Chapter 3
Basic Properties of Indicators

The aim of this section is to review so me of the basic properties of indicators that will be needed later. We let F be a random set (i.e. all the points belonging to one particular facies). Let $\overline{F}$ be its complement (i.e. the points that do not belong to F). The indicator function for the set F takes the value 1 at all the points inside F; it takes the value 0 elsewhere. This indicator function is denoted by $1_F(x)$. In the Fig. 3.1, the set F has been shaded. Its complement $\overline{F}$ includes all of the non-shaded area. The set F could be any shape or form, and need not be a single piece. It could be split into several parts.

The first property of indicator functions relies on the fact that inside F its indicator takes the value 1 whereas the indicator of its complement takes the value zero. Outside F the converse is true. That is, the union of F and its complement fills the whole space. Consequently,

$$1 = 1_{F \cup \overline{F}}(x) = 1_F(x) + 1_{\overline{F}}(x)$$

One property of a geological facies that can be measured experimentally is the proportion of space that it occupies. If we let $P_F(x)$ be the probability that point x lies in F, then it is equal to the mean or mathematical expectation of $1_F(x)$. That is,

$$P_F(x) = E[1_F(x)]$$

As the random function $1_F(x)$ takes the values 0 or 1, its expected value must lie between these two values.

$$0 \leq P_F(x) \leq 1$$

Moreover, because of the linearity of taking expectations in,

$$P_F(x) + P_{\overline{F}}(x) = 1$$

Now we calculate its variance. Since $[1_F(x)]^2 = 1_F(x)$,

$$\mathrm{Var}\,[1_F(x)] = E[1_F(x)] - \{E[1_F(x)]\}^2 = P_F(x)(1 - P_F(x)) \leq 0.25$$

M. Armstrong et al., *Plurigaussian Simulations in Geosciences*,
DOI 10.1007/978-3-642-19607-2_3, © Springer-Verlag Berlin Heidelberg 2011

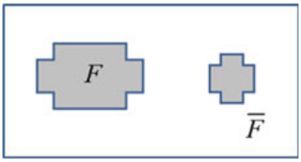

Fig. 3.1 Set F (*shaded grey*) and its complement $\overline{F}$ in *white*

Spatial Covariances, Variograms and Cross-Variograms

In this section we focus on the spatial relation between indicators by studying their centred and non-centred covariances, then their variograms and cross-variograms. After defining these concepts we derive their key properties. The first step is to study the relationship between the facies at point x and the one at y, firstly via the (non-centred) spatial covariance $C_F(x, y)$ and then via the variogram $\gamma_F(x, y)$.

Spatial Covariances

Let x and x + h be any two points. Their non-centred covariance is defined as:

$$\begin{aligned} C_F(x, x+h) &= E[1_F(x)\, 1_F(x+h)] = P[1_F(x) = 1 \,\&\, 1_F(x+h) = 1] \\ &= P[x \in F \,\&\, x+h \in F] \end{aligned}$$

It measures the probability that both points lie inside F (Fig. 3.2). Similarly the centred covariance $\sigma_F(x, x+h)$ and the centred cross covariance between facies F_i and F_j, denoted by $\sigma_{F_iF_j}(x, x+h)$ are respectively defined as

$$\begin{aligned} \sigma_F(x,\ x+h) &= E\{[1_F(x) - P_F(x)]\,[1_F(x+h) - P_F(x+h)]\} \\ &= C_F(x, x+h) - P_F(x)P_F(x+h) \\ \sigma_{Fi\,Fj}(x,\ x+h) &= E\{[1_{Fi}(x) - P_{Fi}(x)][1_{Fj}(x+h) - P_{Fj}(x+h)]\} \\ &= C_{Fi\,Fj}(x, x+h) - P_{Fi}(x)P_{Fj}(x+h) \end{aligned}$$

Variograms and Cross-Variograms

Let A(x) be an arbitrary variable. It could be an ordinary variable Z(x) or an indicator $1_F(x)$. Its variogram, denoted by γ_A, is defined as:

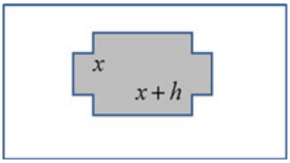

Fig. 3.2 Two points, x and x + h, lying in facies F

$$\gamma_A(x,\ x+h) = \frac{1}{2}\text{Var}\,[A(x) - A(x+h)]$$
$$= \frac{1}{2}\left\{E([A(x) - A(x+h)]^2) - (E[A(x) - A(x+h)])^2\right\}$$

Consequently the indicator variogram of the facies F, denoted by γ_F, is just:

$$\gamma_F(x,\ x+h) = \frac{1}{2}\text{Var}\,[1_F(x) - 1_F(x+h)]$$
$$= \frac{1}{2}\left\{E([1_F(x) - 1_F(x+h)]^2) - (E[1_F(x) - 1_F(x+h)])^2\right\}$$

