\,\|\frac{\partial(\theta\psi)}{\partial t}\|_{H^{-1,0}(Q_{t_0})}\|\psi\|_V.\end{aligned} \tag{10.32}$$

We consider the term $\|\frac{\partial(\theta\psi)}{\partial t}\|_{H^{-1,0}(Q_{t_0})}$ in (10.32). For $\phi \in C_0^1(Q_{t_0})$ we have

$$\frac{\partial(\theta\psi)}{\partial t}(\phi) = -\int_{Q_{t_0}} \theta\psi\frac{\partial\phi}{\partial t}\,dx\,dt = -\int_{Q_{t_0}} \psi\frac{\partial(\theta\phi)}{\partial t}\,dx\,dt + \int_{Q_{t_0}} \theta'\psi\phi\,dx\,dt.$$

Hence, using $\theta\phi \in C_0^1(Q_{t_0})$ we obtain

$$\begin{aligned}\Big|\frac{\partial(\theta\psi)}{\partial t}(\phi)\Big| &\leq \|\frac{\partial\psi}{\partial t}\|_{H^{-1,0}(Q_{t_0})}\|\theta\phi\|_{H^{1,0}(Q_{t_0})} + c\|\psi\|_V\|\phi\|_{H^{1,0}(Q_{t_0})} \\ &\leq c\big(\|\frac{\partial\psi}{\partial t}\|_{H^{-1,0}(Q_{t_0})} + \|\psi\|_V\big)\|\phi\|_{H^{1,0}(Q_{t_0})}.\end{aligned}$$

Using (10.31) this yields

$$\|\frac{\partial(\theta\psi)}{\partial t}\|_{H^{-1,0}(Q_{t_0})} \leq c\big(\|\frac{\partial\psi}{\partial t}\|_{H^{-1,0}(Q_T)} + \|\psi\|_V\big).$$

Using this in (10.32) we obtain

$$\Big|\frac{\partial(\theta\psi)}{\partial t}(\beta\theta\psi)\Big| \leq c\big(\|\frac{\partial\psi}{\partial t}\|_{H^{-1,0}(Q_T)}^2 + \|\psi\|_V^2\big),$$

and combing these results yields

$$\sup_{t\in[\frac{1}{2}T,T]} \|\psi(\cdot,t)\|_{L^2(\Omega)} \leq c\|\psi\|_W.$$

The same bound can be derived for $t_0 \in [0, \frac{1}{2}T]$ by replacing t_0 by $T - t_0$ and applying the same arguments. □

Corollary 10.3.5 The identity (10.27) holds for all $\psi \in W_\beta$.

Proof. This follows from the density of $\mathcal{W}_\beta$ in W_β with respect to $\|\cdot\|_W$, continuity of the bilinear forms on the left-hand side in (10.27) w.r.t. $\|\cdot\|_W$ and the inequality $\|\beta\psi(\cdot,t)\|_{L^2(\Omega)} \leq c\|\psi\|_W$ for all $\psi \in W_\beta$. □

Lemma 10.3.6 *For all $u, v \in W_\beta$ the following holds:*

$$\frac{\partial u}{\partial t}(\beta v) + \frac{\partial v}{\partial t}(\beta u) = \int_\Omega (\beta uv)_{|t=T}\, dx - \int_\Omega (\beta uv)_{|t=0}\, dx - \int_{\Gamma_*} [\beta uv] n_4 \, ds\, dt,$$

with $n_4 = \hat{\mathbf{n}}_4$ the fourth component of the unit normal at Γ_*.

Proof. Take $\psi \in \mathcal{W}_\beta$. Define

$$l_\psi(v) := \int_{Q_T} \frac{\partial(\beta\psi)}{\partial t} v\, dx\, dt - \int_{\Gamma_*} [\beta\psi v] n_4\, ds, \quad v \in V_\beta.$$

From the definition of the distributional derivative and a density argument it follows that $\frac{\partial \psi}{\partial t}(\beta v) = l_\psi(v)$ for all $v \in V_\beta$. For $\phi \in \mathcal{W}_\beta$ we have

$$l_\psi(\phi) = -l_\phi(\psi) + \int_\Omega (\beta\psi\phi)_{|t=T}\, dx - \int_\Omega (\beta\psi\phi)_{|t=0}\, dx - \int_{\Gamma_*} [\beta\psi\phi] n_4\, ds.$$

