Chapter 7
On one Model of Generalized Continuum and its Thermodynamical Interpretation

Elena A. Ivanova

Abstract We consider the mechanical model of a two-component medium whose first component is a classical continuum and the other one is a continuum having only rotational degrees of freedom. We show that the proposed model can be used for the description of thermal and dissipative phenomena. It is the presence of additional rotational degrees of freedom and, accordingly, additional inertia and elastic characteristics which can be interpreted as thermodynamical material parameters that distinguish the proposed model among other continuum models. In special cases the mathematical description of the proposed model is proved to reduce to the well-known equations such as the heat conduction, the self-diffusion and the coupled thermoelastic equations. The mathematical description of the proposed mechanical model includes not only the classical formulation of the coupled problem of thermoelasticity but also the formulation of the coupled problem of thermoelasticity with the hyperbolic type heat conduction equation. In the context of the introduced theory we consider the original model of internal damping.

Key words: Micropolar media. Two-component continuum. Hyperbolic thermoviscoelasticity.

7.1 Introduction

At present thermodynamics covers widespread frame including gas dynamics, thermoelasticity, thermoviscoelasticity, thermoelectric and thermomagnetic phenomena, phase changes and chemical reactions. At the same time it constitutes a set of scientific areas which are not connected to each other and differ by both the inter-

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pretation of the fundamental concepts and the applied mathematical methods. Dealing with the mathematical methods we should refer to the thermodynamic potential theory underlying the chemical and electrochemical thermodynamics, continuum mechanics within the framework of which the models of thermoelastic and thermoviscoelastic media have been developed, the methods of crystal lattice dynamics underlying the description of transport phenomena in solids, and also the classical and quantum statistics [1, 2]. In view of the aforesaid it is important to develop a unified theory for the description of all thermodynamical phenomena which are studied now in different science areas by using various methods. We are firmly convinced that it can be made on the basis of the fundamental laws of mechanics by using the continuum mechanics methods. The idea of the mathematical description of various physical phenomena in microcosm by using the continual models based on rotational degrees of freedom and the moment interactions was repeatedly asserted by P. A. Zhilin [3, 4, 5, 6] and other authors – see e. g. [7, 8, 9]. The model proposed in the present paper is a realization of this idea as applied to the description of thermal and dissipative phenomena.

There exist different macroscopic and microscopic models of internal damping [2, 10, 11, 12, 13]. The point of view that internal damping is concerned with thermal effects is widespread. According to the quantum theory [2], the distribution of phonons is in a local thermodynamical equilibrium and the temperature changes adiabatically, when acoustic wave propagates. Consequently, regions separated by the half-wavelength distance from one another have different temperatures and the irreversible heat flow between these regions arises as a result of the heat conduction phenomena. This process causes transfer of energy of mechanical vibrations into heat energy. We do not call in question the idea about interplay of the internal damping and thermal effects. We emphasize that analysis of the experimental values of the volume (acoustic) viscosity and the shear viscosity of various substances shows that the viscosities are independent substance characteristics which are not related to the heat-conduction coefficient and other thermodynamical parameters [14, 15, 16, 17]. However, we are sure that the internal damping and the heat conduction mechanism have the same physical nature. In our opinion the internal damping and heat conduction should be considered as a result of the interaction of atoms with the infinite surrounding medium which can be called the "thermal field" or the "thermal ether". We propose the mechanical model "thermal ether" which is a continuum of particles having translational and rotational degrees of freedom and interacting by elastic moments. We consider two problems of elastic interaction of the "thermal ether" with the particle having a special structure. As a result of analysis of the problems we show that the influence of the "thermal ether" on the particle can be modeled by the damping moment proportional to the angular moment of the particle. Using of the damping moment in the model of a two-component medium allows us to describe the internal damping and the heat conduction mechanism.

7.2 The Simplest Model of a Body-point

We consider the material system (see Fig. 7.1) consisting of the frame and N rigid bodies attached to the frame by means of elastic springs. For simplicity we suppose that all bodies can move only in the line of axis x and rotate only on axis x. We introduce following notations: m, J, x, φ are the mass, the moment of inertia, the displacement and the angle of rotation of the frame; m_i, J_i are the mass and the moment of inertia of rigid body number i; x_i, φ_i are the displacement and the angle of rotation of rigid body number i relative to the frame. The springs are considered to be elastic helical lines whose properties consist in the fact that when twisting in one direction they become longer and when twisting in the opposite direction they shorten. Conformably, when stretching and pressing the springs they become twisted in different directions. We suppose that the internal energy U_i of spring number i as well as the force F_i and the twisting moment M_i modeling the influence of spring number i on the frame take the form:

$$U_i = U_i(x_i + \chi\varphi_i), \qquad F_i = \frac{\partial U_i}{\partial x_i}, \qquad M_i = \frac{\partial U_i}{\partial \varphi_i}, \tag{7.1}$$

where χ is the coefficient, characterizing the difference of the elastic spring under consideration from analogous spring possessing axial symmetry. Objects similar to considered spring are usually called chiral objects. Therefore we call χ by coefficient of chirality.

As evident from Eqs (7.1), the force and the twisting moment can be represented by means of the derivative of the internal energy with respect to its argument $x_i + \chi\varphi_i$[1]. As a result the simple relation between M_i and F_i can be brought to light:

$$F_i = U_i', \qquad M_i = \chi U_i' \qquad \Rightarrow \qquad M_i = \chi F_i. \tag{7.2}$$

The equations of motion of the frame have the form:

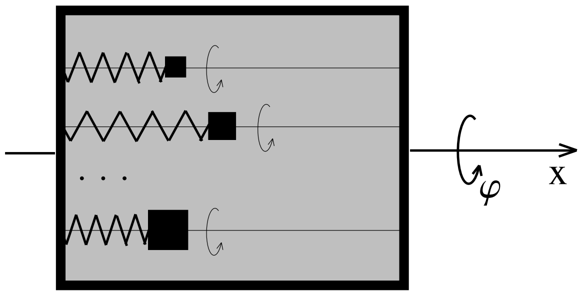

Fig. 7.1 Particle possessing the internal structure

[1] In what follows we denote this derivative by prime.

$$m\ddot{x} = F + \sum_{i=1}^{N} F_i, \qquad J\ddot{\varphi} = M + \sum_{i=1}^{N} M_i, \tag{7.3}$$

where F and M are the external force and the external twisting moment acting on the frame. The equations of motion of the rigid bodies are:

$$m_i(x+x_i)^{\cdot\cdot} = -F_i, \qquad J_i(\varphi+\varphi_i)^{\cdot\cdot} = -M_i, \qquad\qquad i = \overline{1,N}. \tag{7.4}$$

The analysis of Eqs (7.3) and (7.4) shows that translational and rotational motion of the frame are interdependent. If there is no external moment acting on the frame and the initial angular velocity is equal to zero, then because of the internal dynamics of the system the frame starts rotating. If there is no external force acting on the frame and the initial velocity is equal to zero, then because of the internal dynamics of the system the frame starts moving.

Example 1. We consider the free motion of the system, represented in Fig. 7.1. Let us introduce notations for total inertia characteristics of the internal bodies of the system:

$$m_* = \sum_{i=1}^{N} m_i, \qquad J_* = \sum_{i=1}^{N} J_i. \tag{7.5}$$

For simplicity we suppose that all rigid bodies have the same masses $m_i = m_*/N$ and the same moments of inertia $J_i = J_*/N$. Moreover, we suppose that the internal energies of the springs U_i are the quadratic forms of deformations and all springs have stiffness equal c. In that case:

$$U_i = c\,(x_i+\chi\varphi_i)^2 \qquad \Rightarrow \qquad F_i = c\,(x_i+\chi\varphi_i), \qquad M_i = \chi c\,(x_i+\chi\varphi_i). \tag{7.6}$$

Taking into account Eqs (7.6) we rewrite the equations of motion of the rigid bodies (7.4) in the form:

$$\frac{m_*}{N}\,(x+x_i)^{\cdot\cdot} = -c\,(x_i+\chi\varphi_i), \qquad \frac{J_*}{N}\,(\varphi+\varphi_i)^{\cdot\cdot} = -\chi c\,(x_i+\chi\varphi_i). \tag{7.7}$$

From Eqs (7.7) we obtain:

$$(x_i+\chi\varphi_i)^{\cdot\cdot} + k_*^2\,(x_i+\chi\varphi_i) = -(x+\chi\varphi)^{\cdot\cdot}, \qquad k_*^2 = Nc\left(\frac{1}{m_*}+\frac{\chi^2}{J_*}\right). \tag{7.8}$$

Provided that $F=0$, $M=0$ and F_i, M_i satisfy Eqs (7.6) the equations of the motion of the frame (7.3) take the form:

$$m\ddot{x} = c\sum_{i=1}^{N}(x_i+\chi\varphi_i), \qquad J\ddot{\varphi} = \chi c\sum_{i=1}^{N}(x_i+\chi\varphi_i). \tag{7.9}$$