Note that if the indicator is stationary, the second term disappears. The cross variogram between facies F_i and F_j is defined as:

$$\gamma_{F_iF_j}(x, x+h) = \frac{1}{2}E\left\{[1_{F_i}(x) - 1_{F_i}(x+h)][1_{F_j}(x) - 1_{F_j}(x+h)]\right\}$$

When this product is expanded, two of the four terms disappear because 1_{F_i} and 1_{F_j} cannot both take the value 1 at the same point since we have one and only one facies at each point. This gives:

$$\gamma_{F_iF_j}(x, x+h) = -\frac{1}{2}\left\{E[1_{F_i}(x)\ 1_{F_j}(x+h)] + E[1_{F_j}(x)\ 1_{F_i}(x+h)]\right\}$$

Variogram Properties

Property 1

As the indicators can only take the values 0 or 1, the values of the variogram must satisfy the inequality:

$$\gamma_F(x, x+h) = \frac{1}{2}\text{Var}[1_F(x) - 1_F(x+h)] \le 0.5$$

One consequence of this is that indicator variograms must be bounded. So a power function could never be an appropriate model for an indicator variogram.

Property 2: Variograms for two facies (stationary case)

$$\gamma_{F}(h) = \gamma_{\overline{F}}(h)$$

Proof. By definition the variograms of F and $\overline{F}$ are

$$\gamma_{F}(h) = \frac{1}{2}\, E[1_{F}(x+h) - 1_{F}(x)]^{2} \quad \text{and} \quad \gamma_{F}(h) = \frac{1}{2}\, E[1_{F}(x+h) - 1_{F}(x)]$$

The first result follows because

$$1_{F}(x+h) - 1_{F}(x) = -\left(1_{\overline{F}}(x+h) - 1_{\overline{F}}(x)\right)$$

Property 3: Covariances for two facies

If F is a stationary random set, its variogram depends on the vector between the two points, but not on their positions, and similarly for the covariances. The next property follows from this.

$$\sigma_{\overline{F}}(h) = \ \sigma_{\overline{F}}(h) = -\ \sigma_{F\overline{F}}(h) = -\ \sigma_{\overline{F}F}(h)$$

Proof. By definition, we have

$$\sigma_{F}(h) = E\{[1_{F}(x+h) - P_{F}(x+h)][1_{F}(x) - P_{F}(x)]\}$$

Because of the stationarity, $P_{F}(x+h) = P_{F}(x) = P_{F}$ and similarly for its complement. So:

$$\begin{aligned}
\sigma_{F}(h) &= E[1_{F}(x+h) - P_{F}]\,[1_{F}(x) - P_{F}] \\
\sigma_{\overline{F}}(h) &= E[1_{\overline{F}}(x+h) - P_{\overline{F}}]\,[1_{\overline{F}}(x) - P_{\overline{F}}] \\
\sigma_{F\overline{F}}(h) &= E[1_{F}(x) - P_{F}]\,[1_{\overline{F}}(x+h) - P_{\overline{F}}] \\
\sigma_{\overline{F}F}(h) &= E[1_{\overline{F}}(x) - P_{\overline{F}}]\,[1_{F}(x+h) - P_{F}]
\end{aligned}$$

Moreover

$$\begin{aligned}
&1_{F}(x+h) - P_{F} = 1 - 1_{\overline{F}}(x+h) - P_{F} = -\left(1_{\overline{F}}(x+h) - P_{\overline{F}}\right) \\
&1_{F}(x) - P_{F} = -\left(1_{\overline{F}}(x) - P_{\overline{F}}\right)
\end{aligned}$$

Substituting these relations into the covariance formulas gives the second result.

Comments

There are three practical consequences of these results. Firstly two complementary random sets are forcibly correlated. They cannot be independent of each other.

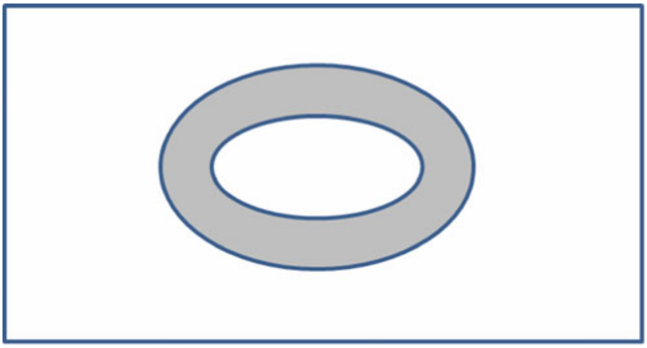

Fig. 3.3 The facies F and its complement do not have the connectivity but their indicator covariances are identical

Secondly their cross covariance is the same as the direct covariance up to a change of sign. Thirdly, the indicator covariance is not a connectivity index. A connectivity index tells us whether it is possible to find a path joining any two points in the facies and lying completely inside that facies, whereas the indicator covariance merely tells us whether to two end points are in the facies. It says nothing about paths joining the two points.

