Supervaluationism and Classical Logic

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Abstract. The supervaluationist theory of vagueness provides a notion of logical consequence that is akin to classical consequence. In the absence of a *definitely* operator, supervaluationist consequence coincides with classical consequence. In the presence of '*definitely*', however, supervaluationist logic gives raise to counterexamples to classically valid patterns of inference. Foes of supervaluationism emphasize the last result to argue against the supervaluationist theory. This paper shows a way in which we might obtain systems of deduction adequate for supervaluationist consequence based on systems of deduction adequate for classical consequence. Deductions on the systems obtained this way adopt a completely classical form with the exception of a single step. The paper reviews (at least part of) the discussion on the non-classicality of supervaluationist logic under the light of this result.

Keywords: Vagueness, Supervaluationism, Global Validity, Deductive Systems, Logical Consequence.

1 Introduction

1.1 Vagueness and Supervaluationism

My youngest daughter, Julia, is 3 months old (at the time I'm writing this paper). She is clearly a baby. Sofia and Carmen are 4 and 6 years old respectively; they are clearly not babies (you can ask them). Julia will probably cease to be a baby, and become, as her older sisters, clearly not a baby. But is there an exact time n such that Julia is a baby at n but Julia is not a baby at time n plus one second? The conclusion seems to be unavoidable if we want to keep to classical logic. Since from the fact that Julia is a baby at t_0 (today) and the supposition that for any time x if Julia is a baby at x then Julia is a baby at time $x+1$ it follows that Julia is a baby at time $t_0 + 10^8$ seconds (t_0 plus three years and a couple of months approx.) by a number 10^8 of applications of universal instantiation and modus ponens. Thus, if we grant that Julia is not a baby at time $t_0 + 10^8$, it

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classically follows that it is not the case that for any time x if Julia is a baby at x then Julia is a baby at time $x + 1$, classically in other words, there is a time x such that Julia is a baby at x but Julia is not a baby at $x + 1$ second.

Epistemicists in vagueness want to retain classical logic and they endorse the somewhat surprising claim that there's actually such an n (they claim we know the existential generalization 'there is an n that such and such' even if there is no particular n of which we know that such and such). Many philosophers, however, find this claim something too hard to swallow and take it as evidence that classical logic should be modified (at least when dealing with vague expressions). One standard way in which we might modify classical logic is by considering some extra value among truth and falsity; we then redefine logical connectives taking into account the new value. This strategy has motivated some philosophers to defend Kleene's strong three-valued logic for the case of vagueness.[1] Under this view, the conclusion that there is an n at which Julia is a baby and such that Julia is not a baby at n plus one second does not follow, since there are times at which the sentence 'Julia is a baby' has not a clearly defined truth-value. Thus, the strategy consists in a suitable weakening of classically valid principles like excluded middle along with other principles at work in the previous paradoxical result like the least number principle (see [4]).

In some sense, supervaluationists take a middle path among these two alternatives. Unlike epistemicists, supervaluationists hold that vague expressions lead to truth-value gaps and, thus, that at some time the sentence 'Julia is a baby' lacks a truth-value. Unlike philosophers endorsing Kleene's strong three-valued logic, however, supervaluationists endorse a non truth-functional semantics that allows them to endorse, broadly speaking, classical logic. How?

The basic thought underlying supervaluationism is that vagueness is a matter of underdetermination of meaning. This thought is captured with the idea that the use we make of an expression does not *decide* between a number of admissible candidates for making the expression precise. According to supervaluationism a vague expression like 'baby' can be made precise in several ways compatible with the actual use we make of the expression. For example, we can make it precise by saying that x is a baby just in case x is less than one year old; but the use of the expression will allow other ways of making precise like 'less than one year plus a second'. If Martin is one year old, the sentence 'Martin is a baby' will be true in some ways of making 'baby' precise and false in others. Since our use does not decide which of the ways of making precise is correct, the truth-value of the sentence 'Martin is a baby' is left unsettled. By supervaluationist standards, a sentence is true just in case it is true in every way of making precise the vague expressions contained in it (that is, 'truth is supertruth').