The system of equations (7.9) can be rewritten in the more convenient form, namely in the form of equation

$$m\ddot{x} = \frac{J}{\chi}\ddot{\varphi}, \tag{7.10}$$

and equation

$$(x+\chi\varphi)^{\cdot\cdot} = \frac{\tilde{k}^2}{N}\sum_{i=1}^{N}(x_i+\chi\varphi_i), \qquad \tilde{k}^2 = Nc\left(\frac{1}{m}+\frac{\chi^2}{J}\right). \tag{7.11}$$

From Eqs (7.8) and (7.11) we obtain the equation in $x+\chi\varphi$:

$$(x+\chi\varphi)^{\cdot\cdot\cdot\cdot} + k^2(x+\chi\varphi)^{\cdot\cdot} = 0, \qquad k^2 = k_*^2 + \tilde{k}^2. \tag{7.12}$$

Taking into account the initial conditions we get the solution of the system (7.10), (7.12) in the form:

$$\begin{aligned} x(t) &= x|_{t=0} + \dot{x}|_{t=0}\,t \\ &+ \frac{c}{mk^2}\sum_{i=1}^{N}\left[(x_i+\chi\varphi_i)\Big|_{t=0}\Big(1-\cos(kt)\Big) + (x_i+\chi\varphi_i)^{\cdot}\Big|_{t=0}\left(t-\frac{\sin(kt)}{k}\right)\right], \end{aligned} \tag{7.13}$$

$$\begin{aligned} \varphi(t) &= \varphi|_{t=0} + \dot{\varphi}|_{t=0}\,t \\ &+ \frac{\chi c}{Jk^2}\sum_{i=1}^{N}\left[(x_i+\chi\varphi_i)\Big|_{t=0}\Big(1-\cos(kt)\Big) + (x_i+\chi\varphi_i)^{\cdot}\Big|_{t=0}\left(t-\frac{\sin(kt)}{k}\right)\right]. \end{aligned} \tag{7.14}$$

Now we introduce the quantities averaged over a period:

$$\begin{aligned} \bar{x}(t) &= \frac{k}{2\pi}\int_{t-\pi/k}^{t+\pi/k} x(\tau)d\tau \\ &= x|_{t=0} + \dot{x}|_{t=0}\,t + \frac{c}{mk^2}\left[\sum_{i=1}^{N}(x_i+\chi\varphi_i)\Big|_{t=0} + \sum_{i=1}^{N}(x_i+\chi\varphi_i)^{\cdot}\Big|_{t=0} t\right], \end{aligned} \tag{7.15}$$

$$\begin{aligned} \bar{\varphi}(t) &= \frac{k}{2\pi}\int_{t-\pi/k}^{t+\pi/k} \varphi(\tau)d\tau \\ &= \varphi|_{t=0} + \dot{\varphi}|_{t=0}\,t + \frac{\chi c}{Jk^2}\left[\sum_{i=1}^{N}(x_i+\chi\varphi_i)\Big|_{t=0} + \sum_{i=1}^{N}(x_i+\chi\varphi_i)^{\cdot}\Big|_{t=0} t\right]. \end{aligned} \tag{7.16}$$

Let us assume that we can observe on average values of the displacement and the angle of rotation of the frame. The motion of the rigid bodies inside the frame is not available for observation. In that case we will interpret the system under consideration as a single whole particle ("body-point"). Then quantities $\bar{x}(t)$ and $\bar{\varphi}(t)$ we will consider as characteristics of the position and the orientation of the particle. Now we discuss two variants of the initial conditions.

Variant 1. The stiffness of springs connecting the internal bodies and the frame is very large. In that case impact on the frame setting it in motion in the initial instant of time will set the internal bodies in the same motion. Then it is reasonable to assume that in the initial instant of time the relative displacements and angles of rotation as well as the relative velocities of the internal bodies are equal to zero:

$$x_i\big|_{t=0} = 0, \qquad \varphi_i\big|_{t=0} = 0, \qquad \dot{x}_i\big|_{t=0} = 0, \qquad \dot{\varphi}_i\big|_{t=0} = 0. \tag{7.17}$$

In that case expressions for $\bar{x}(t)$ and $\bar{\varphi}(t)$ are:

$$\bar{x}(t) = x|_{t=0} + \dot{x}|_{t=0}\, t, \qquad \bar{\varphi}(t) = \varphi|_{t=0} + \dot{\varphi}|_{t=0}\, t. \tag{7.18}$$

It is easy to see that the displacements and the angles of rotation determined by Eqs (7.18) are independent.
Variant 2. The stiffness of springs connecting the internal bodies and the frame is very small. Then impact on the frame setting it in motion in the initial instant of time will not be passed to the internal bodies. Therefore we can assume that in the initial instant of time the absolute displacements and angles of rotation as well as the absolute velocities of the internal bodies are equal to zero:

$$(x+x_i)\big|_{t=0} = 0, \quad (\varphi+\varphi_i)\big|_{t=0} = 0, \quad (\dot{x}+\dot{x}_i)\big|_{t=0} = 0, \quad (\dot{\varphi}+\dot{\varphi}_i)\big|_{t=0} = 0. \tag{7.19}$$

In that case expressions for $\bar{x}(t)$ and $\bar{\varphi}(t)$ take the form:

$$\begin{aligned} \bar{x}(t) &= \left[1 - \frac{\mu}{m}\right] \left(x|_{t=0} + \dot{x}|_{t=0}\, t\right) - \frac{\chi\mu}{m} \left(\varphi|_{t=0} + \dot{\varphi}|_{t=0}\, t\right), \\ \bar{\varphi}(t) &= \left[1 - \frac{\chi^2\mu}{J}\right] \left(\varphi|_{t=0} + \dot{\varphi}|_{t=0}\, t\right) - \frac{\chi\mu}{J} \left(x|_{t=0} + \dot{x}|_{t=0}\, t\right), \end{aligned} \tag{7.20}$$

where parameter μ having the dimension of mass calculated by the formula:

$$\mu = \left(\frac{1}{m} + \frac{1}{m_*} + \chi^2 \left[\frac{1}{J} + \frac{1}{J_*}\right]\right)^{-1}. \tag{7.21}$$

As evident from Eqs (7.20), the initial displacements and translational velocities of the frame have an influence on its rotational motion, and the initial angles of rotation and angular velocities of the frame influence have action upon its translational motion.

Thus, based on the considered example we conclude that the presence or absence of cross effect of the translational and rotational motions depend on the internal structure and the parameters of the system.
Example 2. We consider the motion of the system represented in Fig. 7.1 under the action of the external force and twisting moment being linear time functions:

$$F = A_F t, \qquad M = A_M t, \qquad A_F = \text{const}, \qquad A_M = \text{const}. \tag{7.22}$$

Taking into account Eqs (7.22) we write the equations of the frame motion (7.3) in the form:

$$m\ddot{x} = A_F t + \sum_{i=1}^{N} F_i, \qquad J\ddot{\varphi} = A_M t + \sum_{i=1}^{N} M_i. \tag{7.23}$$

As in preceding example, we suppose that all rigid bodies have the same masses $m_i = m_*/N$ and the same moments of inertia $J_i = J_*/N$. The elastic forces and moments characterizing the interaction of the rigid bodies and the frame are calculated by Eqs (7.6). The equations of motion of the internal bodies (7.7), as well as their sequent Eqs (7.8), are correct in the problem under discussion.

By using Eqs (7.6) for the forces F_i and the moments M_i the equations of motion of the frame (7.23) can be reduced to the equivalent system including the equation

$$m\ddot{x} - \frac{J}{\chi}\ddot{\varphi} = A_F t - \frac{A_M t}{\chi} \tag{7.24}$$

and equation

$$(x+\chi\varphi)^{\cdot\cdot} = \frac{A_F t}{m} + \frac{\chi A_M t}{J} + \frac{\tilde{k}^2}{N}\sum_{i=1}^{N}(x_i + \chi\varphi_i). \tag{7.25}$$

From Eqs (7.8) and (7.25) we obtain the equation in $x+\chi\varphi$:

$$(x+\chi\varphi)^{\cdot\cdot\cdot\cdot} + k^2(x+\chi\varphi)^{\cdot\cdot} = k_*^2\left(\frac{A_F t}{m} + \frac{\chi A_M t}{J}\right). \tag{7.26}$$

Solving Eq. (7.26) we get the following expression for the variable $(x+\chi\varphi)^{\cdot\cdot}$

$$(x+\chi\varphi)^{\cdot\cdot} = (x+\chi\varphi)\Big|_{t=0}\cos(kt) + \frac{1}{k}(x+\chi\varphi)^{\cdot}\Big|_{t=0}\sin(kt) + \frac{k_*^2}{k^2}\left(\frac{A_F t}{m} + \frac{\chi A_M t}{J}\right). \tag{7.27}$$