A facies F and its complement rarely have the same connectivity and yet their indicator covariances are identical. Figure 3.3 illustrates this point. The background is divided into two separate parts but the shaded facies is a single piece

Property 4: Three facies

Instead of two facies we now consider three facies labelled A, B and C, which together fill the whole space under study. See for example Fig. 3.4. At present we assume that the three random sets are stationary.

For these three facies:

$$\sigma_A(h) = -\,\sigma_{AB}(h) - \sigma_{AC}(h)$$
$$\sigma_B(h) = -\,\sigma_{BA}(h) - \sigma_{BC}(h)$$
$$\sigma_C(h) = -\,\sigma_{CA}(h) - \sigma_{CB}(h)$$
$$\sigma_{AB}(h) - \sigma_{BA}(h) = \sigma_{BC}(h) - \sigma_{CB}(h) = \sigma_{CA}(h) - \sigma_{AC}(h)$$

Proof. To prove this result, we note that any point x lies in one and only one of the three sets; that is:

$$1_A(x) + 1_B(x) + 1_C(x) = 1$$

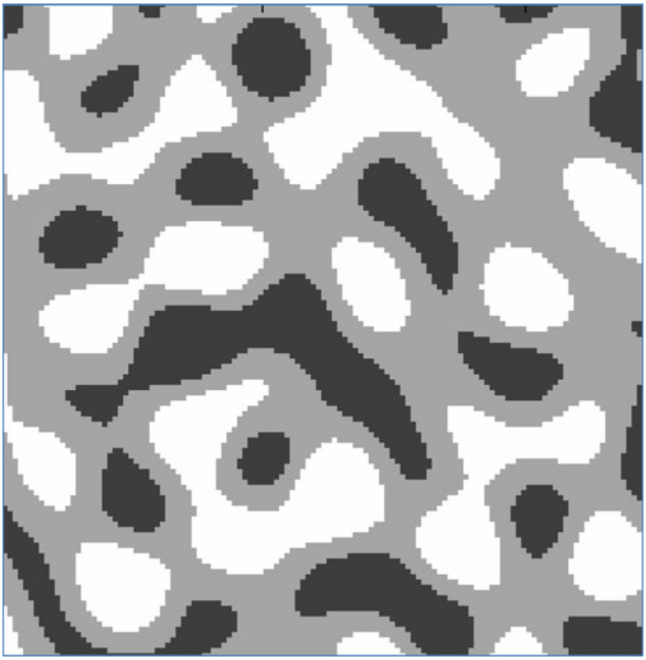

Fig. 3.4 Three facies A, B and C shown in *white*, *light grey* and *dark grey*

Consequently

$$P_A + P_B + P_C = 1$$

Hence

$$1_A(x) - P_A = -[1_B(x) - P_B + 1_C(x) - P_C]$$

We now substitute this into the second term of

$$\sigma_A(h) = E[1_A(x+h) - P_A][1_A(x) - P_A]$$

This gives

$$\sigma_A(h) = -E[1_A(x+h) - P_A][1_B(x) - P_B + 1_C(x) - P_C] = \sigma_{AB}(h) - \sigma_{AC}(h)$$

This proves the first three parts of property #3. Now to get the last part, we start from the definitions:

$$\sigma_{AB}(h) = E[1_A(x) - P_A]\,[1_B(x+h) - P_B]$$
$$\sigma_{BA}(h) = E[1_B(x) - P_B]\,[1_A(x+h) - P_A]$$

As before, we substitute the relation linking 1_A to $1_B + 1_C$ into these two, giving:

$$\sigma_{AB}(h) = -E[1_B(x) - P_B + 1_C(x) - P_C]\,[1_B(x+h) - P_B] = -\ \sigma_B(h) -\ \sigma_{CB}(h)$$
$$\sigma_{BA}(h) = -E[1_B(x) - P_B][1_B(x+h) - P_B + 1_C(x+h) - P_C] = -\ \sigma_B(h) - \sigma_{BC}(h)$$

Subtracting gives the result:

$$\sigma_{AB}(h) - \sigma_{BA}(h) = \ \sigma_{BC}(h) - \sigma_{CB}(h).$$

Property 5: Is it possible for two facies to be independent?

As A, B and C are a partition of the space, they cannot all be uncorrelated. But we can ask whether it is possible for two facies, A and B (say), to be uncorrelated. We are now going to show that this is not possible. If there were more than three facies, we could simply regroup all the other facies (except A and B) into a single facies C. So it suffices to prove the result for three arbitrary facies, A, B and C. Suppose that facies A and B are uncorrelated. Then

$$\sigma_{AB}(h) = E[1_A(x) - P_A]\ E[1_B(x+h) - P_B] = 0.$$

Similarly for $\sigma_{BA}(h)$. So

$$0 = \ \sigma_{AB}(h) - \sigma_{BA}(h),$$

hence

$$\sigma_{BC}(h) = \ \sigma_{CB}(h) \quad \text{and} \quad \sigma_{CA}(h) = \ \sigma_{AC}(h).$$

