Performance of Stabilized Higher-Order Methods for Nonstationary Convection-Diffusion-Reaction Equations

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Abstract We study the performance properties of a class of stabilized higher-order finite element approximations of convection-diffusion-reaction models with nonlinear reaction mechanisms. Streamline upwind Petrov-Galerkin (SUPG) stabilization together with anisotropic shock-capturing as an additional stabilization in crosswind-direction is used. We show that these techniques reduce spurious oscillations in crosswind-direction and increase the accuracy of simulations.

1 Introduction

Time-dependent convection-diffusion-reaction equations

$$\partial_t u + \boldsymbol{b} \cdot \nabla u - \nabla \cdot (a \nabla u) + r(u) = f \qquad (1)$$

are often studied in various technical and environmental applications. Here, $u = u(\boldsymbol{x}, t)$ denotes the unknown where $\boldsymbol{x} \in \Omega \subset \mathbb{R}^d$, with $d \geq 2$, and $t \in (0, T)$ for some $T > 0$. Further, $a \in L^\infty(0, T; W^{1,\infty}(\Omega))$ is the diffusion coefficient, $\boldsymbol{b} \in L^\infty(0, T; \boldsymbol{W}^{1,\infty}(\Omega))$ is the velocity field, $r \in C^1(\mathbb{R}_0^+)$ is the parametrization of the reaction rate and $f \in L^2(0, T; L^2(\Omega))$ is a prescribed right-hand side term. We suppose that $\nabla \cdot \boldsymbol{b}(\boldsymbol{x}, t) = 0$ and $a(x, t) \geq \alpha > 0$ almost everywhere. Throughout the paper we use standard notation.

The accurate numerical approximation of (1) is still a challenging task. In applications, the transport equation (1) is often convection- and/or reaction-dominated and characteristic solutions have sharp layers. In these cases standard finite element methods cannot be applied. Stabilized finite element approaches are required. For a review of these techniques we refer to the recent work of John and Schmeyer [3].

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C. Clavero et al. (eds.), *BAIL 2010 – Boundary and Interior Layers, Computational and Asymptotic Methods*, Lecture Notes in Computational Science and Engineering 81,
DOI 10.1007/978-3-642-19665-2_2,

Stabilization methods are well-understood for linear steady convection-diffusion-reaction problems; cf., e.g., [3, 4]. However, there is still a considerable lack in the analysis, design and application of these methods for unsteady nonlinear problems which is addressed here. Rigorous analyses are rare for the unsteady and nonlinear case.

2 Discretization Scheme

Equipping (1) with initial and homogeneous Dirichlet boundary conditions and discretizing (1) in time by the θ-scheme, with $\theta \in (0,1]$, leads to a sequence of stationary boundary value problems: *Find* $\{u^k\}_{k=1}^N$ *such that*

$$\alpha_k u^k + \theta \boldsymbol{b}(t_k)\cdot\nabla u^k - \theta\nabla\cdot(\boldsymbol{a}(t_k)\nabla u^k) + \theta r(u^k) = \tilde{f}^k \quad \text{in } \Omega\,, \tag{2}$$

with $\tilde{f}^k = \alpha_k u^{k-1} + \theta f(t_k) + (1-\theta) f(t_{k-1}) - (1-\theta)\boldsymbol{b}(t_{k-1})\cdot\nabla u^{k-1} + (1-\theta)\nabla\cdot(\boldsymbol{a}(t_{k-1})\nabla u^{k-1}) - (1-\theta) r(\boldsymbol{u}^{k-1})$, $\alpha_k = 1/(t_k - t_{k-1})$ *and* $u^k = 0$ *on* $\partial\Omega$, $u^0 = u(t_0)$.