This yields

$$\frac{\partial \psi}{\partial t}(\beta\phi) + \frac{\partial \phi}{\partial t}(\beta\psi) = \int_\Omega (\beta\psi\phi)_{|t=T}\, dx - \int_\Omega (\beta\psi\phi)_{|t=0}\, dx - \int_{\Gamma_*} [\beta\psi\phi] n_4\, ds$$

for all $\psi, \phi \in \mathcal{W}_\beta$. By a density argument this even holds for $\psi, \phi \in W_\beta$. □

10.3.2 Space-time weak formulation

In this section we introduce a weak formulation of the mass transport problem (10.1). We restrict to the case with an initial condition $u_0 = 0$. The case $u_0 \neq 0$ can be treated by a shift argument.

For $u \in W_\beta$, due to the result in Lemma 10.3.4 the function $u(\cdot, 0) \in L^2(\Omega)$ is well-defined. We introduce the subspace of W_β of functions with initial data equal to zero:

$$W_{\beta,0} := \{\, u \in W_\beta : u(\cdot,0) = 0 \quad \text{in} \quad \Omega \,\}.$$

The space $(W_{\beta,0}, \|\cdot\|_W)$ is a Hilbert space. We introduce the following space-time weak formulation of the mass transport equation:

Determine $u \in W_{\beta,0}$ such that

$$\frac{\partial u}{\partial t}(v) - \int_{Q_T} u\mathbf{w}\cdot\nabla v\, dx\, dt + \sum_{i=1}^{2} \int_{Q_i} \alpha_i \nabla u_i \cdot \nabla v\, dx\, dt = \int_{Q_T} f v\, dx\, dt \quad (10.33)$$

for all $v \in H_0^{1,0}(Q_T)$.

Remark 10.3.7 The formulation in (10.33) generalizes the one for the stationary interface case given in (10.16). Due to the property (10.19) the test space $H_0^{1,0}(Q_T)$ can be replaced by V_β. For the trial and test space we then have the nice embedding relation $W_{\beta,0} \subset V_\beta$. Using the test space V_β the variational equation (10.33) takes the form

$$\frac{\partial u}{\partial t}(\beta v) - \int_{Q_T} u\mathbf{w} \cdot \nabla(\beta v)\,dx\,dt + \sum_{i=1}^{2} \int_{Q_i} \alpha_i \beta_i \nabla u_i \cdot \nabla v_i\,dx\,dt = \int_{Q_T} \beta f v\,dx\,dt$$

for all $v \in V_\beta$. This generalizes the problem in (10.14).

We show that this problem is consistent with the strong formulation in (10.1):

Lemma 10.3.8 *Assume that the weak formulation* (10.33) *has a solution* $u \in W_{\beta,0}$ *that is sufficiently smooth, namely* $u \in H^1(Q_i)$, $\frac{\partial u}{\partial x_j} \in H^{1,0}(Q_i)$, *for* $j = 1, 2, 3$ *and* $i = 1, 2$. *Then* u *satisfies* (10.1a)–(10.1e) (*in* L^2*-sense*), *with* $u_0 = 0$.

Proof. Due to $u \in W_{\beta,0}$ the properties (10.1c)–(10.1e), with $u_0 = 0$, hold for u. In (10.33) we take $v = \phi \in C_0^1(Q_T)$. Using (10.19), partial integration on the subdomains Q_i and the property (10.25) we get

$$\frac{\partial u}{\partial t}(\phi) - \int_{Q_T} u\mathbf{w} \cdot \nabla\phi\,dx\,dt = \sum_{i=1}^{2} \int_{Q_i} \big(\frac{\partial u_i}{\partial t} + \mathbf{w} \cdot \nabla u_i\big)\phi\,dx\,dt,$$

and

$$\sum_{i=1}^{2} \int_{Q_i} \alpha_i \nabla u_i \cdot \nabla\phi\,dx\,dt = -\sum_{i=2}^{2} \int_{Q_i} \mathrm{div}(\alpha_i \nabla u_i)\phi\,dx\,dt + \int_{\Gamma_*} \nu[\alpha\nabla u \cdot \mathbf{n}]_\Gamma \phi\,ds,$$

