Chapter 2
Hydrodynamic Principles

2.1 General

The hydrostatic phenomenon is simplified by the absence of shear stress within the fluid and in contact with the solid boundary. In contrast, the hydrodynamic phenomenon becomes rather complex. As the fluid flows over a solid boundary, whether stationary or moving, the fluid velocity in contact with the boundary must be the same as that of the boundary, termed *no-slip*. Thus, a velocity gradient is created over the boundary, as the fluid velocity increases with the normal distance from the boundary. The resulting differential velocity normal to the boundary gives rise to the shear stress within the fluid and on the boundary, as already discussed in Newton's law of viscosity, Eq. (1.3). Fluid flows as a result of the action of forces set up by the pressure difference or the gravity. Fluid motion is controlled by the inertia of fluid and the effect of the shear stress exerted by the surrounding fluid. Therefore, the resulting fluid motion cannot be easily analyzed theoretically; and the theories are often essentially supplemented by the experiments. The fluid motion can be defined in the following ways:

The path traced by an individual fluid particle over a period of time is known as *pathline*, which describes the trajectory (position at successive intervals of time) of a particle that started from a given position. On the other hand, *streakline* provides an instantaneous picture of the positions of all the particles which have passed through a particular point at a given time. Streakline is therefore the locus of points of all the fluid particles that have passed continuously through a particular spatial point in the past. Since the flow characteristics may vary from instant to instant, a streakline is not necessarily the same as a pathline.

In analyzing a fluid flow, one often makes use of the idea of a *streamline*, which is an imaginary line whose tangents at every point along the imaginary line represent velocity vectors at that moment. It implies that at a given instant, there is no flow across the streamline. Since there can be no flow through a solid boundary, the streamline in contact with or nearest to the solid boundary is known as *limiting streamline*. Let us consider a particle moves in the direction of the streamline at any instant; it has a displacement d**s** having components dx, dy, and dz and the

S. Dey, *Fluvial Hydrodynamics*, GeoPlanet: Earth and Planetary Sciences,
DOI: 10.1007/978-3-642-19062-9_2,

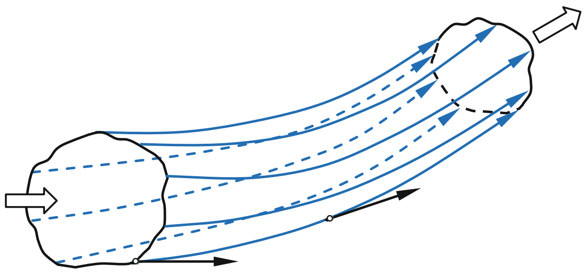


Fig. 2.1 Streamtube, where *curved lines* with *arrows* represent streamlines. The two *arrows* tangential to the lowest streamline show the velocity vectors at those points

velocity vector **v** having components u, v, and w in the x-, y-, and z-direction, respectively. Then, the equation of streamline is

$$\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}y}{v} = \frac{\mathrm{d}z}{w} \tag{2.1}$$

For a continuous stream of fluid, streamlines are continuous lines extending to infinity upstream and downstream or forming closed curves, but they cannot intersect. When flow conditions are steady and laminar, then the pathlines and the streamlines are identical; however, if the flow is fluctuating or turbulent, this is not the case. A family of streamlines through every point on the perimeter of a small area of the fluid flow cross section forms a *streamtube* (Fig. 2.1). Since there is no flow across a streamline, no fluid can enter or leave a streamtube except through its ends. It thus behaves as if it were a solid tube.

The *Lagrangian approach* of the fluid flow is the method of looking at fluid motion, where the observer follows an individual fluid particle as it moves through space and time. To illustrate its use, let ($x_A(x_0, y_0, z_0, t)$, $y_A(x_0, y_0, z_0, t)$, $z_A(x_0, y_0, z_0, t)$) be the position at an instant t of a fluid particle that had an initial position (x_0, y_0, z_0) at time t_0. Hence, by definition, $x_A(x_0, y_0, z_0, t = t_0) = x_0(x_0, y_0, z_0)$. The velocity components are given by

$$\left.\begin{aligned} u(x_0, y_0, z_0, t) &\equiv \lim_{\Delta t \to 0} \frac{x_A(x_0, y_0, z_0, t_0 + \Delta t) - x_A(x_0, y_0, z_0, t_0)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{x_A - x_0}{\Delta t} = \frac{\partial x_A(x_0, y_0, z_0, t)}{\partial t} \\ v(x_0, y_0, z_0, t) &= \frac{\partial y_A(x_0, y_0, z_0, t)}{\partial t} \\ w(x_0, y_0, z_0, t) &= \frac{\partial z_A(x_0, y_0, z_0, t)}{\partial t} \end{aligned}\right\} \tag{2.2}$$

In the above, $\Delta t = t - t_0$. The partial derivatives signify that the differentiation is performed keeping initial position (x_0, y_0, z_0) fixed. Then, the acceleration components are given by

$$\left.\begin{aligned} a_x(x_0, y_0, z_0, t) &= \frac{\partial u(x_0, y_0, z_0, t)}{\partial t} = \frac{\partial^2 x_A(x_0, y_0, z_0, t)}{\partial t^2} \\ a_y(x_0, y_0, z_0, t) &= \frac{\partial v(x_0, y_0, z_0, t)}{\partial t} = \frac{\partial^2 y_A(x_0, y_0, z_0, t)}{\partial t^2} \\ a_z(x_0, y_0, z_0, t) &= \frac{\partial w(x_0, y_0, z_0, t)}{\partial t} = \frac{\partial^2 z_A(x_0, y_0, z_0, t)}{\partial t^2} \end{aligned}\right\} \tag{2.3}$$

On the other hand, the *Eulerian approach* of the fluid flow is the method of looking at fluid motion that focuses on specific locations in the space through which the fluid flows, as over the time. To describe velocity components, it is written as

$$\left.\begin{aligned} u &= u(x, y, z, t) \\ v &= v(x, y, z, t) \\ w &= w(x, y, z, t) \end{aligned}\right\} \tag{2.4}$$

Then, to determine the acceleration, having known that as the acceleration means the rate of change of velocity of a fluid particle at a position while noting that the particle moves from that position at the time it is being studied, the acceleration component in x-direction is

$$\left.\begin{aligned} a_x &= \lim_{\Delta t \to 0} \frac{\Delta u(x + u\Delta t, y + v\Delta t, z + w\Delta t, t + \Delta t)}{\Delta t} \\ &= u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \\ a_y &= u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \\ a_z &= u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} \end{aligned}\right\} \tag{2.5}$$

Note that the first three terms in the right-hand side of Eq. (2.5) are referred to as *convective acceleration* (also occasionally called *advective acceleration*) and the last terms in the right-hand side of Eq. (2.5) are the *local acceleration* (also occasionally called *temporal acceleration*).[1] The convective terms are quadratic in the velocity components and hence they are nonlinear. This introduces the major complexity in having the solution of the equations of fluid motion. On the other hand, as the Lagrangian approach does not have nonlinearity, one might thought that it could be relatively convenient to use. This is, however, not the case, as the

[1] *Convective acceleration* is the acceleration of fluid due to space at a given time, while the *local acceleration* is the acceleration of fluid due to time at a given spatial location.

internal force terms due to viscosity introduced by the Newton's laws become nonlinear in the Lagrangian approach. Further, the physical laws, such as the Newton's laws and the laws of conservation of mass and energy, apply directly to each particle in the Lagrangian approach. However, the fluid flow is a continuum phenomenon, at least down to the molecular level. It is not possible to track each particle in a complex flow field. Thus, the Lagrangian approach is rarely used in hydrodynamics.

In the Eulerian approach, individual fluid particles are not tracked; instead, a control volume is defined. The flow parameters are described as fields within the control volume, expressing them as a function of space and time. Hence, one is not concerned about the location or velocity of a particular fluid particle, but rather about the velocity, acceleration, etc. of whatever particle happens to be at a particular location and at a given time. Since the fluid flow is a continuum phenomenon, the Eulerian approach is usually preferred in hydrodynamics. Notwithstanding that the physical laws, such as the Newton's laws and the laws of conservation of mass and energy, apply directly to particles in a Lagrangian approach, some transformations or reformulations of these laws are required for the use with the Eulerian approach.

2.2 Rates of Deformation

In a fluid flow, if the fluid elements do not undergo rotation as it flows, then the flow is called *irrotational*. In consideration of a frictionless or ideal fluid flow, no shear stresses act on the surface of the elements. Only normal stresses or pressures act following the Pascal's law. Then, the resultant of all surface forces acting on the element should pass through the centroid of the element irrespective of its shape. As a result , there can be no turning moment on the element and it remains in the same orientation at all its locations provided the element remains undisturbed initially. On the other hand, rotation of elements is inevitable, where viscous forces come into play. In a real fluid flow, a fluid element gets distorted as it moves. Note that distortion is not always rotation which is identified by the change in orientation of the diagonal or the axis of the element. An element may, however, get distorted without undergoing rotation or vice versa. A fluid element can undergo four types of motion or deformation: (1) translation, (2) rotation, (3) extensional strain, and (4) shear strain. These types of motion are time dependent.

Consider a rectangular fluid element $ABCD$ at time t and then after elapsing a small interval of time $\mathrm{d}t$, the element undergoes four types of motion to become $A'B'C'D'$ at time $t + \mathrm{d}t$, as shown in Fig. 2.2. The translation is defined by the displacement $u\mathrm{d}t$ and $v\mathrm{d}t$ of the corner B. The rotation of the diagonal BD is expressed as $\omega_z \mathrm{d}t = \theta + \mathrm{d}\alpha - 45°$, where ω_z denotes the angular velocity or rate of rotation about an axis parallel to z-axis.

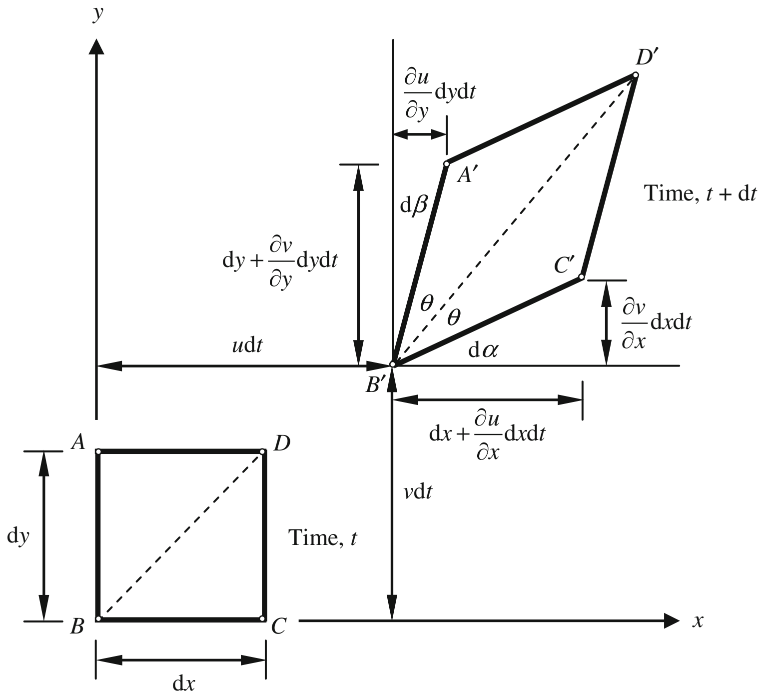


Fig. 2.2 Deformation of a moving fluid element

Using the summation of angles $2\theta + \mathrm{d}\alpha + \mathrm{d}\beta = 90°$ yields

$$\omega_z \mathrm{d}t = \frac{1}{2}(\mathrm{d}\alpha - \mathrm{d}\beta) \tag{2.6}$$

Referring to Fig. 2.2, $\mathrm{d}\alpha$ and $\mathrm{d}\beta$ are expressed as

$$\mathrm{d}\alpha = \lim_{\mathrm{d}t \to 0} \arctan\left(\frac{\frac{\partial v}{\partial x}\mathrm{d}x\mathrm{d}t}{\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}x\mathrm{d}t}\right) \approx \frac{\partial v}{\partial x}\mathrm{d}t \tag{2.7a}$$

$$\mathrm{d}\beta = \lim_{\mathrm{d}t \to 0} \arctan\left(\frac{\frac{\partial u}{\partial y}\mathrm{d}y\mathrm{d}t}{\mathrm{d}y + \frac{\partial v}{\partial y}\mathrm{d}y\mathrm{d}t}\right) \approx \frac{\partial u}{\partial y}\mathrm{d}t \tag{2.7b}$$

Substituting Eqs. (2.7a, b) into Eq. (2.6), the rate of rotation or angular velocity about z-axis is obtained as

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{2.8}$$

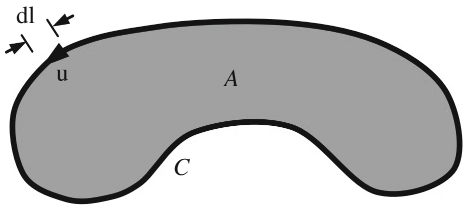


Fig. 2.3 Definition sketch for fluid circulation

Similarly, the rates of rotation about *x*- and *y*-axis are

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \tag{2.9}$$

Note that if the rates of rotational components vanish, then the flow is irrotational, for which the conditions are

$$\omega_x = \omega_y = \omega_z = 0 \Rightarrow \frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \tag{2.10}$$

The *vorticity* is the tendency of a fluid element to spin. The components of vorticity in three dimensions are expressed as follows:

$$\Omega_x = 2\omega_x = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \ \Omega_y = 2\omega_y = \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \ \Omega_z = 2\omega_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{2.11}$$

The *circulation* is the line integral around a closed curve of the fluid velocity (Fig. 2.3). It can be considered as the amount of push along a closed boundary or path. Thus, it provides an estimation of the strength of the rotational flow. Circulation can be related to the vorticity by the *Stokes theorem* as

$$\Gamma = \oint_C \mathrm{u} \cdot \mathrm{dl} = \int\int_A \Omega \mathrm{d}A \tag{2.12}$$

where dl is the linear increment along the contour *C* and *A* is the area within the contour.

The components of circulation in three dimensions are expressed as follows:

$$\Gamma_x = \int_y\int_z \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathrm{d}y\mathrm{d}z, \ \Gamma_y = \int_z\int_x \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathrm{d}z\mathrm{d}x,$$
$$\Gamma_z = \int_x\int_y \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathrm{d}x\mathrm{d}y \tag{2.13}$$

The two-dimensional shear strain rate is generally defined as the average decrease in angle between two lines which are initially perpendicular to each other before the strained state. Taking *AB* and *BC* as initial lines (Fig. 2.2), the *shear strain rate* is

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\mathrm{d}\alpha}{\mathrm{d}t} + \frac{\mathrm{d}\beta}{\mathrm{d}t}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{2.14}$$

Similarly, other components of shear strain rate are

$$\varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right), \ \varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \tag{2.15}$$

The extensional strain in x-direction is defined as the fractional increase in length of horizontal side of the element. The *normal strain rate* in x-direction is

$$\varepsilon_{xx}\mathrm{d}t = \frac{1}{\mathrm{d}x}\left[\left(\mathrm{d}x + \frac{\partial u}{\partial x}\mathrm{d}x\mathrm{d}t\right) - \mathrm{d}x\right] = \frac{\partial u}{\partial x}\mathrm{d}t \Rightarrow \varepsilon_{xx} = \frac{\partial u}{\partial x} \tag{2.16}$$

Similarly, other components of normal strain rate are

$$\varepsilon_{yy} = \frac{\partial v}{\partial y}, \ \varepsilon_{zz} = \frac{\partial w}{\partial z} \tag{2.17}$$

2.3 Conservation of Mass

Except in the relativistic processes ($E = mc^2$, where E is the energy, m is the mass of the matter, and c is the speed of the light) after Albert Einstein in 1905, matter is neither created nor destroyed. This principle of *conservation of mass* can be applied to the fluid flow.

Considering an enclosed region in the flow constituting a control volume (CV) (Fig. 2.4), the equation of conservation of mass can be written in terms of mass flux as

$$\begin{aligned}\text{Mass flux entering} &= \text{Mass flux leaving} \\ &\quad + \text{Change of mass in the CV per unit time}\end{aligned}$$

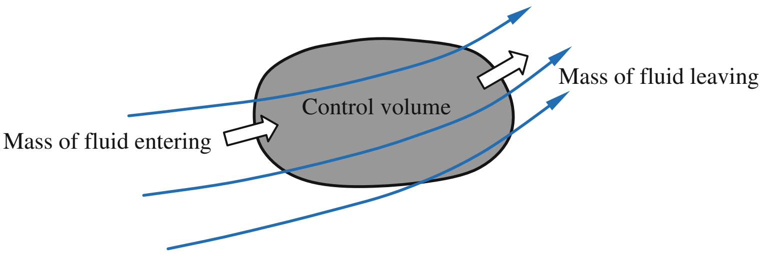
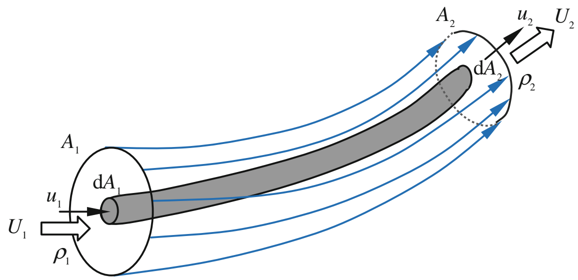


Fig. 2.4 Control volume in a fluid flow

Fig. 2.5 Definition sketch for the fluid flow through a streamtube showing an elementary streamtube



For the steady flow, there is no change of mass of fluid in the *CV* and the relation reduces to

$$\text{Mass flux entering} = \text{Mass flux leaving}$$

Applying this principle to a steady flow in a streamtube (Fig. 2.5) having an elementary cross-sectional area, through which the velocity to be considered as constant across the cross section, since there can be no flow across the wall of a streamtube, the conservation of mass for the region between sections 1 and 2 is

$$\rho_1 u_1 \mathrm{d}A_1 = \rho_2 u_2 \mathrm{d}A_2 = \mathrm{d}\dot{m} = \text{constant} \tag{2.18}$$

The mass influx or mass entering per unit time at section 1 equals the mass efflux or mass leaving per unit time at section 2. In Fig. 2.5, u is the velocity through the elementary cross-sectional area dA, ρ is the mass density of fluid and the subscripts denote sections. Therefore, for a steady flow, it implies that the mass flow rate, termed *mass flux* d$\dot{m}$, across any cross section of the elementary streamtube is constant. This is known as the *continuity equation* for the compressible fluid flow through an elementary streamtube. Therefore, the continuity equation of the fluid flow for the entire cross section of the streamtube can be obtained integrating Eq. (2.18) as

$$\rho_1 U_1 A_1 = \rho_2 U_2 A_2 = \dot{m} = \text{constant} \quad \wedge \quad U = \frac{1}{A}\int_A u\mathrm{d}A \tag{2.19}$$

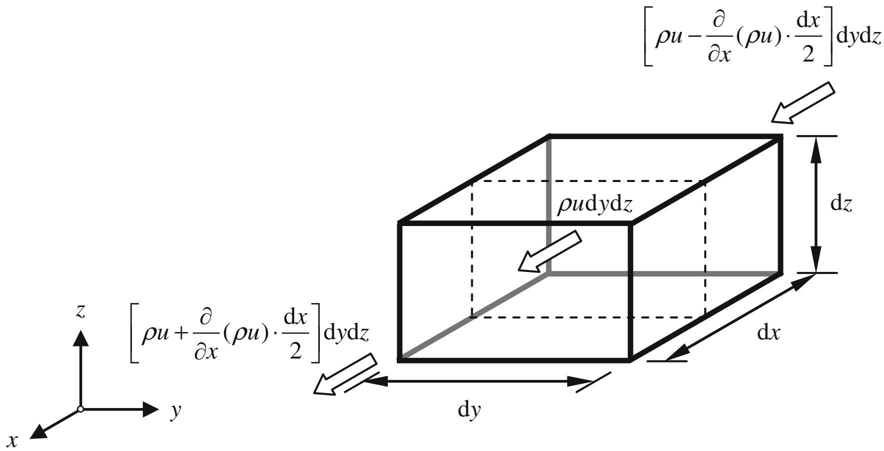


Fig. 2.6 Definition sketch for the derivation of three-dimensional continuity equation of fluid flow in a control volume element

where U is the average velocity through the cross-sectional area A.

