Chapter 13
Dirichlet Energy Integral and Laplace Equation

For a constant $q \in (1, \infty)$, the Dirichlet energy integral is

$$\int_\Omega |\nabla u(x)|^q \, dx.$$

The problem is to find a minimizer for the energy integral among all Sobolev functions with a given boundary value function. The Euler–Lagrange equation of this problem is the q-Laplace equation,

$$\operatorname{div}(|\nabla u|^{q-2} \nabla u) = 0,$$

which has to be understand in the weak sense. The energy integral and q-Laplace equation have been widely studied, see for example [219, 235, 280]. The q-Laplace equation is a prototype of a non-linear elliptic equation. By non-linearity we mean that if $q \neq 2$ then the weak solutions do not form a linear space. However the set of weak solutions is closed under constant multiplication. By celebrated De Giorgi's method and Moser's iteration the minimizers and the weak solutions are locally Hölder continuous and satisfy Harnack's inequality:

$$\sup_B u \leqslant c \inf_B u,$$

where c is independent of u and the ball B.

The Dirichlet energy integral and the Laplace equation can be generalize to the variable exponent case as

$$\int_\Omega |\nabla u(x)|^{p(x)} \, dx \quad \text{and} \quad \operatorname{div}(p(x)|\nabla u(x)|^{p(x)-2} \nabla u) = 0.$$

It turns out that the minimizer exists for a given boundary value function under mild conditions of p. The minimizers and the weak solutions are locally Hölder continuous when p is log-Hölder continuous with $1 < p^- \leqslant p^+ < \infty$. Harnack's inequality does not hold by the mentioned form: in the variable

L. Diening et al., *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics 2017, DOI 10.1007/978-3-642-18363-8_13,

exponent case the constant can not be independent of the function u. The minimizers or the weak solutions are not scalable, i.e. λu need not be a minimizer or a weak solution even if u is. These effects are visible already in the one dimensional case where the minimizers need not to be linear as in the constant exponent case.

In the first section, Sect. 13.1, we study minimizers on an interval with detailed proofs. In Sects. 13.2 and 13.3 we give a rough overview of properties of minimizers and solutions of the prototype equality. In the last section, Sect. 13.4, we generalize, with detailed proofs, Harnack's inequality to all elliptic type Laplace equations with growth conditions of a non-standard form.

The material is selected by the personal taste of the writers; it concentrates to the variable exponent Laplace equation from the potential theoretical viewpoint and all results concerning Harnack's inequality are included. In particular this chapter does not include solutions to the obstacle problems, e.g., [125, 126, 205, 331], systems, e.g., [8, 13, 49, 106, 180, 185, 272, 318, 374, 382, 383, 388], eigenvalues, e.g., [21, 88, 89, 112, 113, 130, 133, 135, 136, 145, 259, 290, 290, 291, 381], parabolic equations, e.g., [9, 36–39, 71, 257, 324, 379, 380]; regularity of solutions, e.g., [3, 5, 10, 19, 31, 72, 124, 131, 146, 147, 150, 156, 182, 276, 370, 395, 396, 398].

Existence and uniqueness of solutions has been studied in a large number of papers. For instance:

- [139, 143, 368, 385, 389, 390] deal with the one-dimensional case.
- [32, 35, 61, 80, 132, 170, 270, 282, 295, 322, 377, 384, 386, 387, 392] deal with existence of solutions to the $p(\cdot)$-Laplacian.
- [17, 30, 33, 33, 42, 55, 56, 64, 68, 78, 85–87, 110, 137, 138, 140, 144, 151, 160, 161, 170–173, 175, 220–222, 273, 274, 292–294, 320, 321, 333, 349, 354, 375–377] deal with existence related to more general equations.

A wider scope can be found from the recent surveys [201, 297].

13.1 The One Dimensional Case

Let us start by stating the Dirichlet energy integral problem on an interval. These results are mainly from [191]. We assume that the bounded interval under consideration is (r, R). Since every element in the space $W^{1,p(\cdot)}(r, R)$ has a continuous representative, we assume that every Sobolev function is continuous. We denote $u \in W_0^{1,p(\cdot)}(r, R)$ and say that u belongs to the *variable exponent Sobolev space with zero boundary values* if it can be continuously continued by 0 outside (r, R) (the extension is again denoted by u). Thus $u \in W_0^{1,p(\cdot)}(r, R)$ if and only if $u(r) = u(R) = 0$.

