Chapter 17
Constitutive Models of Mechanical Behavior of Media with Stress State Dependent Material Properties

Evgeny V. Lomakin

Abstract The behavior of heterogeneous materials is studied. The dependence of the effective elastic properties of micro-heterogeneous materials on the loading conditions are analyzed and corresponding mathematical methods for the description of the observed effects are proposed. The constitutive relations of the theory of elasticity for isotropic solids with stress state dependent deformation properties are considered. The possible approach to the formulation of the constitutive relations for the elastic anisotropic solids that elastic properties depend on the stress state type is considered, and the corresponding constitutive relations are proposed. The method for the determination of material's functions on the base of experimental data is proposed. The quite satisfactory correspondence between the theoretical results and experimental data is shown.

Key words: Micro-heterogeneous materials. Phenomenological approach. Elastic properties. Isotropic materials. Anisotropic materials. Stress state dependent properties.

17.1 Introduction

The experimental studies of deformation properties of many heterogeneous and composite materials display the dependence of their properties on the conditions of loading. There are different mechanisms related to this phenomenon. In the case of granular porous materials, the area of contact between the particles increases under compressive loads. Then one would expect that the elastic characteristics would increase under compression in comparison with values corresponding to the action

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H. Altenbach et al. (eds.), *Mechanics of Generalized Continua,*
Advanced Structured Materials, 7, DOI: 10.1007/978-3-642-19219-7_17,

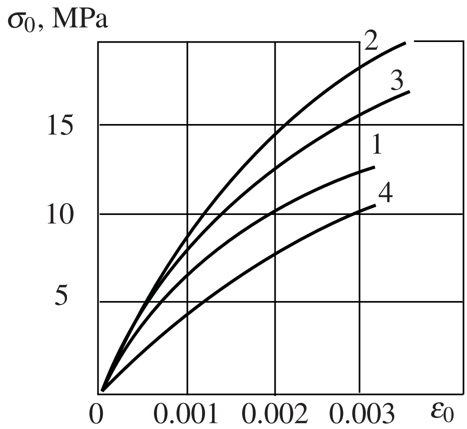

Fig. 17.1 Effective stress-strain diagrams of ARV graphite.

of tensile loads. In the case of cracked materials, the crack opening occurs under tensile load and the effective cross section carrying the load is less than in a solid material. Therefore the effective deformation properties depend on the concentration of microcracks. Under the conditions of compressive loads, it is possible that the closure of microcracks and the contact of crack faces would happen. The mechanical properties in this case depend on the conditions of interactions on the crack faces that are determined in one's turn by the ratios between different components of the stress tensor. This applies equally to an arbitrary type of loading. It means that the material properties are not invariant to the type of external forces but depend on the stress state type. For example, the initial slope of the stress-strain curve under conditions of compression is from 1.3 to 2 times the initial slope of the curve for tension [8].

Similar results have been obtained for structural graphite [1]. The effective stress-strain curves of ARV graphite are shown in the Fig. 17.1, which were obtained by a proportional loading of tubular specimens under plane stress conditions. The effective stress is $\sigma_0 = \sqrt{\frac{3}{2}S_{ij}S_{ij}}$, where $S_{ij} = \sigma_{ij} - \sigma\delta_{ij}$ is the stress deviator and $\sigma = \frac{1}{3}\sigma_{ii}$ is the hydrostatic component of the stress. The effective strain is $\varepsilon_0 = \sqrt{\frac{2}{3}e_{ij}e_{ij}}$, where $e_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon\delta_{ij}$ is the strain deviator and $\varepsilon = \varepsilon_{ii}$ is the bulk strain. Curves 1, 2, 3 and 4 correspond to uniaxial tension, uniaxial compression, shear and uniform biaxial tension, respectively. Instead of the single curve, as usually supposed in different theories of deformation, there is a fan of effective stress-strain curves and their deviation is noticeable. The curves have a weak non-linearity and a linear approximation of them is possible in a certain deformation range. Similar effects can be demonstrated for rocks, concrete, cast-iron and other materials [4].

The opposite effect sometimes can be observed in the case of composites. The fabric based carbon-carbon composites or composites with triaxial weave usually have considerable porosity. The fibers are tightened up under tension but they can buckle into the pores space under compression. The mechanisms of deformation are quite different for these loading conditions. The bending stiffness of fibers is much lower in comparison with the tensile one. Thus the elastic modulus of the composite under tension can be greater by a factor of 4 or 5 then the elastic modulus under compression [2].

