Asymptotic Behavior of a Viscous Fluid Near a Rough Boundary

J. Casado-Díaz, M. Luna-Laynez, and F.J. Suárez-Grau

Abstract The purpose of this paper is to study the asymptotic behavior of a viscous fluid satisfying Navier's condition on a slightly rough boundary. We consider the case of a fluid contained in a domain that has height 1 and the case of a fluid contained in a domain of small height ε. In both cases we show that three different behaviors are possible.

1 Introduction

For a viscous fluid in a three-dimensional domain with a rough boundary, it is known that if the normal velocity vanishes on the boundary (slip condition), then the fluid behaves as if the whole velocity vector vanishes on the boundary (adherence condition). This gives a mathematical explanation of why it is usual for a viscous fluid to impose the adherence condition. The above assertion was proved in [3] for a boundary described by the equation (see Fig. 1 below)

$$x_3 = -\varepsilon\Psi\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right) \quad \forall (x_1, x_2) \in \omega, \tag{1}$$

with $\varepsilon > 0$ devoted to converge to zero, ω a Lipschitz bounded open set of $\mathbb{R}^2$ and Ψ a smooth periodic function such that

$$\mathrm{Span}(\{\nabla\Psi(z') : z' \in \mathbb{R}^2\}) = \mathbb{R}^2. \tag{2}$$

An extension to non-periodic boundaries was obtained in [1].

Our aim in Sect. 2 is to generalize the result given in [3] to the case of weak rugosities of small period ε and amplitude δ_ε described by (see Fig. 2 below)

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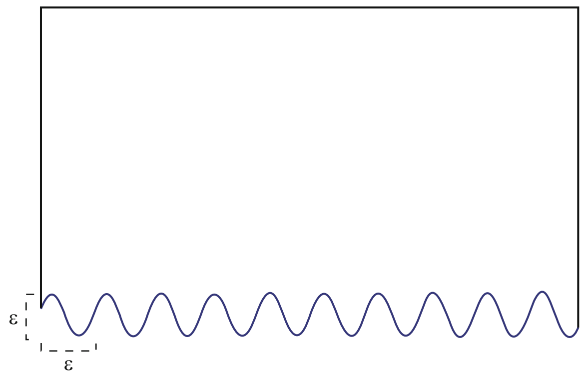

Fig. 1 Rough boundary in 2D described by (1).

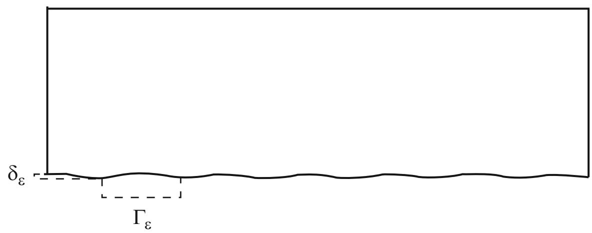

Fig. 2 Rough boundary in 2D described by (3)

$$\Gamma_\varepsilon = \left\{ x = (x_1, x_2, x_3) \in \omega \times \mathbb{R} :\ x_3 = -\delta_\varepsilon \Psi \left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon} \right) \right\}, \qquad (3)$$

where $\omega \subset \mathbb{R}^2$ is a Lipschitz bounded open set, Ψ in $W^{2,\infty}_{loc}(\mathbb{R}^2)$ is a periodic function of period $Z' = (-1/2, 1/2)^2$ and $\delta_\varepsilon > 0$ satisfies $\lim_{\varepsilon \to 0} \frac{\delta_\varepsilon}{\varepsilon} = 0$. Taking the oscillating domain Ω_ε by

$$\Omega_\varepsilon = \left\{ x = (x_1, x_2, x_3) \in \omega \times \mathbb{R} :\ -\delta_\varepsilon \Psi \left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon} \right) < x_3 < 1 \right\}, \qquad (4)$$

we show that if the limit of $\delta_\varepsilon / \varepsilon^{\frac{3}{2}}$ tends to infinity and (2) holds, the slip and the adherence boundary conditions are still asymptotically equivalent. However, this result does not hold if the limit λ of $\delta_\varepsilon / \varepsilon^{\frac{3}{2}}$ belongs to $(0, +\infty)$. In this case we do not have the adherence condition in the limit but the rugosity is large enough to enlarge the friction coefficient in the limit. When $\delta_\varepsilon / \varepsilon^{\frac{3}{2}}$ converges to zero, we prove that the rugosity is so small that it has no effect on the limit problem. For a related result we refer to [2, 4, 5].