Definition 13.1.1. A function $u \in W^{1,p(\cdot)}(r, R)$ is a $p(\cdot)$-*minimizer* for the boundary values a and b if $u(r) = a$, $u(R) = b$ and

$$\int_r^R |u'|^{p(y)}\,dy \leqslant \int_r^R |v'|^{p(y)}\,dy$$

for every v with $u - v \in W_0^{1,p(\cdot)}(r,R)$.

If p is a constant, then the minimizer is linear, $u(x) = \frac{b-a}{R-r}(x-r) + a$. The next example shows that the variable exponent adds some interest to this minimization question.

Example 13.1.2. We define

$$p(x) := \begin{cases} 3, & \text{for } 0 < x \leqslant 1/2; \\ 2, & \text{for } 1/2 < x < 1. \end{cases}$$

Suppose that $u \in W^{1,p(\cdot)}(0,1)$ is the minimizer for the boundary values 0 and $b > 0$. Denote $u(1/2) = \lambda$.

Then $u|_{(0,1/2)}$ is the solution to the classical energy integral problem with boundary values 0 and λ, and $u|_{(1/2,1)}$ is the solution with boundary values λ and b. Therefore these functions are linear, and so

$$u(x) = \begin{cases} 2\lambda x, & \text{for } 0 < x \leqslant 1/2; \\ 2\lambda + 2(b-\lambda)(x - 1/2), & \text{for } 1/2 < x < 1. \end{cases}$$

For this u we have the Dirichlet energy $4\lambda^3 + 2(b-\lambda)^2$. It is easy to see that the function $\lambda \mapsto 2\lambda^3 + (b-\lambda)^2$ has a minimum at $\lambda = (\sqrt{1+12b} - 1)/6$, which determines the minimizer of the variable exponent problem. The minimizing functions for some b's are shown in Fig. 13.1.

As can be seen in the figure, and confirmed by calculation, the minimizer is convex if $b > 2/3$, concave if $b < 2/3$ and linear for $b = 2/3$.

It is in fact possible to give an explicit formula for the minimizer by solving the corresponding Euler–Lagrange equation, as shown in the next theorem. The formula is not quite transparent, however, so we prove some properties of the minimizers later on. We start with preliminary results.

Lemma 13.1.3. *Let $p \in \mathcal{P}(r,R)$ be bounded and strictly greater than one almost everywhere. If $u \in W^{1,p(\cdot)}(r,R)$ is a $p(\cdot)$-minimizer, then $p(x)$ $(u'(x))^{p(x)-1}$ is a constant almost everywhere.*

Proof. Suppose that $p(x)(u'(x))^{p(x)-1}$ is not a constant almost everywhere. Let then $d_1 < d_2$ be such that

$$A_1 := \big\{x \in (r,R)\colon p(x)|u'(x)|^{p(x)-1} < d_1\big\},$$
$$A_2 := \big\{x \in (r,R)\colon p(x)|u'(x)|^{p(x)-1} > d_2\big\}$$

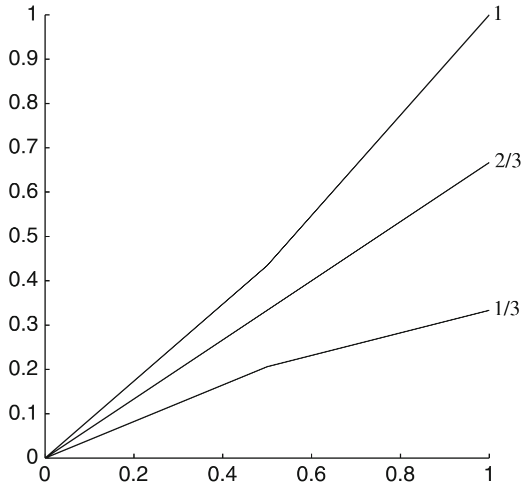

Fig. 13.1 Energy integral minimizers

have positive measure. Let $A_1' \subset A_1$ and $A_2' \subset A_2$ be such that $|A_1'| = |A_2'| > 0$. Define $\xi := \chi_{A_1'} - \chi_{A_2'}$.