A *precisification* is a way of making precise all the expressions of the language so that every sentence gets a truth-value (true or false but not both) in each precisification. In this sense, a precisification is a classical truth-value assignment. However, precisifications should be *admissible* in the sense that some connections must be respected such as analytic relations between expressions. For example,

[1] For example, [7] and [8].

any precisification counting Martin as a baby should not count him as a child. Thus, the sentence 'If Martin is a baby then he is not a child' will be supertrue even if Martin is a borderline-baby. Also comparative relations must be respected by admissible precisifications. For example, any precisification making 'Nicolas is a baby' true (where Nicolas is one year and a month) should also make 'Martin is a baby' true. These restrictions on the admissibility of a precisification enables the supervaluationist theory to endorse Fine's so-called *penumbral connections*, that is, connections that might hold among sentences even if these have a borderline status [5, pp. 269-270]. Taking the previous example, if Nicolas is older than Martin but both are borderline cases of the predicate 'is a baby', the sentence 'If Nicolas is a baby then Martin is a baby' is true in every precisification (and, hence, true *simpliciter* for the supervaluationist) since every precisification in which the antecedent is true, the consequent is also true. At this point supervaluationists have some advantage over some truth-functional approaches such as those endorsing Kleene's strong three-valued logic, since for this semantics, the sentence 'If Nicolas is a baby, then Martin is a baby' comes out as indefinite.[2]

One consequence of supervaluationist semantics is that classical validities are preserved. A sentence φ is valid according to supervaluationist semantics just in case it is supertrue in every model. Since precisifications are classical truth-value assignments, classically valid sentences are true in each precisification and, thus, they are supertrue in every model. For example, though the sentence 'Martin is a baby' lacks a truth-value, the sentence 'Martin is a baby or Martin is not a baby' is supertrue since in each precisification some member of the disjunction is true (though not the same in every precisification). More generally, excluded middle is valid since, for every model, every precisification verifies $p \vee \neg p$. Furthermore, it can be shown that, as long as we stick to the classical language, supervaluationist consequence and classical consequence coincide.[3] At this point, supervaluationists seem to have again the upper hand over truth-functional approaches. In Kleene's strong three-valued logic φ entails φ even if $\varphi \to \varphi$ is not valid (thus, conditional proof is not a valid rule of inference).

The question now is how can supervaluationists explain the sorites paradox without committing themselves to an epistemic explanation of vagueness. If supervaluationist consequence coincides with classical consequence and the existence of an n such that Julia is a baby at n but Julia is not a baby at n plus one second follows by classical reasoning, the supervaluationist must be committed to that consequence as well. The supervaluationist explanation is that though they are committed to the truth of the claim 'there is an n such that Julia is a

[2] Fine claims that supervaluationism is the only view that can accommodate all penumbral connections [5, pp. 278-279]. For more on truth-functionality see [6, pp. 96-100].

[3] See [5, pp. 283-284] and [6, pp. 175-176]. Fine and Keefe identify supervaluationist consequence with what it is more precisely characterize as *global validity* below. The coincidence between supervaluationist and classical consequence is restricted to single conclusions; for example, the truth of a disjunction in a model classically guarantees the truth of some of its disjuncts but according to supervaluationist semantics a disjunction can be supertrue without either disjunct being supertrue. That is, $\{\varphi \vee \psi\} \vDash_{CL} \{\varphi, \psi\}$ but $\{\varphi \vee \psi\} \nvDash_{SpV} \{\varphi, \psi\}$.

baby at n but Julia is not a baby at n plus one second', they are not committed to the truth of any particular instance of that claim. The existential generalization is supertrue since every precisification of the language verifies it; but the n that makes the existential generalization true varies from one precisification to another so that there is no particular instance that is supertrue. (The case is analogous to the truth of the disjunction 'Martin is a baby or Martin is not a baby': the disjunction is verified in every precisification even if neither disjunct is verified in every precisification). The supervaluationist claims that the absence of a verifying instance suffice to show that the theory is not committed to a *sharp transition* from the times in which Julia is a baby to the times in which Julia is not a baby and, thus, to avoid an epistemicist explanation of vagueness while retaining classical logic.

The possibility of endorsing classical logic while avoiding an epistemicist account of vagueness is an appealing feature of the supervaluationist theory. In Fine's words, supervaluationism 'makes a difference to truth, but not to logic' [5, p. 284]. However, it is natural for a theory of vagueness to provide an explanation of the notion of *definiteness* and include a corresponding expression in the language in order to talk about borderline cases. Now supervaluationist logic is no longer classical when we introduce such an expression, and this fact is stressed by foes of supervaluationism to argue that the supposed advantage of supervaluationism over its truth-functional rivals is just an illusion.