Now we suppose that the oscillation period is much smaller than an observing time on the motion process. In that case the characteristics of the motion averaged over a period is of interest for us:

$$\bar{x}(t) = \frac{k}{2\pi}\int_{t-\pi/k}^{t+\pi/k} x(\tau)d\tau, \qquad \bar{\varphi}(t) = \frac{k}{2\pi}\int_{t-\pi/k}^{t+\pi/k} \varphi(\tau)d\tau. \tag{7.28}$$

By averaging over a period Eqs (7.24) and (7.27) we obtain:

$$m\ddot{\bar{x}} - \frac{J}{\chi}\ddot{\bar{\varphi}} = A_F t - \frac{A_M t}{\chi}, \qquad \ddot{\bar{x}} + \chi\ddot{\bar{\varphi}} = \frac{k_*^2}{k^2}\left(\frac{A_F t}{m} + \frac{\chi A_M t}{J}\right). \tag{7.29}$$

Now we transform the system (7.29) to the following form:

$$
\begin{aligned}
&m\left(1+\frac{Jk^2}{\chi^2 m k_*^2}\right)\left(1+\frac{J}{\chi^2 m}\right)^{-1}\ddot{\ddot{x}}+\frac{J\tilde{k}^2}{\chi k_*^2}\left(1+\frac{J}{\chi^2 m}\right)^{-1}\ddot{\ddot{\varphi}}=A_F t,\\
&\frac{\chi m\tilde{k}^2}{k_*^2}\left(1+\frac{J}{\chi^2 m}\right)^{-1}\ddot{\ddot{x}}+J\left(1+\frac{Jk^2}{\chi^2 m k_*^2}\right)\left(1+\frac{J}{\chi^2 m}\right)^{-1}\ddot{\ddot{\varphi}}=A_M t.
\end{aligned}
\tag{7.30}
$$

Let us suppose that the mass and the moment of inertia of the frame are related by the formula

$$
J=\chi^2 m. \tag{7.31}
$$

We introduce following notations:

$$
\hat{m}=\frac{m}{2}\left(1+\frac{k^2}{k_*^2}\right),\qquad \hat{B}=\frac{\chi m\tilde{k}^2}{2k_*^2},\qquad \hat{J}=\frac{J}{2}\left(1+\frac{k^2}{k_*^2}\right). \tag{7.32}
$$

Taking into account Eqs (7.31) and (7.32) we rewrite the system (7.30) in the form:

$$
\hat{m}\ddot{\ddot{x}}+\hat{B}\ddot{\ddot{\varphi}}=A_F t,\qquad \hat{B}\ddot{\ddot{x}}+\hat{J}\ddot{\ddot{\varphi}}=A_M t. \tag{7.33}
$$

By comparison of Eqs (7.33) describing the behavior of the average over a period characteristics of the motion with the starting Eqs (7.23) we see that the influence of the internal structure of the system on the motion of the frame can be taken into account both by means of the internal forces and moments and with the aid of the additional inertial parameters ensuring the interplay of the translational and rotational motions.

Example 3. Now we study the motion of the considered system (see Fig. 7.1) under the action of the external force and twisting moment being periodic time functions:

$$
F=F_0\sin(\omega t),\qquad M=M_0\sin(\omega t),\qquad F_0=\text{const},\qquad M_0=\text{const}. \tag{7.34}
$$

Taking into account Eqs (7.34) we write down the equations of the motion of the frame (7.3) in the form:

$$
m\ddot{x}=F_0\sin(\omega t)+\sum_{i=1}^{N}F_i,\qquad J\ddot{\varphi}=M_0\sin(\omega t)+\sum_{i=1}^{N}M_i. \tag{7.35}
$$

After simple transformations similar to those carried out in the preceding example we reduce the equation of the motion of the frame to the system of equations

$$
\begin{aligned}
m\ddot{x}-\frac{J}{\chi}\ddot{\varphi}&=\left(F_0-\frac{M_0}{\chi}\right)\sin(\omega t),\\
(x+\chi\varphi)^{\cdot\cdot}&=(x+\chi\varphi)\Big|_{t=0}\cos(kt)+\frac{1}{k}(x+\chi\varphi)^{\cdot}\Big|_{t=0}\sin(kt)\\
&\quad+\frac{k_*^2-\omega^2}{k^2-\omega^2}\left(\frac{F_0}{m}+\frac{\chi M_0}{J}\right)\sin(\omega t).
\end{aligned}
\tag{7.36}
$$

Now we suppose that the free period is much smaller than the period of force oscillations. Introducing the average over a period characteristics of the motion (7.28) and averaging Eqs (7.36) over a period we obtain:

$$
\begin{aligned}
m\ddot{\bar{x}} - \frac{J}{\chi}\ddot{\bar{\varphi}} &= \frac{k}{\pi\omega}\sin\left(\frac{\pi\omega}{k}\right)\left(F_0 - \frac{M_0}{\chi}\right)\sin(\omega t),\\
\ddot{\bar{x}} + \chi\ddot{\bar{\varphi}} &= \frac{k(k_*^2-\omega^2)}{\pi\omega(k^2-\omega^2)}\sin\left(\frac{\pi\omega}{k}\right)\left(\frac{F_0}{m} + \frac{\chi M_0}{J}\right)\sin(\omega t).
\end{aligned}
\tag{7.37}
$$

We suppose that $\omega \ll k_*$ and, hence, $\omega \ll k$. Moreover, the mass and the moment of inertia of the frame are assumed to be related by Eq. (7.31). Then by using notations (7.32) we can rewrite the system (7.37) in the form:

$$
\hat{m}\ddot{\bar{x}} + \hat{B}\ddot{\bar{\varphi}} = F_0\sin(\omega t), \qquad \hat{B}\ddot{\bar{x}} + \hat{J}\ddot{\bar{\varphi}} = M_0\sin(\omega t). \tag{7.38}
$$

By comparison of Eqs (7.38) describing the behavior of the average over a period characteristics of the motion with the starting Eqs (7.35) we come to the conclusion that the result is the same to that obtained in the preceding example. Namely, the dynamics of the internal structure of the system has action upon the motion of the frame and the influence in question can be taken into account by means of the additional inertial parameters ensuring the interplay of the translational and rotational motions.

Example 4. Now we study the motion of the considered system (see Fig. 7.1) under the action of conservative load which is modeled by a linear elastic force. In that case the equations of the frame motion (7.3) take the form:

$$
m\ddot{x} = -C_F x + \sum_{i=1}^{N} F_i, \qquad J\ddot{\varphi} = \sum_{i=1}^{N} M_i, \tag{7.39}
$$

where C_F is the stiffness of the elastic spring. The equations of the frame motion (7.39) by using Eqs (7.6) and (7.8) can be reduced to the following system of equations:

$$
m\ddot{x} - \frac{J}{\chi}\ddot{\varphi} = -C_F x, \qquad (x+\chi\varphi)^{\cdot\cdot\cdot\cdot} + k^2(x+\chi\varphi)^{\cdot\cdot} = -\frac{C_F}{m}\left(\ddot{x} + k_*^2 x\right). \tag{7.40}
$$

It is easy to see that at the zero initial conditions lead the system (7.40) to the form:

$$
m\ddot{x} + \hat{B}\ddot{\varphi} = -C_F x, \qquad \hat{B}\ddot{x} + \hat{J}\ddot{\varphi} = -C_M\varphi, \tag{7.41}
$$

where constants $\hat{B}$, $\hat{J}$ and C_M are calculated by the formulae:

$$
\hat{B} = -\frac{J}{\chi}, \quad \hat{J} = J\left[\frac{J}{\chi^2 m} + \frac{C_F}{m\tilde{k}^2} + \frac{C_F J}{\chi^2 m^2 \tilde{k}^2}\right], \quad C_M = \frac{C_F J}{m\tilde{k}^2}\left(k^2 + \frac{J k_*^2}{\chi^2 m}\right). \tag{7.42}
$$

As evident from a comparison of Eqs (7.39) and (7.41) in the case of discussion we can take into account the influence of the dynamics of the internal structure by means of of the additional inertial parameters and of an external elastic moment proportional to the angle of rotation of the frame.

Let us consider Eqs (7.33), (7.38) and (7.41). The quantities on the right-hand side of the equations are the forces and the moments. Hence, the left-hand side of Eqs (7.33), (7.38) and (7.41) can be interpreted as the derivatives of the momentum and the angular momentum. Then the foregoing equations should be regarded as the equation of motion of the particle whose momentum K_1, the angular momentum K_2 and kinetic energy K are:

$$K_1 = \hat{m}\dot{x} + \hat{B}\dot{\varphi}, \qquad K_2 = \hat{B}\dot{x} + \hat{J}\dot{\varphi}, \qquad K = \frac{1}{2}\hat{m}\dot{x}^2 + \hat{B}\dot{x}\dot{\varphi} + \frac{1}{2}\hat{J}\dot{\varphi}^2, \tag{7.43}$$

Consequently, parameter $\hat{B}$ is the moment of inertia. The particle whose dynamic structures are defined by Eqs (7.43) is a special case of the body-point proposed by P. A. Zhilin – see [3].