In the sequel, we suppose that the solution u of (1) is non-negative and bounded from above, i.e., $0 =: u_0 \leq u \leq u_1$ almost everywhere in $\Omega \times (0,T)$, which is admissible from the sake of physical realism, for instance, if u denotes the concentration of a chemical species. We make the assumption that

$$r \in C^1(\mathbb{R}_0^+)\,, \qquad r(0) = 0\,, \qquad r'(s) \geq r_0 \geq 0 \quad \text{for } s \geq 0\,,\ s \in \mathbb{R}\,. \tag{3}$$

To calculate approximations of $\{u^k\}_{k=1}^N$, a standard hp-version of the finite element method is assumed; cf. [1, 4, 7]. For a family of admissible and shape-regular triangulations $\mathcal{T}_h = \{T\}$ of the polyhedral domain $\Omega \subset \mathbb{R}^d$ let

$$V_h^{\boldsymbol{p}} = X_h^{\boldsymbol{p}} \cap H_0^1(\Omega) \quad \text{with} \quad X_h^{\boldsymbol{p}} = \{v \in C(\overline{\Omega}) \mid v_{|T} \circ F_T \in \mathcal{P}_{p_T}(\widehat{T})\ \forall T \in \mathcal{T}_h\}$$

denote the underlying finite element space of piecewise polynomials of local order p_T for all $T \in \mathcal{T}_h$. Here, $\widehat{T}$ is the (open) unit simplex or the (open) unit hypercube in $\mathbb{R}^d$ and $\mathcal{P}_n(\widehat{T})$, with $n \geq 1$, is the set of all polynomials of degree at most n on $\widehat{T}$. We assume that each $T \in \mathcal{T}_h$ is a smooth bijective image of $\widehat{T}$, i.e., $T = F_T(\widehat{T})$. The vector $\boldsymbol{p}$ is defined by $\boldsymbol{p} = \{p_T \mid T \in \mathcal{T}_h\}$. In our analysis the local inverse inequalities

$$\|\nabla w_h\|_{L^2(T)} \leq \mu_{\text{inv}} p_T^2 h_T^{-1} \|w\|_{L^2(T)} \quad \forall w_h \in X_h^{\boldsymbol{p}} \quad \text{on } T \in \mathcal{T}_h \tag{4}$$

are applied. Here, μ_{inv} depends on the shape-regularity parameter; cf. [7].

Skipping for brevity the indices in (2), the SUPG-stabilized approximation of (2) is: *Find* $u_h \in V_h^{\boldsymbol{p}}$ *such that*

$$A_s(u_h, v_h) = L_s(v_h) \tag{5}$$

for all $v_h \in V_h^{\boldsymbol{p}}$, *where*

$$A_s(u,v) = A_{\rm lin}(u,v) + \theta\langle\widehat{r}(u),v\rangle + \sum_{T\in\mathcal{T}_h}\delta_T\langle\widehat{L}u,\boldsymbol{b}\cdot\nabla v\rangle_{L^2(T)}\,, \tag{6}$$

$$L_s(v) = \langle\tilde{f},v\rangle + \sum_{T\in\mathcal{T}_h}\delta_T\langle\tilde{f},\boldsymbol{b}\cdot\nabla v\rangle_{L^2(T)}\,, \tag{7}$$

$$A_{\rm lin}(u,v) = \alpha\langle u,v\rangle + \theta\langle\boldsymbol{b}\cdot\nabla u,v\rangle + \theta\langle a\nabla u,\nabla v\rangle\,, \tag{8}$$

$$\widehat{L}u = \widehat{L}_{\rm lin}u + \theta\widehat{r}(u)\,,\quad \widehat{L}_{\rm lin}u_{|T} = \alpha u + \theta\boldsymbol{b}\cdot\nabla u - \theta\nabla\cdot\Pi_T(a\nabla u)\,. \tag{9}$$

If in addition shock-capturing is applied, we get: *Find* $u_h \in V_h^{\boldsymbol{p}}$ *such that*

$$A_s(u_h,v_h) + A_{\rm sc}(u_h;u_h,v_h) = L_s(v_h) \tag{10}$$

for all $v_h \in V_h^{\boldsymbol{p}}$, *where*

$$A_{\rm sc}(w;u,v) := \sum_{T\in\mathcal{T}_h}\langle\tau_T(w)\boldsymbol{D}_{\rm sc}\nabla u,\nabla v\rangle\,. \tag{11}$$