17.2 Constitutive Relations for Isotropic Materials

The deformation properties of materials under consideration are the stress-state-dependent ones. In the general case, the stress state type can be characterized by two parameters $\xi = \sigma/\sigma_0$ and $S_{\mathrm{III}}/\sigma_0^3$, where $S_{\mathrm{III}} = S_{ij}S_{jk}S_{kj}$ is the third invariant of the deviator of the stress tensor. The hydrostatic component of the stress σ characterizes the mean value of normal stresses at arbitrary point of a continuum and the effective stress or the stress intensity σ_0 defines the mean value of shear stress at the same point of a continuum. The parameter ξ characterizes the stress state type on the average but the parameter $S_{\mathrm{III}}/\sigma_0^3$ specifies the deviation from this average value. For the formulation of the constitutive relations the parameter ξ is used. The potential for the elastic solids with stress state dependent properties can be represented in the form

$$\Phi = \frac{1}{2}\left[1+\zeta(\xi)\right]\left(\mathrm{A}+\mathrm{B}\xi^2\right)\sigma_0^2 \tag{17.1}$$

Differentiating Eq. (17.1) with respect to the stresses σ_{ij}, the strain-stress relations can be obtained

$$\begin{aligned} \varepsilon_{ij} &= \frac{3}{2}\left[\mathrm{A}+\omega(\xi)\right]S_{ij} + \frac{1}{3}\left[\mathrm{B}+\Omega(\xi)\right]\sigma\delta_{ij}, \\ \omega(\xi) &= -\frac{1}{2}\left(\mathrm{A}+\mathrm{B}\xi^2\right)\zeta'(\xi)\xi + \mathrm{A}\zeta(\xi), \\ \Omega(\xi) &= \frac{1}{2}\left(\mathrm{A}+\mathrm{B}\xi^2\right)\zeta'(\xi)/\xi + \mathrm{B}\zeta(\xi). \end{aligned} \tag{17.2}$$

The prime denotes the derivative of function with respect to ξ. The functions $\omega(\xi)$ and $\Omega(\xi)$ and their derivatives are related

$$\begin{aligned} &\omega + \xi^2\Omega = \left(\mathrm{A}+\mathrm{B}\xi^2\right)(1+\zeta), \\ &\omega' + \xi^2\Omega' = 0. \end{aligned} \tag{17.3}$$

From Eqs (17.2) and (17.3), it is possible to obtain the following expressions for the bulk strain ε end the effective strain ε_0:

$$\varepsilon = \left[\mathrm{B}+\Omega(\xi)\right]\sigma, \qquad \varepsilon_0 = \left[\mathrm{A}+\omega(\xi)\right]\sigma_0 \tag{17.4}$$

The Eqs (17.4) determine the relation between the bulk strain and the effective strain

$$\varepsilon = \frac{\mathrm{B} + \Omega(\xi)}{\mathrm{A} + \omega(\xi)} \xi \varepsilon_0 \tag{17.5}$$

Equation (17.5) signifies that the shear strains can cause the volume alteration of a material. The bulk strain ε is proportional to the strain intensity ε_0, but the proportionality factor is not constant but it depends on the stress state type parameter ξ. This factor is the variable quantity according to the type of loading and it has different values for uniaxial tension, uniaxial compression, shear, different types of biaxial and triaxial stress states.

Without loss of generality, we can assume that in the case of pure shear ($\xi = 0$) the function $\omega(\xi)$ has value $\omega(0) = 0$. Then the constant A in Eqs (17.2) is determined by the slope of effective stress-strain curve in the case of pure shear and $\zeta(0) = 0$. The functions $\omega(\xi)$, $\zeta(\xi)$ and $\Omega(\xi)$ can be determined on the base of a series of diagrams of the dependence between the effective strain ε_0 and the effective stress σ_0. According to the Eq. (17.4), the function $\omega(\xi) = -\mathrm{A} + \varepsilon_0/\sigma_0$. The second expression of Eq. (17.2) can be integrated, and it is possible to obtain the following expression for the function $\zeta(\xi)$:

$$1 + \zeta(\xi) = \left(\mathrm{A} + \omega + \mathrm{B}\xi^2 - \xi^2 \int \frac{\omega' \mathrm{d}\xi}{\xi^2}\right)(\mathrm{A} + \mathrm{B}\xi^2)^{-1}$$

As an example of experimental determination of all the parameters in the constitutive relations (17.2), the data obtained for graphite and represented in Fig. 17.1 can be used.