with $\mathbf{n} \in \mathbb{R}^3$ the unit normal at $\Gamma(t)$ and ν a suitable scaling parameter. Hence, we obtain

$$\sum_{i=1}^{2} \int_{Q_i} \left(\frac{\partial u_i}{\partial t} + \mathbf{w} \cdot \nabla u_i - \mathrm{div}(\alpha_i \nabla u_i) - f\right)\phi\,dx\,dt + \int_{\Gamma_*} \nu[\alpha\nabla u \cdot \mathbf{n}]_\Gamma \phi\,ds\,dt = 0$$

for all $\phi \in C_0^1(Q_T)$. Thus (10.1a) and (10.1b) are satisfied. □

10.3.3 Well-posedness of the space-time weak formulation

In this section we prove that the space-time variational problem (10.33) has a unique solution. We assume that the right-hand side $f \in L^2(Q_T)$. For the analysis we apply Theorem 15.1.1. As Hilbert spaces we use $H_1 = W_{\beta,0}$, $H_2 = H_0^{1,0}(Q_T)$. For notational convenience we write $H = H_0^{1,0}(Q_T)$, and thus $\|\cdot\|_{H_0^{1,0}(Q_T)} = \|\cdot\|_{H^{1,0}(Q_T)} = \|\cdot\|_H$. For the bilinear form on the left-hand side in (10.33) we use the notation $a : W_{\beta,0} \times H \to \mathbb{R}$, and this bilinear is split into two parts:

$$
\begin{aligned}
a(u,v) &:= \frac{\partial u}{\partial t}(v) - \int_{Q_T} u\mathbf{w} \cdot \nabla v \, dx\, dt + \sum_{i=1}^{2} \int_{Q_i} \alpha_i \nabla u_i \cdot \nabla v \, dx\, dt \\
&=: a_1(u,v) + a_2(u,v),
\end{aligned}
$$

with the elliptic part

$$
a_2(u,v) = \sum_{i=1}^{2} \int_{Q_i} \alpha_i \nabla u_i \cdot \nabla v \, dx\, dt.
$$

Introduce $f(v) := \int_{Q_T} f v \, dx\, dt$. Using this notation the weak formulation takes the form:

$$
\text{determine } u \in W_{\beta,0} \text{ such that } \; a(u,v) = f(v) \quad \text{for all } \; v \in H. \tag{10.34}
$$

The bilinear form $a(\cdot,\cdot)$ is *continuous* on $W_{\beta,0} \times H$. The problem (10.34) is well-posed iff the following two conditions are satisfied, cf. Theorem 15.1.1:

$$
\exists\, \varepsilon > 0 : \quad \inf_{u \in W_{\beta,0}} \sup_{v \in H} \frac{a(u,v)}{\|u\|_W \|v\|_H} \geq \varepsilon \qquad (\text{"inf-sup condition"}), \tag{10.35a}
$$

$$
[\, a(u,v) = 0 \;\text{ for all } u \in W_{\beta,0}] \;\text{ implies }\; v = 0. \tag{10.35b}
$$

Lemma 10.3.9 *The inf-sup condition* (10.35a) *is fulfilled.*

Proof. Take $u \in W_{\beta,0}$. Note that $W_{\beta,0} \subset W_\beta \subset V_\beta$ and thus $\beta u \in H$, cf. (10.19). Define $g_u := \frac{\partial u}{\partial t}(\cdot) \in H'$. Recall that $\|u\|_W^2 = \|u\|_V^2 + \|g_u\|_{H'}^2$. For $v, w \in H$ we have $a_2(v,w) = \int_{Q_T} \alpha \nabla v \cdot \nabla w \, dx\, dt$, hence using a Friedrichs inequality we conclude that t he bilinear form $a_2(\cdot,\cdot)$ is continuous and elliptic on $H \times H$:

$$
\exists\, \delta > 0 : \quad \delta\, \|v\|_H^2 \leq a_2(v,v) \quad \text{for all } \; v \in H.
$$