1.2 Supervaluationism and Logical Consequence

Supervaluationist semantics for a propositional language containing a *definitely* operator ('$\mathcal{D}$' henceforth) might be modeled along the lines of a possible-worlds semantics for a propositional language with an operator for *necessity*. *Worlds* in a structure are informally read as *admissible precisifications* (admissible ways of making all the expressions of the language precise) and the accessibility between worlds is read as an *admissibility* relation between precisifications.[4] More explicitly, an interpretation for a propositional language with $\mathcal{D}$ is a triple $\langle W, R, \nu \rangle$ where W is a non-empty set of *precisifications*, R is an *admissibility* relation in W and ν is a truth-value assignment to sentences at precisifications. Classical operators have their standard meaning (relative to precisifications) and '$\mathcal{D}$' is defined as the modal operator for necessity:

$\varphi \to \psi$ takes value 1 at w if and only if at w: either φ takes value 0 or ψ takes value 1.

$\neg\varphi$ takes value 1 at w if and only if φ takes value 0 at w.

$\mathcal{D}\varphi$ takes value 1 at w if and only if φ takes value 1 at every precisification admitted by w.[5]

[4] This possible-worlds treatment of supervaluationist semantics is used, for example, in [10] and [11].

[5] When comparing local and global validity I shall talk about *points* instead of *precisifications* to remain neutral on the informal reading. In *Lemma 1* below we will write $\nu_w(\varphi) = 1$ to mean ν assigns value 1 to φ at w.

The question of which system gives the logic of *definiteness* for the supervaluationist reading of this notion depends on the informal reading of the semantics and on questions concerning higher-order vagueness. However, it is uncontroversial for any reading of $\mathcal{D}$ that the principle '$\mathcal{D}\varphi \rightarrow \varphi$' (if something is definite, then it is the case) should be valid and, consequently, that the *admissibility* relation should at least be reflexive.

So far supervaluationist and modal semantics coincide. The difference comes when we look at logical consequence. Logical consequence in modal semantics is standardly defined as *local consequence* [1, p. 31]:

Definition 1 (Local consequence). A sentence φ is a local consequence of a set of sentences Γ, written $\Gamma \vDash_l \varphi$, just in case for every interpretation and any point w in that interpretation: if every $\gamma \in \Gamma$ takes value 1 in w then φ takes value 1 in w.

In some sense, the notion of *local consequence* is a natural way of defining logical consequence in modal semantics. However, local consequence is not well defined in the supervaluationist reading of the semantics since for the supervaluationist that a sentence is true means that it is true *in every precisification* (that is, 'truth is supertruth'); and, thus, local consequence does not preserve the supervaluationist-relevant notion of truth. It is usually accepted in the literature that the supervaluationist is committed to something known as *global consequence*:[6]

Definition 2 (Global consequence). A sentence φ is a global consequence of a set of sentences Γ, written $\Gamma \vDash_g \varphi$, just in case for every interpretation: if every $\gamma \in \Gamma$ take value 1 at every point then φ takes value 1 at every point.

Global consequence preserves the notion of *truth-at-every-point* which is like the counterpart in this semantics of the notion of *supertruth*. In terms of modal semantics, we might see why supervaluationist consequence coincides with classical consequence for the classical language (this is Fine's and Keefe's previously mentioned result). If there are no modal expressions (operators whose truth-conditions depend on what's going on at points different of the evaluation-point) local validity will coincide with classical validity (since the truth conditions of classical expressions depend just on what's going on at the evaluation point which is a classical model). In turn, a language without this kind of operators will not be able to discriminate between global and local consequence. However, in the context of a theory of vagueness in which borderline cases play a key role (as it is the case of supervaluationism) it is natural to consider a '$\mathcal{D}$' operator; and in its presence, global and local validity no longer coincide. In particular, global validity is strictly stronger.

Every locally valid argument is globally valid. For if $\Gamma \nvDash_g \varphi$, then there is an interpretation such that every γ in Γ takes value 1 at every point and φ value 0

[6] I hold that global consequence is not fully adequate for the supervaluationist given the problem of higher-order vagueness (see [2] and [3]). In this paper, however, we will focus just on global validity.

at some. Now the point at which φ takes value 0 shows that $\Gamma \nvDash_l \varphi$. The other direction is not true. In particular the inference from φ to $\mathcal{D}\varphi$ is globally valid (if φ takes value 1 everywhere, so does $\mathcal{D}\varphi$) but not locally valid (φ and $\neg\mathcal{D}\varphi$ might both take value 1 at the same point in an interpretation).