Consequently

$$\sigma_A(h) = 0 - \sigma_{AC}(h)$$
$$\sigma_B(h) = 0 - \sigma_{BC}(h)$$
$$\sigma_C(h) = \ \sigma_{CA}(h) - \sigma_{CB}(h).$$

From this it is easy to show that

$$\sigma_C(h) = \ \sigma_A(h) + \ \sigma_B(h).$$

The next step is to substitute $h = 0$ into this equation. Since $\sigma(0)$ is equal to the variance, we get

$$P_C(1 - P_C) = P_A(1 - P) + P_B(1 - P_B)$$

and because $P_C = 1 - (P_A - P_B)$

$$P_A P_B = 0.$$

This means that one of the two facies A or B is not present, which contradicts the initial hypothesis. This proves that facies cannot be independent.

Property 6: One consequence of the triangle inequality

Matheron (1987) proved that all indicator variograms must satisfy the triangle inequality for any two distances, h_1 and h_2:

$$\gamma(h_1 + h_2) \leq \gamma(h_1) + \gamma(h_2)$$

This tells us about the behaviour of indicator variograms near the origin. Suppose that the behaviour was a power function near $h = 0$; say $\gamma(h) \approx |h|^{\alpha}$. What values are possible for the power α? Let $h_1 = h = h_2$. Then

$$\gamma(h_1 + h_2) \approx |2h|^{\alpha} \text{ and } \gamma(h_1) + \gamma(h_2) \approx 2|h|^{\alpha}$$

Consequently α must be less than or equal to 1. *This means that the gaussian variogram should not be used as a model for an indicator variogram.* Two examples of the types of inconsistencies that can arise when unsuitable variogram models are used for indicators, are given in exercises 2.3 and 2.4 at the end of the chapter. The point in including these examples is to show just how difficult it is to give a general characterisation for the variogram of an indicator random function, or in other words, of a random set. Matheron (1987, 1989, 1993) produced several papers on this subject. The third one gives results concerning groups of 3, 4 and 5 points and then makes a conjecture to generalise these. These results are relevant in "multi-point" geostatistics.

Need for a Mathematically Consistent Method

In the previous section we saw that although indicators are very simple themselves, their properties are far from intuitive. For example, most people believe that it would be possible for some of the facies or lithotypes to be independent of each other, until they have seen the proof to the contrary. Nor do they realise that power functions and the gaussian model are not suitable models for indicator variograms. Although we have proved that these two models are not acceptable, this does not mean that the others can be used. It merely means that no counter examples have yet found to them. Perhaps someone will one tomorrow. This raises a troubling question: which variogram models can be used for indicators. Or putting it more generally, Bochner's theorem provides a general characterisation for positive definite functions (i.e. for the covariances of gaussian random functions, and hence their variograms). Is there an equivalent theorem for random sets and indicator variograms? To the best of our knowledge there is not.

So we are faced with two options: make ad hoc choices about the indicator variograms with the risk of ending up with mathematical inconsistencies, or construct a mathematically consistent model. We prefer the second option. The way that we have chosen to do this is, as we saw in Chap. 1, by truncating simulations of gaussian random functions in a suitable way. The advantages of using underlying gaussian random functions are clear: firstly, the normal (gaussian) distribution has many nice statistical properties; secondly, there are many ways of simulating gaussian random functions. The good properties, from a theoretical point of view, of the normal distribution and gaussian random functions include

- In simple kriging the error is orthogonal to the estimate, so it is independent of it. This allows us to condition simulations by kriging.
- Any positive definite function can be used as the covariance model of a gaussian random function. This is not true for other distributions, either discrete or continuous. We have already seen this for indicators which are discrete variables. In a similar vein, Armstrong (1992) shows that the spherical variogram is not always compatible with a lognormal distribution.

Transition Probabilities

From experimental data, it is easy to compute the proportion of samples belonging to a particular facies. Although it is a little more complex, we can also calculate the conditional probability of going from one facies to another, or of staying in the same facies for points a certain distance apart. We are now going to express these transition probabilities in terms of the indicator covariance and the proportions.

First Type of Transition Probability

The first type of transition probability is just the probability of being in facies F_j at point $x + h$ knowing that point x is in facies F_i:

$$\begin{aligned} P(x + h \in F_j | x \in F_i) &= P\left[1_{F_j}(x + h) = 1 \,|\, 1_{F_i}(x) = 1\right] \\ &= \frac{P\left[1_{F_j}(x + h) = 1 \;\&\; 1_{F_i}(x) = 1\right]}{P[1_{F_i}(x) = 1]} \\ &= \frac{E\left[1_{F_j}(x + h)\, 1_{F_i}(x)\right]}{E[1_{F_i}(x)]} = \frac{C_{ij}(x, x + h)}{p_{F_i}(x)} \end{aligned}$$

This is the probability of going from facies F_i to facies F_j, not being sure that we have a transition between x and x + h. This formula is also correct if j = i. If the facies are stationary, then this probability contains essentially the same information as the indicator variogram.