If the fluid is incompressible, then $\rho_1 = \rho_2 = \rho$; and Eq. (2.19) reduces to

$$U_1A_1 = U_2A_2 = Q = \text{constant} \tag{2.20}$$

where Q is the discharge or volume rate of flow.

2.3.1 Continuity Equation in Three Dimensions

Differential mode of continuity equation is used to analyze two- and three-dimensional flows. To derive three-dimensional continuity equation of fluid flow, a control volume element of fluid $dxdydz$, having a mass density ρ, with a center at (x, y, z) in a Cartesian coordinate system is considered (Fig. 2.6). The velocity components in x-, y- and z-direction are u, v, and w, respectively. The mass influx of fluid flow through the back face of the control volume by advection in the x-direction is given by

$$\left[\rho u - \frac{\partial}{\partial x}(\rho u)\cdot\frac{dx}{2}\right]dydz$$

In the above expression, the first term, $\rho u dydz$, is the mass influx through the central plane normal to the x-axis, as shown by the broken line in Fig. 2.6. The second term, $[\partial(\rho u)/\partial x](dx/2)dydz$, is the change of mass flux with respect to distance in x-direction multiplied by the distance $dx/2$ to the back face.

Similarly, the mass efflux through the front face of the control volume in x-direction is given by

$$\left[\rho u + \frac{\partial}{\partial x}(\rho u)\cdot\frac{\mathrm{d}x}{2}\right]\mathrm{d}y\mathrm{d}z$$

Therefore, the net mass flux out in x-direction through these two faces is obtained as

$$\frac{\partial}{\partial x}(\rho u)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

The other two directions yield similar expressions; and hence, the net mass flux out of the control volume is

$$\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

From the concept of conservation of mass, the net mass flux out of the control volume plus the rate of change of mass in the control volume, given by $(\partial\rho/\partial t)\mathrm{d}x\mathrm{d}y\mathrm{d}z$, equals the rate of production of mass in the control volume, which is zero, by definition of the conservation of mass. Thus, the three-dimensional continuity equation of fluid flow is given by

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = -\frac{\partial\rho}{\partial t} \tag{2.21}$$

which must hold for every point in the flow of a compressible fluid. For incompressible fluid flow (ρ = constant), Eq. (2.21) simplifies to

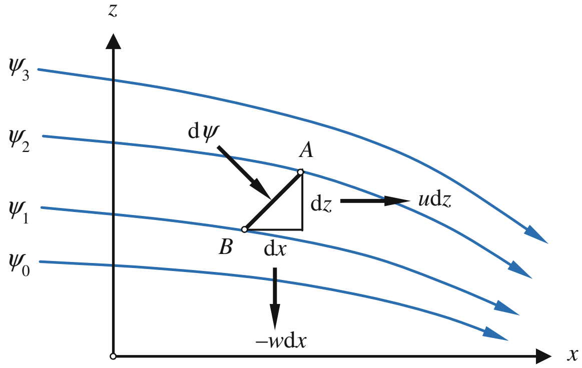
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.22}$$

The kinematic relation in terms of the components of normal strain rate can be obtained using Eqs. (2.16) and (2.17) into Eq. (2.22). It is

$$\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = 0 \tag{2.23}$$

Thus, the sum of the components of normal strain rate vanishes to satisfy the continuity. If Eq. (2.22) reduces to only two terms, regardless of the coordinate system, a useful device is to introduce the so-called *stream function* ψ, defined so as to satisfy the continuity identically. For example, for two-dimensional incompressible flow in xz plane, the continuity equation (Eq. 2.22) reduces to

Fig. 2.7 The stream function



$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{2.24}$$

If the stream function is defined as ψ such that

$$u = \frac{\partial \psi}{\partial z}, \ w = -\frac{\partial \psi}{\partial x} \tag{2.25}$$

By direct substitution of Eq. (2.25), Eq. (2.24) is satisfied identically, assuming that the ψ is continuous to the second-order derivatives. The stream function has a useful physical significance:

$$\mathrm{d}\psi = \frac{\partial \psi}{\partial x}\mathrm{d}x + \frac{\partial \psi}{\partial z}\mathrm{d}z = -w\mathrm{d}x + u\mathrm{d}z \Rightarrow \psi = -\int w\mathrm{d}x + \int u\mathrm{d}z \tag{2.26}$$

Equation (2.26) implies that the line of constant $\psi(\mathrm{d}\psi = 0)$ is the line across which no flow takes place; that means it is a streamline. However, the difference between the values of stream functions ψ_1 and ψ_2 of two neighboring streamlines is numerically equal to the flow rate per unit width (denoted by Δq) between those two streamlines.

$$\psi_2 - \psi_1 = \Delta q \tag{2.27}$$

It is illustrated in Fig. 2.7, where the flow rate across section AB is $\mathrm{d}\psi$ explaining now the flow across the two streamlines $\mathrm{d}\psi = \Delta q = -w\mathrm{d}x + u\mathrm{d}z$.

2.3.2 *Continuity Equation for Open-Channel Flow*

The continuity equation of unsteady flow in open channel states that the difference of mass influx into and mass efflux out of the control volume must be equal to the

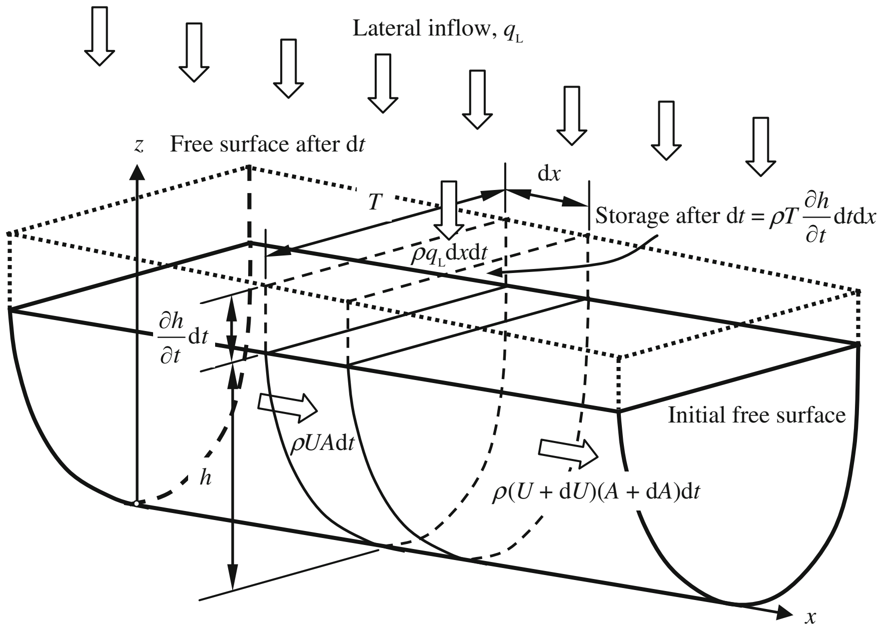


Fig. 2.8 Continuity of an unsteady flow in an open channel

rate of increase in fluid mass within the control volume. In Fig. 2.8, the initial free surface is shown by the solid lines, while the final free surface after a small interval of time dt is shown by the dotted lines. The flow is analyzed through a space between two sections having an elementary distance dx apart to form a control volume. The flow in the channel is fed laterally with a uniform flow rate q_L. Note that q_L may also arise in the form of seepage flow (injection) normal to the wetted perimeter of the channel. The mass influx in time dt into the control volume is

$$\underbrace{\rho U A \mathrm{d}t}_{\text{Main flow}} + \underbrace{\rho q_L \mathrm{d}x \mathrm{d}t}_{\text{Lateral flow}}$$

where U is the area-averaged flow velocity through left section and A is the flow area of the left section.

The mass efflux in time dt out of the control volume is

$$\rho(U + \mathrm{d}U)(A + \mathrm{d}A)\mathrm{d}t \Rightarrow \rho\left(U + \frac{\partial U}{\partial x}\mathrm{d}x\right)\left(A + \frac{\partial A}{\partial x}\mathrm{d}x\right)\mathrm{d}t$$

where $U + \mathrm{d}U$ is the area-averaged flow velocity through right section and $A + \mathrm{d}A$ is the flow area of the right section. Note that $\mathrm{d}U = (\partial U/\partial x)\mathrm{d}x$ and $\mathrm{d}A = (\partial A/\partial x)\mathrm{d}x$.

The rate of increase in fluid mass in time dt within the control volume is

$$\rho T \frac{\partial h}{\partial t} \mathrm{d}t \mathrm{d}x$$

where T is the top width of the flow and h is the initial flow depth.

The continuity of flow in an open channel is therefore given by

$$\rho U A \mathrm{d}t + \rho q_{\mathrm{L}} \mathrm{d}x \mathrm{d}t - \rho \left(U + \frac{\partial U}{\partial x} \mathrm{d}x \right) \left(A + \frac{\partial A}{\partial x} \mathrm{d}x \right) \mathrm{d}t = \rho T \frac{\partial h}{\partial t} \mathrm{d}t \mathrm{d}x \tag{2.28}$$

Simplifying,

$$U \frac{\partial A}{\partial x} + A \frac{\partial U}{\partial x} + T \frac{\partial h}{\partial t} = q_{\mathrm{L}} \tag{2.29}$$

Using $Q = UA$ and $\partial A = T \partial h$ at a given section, Eq. (2.29) becomes

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = q_{\mathrm{L}} \tag{2.30}$$

Further, using hydraulic depth $h_{\mathrm{d}} = A/T$ and $\partial A = T \partial h$ at a given section, Eq. (2.29) can be given as

$$U \frac{\partial h}{\partial x} + h_{\mathrm{d}} \frac{\partial U}{\partial x} + \frac{\partial h}{\partial t} = \frac{q_{\mathrm{L}}}{T} \tag{2.31}$$

Equations (2.30) and (2.31) are the two different forms of the *continuity equation for an unsteady flow in open channels*. For a rectangular channel with no lateral flow ($q_{\mathrm{L}} = 0$), Eq. (2.30) reduces to

$$\frac{\partial q}{\partial x} + \frac{\partial h}{\partial t} = 0 \tag{2.32}$$

where q is the discharge per unit width (=Uh). This equation was first introduced by de Saint-Venant (1871).

2.4 Conservation of Momentum

The momentum equation is a statement of Newton's second law of motion and relates the sum of the forces acting on a fluid element to its acceleration or the rate of change of momentum in the direction of the resultant force. The change of momentum flux in the control volume is obtained from the difference between the momentum efflux and the momentum influx in the control volume. Let us consider

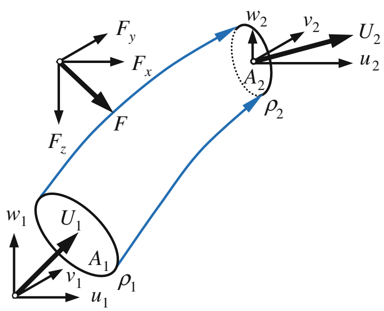


Fig. 2.9 Control volume as a streamtube with influx and efflux normal to the control sections

a simple case of flow through a streamtube, as shown in Fig. 2.9, denoting the flow parameters with subscripts 1 and 2 at the entrance and the exit, respectively. The rate of change of momentum in the horizontal direction according to Newton's second law of motion is caused a horizontal force component F_x, such that

$$F_x = \rho_2 A_2 U_2 u_2 - \rho_1 A_1 U_1 u_1 = \dot{m}(u_2 - u_1) \quad \wedge \quad \rho_1 A_1 U_1 = \rho_2 A_2 U_2 = \dot{m} \tag{2.33}$$

The value of F_x is positive in the direction in which u is assumed to be positive. Similarly, in three dimensions, F_y and F_z can be given as follows:

$$F_y = \dot{m}(v_2 - v_1), \; F_z = \dot{m}(w_2 - w_1) \tag{2.34}$$

To summarize the position, the total force exerted on the fluid in a control volume in a given direction equals the rate of change of momentum in the given direction of the fluid passing through the control volume. Therefore,

$$F = \dot{m}(U_{\text{out}} - U_{\text{in}}) \tag{2.35}$$

The nonuniform distribution (variation with the vertical distance) of velocity affects the computation of momentum in the flow based on the area-averaged velocity U (=Q/A). The actual and the area-averaged velocity distributions are illustrated in Fig. 2.10. Based on the area-averaged velocity, the momentum flux of the fluid passing through a section is expressed as $\beta \, \dot{m}U$, where β is known as the *momentum coefficient* or *Boussinesq coefficient*. The equation balancing the momentum flux calculated from the actual velocity distribution and that obtained from the area-averaged velocity corrected by β is used to determine momentum coefficient β as

$$\int_A (\rho u \mathrm{d}A \cdot u) = \beta \dot{m} U \Rightarrow \int_A \rho u^2 \mathrm{d}A = \beta \rho U^2 A \tag{2.36}$$

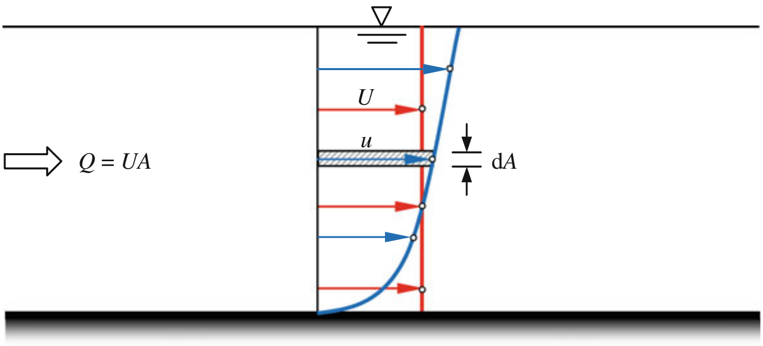


Fig. 2.10 Velocity distribution and area-averaged velocity in an open-channel flow

where u is the velocity through an elementary area dA. Solving for β yields

$$\beta = \frac{1}{A}\int_A \frac{u^2}{U^2}\mathrm{d}A \tag{2.37}$$

In straight prismatic channels, β roughly varies from 1.01 to 1.12 (Chow 1959).

2.4.1 Momentum Equation in Three Dimensions

2.4.1.1 Equations of Motion for Inviscid Flow (Euler Equations)

In Euler equations of motion, the resultant force on a fluid element equals the product of the fluid mass and its acceleration, acting in the direction of the resultant. A control volume element of fluid dxdydz, having a mass density ρ, with a center at (x, y, z) in a Cartesian coordinate system is considered (Fig. 2.11). Assuming that the fluid is inviscid (frictionless), the contact forces are pressure forces acting normally on the faces of the element. The pressure intensity at the center of the element is p. Let the component of the body force per unit mass in the x-direction be g_x. The extraneous force in the x-direction acting on the element is $g_x\rho$dxdydz. The net force in the x-direction is then

$$\begin{aligned} F_x &= \left(p - \frac{\partial p}{\partial x}\cdot\frac{\mathrm{d}x}{2}\right)\mathrm{d}y\mathrm{d}z - \left(p + \frac{\partial p}{\partial x}\cdot\frac{\mathrm{d}x}{2}\right)\mathrm{d}y\mathrm{d}z + g_x\rho\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \left(-\frac{\partial p}{\partial x} + g_x\rho\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z \end{aligned} \tag{2.38}$$

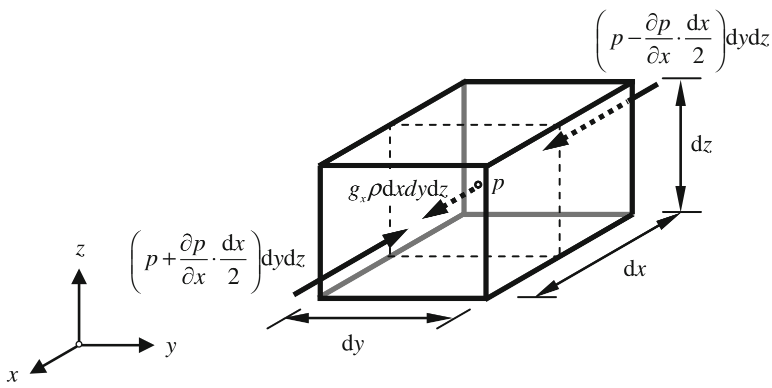


Fig. 2.11 Forces acting on a fluid element in x-direction

According to Newton's second law of motion, the net force F_x in x-direction equals the product of the mass and acceleration, that is, $(\rho dxdydz)a_x$. Hence, using Eq. (2.5) into Eq. (2.38) yields

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}=-\frac{1}{\rho}\cdot\frac{\partial p}{\partial x}+g_x \tag{2.39}$$

Similarly,

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{\partial v}{\partial t}=-\frac{1}{\rho}\cdot\frac{\partial p}{\partial y}+g_y \tag{2.40}$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\frac{\partial w}{\partial t}=-\frac{1}{\rho}\cdot\frac{\partial p}{\partial z}+g_z \tag{2.41}$$

where g_y and g_z are the body forces per unit mass in y- and z-direction, respectively. Equations (2.39)–(2.41) are known as the *Euler equations of motion*.

2.4.1.2 Equations of Motion for Viscous Flow (Navier–Stokes Equations)

Stress Components in Cartesian Coordinates: Nine stress components, as shown in Fig. 2.12, are acting on the faces of the three-dimensional fluid element, whose each face is normal to the coordinate axis of a Cartesian coordinate system. The normal stresses are denoted by σ, considering positive in the outwards and having a subscript to indicate its direction according to the axis. The effects of viscosity are to cause shear or tangential stresses in the fluid. The shear stresses are denoted by τ, having first subscript to indicate the direction of the normal to the plane over which the stress acts and the second subscript to indicate the direction.

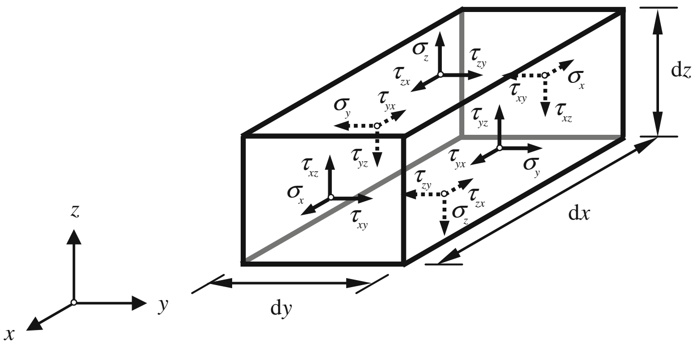


Fig. 2.12 Stress components on a fluid element in Cartesian coordinates

Let the moment be taken about an axis through the center of the element parallel to the z-axis to show that

$$\underbrace{\tau_{xy}\mathrm{d}y\mathrm{d}z}_{\text{Shear force}}\,\mathrm{d}x - \underbrace{\tau_{yx}\mathrm{d}x\mathrm{d}z}_{\text{Shear force}}\,\mathrm{d}y = 0 \Rightarrow \tau_{xy} = \tau_{yx} \tag{2.42}$$

Similarly,

$$\tau_{yz} = \tau_{zy},\ \tau_{zx} = \tau_{xz} \tag{2.43}$$

Hence, the stress components that define the state of stress at a point can be conveniently written in a matrix format as

$$\begin{matrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{matrix}$$

Equations of Motion in Terms of Stress Components: Referring to Fig. 2.13, the shear stresses are included in the equations of motion. Let the stress components at the center (x, y, z) of the fluid element be τ_{xy}, τ_{yz}, τ_{zx}, σ_x, σ_y, and σ_z that follow above matrix. Accordingly, the stresses are obtained on six faces of the fluid element shifting the positions by a distance of one half of the length of the element.