1.3 Counterexamples to Classically Valid Patterns of Inference

The characteristic inference of global validity, the inference from φ to $\mathcal{D}\varphi$, might be used to show that global validity leads to some counterexamples to classically valid patterns of inference as, for example, *conditional proof*:

Definition 3 (Conditional proof). $\Gamma \cup \{\psi\} \vdash \varphi \Longrightarrow \Gamma \vdash \psi \rightarrow \varphi$.

for $\varphi \vDash_g \mathcal{D}\varphi$, but it is not the case $\vDash_g \varphi \rightarrow \mathcal{D}\varphi$ (since the last would render the modality trivial, assuming reflexivity). In a similar manner, always making use of the inference from φ to $\mathcal{D}\varphi$, we might find counterexamples to other classically valid patterns of inference such as contraposition, argument by cases and *reductio ad absurdum* [10, pp. 151-152].

The next section reviews some discussion concerning these counterexamples to classically valid forms of reasoning. But before we proceed, there is a small remark concerning 'classical logic'. The '$\mathcal{D}$' operator is not a classical notion; in this sense any logic for a language containing the operator is not, strictly speaking, *classical logic.* However, it is assumed in the literature that the most standard logic of definiteness corresponds to some of the various normal modal systems, since standard rules like the ones mentioned above (conditional proof, contraposition etc.) are correct for this sort of systems. That's why in the following we will assume that, in the present context, 'classical logic' means local validity.

2 Problems with Global Validity

2.1 The Keefe-Varzi Debate

In her 2000 book on vagueness, Rosanna Keefe considers the issue of counterexamples to classically valid patterns of inference [6, pp. 178-181]. Keefe argues that the failure of those rules is a natural outcome of any non-epistemic reading of '$\mathcal{D}$' and suggests an alternative set of rules that are always *global-truth* preserving.[7] For example, instead of the standard rule of conditional proof, Keefe suggests the use of the following rule:

Definition 4 (Conditional proof*). $\Gamma \cup \{\psi\} \vdash \varphi \Longrightarrow \Gamma \vdash \mathcal{D}\psi \rightarrow \varphi$.

In an analogous way Keefe proposes other rules to deal with the other counterexamples [6, pp. 179-180].

[7] I've got some doubts, however, concerning the soundness of the proposed rule to substitute conditional proof, since it seems we might actually derive $\vdash \mathcal{D}\mathcal{D}\varphi \rightarrow \mathcal{D}\varphi$ which is not always globally true when R is not required to be transitive.

In a recent paper Achille Varzi discusses this suggestion of Keefe. In the first place, Varzi notes that the suggestion, as it has been presented, cannot be acceptable. The problem is that if we replace the old rules by Keefe's new rules the resulting system is doom to be incomplete. For example, the following consequence assertions are correct but not provable making use only of Keefe's rules:

(a) $\vDash_g p \rightarrow p$
(b) $p \vDash_g \neg\neg p$
(c) $p \vee q \vDash_g q \vee p$
(d) $p \rightarrow q \vDash_g (p \wedge r) \rightarrow q$ [9, p. 657].

Keefe's suggestion, however, can be understood in a broader sense. Keefe notes that the classical rules are perfectly sound when the $\mathcal{D}$ operator is not at play; her suggestion is, thus, that we should make use of both kind of rules depending on the presence or absence of the $\mathcal{D}$ operator in the premises: 'so when the $\mathcal{D}$ operator is involved, supervaluationism needs to modify some classical rules of inference, but the new rules are reasonable, and when no $\mathcal{D}$ operator is involved normal classical rules of inference remain intact.' [6, p. 180]. But Varzi does not find this strategy very convincing:

> I am not sure this would work, but even if it did, things would again begin to look ugly and one might as well think that the right thing to do is to bite the bullet and give up [global validity] altogether. [9, p. 657].

I understand that Varzi's objection to Keefe's strategy point out with certain pessimism to the difficulty of providing an adequate system of deduction for global validity based on classical rules in a simple and straightforward way. To some extent, this pessimism on Keefe's suggestion is reasonable since the suggestion is too general to provide any intuition on whether it really works. In order to avoid Varzi's pessimism, Keefe should provide precise constraints on the applicability of old rules; explaining when can we make use of the new ones and showing that the resulting system is adequate (correct and complete). We will consider this question later. Now we turn to a different objection based directly on the non-classicality of supervaluationist logic in the presence of $\mathcal{D}$.