Together, the last terms on the right-hand sides of (6) and (7), respectively, represent the SUPG-stabilization. The choice of the stabilization parameter δ_T is given in Remark 2 below. In (8) we changed $r(\cdot)$ to $\widehat{r}(\cdot)$ where $\widehat{r}(u) = r(u_0)+r'(u_0)(u-u_0)$ for $u \le u_0$, $\widehat{r}(u) = r(u)$ for $u_0 \le u \le u_1$ and $\widehat{r}(u) = r(u_1) + r'(u_1)(u-u_1)$ for $u \ge u_1$. This modification is necessary to prove an error estimates when r' grows with $|u|$. It is probably not necessary in practical computations but easy to incorporate if desired. It holds that

$$|\widehat{r}(u) - \widehat{r}(v)| \le L_r|u-v| \quad \forall u,v\in\mathbb{R}\,, \tag{12}$$

with some constant $L_r > 0$. In (9), the mapping $\Pi_T : \boldsymbol{L}^2(\Omega) \mapsto (\mathcal{P}_{p_T}(T))^d$ is the (elementwise) orthogonal projection onto $(\mathcal{P}_{p_T}(T))^d$. In (11), we use an anisotropic variant of shock-capturing (cf. [2]):

$$\boldsymbol{D}_{\rm sc} := \begin{cases}\boldsymbol{I} - \dfrac{\boldsymbol{b}\otimes\boldsymbol{b}}{|\boldsymbol{b}|^2}\,, & \boldsymbol{b}\neq\boldsymbol{0}\\ \boldsymbol{0}\,, & \boldsymbol{b}=\boldsymbol{0}\end{cases}\,,\quad \tau_T(w) := l_T(w)R_T^*(w) \equiv \frac{l_T(w)R_T(w)}{|w|_{H^1(T)}+\kappa}\,,$$

$$R_T(w) := \|\widehat{L}w - f\|_{L^2(T)}\,,\qquad l_T(w) := l_0 h_T \max\left\{0,\beta - \frac{2\|a\|_{L^\infty(T)}}{h_T R_T^*(w)}\right\}\,. \tag{13}$$

The non-negative limiter function $\tau_T(w)$ aims to restrict the effect of shock-capturing to subregions where the residual $\widehat{L}w - f$ is too large. The term $\frac{h_T R_T^*(w)}{2\|a\|_{L^\infty(T)}}$

can be seen as a pseudo mesh Peclet number. The choice of l_0, κ and β is given in Sect. 3. We note that $\tau_T(u_h)$ depends nonlinearly on u_h. Since $r(\cdot)$ is assumed to be nonlinear, the shock-capturing term (11) does not change the type of the discrete problem. This is in contrast to linear convection-diffusion-reaction models that become nonlinear by adding (11).

For the limiter function $\tau_T(\cdot)$ we suppose that

$$0 \le \tau_T(w) \le M_T(h_T) \quad \text{with} \lim_{h \to 0} M_T(h) = 0 \tag{14}$$

holds for $w \in V_h^{\boldsymbol{p}}$ and all $T \in \mathcal{T}_h$. The shock-capturing approach discussed here is covered by the class of methods satisfying (14); cf. [5, Example 3.2].

The existence and stability of a discrete solution $u_h \in V_h^{\boldsymbol{p}}$ of (10) is ensured. To show this, the following norm is introduced:

$$|||v||| := \left(\sum_{T \in \mathcal{T}_h} \left(\|\sqrt{a} \nabla v\|^2_{L^2(T)} + (\alpha + r_0)\|v\|^2_{L^2(T)} + \delta_T \|\boldsymbol{b} \cdot \nabla v\|^2_{L^2(T)} \right) \right)^{1/2} .$$

Moreover, the following auxiliary estimates are needed.