According to Newton's second law of motion, the product of the mass and acceleration of the element, that is, $(\rho\mathrm{d}x\mathrm{d}y\mathrm{d}z)a_x$, equals the summation of the forces (net force) acting on the element in the x-direction. Thus,

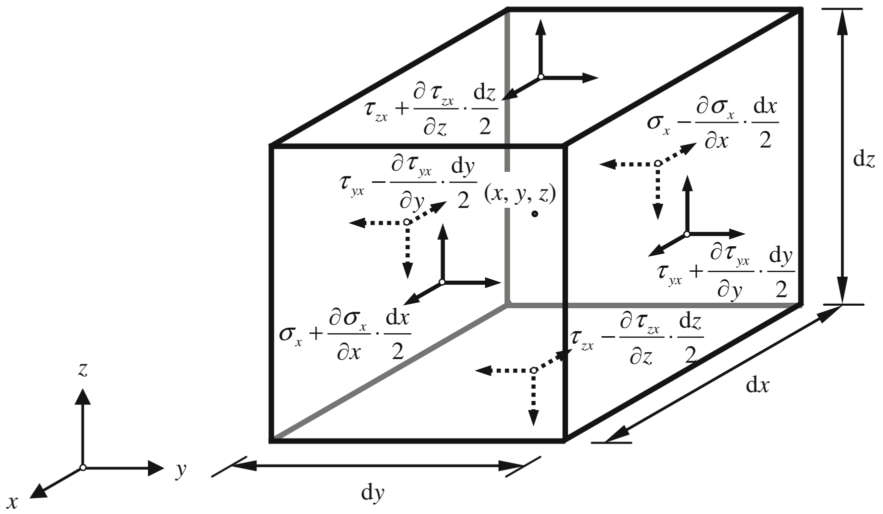


Fig. 2.13 Stress components on a fluid element in *x*-direction

$$
\begin{aligned}
(\rho \mathrm{d}x\mathrm{d}y\mathrm{d}z)a_x = & \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} \cdot \frac{\mathrm{d}x}{2}\right)\mathrm{d}y\mathrm{d}z - \left(\sigma_x - \frac{\partial \sigma_x}{\partial x} \cdot \frac{\mathrm{d}x}{2}\right)\mathrm{d}y\mathrm{d}z \\
& + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} \cdot \frac{\mathrm{d}y}{2}\right)\mathrm{d}x\mathrm{d}z - \left(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y} \cdot \frac{\mathrm{d}y}{2}\right)\mathrm{d}x\mathrm{d}z \\
& + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} \cdot \frac{\mathrm{d}z}{2}\right)\mathrm{d}x\mathrm{d}y - \left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z} \cdot \frac{\mathrm{d}z}{2}\right)\mathrm{d}x\mathrm{d}y \\
& + g_x \rho \mathrm{d}x\mathrm{d}y\mathrm{d}z
\end{aligned}
\tag{2.44}
$$

Dividing both sides of Eq. (2.44) by the element mass, $\rho \mathrm{d}x\mathrm{d}y\mathrm{d}z$, and taking the limit as the element reduces to a point, that is, $\mathrm{d}x\mathrm{d}y\mathrm{d}z \to 0$, the general form of equations of motion in three dimensions can be written using Eqs. (2.5) and (2.44) as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} = g_x + \frac{1}{\rho}\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) \tag{2.45a}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} = g_y + \frac{1}{\rho}\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) \tag{2.45b}$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} = g_z + \frac{1}{\rho}\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z}\right) \tag{2.45c}$$

In Newtonian fluid, both the normal and the shear stress components are related to the velocity gradients so that the viscous stresses are proportional to the shear strain rates. The normal stresses can be defined in terms of a linear deformation by the dynamic viscosity μ (=$\upsilon\rho$, where υ is the coefficient of kinematic viscosity) and a second viscosity μ_S to account for the volumetric deformation, defined as the sum of the velocity gradients or the components of normal strain rate along each of the three coordinate axes (Streeter 1948). The normal stresses are as follows:

$$\sigma_x = -p + 2\mu\frac{\partial u}{\partial x} + \mu_S\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \tag{2.46a}$$

$$\sigma_y = -p + 2\mu\frac{\partial v}{\partial y} + \mu_S\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \tag{2.46b}$$

$$\sigma_z = -p + 2\mu\frac{\partial w}{\partial z} + \mu_S\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \tag{2.46c}$$

In a three-dimensional case, extending the Newton's law of viscosity, the components of shear stress are

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right),\ \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right),\ \tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \tag{2.47}$$

The effect of the second viscosity μ_S is of secondary importance being small in practice. A good approximation is to set

$$\mu_S = -2\mu/3$$

that is, the *Stokes hypothesis*; and the pressure may be seen to be the average from Eqs. (2.46a–c) as

$$p = -\frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) \tag{2.48}$$

As an exemplar, using Eqs. (2.46a), (2.47) and (2.48) into Eq. (2.45a) and applying the Stokes hypothesis, the equation of motion in x-direction is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} = g_x - \frac{1}{\rho}\cdot\frac{\partial p}{\partial x} + \upsilon\nabla^2 u + \frac{\upsilon}{3}\cdot\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \tag{2.49}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Similarly,

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{\partial v}{\partial t}=g_y-\frac{1}{\rho}\cdot\frac{\partial p}{\partial y}+\upsilon\nabla^2 v+\frac{\upsilon}{3}\cdot\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) \quad (2.50a)$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\frac{\partial w}{\partial t}=g_z-\frac{1}{\rho}\cdot\frac{\partial p}{\partial z}+\upsilon\nabla^2 w+\frac{\upsilon}{3}\cdot\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) \quad (2.50b)$$

For unsteady and incompressible flow, by reference to the continuity equation (Eq. 2.22), Eqs. (2.49) and (2.50a, b) reduce to

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}=g_x-\frac{1}{\rho}\cdot\frac{\partial p}{\partial x}+\upsilon\nabla^2 u \quad (2.51a)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{\partial v}{\partial t}=g_y-\frac{1}{\rho}\cdot\frac{\partial p}{\partial y}+\upsilon\nabla^2 v \quad (2.51b)$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\frac{\partial w}{\partial t}=g_z-\frac{1}{\rho}\cdot\frac{\partial p}{\partial z}+\upsilon\nabla^2 w \quad (2.51c)$$

The above equations are known as the *equations of motion for viscous fluid flow* or the *Navier–Stokes equations*. These equations in other coordinate systems are given in Appendix (Sect. 2.9).

2.4.2 Momentum Equation for Open-Channel Flow

2.4.2.1 Momentum Equation for Gradually Varied Steady Flow

A gradually varied steady flow through an open channel, whose bed is inclined at an angle θ with the horizontal, is considered. Figure 2.14 shows the forces acting on the flow within the control volume between sections 1 and 2. The application of Newton's second law of motion, in a one-dimensional flow case, to this control volume along the streamwise direction is made equating the resultant of all the external forces acting on the fluid body with the rate of change of momentum in the flowing fluid body. Thus,

$$F_1-F_2+F_{\mathrm{W}}\sin\theta-F_{\mathrm{f}}=\rho Q(\beta_2U_2-\beta_1U_1) \quad (2.52)$$

where F_1 and F_2 are the resultants of the hydrostatic pressure forces in the direction of flow acting at sections 1 and 2, respectively, F_{w} is the weight of the

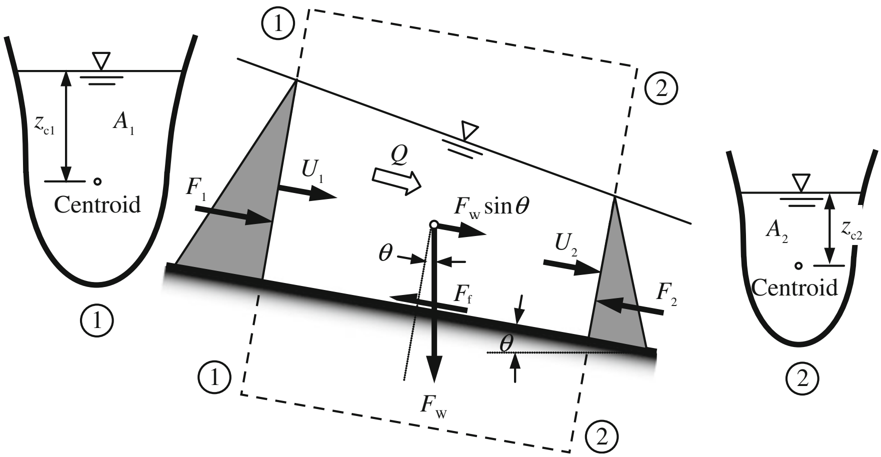


Fig. 2.14 Momentum principle applied to a gradually varied steady flow in an open channel. Forces acting on a control volume are shown

fluid within the control volume, F_f is the total external force due to frictional resistance acting along the contact surface between the fluid and the channel boundary, Q is the total flow discharge, U_1 and U_2 are the area-averaged flow velocities at sections 1 and 2, respectively, and β_1 and β_2 are the momentum coefficients at sections 1 and 2, respectively.

Assuming θ to be small or a horizontal bed ($\theta \approx 0$), F_f to be negligible for a short reach of a prismatic channel and also $\beta_1 = \beta_2 = 1$, Eq. (2.52) reduces to

$$F_1 - F_2 = \rho Q(U_2 - U_1) \tag{2.53}$$

The resultants of the hydrostatic pressure forces in the streamwise direction (that is, the horizontal direction for $\theta \approx 0$) acting on the plane flow areas A_1 and A_2 are expressed as

$$F_1 = \rho g z_{c1} A_1, \; F_2 = \rho g z_{c2} A_2 \tag{2.54}$$

where g is the acceleration due to gravity, and z_{c1} and z_{c2} are the distances to the centroid of respective flow areas A_1 and A_2 from the free surface. Substituting $U_1 = Q/A_1$, $U_2 = Q/A_2$, and Eq. (2.54) into Eq. (2.53) yields

$$\frac{Q^2}{gA_1} + z_{c1}A_1 = \frac{Q^2}{gA_2} + z_{c2}A_2 \tag{2.55}$$

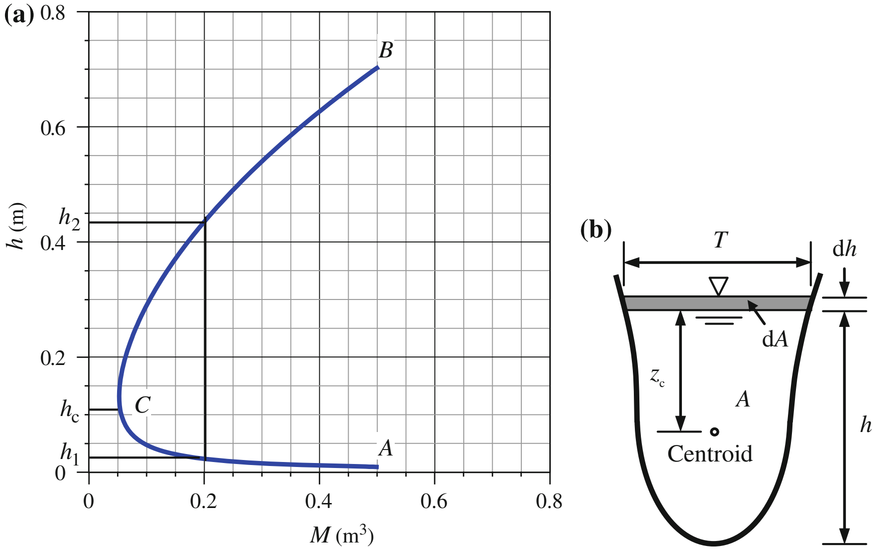


Fig. 2.15 **a** The specific force diagram and **b** a channel section

The left-hand and right-hand sides of Eq. (2.55) are analogous and therefore may be expressed by a general momentum or force function for any flow cross section as

$$M = \frac{Q^2}{gA} + z_c A \tag{2.56}$$

The first term of the right-hand side of Eq. (2.56) represents the momentum flux passing through the channel section per unit weight of fluid, and the second term is the force per unit weight of fluid. The sum of these two terms is called *specific force* and is denoted by M.

To illustrate the variation of specific force M with flow depth h given by Eq. (2.56), the *specific force diagram* [that is, $M(h)$ curve] for a given rectangular channel having width of 2 m carrying a flow discharge of 0.3 m^3 s^{-1} is drawn as shown in Fig. 2.15a. The $M(h)$ curve has two limbs, AC and BC. The lower limb AC asymptotically approaches the abscissa, while the upper limb BC rises upwards and extends indefinitely to the right. Thus, for a given value of specific force M (say $M = 0.2$ m^3 of water as shown in Fig. 2.15), the $M(h)$ curve predicts two possible flow depths, a low stage h_1 (=0.023 m) and a high stage h_2 (=0.436 m). These depths are termed *sequent depths*. For instance, h_1 is the sequent depth of h_2 and vice versa. However, at point C on the $M(h)$ curve, the two depths merge and the specific force becomes a minimum [$M_{\min}(h_c = 0.132 \text{ m}) = 0.052$ m^3, where h_c is the *critical depth*], termed *critical flow condition*. Mathematically, the minimum value of the specific force can be obtained from Eq. (2.56) by taking the first derivative of M with respect to h and setting the resulting expression equal to zero. Thus,

$$\frac{dM}{dh} = 0 \Rightarrow -\frac{Q^2}{gA^2} \cdot \frac{dA}{dh} + \frac{d}{dh}(z_c A) = 0 \tag{2.57}$$

Referring to Fig. 2.15b, note that for a change in flow depth dh, the corresponding change of the moment of the flow area, d($z_c A$), can be obtained as

$$d(z_c A) = \left[(z_c + dh)A + \frac{T(dh)^2}{2} \right] - z_c A \approx A dh \Rightarrow \frac{d}{dh}(z_c A) = A \tag{2.58a}$$

or

$$\frac{d}{dh}(z_c A) = \frac{d}{dh} \int_A h dA = \int_A \frac{dh}{dh} dA = A \tag{2.58b}$$

Using Eq. (2.58a) or Eq. (2.58b), $T = dA/dh$, $U = Q/A$, and $h_d = A/T$ into Eq. (2.57) yields

$$\frac{U_c^2}{2g} = \frac{h_d}{2} \Rightarrow Fr_c \left(= \frac{U_c}{\sqrt{g h_d}} \right) = 1 \tag{2.59}$$

where Fr is the flow Froude number and subscript "c" refers to the quantity for the critical condition. The above equation provides the criterion for the *critical flow*, which states that at the critical flow condition, the velocity head is equal to half the hydraulic depth or the flow Froude number is unity. In conclusion, at the critical flow condition, the specific force is a minimum for a given discharge, and the corresponding flow depth is termed *critical depth*, h_c. More discussion on critical flow is available in Sect. 2.5.1.

2.4.2.2 Momentum Equation for Gradually Varied Unsteady Flow

One can proceed to obtain equations describing an unsteady open-channel flow considering a control volume with a short reach of dx (Fig. 2.16). The bed is inclined at an angle θ with the horizontal. Applying Newton's second law of motion in the streamwise direction (x-direction), one gets

$$pA - \left[pA + \frac{\partial (pA)}{\partial x} dx \right] + F_W \sin\theta - F_f = m a_x \quad \wedge \quad a_x = U \frac{\partial U}{\partial x} + \frac{\partial U}{\partial t} \tag{2.60}$$

where m is the mass of the fluid element (=$\rho A dx$). Using the expressions for the weight of fluid in the control volume $F_W = \rho g A dx$, the bed frictional resistance $F_f = \tau_0 P dx$, m, and a_x into Eq. (2.60) yields

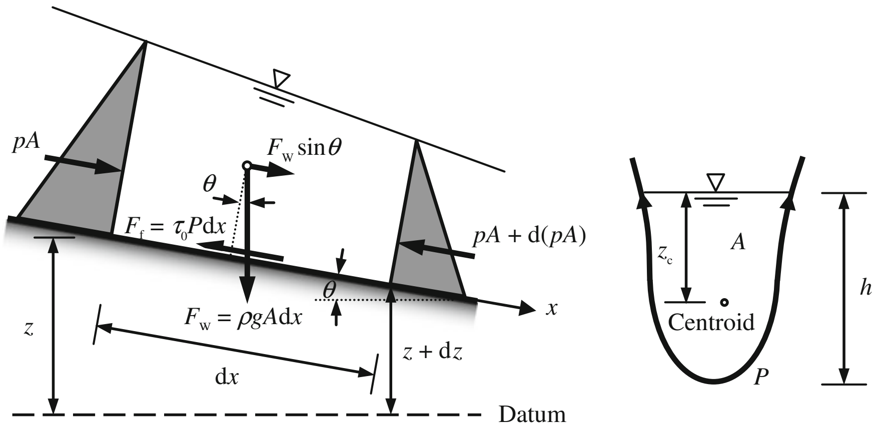


Fig. 2.16 Definition sketch for the derivation of momentum equation of a gradually varied unsteady flow in open channel

$$-\frac{\partial(pA)}{\partial x}\mathrm{d}x + \rho gA\mathrm{d}x\sin\theta - \tau_0 P\mathrm{d}x = \rho A\mathrm{d}x\left(U\frac{\partial U}{\partial x} + \frac{\partial U}{\partial t}\right) \tag{2.61}$$

where τ_0 is the bed shear stress and P is the wetted perimeter of the channel. For a small bed slope (θ = small), one can assume $\sin\theta \approx \tan\theta = -\partial z/\partial x = S_0$ (say). Dividing both sides of Eq. (2.61) by the weight of fluid, $\rho gA\mathrm{d}x$, in the control volume and rearranging, one gets

$$-\frac{1}{\rho gA}\cdot\frac{\partial(pA)}{\partial x} + S_0 - S_\mathrm{f} = \underbrace{\frac{U}{g}\cdot\frac{\partial U}{\partial x}}_{\text{Dynamic pressure slope}} + \underbrace{\frac{1}{g}\cdot\frac{\partial U}{\partial t}}_{\text{Acceleration slope}} \quad \wedge \quad \frac{\tau_0}{\rho gR_\mathrm{b}} = S_\mathrm{f} \tag{2.62}$$

where S_f is the friction slope and R_b is the hydraulic radius (=A/P). The first term of the left-hand side of Eq. (2.62) can be expressed in a more general way as

$$\frac{1}{\rho gA}\cdot\frac{\partial(pA)}{\partial x} = \frac{1}{\rho gA}\cdot\frac{\partial}{\partial x}(\rho g z_\mathrm{c} A) = \frac{1}{A}\cdot\frac{\partial}{\partial x}(z_\mathrm{c}A) = \frac{1}{A}\cdot\frac{\partial}{\partial x}\int_A h\mathrm{d}A = \frac{1}{A}\int_A \frac{\partial h}{\partial x}\mathrm{d}A = \frac{\partial h}{\partial x} \tag{2.63}$$

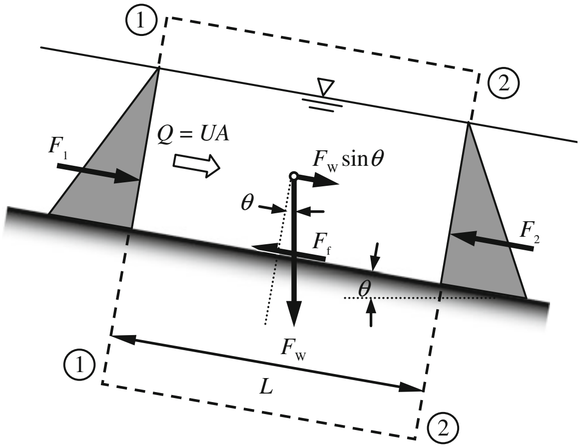


Fig. 2.17 Definition sketch for a steady uniform flow

Then, Eq. (2.62) becomes

$$\underbrace{\underbrace{\underbrace{\underbrace{S_f = S_0}_{\frac{\text{Kinematic}}{\text{Uniform flow}}} - \frac{\partial h}{\partial x}}_{\text{Diffusive}} - \frac{U}{g} \cdot \frac{\partial U}{\partial x}}_{\frac{\text{Quasi-steady}}{\text{Steady-nonuniform flow}}} - \frac{1}{g} \cdot \frac{\partial U}{\partial t}}_{\frac{\text{Dynamic}}{\text{Unsteady-nonuniform flow}}} \tag{2.64}$$

This equation is called the *general dynamic equation for gradually varied unsteady flow*. It is applicable as indicated in Eq. (2.64). It shows how nonuniformity and unsteadiness contribute to the equation of motion. Equation (2.64), also called *de Saint-Venant equation*, was first introduced by de Saint-Venant (1871).

