Vibrations

Vibrations of membranes and beams are important in microtechnique. The frequency range of pressure sensors and microphones is limited by the resonance frequencies of their membrane. In a similar way, the resonance frequency of beams limits the possible applications of acceleration sensors. On the other hand, the resonance frequency of a vibrating element may be proportional to the measurand and allow measurements less affected by noise.

Figure 59 shows the amplitude of a vibrating structure as a function of the exciting frequency. There are two frequency ranges which usually are important for the designer: The range well below the lowest resonance frequency – the so-called fundamental frequency f_1, and the range in the near of the fundamental frequency. The former range is desirable when a sensor or an actuator shall be independent of the frequency. For example, the deflection of the membrane of a pressure sensor shall not be enhanced by resonance effects. Therefore, pressure sensors are limited to a certain frequency range. On the other hand, actuators may achieve larger deflections when driven at their resonance and the resonance frequency may be employed to measure certain quantities such as forces and masses.

The classical way to calculate the resonance frequencies of a body is to set up the equilibrium of forces in the form of a differential equation and to solve that equation. For example, the differential equation for calculating the *vibrations of a membrane* neglecting bending moments and stress due to straining of the neutral fiber is:

$$\sigma_0 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) = \rho_M \frac{\partial^2 w}{\partial t^2}. \tag{127}$$

The left side of this equation is the elastic force due to the residual stress σ_0 of a rectangular membrane with a deflection w(x, y, t) which is a function of x- and y-coordinate and time t. On the right side of the equation is the inertial force according to Newton's law where the mass is represented by the density ρ_M of the membrane. This differential equation is solved by the following ansatz:

$$w_{n,m}(x, y, t) = A_{n,m} \sin\left(\frac{n\pi}{a_M} \left[x + \frac{a_M}{2} \right] \right) \sin\left(\frac{m\pi}{b_M} \left[y + \frac{b_M}{2} \right] \right) \sin\left(\omega_{n,m} t + \Delta\varphi_{n,m} \right). \tag{128}$$

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DOI 10.1007/978-3-642-19489-4_9,

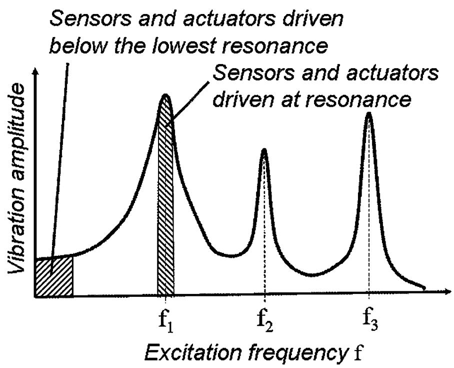


Fig. 59 Vibration amplitude as a function of excitation frequency

In this ansatz, a_M and b_M denote the length of the membrane in x- and y-direction, respectively, and $A_{n,m}$, $\omega_{n,m}$, and $\Delta\varphi_{n,m}$ are the amplitude, angular frequency, and phase of the vibration in the mode n,m, respectively. This ansatz can fulfill the above differential equation only if the frequency f adopts one of the following values:

$$f_{n,m} := \frac{\omega_{n,m}}{2\pi} = \frac{1}{2}\sqrt{\frac{\sigma_0}{\rho_M}}\sqrt{\frac{n^2}{a_M^2} + \frac{m^2}{b_M^2}}. \tag{129}$$

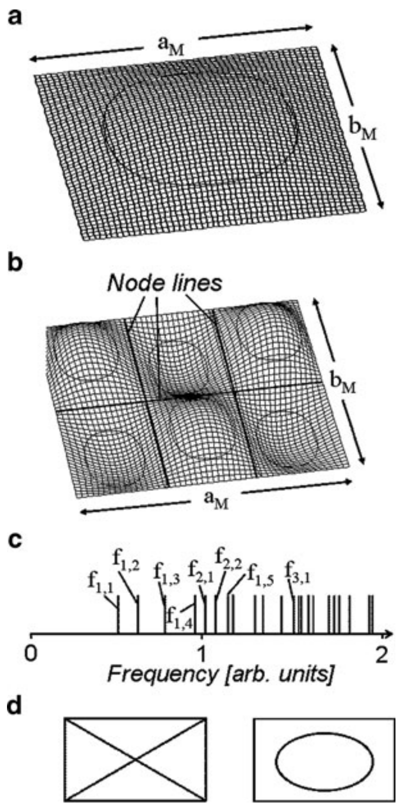
These values are the resonance frequencies of the membrane. The lowest possible frequency $f_{1,1}$ with $n = m = 1$ is the fundamental frequency. Each pair of natural numbers n and m denote a so-called vibration mode of the membrane and every mode corresponds to a certain resonance frequency. Figure 60 shows a deflected membrane vibrating at its fundamental frequency $f_{1,1}$ and at $f_{3,2}$. The numbers n and m correspond to the numbers of antinodes in x- and y-direction, respectively. In Fig. 60c, there are shown the resonance frequencies calculated with (129) with $a_M = 1$ and $b_M = 2.4$. There are even more resonances, because there are further solutions of (127) which are not included in the ansatz (128). The node lines (lines which are not deflected during vibration) of such resonances are shown in Fig. 60d.

In general, a membrane will vibrate at several resonance frequencies at the same time. The general solution of (127) is the sum of all solutions possible. If we restrict ourselves to solutions according to the ansatz (128), the general deflection w(x, y, t) is given by:

$$w(x, y, t) = \sum_{n,m} w_{n,m}(x, y, t). \tag{130}$$

The variety of frequencies that appear above the fundamental frequency impede employing higher resonances for sensors and actuators, because it is difficult to make sure that the desired resonance is in use and not a neighboring one. Therefore,

Fig. 60 Vibrations of rectangular membranes: (**a**) fundamental frequency $f_{1,1}$; (**b**) $f_{3,2}$; (**c**) frequency spectrum calculated with (129); and (**d**) additional nodal lines



in most applications in microtechnique, it is enough to know the fundamental frequency and the effects which may change it.

Equation (129) shows that, besides the geometry, the fundamental frequency is a function of membrane stress and density, i.e., changes of the geometry, density, and stress can be determined by measuring the fundamental resonance frequency of a membrane. On the other hand, changes of these properties may interfere measurements of this kind. An example is *micromembrane sensors* [39]. Figure 61 shows a break through such a sensor. A membrane is separated by a microstructure into micromembranes which due to their small dimensions show a fundamental resonance frequency of some MHz, which is in the range of ultrasound used for medical diagnosis.