Lemma 1. *Suppose that assumption (3) and the condition*

$$0 \le \delta_T \le \frac{1}{4} \min \left\{ \frac{h_T^2}{p_T^4 \mu_{\text{inv}}^2 \|a\|_{L^\infty(\Omega)}} ; \frac{1}{\alpha} ; \frac{\alpha + r_0}{L_r^2} \right\} \tag{15}$$

are satisfied. Then, the semilinear form A_s in (6) satisfies

$$A_s(v_h, v_h) \ge \frac{1}{4} |||v_h|||^2 \quad \forall v_h \in V_h^{\boldsymbol{p}} . \tag{16}$$

For $u \in H_0^1(\Omega)$ with $(\nabla \cdot (a \nabla u))_{|T} \in L^2(T)$ and $v_h \in V_h^{\boldsymbol{p}}$ it holds that

$$\begin{aligned} & A_{\text{lin}}(u, v_h) + \sum_{T \in \mathcal{T}_h} \delta_T \langle \widehat{L}_{\text{lin}} u, \boldsymbol{b} \cdot \nabla v_h \rangle \le Q_s(u) |||v_h||| , \\ & Q_s(u) := |||u||| + \left(\sum_{T \in \mathcal{T}_h} \min \left\{ \frac{1}{\delta_T} ; \frac{\|\boldsymbol{b}\|_{L^\infty(T)}}{\alpha_T} \right\} \|u\|^2_{L^2(T)} \right. \\ & \qquad \left. + \sum_{T \in \mathcal{T}_h} \left(\delta_T \| - \nabla \cdot (\Pi_T (a \nabla u) + \alpha u\|^2_{L^2(T)} \right) \right)^{1/2} , \end{aligned} \tag{17}$$

where $\alpha_T := \inf_{x \in T} a(x)$.

The proof of Lemma 1 is given in [1, Lemma 2.1].

Theorem 1. *Let (3), (14) and (15) be satisfied. Suppose that the limiter function $\tau(w)$ is continuous with respect to w. Then, the SUPG scheme with shock-capturing stabilization (10) admits a solution $u_h \in V_h^p$ satisfying*

$$|||u_h|||^2 + \sum_{T \in \mathcal{T}_h} \left\| \sqrt{\tau_T(u_h)} \boldsymbol{D}_{\mathrm{sc}}^{1/2} \nabla u_h \right\|_{L^2(T)}^2 \leq C |||\tilde{f}|||_*^2 \tag{18}$$

with the dual norm $|||\tilde{f}|||_* := \sup_{v_h \in V_h^p \setminus \{0\}} L(v_h)/|||v_h|||$.

Proof. To prove Theorem 1 we use a variant of Brouwer's fixed point theorem; cf. [9, II Lemma 1.4]. For this, let V_h^p be equipped with the inner product $[u_h, v_h] = \langle \nabla u_h, \nabla v_h \rangle$. Let $P : V_h^p \mapsto V_h^p$ be defined by

$$[P(u_h), v_h] = \langle \nabla P(u_h), \nabla v_h \rangle = A_{\mathrm{sc}}(u_h; u_h, v_h) + A_s(u_h, v_h) - L_s(v_h)\,.$$

First, we note that

$$\begin{aligned}
&[P(u_h) - P(v_h), P(u_h) - P(v_h)] = \|\nabla(P(u_h) - P(v_h))\|_{L^2(\Omega)}^2 \\
&= A_{\mathrm{sc}}(u_h; u_h - v_h, P(u_h) - P(v_h)) \\
&\quad + \sum_{T \in \mathcal{T}_h} (\langle \tau_T(u_h) - \tau_T(v_h)) \boldsymbol{D}_{\mathrm{sc}} \nabla v_h, \nabla(P(u_h) - P(v_h))) \rangle_T \\
&\quad + A_{\mathrm{lin}}(u_h - v_h, P(u_h) - P(v_h)) + \langle \widehat{r}(u_h) - \widehat{r}(v_h), P(u_h) - P(v_h) \rangle \\
&\qquad\qquad + \sum_{T \in \mathcal{T}_h} \delta_T \langle \widehat{L}_{\mathrm{lin}}(u_h - v_h), \boldsymbol{b} \cdot \nabla(P(u_h) - P(v_h)) \rangle_T \\
&\qquad\qquad + \sum_{T \in \mathcal{T}_h} \delta_T \langle \widehat{r}(u_h) - \widehat{r}(v_h), \boldsymbol{b} \cdot \nabla(P(u_h) - P(v_h)) \rangle_T\,.
\end{aligned}$$