2.2 Williamson's Objection

The cleanest exposition of the counterexamples to classically valid rules of inference is [10, pp. 150-152]. Based on this fact, Williamson argues against the supervaluationist theory. According to Williamson, patterns of inference such as conditional proof, contraposition, argument by cases and *reductio* play a central role in formal systems of deduction that are closer to our informal way of reasoning. Given that these rules of inference are not always correct under the global reading of logical consequence, Williamson draws the conclusion that 'supervaluations invalidates our natural mode of deductive thinking.' [10, p. 152].

It seems to me that this last claim is not completely fair. I concede to Williamson that the mentioned rules play a key role in those formal systems closer to our informal way of reasoning.[8] And, of course, there is a sense in which global validity invalidates these forms of deduction (since there are counterexamples to the corresponding patterns of inference). But there is still a sense in which the claim is unfair since we have not considered yet particular systems of deduction for global validity. My point is that, perhaps, these systems (or at least some of them) are relatively simple extensions of classical systems in which the applicability of the *controversial rules* have clearly defined restrictions and such that the form of deductions is, to certain extent, standard.

2.3 Two Questions

The foregoing discussion raise two related questions, one of technical character, the other more philosophical. The first question concerns the possibility of providing adequate systems of deduction for global validity. The second question concerns the aspect of these systems, whether we might include rules of inference such as conditional proof and whether the form of the corresponding deductions in those systems is relatively standard.

The following section aims to provide an answer to both questions. With respect to Varzi's pessimism, the section shows that there is a simple way to extend a deductive system for local validity to a deductive system for global validity. With respect to Williamson's claim that supervaluationism invalidates our natural modes of reasoning, it is shown a way to restrict the applicability of the relevant rules that provide to the deduction in these systems an *almost* classical form.

3 Deduction for Global Validity

This section presents a procedure to extend a given notion of deduction $\vdash_l$ for local consequence to a notion of deduction $\vdash_g$ for global consequence. We provide an argument to show that if $\vdash_l$ is complete with respect to local validity, $\vdash_g$ is complete with respect to global validity (section 3.1). Whether $\vdash_g$ is also correct with respect to global validity (that is, wether $\Gamma \vdash_g \varphi$ entails $\Gamma \vDash_g \varphi$) will depend on the original system defining $\vdash_l$. For the reasons given before, we are

[8] A qualification: Williamson's claim is not uncontroversial. For a start, conditional proof is not unrestrictedly valid in some presentations of first-order logic (in these accounts, $Px \vdash \forall x Px$ but $\nvdash Px \rightarrow \forall x Px$). But more generally, one might well doubt whether there is really anything like 'our natural mode' when we talk about deductive thinking. However, I concede to Williamson that claim in the text for the following reason. Classicality is one of the supposed advantages of supervaluationism over truth-functional approaches but the failure of those rules of inference in the presence of $\mathcal{D}$ calls this point into question. Thus, even if Williamson's claim is not uncontroversial, supervaluationists need to address the objection of non-classicality in the presence of $\mathcal{D}$ anyway.

interested in systems of deduction that make use of rules like conditional proof, contraposition, argument by cases and *reductio*. Section 3.2 shows a straightforward way to restrict the applicability of these rules to render $\vdash_g$ correct with respect to global validity (and that do not destroy the completeness argument in section 3.1!). Section 3.3 evaluates in which way these results shed some light on the discussion in section 2.

3.1 Completeness

The completeness argument below will make use of the following connection between local and global validity:

Lemma 1 (Global-local connection). $\Gamma \vDash_g \varphi$ iff $\{\mathcal{D}^n\gamma \mid \gamma \in \Gamma, n \in \omega\} \vDash_l \varphi$.

The intuitive idea is that φ *globally* follows from Γ just in case φ *locally* follows from the *absolute definitization* of Γ, that is, the set containing: all the γ's plus all the $\mathcal{D}\gamma$'s plus all the $\mathcal{D}\mathcal{D}\gamma$'s etc.

Proof. (i) Right-to-left
Assume: $\Gamma \nvDash_g \varphi$. Then, there is an interpretation $\Im = \langle W, R, \nu\rangle$ where for all w and all $\gamma \in \Gamma$, γ takes value 1 at w and for some w, φ takes value 0 at w. Name w_0 the precisification at which φ is takes value 0. Since every γ in Γ is takes value 1 everywhere in the interpretation, every γ takes value 1 at w_0 for each iteration of $\mathcal{D}$. Thus, precisifcation w_0 shows that $\{\mathcal{D}^n\gamma \mid \gamma \in \Gamma, n \in \omega\} \nvDash_l \varphi$.