In Sect. 3 we will generalize the results obtained in Sect. 2 to a thin domain of small height ε. Taking ω and Ψ as above, our aim is to study the behavior of the fluid near the rough periodic boundary of period r_ε and amplitude δ_ε defined by

$$\Gamma_\varepsilon^{thin} = \left\{ x = (x_1, x_2, x_3) \in \omega \times \mathbb{R} : x_3 = -\delta_\varepsilon \Psi\left(\frac{x_1}{r_\varepsilon}, \frac{x_2}{r_\varepsilon}\right)\right\}, \tag{5}$$

with $r_\varepsilon,\ \delta_\varepsilon > 0$ satisfying $\lim_{\varepsilon\to 0} \frac{r_\varepsilon}{\varepsilon} = 0,\ \lim_{\varepsilon\to 0} \frac{\delta_\varepsilon}{r_\varepsilon} = 0$. Defining $\Omega_\varepsilon^{thin}$ by

$$\Omega_\varepsilon^{thin} = \left\{ x = (x_1, x_2, x_3) \in \omega \times \mathbb{R} : -\delta_\varepsilon \Psi\left(\frac{x_1}{r_\varepsilon}, \frac{x_2}{r_\varepsilon}\right) < x_3 < \varepsilon\right\}, \tag{6}$$

we show analogous results to those presented in Sect. 2, but in this case the behavior of the fluid near Γ^{thin} depends on the limit of a $\frac{\delta_\varepsilon}{r_\varepsilon}\sqrt{\frac{\varepsilon}{r_\varepsilon}}$. That is, the behavior only depend on the small height of the domain, and not only on the parameters describing the rough boundary. When ε goes to 1 we recover the previous case.

2 Asymptotic Behavior of a Viscous Fluid in a Rough Domain

From now on, the points x of $\mathbb{R}^3$ are supposed to be decomposed as $x = (x', x_3)$ with $x' \in \mathbb{R}^2$, $x_3 \in \mathbb{R}$. We also use the notation x' to denote a generic vector of $\mathbb{R}^2$.

Given a bounded connected Lipschitz open set $\omega \subset \mathbb{R}^2$ and $\Psi \in W^{2,\infty}_{loc}(\mathbb{R}^2)$, periodic of period $Z' = (-1/2, 1/2)^2$, we define the domain Ω_ε by (4) and its rough boundary Γ_ε by (3). Then, for $f \in L^2(\omega\times(-1,1))^3$, we consider the Navier-Stokes system in Ω_ε,

$$\begin{cases} -\mu\Delta u_\varepsilon + \nabla p_\varepsilon + (u_\varepsilon\cdot\nabla)u_\varepsilon = f \text{ in } \Omega_\varepsilon, \qquad \operatorname{div} u_\varepsilon = 0 \text{ in } \Omega_\varepsilon, \\ u_\varepsilon = 0 \text{ on } \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon, \\ u_\varepsilon \cdot \nu = 0 \text{ on } \Gamma_\varepsilon, \qquad \dfrac{\partial u_\varepsilon}{\partial \nu} \text{ parallel to } \nu \text{ on } \Gamma_\varepsilon. \end{cases} \tag{7}$$

Here, $\mu > 0$ corresponds to the viscosity of the fluid and ν denotes the unitary outside normal vector to Ω_ε on Γ_ε. It is well known that (7) has at least a solution $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon)^3 \times L^2_0(\Omega_\varepsilon)$ ($L^2_0(\Omega_\varepsilon)$ denotes the space of functions in $L^2(\Omega_\varepsilon)$ whose integral in Ω_ε is zero). Moreover, we can show the following estimates

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)^3} + \|p_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C, \quad \forall \varepsilon > 0. \tag{8}$$

Our problem is to describe the asymptotic behavior of the sequences u_ε and p_ε when ε tends to zero. This is given by the following theorem which is the main result of this section.