Under the hypotheses of Theorem 1, using Lemma 1 and the Poincaré inequality we get from this identity that

$$\begin{aligned}
&[P(u_h) - P(v_h), P(u_h) - P(v_h)] = \|\nabla(P(u_h) - P(v_h))\|_{L^2(\Omega)}^2 \\
&\leq \max_{T \in \mathcal{T}_h} \{M_T(h_T)\} C_{\boldsymbol{D}_{\mathrm{sc}}} \|\nabla(u_h - v_h)\|_{L^2(\Omega)}^2 \|\nabla(P(u_h) - P(v_h))\|_{L^2(\Omega)} \\
&\quad + C_{\boldsymbol{D}_{\mathrm{sc}}} \|v_h\|_{W^{1,\infty}(\Omega)} \|\tau(u_h) - \tau(v_h)\|_{L^2(\Omega)} \|\nabla(P(u_h) - P(v_h))\|_{L^2(\Omega)} \\
&\quad + (Q_s(u_h - v_h) + L_r |||u_h - v_h|||) |||P(u_h) - P(v_h)||| \\
&\quad + L_r \|u_h - v_h\|_{L^2(\Omega)} \|\nabla(P(u_h) - P(v_h))\|_{L^2(\Omega)}\,.
\end{aligned}$$

Recalling (14), using the Poincaré inequality and (4) we get that

$$\|\nabla(P(u_h) - P(v_h))\|_{L^2(\Omega)} \leq C(\|\tau(u_h) - \tau(v_h)\|_{L^2(\Omega)} + \|\nabla(u_h - v_h)\|_{L^2(\Omega)}),$$

which proves the continuity of P, since $\tau(\cdot)$ is assumed to be continuous.

Further, by Lemma 1 and the inequality of Cauchy-Young we get that

$$\begin{aligned}[P(v_h), v_h] &= A_{\mathrm{sc}}(v_h; v_h, v_h) + A_s(v_h, v_h) - L_s(v_h) \\ &\geq A_{\mathrm{sc}}(v_h; v_h, v_h) + \frac{1}{4}|||v_h|||^2 - |||f|||_* |||v_h||| \geq \frac{1}{8}|||v_h|||^2 - 2|||f|||_*^2 .\end{aligned} \tag{19}$$

From (19) we conclude that $[P(v_h), v_h] > 0$ for all $v_h \in V_h^p$ with $\|\nabla v_h\|_{L^2(\Omega)} > 4a_0^{-1/2}|||f|||_*$. Brouwer's fixed point theorem (cf. [9, II Lemma 1.4]) now yields the existence of at least one solution $u_h \in V_h^p$ of $P(u_h) = 0$ and, therefore, of (10). Estimate (18) follows from (19), using that $P(u_h) = 0$.

Remark 1. A (local) uniqueness result for the shock-capturing method is still open. The application of Banach's fixed point theorem would require Lipschitz continuity of the parameters $\tau_T(\cdot)$. But such condition is seemingly too restrictive in practice. Another possibility is to apply a uniqueness result for the Brouwer fixed point theorem which could be proved similarly as a corresponding result for the Schauder fixed point theorem; cf. [5] and the references therein. Unfortunately, the assumptions on $\tau_T(\cdot)$ are too severe.

Remark 2. In [1] the quasi-optimal error estimate

$$||u - u_h|||^2 \leq C \sum_{T \in \mathcal{T}_h} \frac{h_T^{2(l_T-1)}}{p_T^{2(k_T-1)}} \|u\|^2_{H^{k_T}(T)}, \quad C > 0,$$

with $l_T = \min\{p_T + 1, k_T\}$ is proved for the SUPG-stabilized finite element approximation of (2) with and without additional shock-capturing stabilization. It is supposed that the parameter δ_T in (6), (7) is chosen of the order of magnitude $\delta_T \sim \min\left\{\frac{h_T}{p_T\|\boldsymbol{b}\|_{L^\infty(T)}}; \frac{h_T^2}{p_T^4 \mu_{\mathrm{inv}}^2 \|a\|_{L^\infty(\Omega)}}; \frac{1}{\alpha + r_0}; \frac{\alpha + r_0}{L_r^2}\right\}$. Thus, the additional diffusion term (11) does not perturb the asymptotic convergence behaviour. Our numerical studies will show that shock-capturing stabilization reduces at the same time spurious oscillations in crosswind direction.