2.4.2.3 Momentum Equation for Steady Uniform Flow

Referring to Fig. 2.17, for a steady uniform flow, $F_1 = F_2$ and $U_1 = U_2 = U$. Then, Eq. (2.52) reduces to

$$F_W \sin\theta - F_f = 0 \tag{2.65}$$

Using $F_W = \rho g A L$, $F_f = \tau_0 P L$, and $\sin\theta = S_0$ yields

$$\rho g A L S_0 - \tau_0 P L = 0 \tag{2.66}$$

Experiments revealed that the bed shear stress τ_0 is a function of dynamic pressure, $\lambda_f \rho U^2/2$, where λ_f is a friction parameter. Hence, rearranging Eq. (2.66) yields

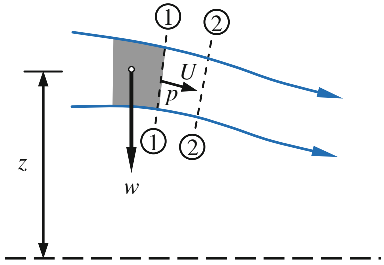


Fig. 2.18 Definition sketch for the derivation of energy equation of fluid flow

$$U = C_R (R_b S_0)^{0.5} \quad \wedge \quad C_R = \left(\frac{2g}{\lambda_f}\right)^{0.5} \tag{2.67}$$

where C_R is the Chézy coefficient. The above equation, that defines the flow resistance, is called the *Chézy equation*, which is applicable for uniform flow in open channels. The flow depth in uniform flow is called *normal flow depth* and is denoted by h_0.

However, the most widely used flow resistance equation for steady uniform flow is the *Manning equation*. It is

$$U = \frac{1}{n} R_b^{2/3} S_0^{0.5} \tag{2.68}$$

where n is Manning roughness coefficient. Note that the Manning equation is an empirical equation.

2.5 Conservation of Energy

An element of fluid, as shown in Fig. 2.18, acquires the potential energy due to its elevation z above the datum and the kinetic energy due to its velocity U.

If weight of the element is w, then the potential energy is wz. Thus,

$$\text{potential energy per unit weight} = z \tag{2.69}$$

Then, the kinetic energy is $wU^2/(2g)$. Thus,

$$\text{kinetic energy per unit weight} = \frac{U^2}{2g} \tag{2.70}$$

A steady fluid flow also does work due to hydrostatic pressure force acting on the cross section of fluid, as the fluid flows. If the hydrostatic pressure p acting at the section 1–1 having a cross-sectional area A, then the pressure force exerted on

1–1 is pA. The section 1–1 moves to 2–2 after a weight of fluid w transported along the streamtube. Then, the volume of fluid passing through the section 1–1 is $w/(\rho g)$. The distance between 1–1 and 2–2 is $w/(\rho gA)$. The pressure energy or the work done by the pressure is $pA \times w/(\rho gA) = pw/(\rho g)$. Therefore,

$$\text{pressure energy per unit weight} = \frac{p}{\rho g} \tag{2.71}$$

Equations (2.69)–(2.71) together represent the total energy per unit weight H in the fluid flow.

Each of the equations has the dimension of a length, called the *head*; and they are often referred to as the *hydrostatic or piezometic pressure head*, $p/(\rho g)$; the *velocity head*, $U^2/(2g)$; the *potential head*, z; and the *total head*, H. Therefore, for a steady flow of an inviscid fluid along a streamline, the energy equation is as follows:

$$\frac{p}{\rho g} + \frac{U^2}{2g} + z = H \tag{2.72}$$

This equation is also commonly called *Bernoulli's equation*. Interestingly, Bernoulli derived it from the integration of the Euler equation along a streamline containing same terms as in Eq. (2.72).

The nonuniform distribution (variation with the vertical distance) of velocity affects the computation of kinetic energy in the flow based on the area-averaged velocity U (=Q/A). Figure 2.10 illustrates the nonuniform and area-averaged velocity distributions and was already used in the context of momentum calculation. Based on the area-averaged velocity, the kinetic energy flux of the fluid passing through a section is expressed as $\alpha\, \dot{m}U^2/2$, where α is known as the *energy coefficient* or *Coriolis coefficient*. The equation balancing the kinetic energy flux calculated from the actual velocity distribution and that obtained from the area-averaged velocity corrected by α is used to determine energy coefficient α as

$$\int_A \left(\rho u \mathrm{d}A \frac{u^2}{2}\right) = \alpha \dot{m}\frac{U^2}{2} \Rightarrow \int_A \rho \frac{u^3}{2}\mathrm{d}A = \alpha\rho\frac{U^3}{2}A \tag{2.73}$$

Solving for α yields

$$\alpha = \frac{1}{A}\int_A \frac{u^3}{U^3}\mathrm{d}A \tag{2.74}$$

In straight prismatic channels, α varies approximately from 1.03 to 1.36 (Chow 1959).

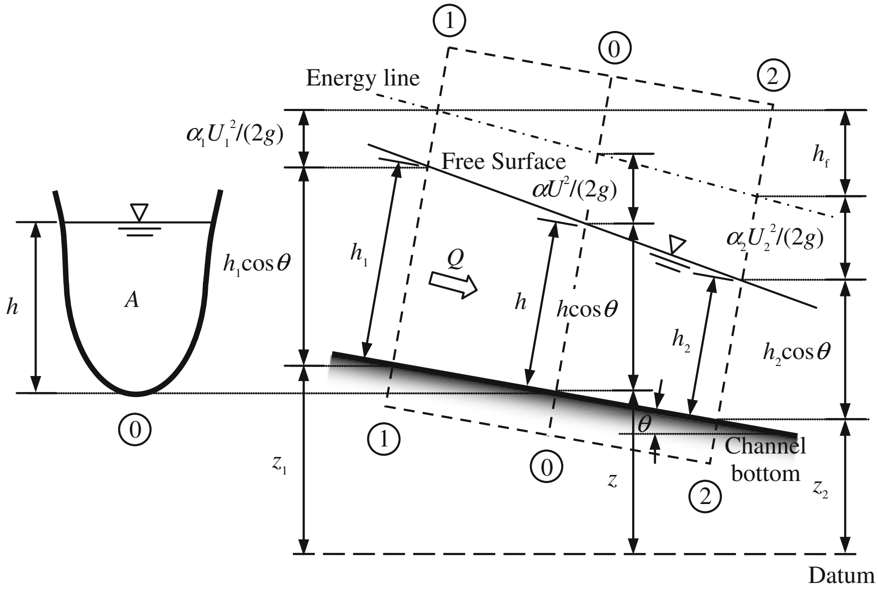


Fig. 2.19 Definition sketch for the derivation of energy equation of a gradually varied steady flow in an open channel

2.5.1 Energy Equation for Open-Channel Flow

Figure 2.19 illustrates the energy heads in a gradually varied steady flow in an open channel, whose bed is inclined at an angle θ with the horizontal. In an open-channel flow, the free surface represents the hydrostatic pressure head, provided there is no curvilinearity in the streamlines in the flow. It implies that $p/(\rho g) = h\cos\theta$, where h is the flow depth. Considering a suitable datum, the Bernoulli's equation is applied to the flow section 0–0, and the total energy head H is given by

$$z + h\cos\theta + \alpha\frac{U^2}{2g} = H \tag{2.75}$$

where α is the energy coefficient, U is the area-averaged velocity, and z is the elevation of the channel bottom above the datum. It is pertinent to mention that as the velocity distribution along the vertical distance varies, the velocity head, which is based on the constant velocity distribution U, that is truly identical for all points across the flow section, is corrected by α.

According to Bernoulli's equation, the total energy head at the upstream section 1 should be equal to the total energy head at the downstream section 2 plus the loss of energy head h_f between the two sections (Fig. 2.19). Thus,

$$z_1 + h_1 \cos\theta + \alpha_1 \frac{U_1^2}{2g} = z_2 + h_2 \cos\theta + \alpha_2 \frac{U_2^2}{2g} + h_f \tag{2.76}$$

This is the *energy equation for gradually varied flow* in an open channel. Assuming θ to be small or a horizontal bed ($\cos\theta \approx 1$) and h_f to be negligible for a short reach of a prismatic channel and also $\alpha_1 = \alpha_2 = 1$, Eq. (2.76) becomes

$$z_1 + h_1 + \frac{U_1^2}{2g} = z_2 + h_2 + \frac{U_2^2}{2g} = \text{constant } (=H) \tag{2.77}$$

Either of the above equations is known as the *energy equation for open-channel flow*.

2.5.1.1 The Specific Energy

Specific energy at a channel section is defined as the total energy head or the total energy per unit weight of the flow at the section with respect to the channel bottom. It means $z = 0$ in Eq. (2.75). Therefore, for a given channel section, the specific energy, denoted by E, is

$$E = h\cos\theta + \alpha\frac{U^2}{2g} \tag{2.78}$$

For θ to be small and $\alpha \approx 1$ (for simplicity), Eq. (2.78) becomes

$$E = h + \frac{U^2}{2g} \tag{2.79}$$

The specific energy, as indicated by Eq. (2.79), is the sum of the flow depth and the velocity head. Substituting $U = Q/A$, Eq. (2.79) becomes

$$E = h + \frac{Q^2}{2gA^2} \tag{2.80}$$

Since for a given channel section, $Q = Q(h)$ and $A = A(h)$, the specific energy E is a function of flow depth h only.

To illustrate the variation of specific energy E with flow depth h given by Eq. (2.80), the *specific energy diagram* [that is, $E(h)$ curve] for a given rectangular channel having a width of 2 m carrying a flow discharge of $Q = 0.3\ \text{m}^3\ \text{s}^{-1}$ is drawn, as shown in Fig. 2.20. The $E(h)$ curve has two limbs, AC and BC. The lower limb AC asymptotically approaches the abscissa toward the right, while the upper limb BC rises upwards and approaches the line OD as it extends to the right. The line OD that passes through the origin and is inclined at an angle 45° represents the hydrostatic pressure head or the flow depth h. Thus, for a given value of

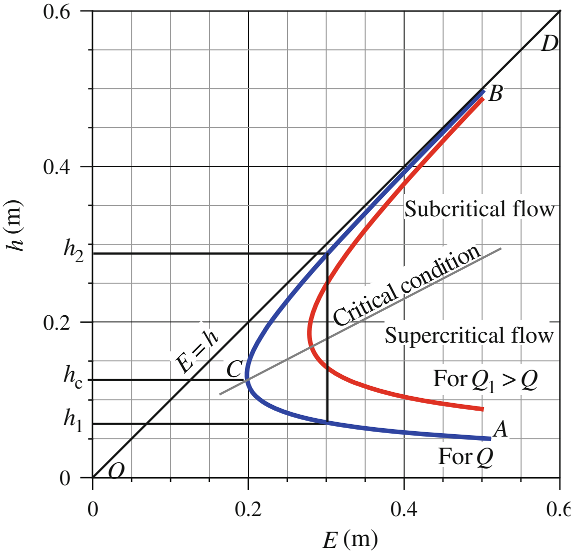


Fig. 2.20 The specific energy diagram

specific energy E (say $E = 0.3$ m of water as shown in Fig. 2.20), the $E(h)$ curve predicts two possible flow depths, a low stage h_1 (=0.071 m) and a high stage h_2 (=0.286 m). These depths are termed *alternate depths*. For instance, h_1 is the alternate depth of h_2, and vice versa. However, at point C on the curve, the alternate depths merge and the specific energy becomes a minimum. The flow corresponding to a minimum specific energy is known as *critical flow* and the resulting flow depth is termed *critical depth*, h_c. In Fig. 2.20, the minimum specific energy $E_{min} = 0.198$ m corresponds to a critical depth $h_c = 0.132$ m. When the flow depth is greater than the critical depth, the flow velocity is less than the critical velocity for a given discharge, and hence, the flow is called *subcritical*. On the other hand, when flow depth is less than the critical depth, the flow is *supercritical*. Hence, h_1 is the supercritical flow depth and h_2 is the subcritical flow depth. With the change in discharge, the $E(h)$ curve changes its position. Figure 2.20 also shows another $E(h)$ curve for a discharge $Q_1 = 0.5\ m^3\ s^{-1}$, which is greater than the previous discharge $Q = 0.3\ m^3\ s^{-1}$. The $E(h)$ curve of Q_1 lies on the right side of $E(h)$ curve of Q. Similarly, the $E(h)$ curve of a discharge less than Q will lie on the left side of $E(h)$ curve of Q.

Mathematically, the minimum value of the specific energy can be obtained from Eq. (2.80) by taking the first derivative of E with respect to h and setting the resulting expression equal to zero. Thus,

$$\frac{dE}{dh} = 0 \Rightarrow 1 - \frac{Q^2}{gA^3} \cdot \frac{dA}{dh} = 0 \tag{2.81}$$

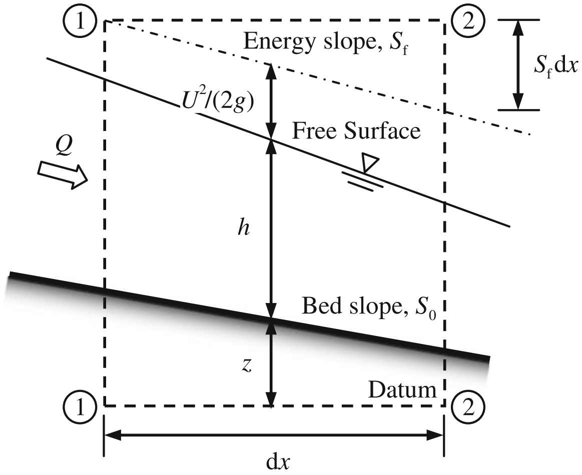


Fig. 2.21 Schematic of a gradually varied flow in an open channel

Using $T = dA/dh$ and $h_d = A/T$ into Eq. (2.81) yields

$$\frac{Q^2T}{gA^3} = 1 \Rightarrow \frac{U_c^2}{2g} = \frac{h_d}{2} \Rightarrow Fr_c\left(= \frac{U_c}{\sqrt{gh_d}}\right) = 1 \qquad (2.82)$$

Equation (2.82) provides the criterion for the *critical flow*, which is similar to that discussed in Sect. 2.4.2. In summary, at the critical flow condition, the specific energy is minimum for a given discharge. Hence, for critical condition, $h = h_c$, $U = U_c$, and $Fr = Fr_c = 1$; for subcritical condition, $h > h_c$, $U < U_c$, and $Fr < 1$; and for supercritical condition, $h < h_c$, $U > U_c$, and $Fr > 1$.

2.5.1.2 The Gradually Varied Flow

Figure 2.21 shows a schematic of a *gradually varied flow* (GVF) in a prismatic open channel. The definition of a GVF indicates two conditions: (1) steady flow and (2) practically parallel streamline flow, that is, a hydrostatic pressure distribution prevailing along the depth. The derivation of a GVF profile is based on the following assumptions:

(a) The channel is prismatic.
(b) The flow depth is indifferent whether it is measured in the vertical or normal (to the channel bed) direction. It means the bed slope is small; and hence, $h \approx h\cos\theta$ such that $\cos\theta \approx 1$.
(c) The head loss at a channel section is identical as for a uniform flow having the same velocity and hydraulic radius of the section. Thus, the resistance equation, such as the Manning equation, for the uniform flow can be used to determine the energy slope of a GVF.

(d) The friction coefficient is independent of the flow depth and unchanged throughout the channel reach, that is under consideration.

Considering $\cos\theta \approx 1$ and $\alpha \approx 1$ and differentiating Eq. (2.75) with respect to x, one gets

$$\frac{\mathrm{d}H}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}x} + \frac{\mathrm{d}h}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{U^2}{2g}\right) \tag{2.83}$$

Using $U = Q/A$ and $T = \mathrm{d}A/\mathrm{d}h$, the last term of the right-hand side of Eq. (2.83) is developed as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{U^2}{2g}\right) = \frac{\mathrm{d}}{\mathrm{d}h}\left(\frac{Q^2}{2gA^2}\right)\frac{\mathrm{d}h}{\mathrm{d}x} = -\frac{Q^2}{gA^3}\cdot\frac{\mathrm{d}A}{\mathrm{d}h}\cdot\frac{\mathrm{d}h}{\mathrm{d}x} = -\frac{Q^2T}{gA^3}\cdot\frac{\mathrm{d}h}{\mathrm{d}x} \tag{2.84}$$

Substituting Eq. (2.84) into Eq. (2.83) and rearranging yield

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{S_0 - S_\mathrm{f}}{1 - \frac{Q^2T}{gA^3}} \quad \wedge \quad \frac{\mathrm{d}H}{\mathrm{d}x} = -S_\mathrm{f} \quad \vee \quad \frac{\mathrm{d}z}{\mathrm{d}x} = -S_0 \tag{2.85}$$

where S_f is the energy or friction slope and S_0 is the bed slope. Further, using $h_\mathrm{d} = A/T$ and $Q = UA$ into Eq. (2.85), it produces

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{S_0 - S_\mathrm{f}}{1 - Fr^2} \quad \wedge \quad Fr = \frac{U}{\sqrt{gh_\mathrm{d}}} \tag{2.86}$$

This is the general differential equation of a *GVF* and predicts the free surface profiles. Flow with a positive value of $\mathrm{d}h/\mathrm{d}x$ refers to an increase in flow depth along the streamwise direction and is called *backwater curve*. On the other hand, flow with a negative value of $\mathrm{d}h/\mathrm{d}x$ refers to a decrease in flow depth along the streamwise direction and is called *drawdown curve*. However, for a uniform flow, $\mathrm{d}h/\mathrm{d}x = 0$ or $S_0 = S_\mathrm{f}$.

Classification of Bed Slope: A downward bed slope (positive value of S_0) is classified as *steep* if the normal depth is less than the critical depth (that is, the normal flow is supercritical) and *mild* if the normal depth is greater than the critical depth (that is, the normal flow is subcritical).[2] Other types of slopes are *critical* ($S_0 = S_\mathrm{c} > 0$ and $h_0 = h_\mathrm{c}$), *horizontal* ($S_0 = 0$ and $h_0 \to \infty$), and *adverse* ($S_0 < 0$ and $h_0 =$ imaginary). The slopes are designated using the first characters as *S*, *M*, *C*, *H*, and *A* for steep, mild, critical, horizontal, and adverse slopes, respectively. Further, to designate the flow profiles (that is, free surface profiles) corresponding to a given slope, the second characters 1, 2, and 3 are used as the subscript of the

[2] Alternatively, a downward slope is *steep* if it exceeds the critical slope S_c (that is the slope at which the normal depth of flow is critical depth). Hence, $S_0 > S_\mathrm{c}$. Similarly, *mild slope* can be explained.

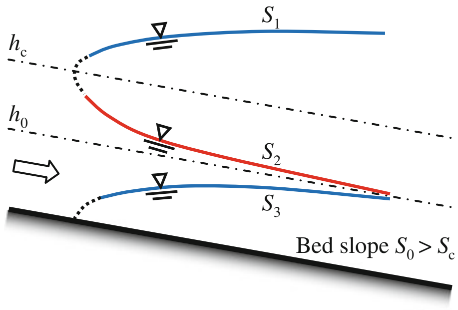


Fig. 2.22 Steep slope profiles (*S*-profiles)

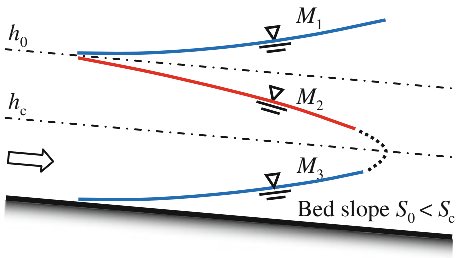


Fig. 2.23 Mild slope profiles (*M*-profiles)

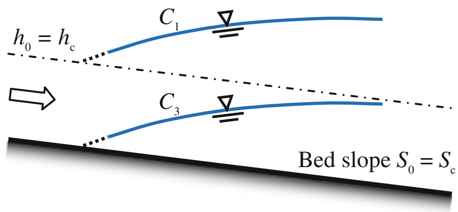


Fig. 2.24 Critical slope profiles (*C*-profiles)

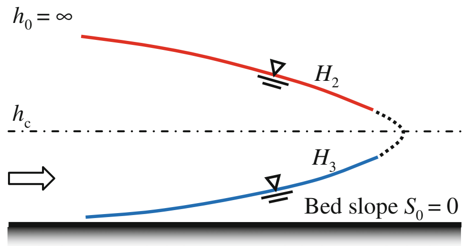


Fig. 2.25 Horizontal slope profiles (*H*-profiles)

first characters referring to the zone, where the actual depth h lies with respect to the flow depth lines for h_c and h_0 and the channel bed. By convention, zone 1 refers to the zone above the upper line, whichever (either h_c or h_0) it may be; zone 2 is the zone between the two lines; and zone 3 is the zone between the bed and the lower line. Figures 2.22, 2.23, 2.24, 2.25 and 2.26 show various flow profiles.