7.3 Continuum of One-rotor Gyrostats

The material medium (see Fig. 7.2) consisting of one-rotor gyrostats is considered. A one-rotor gyrostat consists of a rotor concealed in a rigid body which is called "carrier body". A rotor can rotate independently of the carrier body rotation, but a rotor can not move independently the carrier body motion. A carrier body of the gyrostat is a classical rigid body, and a rotor is a non-classical particle whose properties will be defined in what follows.

To derive the dynamic equations of the continuum we apply the spatial description. Let vector $\mathbf{r}$ determine the position of some point of space. We introduce following notations: $\rho(\mathbf{r},t)$ is the mass density of the material medium at a given point of space; $\mathbf{v}(\mathbf{r},t)$ is the velocity field; $\mathbf{u}(\mathbf{r},t)$ is the displacement field; $\tilde{\mathbf{P}}(\mathbf{r},t)$, $\tilde{\omega}(\mathbf{r},t)$ are the fields of the rotation tensors and the angular velocity vectors of the carrier bodies; $\mathbf{P}(\mathbf{r},t)$ and $\boldsymbol{\omega}(\mathbf{r},t)$ are fields of the rotation tensors and the angular velocity vectors of the rotors.

The particles of continuum under consideration possess the internal degrees of freedom. Therefore, in order to describe the motion of this continuum it is not sufficient to formulate the balance equations of the momentum and the angular momentum for the control volume of the continuum. It is necessary to add these equations to the balance equation of the angular momentum for the rotors in control volume of the continuum. Therefore below we need the densities of the momentum and the angular momentum of the carrier bodies

$$\rho\mathbf{K}_1^{(cb)} = \rho(1-\zeta)\mathbf{v}, \qquad \rho\mathbf{K}_2^{(cb)} = \rho\left[\mathbf{r}\times(1-\zeta)\mathbf{v} + \mathbf{I}_*\cdot\tilde{\boldsymbol{\omega}}\right], \tag{7.44}$$

and the momentum and the angular momentum of the rotors

$$\rho\mathbf{K}_1^{(\mathrm{rot})} = \rho(\zeta\mathbf{v} + B\boldsymbol{\omega}), \qquad \rho\mathbf{K}_2^{(\mathrm{rot})} = \rho\Big[\mathbf{r}\times(\zeta\mathbf{v} + B\boldsymbol{\omega}) + B\mathbf{v} + J\boldsymbol{\omega}\Big]. \tag{7.45}$$

Here $\mathbf{I}_*$ is the inertia tensor of the carrier body of the gyrostat, B and J are the moments of inertia of the rotor. Dimensionless parameter ζ in Eqs (7.44) and (7.45) characterizes the distribution of mass in the gyrostat: if m is the mass of the gyrostat then $(1-\zeta)m$ is the mass of its carrier body and ζm is the mass of its rotor. Below we will see that the value of parameter ζ is not important. The densities of the momentum and the angular momentum of the gyrostats are

$$\rho\mathbf{K}_1 = \rho\mathbf{K}_1^{(\mathrm{cb})} + \rho\mathbf{K}_1^{(\mathrm{rot})}, \qquad \rho\mathbf{K}_2 = \rho\mathbf{K}_2^{(\mathrm{cb})} + \rho\mathbf{K}_2^{(\mathrm{rot})}. \tag{7.46}$$

We assume that in the reference configurations the tensors $\tilde{\mathbf{P}}(\mathbf{r},t)$ and $\mathbf{P}(\mathbf{r},t)$ are equal to the unit tensor. Therefore, upon the linearization near the reference position they take the form

$$\tilde{\mathbf{P}}(\mathbf{r},t) = \mathbf{E} + \boldsymbol{\varphi}(\mathbf{r},t)\times\mathbf{E}, \qquad \mathbf{P}(\mathbf{r},t) = \mathbf{E} + \boldsymbol{\theta}(\mathbf{r},t)\times\mathbf{E}, \tag{7.47}$$

where $\boldsymbol{\varphi}(\mathbf{r},t)$, $\boldsymbol{\theta}(\mathbf{r},t)$ are the rotation vector fields of carrier bodies and rotors, respectively, $\mathbf{E}$ is the unit tensor. Kinematic relations in the linear approximation are

$$\mathbf{v} = \frac{d\mathbf{u}}{dt}, \qquad \tilde{\boldsymbol{\omega}} = \frac{d\boldsymbol{\varphi}}{dt}, \qquad \boldsymbol{\omega} = \frac{d\boldsymbol{\theta}}{dt}. \tag{7.48}$$

The mass balance equation in the linear approximation takes the form

$$\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{v} = 0 \qquad \Rightarrow \qquad \rho = \rho_*\big(1 - \nabla\cdot\mathbf{u}\big). \tag{7.49}$$

Here ρ_* is the mass density per unit volume in the reference position. Note that mass density at the initial time instant ρ_0 may not coincide with the mass density in the reference position ρ_*. These two quantities are related with each other by the

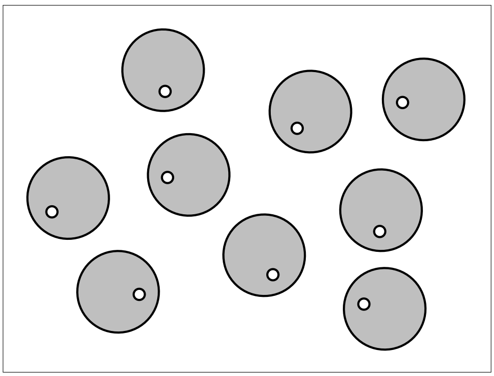

Fig. 7.2 Elementary volume of continuum consisting of one-rotor gyrostats

formula

$$\rho_0 = \rho_* \left(1 - \nabla \cdot \mathbf{u}_0\right), \tag{7.50}$$

and they coincide only if the medium is not deformable at the initial time instant.

The equations of motion of the material continuum can be written in the form

$$\nabla \cdot \boldsymbol{\tau} + \rho_* \mathbf{f} = \rho_* \frac{d}{dt}\big(\mathbf{v} + B\boldsymbol{\omega}\big), \qquad \nabla \cdot \boldsymbol{\mu} + \boldsymbol{\tau}_\times + \rho_* \mathbf{m} = \rho_* \frac{d}{dt}\big(\mathbf{I}_*^{(0)} \cdot \tilde{\boldsymbol{\omega}}\big), \tag{7.51}$$

where inertia tensor $\mathbf{I}_*^{(0)}$ is calculated in the reference configuration. tensor $\boldsymbol{\tau}$ is the stress tensor, and tensor $\boldsymbol{\mu}$ is the moment stress tensor modeling the influence of surrounding medium on the carrier bodies of gyrostats. The second equation in Eqs (7.51) is the equation of the motion of the carrier bodies. That is why the right-hand part of this equation does not depend on the velocity $\mathbf{v}$. The equation of motion of the rotors takes the form

$$\nabla \cdot \mathbf{T} + \rho_* \mathbf{L} = \rho_* \frac{d}{dt}\big(B\mathbf{v} + J\boldsymbol{\omega}\big), \tag{7.52}$$

where $\mathbf{T}$ is the moment stress tensor modeling the influence of surrounding medium on the rotors of gyrostats.

After simple transformations the equation of energy balance is written as follows:

$$\rho_* \frac{dU}{dt} = \boldsymbol{\tau}^T \cdot\cdot \frac{d\boldsymbol{\varepsilon}}{dt} + \boldsymbol{\mu}^T \cdot\cdot \frac{d\boldsymbol{\kappa}}{dt} + \mathbf{T}^T \cdot\cdot \frac{d\boldsymbol{\vartheta}}{dt}, \tag{7.53}$$

where U is the internal energy density per unit mass and the strain tensors $\boldsymbol{\varepsilon}$, $\boldsymbol{\kappa}$, $\boldsymbol{\vartheta}$ are introduced into consideration. These tensors are calculated by the formulas

$$\boldsymbol{\varepsilon} = \nabla \mathbf{u} + \mathbf{E} \times \boldsymbol{\varphi}, \qquad \boldsymbol{\kappa} = \nabla \boldsymbol{\varphi}, \qquad \boldsymbol{\vartheta} = \nabla \boldsymbol{\theta}. \tag{7.54}$$

In what follows we consider the elastic material i. e. a material whose density of internal energy and the tensors of force and moment stresses depend only on the strain tensors and do not depend on the velocities. For the elastic material the Cauchy–Green relations follow from the equation of energy balance (7.53):

$$\boldsymbol{\tau} = \rho_* \frac{\partial U}{\partial \boldsymbol{\varepsilon}}, \qquad \boldsymbol{\mu} = \rho_* \frac{\partial U}{\partial \boldsymbol{\kappa}}, \qquad \mathbf{T} = \rho_* \frac{\partial U}{\partial \boldsymbol{\vartheta}}. \tag{7.55}$$

To close the system of differential equations it is necessary to express the internal energy as a function of the strain tensors

$$\rho_* U = \rho_* U(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}, \boldsymbol{\vartheta}). \tag{7.56}$$