The micromembranes can be excited to vibrations by an acoustic pulse generated by an ultrasonic transducer. After this, the membranes emit ultrasound at their resonance frequency. Figure 62a shows the setup of an experiment in which micromembranes were placed on the bottom of a vessel filled with water. The micromembranes were excited by a transducer pulse and the same transducer picked up the acoustic response from the membranes which is shown in Fig. 62d.

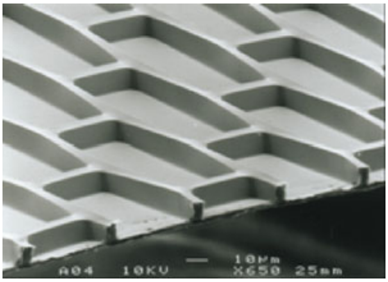


Fig. 61 Break through a micromembrane sensor. (Courtesy of Karlsruhe Institute of Technology, KIT)

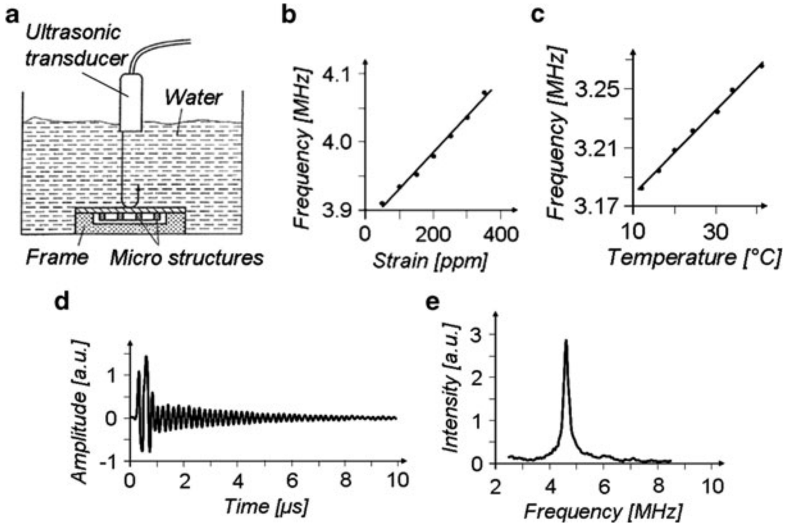


Fig. 62 Experimental test setup for micromembrane sensors [39] (**a**); Measured resonance frequency as a function of strain (**b**) and temperature (**c**); Measured signal from the micromembranes (**d**); and Fourier transformation of the second part of that signal (**e**). (Courtesy of Karlsruhe Institute of Technology, KIT)

The first part of the signal is the excitation pulse reflected from the membranes and the second part of the signal is the sound emission from the micromembranes. From this second part of the signal, a Fourier transformation was calculated which yields the resonance frequency (cf. Fig. 62e).

According to (129), the resonance frequency is a function of membrane stress which may be a function of temperature also, if the thermal expansion of the housing of the membranes is different from the one of the membranes. Figure 62b, c shows the resonance frequency measured as a function of straining and temperature [39].

When (129) was derived, residual stress was taken into account only. Stress due to straining of the neutral fiber and bending moments influences the resonance frequency also. Therefore, this equation is only correct for thin membranes with comparatively small deflections and large residual stress. The stress due to straining of the neutral fiber is only important for membranes vibrating with large amplitudes. This does not occur often, and, therefore, here is not discussed further. The bending moments are included employing the *Rayleigh method* [40]. Similar to the Ritz method (cf. page 34), the energy of the vibrating membrane is calculated. When an elastic body or structure is vibrating, its energy is transformed back and forth between potential and kinetic energy. If damping is neglected, the maxima of potential and kinetic energy are equal. From this equation, the resonance frequency can be calculated.

Figure 63 shows a simple example. A mass affixed to the end of a spring (spring constant k) is vibrating around its rest position. The maximum potential energy $V_{p,m}$ of this system is reached at the extreme positions of the deflections where the velocity is zero and the maximum kinetic energy $E_{kin,m}$ is achieved while passing its idle position with maximum velocity v_{max}:

$$V_{p,m} = \frac{k}{2} w_0^2 = E_{kin,m} = \frac{m_K}{2} v_{max}^2. \tag{131}$$

The position of every vibrating body is described by an equation of the following type:

$$w(t) = w_0 \sin(\omega\, t + \varphi) \Rightarrow v(t) = \frac{\partial w}{\partial t} = w_0\, \omega \cos(\omega\, t + \varphi). \tag{132}$$

The maximum of the velocity is reached when the cosine function equals 1. Therefore, (131) becomes now:

$$\frac{k}{2} w_0^2 = \frac{m_K}{2} w_0^2\, \omega^2 \Rightarrow \omega = \sqrt{\frac{k}{m_K}}. \tag{133}$$

This result could have been obtained from the calculation of the equilibrium of forces with the inertial force of Newton F = a m also, but for more complicated cases such as a membrane subject to both residual stress and bending moments, it is nearly impossible to set up the equilibrium of forces. Therefore, it is an advantage that it is enough to calculate the potential and kinetic energy of such a membrane.

The kinetic energy $E_{kin,m}$ is calculated from the integral over the kinetic energies of all infinitesimal elements of the membrane. The kinetic energy dE_{kin} of an

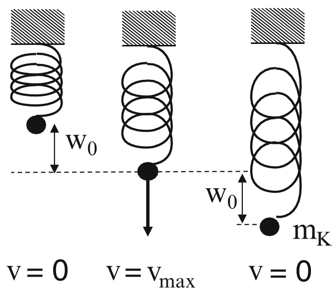
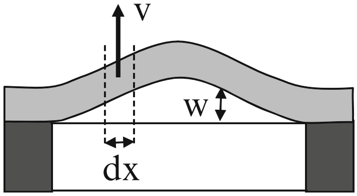


Fig. 63 Vibrating spring loaded with a mass

Fig. 64 Vibrating membrane



infinitesimal volume element d_M dx dy of a membrane with thickness d_M and density ρ_M is given by (cf. Fig. 64):

$$dE_{kin} = \frac{\rho_M\ d_M\ dx\ dy}{2} v^2 = \frac{\rho_M\ d_M\ dx\ dy}{2} \left(\frac{\partial w}{\partial t}\right)^2. \tag{134}$$

