Chapter 1
A Historical Perspective of Generalized Continuum Mechanics

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Abstract In a period of forty years the author has had the opportunity to work, or to entertain friendly connections, with many actors of the scene of generalized continuum mechanics (GCM). This training and knowledge here is used to the benefit of the readers as an overview of this scene with the aim to delineate further avenues of development within the framework of the trilateral seminar held in Wittenberg (2010). Starting essentially with Pierre Duhem and the Cosserat brothers, this specialized, albeit vast, field of continuum mechanics has developed by successive abandonments of the working hypotheses at the basis of standard continuum mechanics, that mechanics masterly devised by Euler and Cauchy and some of their successors in the 19^{th} century (Piola, Kirchhoff, *etc.*). In the present survey we briefly analyze successive steps such as the introduction of nonsymmetric stresses, couple stresses, internal degrees of freedom and microstructure, the introduction of strain gradient theories, and material inhomogeneities with a length scale, nonlocality of the weak and strong types, the loss of Euclidean geometry to describe the material manifold, and finally the loss of classical differentiability of basic operations as can occur in a deformable fractal material object.

Key words: Generalized continua. Nonsymmetric stress. Couple stress. Micromorphic bodies. Micropolar materials. Nonlocality. Strain-gradient materials. Non-Euclidean manifold.

1.1 Introduction

At a recent colloquium [73] we have given a historical view of the development of so-called "generalized continuum mechanics". The thesis presented was that gener-

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alization occurs through the successive abandonment of the basic working hypotheses of standard continuum mechanics of Cauchy: that is, introduction of a rigidly rotating microstructure and *couple stresses* (Cosserat continua or *micropolar* bodies, nonsymmetric stresses), introduction of a truly deformable microstructure (*micromorphic* bodies), "weak" *nonlocalization* with *gradient theories* and the notion of *hyperstresses*, and the introduction of characteristic lengths, "strong nonlocalization" with space functional constitutive equations and the loss of the Cauchy notion of stress, and finally giving up the Euclidean and even Riemannian material background.

This evolution is paved by landmark papers and timely scientific gatherings (*e.g.*, Freudenstadt in 1967, Udine in 1970, Warsaw in 1977) to which the Paris colloquium of 2009 must now be added (Maugin and Metrikine, editors [76]). This will be examined in some detail in the following sections. Here we simply note that the publication of the book of the Cosserat brothers in 1909 [10] was a true initial landmark, although at the time noticed by very few people – among them Élie Cartan and Ernst Hellinger [44]. In passing we also emphasize that this was one of the first attempts to exploit some group theoretical argument (so-called Euclidean action) in the general formulation of continuum mechanics. Thus a real "generalized continuum mechanics" developed first slowly and rather episodically and then with a real acceleration in the 1960s. Accordingly, a new era was born in the field of continuum mechanics.

1.2 From Cauchy and the 19$^{\text{th}}$ Century

Here we consider as a classical standard the basic model considered by engineers in solid mechanics and the theory of structures. This essentially is the theory of continua set forth by A.L. Cauchy in the early 19$^{\text{th}}$ century for isotropic homogeneous elastic solids in small strains. The theory of continua respecting Cauchy's axioms and simple working hypotheses is such that the following holds true:

1. **Cauchy's postulate**: The traction $\mathbf{T}^d$ on a facet cut in the solid depends on the geometry of that facet only at the *first order* (the local unit normal of components n_j); it will be linear in that normal. From this follows the notion of *stress tensor* $\boldsymbol{\sigma} = \{\sigma_{ji}; i,j = 1,2,3\}$, the so-called stress being the only "internal force" in the theory. That is, using a classical Cartesian tensor notation:

$$T_i^d = n_j \sigma_{ji}. \tag{1.1}$$

2. **It being understood** that both physical space (of Newton) and material manifold (the set of material particles constituting the body) are Euclidean and connected, hence the notion of displacement $\mathbf{u} = \{u_i\}$ is well defined.
3. **Working hypotheses**

   (i) There are no applied couples in both volume and surface.

(ii) There exists no "microstructure" described by additional internal degrees of freedom.