Let $0 < \varepsilon < 1$. Using $\big||x+h|^p - |x|^p\big| \leqslant p\big||x+h| - |x|\big|\big(|x+h|^{p-1} + |x|^{p-1}\big)$ and $\varepsilon^{-1}\big||u' + \varepsilon\xi| - |u'|\big| \leqslant c$ we obtain

$$\begin{aligned}
&\left|\frac{|u'(x) + \varepsilon\xi(x)|^{p(x)} - |u'(x)|^{p(x)}}{\varepsilon}\right| \\
&\quad \leqslant \frac{p(x)\Big||u'(x) + \varepsilon\xi(x)| - |u'(x)|\Big|\Big(|u'(x) + \varepsilon\xi(x)|^{p(x)-1} + |u'(x)|^{p(x)-1}\Big)}{\varepsilon} \\
&\quad \leqslant c\big(|u'(x)|^{p(x)-1} + \varepsilon^{p(x)-1}\big) \leqslant c\big(|u'(x)|^{p(x)} + \varepsilon + 1\big).
\end{aligned}$$

Since p is bounded $|u'|^{p(\cdot)} \in L^1(r, R)$ and the dominated convergence theorem yields

$$\begin{aligned}
\lim_{\varepsilon \to 0} \int_r^R \frac{|u' + \varepsilon\xi|^{p(x)} - |u'|^{p(x)}}{\varepsilon}\, dx &= \int_r^R p(x)|u'|^{p(x)-1}\xi\, dx \\
&\leqslant (d_1 - d_2)|A_1'| < 0.
\end{aligned}$$

Note that ξ is the classical derivative of $v(x) := \int_r^x \chi_{A_1'} - \chi_{A_2'}\, dy$. Since v' is bounded and $v(r) = v(R) = 0$ we obtain that $v \in W_0^{1,p(\cdot)}(r, R)$. Thus the previous inequality contradicts that u is the minimizer and hence $p(x)(u'(x))^{p(x)-1}$ is a constant almost everywhere. $\square$

Lemma 13.1.4. *Let $p \in \mathcal{P}(r, R)$ be bounded and strictly greater than one almost everywhere. Let $u \in W^{1,p(\cdot)}(r, R)$. If $p(x)(u'(x))^{p(x)-1}$ is a constant almost everywhere then u is a $p(\cdot)$-minimizer for its own boundary values.*

Proof. Since $p(x)(u'(x))^{p(x)-1}$ is a constant almost everywhere we have for every v with $v - u \in W_0^{1,p(\cdot)}(r, R)$ that

$$\int_r^R p(x)(u')^{p(x)-1}(v' - u')\,dx = 0.$$

By the inequality $|b|^p \geqslant |a|^p + p|a|^{p-1}(b - a)$ we obtain

$$\int_r^R |v'|^{p(x)}\,dx \geqslant \int_r^R |u'|^{p(x)}\,dx + \int_r^R p(x)(u')^{p(x)-1}(v' - u')\,dx.$$

Since the last integral is zero we have

$$\int_r^R |u'|^{p(x)}\,dx \leqslant \int_r^R |v'|^{p(x)}\,dx.$$ □

The next theorem gives an explicit form for the minimizers. The original formulation in [191] included a mistake, so we reformulate the theorem here.

Theorem 13.1.5 ([191, Theorem 3.2]). *Let $p \in \mathcal{P}(r, R)$ be bounded and strictly greater than one almost everywhere and let $a, b \in \mathbb{R}$, $a < b$, be the boundary values at r and R. Then there exists a unique minimizer for these boundary values if and only if there exists $\tilde{m} \geqslant 0$ such that*

$$b - a \leqslant \int_r^R \left(\frac{\tilde{m}}{p(x)}\right)^{\frac{1}{p(x)-1}} dx < \infty. \tag{13.1.6}$$

In this case the minimizer is given by

$$u(x) := \int_r^x \left(\frac{m}{p(y)}\right)^{\frac{1}{p(y)-1}} dy + a,$$

for appropriate $m \in (0, \tilde{m}]$.