The total kinetic energy is the integral over this equation:

$$E_{kin} = \iint \frac{\rho_M\ d_M}{2} \left(\frac{\partial w}{\partial t}\right)^2 dx\ dy. \tag{135}$$

The membrane deflection is approximated now with the following ansatz:

$$w(x, y, t) = w_0\, u(x, y) \cos(\omega t + \varphi). \tag{136}$$

In this equation, u(x, y) denotes the part of the ansatz which is a function of the position and cos $(\omega t + \varphi)$ is the part which is a function of time. The kinetic energy in (135) is expressed now with the derivative of (136):

$$E_{kin} = \frac{\rho_M\ d_M}{2} w_0^2\ \omega^2\ \sin^2(\omega t + \varphi) \iint u^2\ dx\ dy. \tag{137}$$

The kinetic energy is maximal when the sine function is equal to one, and this needs to be equal to the maximum of the potential energy $V_{p,m:}$

$$E_{kin,m} = \frac{\rho_M\ d_M}{2} w_0^2\ \omega^2 \iint u^2\ dx\ dy = V_{p,m}. \tag{138}$$

This equation is solved for ω achieving the resonance frequency f of the membrane:

$$f = \frac{\omega}{2\ \pi} = \frac{1}{2\ \pi} \sqrt{\frac{2\ V_{p,m}(u)}{\rho_M\ d_M\ w_0^2 \iint u^2\ dx\ dy}}. \tag{139}$$

The maximum potential energy $V_{p,m}$ can be calculated with the Ritz method (44 or 45 on page 35). The unknown shape u(x, y) of the vibrating membrane is approximated with a reasonable ansatz containing unknown parameters. The unknown parameters are found then by calculating the minimum of the resonance frequency as a function of these parameters. Every deviation from the real shape of the vibrating membrane results in a higher frequency, and, therefore, the minimum approximates the real frequency [41]. With a suitable ansatz more than one relative minimum may be found. The least of these minima is an upper limit of the fundamental frequency and the larger ones correspond to higher-order resonances.

Typically, the static deflection shape $u_1(x, y)$ of the membrane is a good ansatz for the deflection shape of the fundamental resonance frequency. Here, the shape of a thick membrane statically deflected by a constant pressure difference is assumed as described by (23) (page 29):

$$u_1(r) = \left(1 - \frac{r^2}{R_M^2}\right)^2. \tag{140}$$

With this ansatz, the only unknown parameter in (139) is the maximum deflection w_0 of the vibrating membrane also called the amplitude of the vibration. When the potential energy is inserted into (139) from (44) or (45) (page 35) neglecting the term of stress due to straining of the neutral fiber, the vibration amplitude is canceling out. Thus, the resonance frequency is obtained directly from (139) and no further calculation of the minimum is required. For a circular membrane it is obtained:

$$f_1 = \sqrt{\frac{5}{3}}\frac{1}{\pi\, R_M\sqrt{\rho_M}}\sqrt{\frac{4}{3}\frac{E_M}{1-\nu_M^2}\frac{d_M^2}{R_M^2} + \sigma_0}. \tag{141}$$

The fundamental frequency of a rectangular membrane is found with the ansatz, which was employed for the static deflection of square membranes also [(65) on page 45]:

$$u_1(x, y) = \left(1 - 4\frac{x^2}{a_M^2}\right)^2\left(1 - 4\frac{y^2}{b_M^2}\right)^2 \tag{142}$$

and yields:

$$f_{1,1} = \frac{\sqrt{3}}{\pi\sqrt{\rho_M}}\sqrt{\frac{1}{2}\frac{E_M}{1-\nu_M^2}d_M^2\left(\frac{7}{a_M^4} + \frac{4}{a_M^2\, b_M^2} + \frac{7}{b_M^2}\right) + \left(\frac{\sigma_a}{a_M^2} + \frac{\sigma_b}{b_M^2}\right)}. \tag{143}$$

In these equations, σ_a and σ_b denote the stresses in the direction of the edges of the rectangular membrane with the lengths a_M and b_M, respectively. The stresses are assumed to be constant all over the membrane.

Figure 65 shows the fundamental frequency calculated with (141) as a function of membrane stress. A circular membrane with a radius and thickness of 0.5 mm and 5 µm, respectively, and a density, Young's modulus, and Poisson's ratio of 1 kg/L,

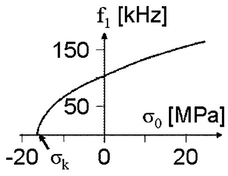


Fig. 65 Fundamental frequency of a circular membrane calculated with (141)

109 GPa, and 0.3, respectively, were assumed for this calculation. The figure and the equation show that the resonance frequency is decreasing with decreasing stress. At a certain compressive stress, the frequency becomes zero. This is the critical stress σ_k already known from (54) on page 41. Thus, (141) can be written in the form:

$$f_1 = \sqrt{\frac{5}{3}}\frac{1}{\pi\,R_M}\sqrt{\frac{\sigma_0-\sigma_k}{\rho_M}} \quad \text{with} \quad \sigma_k = -\frac{4}{3}\frac{E_M}{1-\nu_M^2}\frac{d_M^2}{R_M^2}. \tag{144}$$

Up to now, the medium surrounding the membrane has not been taken into account. Thus, the equations above are valid for membranes vibrating in vacuum only. If the membrane is in contact to a medium such as air or water, this fluid will vibrate together with the membrane and contribute to the moving mass. This is introduced by a so-called *additive mass* β_m. The following equation includes the effect of a fluid with density ρ_F which is in contact to one side of a circular membrane with density ρ_M [42]:

$$f_1 = \sqrt{\frac{5}{3}}\frac{1}{\pi\,R_M}\sqrt{\frac{\sigma_0-\sigma_k}{\rho_M(1+\beta_m)}} \quad \text{with} \quad \beta_m = \frac{2}{3}\frac{\rho_F}{\rho_M}\frac{R_M}{d_M}. \tag{145}$$

Most often the membrane will be in contact to the fluid on both sides, and, as a consequence, the additive mass needs to be doubled.

The effect of the additive mass was observed with micromembrane sensors as well. Figure 66a shows the measured frequency as a function of the density of a water solution. Potassium iodide (KI) was added stepwise enhancing the density with only little changes of the viscosity [39].