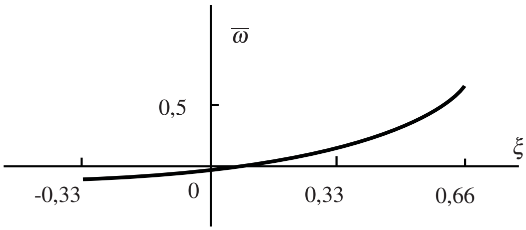

Fig. 17.2 The graph of function $\overline{\omega}(\xi)$ for ARV graphite.

In consequence of the weak non-linearity of effective stress-strain curves in Fig. 17.1, it is possible to approximate them by linear functions in the range of deformation 0.001. For this approximation the following values of elastic modulus and Poisson's ratio are obtained: $E^+ = 5.1 \cdot 10^3$ MPa, $\nu^+ = 0.2$ for uniaxial tension and $E^- = 7.83 \cdot 10^3$ MPa, $\nu^- = 0.3$ for uniaxial compression, respectively. The constant A can be obtained on the base of the curve 3 in Fig. 17.1 corresponding pure shear, and it has the value $\mathrm{A} = 1.35 \cdot 10^{-4}\,\mathrm{MPa}^{-1}$. The graph of function $\overline{\omega} = \omega(\xi)/\mathrm{A}$ is shown in Fig. 17.2. The piecewise linear approximation $\overline{\omega} = C\xi$ can be used for this function with $C = 0.45$ for $\xi > 0$ and $C = 0.3$ for $\xi < 0$, respectively. The constant B can be determined on the base of the value of elastic modulus for tension E^+, and this value is $\mathrm{B} = 1.76 \cdot 10^{-4}\,\mathrm{MPa}^{-1}$. The calculated value of elastic modulus under compression is $E^- = 7.81 \cdot 10^3$ MPa. The calculated values of Poisson's ratio under tension and compression are $\nu^{+} = 0.195$ and $\nu^{-} = 0.42$, respectively. The correspondence between the experimental and calculated values of graphite deformation properties is quite satisfactory.

The Eqs (17.2) can be solved for the stresses by introducing the parameter $\gamma = \varepsilon/\varepsilon_0$. The Eq. (17.5) gives the possibility to express the parameter ξ as a function of parameter γ, The potential can be represented in the form

$$\mathrm{U} = \frac{1}{2}\left[1+\eta(\gamma)\right]\left(\frac{1}{\mathrm{A}}+\frac{\gamma^2}{\mathrm{B}}\right)\varepsilon_0^2 \tag{17.6}$$

The stress-strain relations can be obtained by differentiating Eq. (17.6) with respect to the strains ε_{ij}

$$\begin{aligned}
\sigma_{ij} &= \frac{2}{3}\psi(\gamma)e_{ij} + \Psi(\gamma)\varepsilon\delta_{ij}, \\
\psi(\gamma) &= -\frac{1}{2}\left(\frac{1}{A}+\frac{\gamma^2}{\mathrm{B}}\right)\eta(\gamma)\gamma + \frac{1}{A}[1+\eta'(\gamma)], \\
\Psi(\gamma) &= \frac{1}{2}\left(\frac{1}{A}+\frac{\gamma^2}{\mathrm{B}}\right)\eta(\gamma)\gamma^{-1} + \frac{1}{B}[1+\eta'(\gamma)].
\end{aligned} \tag{17.7}$$

Some properties of the constitutive relations (17.2) and (17.7) are analyzed in [4, 5, 6]. It is shown that some traditional approaches to the solution of boundary value problems can not be accepted and new methods are proposed [7].