Theorem 2.1 *We assume that $(u_\varepsilon, p_\varepsilon)$ is a solution of (7). Then, there exists $(u, p) \in H^1(\Omega)^3 \times L^2_0(\Omega)$, such that, up to a subsequence,*

$$u_\varepsilon \rightharpoonup u \ \text{ in } H^1(\Omega)^3, \qquad p_\varepsilon \rightharpoonup p \ \text{ in } L^2(\Omega), \qquad \text{where } \Omega = \omega \times (0,1). \tag{9}$$

The pair (u, p) satisfies the Navier-Stokes system

$$-\mu\Delta u + \nabla p + (u \cdot \nabla)u = f \ \text{ in } \Omega, \qquad \text{div } u = 0 \text{ in } \Omega, \tag{10}$$

the adherence condition $u = 0$ on $\partial\Omega \setminus \Gamma$ and the vertical component of the limit velocity satisfies $u_3 = 0$ on Γ, where $\Gamma = \omega \times \{0\}$. Moreover, denoting (this limit exists at least for a subsequence)

$$\lambda = \lim_{\varepsilon \to 0} \frac{\delta_\varepsilon}{\varepsilon^{\frac{3}{2}}} \in [0, +\infty], \tag{11}$$

the tangential component of the limit velocity, u', also satisfies the following boundary condition on Γ

i) If $\lambda = 0$, then

$$\partial_3 u' = 0 \ \text{ on } \Gamma. \tag{12}$$

ii) If $\lambda \in (0, +\infty)$, then defining $(\widehat{\phi}^i, \widehat{q}^i)$, $i = 1, 2$ as a solution of

$$\begin{cases} -\mu\Delta_z\widehat{\phi}^i + \nabla_z\widehat{q}^i = 0 \ \text{ in } \mathbb{R}^2 \times (0, +\infty), \\ \text{div}_z\, \widehat{\phi}^i = 0 \ \text{ in } \mathbb{R}^2 \times (0, +\infty), \\ \widehat{\phi}^i_3(z', 0) + \partial_{z_i}\Psi(z') = 0, \ \partial_{z_3}(\widehat{\phi}^i)'(z', 0) = 0, \\ \widehat{\phi}^i(., z_3), \ \widehat{q}^i(., z_3) \text{ periodic of period } Z', \\ D_z\widehat{\phi}^i \in L^2(Z' \times (0, +\infty))^{3\times 3}, \ \widehat{q}^i \in L^2(Z' \times (0, +\infty)), \end{cases} \tag{13}$$

and $R \in \mathbb{R}^{2\times 2}$ by $$R_{ij} = \mu \int_{Z' \times (0,+\infty)} D_z\widehat{\phi}^i : D_z\widehat{\phi}^j \, dz, \quad \forall\, i, j \in \{1, 2\} \tag{14}$$