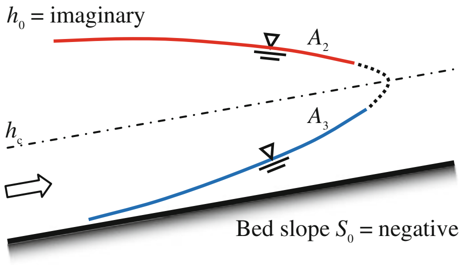


Fig. 2.26 Adverse slope profiles (*A*-profiles)

A number of methods to compute steady GVF from Eq. (2.86) are furnished in Chow (1959). The direct and standard step methods solve the energy equation between two consecutive channel sections. Dey (2000) presented a generalized numerical solution in the *Chebyshev form* for the standard step method. Then, a number of numerical methods are also available to integrate the differential equation, Eq. (2.86). These methods do not allow a direct solution, and therefore, trial-and-error method of solution is to be used (Chaudhry 2008).

2.5.1.3 Pressure Distribution in Curvilinear Flow

In the preceding cases, the streamlines were straight and parallel to the channel bottom. For instance, streamlines in a uniform flow are practically parallel and those in GVF may also be regarded as parallel, since the variation of flow depth is gradual that the streamlines have neither considerable curvature nor steep divergence/convergence. However, in real-life cases, the streamlines in several flow situations have pronounced curvature and/or divergence/convergence that the effects of acceleration components on the flow section are significant.

When the streamlines in a fluid flow have substantial curvature, the flow is called *curvilinear flow*. The curvature of streamlines is to induce considerable acceleration component normal to the direction of flow, called *centrifugal acceleration*. Thus, the pressure distribution in a curvilinear flow over the flow depth departs from the hydrostatic law, that is, $p = \rho gh$. Such curvilinear flows may be either convex or concave as shown in Figs. 2.27a, b. In a convex flow situation guided by a convex boundary, the centrifugal acceleration acts upward against the gravity and the resulting pressure is less than the hydrostatic pressure. On the other hand, in a concave flow situation guided by a concave boundary, the centrifugal acceleration acts downward to add to the gravity and the resulting pressure is greater than the hydrostatic pressure. Likewise, when streamlines have considerable divergence/convergence to develop appreciable acceleration normal to the flow direction, the pressure distribution again departs from the hydrostatic law. The distribution of pressure can be obtained by the Euler equations (Eqs. 2.39–2.41). In normal or radial direction of flow, it is

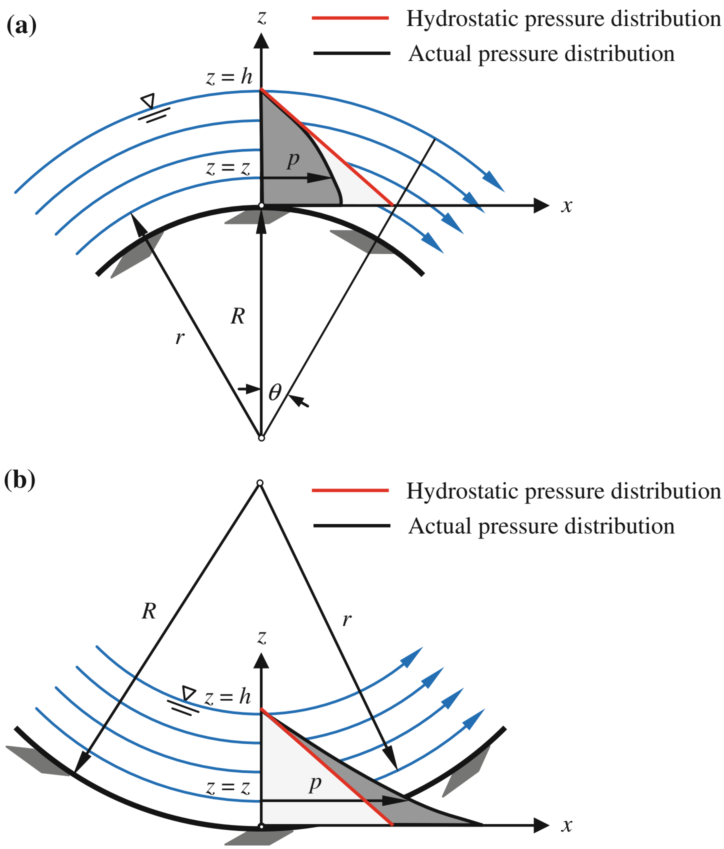


Fig. 2.27 Pressure distributions in curvilinear flows: **a** convex flow and **b** concave flow

$$\frac{\partial}{\partial z}\left(\frac{p}{\rho g} + z\cos\theta\right) = \pm\frac{a_r}{g} \tag{2.87}$$

where a_r is the centrifugal acceleration at the radius of curvature r of the streamline and θ is the angle between the section of interest and the vertical line. The centrifugal acceleration at any point in a curvilinear flow is $a_r = u^2/r$, where u is the tangential velocity at r. It is positive for the concave flow and negative for the convex flow. Integrating Eq. (2.87) within limits $z = z$ and $z = h$ yields

$$\frac{p}{\rho g} = (h - z)\cos\theta \pm \frac{1}{g}\int_h^z \frac{u^2}{r}\mathrm{d}z \tag{2.88}$$

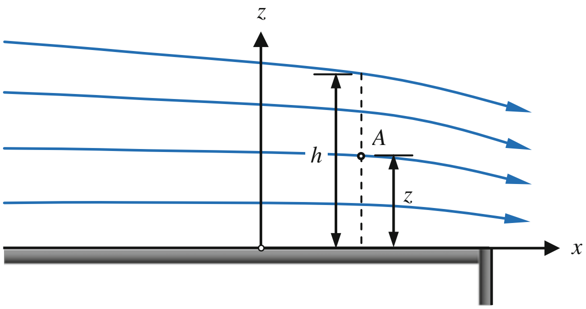


Fig. 2.28 Schematic of flow with a small free surface curvature

The above expression can be evaluated if $u = u(r)$ is known using $r = R \pm z$, where positive z is for convex flow and negative z for concave flow. Here, R is the radius of curvature of the channel boundary. For instance, (1) u can be invariant of r, as an average velocity; (2) u is proportional to r for forced vortex type of flow; and (3) u is proportional to r^{-1} for free vortex type of flow.

2.5.1.4 Pressure Distribution in Flow with Small Free Surface Curvature

Figure 2.28 shows a schematic of a free overfall whose free surface curvature is relatively small varying from a finite value at the free surface to zero at the channel bottom. According to Boussinesq approximation (Jaeger 1957), a linear variation of the streamline curvature at any point A at a vertical distance z is assumed. Hence, the radius of curvature r of a streamline at A is expressed as

$$\frac{1}{r} = \frac{z}{h} \cdot \frac{1}{r_s} \tag{2.89}$$

where r_s is the radius of curvature of the free surface. For small free surface curvature, it can be approximated as

$$\frac{1}{r_s} = \frac{d^2 h}{dx^2} \tag{2.90}$$

where x is the streamwise distance. The normal acceleration a_z based on the aforementioned assumption is given by

$$a_z = Kz \quad \wedge \quad K = \frac{U^2}{h} \cdot \frac{d^2 h}{dx^2} \tag{2.91}$$

where U is the average flow velocity over depth h; hence, it is constant along the vertical distance. Considering the bottom as a datum and then integrating Eq. (2.87), the hydrostatic pressure head h_p at point A is obtained as

$$h_p = \left(\frac{p}{\rho g} + z\right) = -\frac{1}{g}\int Kz\mathrm{d}z + C = -\frac{K}{g}\cdot\frac{z^2}{2} + C \tag{2.92}$$

Using the boundary condition, at $z = h$, $p = 0$, and $h_p = h$, it leads to

$$C = h + \frac{K}{g}\cdot\frac{h^2}{2} \tag{2.93}$$

Hence, from Eq. (2.92), the hydrostatic pressure head h_p is obtained as

$$h_p = h + \frac{K(h^2 - z^2)}{2g}, \quad \therefore \quad h_p = h + \Delta h \tag{2.94}$$

It indicates that the variation of hydrostatic pressure head is given by Δh. Therefore, the depth-averaged value of Δh can be determined as

$$\overline{\Delta h} = \frac{1}{h}\int_0^h \Delta h\mathrm{d}z = \frac{1}{h}\cdot\frac{K}{2g}\int_0^h (h^2 - z^2)\mathrm{d}z = \frac{Kh^2}{3g} \tag{2.95}$$

The effective hydrostatic pressure head h_{ep} is therefore

$$h_{ep} = h + \frac{Kh^2}{3g} = h + \frac{U^2 h}{3g}\cdot\frac{\mathrm{d}^2 h}{\mathrm{d}x^2} \tag{2.96}$$

Note that $\mathrm{d}^2h/\mathrm{d}x^2$ is negative for convex flow and positive for concave flow.

2.6 The Boundary Layer

According to the concept of the ideal fluid flow (that is, the potential flow), a streamline follows the solid boundary, termed *limiting streamline*, involving a finite fluid velocity at the boundary. It, in fact, implies that the fluid particles slip at the boundary, as a result of which, the no-slip condition is not preserved in the ideal fluid flow. However, in real fluid flow, the viscosity causes the fluid particles to have no motion at the boundary preserving a no-slip condition. In reality, the velocity, that is zero at the boundary, keeps increasing with the perpendicular distances away from the boundary. The change in velocity is discernible only within a layer adjacent to the boundary. The layer close to the solid boundary affected by the boundary shear is called *boundary layer*, where the viscous effects

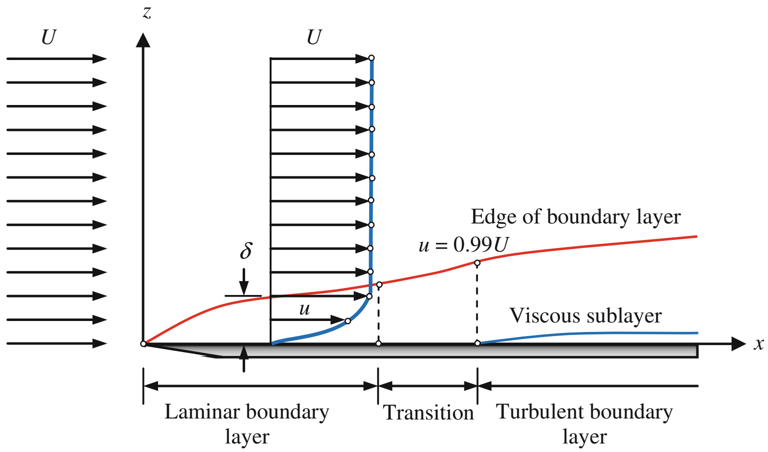


Fig. 2.29 Details of a boundary layer developed over a flat plate

are prominent. This phenomenon was discovered by Prandtl (1904). He, however, arbitrarily suggested the boundary layer to extend up to 99 % of the free stream velocity U. Hence, it is possible to define the *boundary layer thickness* δ as that the distance from the boundary where the local velocity u equals $0.99U$:

$$\delta = z|_{u=0.99U}$$

In fluid flow outside the boundary layer, the effects of viscosity may be vanishingly small that the theory of ideal fluid flow is applicable. Importantly, *boundary layer is not a streamline*. It is worth mentioning that the concept of boundary layer is the most significant contribution to the development of hydrodynamics.

2.6.1 Characteristics of Boundary Layer

Consider a fluid flow over a flat plate aligned parallel to the approaching free stream, as shown in Fig. 2.29. The approaching free stream that has a velocity U suffers retardation in the vicinity of the plate due to the viscous resistance offered by the solid boundary. The boundary layer starts growing from the leading edge of the plate. Its thickness increases with distance from the leading edge as more and more fluid is to decelerate by the viscous resistance near the solid boundary. Near the leading edge, the flow in the boundary layer is entirely laminar. With an increase in distance, the laminar boundary layer thickness grows becoming progressively unstable and eventually changes to a turbulent boundary layer over a *transition region*. The transition occurs in the range $R_x = 3 \times 10^5$ to 10^6,

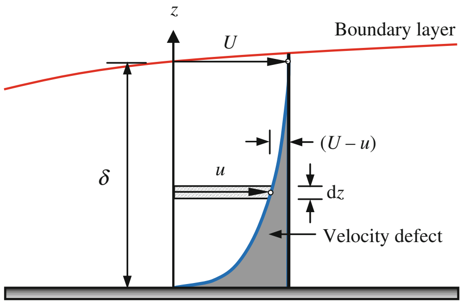


Fig. 2.30 The velocity defect

where $R_x = Ux/\upsilon$. Even in the region of the turbulent boundary layer, the turbulence becomes suppressed to such a degree that the viscous effects predominate and a very thin layer adjacent to the solid boundary remains laminar, which is called the *viscous sublayer*.

The boundary layer thickness is the distance from the boundary to a point where the velocity is $0.99U$, which has already been discussed. It is based on the fact that beyond this arbitrary limit of vertical distance $z|_{u=0.99U}$, the viscous stresses are practically absent. Other definitions of thickness, such as *displacement thickness* and *momentum thickness*, are also useful in boundary layer theory.

The presence of a boundary introduces a retardation to the free stream velocity in the vicinity of the boundary. The difference $(U - u)$, called the *velocity defect*, causes a decrease in the mass flux as compared to the mass flux of the free stream that would pass through the same section in the absence of the boundary layer (see Fig. 2.30). To compensate for this defect, the actual boundary may be imagined to have been displaced by a *displacement thickness* $\delta*$ such that the mass flux would be the same as that of an ideal fluid flowing over the displaced boundary. The equivalence of the two mass fluxes yields the displacement thickness in incompressible flow (ρ = constant) as

$$\int_0^{\delta} \rho u \mathrm{d}z = \int_{\delta^*}^{\delta} \rho U \mathrm{d}z \Rightarrow \int_0^{\delta} u \mathrm{d}z = \int_0^{\delta} U \mathrm{d}z - U\delta^*, \quad \therefore \quad \delta^* = \int_0^{\delta} \left(1 - \frac{u}{U}\right) \mathrm{d}z \tag{2.97}$$

Further, the retardation of flow within the boundary layer causes a reduction in the momentum flux as well. The *momentum thickness* θ is defined as the thickness of an imaginary layer in free stream flow which has a momentum flux equals the deficiency of momentum flux over the entire section caused to the actual mass flux within the boundary layer. Mathematically, for an incompressible flow (ρ = constant), it can be developed as

$$\rho U^2 \theta = \int_0^{\delta} \rho u (U - u) \mathrm{d}z, \quad \therefore \quad \theta = \int_0^{\delta} \frac{u}{U} \left(1 - \frac{u}{U}\right) \mathrm{d}z \tag{2.98}$$

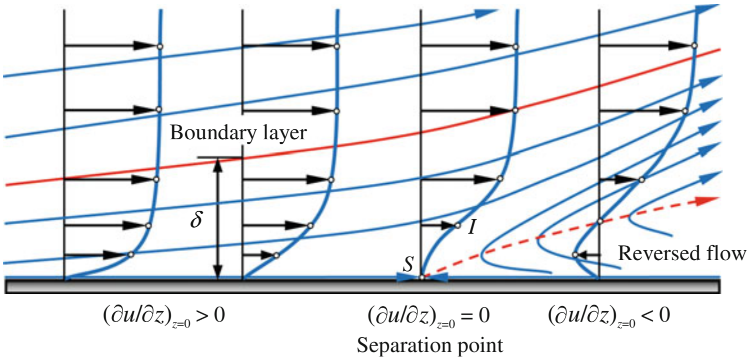


Fig. 2.31 Boundary layer separation

The *shape factor* H_s, that is defined as the ratio of the displacement thickness to the momentum thickness is used to determine the nature of the flow.

$$H_s = \frac{\delta^*}{\theta} \tag{2.99}$$

For a higher value of shape factor H_s, a stronger adverse pressure gradient ($\partial p/\partial x > 0$) is indicated. Conventionally, $H_s = 2.59$ is typical for laminar flow, while $H_s = 1.3 - 1.4$ is typical for turbulent flow.

2.6.1.1 Boundary Layer Separation

In a favorable streamwise pressure gradient ($\partial p/\partial x < 0$), the flow is accelerated by the pressure force and thereby the boundary layer thickness keeps thin. In contrast, when the flow encounters an adverse streamwise pressure gradient ($\partial p/\partial x > 0$) along the solid boundary, the flow is decelerated by the pressure force, thereby causing the boundary layer to thicken. Then, the flow cannot advance too far in the region of adverse pressure gradient due to the insufficient kinetic energy that the fluid flow possesses. As a result, the boundary layer is deflected from the wall, known to be the *separated boundary layer*, which progresses into the main flow, as shown in Fig. 2.31. In general, the flow downstream the *separation point* (point *S*) experiences the adverse pressure gradient and turns to the reverse direction of the main flow that exists in the upper region of the separation line. As a result of the flow reversal, the boundary layer is thickened rapidly. The separation point is defined as the limit between the main and the reverse flow in the immediate vicinity of the wall. Further, in explaining the separation phenomenon by the potential flow theory, the streamlines within the boundary layer in the vicinity of the boundary layer separation are shown in Fig. 2.31. At the separation point, a streamline originates from the wall at a certain angle due to the merger of two limiting streamlines moving in the opposite direction. The separation point can be determined by the condition that the velocity gradient normal to the wall becomes zero on the wall, that is, $\partial u/\partial z|_{z=0}^{S} = 0$.

The integral equation of the boundary layer to be discussed in the Sect. 2.6.2 is only applicable to the extent where the separation point occurs. At a short distance downstream the separation point, the boundary layer becomes so thick that the assumptions that are made in deriving the boundary layer equation no longer apply. In a steady flow, the event of separation that occurs only in a decelerated flow can be obtained from the relation between the pressure gradient $\partial p/\partial x$ and the velocity distribution $u(z)$ with the aid of the Navier–Stokes equations. From Eq. (2.51a) with the boundary condition $u = w = 0$ (no-slip at the wall, $z = 0$) in a two-dimensional flow, one can have at $z = 0$

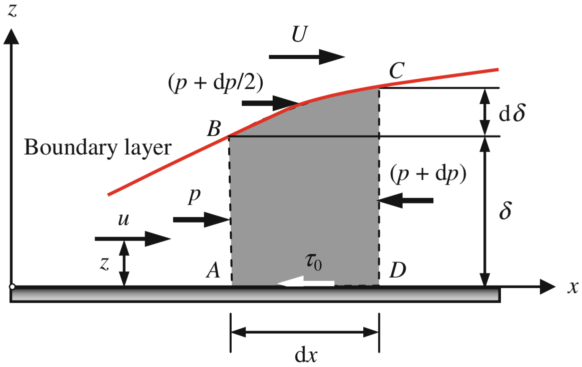
$$\mu \frac{\partial^2 u}{\partial z^2}\bigg|_{z=0}^{S} = \frac{\partial p}{\partial x} \Rightarrow \frac{\partial^3 u}{\partial z^3}\bigg|_{z=0}^{S} = 0$$

In the vicinity of the wall, the curvature of the velocity distribution $\partial^2 u/\partial z^2$ depends only on the pressure gradient $\partial p/\partial x$. The curvature $\partial^2 u/\partial z^2|_{z=0}$ at the wall does changeover its sign with $\partial p/\partial x$. In flow with a decreasing pressure (accelerated flow, $\partial p/\partial x < 0$), the prevailing condition is $\partial^2 u/\partial z^2|_{z=0} < 0$; and therefore, $\partial^2 u/\partial z^2$ is negative over the entire boundary layer thickness (Fig. 2.31). On the other hand, in flow within the near-wall layer of increasing pressure (decelerated flow, $\partial p/\partial x > 0$), the prevailing condition is $\partial^2 u/\partial z^2 > 0$. In flow with $\partial^2 u/\partial z^2 < 0$ at some distance above the wall, there must exist a point (point I) for which $\partial^2 u/\partial z^2 = 0$, which is an *inflexion point* of the velocity distribution within the boundary layer (Fig. 2.31). It suggests that in the region of decelerating potential flow, the velocity distribution within the boundary layer always displays an inflexion point. Since there exists $\partial^2 u/\partial z^2 < 0$ at the edge of the boundary layer, the velocity distribution that has a separation point with $\partial u/\partial z|_{z=0} = 0$ must have an inflexion point.