According to items 3 (i) and (ii) the Cauchy stress tensor is symmetric:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T \quad i.e. \quad \sigma_{ji} = \sigma_{ij}. \tag{1.2}$$

This results from the application of the balance of angular momentum. Isotropy, homogeneity, and small strains are further hypotheses but they are not so central to our argument.

Then generalizations of various degrees consist in relaxing more or less these different items above, hence the notion of *generalized continuum*. This notion of generalization depends also on the culture and physical insight of the scientists. For instance the following generalizations are "weak" ones:

- "Generalized" Hooke's law (linear, homogeneous, but *anisotropic* medium);
- Hooke–Duhamel law in thermoelasticity;
- Linear homogeneous piezoelectricity in obviously anisotropic media (no center of symmetry)

These are "weak" generalizations because they do not alter the main mathematical properties of the system. Of course, thermoelasticity and linear piezoelectricity require adding new independent variables (*e.g.*, temperature θ or scalar electric potential ϕ). In some sense, the problem becomes four-dimensional for the basic field (elastic displacement and temperature in one case, elastic displacement and electric potential in the other). The latter holds in this mere simplicity under the hypothesis of weak electric fields, from which there follows the neglect of the so-called ponderomotive forces and couples, *e.g.*, the couple

$$(\mathbf{P} \times \mathbf{E})_i = \varepsilon_{ijk} P_j E_k \tag{1.3}$$

with ε_{ijk} as the permutation symbol in Cartesian tensor index notation, and this will yield (square brackets denote anti-symmetrization)

$$\sigma_{[ji]} = C_{ji}, \quad e.g., \quad C_{ji} = P_{[j} E_{i]}, \tag{1.4}$$

when electric field $\mathbf{E}$ and electric polarization $\mathbf{P}$ are not necessarily aligned; see Eringen and Maugin [30]. Such theories, just like standard elasticity, do not involve a *length scale*. But classical linear *inhomogeneous* elasticity presents a higher degree of generalization because a characteristic length intervenes necessarily, the characteristic length over which the material properties vary in the absence of loading.

1.3 True Generalizations

From here on we envisage three true (in our view) generalizations. The first of these is that the Cauchy stress tensor becomes nonsymmetric. The second one is that the validity of the Cauchy postulate can be lost. And last but not least that the Euclidean nature of the material manifold can be lost. In what follows all three items will be discussed in detail.

1.3.1 Various Reasons of the Nonsymmetry of the Cauchy Stress Tensor

The nonsymmetry may be due to

(i) the existence of body couples (*e.g.*, just as above in electromagnetism:

$$\mathbf{P}\times\mathbf{E} \quad \text{or/and} \quad \mathbf{M}\times\mathbf{H}$$

if **M** and **H** denote volume magnetization and magnetic field; case of intense electromagnetic fields or linearization about intense bias fields);

(ii) the existence of surface couples (introduction of "internal forces" of a new type: so-called *couple stresses*); the medium possesses internal degrees of freedom that modify the balance of angular momentum;

(iii) the existence of internal degrees of freedom (of a nonmechanical nature in origin, *e.g.*, polarization inertia in ferroelectrics, intrinsic spin in ferromagnetics (see [67]);

(iv) the existence of internal degrees of freedom of "mechanical" nature.

This is where the Cosserats' model comes into the picture. The first example in this class pertains to a *rigid microstructure* (three additional degrees of freedom corresponding to an additional rotation at each material point, independently of the vorticity). Examples of media of this type go back to the early search for a continuum having the capability to transmit transverse waves (as compared to acoustics in a pure fluid), *i.e.*, in relation to optics. The works of McCullagh [77] and Lord Kelvin must be singled out (*cf.* Whittaker [106]). Pierre Duhem [15] proposes to introduce a triad of three rigidly connected directors (unit vectors) to represent this rotation. In modern physics there are other tools for this including Euler's angles (not very convenient), quaternions and spinors. It is indeed the Cosserats, among other studies in elasticity, who really introduced internal degrees of freedom of the rotational type (these are *micropolar continua* in the sense of Eringen [26, 27]) and the dual concept of *couple stress*. Hellinger, in a brilliant essay [44], recognized at once the new potentialities offered by this generalization but did not elaborate on these.