17.3 Constitutive Relations for Anisotropic Materials

The formulation of the constitutive relations for the anisotropic materials is much more complex in comparison with the isotropic ones. In general, it is necessary to suppose that all the anisotropy coefficients are the functions of the stress state parameter ξ. The potential for an anisotropic solid with stress state dependent deformation properties can be represented in the following form:

$$\Phi = \frac{1}{2}\mathrm{A}_{ijkl}(\xi)\sigma_{ij}\sigma_{kl} \tag{17.8}$$

Equation (17.8) represents some generalization of classic elastic potential. Differentiating Eq. (17.8) with respect to the stresses and taking into account that

$$\frac{\partial\sigma_0}{\partial\sigma_{ij}} = \frac{3}{2}\frac{S_{ij}}{\sigma_0}$$

and

$$\frac{\partial\sigma}{\partial\sigma_{ij}} = \frac{1}{3}\delta_{ij},$$

we obtain the dependence of the strains on the stresses:

$$\varepsilon_{ij} = \mathrm{A}_{ijkl}(\xi)\sigma_{kl} + \frac{1}{2}\mathrm{A}'_{mnpq}(\xi)\sigma_{mn}\sigma_{pq}\left[\left(\frac{1}{3}+\frac{3}{2}\xi^2\right)\delta_{ij} - \frac{3}{2}\xi\sigma_{ij}\sigma_0^{-1}\right]\sigma_0^{-1} \tag{17.9}$$

From Eq. (17.9) it follows that the strains consist of two parts, one corresponds to deformations of an anisotropic solid, the second one represents the deformations of some isotropic solid because this part has isotropic nature. The strain potential represents the homogeneous function of second order of the components of stress tensor and according to the Euler theorem one can obtain

$$2\Phi = \sigma_{ij}\varepsilon_{ij} \tag{17.10}$$

From Eq. (17.10) it follows that the Clapeyron theorem is valid for materials under consideration, namely the work of external forces

$$\mathrm{A} = \frac{1}{2}\int_V \sigma_{ij}\varepsilon_{ij}\mathrm{d}V. \tag{17.11}$$

As distinct from classic anisotropic solid, the problem of determination an anisotropy coefficients as the functions of the stress state parameter ξ arises. Thus, in general for each stress state type, it is necessary to determine the set of coefficients $\mathrm{A}_{ijkl}(\xi)$. The constitutive relations (17.9) seem to be very complex but their nature is clear and simple and the procedure for the determination of the anisotropy functions can be proposed. Analyzing these constitutive relations one can discover that the complex expression in the square brackets reduces to zero in the case of uniaxial tension and uniaxial compression when the parameter ξ is equal $1/3$ and $-1/3$, respectively. Let us consider plane stress conditions of an anisotropic solid. Then relations (17.9) reduce to the following:

$$\begin{aligned}
\varepsilon_x &= \mathrm{a}_{11}(\xi)\sigma_x + \mathrm{a}_{12}(\xi)\sigma_y + \left[\left(\frac{1}{3\xi}+\frac{3}{2}\xi\right)\sigma - \frac{3}{2}\xi\sigma_x\right]\Phi_1\sigma_0^{-2},\\
\varepsilon_y &= \mathrm{a}_{12}(\xi)\sigma_x + \mathrm{a}_{22}(\xi)\sigma_y + \left[\left(\frac{1}{3\xi}+\frac{3}{2}\xi\right)\sigma - \frac{3}{2}\xi\sigma_y\right]\Phi_1\sigma_0^{-2},\\
\gamma_{xy} &= \left[\mathrm{a}_{66}(\xi) - \frac{3}{2}\xi\Phi_1\sigma_0^{-2}\right]\tau_{xy},\\
\Phi_1 &= \frac{1}{2}\left[\mathrm{a}'_{11}(\xi)\sigma_x^2 + \mathrm{a}'_{22}(\xi)\sigma_y^2 + 2\mathrm{a}'_{12}(\xi)\sigma_x\sigma_y + \mathrm{a}'_{66}(\xi)\tau_{xy}^2\right].
\end{aligned} \tag{17.12}$$

In the case of potential (17.8), it is necessary to determine the character of dependence of coefficients A_{ijkl} on the parameter ξ on the base of the experimental data. The most simple and useful one is a piecewise approximation. In the case of linear dependence of the coefficients A_{ijkl} on the parameter ξ, it is possible to write

$$\mathrm{A}_{ijkl} = \mathrm{A}^0_{ijkl} + C_{ijkl}(\xi - 1/3), \tag{17.13}$$

where A^0_{ijkl} are the values of anisotropy coefficients under uniaxial tension ($\xi = 1/3$). Then the coefficients $\mathrm{a}_{ij}(\xi)$ and the function Φ_1 can be represented in the form of

$$\begin{aligned} &\mathrm{a}_{ij}(\xi) = \mathrm{a}^0_{ij} + c_{ij}(\xi - 1/3), \qquad \mathrm{a}^0_{ij} = \mathrm{a}_{ij}(1/3), \\ &\Phi_1 = \frac{1}{2}(c_{11}\sigma_x^2 + c_{22}\sigma_y^2 + 2c_{12}\sigma_x\sigma_y + \sigma_{66}\tau_{xy}^2). \end{aligned} \tag{17.14}$$