we have

$$-\mu\partial_3 u' + \lambda^2 R u' = 0 \ \text{ on } \Gamma. \tag{15}$$

iii) If $\lambda = +\infty$, then defining

$$W = Span(\{(\nabla\Psi(z'), 0) : \ z' \in Z'\}), \tag{16}$$

we have

$$u' \in W^\perp \ \text{ on } \Gamma, \quad \partial_3 u' \in W. \tag{17}$$

Remark 2.2 *For $\lambda = 0$, the rugosity of Γ_ε is very slight and the solution $(u_\varepsilon, p_\varepsilon)$ of (7) behaves as if Γ_ε coincides with the plane boundary Γ. For $0 < \lambda < +\infty$ (critical size), the boundary condition satisfied by the limit u of u_ε on the tangent* *space to Γ contains the new term $\lambda^2 Ru'$. The effect of the rugosity of the wall Γ_ε is not worthless in this case. Finally, for $\lambda = +\infty$ the rugosity is so strong that the limit u of u_ε does not only satisfies the condition $u_3 = 0$ on Γ, but it is also such that its tangent velocity on Γ, u', is in $W^\perp$, for every $z' \in Z'$. In particular, if the linear space spanned by W has dimension 2 (this holds if and only if Ψ is not constant in any straight line of $\mathbb{R}^2$, see [3–5]), we get that u satisfies the adherence condition $u = 0$ on Γ, i.e. although we have imposed a slip condition on Γ_ε, the rugosity forces u to satisfy a no-slip (adherence) condition on Γ. This result extends to the case where*

$$\lim_{\varepsilon\to 0} \frac{\delta_\varepsilon}{\varepsilon} = 0, \qquad \lim_{\varepsilon\to 0} \frac{\delta_\varepsilon}{\varepsilon^{\frac{3}{2}}} = +\infty,$$

the results obtained in [3] for $\delta_\varepsilon = \varepsilon$ (see also [2] for the nonperiodic case).

The limit equation (15) corresponding to the critical size $\lambda \in (0, +\infty)$ can be considered as the general one. In fact, if λ is tending to zero or $+\infty$ in (15) we get (12) and (17) respectively.

Remark 2.3 *In the cases $\lambda = 0$ or $+\infty$, we can prove that the convergences in (9) are strong. In fact, assuming ω smooth enough (for example C^2), we can show that we have*

$$\int_{\Omega_\varepsilon} |u_\varepsilon - u|^2 dx \to 0, \quad \int_{\Omega_\varepsilon} |D(u_\varepsilon - u)|^2 dx \to 0, \quad \int_{\Omega_\varepsilon} |p_\varepsilon - p|^2 dx \to 0.$$

In the critical case $\lambda \in (0, +\infty)$, defining $\bar{u}_\varepsilon$ and $\bar{p}_\varepsilon$ by

$$\bar{u}_\varepsilon(x) = u(x) - \lambda\sqrt{\varepsilon}\left(u_1(x',0)\widehat{\phi}^1(\frac{x}{\varepsilon}) + u_2(x',0)\widehat{\phi}^2(\frac{x}{\varepsilon})\right),$$

$$\bar{p}_\varepsilon(x) = p(x) - \frac{\lambda}{\sqrt{\varepsilon}}\left(u_1(x',0)\widehat{q}^1(\frac{x}{\varepsilon}) + u_2(x',0)\widehat{q}^2(\frac{x}{\varepsilon})\right),$$

then the above assertion still holds by replacing u and p by $\bar{u}_\varepsilon$ and $\bar{p}_\varepsilon$, respectively.

3 Asymptotic Behavior of a Viscous Fluid in a Rough Thin Domain

In this section we will generalize the results given in Sect. 2 to the thin domain $\Omega_\varepsilon^{thin}$ given by (6) with a rough boundary $\Gamma_\varepsilon^{thin}$ described by (5). Then, for $f = (f', f_3) \in L^2(\omega)^3$ we consider the Navier-Stokes system

$$\begin{cases} -\mu\Delta u_\varepsilon + \nabla p_\varepsilon + (u_\varepsilon\cdot\nabla)u_\varepsilon = f \ \text{ in } \Omega_\varepsilon^{thin}, \qquad \operatorname{div} u_\varepsilon = 0 \ \text{ in } \Omega_\varepsilon^{thin}, \\ u_\varepsilon = 0 \ \text{ on } \partial\Omega_\varepsilon^{thin}\setminus\Gamma_\varepsilon^{thin}, \\ u_\varepsilon\cdot\nu = 0 \ \text{ on } \Gamma_\varepsilon^{thin}, \qquad \dfrac{\partial u_\varepsilon}{\partial\nu} \text{ parallel to } \nu \ \text{ on } \Gamma_\varepsilon^{thin}. \end{cases} \tag{18}$$