2.6.2 von Kármán Momentum Integral Equation

Consider a control volume $ABCD$ of elementary length dx having a boundary layer thickness δ, as shown in Fig. 2.32. For a steady flow, the forces on the control surface are caused by the pressure and the wall or boundary shear stress. As the flow is almost parallel, a uniform pressure at a section can be assumed, neglecting the hydrostatic pressure. The components of force (per unit area and width) in x-direction are shown in Fig. 2.32. Since the boundary layer is thin, the pressure within the boundary layer at a section equals the pressure in the free stream portion at that section outside the boundary layer. The summation of forces in x-direction is

Fig. 2.32 Control volume in the boundary layer



$$\sum F_x = p\delta - (p + dp)(\delta + d\delta) + \left(p + \frac{dp}{2}\right)d\delta - \tau_0 dx \ \wedge \ dp = \frac{\partial p}{\partial x}dx \quad (2.100)$$

Simplifying and neglecting second-order terms, Eq. (2.100) becomes

$$\sum F_x = -\left(\delta\frac{\partial p}{\partial x} + \tau_0\right)dx \quad (2.101)$$

Change of momentum flux in the control volume is

$$\underbrace{\int_0^\delta \rho u^2 dz + \frac{\partial}{\partial x}\left(\int_0^\delta \rho u^2 dz\right)dx}_{\text{Through } CD} - \underbrace{\int_0^\delta \rho u^2 dz}_{\text{Through } AB} + \underbrace{\rho U(w dx - u d\delta)}_{\text{Through } BC}$$
$$= \frac{\partial}{\partial x}\left(\int_0^\delta \rho u^2 dz\right)dx + \rho U(w dx - u d\delta)$$

By Newton's second law of motion, one can write

$$-\left(\delta\frac{\partial p}{\partial x} + \tau_0\right)dx = \frac{\partial}{\partial x}\left(\int_0^\delta \rho u^2 dz\right)dx + \rho U(w dx - u d\delta) \quad (2.102)$$

The continuity of flow for the control volume that constitutes the equation is[3]

[3] The mass flux through BC can be obtained as $\rho(u\hat{i} + w\hat{k})\cdot(-d\delta\hat{i} + dx\hat{k}) = \rho(-u d\delta + w dx)$.

$$\underbrace{-\int_0^\delta \rho u \mathrm{d}z}_{\text{Through } AB} + \underbrace{\int_0^\delta \rho u \mathrm{d}z + \frac{\partial}{\partial x}\left(\int_0^\delta \rho u \mathrm{d}z\right)\mathrm{d}x}_{\text{Through } CD} + \underbrace{\rho(w\mathrm{d}x - u\mathrm{d}\delta)}_{\text{Through } BC} = 0,$$
$$\therefore \quad \rho(w\mathrm{d}x - u\mathrm{d}\delta) = -\frac{\partial}{\partial x}\left(\int_0^\delta \rho u \mathrm{d}z\right)\mathrm{d}x \tag{2.103}$$

Using Eq. (2.103) into Eq. (2.102) and replacing partial differential by total differential yield

$$-\delta\frac{\mathrm{d}p}{\mathrm{d}x} - \tau_0 = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta \rho u^2 \mathrm{d}z - U\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta \rho u \mathrm{d}z \tag{2.104}$$

Further, the pressure gradient dp/dx can be determined from the Bernoulli's equation (Eq. 2.72), considering the potential head $z = 0$. Hence,

$$\frac{p}{\rho g} + \frac{U^2}{2g} = H \Rightarrow p + \frac{\rho}{2}U^2 = \text{constant} \tag{2.105}$$

Differentiating with respect to x and rearranging, Eq. (2.105) becomes

$$\frac{\mathrm{d}p}{\mathrm{d}x} = -\rho U\frac{\mathrm{d}U}{\mathrm{d}x} \tag{2.106}$$

Substituting the expression of dp/dx into Eq. (2.104), the wall shear stress τ_0 for incompressible flow (ρ = constant) is obtained as

$$\tau_0 = -\rho\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta u^2 \mathrm{d}z + \rho U\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta u \mathrm{d}z + \rho U\delta\frac{\mathrm{d}U}{\mathrm{d}x},$$
$$\therefore \quad \tau_0 = \rho\frac{\mathrm{d}}{\mathrm{d}x}(U^2\theta) + \rho U\delta\frac{\mathrm{d}U}{\mathrm{d}x} \tag{2.107}$$

Equation (2.107) is the generalized *von Kármán momentum integral equation*, which can be applicable for both laminar and turbulent boundary layer flows.

If the flow has a zero-pressure gradient dp/d$x = 0$, then dU/d$x = 0$; and Eq. (2.107) reduces to

$$\tau_0 = \rho U^2\frac{\mathrm{d}\theta}{\mathrm{d}x} \tag{2.108}$$

2.6.2.1 Laminar Boundary Layer Over a Flat Plate in a Zero-Pressure Gradient Flow

The *laminar boundary layer* is formed as a result of flow over a short reach of the leading edge of a flat plate. In practice, it always prevails, even in flows that are evidently turbulent. To apply von Kármán momentum integral equation for such a flow situation, the assumption on the velocity u distribution, which is reasonably a function of η (=z/δ) and invariant of x, is an essential prerequisite. For an approximate analysis of a laminar boundary layer, a third-order polynomial law ($u/U = A + B\eta + C\eta^2 + D\eta^3 = f$, where A, B, C, and D are the coefficients) of velocity distribution that satisfies the boundary conditions (1) $u(z = 0) = 0$, (2) $\partial^2u/\partial z^2(z = 0) = 0$, (3) $u(z = \delta) = U$, and (4) $\partial u/\partial z(z = \delta) = 0$ was assumed by Prandtl within the boundary layer ($0 \leq z \leq \delta$). The coefficients are obtained as $A = C = 0$, $B = 3/2$ and $D = -1/2$. Therefore, the velocity distribution is

$$\frac{u}{U} = f(\eta) \quad \wedge \quad f(0 \leq \eta < 1) = \frac{3}{2}\eta - \frac{1}{2}\eta^3, \quad f(\eta \geq 1) = 1 \tag{2.109}$$

Inserting Eq. (2.109) into Eq. (2.108), one can obtain

$$\tau_0 = \rho U^2 \frac{d\theta}{dx} = \rho U^2 \frac{d\delta}{dx} \int_0^1 f(1-f)d\eta = \rho U^2 \frac{d\delta}{dx} \int_0^1 \left(\frac{3}{2}\eta - \frac{1}{2}\eta^3\right)\left(1 - \frac{3}{2}\eta + \frac{1}{2}\eta^3\right)d\eta$$
$$= 0.139\rho U^2 \frac{d\delta}{dx} \quad \wedge \quad \theta = \delta \int_0^1 f(1-f)d\eta \tag{2.110}$$

Further, applying Newton's law of viscosity at the boundary, one gets

$$\tau_0 = \mu \frac{du}{dz}\bigg|_{z=0} = \mu \frac{U}{\delta} \cdot \frac{df}{d\eta}\bigg|_{\eta=0} = \mu \frac{U}{\delta} \cdot \frac{d}{d\eta}\left(\frac{3}{2}\eta - \frac{1}{2}\eta^3\right)\bigg|_{\eta=0} = \frac{3}{2}\mu \frac{U}{\delta} \tag{2.111}$$

Equating Eqs. (2.110) and (2.111) and rearranging yield

$$\delta d\delta = 10.79 \frac{\upsilon}{U} dx \tag{2.112}$$

The above equation is integrated as follows:

$$\int_0^\delta \delta d\delta = 10.79 \frac{\upsilon}{U} \int_0^x dx \quad \wedge \quad \frac{\delta^2}{2} = 10.79 \frac{\upsilon}{U} x, \quad \therefore \quad \delta = 4.643 x R_x^{-0.5} \tag{2.113}$$

Equation (2.113) can be used to determine the boundary layer thickness $\delta(x)$, which varies directly as a square root of distance x. Substituting the expression for δ into Eq. (2.111), the wall shear stress expression $\tau_0(x)$ is obtained. It is

$$\tau_0 = 0.323\rho U\left(\upsilon\frac{U}{x}\right)^{0.5} = 0.323\mu\frac{U}{x}R_x^{0.5} \tag{2.114}$$

It indicates that the wall shear stress varies inversely as a square root of distance x. Equation (2.114) can be used to determine the wall shear resistance per unit width F_τ on the surface of the plate for a given length $x = 0$ to L as

$$F_\tau = \int_0^L \tau_0 \mathrm{d}x = 0.646\rho U(\upsilon UL)^{0.5} = 0.646\rho U\upsilon R_L^{0.5} \quad \wedge \quad R_L = \frac{UL}{\upsilon} \tag{2.115}$$

2.6.2.2 Turbulent Boundary Layer Over a Flat Plate in a Zero-Pressure Gradient Flow

For the approximation of a turbulent boundary layer, a 1/7-th power law of velocity distribution, which is a good replacement of the logarithmic law (Sect. 3.7.2), as proposed by Prandtl, can be assumed within the boundary layer ($0 \le z \le \delta$). Thus,

$$\frac{u}{U} = f(\eta) \quad \wedge \quad f(0 \le \eta < 1) = \eta^{1/7}, \quad f(\eta \ge 1) = 1 \tag{2.116}$$

Inserting Eq. (2.116) into Eq. (2.108), one can obtain

$$\tau_0 = \rho U^2\frac{\mathrm{d}\delta}{\mathrm{d}x}\int_0^1 f(1-f)\mathrm{d}\eta = \rho U^2\frac{\mathrm{d}\delta}{\mathrm{d}x}\int_0^1 \eta^{1/7}(1-\eta^{1/7})\mathrm{d}\eta = \frac{7}{72}\rho U^2\frac{\mathrm{d}\delta}{\mathrm{d}x} \tag{2.117}$$

Blasius (1912, 1913) obtained the wall shear stress for hydraulically smooth flow as

$$\tau_0 = 2.28\times 10^{-2}\rho U^2\left(\frac{\upsilon}{U\delta}\right)^{0.25} \tag{2.118}$$

Equating Eqs. (2.117) and (2.118) and rearranging yield

$$\delta^{0.25}\mathrm{d}\delta = 0.235\left(\frac{\upsilon}{U}\right)^{0.25}\mathrm{d}x \tag{2.119}$$

The above equation is integrated as follows:

$$\int_0^{\delta} \delta^{0.25} \mathrm{d}\delta = 0.235\left(\frac{\upsilon}{U}\right)^{0.25}\int_0^{x} \mathrm{d}x \quad \wedge \quad \delta^{1.25} = 0.294\left(\frac{\upsilon}{U}\right)^{0.25} x, \qquad (2.120)$$
$$\therefore \quad \delta = 0.376 x R_x^{-0.2}$$

Hence, the turbulent boundary layer thickness increases with distance as $x^{0.8}$, as compared to the laminar boundary layer thickness that varies as $x^{0.5}$. It indicates that the turbulent boundary layer thickness grows faster. Substituting the expression for δ into Eq. (2.118) yields

$$\tau_0 = 2.91 \times 10^{-2} \rho U^2 \left(\frac{\upsilon}{Ux}\right)^{0.2} = 2.91 \times 10^{-2} \rho U^2 R_x^{-0.2} \qquad (2.121)$$

The wall shear resistance per unit width F_τ on the surface of the plate for a given length $x = 0$ to L is

$$F_\tau = \int_0^{L} \tau_0 \mathrm{d}x = 3.638 \times 10^{-2} \rho U^2 L \left(\frac{\upsilon}{UL}\right)^{0.2} = 3.638 \times 10^{-2} \rho U^2 L R_L^{-0.2} \qquad (2.122)$$

2.7 Flow in Curved Channels

Flow in a curved channel is influenced by the centrifugal acceleration, which induces a three dimensionality in the flow characterized by a helical (spiral) motion with a superelevated free surface. The helical motion can be viewed across a cross section as a transverse circulation. The differential centrifugal acceleration u^2/r along a vertical line due to vertical variation of streamwise velocity u in open channel is the cause of the transverse circulation. As a result, a helical motion is initiated when the flow enters the curved (bend) portion of the channel. The helicoidal flow is gradually fully developed becoming in an equilibrium state, where the flow structure remains unchanged from cross section to cross section. Such a flow situation eventually prevails, if a prismatic channel has an adequately long curved reach. The streamlines near the free surface are deflected toward the outer bank, whereas those near the bed are inclined toward the inner bank (Fig. 2.33). Hence, the near-bed velocity and the bed shear stress are generally directed toward the inner bank.

The flow in a curved channel is analyzed in cylindrical polar coordinates restricting to a subcritical flow having a hydrostatic pressure distribution (Fig. 2.34). In natural channels, the flow depth is in general much smaller than the

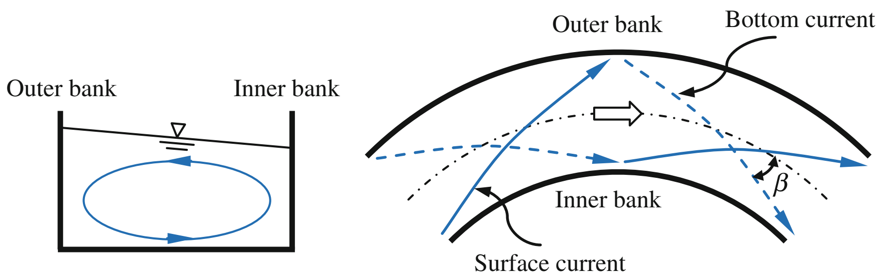


Fig. 2.33 Flow in a curved channel

width and the radius of curvature. In cylindrical polar coordinates, the velocity components (u_r, u_θ, u_z) are in r-, θ-, and z-direction, respectively. Note that $\partial s = r\partial\theta$. Referring to Fig. 2.34, the forces in the tangential direction, that is, θ-direction, is given by

$$[(\tau_\theta + \mathrm{d}\tau_\theta) - \tau_\theta + \rho g S_\theta \mathrm{d}z]\mathrm{d}r\mathrm{d}s = \left(\frac{\partial \tau_\theta}{\partial z} + \rho g S_\theta\right)\mathrm{d}r\mathrm{d}s\mathrm{d}z \quad \wedge \quad \mathrm{d}\tau_\theta = \frac{\partial \tau_\theta}{\partial z}\mathrm{d}z$$

where τ_θ and S_θ are the shear stress and the slope of the channel in θ-direction, respectively. Applying Newton's second law of motion in θ-direction yields

$$\begin{aligned} &a_\theta \rho \mathrm{d}r\mathrm{d}s\mathrm{d}z = \left(\frac{\partial \tau_\theta}{\partial z} + \rho g S_\theta\right)\mathrm{d}r\mathrm{d}s\mathrm{d}z \;\Rightarrow\; a_\theta = \frac{1}{\rho}\cdot\frac{\partial \tau_\theta}{\partial z} + gS_\theta \\ &a_\theta = u_\theta\frac{\partial u_\theta}{\partial s} + \frac{u_\theta u_r}{r} + u_r\frac{\partial u_\theta}{\partial r} + u_z\frac{\partial u_\theta}{\partial z} + \frac{\partial u_\theta}{\partial t}, \\ &\therefore \quad u_\theta\frac{\partial u_\theta}{\partial s} + u_r\frac{\partial u_\theta}{\partial r} + u_z\frac{\partial u_\theta}{\partial z} + \frac{\partial u_\theta}{\partial t} = \frac{1}{\rho}\cdot\frac{\partial \tau_\theta}{\partial z} + gS_\theta - \frac{u_\theta u_r}{r} \end{aligned} \tag{2.123}$$

where a_θ is the total acceleration in θ-direction. On the other hand, the forces in the radial direction, that is, r-direction, is given by

$$\begin{aligned} &[(\tau_r + \mathrm{d}\tau_r) - \tau_r]\mathrm{d}r\mathrm{d}s + [p - (p + \mathrm{d}p)]\mathrm{d}s\mathrm{d}z \\ &= \left(\frac{\partial \tau_r}{\partial z} - \rho g S_r\right)\mathrm{d}r\mathrm{d}s\mathrm{d}z \quad \wedge \quad \mathrm{d}\tau_r = \frac{\partial \tau_r}{\partial z}\mathrm{d}z \quad \vee \quad \mathrm{d}p = \frac{\partial p}{\partial r}\mathrm{d}r = \rho g S_r \mathrm{d}r \end{aligned}$$

where τ_r and S_r are the shear stress and the slope of the free surface in r-direction, respectively. Applying Newton's second law of motion in r-direction yields

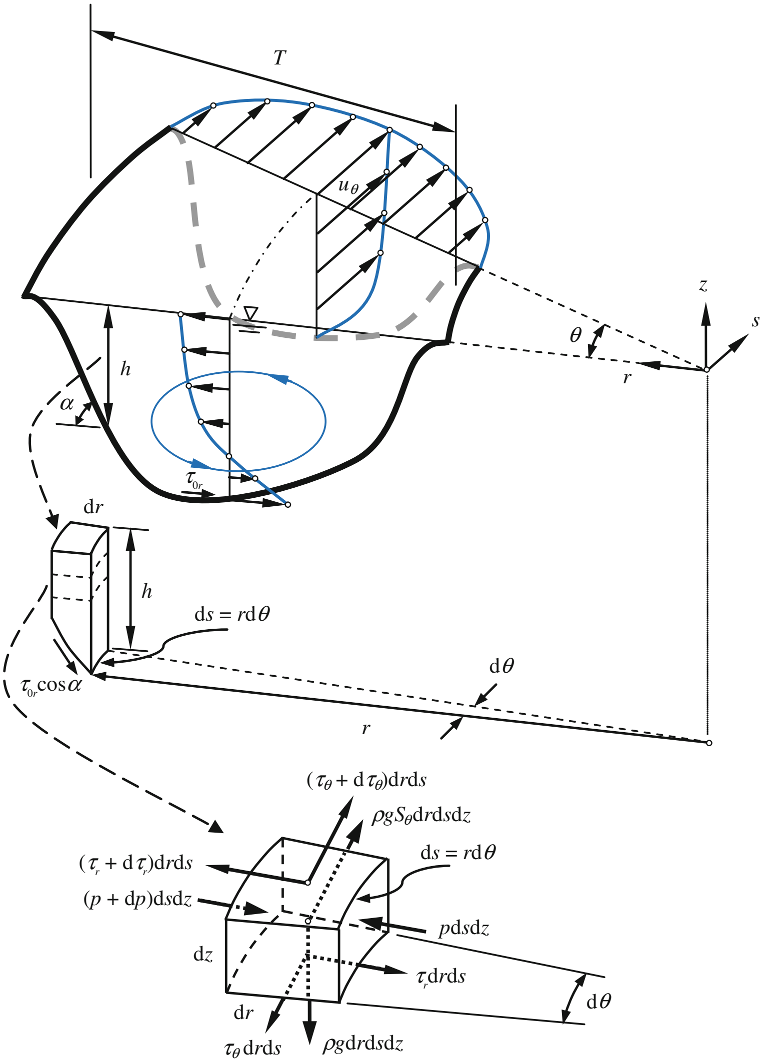


Fig. 2.34 Velocity and force distributions in flow through a curved channel

$$a_r \rho \mathrm{d}r\mathrm{d}s\mathrm{d}z = \left(\frac{\partial \tau_r}{\partial z} - \rho g S_r\right)\mathrm{d}r\mathrm{d}s\mathrm{d}z \Rightarrow a_r = \frac{1}{\rho}\cdot\frac{\partial \tau_r}{\partial z} - gS_r$$

$$a_r = u_\theta\frac{\partial u_r}{\partial s} + u_r\frac{\partial u_r}{\partial r} - \frac{u_\theta^2}{r} + u_z\frac{\partial u_r}{\partial z} + \frac{\partial u_r}{\partial t}, \tag{2.124}$$

$$\therefore \quad u_\theta\frac{\partial u_r}{\partial s} + u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z} + \frac{\partial u_r}{\partial t} = \frac{1}{\rho}\cdot\frac{\partial \tau_r}{\partial z} - gS_r + \frac{u_\theta^2}{r}$$

where a_r is the total acceleration in r-direction. The continuity equation is

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_\theta}{\partial s} + \frac{\partial u_z}{\partial z} = 0 \tag{2.125}$$

2.7.1 Superelevation in Curved Channels

The *superelevation* Δz of the free surface is the difference between the free surface level at the outer and the inner banks. It can be approximated as

$$\Delta z = \int_{r_\mathrm{i}}^{r_\mathrm{o}} S_r \mathrm{d}r \tag{2.126}$$

where r_o and r_i are the radii of curvature of outer and inner banks, respectively. The slope of the free surface in radial direction can be obtained by balancing the radial force components acting on the column of fluid with depth h. Neglecting the bed resistance, the net pressure force due to the free surface slope in r-direction is balanced by the centripetal force. It yields

$$\int_0^h \frac{u_\theta^2}{r} \rho \mathrm{d}r \mathrm{d}s \mathrm{d}z = \rho g h S_r \mathrm{d}r \mathrm{d}s \Rightarrow S_r = \frac{1}{gh} \int_0^h \frac{u_\theta^2}{r} \mathrm{d}z = \beta_r \frac{U^2}{gr} \quad \wedge \quad \beta_r U^2 h = \int_0^h u_\theta^2 \mathrm{d}z \tag{2.127}$$

where β_r is the correction factor and U is the depth-averaged tangential velocity. Then, using Eq. (2.127) into Eq. (2.126), the superelevation Δz is obtained as

$$\Delta z = \int_{r_\mathrm{i}}^{r_\mathrm{o}} \beta_r \frac{U^2}{gr} \mathrm{d}r \approx \beta_r \frac{U_\mathrm{a}^2 T}{g r_\mathrm{c}} \tag{2.128}$$

where U_a is the cross-sectional averaged tangential velocity, T is the width of the free surface, and r_c is the radius of curvature of the centerline of the channel. In Eq. (2.128), β_r can be assumed as unity.