Example of the First Type of Transition Probability

To illustrate how to calculate this type of transition probability, we have selected four neighbouring drill-holes extending from one chronostratigraphic marker (N°11) down to the following one (N°10). Only two facies are present, A and B. Observations of the facies were made at 2 m intervals down the holes (Table 3.1).

As the drill-hole sections are of different lengths and as Marker N°10 is considered as the reference level (see Chap. 3 for more information on reference levels), the sections are given from this level upward rather than from marker N°11 downward, with blanks at the top. Our objective is to calculate the transition probabilities. By definition, these depend on the two points x and x + h. If the data are stationary horizontally (which is often the case), we can average along the rows. Similarly, if they are stationary vertically we can average in that direction but vertical stationarity is rare in sedimentary rock-types. Here as we only have four

Table 3.1 Facies observed at 2 m spacing going from marker N°11 down to marker N°10

Drill-hole N°92	Drill-hole N°58	Drill-hole N°39	Drill-hole N°84
A	–	–	–
B	–	–	–
A	A	–	–
B	A	–	–
A	A	B	–
A	A	A	–
A	A	B	–
B	A	B	–
B	A	A	–
B	B	A	–
B	A	A	–
B	A	B	B
B	A	B	B
B	B	B	A
B	B	B	A
B	B	B	B
B	A	B	A
B	A	A	B
B	B	B	A
A	A	B	A
B	A	B	B
A	B	B	B
B	B	B	B

Table 3.2 (a) Observed number of transitions for a distance of 2 m upward (out of a total of 71), (b) expected number of transitions assuming independence

(a)			(b)		
	Up to A	Up to B		Up to A	Up to B
From A	16	13	From A	13.7	17.5
From B	15	27	From B	17.5	22.3

Table 3.3 (a) Observed number of transitions for a distance of 10 m upward (out of a total of 55), (b) expected number of transitions assuming independence

(a)			(b)		
	Up to A	Up to B		Up to A	Up to B
From A	11	8	From A	10.7	13.5
From B	14	22	From B	13.5	17.3

drill-holes in this example, we will assume vertical and horizontal stationarity for simplicity.

Table 3.2a shows the numbers of each type of transition (i.e. A → A, A → B, B → A and B → B) for a distance of 2 m upward while Table 3.2b shows the number of transitions that would be expected, assuming a random distribution of facies. As 33 of the 75 observations are facies A, the Pr(A) = 0.44 and Pr(B) = 0.56. Consequently since there are 71 transitions each 2 m long, the expected number of transitions A → B or B → A is just 0.44 × 0.56 × 71. In fact, the observed number of transitions A → A and B → B is higher than would be expected if the facies were arranged randomly, which was expected because of the sequential structure of sedimentary facies at this scale. Similarly, Table 3.3a shows the numbers of each type of transition for a distance of 10 m upward while Table 3.3b shows the corresponding number of transitions that would be expected, assuming independence.

One difference between the observed values in the previous tables is that Table 3.2 is almost symmetric around the diagonal but Table 3.3 is definitely not. The lower line representing sections starting with facies B contains 36 out of 55 observations (i.e. 65% rather than 56%). This indicates that facies B tends to be lower in the stratigraphic sequence than facies A; that is, this is evidence of non-stationarity. So averaging along the vertical direction is not a valid operation in this case. This problem will be encountered again when calculating variograms.

Second Type of Transition Probability

Here we consider the case where there are three or more facies. We are interested in the probability of being in facies F_j at point x + h knowing that we are in facies F_i at point x but not at x + h. This probability is:

$$
\begin{aligned}
&P\left(x+h \in F_j \,|\, x \in F_i \ \ \&\ x+h \notin F_i\right)\\
&\quad = P\left[1_{Fj}(x+h)=1 \,|\, 1_{Fi}(x)=1 \ \&\ 1_{Fi}(x+h)=0\right]\\
&\quad = \frac{P\left[1_{Fj}(x+h)=1 \ \&\ 1_{Fi}(x)=1 \ \&\ 1_{Fi}(x+h)=0\right]}{P[1_{Fi}(x)=1 \ \&\ 1_{Fi}(x+h)=0]}\\
&\quad = \frac{E\left[1_{Fj}(x+h)\,1_{Fi}(x)[1-1_{Fi}(x+h)]\right]}{E\{1_{Fi}(x)[1-1_{Fi}(x+h)]\}}\\
&\quad = \frac{E\left[1_{Fj}(x+h)\,1_{Fi}(x)\right]}{E\{1_{Fi}(x)[1-1_{Fi}(x+h)]\}} = \frac{C_{ij}(x,x+h)}{P_{Fi}(x)-C_{ii}(x,x+h)},
\end{aligned}
$$

where $C_{ij}(x, x+h)$ is the non centred covariance between $1_{F_i}(x)$ and $1_{F_j}(x+h)$. This is the probability of going from facies F_i to facies F_j, when we know that the facies at x + h is not the same as at x. Clearly this probability cannot be computed if $j = i$.