Now we consider the physically linear theory and therefore we represent the density of internal energy in the following form:

$$\rho_* U = \boldsymbol{\tau}_0^T \cdot\cdot \boldsymbol{\varepsilon} + \boldsymbol{\mu}_0^T \cdot\cdot \boldsymbol{\kappa} + \mathbf{T}_*^T \cdot\cdot (\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_*) + \frac{1}{2} \boldsymbol{\varepsilon} \cdot\cdot {}^4\mathbf{C}_1 \cdot\cdot \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} \cdot\cdot {}^4\mathbf{C}_2 \cdot\cdot \boldsymbol{\kappa} +$$
$$+ \frac{1}{2} \boldsymbol{\kappa} \cdot\cdot {}^4\mathbf{C}_3 \cdot\cdot \boldsymbol{\kappa} + \boldsymbol{\varepsilon} \cdot\cdot {}^4\mathbf{C}_4 \cdot\cdot (\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_*) + \boldsymbol{\kappa} \cdot\cdot {}^4\mathbf{C}_5 \cdot\cdot (\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_*) +$$
$$+ \frac{1}{2} (\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_*) \cdot\cdot {}^4\mathbf{C}_6 \cdot\cdot (\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_*). \tag{7.57}$$

Coefficients $\boldsymbol{\tau}_0$, $\boldsymbol{\mu}_0$ and $\mathbf{T}_*$ are called the initial stresses, $\boldsymbol{\vartheta}_*$ is the reference value of $\boldsymbol{\vartheta}$. Coefficients of the quadratic form ${}^4\mathbf{C}_i$ are called the stiffness tensors. In the linear theory the stiffness tensors do not depend on time. The only restriction imposed on the stiffness tensors is concerned with the requirement of positive definiteness of the quadratic form (7.57). The structure of the stiffness tensors and the values of the coefficients of elasticity are determined by the physical properties of the material medium.

After substituting expression for the density of internal energy (7.57) in the Cauchy–Green relations (7.55) we obtain the following constitutive equations:

$$\begin{aligned}
\boldsymbol{\tau}^T &= \boldsymbol{\tau}_0^T + {}^4\mathbf{C}_1 \cdot\cdot \boldsymbol{\varepsilon} + {}^4\mathbf{C}_2 \cdot\cdot \boldsymbol{\kappa} + {}^4\mathbf{C}_4 \cdot\cdot (\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_*), \\
\boldsymbol{\mu}^T &= \boldsymbol{\mu}_0^T + \boldsymbol{\varepsilon} \cdot\cdot {}^4\mathbf{C}_2 + {}^4\mathbf{C}_3 \cdot\cdot \boldsymbol{\kappa} + {}^4\mathbf{C}_5 \cdot\cdot (\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_*), \\
\mathbf{T}^T &= \mathbf{T}_*^T + \boldsymbol{\varepsilon} \cdot\cdot {}^4\mathbf{C}_4 + \boldsymbol{\kappa} \cdot\cdot {}^4\mathbf{C}_5 + {}^4\mathbf{C}_6 \cdot\cdot (\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_*).
\end{aligned} \tag{7.58}$$

According to Eqs (7.58) all stress tensors can depend on all strain tensors. It means, in particular, that the moment stress tensor of rotors can depend not only on their relative orientation, but also on the relative orientation and relative position of the carrier bodies.

7.4 The Simplest Theory of One-rotor Gyrostats Continuum

We consider the material continuum (see Fig. 7.3) that consists of one-rotor gyrostats. In limits of linear theory the motion of this continuum is described by Eqs (7.48), (7.49), (7.51), (7.52), (7.54) and (7.58). Free space between the gyrostats is filled up by body-points whose structure coincides with the structure of rotors belonging to the gyroststs. The body-points in the space between the gyrostats are the elementary particles of a continuum which will be called the “thermal ether” in what follows. In fact, the material continuum represented in Fig. 7.3 is a two-component medium. We are not going to study in detail the motion of the body-points continuum (“thermal ether”) and the interaction between the gyrostats continuum and the body-points continuum. We consider only the gyrostats continuum as an object under study. The interaction between the carrier bodies of the gyrostats and the interaction between rotors of the gyrostats are characterized by tensors of the force and moment stresses (7.58). The body-points continuum (“thermal ether”) positioned in space between gyrostats is considered to be an external factor with respect to the

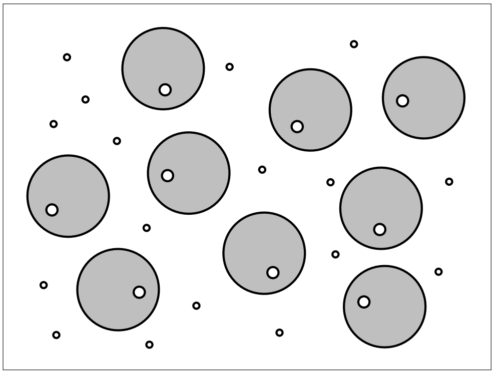

Fig. 7.3 Elementary volume of continuum interacting with environment

continuum under study. That is why we will model the influence of the "thermal ether" on the gyrostats by an external moment in the equation of the rotors motion (7.52).

Accepting two important hypotheses we consider a special case of the linear theory of one-rotor gyrostats continuum.

Hypothesis 1. Vector $\mathbf{L}$ (the mass density of external actions on the rotors of gyrostats) is a sum of the moment $\mathbf{L}_h$ characterizing external actions of all sorts and the moment of linear viscous damping

$$\mathbf{L}_f = -\beta(B\mathbf{v}+J\boldsymbol{\omega}). \tag{7.59}$$

The moment (7.59) characterizes the influence of the "thermal ether". Structure of the moment is chosen in accordance with the results of solving some model problems. One of these problems is considered in Sect. 7.8. Now we explain the physical meaning of the moment of linear viscous damping (7.59). We suppose that the rotors of the quasi-rigid bodies interact with body-points of the "thermal ether" and this interaction is described by the elastic moments analogous to the moments characterizing the interaction of the rotors with each other. The "thermal ether" having infinite extent eliminates energy of the oscillating rotors. The solution of modeling problems reveals that in the case of an infinite surrounding medium the dissipative moment arising due to the interaction with this medium is proportional to the proper angular momentum vector (dynamic spin).

Hypothesis 2. The moment stress tensor $\mathbf{T}$ characterizing the interactions between rotors is the spherical tensor

$$\mathbf{T} = T\mathbf{E}. \tag{7.60}$$

In view of assumptions (7.59) and (7.60) the equation of the rotors motion (7.52) takes the form

$$\nabla T - \rho_*\beta\,(B\mathbf{v}+J\boldsymbol{\omega}) + \rho_*\mathbf{L}_h = \rho_*\frac{d}{dt}\big(B\mathbf{v}+J\boldsymbol{\omega}\big), \tag{7.61}$$

In view of assumption (7.60) the last term on the right-hand side of the energy balance equation (7.53) can be reduced to the more simple form. By using notation $\vartheta = \mathrm{tr}\,\boldsymbol{\vartheta}$ the energy balance equation (7.53) is written as

$$\rho_* \frac{dU}{dt} = \boldsymbol{\tau}^T \cdot\cdot \frac{d\boldsymbol{\varepsilon}}{dt} + \boldsymbol{\mu}^T \cdot\cdot \frac{d\boldsymbol{\kappa}}{dt} + T \frac{d\vartheta}{dt}. \tag{7.62}$$

Since the material medium under consideration is an elastic one, we obtain from Eq. (7.62) the Cauchy–Green relations of which the first and the second ones coincide with the first and the second relations of (7.55) respectively and the third one has a simpler form:

$$\boldsymbol{\tau} = \rho_* \frac{\partial U}{\partial \boldsymbol{\varepsilon}}, \qquad \boldsymbol{\mu} = \rho_* \frac{\partial U}{\partial \boldsymbol{\kappa}}, \qquad T = \rho_* \frac{\partial U}{\partial \vartheta}. \tag{7.63}$$

According to Eq. (7.62) the density of internal energy is a function of arguments $\boldsymbol{\varepsilon}$, $\boldsymbol{\kappa}$ and ϑ. Let us construct the physically linear theory based on representation of the internal energy density in the following form:

$$\rho_* U = \boldsymbol{\tau}_0 \cdot\cdot \boldsymbol{\varepsilon} + T_* (\vartheta - \vartheta_*) + \frac{1}{2} \boldsymbol{\varepsilon} \cdot\cdot {}^4\mathbf{C}_1 \cdot\cdot \boldsymbol{\varepsilon} + \Upsilon \,\mathrm{tr}\boldsymbol{\varepsilon}\, (\vartheta - \vartheta_*) + \frac{1}{2} K (\vartheta - \vartheta_*)^2. \tag{7.64}$$

Then the constitutive equations (7.58) take the form

$$\boldsymbol{\tau}^T = \boldsymbol{\tau}_0^T + {}^4\mathbf{C}_1 \cdot\cdot \boldsymbol{\varepsilon} + \Upsilon (\vartheta - \vartheta_*) \mathbf{E}, \qquad \boldsymbol{\mu} = 0, \qquad T = T_* + \Upsilon \,\mathrm{tr}\boldsymbol{\varepsilon} + K(\vartheta - \vartheta_*). \tag{7.65}$$

Thus the simplest linear theory of the material continuum consisting of one-rotor gyrostats is described by Eqs (7.48), (7.51), (7.54), (7.61) and (7.65).