2.7.2 Velocity Distributions in Curved Channels

In a steady fully developed flow, $\partial u_\theta/\partial t = \partial u_r/\partial t = 0$ and $\partial u_\theta/\partial s = \partial u_r/\partial s = 0$. Further, the radial and the vertical velocity components are negligible as compared to the tangential velocity components. Hence, $u_\theta = u_\theta(z)$ and $u_r = u_z = 0$. From Eqs. (2.123) and (2.124), one can obtain

$$\frac{1}{\rho} \cdot \frac{\partial \tau_\theta}{\partial z} + gS_\theta = 0 \tag{2.129a}$$

$$\frac{1}{\rho} \cdot \frac{\partial \tau_r}{\partial z} - gS_r + \frac{u_\theta^2}{r} = 0 \tag{2.129b}$$

Integration of Eq. (2.129a) produces a linear distribution of tangential shear stress as

$$\tau_\theta = \rho g h S_\theta \left(1 - \frac{z}{h}\right) \tag{2.130}$$

Following the concept of the mixing length (see Sect. 3.5), it can be written as

$$\tau_\theta = \rho l^2 \left|\frac{\mathrm{d}u_\theta}{\mathrm{d}z}\right| \frac{\mathrm{d}u_\theta}{\mathrm{d}z} = \rho \varepsilon_\mathrm{t} \frac{\mathrm{d}u_\theta}{\mathrm{d}z} \tag{2.131}$$

where l is the mixing length and ε_t is the eddy viscosity or turbulent diffusivity. Equating Eqs. (2.130) and (2.131), ε_t can be determined from

$$\varepsilon_\mathrm{t} = ghS_\theta \left(1 - \frac{z}{h}\right) \left(\frac{\mathrm{d}u_\theta}{\mathrm{d}z}\right)^{-1} \tag{2.132}$$

if a suitable velocity distribution, $u_\theta = u_\theta(z)$, is assumed. Note that by the concept of the isotropic turbulence

$$\tau_r = \rho \varepsilon_\mathrm{t} \frac{\mathrm{d}u_r}{\mathrm{d}z} \tag{2.133}$$

Using Eqs. (2.132) and (2.133) yields

$$\tau_r = \rho g h S_\theta \left(1 - \frac{z}{h}\right) \left(\frac{\mathrm{d}u_\theta}{\mathrm{d}z}\right)^{-1} \frac{\mathrm{d}u_r}{\mathrm{d}z} \tag{2.134}$$

Equation (2.134) can be used in Eq. (2.129b) to determine the radial velocity distribution, as $u_\theta = u_\theta(z)$.

For tangential velocity distribution, Rozovskii (1957) assumed

$$\frac{u_\theta}{U} = 1 + \frac{g^{0.5}}{\kappa C_\mathrm{R}} (1 + \ln \tilde{z}) \tag{2.135}$$

where κ is the von Kármán constant, C_R is the Chézy coefficient, and $\tilde{z} = z/h$. Using Eq. (2.135), Rozovskii (1957) derived the radial velocity distribution in case of a hydraulically smooth flow as

$$\frac{u_r}{U} = \frac{h}{r} \cdot \frac{1}{\kappa^2}\left(\phi_1 - \frac{g^{0.5}}{\kappa C_R}\phi_2\right) \quad \wedge \quad \phi_1 = \int \frac{2\ln\tilde{z}}{\tilde{z}-1}\mathrm{d}\tilde{z} \quad \vee \quad \phi_2 = \int \frac{\ln^2\tilde{z}}{\tilde{z}-1}\mathrm{d}\tilde{z} \tag{2.136}$$

On the other hand, in case of a hydraulically rough flow, Rozovskii (1957) derived the radial velocity distribution as

$$\frac{u_r}{U} = \frac{h}{r} \cdot \frac{1}{\kappa^2}\left\{\phi_1 - \frac{g^{0.5}}{\kappa C_R}[\phi_2 + 0.8(1 + \ln\tilde{z})]\right\} \tag{2.137}$$

The angle β of the velocity vector at any depth with the tangential direction, as shown in Fig. 2.33, can be obtained from $\beta = \arctan(u_r/u_\theta)$.

Kikkawa et al. (1976) also derived the velocity distributions from the equation of motion, where the eddy viscosity was assumed to be same as that of a two-dimensional flow in a straight channel. They suggested the equation of motion that governs by the secondary flow as

$$\frac{\partial^4\psi}{\partial z^4} = u_\theta\frac{\partial u_\theta}{\partial z} \tag{2.138}$$

where ψ is the stream function. Neglecting the nonlinear interaction between the secondary flow and the main flow, the tangential velocity distribution could be shown as

$$\frac{u_\theta - u_s}{u_*} = -\frac{1}{\kappa}\ln\tilde{z} \tag{2.139}$$

where u_s is the tangential velocity at the free surface and u_* is the shear velocity. Kikkawa et al. (1976) derived the radial velocity distribution in a fully developed flow by integrating Eq. (2.138) as

$$\frac{u_r}{U_a} = \frac{U^2}{U_a^2} \cdot \frac{h}{r} \cdot \frac{1}{\kappa}\left(\phi_A - \frac{1}{\kappa} \cdot \frac{u_*}{U_a}\phi_B\right) \tag{2.140}$$

where U is the depth-averaged tangential velocity, which is a function of r, and ϕ_A and ϕ_B are as follows:

$$\phi_A = -15\left(\tilde{z}^2\ln\tilde{z} - \frac{\tilde{z}^2}{2} + \frac{15}{54}\right) \tag{2.141a}$$

$$\phi_{\mathrm{B}} = \frac{15}{2}\left(\tilde{z}^2 \ln^2 \tilde{z} - \tilde{z}^2 \ln \tilde{z} + \frac{\tilde{z}^2}{2} - \frac{19}{54}\right) \tag{2.141b}$$

Equation (2.140) indicates that the radial velocity distribution $u_r(z)$ is proportional to U^2 and h/r.

Odgaard (1989) assumed the tangential and the radial velocity distributions as

$$\frac{u_\theta}{U} = \frac{m+1}{m}\tilde{z}^{1/m} \quad \wedge \quad m = \kappa\frac{U}{u_*} = \kappa\left(\frac{8}{\lambda_{\mathrm{D}}}\right)^{0.5} = \kappa\frac{C_{\mathrm{R}}}{g^{0.5}} \tag{2.142a}$$

$$u_r = \frac{1}{h}\int_0^h u_r \mathrm{d}z + 2u_r|_{\mathrm{ca}}\left(\tilde{z} - \frac{1}{2}\right) \tag{2.142b}$$

where m is the reciprocal of exponent, λ_{D} is the Darcy-Weisbach friction factor, and $u_r|_{\mathrm{ca}}$ is the centrifugally induced component. For a fully developed flow, Odgaard (1989) gave

$$U = \frac{m}{\kappa}(ghS)^{0.5} \tag{2.143a}$$

$$\frac{u_r|_{\mathrm{ca}}}{U} = \frac{1}{\kappa^2}\cdot\frac{(m+1)(2m+1)}{1+m+2m^2}\cdot\frac{h}{r} \tag{2.143b}$$

Odgaard argued that the ratio h/r is nearly a constant varying between 7.2 and 8, while m can vary between 3 and 6 in a curved channel.

2.7.3 Bed Shear Stress Distribution in Curved Channels

The bed shear stress in a curved channel is decomposed into tangential $\tau_{0\theta}$ and radial τ_{0r} components. The tangential component of the bed shear stress can be given by

$$\tau_{0\theta} = \rho g\frac{U^2}{C_{\mathrm{R}}^2} \tag{2.144}$$

From a radial velocity distribution similar to that of Rozovskii (1957) (Eq. 2.137), Jansen et al. (1979) derived the radial component of bed shear stress as

$$\tau_{0r} = -\frac{2\rho gh}{r\kappa^2}\cdot\frac{U^2}{C_{\mathrm{R}}^2}\left(1 - \frac{g^{0.5}}{\kappa C_{\mathrm{R}}}\right) \tag{2.145}$$

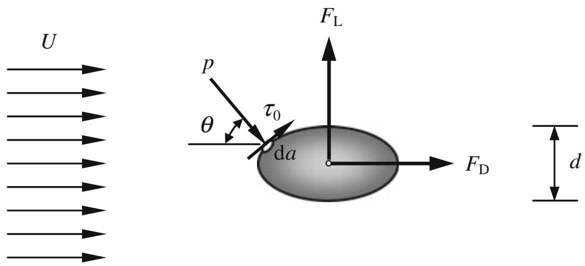


Fig. 2.35 Hydrodynamic drag and lift due to flow past a particle

2.8 Hydrodynamic Drag and Lift on a Particle

When a real fluid flows past a solid particle, the hydrodynamic force of resistance contributes in two ways. Firstly, resistance due to viscosity is developed at the wall of the particle in the form of shear stresses. Secondly, differential pressure intensities act normal to the wall. The integration of both the forces over the entire surface of the particle composes the total hydrodynamic force. The component of the hydrodynamic force in the flow direction is called *drag*, which is the force by which the fluid tends to drag the particle. On the other hand, the component normal to the flow direction is called *lift*, which is the force by which the fluid tends to lift the particle (Fig. 2.35).

2.8.1 The Drag

The drag on the body of the particle is made up of two contributions, namely *skin friction drag* and *form* or *pressure drag*. Thus, drag is the sum of the components of the wall shear stress τ_0 and the pressure p in the flow direction, respectively. Thus, referring to Fig. 2.35, τ_0 and p act on an elementary area da tangentially and normally, respectively; and the drag is given by

$$F_D = \underbrace{\int_a \tau_0 \sin\theta \mathrm{d}a}_{\text{Skin friction drag}} + \underbrace{\int_a p\cos\theta \mathrm{d}a}_{\text{Form drag}} \quad (2.146)$$

where a is the total surface area. The above equation suggests that both the contributions to the drag can therefore be theoretically calculated. It, however, requires knowledge of the wall shear stress distribution on the surface of the particle and the pressure distribution around the particle. Nevertheless, the integrals of Eq. (2.146) cannot be evaluated easily, as the description of τ_0 and p becomes uncertain due to the boundary layer separation phenomenon, as described in the preceding section. It is therefore simpler to measure the drag

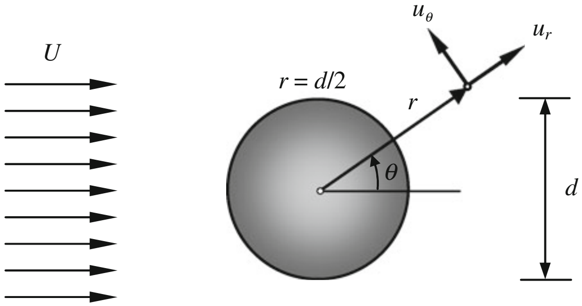


Fig. 2.36 Creeping flow past a spherical particle

experimentally and express it as a function of dynamic pressure force $(\rho U^2/2)A$, where A is the projected area of the particle on a plane, that is normal to the flow direction. Thus,

$$F_D = \frac{1}{2} C_D \rho U^2 A \tag{2.147}$$

where C_D is the drag coefficient being determined experimentally (see Sect. 1.7).

2.8.1.1 Creeping Flow About a Spherical Particle (Stokes' Law)

The basic assumption for a *creeping flow* is that the inertia terms are negligible in the momentum equation if the particle Reynolds number R_e is very small ($R_e \ll 1$, where $R_e = Ud/\upsilon$ and d is the size or diameter of the particle). This is the special case of creeping viscous flow, where viscous effects predominate.

Let a creeping flow of free stream velocity U about a solid spherical particle of diameter d be considered (Fig. 2.36). Using a spherical polar coordinates (r, θ), the radial and tangential velocity components u_r and u_θ are related to the Stokes stream function ψ by the relations

$$u_r = \frac{1}{r^2 \sin\theta} \cdot \frac{\partial \psi}{\partial \theta}, \; u_\theta = -\frac{1}{r \sin\theta} \cdot \frac{\partial \psi}{\partial r} \tag{2.148}$$

For a creeping flow, the Navier–Stokes equations in two-dimensional spherical coordinates reduce to

$$\frac{1}{\rho} \cdot \frac{\partial p}{\partial r} = \upsilon \nabla^2 u_r, \quad \frac{1}{\rho} \cdot \frac{1}{r} \cdot \frac{\partial p}{\partial \theta} = \upsilon \nabla^2 u_\theta \quad \wedge \quad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \cdot \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta} \cdot \frac{\partial}{\partial \theta}\right) \tag{2.149}$$

Using Eq. (2.148) into Eq. (2.149) yields

$$\frac{\partial p}{\partial r} = \frac{\mu}{r^2 \sin\theta} \cdot \frac{\partial}{\partial \theta}(\nabla^2 \psi), \ \frac{\partial p}{\partial \theta} = -\frac{\mu}{\sin\theta} \cdot \frac{\partial}{\partial r}(\nabla^2 \psi) \tag{2.150}$$

Eliminating p, one finds

$$\nabla^4 \psi = 0 \text{ or } \left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \cdot \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta} \cdot \frac{\partial}{\partial \theta}\right)\right]^2 \psi = 0 \tag{2.151}$$

Making substitution $\psi = f(r)\sin^2\theta$, it allows to reduce Eq. (2.151) to a fourth-order ordinary differential equation

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)\left(\frac{d^2 f}{dr^2} - \frac{2f}{r^2}\right) = 0 \tag{2.152}$$

A substitution of $f = r^k$ leads to fourth-order polynomial, whose roots are $k = -1$, 1, 2, and 4. Thus, the general solution for f becomes

$$f(r) = Ar^{-1} + Br + Cr^2 + Dr^4 \tag{2.153}$$

where A, B, C, and D are the constants of integration. The boundary conditions are the following: (1) at $r = d/2$ (at surface), $\psi = 0$ ($u_r = 0$ at surface), and $\partial\psi/\partial r = 0$ ($u_\theta = 0$ at surface) and (2) at $r \to \infty$, $\psi \to (Ur^2/2)\sin^2\theta$. It leads to $A = Ud^3/32$, $B = -3Ud/8$, $C = U/2$, and $D = 0$. Then, the desired stream function for a creeping flow is obtained as

$$\psi(r,\theta) = \frac{1}{16} Ud^2 \sin^2\theta \left(\frac{d}{2r} - \frac{6r}{d} + \frac{8r^2}{d^2}\right) \tag{2.154}$$

Then, the velocity components are obtained from Eq. (2.148) as

$$u_r = U\cos\theta\left(1 + \frac{d^3}{16r^3} - \frac{3d}{4r}\right), \ u_\theta = U\sin\theta\left(-1 + \frac{d^3}{32r^3} + \frac{3d}{8r}\right) \tag{2.155}$$

With known u_r and u_θ, the pressure is determined by integrating Eq. (2.150). The result is

$$p = p_\infty - \frac{3\mu dU}{4r^2}\cos\theta \tag{2.156}$$

where p_∞ is the uniform free stream pressure. This exerts a pressure drag on the spherical particle. In addition, a wall shear stress exerts a drag. The shear stress distribution is given by

$$\tau_{r\theta} = \mu\left(\frac{1}{r}\cdot\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) = -\mu U\frac{\sin\theta}{r}\cdot\frac{3d^3}{16r^3} \tag{2.157}$$

Then, the total drag can be obtained from Eq. (2.146) as

$$F_\mathrm{D} = -\int_0^\pi \tau_{r\theta}|_{r=d/2}\sin\theta\mathrm{d}a - \int_0^\pi p|_{r=d/2}\cos\theta\mathrm{d}a \quad \wedge \quad \mathrm{d}a = \pi d\frac{d}{2}\sin\theta\mathrm{d}\theta,$$

$$\therefore \quad F_\mathrm{D} = 2\pi\mu dU + \pi\mu dU = 3\pi\mu dU \tag{2.158}$$

This is known as the *Stokes' law* (Stokes 1851). Note that viscous shear force contributes two-third and pressure force one-third. Equation (2.158) is strictly valid only for $R_\mathrm{e} \ll 1$, but satisfactorily agrees with the experimental data up to $R_\mathrm{e} \approx 1$.

2.8.2 The Lift

As already discussed, fluid flowing past a particle exerts hydrodynamic force on the surface of the particle. Lift is the component of this force. It acts normal to the flow direction. The total lift force is the integral of the pressure forces normal to the flow direction. Thus,

$$F_\mathrm{L} = \int_a p\sin\theta\mathrm{d}a \tag{2.159}$$

Analogous to Eq. (2.147), lift can also be expressed as a function of dynamic pressure force $(\rho U^2/2)A$, where A is the planform area. Thus,

$$F_\mathrm{L} = \frac{1}{2}C_\mathrm{L}\rho U^2 A \tag{2.160}$$

where C_L is the lift coefficient being determined experimentally.

Further, when a small spherical particle spinning with an angular velocity ω is placed in a uniform free stream, in addition to drag a lift due to *Magnus effect* acts on the particle. Rubinow and Keller (1961) formulated it as

$$F_\mathrm{L} = \frac{\pi}{8}\rho d^3 U\omega \tag{2.161}$$

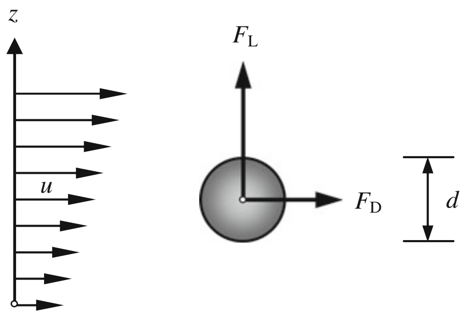


Fig. 2.37 Shear flow past a spherical particle

It is pertinent to mention that the lift due to *Magnus effect* on a rotating cylinder in an ideal (inviscid) fluid flow is given by $F_L = \rho U\Gamma$, where Γ is the circulation of the cylinder.