3 Numerical Studies

Now we study the numerical performance properties of the schemes. We show that higher order finite element methods combined with SUPG and shock-capturing stabilization are able to resolve interior layers and lead to accurate approximations of solutions of problem (1). For the discretization in time we use the Crank-Nicholson method corresponding to $\theta = 1/2$ in (2). All computations were done with the Toolbox ALBERTA [6] on triangular meshes.

Example 3.1. Our first numerical study is devoted to the quasi-stationary case. It is a stationary nonlinear variant of the second example in [3, Sect. 7]. We consider (2) for fixed k and $\theta = 1$ on $\Omega = (0,1)^2$ with $\alpha = 1.0$, $a = 10^{-6}$ and $\boldsymbol{b}(\boldsymbol{x}) = (2,3)^\top$. The reaction mechanism is governed by an Arrhenius law $r(u) = k_{Ar} e^{\frac{b_{Ar}(u-1)}{1+a_{Ar}(u-1)}}$ with $k_{Ar} = 50$, $b_{Ar} = 10$ and $a_{Ar} = 0.8$. Studies for polynomial reaction rates are given in [1,8]. The source $\tilde{f}$ is chosen in such a way that $u(\boldsymbol{x}) = 16x_1(1-x_1)x_2(1-x_2) \cdot \left[0.5 + \pi^{-1} \arctan\left(2a^{-1/2}(0.25^2 - (x_1 - 0.5)^2 - (x_2 - 0.5)^2)\right)\right]$ is the exact solution. It is characterized by an interior layer of thickness $\mathcal{O}(\sqrt{a})$. We study solutions of (5) and (10). In (13), we put $l_0 = 0.2$, $\kappa = 10^{-4}$, $\beta = 0.7$.

In Table 1 we summarize the calculated errors for different P_p-elements for $p \in \{2,4\}$. Although the SUPG-scheme shows a slightly smaller error, we observe that the errors of the either schemes are of the same order of magnitude, as claimed in Remark 2. The larger errors of the shock-capturing approach are due to its additional artificial crosswind-diffusion that however reduces spurious oscillations. In Table 1 we do not observe the optimal uniform convergence rates since u depends on the diffusion parameter a. In such cases optimal convergence rates are observed for very small step sizes only.

To study the effects of additional crosswind-diffusion more precisely, cross-section plots of the solution at the outflow boundary without and with shock-capturing stabilization are presented in Fig. 1. Significant over- and undershoots of the SUPG-solution without shock-capturing in the neighborhood of the layer are

Table 1 Mesh size, number of degrees of freedom, errors in $|||\cdot|||$ and convergence rates for the SUPG-scheme without (SUPG) and with (SC-CD) shock-capturing/crosswind diffusion and h-refinement for Example 3.1

h	d.o.f.	$p=2$, $\|\|\|\cdot\|\|\|$				d.o.f.	$p=4$, $\|\|\|\cdot\|\|\|$			
		SUPG		SC-CD			SUPG		SC-CD	
8.839e-2	545	3.721e-2	–	3.726e-2	–	2113	2.444	–	2.453	–
4.419e-2	2113	2.456e-2	0.60	2.459e-2	0.60	8321	1.507	0.70	1.512	0.70
2.210e-2	8321	1.466e-2	0.74	1.468e-2	0.74	33025	6.475e-1	1.22	6.485e-1	1.22
1.105e-2	33025	6.469e-3	1.18	6.475e-3	1.18	131585	3.351e-1	0.95	3.350e-1	0.95
5.524e-3	131585	2.501e-3	1.37	2.504e-3	1.37	525313	1.346e-1	1.32	1.346e-1	1.32