A modern rebirth of the field had to await works in France by crystallographs (Laval [53, 54, 55]; Le Corre [59]), in Russia by Aero and Kuvshinskii [1], and Palmov [87], in Germany by Schaefer [94], Günther [43], Neuber [82], and in Italy by Grioli [42] and Capriz – see his book [7]. But the best formulations are those obtained by considering a field of orthogonal transformations (rotations) and not the directors themselves: Eringen [23], Kafadar and Eringen [45], Nowacki [86], although we note some obvious success of the "director" representation, *e.g.*, in *liquid crystals* (Ericksen [21]; Leslie [62], Stokes [98]) and the kinematics of the deformation of slender bodies (works by Ericksen, Truesdell, Naghdi).

But in the mid 1960s a complete revival of continuum mechanics took place which, by paying more attention to the basics, favored the simultaneous formulation of many more or less equivalent theories of generalized continua in the line of thought of the Cosserats (works by Mindlin [78], Mindlin and Tiersten [81], Mindlin and Eshel [80], Green and Rivlin [41], Green and Naghdi [40], Toupin [100, 101], Truesdell and Toupin [103], Truesdell and Noll [102], and Eringen and Suhubi [31, 32], *etc.*).

More precisely, in the case of a *deformable microstructure* at each material point, the vector triad of directors of Duhem-Cosserats becomes deformable and the additional degree of freedom at each point, or micro-deformation, is akin to a general linear transformation (nine degrees of freedom). These are **micromorphic continua** in Eringen's classification [26, 27]. A particular case is that of **continua with microstretch** [24]. A truly new notion here is that of the existence of a conservation law of *micro-inertia* (Eringen [2]). We illustrate these various generalizations by giving the relevant form of the local equation of moment of momentum in quasi-statics:

- **Micromorphic bodies** (Eringen [2, 31, 32], Mindlin [78, 80, 81]; Years 1962-1966) [Notation: μ_{kji} is the hyperstress tensor, s_{ji} is the so-called symmetric micro-stress, and ℓ_{ij} is the body-moment tensor of which the skew part represents a body couple $C_{ji} = -C_{ij}$]:

$$\begin{aligned} &\mu_{kij,k} + \sigma_{ji} - s_{ji} + \ell_{ij} = 0, &\quad& \sigma_{ji} = \sigma_{(ji)} + \sigma_{[ji]}, \\ &s_{[ji]} = 0, && \ell_{ji} = C_{ji} + \ell_{(ji)}. \end{aligned} \tag{1.5}$$

- **Micropolar bodies** (Cosserat brothers [10], *etc.*) [Notation: $\mu_{k[ji]}$ is the couple-stress tensor; C_i is the axial vector uniquely associated with C_{ji} while m_{ji} is associated in the same way with $\mu_{k[ji]}$]:

$$\mu_{k[ji],k} + \sigma_{[ji]} + C_{ij} = 0 \quad \text{or} \quad m_{ji,j} + \varepsilon_{ikj}\,\sigma_{kj} + C_i = 0. \tag{1.6}$$

- **Bodies with microstretch** (Eringen [24]) [Notation: m_k denotes the intrinsic dilatational stress or microstretch vector; ℓ is the body microstretch force such that $\ell_{ij} = (\ell/3)\,\delta_{ij}$, and σ and s are intrinsic and micro scalar forces]:

$$\mu_{klm} = \frac{1}{3} m_k\,\delta_{lm} - \frac{1}{2}\,\varepsilon_{lmr}\,m_{kr} \tag{1.7}$$

  so that

$$m_{kl,k} + \varepsilon_{lmn}\,\sigma_{mn} + C_l = 0, \quad m_{k,k} + \sigma - s + \ell = 0. \tag{1.8}$$