2.8.2.1 Lift in a Shear Flow

Particle in a shear flow, that has a spatially nonuniform velocity distribution, experiences a transverse force on the particle, even when the particle is prevented from a spinning motion (Fig. 2.37). The shear lift is originated from the inertia effects in the viscous flow around the particle and is fundamentally different from the hydrodynamic lift. The expression for the inertia *shear lift* was first obtained by Saffman (1965, 1968). It is

$$F_L = \alpha_L \rho d^2 u \left(\upsilon \frac{\partial u}{\partial z} \right)^{0.5} \tag{2.162}$$

where α_L is the Saffman lift coefficient, being equal to 1.615.

2.9 Appendix

2.9.1 Navier–Stokes and Continuity Equations in a Cylindrical Polar Coordinate System

Navier–Stokes equations in cylindrical polar coordinates (r, θ, z) with corresponding velocity components (u, v, w) for an incompressible fluid flow are given as follows:

$$u\frac{\partial u}{\partial r}+\frac{v}{r}\cdot\frac{\partial u}{\partial \theta}-\frac{v^2}{r}+w\frac{\partial u}{\partial z}+\frac{\partial u}{\partial t}=g_r-\frac{1}{\rho}\cdot\frac{\partial p}{\partial r}$$
$$+\upsilon\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\cdot\frac{\partial}{\partial r}(ru)\right]+\frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial \theta^2}-\frac{2}{r^2}\cdot\frac{\partial v}{\partial \theta}+\frac{\partial^2 u}{\partial z^2}\right\} \tag{2.163a}$$

$$u\frac{\partial v}{\partial r}+\frac{v}{r}\cdot\frac{\partial v}{\partial \theta}+\frac{uv}{r}+w\frac{\partial v}{\partial z}+\frac{\partial v}{\partial t}=g_\theta-\frac{1}{\rho}\cdot\frac{1}{r}\cdot\frac{\partial p}{\partial \theta}$$
$$+\upsilon\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\cdot\frac{\partial}{\partial r}(rv)\right]+\frac{1}{r^2}\cdot\frac{\partial^2 v}{\partial \theta^2}+\frac{2}{r^2}\cdot\frac{\partial u}{\partial \theta}+\frac{\partial^2 v}{\partial z^2}\right\} \tag{2.163b}$$

$$u\frac{\partial w}{\partial r}+\frac{v}{r}\cdot\frac{\partial w}{\partial \theta}+w\frac{\partial u}{\partial z}+\frac{\partial w}{\partial t}=g_z-\frac{1}{\rho}\cdot\frac{\partial p}{\partial z}$$
$$+\upsilon\left[\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)+\frac{1}{r^2}\cdot\frac{\partial^2 w}{\partial \theta^2}+\frac{\partial^2 w}{\partial z^2}\right] \tag{2.163c}$$

The continuity equation for an incompressible flow is

$$\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{1}{r}\cdot\frac{\partial v}{\partial \theta}+\frac{\partial w}{\partial z}=0 \tag{2.164}$$

2.9.2 Navier–Stokes and Continuity Equations in a Spherical Polar Coordinate System

Navier–Stokes equations in spherical polar coordinates (r, θ, φ) with corresponding velocity components (u, v, w) are as follows:

$$u\frac{\partial u}{\partial r}+\frac{v}{r}\cdot\frac{\partial u}{\partial \theta}+\frac{w}{r\sin\theta}\cdot\frac{\partial u}{\partial \varphi}-\frac{v^2+w^2}{r}+\frac{\partial u}{\partial t}=g_r-\frac{1}{\rho}\cdot\frac{\partial p}{\partial r}$$
$$+\upsilon\left(\nabla^2 u-\frac{2u}{r^2}-\frac{2}{r^2}\cdot\frac{\partial^2 v}{\partial \theta^2}-\frac{2v}{r^2}\cot\theta-\frac{2}{r^2\sin\theta}\cdot\frac{\partial w}{\partial \varphi}\right) \tag{2.165a}$$

$$u\frac{\partial v}{\partial r}+\frac{v}{r}\cdot\frac{\partial v}{\partial \theta}+\frac{w}{r\sin\theta}\cdot\frac{\partial v}{\partial \varphi}+\frac{uv}{r}-\frac{w^2}{r}\cot\theta+\frac{\partial v}{\partial t}=g_\theta-\frac{1}{\rho}\cdot\frac{1}{r}\cdot\frac{\partial p}{\partial \theta}$$
$$+\upsilon\left(\nabla^2 v+\frac{2}{r^2}\cdot\frac{\partial u}{\partial \theta}-\frac{v}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\cdot\frac{\partial w}{\partial \varphi}\right) \tag{2.165b}$$

$$u\frac{\partial w}{\partial r}+\frac{v}{r}\cdot\frac{\partial w}{\partial \theta}+\frac{w}{r\sin\theta}\cdot\frac{\partial w}{\partial \varphi}+\frac{wu}{r}+\frac{vw}{r}\cot\theta+\frac{\partial w}{\partial t}=g_\varphi-\frac{1}{\rho}\cdot\frac{1}{r\sin\theta}\cdot\frac{\partial p}{\partial \varphi}$$
$$+\upsilon\left(\nabla^2 w-\frac{w}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\cdot\frac{\partial u}{\partial \varphi}+\frac{2\cos\theta}{r^2\sin\theta}\cdot\frac{\partial v}{\partial \varphi}\right) \tag{2.164c}$$

where

$$\nabla^2 = \frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{r^2\sin^2\theta}\cdot\left(\frac{\partial^2}{\partial\varphi^2}\right)$$

The continuity equation for an incompressible flow is

$$\frac{1}{r^2}\cdot\frac{\partial}{\partial r}(r^2u) + \frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial\theta}(v\sin\theta) + \frac{1}{r\sin\theta}\cdot\frac{\partial w}{\partial\varphi} = 0 \tag{2.166}$$

2.10 Examples

Example 2.1 The velocity distribution in a wide channel is given by $u/u_{\max} = (z/h)^{1/n}$, where $u_{\max}$ is the maximum velocity at a flow depth h. Find the depth-averaged velocity, momentum coefficient β and energy coefficient α.

Solution

The depth-averaged velocity U is obtained as

$$U = \frac{1}{h}\int_0^h u\mathrm{d}z = \frac{1}{h}\int_0^h u_{\max}\left(\frac{z}{h}\right)^{1/n}\mathrm{d}z = \frac{n}{1+n}u_{\max}$$

Therefore, the velocity distribution can be expressed in terms of depth-averaged velocity as

$$u = u_{\max}\left(\frac{z}{h}\right)^{1/n} = \frac{1+n}{n}U\left(\frac{z}{h}\right)^{1/n}$$

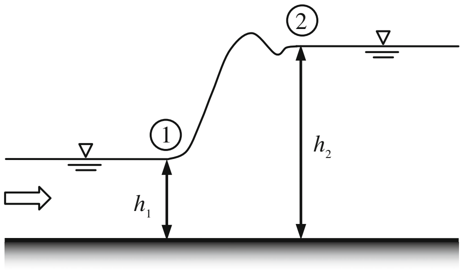
For a wide channel, the momentum coefficient β given by Eq. (2.37) can be expressed as

$$\beta = \frac{1}{h}\int_0^h \frac{u^2}{U^2}\mathrm{d}z = \frac{1}{h}\int_0^h \frac{1}{U^2}\left(\frac{1+n}{n}\right)^2 U^2\left(\frac{z}{h}\right)^{2/n}\mathrm{d}z = \frac{(1+n)^2}{(2+n)n}$$

Again, for a wide channel, the energy coefficient α given by Eq. (2.74) can be expressed as

$$\alpha = \frac{1}{h}\int_0^h \frac{u^3}{U^3}\mathrm{d}z = \frac{1}{h}\int_0^h \frac{1}{U^3}\left(\frac{1+n}{n}\right)^3 U^3\left(\frac{z}{h}\right)^{3/n}\mathrm{d}z = \frac{(1+n)^3}{(3+n)n^2}$$

Fig. E2.1 Hydraulic jump



Example 2.2 Determine the critical depth for a water discharge of 8 m^3 s^{-1} flowing in a trapezoidal channel with a base width of 3 m and a side slope of 1 horizontal to 2 vertical.

Solution

For base width, $b = 3$ m and side slope, $z = 1/2 = 0.5$, the area A and the top width T are

$$A = (b + zh)h = 3h + 0.5h^2, \; T = b + 2zh = 3 + h$$

The condition of a critical flow is

$$f(h) = \frac{Q^2T}{gA^3} = 1 \Leftarrow \text{Eq. (2.82)}$$

$$\therefore \quad f(h) = \frac{Q^2T}{gA^3} = \frac{8^2(3+h)}{9.81(3h + 0.5h^2)^3} = \frac{6.524(3+h)}{(3h+0.5h^2)^3}$$

Adopting the trial-and-error method, the solution of the above equation is $f(h = 0.854) \approx 1$. Therefore, the critical depth is 0.854 m.

Example 2.3 Derive the relationship for the sequent depth ratio of a *hydraulic jump*[4] on a horizontal floor of a rectangular channel, as shown in Fig. E2.1. Also determine the energy loss.

Solution

The specific force equation between sections 1 and 2 for a prismatic rectangular channel having a width b can be given by

$$\frac{Q^2}{gA_1} + z_{c1}A_1 = \frac{Q^2}{gA_2} + z_{c2}A_2 \Leftarrow \text{Eq. (2.55)}$$

[4] *Hydraulic jump* occurs when there is a rapid change in flow depth resulting from a low stage (supercritical) to a high stage (subcritical) with an abrupt rise in free surface elevation. It is therefore a local phenomenon due to a transition from a supercritical flow to a subcritical flow.

Substituting $Q = U_1A_1 = U_2A_2$, $A_1 = bh_1$, $A_2 = bh_2$, $z_{c1} = h_1/2$, $z_{c2} = h_2/2$, and $F_1 = U_1/(gh_1)^{0.5}$, the above equation becomes

$$\left(\frac{h_2}{h_1}\right)^2 + \frac{h_2}{h_1} - 2F_1^2 = 0$$

where h_1 and h_2 are the sequent depths and F_1 is the Froude number in low stage. The feasible solution for the sequent depth ratio of the quadratic equation is

$$\frac{h_2}{h_1} = \frac{1}{2}\left[(1 + 8F_1^2)^{0.5} - 1\right]$$

Bélanger (1828) was the first to apply the momentum equation across a hydraulic jump to obtain the above equation, which is often called the *Bélanger equation*. Applying the specific energy concept, the energy loss ΔE in a hydraulic jump can be expressed as

$$\Delta E = E_1 - E_2 = \left(h_1 + \frac{U_1^2}{2g}\right) - \left(h_2 + \frac{U_2^2}{2g}\right)$$

where E_1 and E_2 are the specific energies at sections 1 and 2, respectively. In the above, the energy coefficients are assumed to be unity, that is, $\alpha_1 = \alpha_2 = 1$. Applying the continuity equation, the discharge per unit width q is given by $q = U_1h_1 = U_2h_2$, and then, the above equation becomes

$$\Delta E = -(h_2 - h_1) + \frac{q^2}{2g}\left(\frac{1}{h_1^2} - \frac{1}{h_2^2}\right)$$

Further, applying $q = U_1h_1$ and $F_1 = U_1/(gh_1)^{0.5}$, the equation of sequent depth ratio or the Bélanger equation is expressed as

$$\frac{q^2}{g} = \frac{h_1h_2}{2}(h_1 + h_2)$$

Therefore, the energy loss ΔE in a hydraulic jump is formulated as

$$\Delta E = \frac{(h_2 - h_1)^3}{4h_1h_2}$$

Example 2.4 Derive the relationship of the critical depth in terms of alternate depths in the flow through a rectangular channel.

Solution

Since the specific energy is same at two sections (low and high stages), assuming $\alpha_1 = \alpha_2 = 1$, it gives

$$h_1 + \frac{U_1^2}{2g} = h_2 + \frac{U_2^2}{2g}$$

Then, applying $q = U_1 h_1$ yields

$$h_2 - h_1 = \frac{q^2}{2g}\left(\frac{1}{h_1^2} - \frac{1}{h_2^2}\right)$$

Again, the condition of a critical flow in a rectangular channel can be obtained from Eq. (2.82) as

$$h_c^3 = \frac{q^2}{g}$$

Thus, the relationship of the critical depth in terms of alternate depths is obtained as follows:

$$h_c = \left(\frac{2h_1^2 h_2^2}{h_1 + h_2}\right)^{1/3}$$

Example 2.5 Oil with a free stream velocity of 1 m s^{-1} flows over a thin plate of 1.5 m wide and 2.5 m long. Determine the boundary layer thickness and the wall shear stress at a distance of 1.5 m from the leading edge of the plate and also calculate the total resistance on one side of the plate. Consider coefficient of kinematic viscosity of oil $\upsilon = 10^{-5}$ m^2 s^{-1} and relative density of oil $s = 0.8$.

Solution

Given data are as follows:

Free stream velocity, $U = 1$ m s^{-1}; plate width, $b = 1.5$ m; plate length, $L = 2.5$ m; coefficient of kinematic viscosity of oil, $\upsilon = 10^{-5}$ m^2 s^{-1}; and relative density of oil, $s = 0.8$.

The mass density of oil, $\rho = 0.8 \times 10^3$ kg m^{-3}

The Reynolds number R_x at $x = 1.5$ m is

$$R_x = \frac{Ux}{\upsilon} = \frac{1 \times 1.5}{10^{-5}} = 1.5 \times 10^5 < 3 \times 10^5$$

It is low enough to allow a laminar boundary layer

The boundary layer thickness δ at $x = 1.5$ m is

$$\delta = 4.643xR_x^{-0.5} = 4.643 \times 1.5(1.5 \times 10^5)^{-0.5} = 0.018 \text{ m} \Leftarrow \text{Eq. (2.113)}$$

The wall shear stress τ_0 at $x = 1.5$ m is

$$\tau_0 = 0.323\mu\frac{U}{x}R_x^{0.5} = 0.323(10^{-5} \times 0.8 \times 10^3)\frac{1}{1.5}(1.5 \times 10^5)^{0.5} = 0.667 \text{ Pa} \Leftarrow \text{Eq. (2.114)}$$

The Reynolds number R_L at the end of the plate having a length $L = 2.5$ m is

$$R_L = \frac{UL}{\upsilon} = \frac{1 \times 2.5}{10^{-5}} = 2.5 \times 10^5 < 3 \times 10^5$$

The R_L is to allow a laminar boundary layer.
The wall shear resistance per unit width F_τ on one side of the plate of length $L = 2.5$ m can be obtained from

$$F_\tau = 0.646\rho U\upsilon R_L^{0.5} = 0.646 \times 0.8 \times 10^3 \times 1 \times 10^{-5}(2.5 \times 10^5)^{0.5} = 2.584 \text{ N m}^{-1} \Leftarrow \text{Eq. (2.115)}$$

Therefore, the total resistance is $F_R = F_\tau\, b = 2.584 \times 1.5 = 3.876$ N

Example 2.6 Water that has a free stream velocity of 1.5 m s^{-1} at the entrance flows through a 2.5 m wide rectangular channel. Determine the length of the channel required to obtain a fully developed turbulent flow for the flow depth of 0.5 m and also calculate the wall shear stress at the location of the fully developed flow and the total resistance on the channel base up to that location. Consider coefficient of kinematic viscosity of water $\upsilon = 10^{-6}$ m^2 s^{-1}. Assume the length of the channel over which the laminar boundary layer exists is negligibly small as compared to that over which the turbulent boundary layer exists.

Solution

Given data are as follows:
Free stream velocity, $U = 1.5$ m s^{-1}; channel width, $b = 2.5$ m; and coefficient of kinematic viscosity of water, $\upsilon = 10^{-6}$ m^2 s^{-1}.
Fully developed flow depth, $h = \delta = 0.5$ m.
Considering a turbulent boundary layer, the equation of boundary layer is used as

$$\delta = 0.376xR_x^{-0.2} = 0.376x\left(\frac{Ux}{\upsilon}\right)^{-0.2} \Leftarrow \text{Eq. (2.120)}$$

$$\therefore \quad x = 3.396\delta^{1.25}\left(\frac{U}{\upsilon}\right)^{0.25} = 3.396 \times 0.5^{1.25}\left(\frac{1.5}{10^{-6}}\right)^{0.25} = 49.97 \text{ m}$$

The Reynolds number R_x at $x = 49.97$ m is

$$R_x = \frac{Ux}{\upsilon} = \frac{1.5 \times 49.97}{10^{-6}} = 7.496 \times 10^7 > 10^6$$

Noting that the transition occurs in the range $R_x = 3 \times 10^5$ to 10^6, the $R_x = 7.496 \times 10^7$ is therefore large enough to allow a turbulent boundary layer. However, the length up to the transition is $x = R_x(\upsilon/U) = 10^6(10^{-6}/1.5) = 0.667$ m, which is negligibly small in comparison to the length required to form the fully developed turbulent flow, that is, $x = 49.97$ m.
The wall shear stress τ_0 at $x = 49.97$ m is

$$\tau_0 = 2.91 \times 10^{-2} \rho U^2 R_x^{-0.2} = 2.91 \times 10^{-2} \times 10^3 \times 1.5^2 (7.496 \times 10^7)^{-0.2}$$
$$= 1.742 \text{ Pa} \Leftarrow \text{Eq. (2.121)}$$

The wall shear resistance per unit width F_τ for $L = x = 49.97$ m and $R_L = R_x = 7.496 \times 10^7$ is

$$F_\tau = 3.638 \times 10^{-2} \rho U^2 L R_L^{-0.2}$$
$$= 3.638 \times 10^{-2} \times 10^3 \times 1.5^2 \times 49.97 (7.496 \times 10^7)^{-0.2} = 108.84 \text{ N m}^{-1}$$
$$\Leftarrow \text{Eq. (2.122)}$$

Therefore, the total resistance is $F_R = F_\tau\, b = 108.84 \times 2.5 = 272.1$ N

Example 2.7 A spherical particle having a diameter $d = 4$ mm is placed in a free stream of water with a velocity $U = 1.2$ m s^{-1}. Determine the drag and the lift acting on the particle. The drag and lift coefficients are given by $C_D = 24R_e^{-1}(1 + 0.15R_e^{0.687})$ and $C_L = 0.85C_D$, where $R_e = Ud/\upsilon$. Consider coefficient of kinematic viscosity of water $\upsilon = 10^{-6}$ m^2 s^{-1}.

Solution

Given data are as follows:
Particle diameter, $d = 4$ mm; free stream velocity, $U = 1.2$ m s^{-1} and coefficient of kinematic viscosity of water, $\upsilon = 10^{-6}$ m^2 s^{-1}.
The particle Reynolds number R_e is

$$R_e = \frac{Ud}{\upsilon} = \frac{1.2 \times 4 \times 10^{-3}}{10^{-6}} = 4.8 \times 10^3$$

The drag and lift coefficients are

$$C_D = \frac{24}{R_e}(1 + 0.15R_e^{0.687}) = \frac{24}{4.8 \times 10^3}[1 + 0.15(4.8 \times 10^3)^{0.687}] = 0.259$$
$$C_L = 0.85C_D = 0.85 \times 0.259 = 0.22$$

The drag is

$$F_{\mathrm{D}} = \frac{1}{2} C_{\mathrm{D}} \rho U^2 A = \frac{1}{2} \times 0.259 \times 10^3 \times 1.2^2 \times \frac{\pi}{4} (4 \times 10^{-3})^2$$
$$= 2.343 \times 10^{-3}\ \mathrm{N} \Leftarrow \text{Eq. (2.147)}$$

The lift is

$$F_{\mathrm{L}} = \frac{1}{2} C_{\mathrm{L}} \rho U^2 A = \frac{1}{2} \times 0.22 \times 10^3 \times 1.2^2 \times \frac{\pi}{4} (4 \times 10^{-3})^2$$
$$= 1.991 \times 10^{-3}\ \mathrm{N} \Leftarrow \text{Eq.(2.160)}$$

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