Example

The second type of transition probabilities were calculated in the vertical direction for the image shown in Fig. 3.5, which was obtained by truncating a gaussian with an anisotropic gaussian variogram. The EW range was 50 units compared to 10 units in the NS direction. The graph shows the probability that the second point (x + h) lies in facies 2 (or facies 3) given that the first point (x) is in facies 1 and that the second one is not in facies 1 (right). We see that for short distance up to about 4 units the only possible transition is from facies 1 to 2. As the distance increases the probability that the second point lies in facies 3 rises steadily to about 0.35.

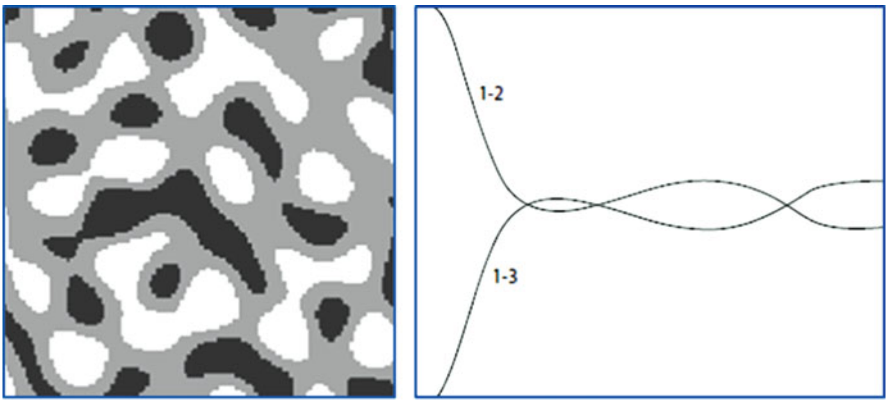

Fig. 3.5 Simulation of a gaussian random function with a gaussian variogram with an EW range of 50 units and a NS range of 10 units (*left*), and the probability of a point at position x + h being in facies 2 (or facies 3, respectively) given that the first point at position x lies in facies 1 and that the second one does not, plotted as a function of the distance between the two points (*right*)

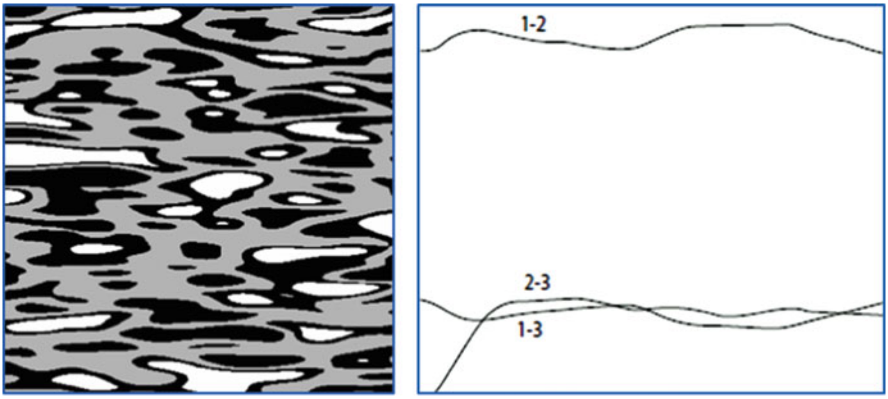

Fig. 3.6 Simulation of a gaussian random function with a gaussian variogram with an EW range of 50 units and a NS range of 10 units (*left*), and the probability of a point at position x + h being in facies 2 (or facies 3, respectively) given that the first point at position x lies in facies 1 and that the second one does not, plotted as a function of the distance between the two points (*right*)

As a contrast, the image shown in Fig. 3.6 was obtained by truncating the same gaussian random function using a different rock type rule. In the previous case, the rock type rule consisted of three intervals in the order: black, then grey and then white. Now there are four intervals: black, grey, black and white. The graph shows the corresponding transition probability. The essential difference between these two is that in the first case the black never touches the white whereas it does in the second case.

Exercises

The first two exercises are designed to familiarise readers with indicator random functions. The next two illustrate the types of inconsistencies that arise with unsuitable models for indicator covariances. The last exercise gives some rather interesting results on multi-point statistics.

Exercise 3.1

- Figure 3.7 shows 12 regularly spaced points in a 2D zone containing two facies (coloured grey and white). Let F denote the white facies. Write the values of the indicators of F and $\overline{F}$ in Table 3.4.
- Calculate the experimental mean and variance of the indicator for F and of its complement. Check that the means lie between 0 and 1, and that the variances are no greater than 0.25. Under what circumstances would the variances equal 0.25?
- Calculate the experimental variogram for facies F, and then for its complement for up to 3 lags in the EW direction. Plot these. Why are they the same? What shape do they have near the origin?