7.5 Temperature and Entropy

Let us consider the foregoing mathematical model of elastic continuum of one-rotor gyrostats. Suppose that the model describes the behavior of a classical medium which possesses not only elastic properties but also the viscous and thermic properties. Now we can give a thermodynamic interpretation of the variables describing motion and interaction of the rotors and next we can carry out the identification of the model parameters and well-known thermodynamic constants.

Let us consider the energy balance equation (7.62). Conceive that Eq. (7.62) is the equation of energy balance for a classical moment medium (medium without rotors). Then the last term on the right-hand side of Eq. (7.62) can be treated as a thermodynamical one. The physical quantities T and ϑ acquire the meaning of temperature and volume density of entropy, respectively.

It is evident, that the dimensions of the temperature and the entropy defined by formula (7.62) are different from the dimensions of those in classical thermodynamics of the present simple case. This problem can be solved by introduction of a normalization factor:

$$T = aT_a, \qquad \vartheta = \frac{1}{a}\,\vartheta_a. \tag{7.66}$$

Here a is the normalization factor; T_a is the absolute temperature measured by a thermometer; ϑ_a is volume density of the absolute entropy. Let us introduce the similar relations for the remaining variables:

$$\boldsymbol{\theta} = \frac{1}{a}\,\boldsymbol{\theta}_a, \qquad \boldsymbol{\omega} = \frac{1}{a}\,\boldsymbol{\omega}_a, \qquad \mathbf{L}_h = a\mathbf{L}_h^a, \qquad \mathbf{L}_f = a\mathbf{L}_f^a. \tag{7.67}$$

Now rewriting all equations for new variables and using new parameters

$$B_a = \frac{B}{a}, \qquad J_a = \frac{J}{a^2}, \qquad \Upsilon_a = \frac{\Upsilon}{a}, \qquad K_a = \frac{K}{a^2}, \tag{7.68}$$

we can eliminate the normalization factor a from these equations at least in the linear formulation of the problem and in some particular cases of physical nonlinearity.

7.6 Linear Theory of Thermoelasticity

Classical theory of thermoelasticity is a momentless one. Therefore considering the problem of thermoelasticity in the context of proposed model we assume only the force interaction between carrier bodies of the gyrostats and only the force action of external factors upon them:

$$\boldsymbol{\mu} = \mathbf{0}, \qquad \mathbf{m} = \mathbf{0}. \tag{7.69}$$

In the static problems from the second equation of (7.51) under the assumption (7.69) it follows that $\boldsymbol{\tau}_\times = \mathbf{0}$. In the dynamic problems the stress tensor can be nonsymmetric in spite of assumption (7.69). In this case it is necessary to take into account the dependence of the strain tensor $\boldsymbol{\varepsilon}$ on the angle of rotation of carrier bodies $\boldsymbol{\varphi}$. Thus, assumption (7.69) does not imply the transition to the momentless theory of elasticity for carrier bodies. In addition let us assume that $\mathbf{I}_*^{(0)} = 0$. In this case tensor $\boldsymbol{\tau}$ will be symmetrically both in the static and dynamic problems and all equations concerned with rotational motions of the carrier bodies of gyrostats can be excluded.

Applying the linear theory it is admissible in certain range of temperatures and entropy densities to change some reference values T_a^* and ϑ_a^*. Let us introduce deviations of the temperature and the density of entropy from their reference values:

$$T_a = T_a^* + \tilde{T}_a, \qquad \vartheta_a = \vartheta_a^* + \tilde{\vartheta}_a. \tag{7.70}$$

Resume of the basic equations of linear theory of the elastic medium consisting of the one-rotor gyrostats includes the dynamic equations (7.51), (7.61) which under notations (7.66) – (7.70) take the form

$$\nabla\cdot\boldsymbol{\tau}+\rho_*\mathbf{f}=\rho_*\frac{d}{dt}\left(\mathbf{v}+B_a\boldsymbol{\omega}_a\right),$$
$$\nabla\tilde{T}_a-\rho_*\beta\left(B_a\mathbf{v}+J_a\boldsymbol{\omega}_a\right)+\rho_*\mathbf{L}_h^a=\rho_*\frac{d}{dt}\left(B_a\mathbf{v}+J_a\boldsymbol{\omega}_a\right), \tag{7.71}$$

the mass balance equation (7.49), the kinematical and geometrical relations (7.48) and (7.54) which under notations (7.66) and (7.67) and condition of symmetry of the stress tensor are reduced to

$$\rho=\rho_*\left(1-\varepsilon\right),\qquad \mathbf{v}=\frac{d\mathbf{u}}{dt},\qquad \boldsymbol{\omega}_a=\frac{d\boldsymbol{\theta}_a}{dt},$$
$$\boldsymbol{\varepsilon}=\frac{1}{2}\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right),\qquad \varepsilon=\mathrm{tr}\,\boldsymbol{\varepsilon},\qquad \vartheta_a=\mathrm{tr}\,\boldsymbol{\vartheta}_a=\nabla\cdot\boldsymbol{\theta}_a, \tag{7.72}$$

and the constitutive equations (7.65) which under notations (7.66) – (7.70) and the condition of symmetry of the stress tensor are written as

$$\boldsymbol{\tau}=\left(K_{\mathrm{ad}}-\frac{2}{3}G\right)\varepsilon\mathbf{E}+G\boldsymbol{\varepsilon}+\Upsilon_a\,\tilde{\vartheta}_a\,\mathbf{E},\qquad \tilde{T}_a=\Upsilon_a\,\varepsilon+K_a\,\tilde{\vartheta}_a, \tag{7.73}$$

where K_{ad} is the adiabatic modulus of compression (the adiabatic bulk modulus), G is the shear modulus.

Let us suppose that $B_a=0$ and other parameters take the values

$$\beta J_a=\frac{T_a^*}{\rho_*\lambda},\qquad K_a=\frac{T_a^*}{\rho_*c_v},\qquad \Upsilon_a=-\frac{\alpha K_{\mathrm{is}}T_a^*}{\rho_*c_v}, \tag{7.74}$$

where c_v is the specific heat at constant volume, λ is the heat-conduction coefficient, K_{is} is the isothermal modulus of compression (the isothermal bulk modulus), α is the volume coefficient of thermal expansion,

$$K_{\mathrm{ad}}=K_{\mathrm{is}}\frac{c_p}{c_v},\qquad c_p-c_v=\frac{\alpha^2K_{\mathrm{is}}T_a^*}{\rho_*}\qquad\Rightarrow\qquad K_{\mathrm{ad}}=K_{\mathrm{is}}+\frac{\alpha^2K_{\mathrm{is}}^2T_a^*}{\rho_*c_v}, \tag{7.75}$$

where c_p is the specific heat at constant pressure. In this case we can transform the system of equation (7.71) – (7.73) to the following form:

$$\nabla\cdot\boldsymbol{\tau}+\rho_*\mathbf{f}=\rho_*\frac{d^2\mathbf{u}}{dt^2},\qquad \boldsymbol{\tau}=\left(K_{\mathrm{is}}-\frac{2}{3}G\right)\varepsilon\mathbf{E}+2G\boldsymbol{\varepsilon}-\alpha K_{\mathrm{is}}\tilde{T}_a\,\mathbf{E},$$
$$\Delta\tilde{T}_a-\frac{\rho_*c_v}{\lambda}\left(\frac{d\tilde{T}_a}{dt}+\frac{1}{\beta}\frac{d^2\tilde{T}_a}{dt^2}\right)=\frac{\alpha K_{\mathrm{is}}T_a^*}{\lambda}\left(\frac{d\varepsilon}{dt}+\frac{1}{\beta}\frac{d^2\varepsilon}{dt^2}\right)-\rho_*\nabla\cdot\mathbf{L}_h^a, \tag{7.76}$$

Thus, the mathematical description of the proposed mechanical model includes as a special case the formulation of the coupled problem of thermoelasticity with the hyperbolic type heat conduction equation.

7.7 Model of Internal Damping

There exist different macroscopic and microscopic models of internal damping. At present, however, viscoelasticity is not a well-developed science for the treatment of thermodynamical and dissipative phenomena. The point of view that internal damping is concerned with thermal effects is widespread. The distribution of phonons is in a local thermodynamical equilibrium, i. e. the temperature changes adiabatically, when acoustic wave propagates. Consequently, regions separated by the half-wavelength distance from one another have different temperatures and the irreversible heat flow between these regions arises as a result of the heat conduction phenomena. This process causes transfer of the energy of mechanical vibrations into heat energy. Now we do not call in question the idea about interplay of the internal damping and thermal effects. We emphasize that the analysis of the experimental values of the volume (acoustic) viscosity of various substances shows that the volume viscosity is an independent substance characteristic which is not related to the heat-conduction coefficient and other thermodynamical parameters. This means that we should not consider the nature of the acoustic viscosity to be directly connected with heat conduction mechanisms. Let us emphasize that by discussing the internal damping we mean only the volume (acoustic) viscosity. In our opinion the shear viscosity has an absolutely different nature and it is not discussed here.