This system has at least a solution $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon)^3 \times L^2_0(\Omega_\varepsilon)$. Moreover it satisfies

$$\fint_{\Omega_\varepsilon^{thin}} |u_\varepsilon|^2 dx \leq C\varepsilon^4, \quad \fint_{\Omega_\varepsilon^{thin}} |Du_\varepsilon|^2 dx \leq C\varepsilon^2, \quad \fint_{\Omega_\varepsilon^{thin}} |p_\varepsilon|^2 dx \leq C. \tag{19}$$

As in the previous section, our aim is to study the asymptotic behavior of u_ε and p_ε when ε tends to zero. For this purpose, as usual, we use a dilatation in the variable x_3 in order to have the functions defined in an open set of fixed height. Namely, we define $\tilde{u}_\varepsilon \in H^1(\Omega)^3$, $\tilde{p}_\varepsilon \in L^2_0(\Omega)$ by

$$\tilde{u}_\varepsilon(y) = u_\varepsilon(y', \varepsilon y_3), \quad \tilde{p}_\varepsilon(y) = p_\varepsilon(y', \varepsilon y_3), \quad \text{a.e. } y \in \Omega = \omega \times (0,1). \tag{20}$$

Then, our problem is to describe the asymptotic behavior of these sequences $\tilde{u}_\varepsilon$, $\tilde{p}_\varepsilon$. This is given by the following theorem.

Theorem 3.1 *Let $(u_\varepsilon, p_\varepsilon) \in H^1(\Omega_\varepsilon)^3 \times L^2_0(\Omega_\varepsilon)$ be a solution of (18) and let $\tilde{u}_\varepsilon$, $\tilde{p}_\varepsilon$ be defined by (20). Then, there exist $v \in H^1(0,1; L^2(\omega))^2$, $w \in H^2(0,1; H^{-1}(\omega))$ and $p \in L^2_0(\Omega)$, where p does not depend on y_3, such that, up to a subsequence,*

$$\begin{gathered}\frac{\tilde{u}_\varepsilon}{\varepsilon} \rightharpoonup 0 \text{ in } H^1(\Omega)^3, \quad \frac{\tilde{u}_\varepsilon}{\varepsilon^2} \rightharpoonup (v, 0) \text{ in } H^1(0,1; L^2(\omega))^3, \\ \frac{\tilde{u}_{\varepsilon,3}}{\varepsilon^3} \rightharpoonup w \text{ in } H^2(0,1; H^{-1}(\omega)),\end{gathered} \tag{21}$$

$$\tilde{p}_\varepsilon \rightharpoonup p \text{ in } L^2(\Omega), \quad \frac{\partial_{y_3}\tilde{p}_\varepsilon}{\varepsilon} \rightharpoonup f_3 \text{ in } H^{-1}(\Omega). \tag{22}$$

According to the value of λ^{thin} defined by

$$\lambda^{thin} = \lim_{\varepsilon \to 0} \frac{\delta_\varepsilon}{r_\varepsilon}\sqrt{\frac{\varepsilon}{r_\varepsilon}} \in [0, +\infty], \tag{23}$$

the functions v, w and p are given by

(i) If $\lambda^{thin} = +\infty$, then denoting by $P_{W^\perp}$ the orthogonal projection from $\mathbb{R}^2$ to the orthogonal of the space W defined by (16), we have that v is given by

$$v(y) = \frac{(y_3 - 1)}{2\mu}\Big(y_3 I + P_{W^\perp}\Big)\big(\nabla_{y'} p(y') - f'(y')\big), \quad \text{a.e. } y \in \Omega,$$

where p satisfies

$$\begin{cases} -div_{y'}\left(\left(\frac{1}{3}I + P_{W^\perp}\right)(\nabla_{y'} p - f')\right) = 0 \text{ in } \omega, \\ \left(\frac{1}{3}I + P_{W^\perp}\right)(\nabla_{y'} p - f') \cdot \nu = 0 \text{ on } \partial\omega. \end{cases}$$