(ii) Left-to-right[9]
Assume: $\{\mathcal{D}^n\gamma \mid \gamma \in \Gamma, n \in \omega\} \nvDash_l \varphi$. Then there is an interpretation $\Im = \langle W, R, \nu\rangle$ and a precisification w_0 in it such that, for every γ in Γ and any iteration of $\mathcal{D}$, $\mathcal{D}^n\gamma$ takes value 1 at w_0 and φ takes value 0 at w_0. Let W' be $\{w|w_0R^nw\} \cup \{w_0\}$ (that is, w_0 plus the precisifications reachable from w_0 in any number of R-steps) and R', ν' the restrictions of R, ν to W'. We should demonstrate that the modified interpretation is a countermodel showing $\Gamma \nvDash_g \varphi$. We show first i) that both interpretations agree in the truth-values assigned to any formula in any w' in W'.

To show i), note first that if $w' \in W'$ then R' and R relate w' exactly to the same worlds, that is, if $w' \in W'$ then $w'R'w$ iff $w'Rw$. For if $w'R'w$ then both $w \in W'$ and $w'Rw$. On the other hand, if $w'Rw$, as $w' \in W'$, w_0R^mw' and thus $w_0R^{m+1}w$, that is, $w \in W'$. Thus, $w'R'w$.

i) is proved by induction over the set of wff. The case for propositional variables holds by definition. The case for non-modal operators is straightforward. For $\psi = \mathcal{D}\alpha$, suppose that $w' \in W'$:

$\nu'_{w'}(\mathcal{D}\alpha) = 1$ iff $\forall w{*}_{\in W'}$ such that $w'R'w{*}$, $\nu'_{w*}(\alpha) = 1$
iff $\forall w{*}_{\in W'}$ such that $w'R'w{*}$, $\nu_{w*}(\alpha) = 1$ (by IH)
iff $\forall w{*}_{\in W}$ such that $w'Rw{*}$, $\nu_{w*}(\alpha) = 1$ (by the fact noted above)

[9] The result is based on the fact that $\langle W', R', \nu'\rangle$ is a generated submodel of $\langle W, R, \nu\rangle$ [1, p. 56].

To show that the modified interpretation is a countermodel showing $\Gamma \nvDash_g \varphi$ note that w_0 has access to every world in W' (excluding, perhaps, w_0 itself) thorough some number of R-steps. Since for every γ in Γ and every $n \in \omega$, $\nu_{w_0}(\mathcal{D}^n\gamma) = 1$, every member of Γ takes value 1 at every world in W'. On the other hand, as $\nu_{w_0}(\varphi) = 0$, there is at least one world in W' in which φ takes the value 0. Thus, the modified interpretation shows that $\Gamma \nvDash_g \varphi$.

Since local consequence is standard we might assume that there are adequate systems of deduction for it. Let $\vdash_l$ be an adequate deductive relation for local consequence. Among other rules the following are locally valid (sometimes called *structural rules*):[10]

Definition 5 (Reflexivity). $\varphi \in \Gamma \Longrightarrow \Gamma \vdash \varphi$.

Definition 6 (Cut). $\Gamma \vdash \varphi,\ \Delta \vdash \gamma_1, \ldots \Delta \vdash \gamma_n \Longrightarrow \Delta \vdash \varphi$.

We consider an extra rule that is **not** locally valid:

Definition 7 ($\mathcal{D}$-introduction). $\Gamma \vdash \varphi \Longrightarrow \Gamma \vdash \mathcal{D}\varphi$[11].

The addition of this rule to $\vdash_l$ leads to a new notion of deductive consequence, $\vdash_g$. With the help of *Lemma 1* we can now show that $\vdash_g$ is complete with respect to global consequence:

Theorem 1 ($\vdash_g$-completeness). If $\Gamma \vDash_g \varphi$ then $\Gamma \vdash_g \varphi$.

Proof. (i) Assume: $\Gamma \nvdash_g \varphi$.

$$\Downarrow$$

(ii) $\{\mathcal{D}^n\gamma \mid \gamma \in \Gamma, n \in \omega\} \nvdash_g \varphi$.

$$\Downarrow$$

(iii) $\{\mathcal{D}^n\gamma \mid \gamma \in \Gamma, n \in \omega\} \nvdash_l \varphi$.

$$\Downarrow$$

(iv) $\{\mathcal{D}^n\gamma \mid \gamma \in \Gamma, n \in \omega\} \nvDash_l \varphi$

$$\Downarrow$$

(v) $\Gamma \nvDash_g \varphi$.