Up to now the damping of a vibrating membrane has not been taken into account here. The effect of the damping of a membrane (and any other vibrating body) is described by an exponential function. Thus, the maximum deflection is a function of time and the overall deflection becomes (cf. 136):

$$w(x,y,t) = A_0\, e^{-\alpha_D\, t}\, u(x,y)\cos(\omega\, t+\varphi), \tag{146}$$

$$w_0 = A_0\, e^{-\alpha_D\, t}, \tag{147}$$

where A_0 and α_D denote the vibration amplitude and the damping constant, respectively. Figure 67 shows the maximum deflection as a function of time according to the above equation.

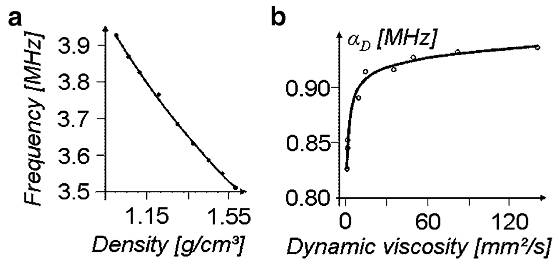


Fig. 66 (**a**) Measured resonance frequency of micromembrane sensors as a function of the density of the surrounding fluid and (**b**) damping coefficient as a function of the dynamic viscosity [39]

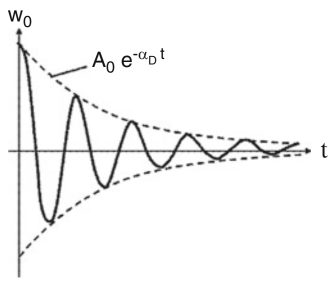


Fig. 67 Deflection of a vibrating body as a function of time

Damping lowers the resonance frequency also. The angular frequency ω of the damped vibration is described as a function of the undamped angular velocity ω_0 by the following equation:

$$2\,\pi\,f = \omega = \sqrt{\omega_0^2 - \alpha_D^2}. \tag{148}$$

There are several reasons for damping such as the emission of sound and the friction in the surrounding fluid. The friction of the fluid is a function of its viscosity as shown in Fig. 66b, which displays the measured frequency of micromembrane sensors as a function of the viscosity enhanced by stepwise adding polyvinylmethylether (PVM) which has approximately the same density as water but a much larger viscosity. The effect of friction may be enhanced by the squeeze film effect also (cf. page 119ff).

The *damping constant due to the emission of sound* into a fluid on one side of a circular membrane is calculated by the following equation [42]:

$$\alpha_D = \frac{5}{36}\,\frac{\rho_F}{\rho_M}\,\frac{R_M^2}{c\;d_M(1+\beta_m)^2}\,\frac{f_0^2}{4\,\pi^2}. \tag{149}$$

In this equation, c denotes the velocity of sound in the fluid and f_0 is the resonance frequency without damping due to sound emission. Often the membrane will be in contact to the fluid on both sides. As a consequence, both the additive mass needs to be doubled and the entire equation needs to be multiplied by two. This is illustrated very well by an experiment with micromembrane sensors. Instead of a sensor with entrapped air as shown in Fig. 62a (page 88), a sensor with a frame open at the reverse side was used (cf. Fig. 68a). As a consequence, the micromembranes were emitting ultrasound to both sides and damping was larger. As shown in Fig. 68b, the vibrations of this sensor were damped out much quicker compared with Fig. 62d.

Damping of a vibrating element may be an advantage. For example, micromembrane sensors and the membranes of loud speakers of hearing aids need to emit enough sound for good performance. An element which emits sound also absorbs it very well. Thus microphones need to show large damping by sound emission also. Equation (149) shows that a larger damping can be achieved by increasing the diameter of a membrane and employing thin membranes with a small density.

The resonance frequencies of *vibrating beams* are calculated in a similar way as for membranes. Again there is a differential equation which needs to be solved with the required boundary conditions to obtain the frequencies. This differential equation is:

$$E_B\ I \frac{\partial^4 w}{\partial x^4} + \rho_B\ A_B \frac{\partial^2 w}{\partial t^2} = 0, \tag{150}$$

where E_B, A_B, ρ_B, I, and x are Young's modulus, cross-sectional area, density, area moment of inertia of the beam, and the coordinate in the direction of the beam, respectively.

The solutions of this differential equation are:

$$w_i(x,t) = u_i(x) \cos(2\ \pi\ f_i\ t + \varphi_i). \tag{151}$$

In this equation, $w_i(x, t)$ describes the deflection of the beam as a function of the position x along the beam, the time t, and the parameter i which denotes the

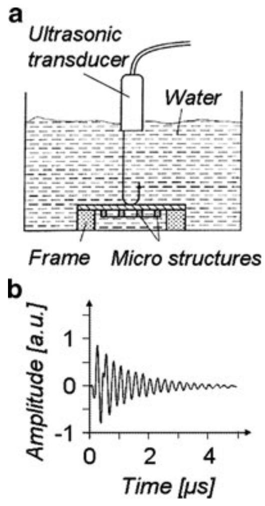


Fig. 68 Micromembrane sensor with an open frame (**a**) and signal measured with this sensor (**b**)

so-called mode of vibration. Every mode corresponds to a certain deflection shape $u_i(x)$ which needs to fulfill both the boundary conditions and the differential equation. The resonance frequencies are the distinct values at which the differential equation is solved. The general solution of the differential equation includes all resonance frequencies and is the sum of all possible solutions:

$$w(x,t) = \sum_i A_i\ w_i(x,t) = \sum_i A_i\ u_i(x) \cos(2\ \pi\ f_i\ t + \varphi_i). \tag{152}$$

The amplitudes A_i in this equation are a function of the excitation of the vibration.