Let $z \in H$ be such that

$$
a_2(z,v) = g_u(v) \quad \text{for all } \; v \in H.
$$

Using

$$
\|g_u\|_{H'} = \sup_{v \in H} \frac{a_2(z,v)}{\|v\|_H} \leq c\|z\|_H
$$

it follows that there is a constant $\xi > 0$ independent of u such that

$$
\xi \|g_u\|_{H'}^2 \leq \delta \|z\|_H^2 \leq g_u(z) \leq \|g_u\|_{H'} \|z\|_H. \tag{10.36}
$$

Take $v := z + \mu\beta u \in H$, with a fixed, sufficiently large $\mu > 0$. Using (10.36) we obtain

$$\|v\|_H \le \|z\|_H + \mu\|\beta u\|_H \le c_\mu(\|g_u\|_{H'} + \|u\|_V) \le \frac{1}{2}c_\mu\|u\|_W. \tag{10.37}$$

Substitution of this $v \in H$ in the bilinear form yields

$$\begin{aligned} a(u,v) &= \frac{\partial u}{\partial t}(z+\mu\beta u) - \int_{Q_T} u\mathbf{w}\cdot\nabla(z+\mu\beta u)\ dxdt + a_2(u, z+\mu\beta u) \\ &= g_u(z) + \mu\left[\frac{\partial u}{\partial t}(\beta u) - \int_{Q_T} u\mathbf{w}\cdot\nabla(\beta u)\,dx\,dt\right] \\ &\quad - \int_{Q_T} u\mathbf{w}\cdot\nabla z\,dx\,dt + a_2(u,z) + \mu a_2(u,\beta u) \\ &\ge g_u(z) - \int_{Q_T} u\mathbf{w}\cdot\nabla z\ dxdt + a_2(u,z) + \mu a_2(u,\beta u), \end{aligned}$$

where in the last inequality we used Corollary 10.3.5 and $u(x,0) = 0$, since $u \in W_{\beta,0}$. Now note that

$$\Big|\int_{Q_T} u\mathbf{w}\cdot\nabla z\,dx\,dt + a_2(u,z)\Big| \le \tilde{c}\|u\|_V\|z\|_H$$

holds with $\tilde{c}$ independent of u. For the term $a_2(u,\beta u)$ we have

$$\begin{aligned} a_2(u,\beta u) &\ge \min\{\beta_1^{-1},\beta_2^{-1}\}\ a_2(\beta u,\beta u) \ge \min\{\beta_1^{-1},\beta_2^{-1}\}\ \delta\|\beta u\|_H^2 \\ &\ge \delta\min\{\beta_1^{-1},\beta_2^{-1}\}\ \min\{\beta_1^2,\beta_2^2\}\ \|u\|_V^2 =: \hat{c}\|u\|_V^2. \end{aligned} \tag{10.38}$$

Hence, for μ sufficiently large, independent of u, we get

$$\begin{aligned} a(u,v) &\ge g_u(z) - (\tilde{c}\delta^{-\frac{1}{2}}\|u\|_V)(\delta^{\frac{1}{2}}\|z\|_H) + \mu\hat{c}\|u\|_V^2 \\ &\ge g_u(z) - \frac{1}{2}\delta\|z\|_H^2 + \big(\mu\hat{c} - \frac{1}{2}\tilde{c}^2\delta^{-1}\big)\|u\|_V^2 \\ &\ge \frac{1}{2}g_u(z) + \big(\mu\hat{c} - \frac{1}{2}\tilde{c}^2\delta^{-1}\big)\|u\|_V^2 \\ &\ge \frac{1}{2}\xi(\|g_u\|_{H'}^2 + \|u\|_V^2) = \frac{1}{2}\xi\|u\|_W^2. \end{aligned}$$

Combining this with (10.37) we obtain

$$a(u,v) \ge \varepsilon\,\|u\|_W\|v\|_H,$$

with $\varepsilon > 0$ independent of u, which proves the inf-sup property. □

We now consider the second condition (10.35b). In the proof of the next lemma we use Proposition 10.3.2.