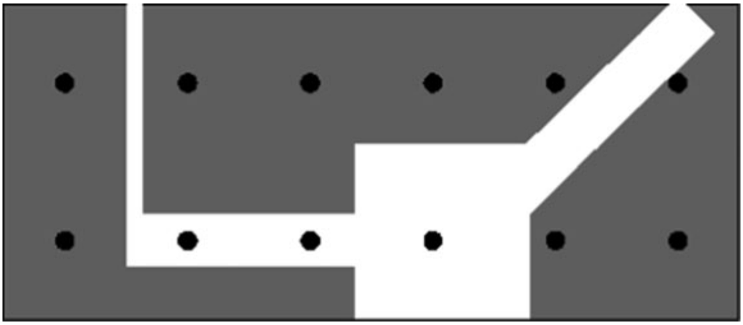

Fig. 3.7 Twelve samples are available in an area containing a facies *F* that is coloured in *white*. Its complement $\overline{F}$ is *shaded grey*

Table 3.4 Values of the indicator for the facies F and its complement

Point N°	1	2	3	4	5	6
$1_F(x)$						
$1_{\overline{F}}(x)$						
$1_F(x) + 1_{\overline{F}}(x)$						
Point N°	7	8	9	10	11	12
$1_F(x)$						
$1_{\overline{F}}(x)$						
$1_F(x) + 1_{\overline{F}}(x)$						

- The next step is to calculate the centred spatial covariances for F and its complement, then their cross covariances. Check that the direct covariances are identical and that the cross covariances are the negative of these.

Exercise 3.2

Assume that a facies F is stationary with probability $P_F = 0.6$ in the area under study. Show that the variance of the indicator $1_F(x)$ equals 0.24. Suppose that its variogram can be modelled as an exponential with a scale parameter equal "a".

- What is the value of its sill?
- Write down the equation for the variogram of its complement.
- Write down the equations for the centred covariance of the indicator for F and for its complement.
- Plot the non-centred covariances for these two indicators.

Exercise 3.3

Inappropriate models for indicator variograms. This example taken from Matheron (1987) and Armstrong (1992) highlights the types of inconsistencies that can arise when inappropriate models are chosen as indicator variograms. In this case we are going to show again that the gaussian variogram is unsuitable as a model for the variogram of the indicators.

Consider two points x_1 and x_3. Let x_2 be their midpoint. As each of the three indicators can take two values, 0 and 1, there are 8 possible combinations for the values of the three indicators. Let ω denote the probability that all three indicators take the value 1. That is,

$$\omega = \Pr[1_F(x_1) = 1_F(x_2) = 1_F(x_3) = 1]$$

As usual the non-centred covariance is:

$$C_{ij} = C_F(x_i, x_j) = E\left[1_F(x_i)1_F(x_j)\right] = \Pr\left[1_F(x_i) = 1_F(x_j) = 1\right]$$

Show that:

$$\begin{aligned}
\Pr[1_F(x_1) = 1_F(x_2) = 1 \ \& \ 1_F(x_3) = 0] &= C_{12} - \omega \\
\Pr[1_F(x_2) = 1_F(x_3) = 1 \ \& \ 1_F(x_1) = 0] &= C_{23} - \omega \\
\Pr[1_F(x_1) = 1_F(x_3) = 1 \ \& \ 1_F(x_2) = 0] &= C_{13} - \omega
\end{aligned}$$

Hint. Split the event $[1_F(x_1) = 1_F(x_3) = 1]$ into two mutually exclusive events according to the value of the third indicator e.g.

$$\Pr[1_F(x_1) = 1_F(x_2) = 1] = \Pr[1_F(x_1) = 1_F(x_2) = 1_F(x_3) = 1] + \Pr[1_F(x_1) = 1_F(x_2) = 1 \ \& \ 1_F(x_3) = 0]$$

Similarly show that

$$\begin{aligned}
\Pr[1_F(x_1) = 1 \ \& \ 1_F(x_2) = 0 \ \& \ 1_F(x_3) = 0] &= C_{11} - C_{12} - C_{13} + \omega \\
\Pr[1_F(x_1) = 0 \ \& \ 1_F(x_2) = 1 \ \& \ 1_F(x_3) = 0] &= C_{22} - C_{12} - C_{23} + \omega \\
\Pr[1_F(x_1) = 0 \ \& \ 1_F(x_2) = 0 \ \& \ 1_F(x_3) = 1] &= C_{33} - C_{13} - C_{23} + \omega
\end{aligned}$$

Hint. consider the event $[1_F(x_1) = 1 \ \& \ 1_F(x_2) = 1_F(x_3) = 0]$.

The contradiction becomes apparent if the three points are set 0.15 apart (compared to a scale parameter of 1) and if the probability that $1_{F_i}(x) = 1$ is 0.1.

$$C_{11} = C_{22} = C_{33} = 0.9$$

Calculate the value of the gaussian variogram for a distance of 0.15 and show that the following values are obtained for the noncentred covariance:

$$C_{12} = C_{23} = 0.1 \times 0.9 \times 0.494 = 0.0396 \ \text{and} \ C_{13} = 0.1 \times 0.9 \times 0.478 = 0.043$$

Show that $0 \leq 0.043 - \omega$ since $\Pr[1_F(x_1) = 1_F(x_3) = 1 \ \& \ 1_F(x_2) = 0] \geq 0$.