The deflection shape solves the differential equation (150), if it is of the following form:

$$u_i(x) = a_1 \left[\cosh\left(\frac{\lambda_i\ x}{L_B}\right) + a_2 \cos\left(\frac{\lambda_i\ x}{L_B}\right) - \sigma_i \sinh\left(\frac{\lambda_i\ x}{L_B}\right) + a_3 \sin\left(\frac{\lambda_i\ x}{L_B}\right) \right], \tag{153}$$

In this equation, a_1 through a_3, σ_i, and λ_i are coefficients which are to be chosen such that the boundary conditions are fulfilled. These boundary conditions are that the deflection $u_i(x)$ of the beam and its first derivative are zero at $x = 0$ where the beam is clamped and that the curvature of the beam (its second derivative, cf. page 6) is zero at its free end:

$$u_i(0) = 0 \Rightarrow a_2 = -1 \tag{154}$$

$$\left.\frac{\partial w}{\partial t}\right|_{x=0} = 0 \Rightarrow a_3 = \sigma_i \tag{155}$$

$$\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=L_B} = 0 \Rightarrow \sigma_i = \frac{\cosh(\lambda_i) + \cos(\lambda_i)}{\sinh(\lambda_i) + \sin(\lambda_i)} = \frac{\sinh(\lambda_i) - \sin(\lambda_i)}{\cosh(\lambda_i) + \cos(\lambda_i)}. \tag{156}$$

A pair of shear forces F_S acting on a beam generates a bending moment (Fig. 69). If the axial distance at which the shear forces act is dx, the generated bending moment dM is:

$$dM = F_S\, dx \Rightarrow F_S = \frac{\partial M}{\partial x}. \tag{157}$$

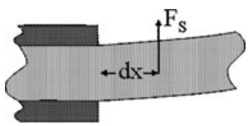


Fig. 69 Shear force generating a bending moment

The bending moment of a beam is described by (7) on page 7. Introducing this into the above equation, yields:

$$F_S = -E_B\, I \frac{\partial^3 w}{\partial x^3}. \tag{158}$$

Thus, as a consequence of the fact that no forces are acting on the free end of a beam, the third derivative of the deflection curve needs to be zero at the free end:

$$\left.\frac{\partial^3 w}{\partial x^3}\right|_{x=L_B} = 0 \Rightarrow \cos(\lambda_i)\cosh(\lambda_i) = -1. \tag{159}$$

The frequency parameters λ_i and σ_i calculated with these equations are shown in Table 7 with the precision of five digits. It is important to perform the calculations of the deflection shape with this accuracy, because otherwise the boundary conditions are not fulfilled.

a_1 is chosen such that the largest deflection which occurs along the beam is one. This way, the amplitude A_i in (152) corresponds to the largest deflection of the beam. As a consequence, the deflection shape of a beam clamped at one end is:

$$u_i(x) = \frac{1}{2}\left\{\cosh\left(\frac{\lambda_i\, x}{L_B}\right) - \cos\left(\frac{\lambda_i\, x}{L_B}\right) - \sigma_i\left[\sinh\left(\frac{\lambda_i\, x}{L_B}\right) - \sin\left(\frac{\lambda_i\, x}{L_B}\right)\right]\right\}. \tag{160}$$

When (151) and (160) are inserted into the differential equation (150), the following equation results, which yields the resonance frequencies of the beam:

$$\frac{\lambda_i^4}{L_B^4} - \frac{\rho\, A_B}{E_B\, I}(2\,\pi\, f_i)^2 = 0 \Rightarrow f_i = \frac{\lambda_i^2}{2\,\pi\, L_B^2}\sqrt{\frac{E_B\, I}{\rho_B\, A_B}}. \tag{161}$$

The deflection shape $u_i(x)$ of a vibrating beam, 1 mm in length, is shown in Fig. 70a for the first three modes as calculated with (160) and Table 7.

If a vibrating beam is to be employed as a sensor, strain gauges are a good way to measure the vibration amplitude and frequency. The resistance change as a function of the position along the beam can be calculated when the strain on the beam surface is known. The strain ε_B on the surface of the beam generated by bending is according to (5) on page 7:

$$\varepsilon_B = -\frac{d_B}{2}\frac{\partial^2 w}{\partial x^2}. \tag{162}$$

Table 7 Frequency parameters of beams clamped at one end

Mode i	λ_i	σ_i
1	1.8751	0.7341
2	4.6941	1.0185
3	7.8548	0.9992

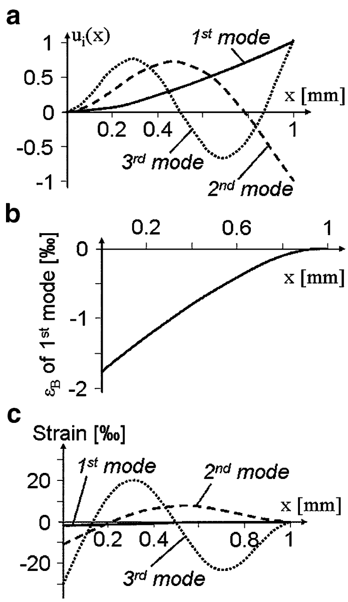


Fig. 70 Deflection shape (**a**) and surface strain (**b**) and (**c**) of a vibrating beam clamped at one end

Inserting (152) and (160) yields:

$$\varepsilon_B = -\frac{d_B}{2\,L_B^2}\sum_i\left[A_i\,\lambda_i^2\frac{1}{2}\left\{\cosh\left(\frac{\lambda_i\,x}{L_B}\right)+\cos\left(\frac{\lambda_i\,x}{L_B}\right)\right.\right.$$
$$\left.\left.-\sigma_i\left[\sinh\left(\frac{\lambda_i\,x}{L_B}\right)+\sin\left(\frac{\lambda_i\,x}{L_B}\right)\right]\right\}\cos(2\,\pi\,f_i\,t+\varphi_i)\right]. \qquad (163)$$

In Fig. 70b, c, the strain ε_B of the first three modes i is shown as calculated with this equation for the beam mentioned assuming a thickness of 100 μm and an amplitude of 10 μm. The resistance change of strain gauges mounted on top of the surface of the beam can be calculated by inserting (163) into (99) and (100) on page 68. If strain gauges are produced by doping of monocrystalline silicon, the resistance change is a function of crystal orientation also. The resistance changes then need to be calculated with (21) on page 22.

As both for strain gauges from silicon and from other materials, the resistance change is proportional to the strain, the largest effect may be expected from placing them in the near of the clamped end. As usual, strain gauges parallel and perpendicular to the beam may be combined compensating for temperature effects.

If the contribution of other modes than the desired one needs to be excluded from the measurement, an electronic band pass filter can be employed and/or the strain gauges are placed around the zeros of the strain of undesired modes. In Fig. 70c, this would be at 0.2 and 0.5 mm for suppressing the second and third mode, respectively. It could be possible also to measure the strain near the clamping point and to subtract values proportional to the signal of strain gauges at the antinodes of undesired modes.