Let us consider the energy dissipation caused by heat conduction phenomena. It is well-known that this energy dissipation takes place only in the case when the process is not isothermal and not adiabatic. Now let us consider the energy dissipation caused by the viscosity. This energy dissipation always takes place processes included adiabatic processes. Proceeding from this fact we assume that dissipation is caused only by viscosity and the process is adiabatically, i.e. the volume density of entropy is constant:

$$\vartheta_a = \vartheta_a^* = \text{const} \qquad \Rightarrow \qquad \tilde{\vartheta}_a = 0 \qquad \Rightarrow \qquad \tilde{T}_a = \Upsilon_a \, \varepsilon. \tag{7.77}$$

By comparison of the equations describing the dynamics of one-rotor gyrostat continuum with the classical equations of thermoelasticity we assumed that $B_a = 0$. Now we reject this restriction. We suppose that the terms containing parameter B_a are concerned with the internal damping mechanism. In order to argue in favor of this hypothesis we consider the heat conduction equation

$$\begin{aligned} \Delta \tilde{T}_a - \frac{\rho_* \beta J_a}{K_a} \frac{d\tilde{T}_a}{dt} - \frac{\rho_* J_a}{K_a} \frac{d^2 \tilde{T}_a}{dt^2} \\ = \beta \rho_* \left(B_a - \frac{\Upsilon_a J_a}{K_a} \right) \frac{d\varepsilon}{dt} + \rho_* \left(B_a - \frac{\Upsilon_a J_a}{K_a} \right) \frac{d^2 \varepsilon}{dt^2} - \rho_* \nabla \cdot \mathbf{L}_h^a. \end{aligned} \tag{7.78}$$

Let us transform this equation by using the adiabatic condition (7.77). As a result we obtain

$$\Upsilon_a \Delta \varepsilon - \rho_* \beta B_a \frac{d\varepsilon}{dt} - \rho_* B_a \frac{d^2 \varepsilon}{dt^2} = -\rho_* \nabla \cdot \mathbf{L}_h^a. \tag{7.79}$$

It is easy to see that Eq. (7.79) contains a dissipative term. This dissipative term is in no way concerned with the heat conduction phenomena.

In order to clarify the physical meaning of the coefficients in Eq. (7.79) we stop the discussion of the proposed model and consider the motion of a viscous fluid in which the pressure obeys the Stokes law. The liquid state (in the case of no external mass forces) is described by the following equations:

$$\nabla p = \rho_* \frac{d\mathbf{v}}{dt}, \qquad p = \eta_v \frac{d\varepsilon}{dt}, \tag{7.80}$$

where η_v is the volume (acoustic) viscosity. From Eqs. (7.80) we obtain the relation between the flow of matter $\rho_*\mathbf{v}$ and the volume strain gradient

$$\eta_v \nabla \varepsilon = \rho_* \mathbf{v}. \tag{7.81}$$

By taking the divergence of both sides of Eqs. (7.81) we obtain the self-diffusion equation which can be generalized by adding the source term $\rho_* \Psi$ to it:

$$\eta_v \Delta \varepsilon - \rho_* \frac{d\varepsilon}{dt} = -\rho_* \Psi. \tag{7.82}$$

Comparing Eq. (7.79) with the self-diffusion equation (7.82) we find these two equations to be equivalent with the only difference that the former contains the inertial term if

$$\frac{\Upsilon_a}{\beta B_a} = \eta_v, \qquad \frac{1}{\beta B_a} \nabla \cdot \mathbf{L}_h^a = \Psi. \tag{7.83}$$

From the first equation of (7.83) by using the third equation of (7.74) we get

$$\beta B_a = -\frac{\alpha K_{\mathrm{is}} T_a^*}{\rho_* c_v \eta_v}. \tag{7.84}$$

As evident from Eq. (7.84), parameter B_a is negative for finite values of the volume viscosity η_v and is equal to zero when $\eta_v \to \infty$.

In order to clarify the physical meaning of the obtained result we now consider the dissipative term in equation (7.71) for the rotor dynamics

$$\rho_* \mathbf{L}_f^a = -\beta \rho_* (B_a \mathbf{v} + J_a \boldsymbol{\omega}_a). \tag{7.85}$$

Upon substituting expressions for parameters (7.74), (7.84) into Eq. (7.85) we get

$$\rho_* \mathbf{L}_f^a = \frac{\alpha K_{\mathrm{is}} T_a^*}{c_v \eta_v} \mathbf{v} - \frac{T_a^*}{\lambda} \boldsymbol{\omega}_a. \tag{7.86}$$

Let us calculate the power of the dissipative moment (7.86):

$$\rho_* \mathbf{L}_f^a \cdot \boldsymbol{\omega}_a = \frac{\alpha K_{\mathrm{is}} T_a^*}{c_v \eta_v} \mathbf{v} \cdot \boldsymbol{\omega}_a - \frac{T_a^*}{\lambda} \boldsymbol{\omega}_a \cdot \boldsymbol{\omega}_a. \tag{7.87}$$

The second term in expression (7.87) is a dissipative one. When the heat-conduction coefficient decreases the dissipation increases. The first term in expression (7.87) determines the process which under the certain conditions can become inverse to the dissipative one. In particular, in the isothermal case the inequality $\mathbf{v} \cdot \boldsymbol{\omega}_a > 0$ is valid and, therefore, the first term in expression (7.87) determines the process of energy supply from the thermal ether. When the volume viscosity decreases the energy supply in the body from the thermal ether increases.

Let us transform Eq. (7.87) by separating the total squares in it:

$$\rho_* \mathbf{L}_f^a \cdot \boldsymbol{\omega}_a = \frac{\lambda \alpha^2 K_{is}^2 T_a^*}{4\eta_v^2 c_v^2} \mathbf{v} \cdot \mathbf{v} - \frac{T_a^*}{\lambda} \left(\boldsymbol{\omega}_a - \frac{\lambda \alpha K_{\text{is}}}{2\eta_v c_v} \mathbf{v} \right)^2 . \qquad (7.88)$$

It is easy to see that the second term in expression (7.88) determines the dissipative process and the first term characterizes the process of the energy supply from the thermal ether. The first term is inversely as the square of the viscosity. Therefore, when the volume viscosity decreases the supply of energy of the thermal ether into the body increases. The second term defining the dissipative process also depends on the volume viscosity. As a result the energy interchange between the body and the thermal ether depends on the volume viscosity in a complicated manner. Thus the volume viscosity characterizes the natural ability of a substance to absorb the energy of the thermal ether. Will this ability be realized? It depends on other properties of the substance and external circumstances. The volume viscosity of gases is very small and therefore gases possess a good ability to absorb the energy of the thermal ether. Therefore the gas particles are in a state of intense mgotion in spite of the energy dissipation caused by the heat conduction phenomena. The volume viscosity of fluids (even inviscid fluid) is much larger than the volume viscosity of gases. The volume viscosity of solids is as large as that it can be considered to approach infinity. In this case parameter B_a is negligible. Thus the problem of thermoelasticity is admissible for solids while for fluids and gases it is important to take into account the terms dependint on the volume viscosity.

7.8 Interaction of Body-point and "Thermal Ether"

In what follows we consider a model problem which solution allows us to substantiate the choice of the low of viscous damping (7.59).

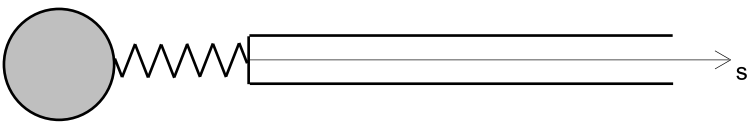

Fig. 7.4 Interaction of the body-point with the semi-infinite continuum

Let us consider a semi-infinite inertial rod (see Fig. 7.4), consisting of the body-points which are similar to the rotors of the one-rotor gyrostats. The rod is connected with the analogous body-point by means of an inertialess spring working in torsion (rotation about the axis of the rod). The inertia of the rod is characterized by the moments of inertia $\hat{B}$, $\hat{J}$ and the linear density $\sigma\tilde{\rho}$, where σ is "the area of rod section" and $\tilde{\rho}$ is the volume density of mass. The elastic properties of the rod are characterized by the torsional stiffness $\sigma\tilde{k}$, where the coefficient σ is introduced in order that stiffness $\tilde{k}$ possesses the dimension in 3D problems. The inertia of the body-point is characterized by the mass m and the moments of inertia B, J. The torsional stiffness of the spring connecting the body-point with the rod is equal to $\sigma k_*/r_0$, where r_0 is "the length" of the spring. The coefficients σ and r_0 are introduced in order that stiffness k_* possesses the dimension like $\tilde{k}$. The motion of the system is described by the following quantities: $u(s,t)$ is the longitudinal displacement of the rod, $\theta(s,t)$ is the rotation angle of the rod particles, $y(t)$ is the displacement of the body-point along the axis of the rod, $\psi(t)$ is the angle of rotation of the body-point about the axis of the rod. We suppose that the particles of the rod interact only by the moment. The force interaction of the rod particles is assumed to be zero. At the initial instant of time the displacements and the rotation angles as well as the translational and angular velocities are equal to zero. The body-point possesses a non-zero initial angular velocity directed along the axis of the rod and a non-zero initial angle of rotation about the axis of the rod. It is evident that under such an initial condition the system will be in motion which are longitudinal–torsional oscillations.