The step from (i) to (ii) is guaranteed by the rules of $\mathcal{D}$-introduction, Reflexivity and Cut. For assume that $\{\mathcal{D}^n\gamma \mid \gamma \in \Gamma, n \in \omega\} \vdash_g \varphi$; then, since formal proofs are finite, there's a finite $\Gamma^* \subseteq \{\mathcal{D}^n\gamma \mid \gamma \in \Gamma, n \in \omega\}$ such that $\Gamma^* \vdash_g \varphi$. Now,

[10] A third structural rule not used in the proof is *Monotonicity*: $\Gamma \vdash \varphi \Longrightarrow \Delta \vdash \varphi$ for all Δ such that $\Gamma \subseteq \Delta$.

[11] Given the Reflexivity rule, the inference from φ to $\mathcal{D}\varphi$ is a special case of this rule. This rule must not be confused with the *Necessitation* rule of standard modal logics that can be stated this way: $\Gamma \vdash \varphi \Longrightarrow \mathcal{D}(\Gamma) \vdash \mathcal{D}\varphi$, where $\mathcal{D}(\Gamma)$ is $\{\mathcal{D}\gamma \mid \gamma \in \Gamma\}$.

each $\gamma^* \in \Gamma^*$ is either an element of Γ or an element of Γ with a finite number of $\mathcal{D}$'s attached to it. Thus, making use of Reflexivity and $\mathcal{D}$-introduction, $\Gamma \vdash_g \gamma^*$ for any $\gamma^* \in \Gamma^*$ and thus, making use of Cut, $\Gamma \vdash_g \varphi$.

The step from (ii) to (iii) is guaranteed by the way we have defined $\vdash_g$: since this notion extends $\vdash_l$ (every local proof is a global proof), if we cannot provide a global proof, we cannot provide a local proof either. The step from (iii) to (iv) is based on the assumption that $\vdash_l$ is complete with respect to local consequence. The step from (iv) to (v) is based on the left-to-right direction of *Lemma 1*.

In order to prove the theorem we've had just to add the rule of $\mathcal{D}$-introduction. The intuitive explanation is (if any) as follows. *Lemma 1* shows that global consequence adds to local consequence the supposition that the premises are *absolutely definite*. For this reason we need to strengthen a system for local consequence with a rule reflecting the supposition that the premises are absolutely definite; and this is precisely what the $\mathcal{D}$-introduction rule does.

3.2 Correctness

The previous subsection shows that in order to obtain a complete notion of deduction for global validity all we need to do is to add the rule of $\mathcal{D}$-introduction to a complete system for local validity. But at this point we must be careful since a system obtained by the addition of $\mathcal{D}$-introduction might turn to be *too complete* as it is shown in the counterexamples to classically valid patterns of inference in section 1.3. If the system for local validity that we take to define the system for global validity contains rules that are not always globally valid (such as conditional proof), the addition of $\mathcal{D}$-introduction will render a system complete but not correct (we will be able to prove, for example, $\vdash_g \varphi \rightarrow \mathcal{D}\varphi$ which is not valid by supervaluationist standards).

At this point there are two possible alternatives. The first one would be focussing on rather succinct axiomatic systems in which the rules of deduction are always globally valid. Though this alternative might be logically satisfactory, it is not satisfactory from a more philosophical point of view. In particular, in order to address Williamson's objection, we should show how to incorporate deductive systems with rules like conditional proof etc. In order to incorporate such systems we should put some restriction on the applicability of *problematic* rules. Now, it should be noted that restrictions on the applicability of rules is a common place in formal logic (think, for example, on the rules of $\forall$-*introduction* and $\exists$-*elimination* in standard formulations for first-order logic) and so, it seems to me, that the fact that we should make use of restrictions does not constitute an objection *per se*.