Rotating the coordinate system the coefficients of anisotropy a_{ij} are transformed according to the usual equations for the transformation of components of a fourth rank tensor [3]. In principal stress axes p and q, Eq. (17.12) can be represented in the form

$$\begin{aligned} &\varepsilon_p = \mathrm{b}_{11}(\xi)\sigma_p + \mathrm{b}_{12}\sigma_q + \left[\left(\frac{1}{9\xi} - \xi\right)\sigma_p + \left(\frac{1}{9\xi} + \frac{1}{2}\xi\right)\sigma_q\right]\Phi_1\sigma_0^{-2}, \\ &\varepsilon_q = \mathrm{b}_{21}(\xi)\sigma_p + \mathrm{b}_{22}\sigma_q + \left[\left(\frac{1}{9\xi} - \xi\right)\sigma_q + \left(\frac{1}{9\xi} + \frac{1}{2}\xi\right)\sigma_p\right]\Phi_1\sigma_0^{-2}, \\ &\gamma_{pq} = \mathrm{b}_{61}(\xi)\sigma_p + \mathrm{b}_{62}(\xi)\sigma_q, \\ &\sigma_0^2 = \sigma_p^2 + \sigma_q^2 - \sigma_p\sigma_q, \end{aligned} \tag{17.15}$$

where b_{ij} are the coefficients of anisotropy in the principal stress axes. Coefficients a_{ij} and b_{ij} are related by the following formulae:

$$\begin{aligned} &\mathrm{b}_{11} = \mathrm{a}_{11}\cos^4\varphi + (2\mathrm{a}_{12} + \mathrm{a}_{66})\sin^2\varphi\cos^2\varphi + \mathrm{a}_{22}\sin^4\varphi, \\ &\mathrm{b}_{22} = \mathrm{a}_{11}\sin^4\varphi + (2\mathrm{a}_{12} + \mathrm{a}_{66})\sin^2\varphi\cos^2\varphi + \mathrm{a}_{22}\cos^4\varphi, \\ &\mathrm{b}_{12} = (\mathrm{a}_{11} + \mathrm{a}_{22} - 2\mathrm{a}_{12} - \mathrm{a}_{66})\sin^2\varphi\cos^2\varphi + \mathrm{a}_{12}. \end{aligned} \tag{17.16}$$

Similar equations can be written for the coefficients b_{61}, b_{62} and b_{66}, too. The coefficients a^0_{ij} and c_{ij} can be determined on the base of experimental data for the conditions of uniaxial tension and uniaxial compression along the principal axes of orthotropy and along the directions at some angles with them. Under conditions of uniaxial tension, $\sigma = \frac{1}{3}\sigma_p$, $\sigma_0 = \sigma_p$, $\xi = \frac{1}{3}$ and for the strains along the x and y axes from Eqs (17.12) we obtain

$$\begin{aligned} &\varepsilon_x = \mathrm{a}^0_{11}\sigma_x, \qquad \varepsilon_y = \left(\mathrm{a}^0_{12} + \frac{1}{4}c_{11}\right)\sigma_x \\ &\varepsilon_y = \mathrm{a}^0_{22}\sigma_y, \qquad \varepsilon_x = \left(\mathrm{a}^0_{12} + \frac{1}{4}c_{22}\right)\sigma_y \end{aligned} \tag{17.17}$$

Under conditions of uniaxial compression ($\xi = -1/3$) along the same axes, it can be obtained

$$\varepsilon_x = \left(a_{11}^0 - \frac{2}{3}c_{11}\right)\sigma_x, \qquad \varepsilon_y = \left(a_{12}^0 - \frac{2}{3}c_{12} - \frac{1}{4}c_{11}\right)\sigma_x$$
$$\varepsilon_y = \left(a_{22}^0 - \frac{2}{3}c_{22}\right)\sigma_y, \qquad \varepsilon_x = \left(a_{12}^0 - \frac{2}{3}c_{12} - \frac{1}{4}c_{22}\right)\sigma_y \tag{17.18}$$

The coefficients a_{11}^0 and a_{22}^0 are determined as ratio of the axial strain to the axial stress according to Eqs (17.17). The coefficients c_{11} and c_{22} can be determined from Eq. (17.18). The coefficients a_{12}^0, a_{66}^0 c_{12} and c_{66} can be determined on the base of the results of experiments under conditions of uniaxial tension and uniaxial compression at some angle to the principal axes using Eqs (17.13)–(17.18).