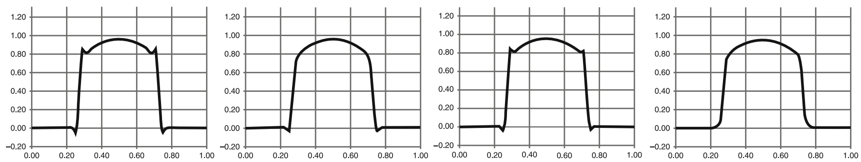

Fig. 1 Cross-section plot for the SUPG-scheme without shock-capturing/crosswind diffusion ($p = 1$, $p = 4$) and with shock-capturing/crosswind diffusion ($p = 1$, $p = 3$) for Example 3.1 with $h = 2.21$e-2; *from left to right*

observed. This also holds for higher order finite element methods. The unphysical oscillations are clearly damped by higher order finite element approximations along with SUPG and shock-capturing stabilization. No significant improvement is obtained by shock-capturing stabilization if linear finite element methods are applied. This was also observed in [3]. We note that the strong gradient of the SUPG solution in the layer is preserved by the shock-capturing technique. This underlines its proper construction.

Example 3.2. Now we study the performance properties of the schemes for the nonstationary counterpart of Example 3.1; cf. [3, Sec. 7]. We consider (1) on $\Omega = (0,1)^2$. We put $a = 10^{-6}$, $\boldsymbol{b}(\boldsymbol{x}) = (2,3)^\top$ and $r(u) = u^2$. The given solution with corresponding right-hand side f is $u(\boldsymbol{x},t) = 16\sin(\pi t)x_1(1-x_1)x_2(1-x_2)\cdot \left[0.5 + \pi^{-1}\arctan\left(2a^{-\frac{1}{2}}(0.25^2 - (x_1-0.5)^2 - (x_2-0.5)^2)\right)\right]$. Now the hump changes its height in time. We use the time step size $\Delta t = 10^{-3}$. The final time is $T = 1.0$. The finite element mesh size is $h = 5.52\text{e-}3$. In Table 2 we present $\|e_h\|_{L^\infty(L^2)} := \|e_h\|_{L^\infty(0,T;L^2(\Omega))}$, $e_h := u - u_h$, and var(t) $:= \max_{(x,y)\in\Omega} u_h(x_1,x_2,t) - \min_{(x,y)\in\Omega} u_h(x_1,x_2,t)$, where the maximum and minimum were computed in the degrees of freedom of the mesh cells. The numbers $\|e_h\|_{L^\infty(L^2)}$ give some indication of the accuracy of the methods whereas var(t) measures the size of the spurious oscillations. Here we use the variation of the discrete solution at $t = 0.5$ (maximal height of the hump). The value for u is var(0.5) = 0.997453575. In Fig. 2 we further visualize the computed profiles of the solution for $t = 0.5$.

Similarly to the stationary case, a positive impact on the accuracy is obtained by using higher order finite element methods together with SUPG and shock-capturing stabilization. Oscillations close to the layer are completely eliminated. The profiles of the numerical solutions show oscillations behind the hump in the direction of convection. They are damped by higher order approaches. If fourth order finite elements are used, they vanish almost completely. In [3], all schemes show significant oscillations behind the hump.

Table 2 Results with SUPG and shock-capturing stabilization for Example 3.2

p	$\|e_h\|_{L^\infty(L^2)}$	var(0.5)	p	$\|e_h\|_{L^\infty(L^2)}$	var(0.5)
1	2.8090e-2	1.2088	3	6.3204e-3	1.0319
2	1.3973e-2	1.1019	4	4.5312e-3	1.0203

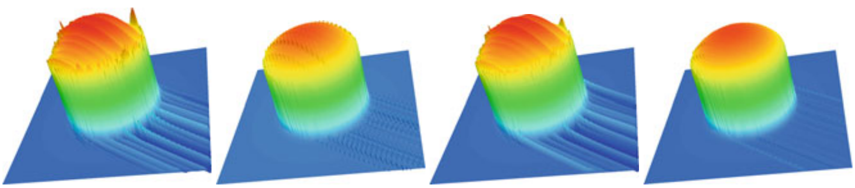

Fig. 2 Profile of the computed solution at $t = 0.5$ without ($p = 1$, $p = 4$) and with shock-capturing stabilization ($p = 1$, $p = 4$) for Example 3.2 with $h = 5.52\text{e-}3$; *from left to right*

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