- **Dilatational elasticity** (Cowin and Nunziato [11]) [only the second of Eqs (1.8) is relevant]:

$$m_{k,k} + \sigma - s + \ell = 0. \tag{1.9}$$

In these equations given in Cartesian components in order to avoid any misunderstanding (note that the divergence is always taken on the first index of the tensorial object to which it applies), μ_{kij} is a new internal force having the nature of a third-order tensor. It has no specific symmetry in Eqs (1.5) and it may be referred to as a *hyperstress*. In the case of Eqs (1.6) this quantity is skewsymmetric in its last two indices and a second order tensor – called a *couple stress* – of components m_{ji} can be introduced having *axial* nature with respect to its second index. The fields s_{ji} and ℓ_{ij} are, respectively, a symmetric second order tensor and a general second order tensor. The former is an *intrinsic interaction stress*, while the latter refers to an external source of *both* stress and couple according to the last of Eqs (1.5). Only the skew part of the later remains in the special case of micropolar materials (Eqs (1.6) in which C_i represents the components of an *applied couple*, an axial vector associated with the skewsymmetric C_{ji}). The latter can be of electromagnetic origin, and more rarely of pure mechanical origin. Equations (1.7) and (1.8) represent a kind of intermediate case between micromorphic and micropolar materials. The case of dilatational elasticity in Eq. (1.9) appears as a further reduction of that in Eqs (1.8). This will be useful in describing the mechanical behavior of media exhibiting a distribution of holes or cavities in evolution.

Concerning the micromorphic case, a striking example is due to Drouot and Maugin [14] while dealing with fluid solutions of macromolecules, while Pouget and Maugin [89] have provided a fine example of truly micromorphic solids with the case of piezoelectric powders treated as continua.

Remark 1.1. Historical moments in the development of this avenue of generalization have been the IUTAM symposium organized by E. Kröner in Freudenstadt in 1967 (see Kröner [49]) and the CISM Udine summer course of 1970 (were present: Mindlin [79], Eringen [25], Nowacki [85], Stojanovic [97], Sokolowski, Maugin, Jaric, Micunovic, *etc.*).

Remark 1.2. Strong scientific initial motivations for the studies of generalized media at the time (1960s-1970s) were (i) the expected elimination of field singularities in many problems with standard continuum mechanics, (ii) the continuum description of *real* existing materials such as granular materials, suspensions, blood flow, *etc.* But further progress was hindered by a notorious lack of knowledge of new (and too numerous) material coefficients despite trials at estimates of such coefficients *e.g.*, by Gauthier and Jashman [37] at the Colorado School of Mines by building artificially microstructured solids.

Remark 1.3. The intervening of a rotating microstructure allows for the introduction of wave modes of rotation of the "optical" type with an obvious application to many solid crystals (*e.g.*, crystals equipped with a polar group such as $NaNO_2$; *cf.* Pouget and Maugin [89, 90]).

Remark 1.4. In some physical theories (micromagnetism, *cf.* Maugin [64]), an equation such as the first of Eqs (1.6) can be obtained in full dynamics:

$$m_{kij,k} + \sigma_{[ji]} + C_{ij} = \dot{S}_{ij}, \tag{1.10}$$

where m_{kij} (Heisenberg exchange-force tensor that is skewsymmetric in its last two indices), C_{ij} (interaction couple between material and electronic-spin continua) and S_{ij} (magnetic spin) all have a magnetic origin.

1.3.2 Loss of Validity of the Cauchy Postulate

Then the geometry of a cut intervenes at a higher order than one (variation of the normal unit, role of the curvature, edges, apices and thus capillarity effects). We may consider two different cases referred to as the *weakly nonlocal theory* and the *strongly nonlocal theory* (distinction introduced by the author at the Warsaw meeting of 1977; *cf.* Maugin [65]). Only the first type does correspond to the exact definition concerning a cut and the geometry of the cut surface. This is better referred to as *gradient theories of the n-th order*, it being understood that the standard Cauchy theory in fact is a *theory of the first gradient* (meaning by this first gradient of the displacement or theory involving just the strain and no gradient of it in the constitutive equations).

1.3.2.1 Gradient Theories

Now, as a matter of fact, gradient theories abound in physics, starting practically with all continuum theories in the 19^{th} century. Thus, Maxwell's electromagnetism is a first-gradient theory (of the electromagnetic potentials); the Korteweg theory of fluids [47] is a theory of the first gradient of density (equivalent to a second-gradient theory of displacement in elasticity); Einstein's theory of gravitation (general relativity [16, 17]) is a second-gradient theory of the metric of curved space-time, and Le Roux [60, 61] seems to be the first public exhibition of a second-gradient theory of (displacement) elasticity in small strains (using a variational formulation). There was a renewal of such theories in the 1960s with the works of Casal [8] on capillarity, and of Toupin [100], Mindlin and Tiersten [81], Mindlin and Eshel [80], and Grioli [42] in elasticity.