In the case of similar dependence of anisotropy coefficients on the stress state type parameter ξ, it could be considered a simplified potential

$$\Phi = \frac{1}{2}[1+\zeta(\xi)]A_{ijkl}\sigma_{ij}\sigma_{kl} \tag{17.19}$$

In this case, the strain-stress relations (17.12) reduce to the following ones:

$$\begin{aligned}
\varepsilon_x &= [1+\zeta(\xi)](a_{11}\sigma_x + a_{12}\sigma_y) + \left[\left(\frac{1}{3\xi}+\frac{3}{2}\xi\right)\sigma - \frac{3}{2}\xi\sigma_x\right]\zeta'(\xi)\Phi_0\sigma_0^{-2}, \\
\varepsilon_y &= [1+\zeta(\xi)](a_{12}\sigma_x + a_{22}\sigma_y) + \left[\left(\frac{1}{3\xi}+\frac{3}{2}\xi\right)\sigma - \frac{3}{2}\xi\sigma_y\right]\zeta'(\xi)\Phi_0\sigma_0^{-2}, \\
\gamma_{xy} &= \left\{[1+\zeta(\xi)]a_{66} - \frac{3}{2}\xi\zeta'(\xi)\Phi_0\sigma_0^{-2}\right\}\tau_{xy}, \\
\Phi_0 &= \frac{1}{2}\left[a_{11}\sigma_x^2 + a_{22}\sigma_y^2 + 2a_{12}\sigma_x\sigma_y + a_{66}\tau_{xy}^2\right], \\
\sigma_0 &= (\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy})^{1/2}
\end{aligned} \tag{17.20}$$

Here the x and y directions coincide with the warp and woof directions of the cloth, respectively. Coefficients b_{ij} in Eqs (17.15) can be represented in the form $b_{ij}(\xi) = [1+\zeta(\xi)]b_{ij}$. Coefficients a_{ij} and b_{ij} are related by the conversion of formulae (17.16). We can denote the coefficients of the transverse deformation for the principal and rotated coordinate systems as $k_{12} = \varepsilon_y/\sigma_x$ and $k'_{12} = \varepsilon_q/\sigma_p$, respectively. Then from Eqs (17.16) it can be found

$$\begin{aligned}
a_{12} &= \left[b_{11}k_{12} - a_{11}(b_{11} - k'_{12} - a_{11}\cos^2\varphi - a_{22}\sin^2\varphi)\right](b_{11} - a_{11})^{-1}, \\
a_{66} &= b_{11}(\cos^2\varphi\sin^2\varphi)^{-1} - a_{11}\cot^2\varphi - a_{22}\tan^2\varphi - 2a_{12}
\end{aligned} \tag{17.21}$$

For uniaxial test conditions the coefficients $a_{11} = \varepsilon_x/\sigma_x$, $a_{22} = \varepsilon_y/\sigma_y$, $b_{11} = \varepsilon_p/\sigma_p$.

For the potential (17.19), the only function of the stress state type $\zeta(\xi)$ has to be determined and all the coefficients of anisotropy can be determined on the base of experiments under conditions of uniaxial tension. The function $\zeta(\xi)$ can be determined on the base of Eqs (17.10) and (17.19), from which it follows:

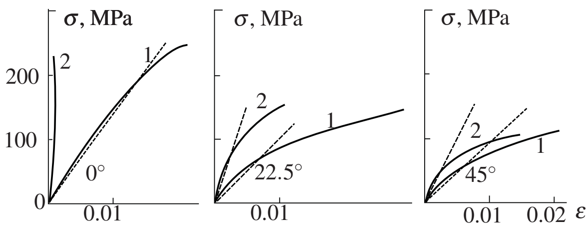

Fig. 17.3 The stress-strain diagrams for the composite fiberglass cloth/polyether resin under conditions of tension at the angles 0°, 22.5° and 45° to the direction of the warp of the cloth.