Moreover, the distribution w is given by

$$w(y) = -\int_0^{y_3} div_{y'} v(y', s)ds, \quad in\ \Omega. \tag{24}$$

(ii) If $\lambda \in (0, +\infty)$, *then defining* $(\widehat{\phi}^i, \widehat{q}^i)$, $i = 1, 2$, *as solutions of the Stokes systems (13) and the matrix R by (14), we have*

$$v(y) = \frac{(y_3 - 1)}{2\mu}\left(y_3 I + \left(I + \frac{\lambda^2}{\mu}R\right)^{-1}\right)\left(\nabla_{y'} p(y') - f'(y')\right), \ a.e.\ y \in \Omega,$$

where p satisfies

$$\begin{cases} -\text{div}_{y'}\left(\left(\frac{1}{3}I + \left(I + \frac{\lambda^2}{\mu}R\right)^{-1}\right)(\nabla_{y'} p - f')\right) = 0 \ \ in\ \omega, \\ \left(\frac{1}{3}I + \left(I + \frac{\lambda^2}{\mu}R\right)^{-1}\right)(\nabla_{y'} p - f') \cdot \nu = 0 \ \ on\ \partial\omega. \end{cases}$$

Moreover, the distribution w is given by (24).

(iii) If $\lambda = 0$, *then* $\quad v(y) = \dfrac{(y_3^2 - 1)}{2\mu}(\nabla_{y'} p(y') - f'(y')), \quad a.e.\ y \in \Omega,$

where p satisfies $\quad -\Delta_{y'} p = -div_{y'} f' \ in\ \omega, \qquad \dfrac{\partial p}{\partial \nu} = f' \cdot \nu \ on\ \partial\omega.$

Moreover, the distribution w is zero.

Remark 3.2 *The role of the parameter* λ^{thin} *in Theorem 3.1 is similar to the one of* λ *in Theorem 2.1. Indeed, we remember that for* $\varepsilon = 1$ *both parameters agree (note that the parameter* ε *in Theorem 2.1 is now called* r_ε*).*

Remark 3.3 *In the cases* $\lambda^{thin} = 0$ *or* $+\infty$, *we can prove that the convergences in (21)–(22) are strong. In fact, assuming* ω *smooth enough, we prove that defining* $\bar{u}_\varepsilon$, $\bar{p}_\varepsilon$ *by*

$$\bar{u}_\varepsilon(x) = \left(\varepsilon^2 v(x', \frac{x_3}{\varepsilon}), 0\right), \quad \bar{p}_\varepsilon(x) = p(x') \ \ a.e.\ x \in \Omega_\varepsilon^{thin},$$

we have

$$\frac{1}{\varepsilon^4}\int_{\Omega_\varepsilon} |u_\varepsilon - \bar{u}_\varepsilon|^2 dx \to 0, \quad \frac{1}{\varepsilon^2}\int_{\Omega_\varepsilon} |D(u_\varepsilon - \bar{u}_\varepsilon)|^2 dx \to 0, \quad \int_{\Omega_\varepsilon} |p_\varepsilon - \bar{p}_\varepsilon|^2 dx \to 0.$$

In the critical case $\lambda^{thin} \in (0, +\infty)$, *the above assertion still holds replacing* $\bar{u}_\varepsilon$ *by*

$$\bar{u}_\varepsilon(x) = \left(\varepsilon^2 v(x', \frac{x_3}{\varepsilon}), 0\right) - \lambda\varepsilon\sqrt{\varepsilon r_\varepsilon}\left(v_1(x', 0)\widehat{\phi}^1(\frac{x}{r_\varepsilon}) + v_2(x', 0)\widehat{\phi}^2(\frac{x}{r_\varepsilon})\right).$$

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