The particular way in which we might formulate these restrictions would depend, partly, on the particular form of the deductive system in question. My proposal is to restrict the applicability of *problematic* rules to proofs that do not make use of the rule of $\mathcal{D}$-introduction. For example, if the proof showing that $\Gamma \cup \{\psi\} \vdash_g \varphi$ involves *any* application of $\mathcal{D}$-introduction, we are not allowed to use conditional proof to get $\Gamma \vdash_g \psi \rightarrow \varphi$. The restriction formulated this way might look a bit drastic and it is perhaps possible to formulate restrictions in a more sensitive way, but the point is that this restriction guarantees the correctness of the system without destroying our previous completeness argument. When we consider restrictions on the applicability of rules of $\vdash_l$, the sensitive cases in the argument above are: the step from (ii) to (iii) and the step from (iii) to (iv). The step from (ii) to (iii) requires that every local proof is a global proof, but the previous restriction respects this fact since local proofs do not make use of $\mathcal{D}$-introduction (any local proof meets the restriction). The step from (iii) to (iv) is justified by analogous reasons: since local proofs do not make use of $\mathcal{D}$-introduction, the restriction on the applicability of rules do not restrict the number of local proofs ($\vdash_l$ is still complete after the restriction). This abstract consideration on the restriction of applicability of *problematic* rules might look a bit mysterious, so let us look to a particular example.

The inference from φ to $\psi \rightarrow \mathcal{D}\varphi$ is globally, but not locally valid. One might think that an appropriate way to provide a global proof would be something like this,

1 $\{\psi, \varphi\} \vdash_g \varphi$	[Reflexivity]
2 $\{\psi, \varphi\} \vdash_g \mathcal{D}\varphi$	[From 1, by $\mathcal{D}$-introduction]
3 $\{\varphi\} \vdash_g \psi \rightarrow \mathcal{D}\varphi$	[From 2, by conditional proof]

however, our restriction on the applicability of rules like conditional proof would render the step from 2 to 3 illegitimate. Now, if our previous remark on the restriction is correct (that is, if the restriction does not destroy the previous completeness argument), there must be a way to write the proof that respects the restriction. The *natural* way to do it (perhaps the only one) is this:

1 $\{\psi, \mathcal{D}\varphi\} \vdash_g \mathcal{D}\varphi$	[Reflexivity]
2 $\{\mathcal{D}\varphi\} \vdash_g \psi \rightarrow \mathcal{D}\varphi$	[From 1, by conditional proof]
3 $\{\varphi\} \vdash_g \mathcal{D}\varphi$	[Reflexivity and $\mathcal{D}$-introduction]
4 $\{\varphi\} \vdash_g \psi \rightarrow \mathcal{D}\varphi$	[From 2 and 3, by Cut]

Note that the restriction gives to any proof the same pattern as the one followed in the previous completeness argument (in particular step from (i) to (ii)). The general strategy to construct a global proof respecting the restriction is this: (i) assume the premises are as definite as you need for the proof and proceed classically (that is, here you might make use of any local rule, this corresponds to steps 1 and 2 in the example); (ii) *reduce* the $\mathcal{D}$'s attached to the premises making use of the rules of $\mathcal{D}$-introduction, Reflexivity and Cut (this corresponds to steps 3 and 4 in the example).

3.3 The Two Questions Revisited

The discussion in section 2 raised two questions concerning global validity. The first, more technical, whether we might provide adequate deductive systems for global validity. The second, concerning the form of these systems, whether it is possible to incorporate rules like conditional proof, contraposition, argument by cases and *reductio* and what is the aspect of formal proofs within these systems. In this third section we have found an answer to these two questions.

In the first place, we have provided a simple procedure to extend any adequate notion of deduction for local validity to a complete notion of deduction for global validity. In the second place, the section provides a positive answer to the second question by showing a way in which we might incorporate the aforementioned rules placing a suitable restriction on their applicability. It is worth noting that the aspect of global proofs respecting this restriction is completely classical, with the exception of the last step.

These answers to the two questions raised in subsection 2.3 contribute to the debate on the non-classicality of supervaluationist logic presented in subsections 2.1 and 2.2. Varzi's pessimism is overcome by providing a simple way to adapt classical systems of deduction for supervaluationist logic. *Problematic* rules are perfectly applicable with the exception of proofs in which the rule of $\mathcal{D}$-introduction has already been used. This last remark provides a precise sense to Keefe's quoting above according to which classical rules can be applied when the $\mathcal{D}$-operator is not involved. Thus, I think that section 3 shows a precise sense in which Keefe's original suggestion works perfectly fine. On the other hand, the result qualifies Williamson's claim according to which supervaluationism invalidates our natural form of deductive thinking. While there's a sense in which Williamson's claim is correct (since supervaluationist logic for a language with $\mathcal{D}$ gives raise to counterexamples to classically valid rules), there is another sense in which the claim must be qualified. Since we might employ systems of deduction correct and complete for global validity in which the problematic rules are present (with restrictions) and such that formal proofs in these systems are completely classical; with the exception of a single last step.

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