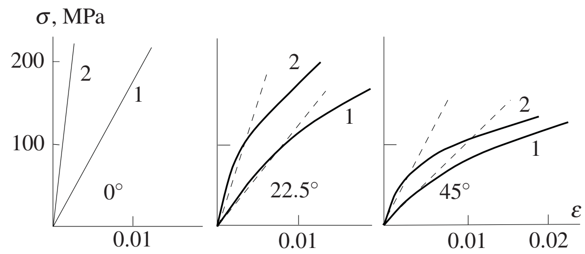

Fig. 17.4 The stress-strain diagrams under conditions of compression of the composite at the angles 0°, 22.5° and 45° to the direction of the warp of the cloth.

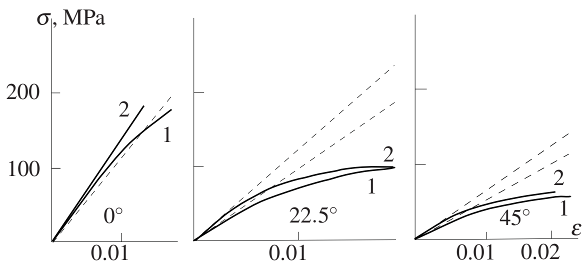

Fig. 17.5 The stress-strain diagrams for the conditions of shear with tension-compression directions 0° – 90°, 22.5° – 112.5° and 45° – 135°.

$$1+\zeta = \sigma_{ij}\varepsilon_{ij}(\mathrm{a}_{ijkl}\sigma_{ij}\sigma_{kl})^{-1} \tag{17.22}$$

For each stress state type or the type of loading, it is possible to define the values of function ζ using the known values of strains and stresses. Without the loss of the generality, one can suppose that $\zeta(1/3) = 0$.

The possibilities of the constitutive relations (17.20) in the description of the mechanical behavior of composite materials can be demonstrated on the base of the comparison of theoretical prediction and experimental data for the composite on the base of glass cloth and polyether matrix [9]. The plain specimens were used in the experiments. Therefore the Eqs (17.15), (17.16) and (17.20) can be used for the analysis of the results of experimental studies of the deformation properties of the composite. The stress-strain diagrams under tension of the composite in the direction of the warp of the cloth and at the angles 22.5° and 45°to this direction are shown in Fig. 17.3. The stress-strain diagrams under conditions of uniaxial compression of the composite are shown in the Figs 17.4 for the same directions. Curves 1 and 2 refer to longitudinal and transverse deformation, respectively. The diagram for compression along the fibers is linear up to failure. The diagram for tension displays some nonlinearity. With a certain degree of accuracy, the diagram for ten-

Direction of loading		0°		22.5°		45°	
	ξ	$\frac{\varepsilon_x}{\sigma_x}$	$\frac{\varepsilon_y}{\sigma_x}$	$\frac{\varepsilon_p}{\sigma_p}$	$\frac{\varepsilon_q}{\sigma_p}$	$\frac{\varepsilon_p}{\sigma_p}$	$\frac{\varepsilon_q}{\sigma_p}$
Experimental	1/3	7.0	−0.64	9.9	−3.14	–	–
Theoretical	1/3	–	–	–	–	13	−5.7
Experimental	−1/3	5.34	−0.97	7.8	−3.23	10.2	−5.4
Theoretical	−1/3	–	−0.94	7.5	−3.04	9.9	−5.2
Experimental	0	8.39	−7.04	14.7	−12.15	20.3	−16.9
Theoretical	0	8.17	−6.08	13.7	−11.37	19.7	−16.2

Table 17.1 Experimental and theoretical values of the deformation coefficients of the composite in $10^{-5}\,\mathrm{MPa}^{-1}$

sion can also be approximated by a linear one. It is possible to accept a measure of deviation from the initial slope of the diagram corresponding to the nonlinear deformation of 0.002. This linear diagram for the tension in the warp direction is shown in the Figs 17.3 by the dotted line. Under this approximation, the elastic modulus of the composite under tension along the fibers is $E^+ = 1.428 \cdot 10^4\,\mathrm{MPa}$. The elastic modulus under the compression in the same direction is $E^- = 1.873 \cdot 10^4\,\mathrm{MPa}$.