Lemma 10.3.10 *Condition* (10.35b) *is fulfilled.*

Proof. Let $v \in H$ be such that $a(u,v) = 0$ for all $u \in W_{\beta,0}$. Define $\hat{v} := \frac{1}{\beta} v \in V_\beta$. Introduce, for $u, v \in H^{1,0}(Q_1 \cup Q_2)$,

$$d(u,v) := \sum_{i=1}^{2} \int_{Q_i} -u_i \mathbf{w} \cdot \nabla v_i + \alpha_i \nabla u_i \cdot \nabla v_i \, dx \, dt,$$

hence, $\frac{\partial u}{\partial t}(\beta \hat{v}) + d(u, \beta \hat{v}) = 0$ for all $u \in W_{\beta,0}$. For arbitrary $u \in \mathcal{V}_\beta \subset W_{\beta,0}$ we have

$$\frac{\partial u}{\partial t}(\beta \hat{v}) = \int_{Q_T} \frac{\partial(\beta u)}{\partial t} \hat{v} \, dx \, dt - \int_{\Gamma_*} [\beta u \hat{v}] n_4 \, ds = -\frac{\partial \hat{v}}{\partial t}(\beta u) - \int_{\Gamma_*} [\beta u \hat{v}] n_4 \, ds,$$

and thus

$$\begin{aligned} \frac{\partial \hat{v}}{\partial t}(\beta u) &= -\frac{\partial u}{\partial t}(\beta \hat{v}) - \int_{\Gamma_*} [\beta u \hat{v}] n_4 \, ds = d(u, \beta \hat{v}) - \int_{\Gamma_*} [\beta u \hat{v}] n_4 \, ds \\ &= d(\beta u, \hat{v}) - \int_{\Gamma_*} [\beta u \hat{v}] n_4 \, ds. \end{aligned}$$

The linear functional $w \to d(w, \hat{v}) - \int_{\Gamma_*} [w \hat{v}] n_4 \, ds$ is bounded on H. Thus it follows that $\frac{\partial \hat{v}}{\partial t}(\cdot) \in H'$, i.e., $\hat{v} \in W_\beta$ and, by a density argument,

$$\frac{\partial \hat{v}}{\partial t}(\beta u) - d(u, \beta \hat{v}) + \int_{\Gamma_*} [\beta u \hat{v}] n_4 \, ds = 0 \quad \text{for all} \quad u \in V_\beta. \tag{10.39}$$

Using $-d(u, \beta \hat{v}) = \frac{\partial u}{\partial t}(\beta \hat{v})$ for all $u \in W_{\beta,0}$ and Lemma 10.3.6 we get

$$0 = \frac{\partial \hat{v}}{\partial t}(\beta u) + \frac{\partial u}{\partial t}(\beta \hat{v}) + \int_{\Gamma_*} [\beta u \hat{v}] n_4 \, ds = \int_\Omega \big(\beta u \hat{v}\big)_{t=T} \, dx \quad \text{for all} \quad u \in W_{\beta,0}.$$

This implies $\hat{v}(\cdot, T) = 0$. In equation (10.39) we take $u = \hat{v} \in W_\beta$, apply partial integration and use Corollary 10.3.5, resulting in

$$\begin{aligned} 0 &= \frac{\partial \hat{v}}{\partial t}(\beta \hat{v}) - d(\hat{v}, \beta \hat{v}) + \int_{\Gamma_*} [\beta \hat{v}^2] n_4 \, ds \\ &= \frac{\partial \hat{v}}{\partial t}(\beta \hat{v}) - \int_{Q_T} \beta \hat{v} \mathbf{w} \cdot \nabla \hat{v} \, dx \, dt + \int_{\Gamma_*} [\beta \hat{v}^2] n_4 \, ds - a_2(\hat{v}, \beta \hat{v}) \\ &= -\frac{1}{2} \|\beta \hat{v}(\cdot, 0)\|^2_{L^2(\Omega)} - a_2(\hat{v}, \beta \hat{v}). \end{aligned}$$

Using ellipticity of $a_2(\cdot, \beta \cdot)$, cf. (10.38), this implies $\hat{v} = 0$ and thus $v = 0$. □.

Theorem 10.3.11 *For each $f \in L^2(Q_T)$ the space-time variational problem* (10.33) *has a unique solution $u \in W_{\beta,0}$ and $\|u\|_W \le c \|f\|_{L^2(Q_T)}$ holds with a constant c independent of f.*

Proof. This follows from Theorem 15.1.1 and the Lemmas 10.3.9 and 10.3.10. □