In the same way, use the fact that

$$\Pr[1_F(x_1) = 0 \ \& \ 1_F(x_2) = 1 \ \& \ 1_F(x_3) = 0] = C_{22} - C_{13} + C_{23} - \ \omega \geq 0$$

to prove that $\omega \geq 0.82$

Since it is impossible for $\omega \leq 0.043$ and $\omega \geq 0.821$, the gaussian variogram model is incompatible with indicator data.

Exercise 3.4

Another unsuitable variogram model. At first one might be tempted to think that the gaussian variogram is not allowable because it is infinitely differentiable or because it is quadratic near the origin. To show that this is not the case, Matheron produced a construction for a variogram model

$$f(x) = [a^2 e^{-ax} - b(2a - b)e^{-bx}]$$

where $a < b < 2a$ and $\omega^2 = 2ab - b^2$.

This is licit for indicators in 1-D provided that $\omega < b$. However he also proved that if $\omega > b$, the triangle inequality is not respected. To see this, let $b = 0.1$, $\omega = 1$ and $h = 0.2$.

Exercise 3.5

Multi-point Statistics. Covariances and variograms are examples of two point statistics. This exercise gives some theoretical results on multi-point statistics. Part (a) shows that once one of the three point covariances is known, all the others can be expressed in terms of it and of lower order statistics. Let

$$C_{FF\overline{F}}(x, y, z) = E(1_F(x) 1_F(y) 1_{\overline{F}}(z))$$

Show that:

$$\begin{aligned}
C_{FF\overline{F}}(x, y, z) &= -C_{FFF}(x, y, z) + C_{FF}(x, y) \\
C_{F\overline{F}\overline{F}}(x, y, z) &= C_{FFF}(x, y, z) - C_{FF}(x, y) - C_{FF}(x, z) + P_F(x) \\
C_{\overline{F}\overline{F}\overline{F}}(x, y, z) &= -C_{FFF}(x, y, z) + C_{FF}(x, y) + C_{FF}(x, z) + C_{FF}(y, z) + 1 \\
&\quad - P_F(x) - P_F(y) - P_F(z)
\end{aligned}$$

By permuting the order of the coordinates show that

$$C_{F\overline{F}F}(x, y, z) = -C_{FFF}(x, y, z) + C_{FF}(x, z)$$

Deduce similar expressions for the remaining three point statistics. (Exercise 2.3 is based on these.) We can now extend this result to higher order multi-point statistics. First we need to define the notation. Let

$$C_{FF}(x_1, x_2) = C_{F^2} \ \text{ and } \ C_{FFF}(x_1, x_2, x_3) = C_{F^3}$$

More generally, let the covariance for the case where the first (p−k) points belong to the facies F and the remaining k points to its complement be

$$C_{\underbrace{F\ldots F}_{(p-k)},\underbrace{\overline{F}\ldots\overline{F}}_{k}}(x_1,\ldots x_p) = C_{F^{p-k}\overline{F}^k}(x_1,\ldots x_p)$$

Show that for any two integers k and p

$$C_{F^{p-k},\overline{F}^k}(x_1,x_2,\ldots,x_p) = (-1)^k C_{F^p}(x_1,x_2,\ldots,x_p) + \Phi\big[(x_1,x_2,\ldots,x_p)\big]$$

The function Φ involves only lower order covariances $C_{F^{p-i}}$ and proportions. This can be deduced by expanding the appropriate product. For example, to find $C_{F\overline{F}\overline{F}}(x,y,z)$, we expand

$$F(x)[1-F(y)][1-F(z)] = F(x)F(y)F(z) - F(x)F(y) - F(x)F(z) + F(x)$$

This gives

$$C_{F\overline{F}\overline{F}}(x,y,z) = C_{FFF}(x,y,z) - C_{FF}(x,y) - C_{FF}(x,z) + P_F(x).$$

By permuting the coordinates we can show that any n-point statistics can be expressed using the basic n-point covariance $C_{F^p}(x_1,x_2,\ldots,x_p)$ and terms involving lower order covariances.

Exercise 3.6

Erosion model. Consider two models with two facies where A and D are the main facies and $\overline{A},\overline{D}$ their complements. Then form a model with three facies, A, B and C, by letting A erode D and its complement.

- Write the indicators of A, B, C using the indicators of A, and D.
- Use the previous results to compute the indicator covariances of A, B, C for the case when the facies A is independent of D.

Exercise 3.7

Porous medium. Consider a porous media. Let A be the random set describing the pores. The porosity of support V is the mean value of the indicator A on the volume V centred at the point x. Compute the analytical form of

- The covariance of the porosity of support V.
- The covariance of another indicator F with the porosity of support V.