However, it is with a neat formulation basing on the *principle of virtual power* that some order was imposed in these formulations with an unambiguous deduction of the (sometimes tedious) boundary conditions and a clear introduction of the notion of *internal forces* of higher order, *i.e.*, *hyperstresses* of various orders (see, Germain [38, 39]; Maugin [66]). Phenomenological theories involving gradients of other physical fields than displacement or density, coupled to deformation, were envisaged consistently by the author in his Princeton doctoral thesis [64] dealing with typical ferroïc electromagnetic materials. This is justified by a microscopic approach, *i.e.*, the continuum approximation of a crystal lattice with medium-range interactions; with distributed magnetic spins or permanent electric dipoles. This also applies to the pure mechanical case (see, for instance, the Boussinesq paradigm in

Christov *et al.* [9, 75]). The following are examples of such theories illustrated by the dependence of the potential energy W per unit volume for small strains:

- **Le Roux** [60, 61]:
$$W = W(u_{i,j}, u_{i,jk}, \ldots), \tag{1.11}$$
where $u_{i,j}$ denotes the displacement gradient, and $u_{i,jk}$ is the second gradient of the displacement.
- **Modern form** (Mindlin [78, 80, 81], Toupin [100, 101], Sedov [95], Germain [38], *etc.*; in the period 1962–1972):
$$W = W(e_{ij}, e_{ij,k}). \tag{1.12}$$
In the last case, *the symmetric first-order stress* $\bar{\sigma}_{ji}$ and the second-order stress or *hyperstress* m_{kji} (symmetric in its last two indices) are given by
$$\bar{\sigma}_{ji} = \frac{\partial W}{\partial e_{ij}} = \bar{\sigma}_{ij}, \quad m_{kji} = \frac{\partial W}{\partial e_{ij,k}} = m_{kij}, \tag{1.13}$$
where e_{ij} is the symmetric small strain, and $e_{ij,k}$ denotes its first gradient. Then the symmetric Cauchy stress reads
$$\sigma_{ji} = \bar{\sigma}_{ji} - m_{kji,k} = \frac{\delta W}{\delta e_{ij}} = \sigma_{ij}. \tag{1.14}$$
This is the functional derivative of the energy W.

Very interesting features of these models are:

F1. Inevitable introduction of characteristic lengths;
F2. Appearance of so-called capillarity effects (surface tension) due to the explicit intervening of curvature of surfaces;
F3. Correlative boundary layers effects,
F4. Dispersion of waves with a possible competition and balance between non-linearity and dispersion, and the existence of solitonic structures (see Maugin [70]);
F5. Intimate relationship with the Ginzburg–Landau theory of phase transitions [12, 93] and, for fluids, van der Waals' theory [4, 104].

Indeed, a typical characteristic length ℓ is introduced by the ratio
$$\ell = \frac{|m_{kji}|}{|\bar{\sigma}_{ji}|}, \tag{1.15}$$
and this is obviously supposed to be much smaller than a typical macroscopic length L, *i.e.*, $\ell \ll L$.

Features F2 and F3 above are typically illustrated by the following set of boundary conditions [38, 99] ($\Omega = -D_j n_j/2$ is the mean curvature)

$$n_j \sigma_{ji} + (n_j D_p n_p - D_j)(n_k m_{kji}) = T_i^d \quad \text{at } \partial B - \Gamma\uparrow, \tag{1.16}$$
$$n_k m_{kji} n_j = R_i \quad \text{at } \partial B - \Gamma\uparrow, \tag{1.17}$$
$$\varepsilon_{ipq} \tau_p [n_k m_{kjq} n_j] = E_i \quad \text{along } \Gamma\uparrow, \tag{1.18}$$

where $\Gamma\uparrow$ is an oriented edge, τ_p denotes its unit tangent, and D indicates a tangential gradient. Here T_i^d, R_i and E_i are, respectively, an applied surface traction, a prescribed double-normal force, and a linear force density.