The analysis of the results of experiments under uniform biaxial loading indicates that the compliances of the composite in the directions of warp and woof of the cloth are almost equal within the frame of the adopted approximation. Thus it can be assumed that $\mathrm{a}_{11} = \mathrm{a}_{22} = 1/E^+ = 7 \cdot 10^{-5}\,\mathrm{MPa}^{-1}$. On the base of Eq. (17.21) using the values of compliances given in the first line of Table 17.1 we can determine the values of other two elastic constants $\mathrm{a}_{12} = -1.6 \cdot 10^{-5}\,\mathrm{MPa}^{-1}$ and $\mathrm{a}_{66} = 41.2 \cdot 10^{-5}\,\mathrm{MPa}^{-1}$. When all the anisotropy coefficients for plane stress conditions are determined, it is possible to calculate the value of function $\zeta(\xi)$ for the conditions of uniaxial compression ($\xi = -1/3$). In accordance with Eq. (17.22) and the value of compliance ε_x/σ_x under compression we obtain $\zeta = -0.24$.

The stress-strain diagrams for the conditions of shear ($\xi = 0$) in the plane of composite layers are shown in Figs 17.5. The experiments were carried out for three directions of tension-compression with respect to the warp of the cloth, that is $0^\circ - 90^\circ$, $22.5^\circ - 112.5^\circ$ and $45^\circ - 135^\circ$. Indexes 1 and 2 refer to the tensile strain and the compressive strain, respectively. Using the values of the strains and the stresses for the $0^\circ - 90^\circ$ conditions, we can determine the value of function $\zeta(\xi)$ for shear, $\zeta(0) = -0.13$. The graph of the function $\zeta(\xi)$ for the range of variation of parameter ξ, $-1/3 \leqslant \xi \leqslant 1/3$, is shown in Fig. 17.6.

Since all the parameters of the constitutive relations represented by Eq. (17.20) are determined, it is possible to compare the theoretical and experimental data obtained for plane stress conditions. In the case of shear,

$$\sigma_y = -\sigma_x, \quad \sigma_0 = \sqrt{3}\sigma_x, \quad \Phi_0 = (\mathrm{a}_{11} - \mathrm{a}_{22})\sigma_x^2, \quad \xi = 0$$

and according to Eq. (17.20) we have

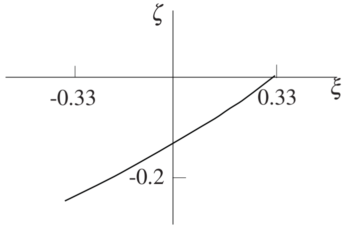

Fig. 17.6 The graph of function $\zeta(\xi)$ for the composite.

$$\varepsilon_x/\sigma_x = (\mathrm{a}_{11} - \mathrm{a}_{12})[1 + \zeta(0) + \zeta'(0)(3\sqrt{3})^{-1}],$$
$$\varepsilon_y/\sigma_x = -(\mathrm{a}_{11} - \mathrm{a}_{12})[1 + \zeta(0) - \zeta'(0)(3\sqrt{3})^{-1}] \tag{17.23}$$

For the case when the stresses are applied at some angle to the fiber direction, the coefficients a_{11} and a_{12} in Eq. (17.23) should be replaced by the coefficients b_{11} and b_{12} in accordance with Eq. (17.16). The theoretical and experimental values of compliances for different types of loading and various loading directions are given in Table 17.1. The correspondence between the theoretical values and experimental data is quite satisfactory. The calculated initial slope of the stress-strain diagrams are shown in Fig. 17.3–Fig. 17.5 by the dotted lines, and the trend of the variation of initial elastic deformation properties of the composite under various external forces is described by the considered constitutive equations satisfactorily.

17.4 Conclusions

The dependence of the effective elastic properties of micro-heterogeneous materials on the conditions of loading or the conditions of deformation is studied. The phenomenological approach to the description of the behavior of the heterogeneous materials under different types of external forces is considered. The constitutive equations of the theory of elasticity for isotropic solids with stress state dependent deformation properties are analyzed. Some properties of constitutive equations are studied. The method for experimental determination of material functions is proposed.

A possible approach to the formulation of the constitutive equations of the theory of elasticity for the anisotropic solids, which deformation properties depend on the stress state type is considered. The mechanical properties of these materials are characterized by a set of the anisotropy functions instead of a set of elastic constants in the case of the classic linear elastic solid. The method for the determination of anisotropy functions on the base of experimental data is described. The results of experimental studies of properties of composite materials on the base of fiberglass cloth and polyether resin obtained under different loading conditions are analyzed. The satisfying correspondence between the calculated values of deformation coefficients and experimental data is demonstrated.

Acknowledgements The work was supported by the Russian Foundation for Basic Research (grant 11-01-00168).

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