The longitudinal–torsional oscillations of the rod are described by the linear equations:

$$\frac{\partial T}{\partial s} = \sigma\tilde{\rho}\left(\hat{B}\ddot{u} + \hat{J}\ddot{\theta}\right), \qquad T = \sigma\tilde{k}\frac{\partial\theta}{\partial s}, \qquad \sigma\tilde{\rho}\left(\ddot{u} + \hat{B}\ddot{\theta}\right) = 0, \tag{7.89}$$

where s is the space coordinate ($0 \le s < +\infty$). After simple transformation the system (7.89) can be reduced to the wave equation in of the unknown θ:

$$\frac{\partial^2\theta}{\partial s^2} - \frac{1}{c^2}\ddot{\theta} = 0, \qquad c^2 = \frac{\tilde{k}}{\tilde{\rho}(\hat{J} - \hat{B}^2)}. \tag{7.90}$$

The boundary conditions for the rod take the form:

$$\sigma\tilde{k}\frac{\partial\theta}{\partial s}\bigg|_{s=0} = -\frac{\sigma k_*}{r_0}\left(\psi - \theta|_{s=0}\right). \tag{7.91}$$

Now we formulate the equations of the body-point motion:

$$m\left(B\ddot{y} + J\ddot{\psi}\right) = -\frac{\sigma k_*}{r_0}\left(\psi - \theta|_{s=0}\right), \qquad m\left(\ddot{y} + B\ddot{\psi}\right) = F. \tag{7.92}$$

Here F is an external force. The initial conditions for the body-point have the form:

$$y(0) = y_0, \qquad \psi(0) = \psi_0, \qquad \dot{y}(0) = v_0, \qquad \dot{\psi}(0) = \omega_0. \tag{7.93}$$

Let us represent the solution of the Eq. (7.90) in the form given by d'Alembert and Euler:

$$\theta(s,t) = f(s-ct) + g(s+ct). \tag{7.94}$$

Since the waves propagate to the right and there are no perturbations at infinity, we can assert that $g(s+ct) = 0$. In view of zero initial conditions for the rod we see that the function $f(s-ct)$ is not equal to zero only on the negative semiaxis. Hence

$$\theta(s,t) = \begin{cases} 0, & s > ct, \\ f(s-ct), & s < ct. \end{cases} \tag{7.95}$$

Let us denote:

$$\theta_*(t) = \theta(s,t)\big|_{s=0} = f(s-ct)\big|_{s=0}. \tag{7.96}$$

Then

$$\dot{\theta}_*(t) = -cf'(s-ct)\big|_{s=0}, \tag{7.97}$$

where the derivation with respect to argument $(s-ct)$ is denoted by the stroke. Hence

$$\left.\frac{\partial \theta}{\partial s}\right|_{s=0} = f'(s-ct)\big|_{s=0} = -\frac{1}{c}\,\dot{\theta}_*(t). \tag{7.98}$$

Subject to (7.96), (7.98) the boundary condition for the rod (7.91) takes the form

$$\frac{\sigma\tilde{k}}{c}\,\dot{\theta}_* = \frac{\sigma k_*}{r_0}\big(\psi - \theta_*\big), \tag{7.99}$$

and the equations of the body-point motion (7.92) can be rewritten as follows

$$m(B\ddot{y} + J\ddot{\psi}) + \frac{\sigma k_*}{r_0}\big(\psi - \theta_*\big) = 0, \qquad m(\ddot{y} + B\ddot{\psi}) = F. \tag{7.100}$$

Let us express the difference $(\psi - \theta_*)$ from Eq. (7.99) and put it in the first equation of (7.100). We obtain:

$$B\ddot{y} + J\ddot{\psi} + \frac{\sigma\tilde{k}}{mc}\,\dot{\theta}_* = 0. \tag{7.101}$$

Now we integrate Eq. (7.101) taking into account the initial conditions. As a result we obtain:

$$B\dot{y} + J\dot{\psi} + \frac{\sigma\tilde{k}}{mc}\,\theta_* = Bv_0 + J\omega_0. \tag{7.102}$$

Let us express θ_* from Eq. (7.102) and substitute it in Eqs (7.100). We obtain the following system of equations:

$$m(B\ddot{y} + J\ddot{\psi}) + m\beta(B\dot{y} + J\dot{\psi}) + \frac{\sigma k_*}{r_0}\psi = m\beta(Bv_0 + J\omega_0), \quad m(\ddot{y} + B\ddot{\psi}) = F, \tag{7.103}$$

where coefficient β is calculated by the formula:

$$\beta = \frac{ck_*}{r_0\tilde{k}} = \frac{k_*/r_0}{\sqrt{\tilde{k}\tilde{\rho}(\hat{J}-\hat{B}^2)}}. \tag{7.104}$$

According to Eqs (7.103), the moment of viscous damping characterizing the radiation of energy in the surrounding medium is proportional to the angular momentum of the body-point, i. e. it depends on both the angular velocity and the translational velocity. If $B = 0$ then the dependence on the translational velocity vanishes. In this case the problem under consideration becomes similar to the problem of the motion of an ordinary oscillator on the elastic waveguide. Analysis of formula (7.104) for the coefficient of damping β allows us to conclude that increasing the torsional stiffness of the spring connecting the body-point and the rod causes increasing of the radiation in the surrounding medium.

7.9 Conclusion

A model of a two-component continuum is suggested which takes into account thermomechanical processes. The mathematical description of this model is developed in the framework of physically and geometrically linear theory. In future we intend to carry out further development of the theory in two directions. The first one is concerned with consideration of nonlinear effects in the context of the same mechanical model. This is necessary for describing the behavior of substances in the states near the phase changes and heat-conduction processes under the circumstances of quickly varying and superhigh temperatures. The second direction deals with a modification of the mechanical model by taking into account the additional degrees of freedom for introducing the chemical potential and a number of additional physical characteristics of the medium. This is necessary to describe the phase changes and chemical reactions and also to take into account the interaction of the substance with the electromagnetic field and to describe thermoelectric and thermomagnetic effects.

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References

[1] Tzou, D. Y.: *Macro- and Microscale Heat Transfer: The Lagging Behavior*. Bristol, 1997
[2] Ziman, J. M.: *Electrons and Phonons. The Theory of Transport Phenomena in Solids*. Oxford, 1960
[3] Zhilin, P. A.: *Theoretical Mechanics. Fundamental Laws of Mechanics*. St. Petersburg (2003) (in Russian)
[4] Zhilin, P. A.: *Advanced Problems in Mechanics. Vol.* 1. St. Petersburg (2006) (in Russian)
[5] Zhilin, P. A.: *Advanced Problems in Mechanics. Vol. 2*. St. Petersburg (2006)
[6] Zhilin, P. A.: *Theoretical Mechanics*. St. Petersburg (2001) (in Russian)
[7] Brown, W. F.: *Magnetoelastic Interactions*. Springer, New York, 1966
[8] Eringen, A. C., Maugin, G. A.: *Electrodynamics of Continua*. Springer, New York, 1990
[9] Maugin, G. A.: *Continuum Mechanics of Electromagnetic Solids*. Elsevier Science Publishers, Oxford, 1988
[10] Truesdell, C. *The Elements of Continuum Mechanics*. Springer, New York, 1965
[11] Eringen, A. C.: *Mechanics of Continua*. Huntington - New York, 1980
[12] Christensen, R. M.: *Theory of Viscoelasticity*. Academic Press, New York and London, 1971
[13] Kondepudi, D., Prigogine, I.: *Modern Thermodynamics. From Heat Engines to Dissipative Structures*. Chichester et al. 1998
[14] Koshkin, N. I., Shirkevich, M. G.: *Handbook of Elementary Physics*. Moscow, 1968
[15] Ebert, H.: *Physikalisches Taschenbuch*. Braunschweig, 1957
[16] Handbook of Physical Quantities. Ed. by I.S. Grigoriev and E.Z. Meilikhov. CRC Press, 1997
[17] Physical Acoustics. Principles and Methods. Ed. by W. Mason. Vol. 2, Part A. Properties of Gases, Liquids and Solutions. New York - London, 1965