Remark 1.5. The principle of virtual power here is an interesting tool to obtain the set (1.16)–(1.18) unambiguously. But it also shows in agreement with Eq. (1.14) that the power of internal forces can be written either as

$$p_{(\text{int})}(\sigma) = -\sigma : \nabla \dot{\mathbf{u}}, \tag{1.19}$$

or as

$$p_{(\text{int})}(\bar{\sigma}, \mathbf{m}) = -(\bar{\sigma} : \nabla \dot{\mathbf{u}} + \mathbf{m} : \mathbb{W} \;\; \dot{\mathbf{u}}), \tag{1.20}$$

so that

$$p_{(\text{int})}(\sigma) = p_{(\text{int})}(\bar{\sigma}, \mathbf{m}) + \nabla \cdot (\mathbf{m} : \nabla \dot{\mathbf{u}}). \tag{1.21}$$

Here we used the convention that

$$\bar{\sigma} : \nabla \dot{\mathbf{u}} = \sigma_{ji} \dot{u}_{i,j}, \quad \mathbf{m} : \mathbb{W} \;\; \dot{\mathbf{u}} = m_{kji} \dot{u}_{i,jk}.$$

Repeated use of the divergence theorem will then directly lead to the set (1.16)–(1.18).

Truly sophisticated examples of the application of these gradient theories are found in

(i) the coupling of a gradient theory (of the carrier fluid) and consideration of a microstructure in the study of the inhomogeneous diffusion of microstructures in polymeric solutions (Drouot and Maugin [14]).
(ii) the elimination of singularities in the study of structural defects (dislocations, disclinations) in elasticity combining higher-order gradients and polar microstructure (*cf.* Lazar and Maugin [58]).

Most recent works consider the application of the notion of gradient theory in elastoplasticity for nonuniform plastic strain fields (works by Aifantis [2, 3, 107], Fleck & Hutchinson [33, 34, 35], and many others) – but see the thermodynamically admissible formulation in Maugin [68].

In so far as general mathematical principles at the basis of the notion of gradient theory are concerned, we note the fundamental works of Noll and Virga [84] and dell'Isola and Seppecher [13], the latter with a remarkable economy of thought.

1.3.2.2 Strongly Nonlocal Theory (Spatial Functionals)

Initial concepts in this framework were etablished by Kröner and Datta [50], Kunin [51, 52], Rogula [92], Eringen and Edelen [29]. As a matter of fact, the Cauchy construct does *not* apply anymore. In principle, only the case of **infinite bodies** should be considered as any cut would destroy the prevailing long-range ordering. Constitutive equations become integral expressions over space, perhaps with a more or less rapid attenuation with distance of the spatial kernel. This, of course, inherits from the action-at-a-distance dear to the Newtonians, while adapting the disguise of a continuous framework. This view is justified by the approximation of an infinite crystal lattice: the relevant kernels can be justified through this discrete approach. Of course this raises the matter of solving integro-differential equations instead of partial-differential equations. What about boundary conditions that are in essence foreign to this representation of matter-matter interaction? There remains a possibility of the existence of a "weak-nonlocal" limit by the approximation by gradient models. Typically one would consider in the linear elastic case a stress constitutive equation in the form

$$\sigma_{ji}(\mathbf{x}) = \int_{\text{all space}} C_{jikl}\left(|\mathbf{x}-\mathbf{x}'|\right) e_{kl}(\mathbf{x}')\, \mathrm{d}^3\mathbf{x}', \tag{1.22}$$

where the constitutive functions C_{jikl} decreases markedly with the distance between material points $\mathbf{x}'$ and $\mathbf{x}$. In space of one dimension, an inverse to Eq. (1.22) may be of the form

$$\sigma - K\nabla^2\sigma = E\, e \tag{1.23}$$

with coefficients K and E, a model that we call "Helmholtz's" one because of the presence of the Laplacian ∇^2 that reflects the equivalence of interactions to the "right" and the "left". It is this kind of relation that allows one to compare the effects of "weakly" and "strongly" nonlocal theories in so far as the degree of singularity of some quantities is concerned (*cf.* Lazar and Maugin [58]).