Often there is a *mass fixed at the free end of a vibrating beam* fixed at one end. For example, most acceleration sensors consist of a beam with a seismic mass at their end (cf. page 83). If the mass is approximated by a point mass m_0 at the end of the beam and the mass of the beam is m_B, deflection shape $u_1(x)$ and fundamental frequency f_1 of this beam are calculated by:

$$u_1(x) = \frac{1}{2}\left(\left(1 - \frac{x}{L_B}\right)^3 + 3\frac{x}{L_B} - 1\right), \tag{164}$$

$$f_1 = \frac{1}{2\,\pi}\sqrt{\frac{3\ E_B\ I}{L_B^3(m_0 + 0.24\ m_B)}}. \tag{165}$$

If a *vibrating beam* is *clamped at both ends* and only bending moments are in action as elastic forces (i.e., there is no residual stress), the same differential equation (150) applies as for a beam clamped only at one end. The only difference is that the boundary conditions are changed. As a consequence, the same equations are valid for the deflection shape (160) and resonance frequencies (161). However, the changed boundary conditions result in a changed factor a_1 (= ½ in 160) and different frequency parameters λ_i and σ_i. They are derived from the boundary condition that both deflection and slope of the beam are zero where it is clamped. Thus, (154) and (155) need to be fulfilled not only at $x = 0$ but also at $x = L_B$. Zero deflection at the end of the beam results in:

$$\sigma_i = \frac{\cos(\lambda_i) - \cosh(\lambda_i)}{\sin(\lambda_i) - \sinh(\lambda_i)} \tag{166}$$

and the condition of zero beam slope at its end yields:

$$\cos(\lambda_i)\cosh(\lambda_i) = 1. \tag{167}$$

Table 8 shows the frequency parameters λ_i and σ_i calculated with these equations and Fig. 71a shows the deflection shape calculated with these parameters and (160).

Table 8 Frequency parameters of beams clamped at both ends

Mode i	λ_i	σ_i
1	4.7300	0.9825
2	7.8532	1.000777
3	10.9956	0.999966

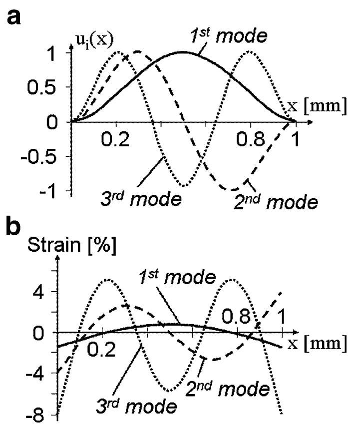


Fig. 71 Deflection shape (**a**) and surface strain (**b**) of a vibrating beam without residual stress clamped at both ends

The factor ½ was replaced by 0.63 for the first mode, and 0.66 for the second and third mode, achieving a maximum deflection of 1.

The strain at the surface of the beam is calculated with (163) as in the case of a beam clamped only at one end, but with the frequency parameters from Table 8. The resistance change of strain gauges on the beam is obtained by inserting (163) into (99) and (100) on page 68. Figure 71b shows the strain on the surface of the beam. The resistance change of strain gauges is proportional to this. Again the largest effect is found in the near of the clamping points. If specific modes are to be selected, as in the case of a beam clamped only at one end, it is possible to place strain gauges at the antinodes or the nodes of the corresponding modes or to use frequency filters.

If a certain mode of vibration shall be excited, it is important to deflect it with the right resonance frequency. As shown later in this book, there are several possibilities to generate a beam deflection such as local heating (page 164f), electrostatic actuation (page 131f), and piezos (page 144f). The vibration at a certain mode is facilitated by generating beam bending at the antinodes or by generating a deflection at these positions.

Sometimes the vibrations of a *beam with a concentrated mass at its center* need to be known for designing. The deflection shape of the fundamental mode of this beam is described by:

$$u_1(x) = 4\left(3\left(\frac{x}{L_B}\right)^2 - 4\left(\frac{x}{L_B}\right)^3\right). \tag{168}$$

This equation describes only half of the symmetrical deflection shape of the beam. The x-coordinate is starting at one clamping point. The fundamental resonance frequency of the beam is:

$$f_1 = \frac{4}{\pi}\sqrt{\frac{3\,E_B\ I}{L_B^3(m_0 + 0.37\ m_B)}}. \tag{169}$$

All the equations shown above for a beam clamped at both ends do not include residual stress. However, *residual stress* has a significant *influence on resonance frequencies* and cannot be avoided, in general. It can be approximately taken into account by a factor multiplied to the resonance frequency without residual stress [43]:

$$f_i(\sigma_0) = \sqrt{1 + \frac{\lambda_1^2}{\lambda_i^2}\frac{\sigma_0}{|\sigma_k|}}\ f_i(\sigma_0 = 0). \tag{170}$$

The frequency shift due to residual stress in a beam is shown in Fig. 72 as calculated with this equation. It is clearly seen that the fundamental frequency approaches zero when the compressive stress of the beam converges against the buckling load σ_k.

If the resonance frequency of a vibrating beam clamped at both ends shall be a measure of a force or a stress, respectively, it appears to be advantageous to employ the fundamental frequency of a beam near its critical stress σ_k, because the slope of the curve in Fig. 72 is largest there. If other parameters such as the mass of substances deposited on the beam shall be determined, it is desirable to avoid the effect of residual stress and a beam with more stress or vibrating at a higher mode may be preferred.

The *resonance frequencies* of all arrangements *of beams with a mass* fixed at their end which is much larger than the mass of the vibrating beam can easily be calculated when the force required for their deflection is known. Table 6 (page 80) shows the forces required for deflecting beams of several cases. The spring constant k of a certain case is the ratio of force F and deflection w_0, and the frequency can be calculated with (133) on page 89.

Torsional vibrations of microbeams occur easily, because the torque required to twist a beam by a certain angle φ (cf. Fig. 57 on page 78) is proportional to the third power of the linear dimensions [cf. (126) on page 79]. Thus, if torsional vibrations are undesired, this needs to be avoided by a suitable design. On the other hand,

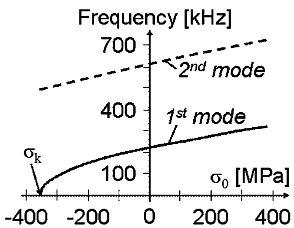


Fig. 72 Resonance frequencies of a vibrating beam as a function of residual stress calculated with (170) and (161)

torsional vibrations may be employed to measure small changes of a force or a torque.