The historical moment in the recognition of the usefulness of strongly nonlocal theories was the EUROMECH colloquium on nonlocality organized by D. Rogula in Warsaw in 1977 (*cf.* Maugin [65]). A now standard reference is Eringen's book [28]. A recent much publicized application of the concept of nonlocality is that to *damage* by Pijaudier-Cabot and Bažant [88].

Note in conclusion to this point that any field theory can be generalized to a non-local one while saving the notions of linearity and anisotropy; but loosing the usual notion of flux. Also, it is of interest to pay attention to the works of Lazar and Maugin [56, 57] for a comparison of field singularities in the neighborhood of structural defects in different "generalized" theories of elasticity (micropolar, gradient-like, strongly non local or combining these).

1.3.3 Loss of the Euclidean Nature of the Material Manifold

Indeed the basic relevant problem emerges as follows. How can we represent **geometrically** the fields of structural defects (such as **dislocations** associated with a loss of continuity of the elastic displacement, or **disclinations** associated with such a loss for rotations)? A similar question is raised for **vacancies and point** defects. One possible answer stems from the consideration of a non-Euclidean material manifold, *e.g.*, a manifold without curvature but with affine connection, or an Einstein-Cartan space with **both** torsion and curvature, *etc.* With this one enters a true "geometrization" of continuum mechanics of which conceptual difficulties compare favorably with those met in modern theories of gravitation. Pioneers in the field in the years 1950-70 were K. Kondo [46] in Japan, E. Kröner [48] in Germany, Bilby [5] in the UK, Stojanovic [96] in what was then Yugoslavia, W. Noll [83] and C.C. Wang [105] in the USA. Modern developments are due to, among others, M. Epstein and the author [18, 19], M. Elzanowski and S. Preston (see the theory of material inhomogeneities by Maugin, [69]). Main properties of this type of approach are

(i) the relationship to the multiple decomposition of finite strains (Bilby, Kröner, Lee) and
(ii) the generalization of theories such as the theory of volumetric growth (Epstein and Maugin [20]) or the theory of phase transitions within the general **theory of local structural rearrangements** (local evolution of reference; see Maugin [72], examining Kröner's inheritance and also the fact that true **material inhomogeneities** (dependence of material properties on the material point) are then seen as **pseudo-plastic effects** [71]).

All local structural rearrangements and other physical effects (*e.g.*, related to the diffusion of a dissipative process) are reciprocally seen as pseudo material inhomogeneities [72]. Many of these advances are first-hand critically expanded in a recent book [74]. An original geometric solution is presented in the book of Rakotomanana [91] which offers a representation of a material manifold that is everywhere dislocated. Introduction of the notion of fractal sets opens new horizons (*cf.* Li and Ostoja–Starzewski, [63]). An antiquated forerunner work of all this may be guessed in Burton [6], but only with obvious good will by a perspicacious reader.

1.4 Conclusions

Since the seminal work of the Cosserats, three more or less successful paths have been taken towards the generalization of continuum mechanics. These were recalled above. An essential difference between the bygone times of the pioneers and now is that artificial materials can be man-made that are indeed generalized continua. In addition, mathematical methods have been developed (homogenization techniques) that allow one to show that generalized continua are deduced as macroscopic continuum limits of some structured materials. This is illustrated by the book of Forest [36].

In conclusion, we can answer three basic questions that are clearly posed:

(1) Do we need GCM at all?
(2) Do we find the necessary tools in what exists nowadays?

(3) What is the relationship between discrete and continuous descriptions if there must exist a consistent relationship between the two?

The first two questions are positively answered in view of the above described developments. The third question is of a different nature because, in principle, continuum theories can be developed independently of any precise microscopic vision, being judged essentially on their inherent logical structure, the possibility to have access through appropriate experiments to the material constants they introduce, and finally their efficiency in solving problems. However, in contrast to those hard-line continuum theoreticians, we personally believe that any relationship that can be established with a sub-level degree of physical description is an asset that no true physicist can discard.

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