Similar as in the case of transversal vibrations of beams [(152) on page 95] in general, the torsional angel θ of a beam clamped only at one end is described by a sum over wave functions θ_i, which need to fulfill both the boundary conditions and the differential equation:

$$\theta(x,t) = \sum_i A_i\ \theta_i(x,t) = \sum_i A_i\ u_i(x)\cos(2\ \pi\ f_i\ t + \varphi_i). \tag{171}$$

In this equation, A_i denotes the amplitude, $u_i(x)$ the deflection shape, f_i the resonance frequency, and φ_i a phase angle of the mode i of the torsional vibration. The boundary condition of a beam clamped only at one end is fulfilled, if

$$u_i(x) = \sin\left(\lambda_i \frac{x}{L_B}\right), \tag{172}$$

where λ_i is a solution of the following transcendental equation:

$$\cot(\lambda_i) = \frac{I_m}{\rho_B\ L_B\ I_t}\lambda_i. \tag{173}$$

In this equation, ρ_i, L_B, I_m, and I_t are the density, length, mass moment of inertia, and torsional constant (cf. Table 5 on page 78) of the beam, respectively. The mass moment of inertia is defined by the following integral:

$$I_m = \int r^2\, dm, \tag{174}$$

where r and dm are the distance from the axis of rotation and the infinitesimal mass element of the beam. The mass moments of inertia of beams often employed in microtechnique are shown in Table 9. For other cross-sections, the corresponding values are found in text books such as [32, 33].

Table 9 Mass momentums of inertia of homogeneous beams rotating around their length axis as a function of their cross-section

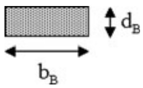
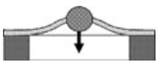
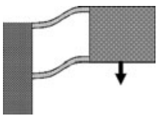
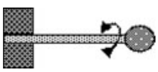
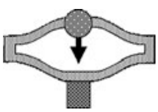
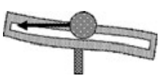
Form of cross-section		Mass momentum of inertia
Rectangle		$\frac{m_B}{12}\left(b_B^2 + d_B^2\right)$
Circle		$\frac{m_B}{2}R_B^2$

Table 10 Equations for the calculation of the resonance frequencies f of membranes and beams

Type	Equation
Circular membrane	$f_1 = \sqrt{\frac{5}{3}}\frac{1}{\pi\ R_M}\frac{1}{\sqrt{\rho_M}}\sqrt{\sigma_0 - \sigma_k}$ with $\sigma_k = -\frac{4}{3}\frac{E_M}{1-\nu_M^2}\frac{d_M^2}{R_M^2}$
Rectangular membrane	$f_{1,1} = \frac{\sqrt{3}}{\pi\sqrt{\rho_M}}\sqrt{\frac{1}{2}\frac{E_M}{1-\nu_M^2}d_M^2\left(\frac{7}{a_M^4}+\frac{4}{a_M^2\ b_M^2}+\frac{7}{b_M^4}\right)+\left(\frac{\sigma_a}{a_M^2}+\frac{\sigma_b}{b_M^2}\right)}$
Square membrane	$f_{1,1} = \frac{\sqrt{6}}{\pi\ a_M\sqrt{\rho_M}}\sqrt{\frac{\sigma_a+\sigma_b}{2}-\sigma_k}$ with $\sigma_k = -\frac{9}{2}\frac{E_M}{1-\nu_M^2}\frac{d_M^2}{a_M^2}$
Beam clamped at both ends, λ_i see Table 8 (page 98)	$f_i = \frac{\lambda_i^2}{2\ \pi\ L_B}\sqrt{\frac{E_B\ I}{L_B\ m_B}}\sqrt{1-\frac{\lambda_1^2}{\lambda_i^2}\frac{\sigma_0}{\sigma_k}}$ with $\sigma_k = -\frac{4\pi^2}{A_B}\frac{E_B\ I}{L_B^2}$
	$f_i = \frac{\lambda_i^2}{2\ \pi\ L_B}\sqrt{\frac{E_B\ I}{L_B\ m_B}}$, λ_i see Table 7 (page 96)
	$f_1 = \frac{\sqrt{3}}{2\ \pi\ L_B}\sqrt{\frac{E_B\ I}{L_B(m_0+0.24\ m_B)}}$
	$f_1 = \frac{4\sqrt{3}}{\pi\ L_B}\sqrt{\frac{E_B\ I}{L_B(m_0+0.37\ m_B)}}$
	$f_1 = \frac{1}{2\ \pi}\sqrt{\frac{E_B\ A_B}{m_0\ L_B}}$
	$f_1 = \frac{1}{2\ \pi}\sqrt{\frac{24\ E_B\ I}{m_0\ L_B^3}}$
	$f_i = \frac{\lambda_i}{2\ \pi\ d_B\ L_B}\sqrt{6\frac{E_B}{1+\nu_B}\frac{L_B\ I_t}{m_B}}$, λ_i from (173)
	$f_1 = \frac{1}{2\ \pi}\sqrt{\frac{96\ E_B\ I}{m_0\ L_B^3}}$
	$f_1 = \frac{1}{2\pi}\sqrt{\frac{(49\pm 5)E_B\ I}{m_0\ L_{Bc}^2\ L_B}}$

For I and I_t see Table 3 (page 67) and Table 5 (page 78), respectively

Table 10 summarizes the resonance frequencies of circular membranes and beams with different boundary conditions.

Exercises

Problem 14

The fluid surrounding a membrane and the temperature have an effect on the fundamental frequency of a membrane.

(a) Calculate the fundamental frequency of two circular membranes in vacuum and in water. One membrane is from silicon and the other one from polyimide. The thickness of the membrane is 25 μm and the radius is 2 mm. The initial stress σ_0 is the same for both membranes ($\sigma_0 = 35$ MPa at 20°C).

(b) What is the fundamental frequency of the polyimide membrane, if the temperature is enhanced from 20 to 95°C? The polyimide membrane is stretched over a frame from steel. Take into account the effect of the thermal strain only, and neglect the geometrical changes of thickness and radius, and the change of the density surrounding the membrane.

(c) The density of the surrounding fluid is changed by the temperature change as well. This has an effect on the additive mass and in this way on the fundamental frequency. Calculate the changed fundamental frequency of the polyimide membrane in air and water, when the temperature is enhanced from 20 to 95°C. Take into account the change of the additive mass and the initial stress. Hint: The change of the density of a gas can be calculated with sufficient accuracy with the fundamental equation for gases which says that the density is proportional to the inverse of the absolute temperature [K].

(d) Which meaning do the above effects calculated by you have for the designer of membranes in microsystems?

Young's modulus of silicon	160 GPa	Young's modulus of polyimide	1.66 GPa
Poisson's ratio of silicon	0.23	Poisson's ratio of polyimide	0.41
Density of air at 20°C	1.2 g/L	Density of water at 20°C	998 g/L
Temperature coefficient of the polyimide membrane	45×10^{-6} 1/K	Temperature coefficient of the frame of the membrane	12×10^{-6} 1/K
Density of water at 95°C	962 g/L	Gas constant R_G	8.314 J/(mol K)
Density of silicon	2.3 g/cm^3	Density of polyimide	1.43 g/cm^3
Temperature coefficient of silicon	2.3×10^{-6} 1/K		

Problem 15

You have started a new job in a company producing sensors for the measurement of machine vibrations. The sensors are fabricated with silicon technology and consist of beams clamped at one end with strain gauges.

(a) First you shall get to know the vibration sensors. You get some sensors from the production line on which the strain gauges are not yet implemented. You investigate the geometrical dimensions of the beams with an SEM (scanning electron microscope) and find out that the beams have a width of 50 μm, a thickness of 25 μm, and are 605 μm long.
Please calculate the resonance frequencies of the first three modes.

(b) By what factor would the thickness or length of the beam need to be altered achieving a fundamental frequency of 5,820 Hz? What is the effect of the width of the beam?

(c) Where would you place strain gauges if the sensor shall be sensitive to the resonance frequency of the three lowest modes?

(d) At one customer of your company damages of the machine occur especially at a frequency which corresponds to the third mode of your beam. When these vibrations occur, the machine shall be switched off. Where would you place the strain gauges on the sensor described before to obtain a sensor which is especially sensitive to the third mode and insensitive to noise from the fundamental mode? Take the value from Fig. 70 on page 97.

(e) The most recent development of your company is a vibration sensor consisting of several silicon beams arranged parallel to each other (cf. Fig. E16). At the end of each beam, there is an enlarged area with a gold layer serving as a seismic mass. The output signal of this sensor as a function of frequency is shown in Fig. E17. Obviously, it is an advantage of this sensor that a wide range of vibration frequencies can be recorded.
The gold covered ends of the beams show dimensions of $200 \times 200\ \mu m^2$.
The silicon of the beam and the end mass show the same thickness as the beams of part (a), but the beam is 2.5 mm long. What mass is needed at the end of the beam to obtain a resonance frequency of 3 kHz? Calculate the thickness of the gold layer.

Hint: Assume for your calculation that the gold covered end would be concentrated as a point mass at the end of the beam.

(f) What is the advantage of the gold covered ends compared with beams without any extrastructure at their end but the same resonance frequency?

Density of silicon	2.32 g/cm^3	Young's modulus of silicon	160 GPa
Density of gold	19.3 g/cm^3		

Hint: Assume for your calculations that the sensor would vibrate in vacuum and that the surrounding medium would show no effect on the vibrations.

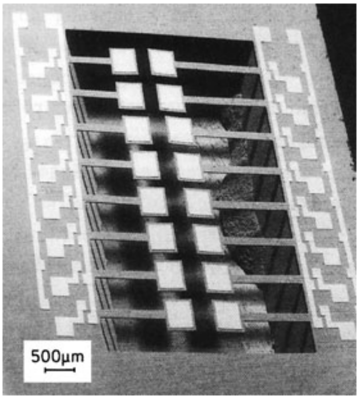


Fig. E16 Vibration sensor with several beams of different lengths and with gold covered seismic masses at their ends [44]

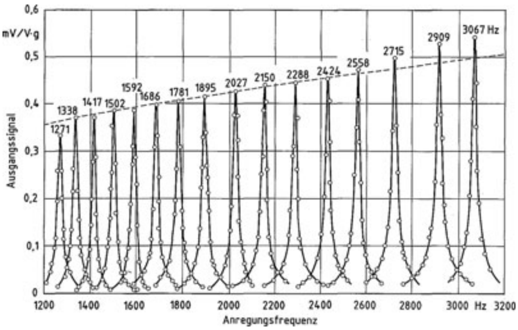


Fig. E17 Output signal of the vibration sensor [44] from Fig. E16

Problem 16

By microgravimetry the mass of very small particles and molecules can be measured. The change of the resonance frequency of membranes or beams as a function of the mass of these structures is employed to measure small masses in microgravimetry. A selective layer on the microstructure allows the binding of a certain kind of molecules only. This allows the analysis of DNA, RNA, and proteins. The performance of such microscales is demonstrated especially by so-called electronic noses: Even the binding of smell molecules on the surface of a vibrating membrane or a vibration beam results in a change of the resonance frequency, and, therefore, can be employed to recognize the presence of a certain smell.

The microscale manufactured by you consists of a silicon beam clamped at one end. With a piezo, the beam can be excited to vibrations. The beam shows a length of 200 µm, a width of 50 µm, and a thickness of 5 µm. The free end of the beam is covered on both sides with proteins on an area of $50 \times 50\ \mu m^2$. On these proteins, only the macromolecules of the type A can adhere.

(a) Please calculate the fundamental resonance frequency of the beam without any molecules on it.
(b) At the beginning of your measurement, you perform a reference measurement. You obtain the resonance frequency calculated at part (a) of this exercise. Now, you conduct an air stream with the macromolecules of interest over the beam and measure the resonance frequency once more. Assume that the area on both sides of the beam becomes covered with a 100-nm thick layer with a density of 1.2 g/cm^3.
What resonance frequency is measured now?

Hint:

- Assume that the molecules are distributed homogeneously on both sides of the beam and all remain to be adhered.
- Assume for the calculation that the additional mass of the macromolecules is attached to the end of the beam.

(c) How much does the resonance frequency change, when a dust particle with a mass of 10 μg would adhere at the end of the beam?

Hint: Assume for all problems of the exercise that you may use the equations of beam vibrations in vacuum. (In a real calculation, the effect of the surrounding air with its damping and additive mass would need to be taken into account.)

Young's modulus of silicon	160 GPa	Density of silicon	2.32 g/cm^3