Note that for m we have

$$\int_r^R \left(\frac{m}{p(y)}\right)^{\frac{1}{p(y)-1}} dy = b - a.$$

Proof. Let $f_m(x) := \big(\frac{m}{p(x)}\big)^{\frac{1}{p(x)-1}}$. We first show that m can be chosen so that $\int_r^R f_m\,dx = b-a$. If $\tilde{m}$ is such that $\int_r^R f_{\tilde{m}}\,dx = b-a$, then $\tilde{m}=m$. Assume then that $\int_r^R f_{\tilde{m}}\,dx > b-a$. It is enough to show that $m \mapsto \int_r^R f_m\,dx$ is continuous on $[0,\tilde{m}]$. Fix $\varepsilon > 0$ and define

$$A_\lambda := \big\{x \in (r,R)\colon p(x) > \lambda\big\}.$$

We choose $\lambda > 1$ such that

$$\int\limits_{(r,R)\setminus A_\lambda} f_{\tilde{m}}\,dx < \varepsilon.$$

In A_λ the exponent $\frac{1}{p(x)-1}$ is bounded from above and so we can choose a small real number d, $m+|d| < \tilde{m}$, such that $\int_{A_\lambda} |f_m - f_{m+d}|\,dx < \varepsilon$. Since $m \mapsto \int_r^R f_m\,dx$ is increasing, we find that

$$\bigg|\int\limits_r^R f_m - f_{m+d}\,dx\bigg| \leqslant \int\limits_{A_\lambda} |f_m - f_{m+d}|\,dx + \int\limits_{(r,R)\setminus A_\lambda} (f_m + f_{m+d})\,dx \leqslant 3\varepsilon.$$

Hence $m \mapsto \int_r^R f_m\,dx$ is continuous.

Let f_m be such that $\int_r^R f_m\,dx = b-a$. If $u \in W^{1,p(\cdot)}(r,R)$ is such that $f_m = u'$ and $u(r) = a$, then by Lemma 13.1.4 the function u is the minimizer we are looking for. Define therefore $u(x) := \int_r^x f_m(y)\,dy + a$ for $x \in (r,R]$. Since $a \leqslant u \leqslant b$, $u \in L^{p(\cdot)}(r,R)$. Further,

$$\overline{\varrho}_{p(\cdot)}(u') = \int\limits_r^R \Big(\frac{m}{p(x)}\Big)^{\frac{p(x)}{p(x)-1}}\,dx \leqslant \tilde{m}\int\limits_r^R \Big(\frac{\tilde{m}}{p(x)}\Big)^{\frac{1}{p(x)-1}}\,dx < \infty.$$

Therefore $u \in W^{1,p(\cdot)}(r,R)$ is a minimizer.

To prove the other direction, let u be a minimizer. Then by Lemma 13.1.3 $f_m = u'$ with $\int_r^R f_m\,dy = b-a$. So then (13.1.6) holds. Therefore the condition is both necessary and sufficient.

Finally we show that the minimizer is unique. Assume that we have two minimizers u and v with same boundary values. Since $y \mapsto y^{p(x)}$ is strictly convex when $p(x) > 1$,

$$\Big|\frac{1}{2}u' + \frac{1}{2}v'\Big|^{p(x)} < \frac{1}{2}|u'|^{p(x)} + \frac{1}{2}|v'|^{p(x)}$$

for almost every $x \in I$ with $u'(x) \neq v'(x)$. The function $\frac{1}{2}u + \frac{1}{2}v$ has the same boundary values than u and v. If the set $\{u'(x) \neq v'(x)\}$ has positive measure then

$$\int_r^R \left|\frac{1}{2}u' + \frac{1}{2}v'\right|^{p(x)} dx < \frac{1}{2}\int_r^R |u'|^{p(x)}\, dx + \frac{1}{2}\int_r^R |v'|^{p(x)}\, dx = \int_r^R |u'|^{p(x)}\, dx.$$

Since u is a minimizer, we have $u' = v'$ almost everywhere. We obtain $u = v$. □

The gradient of the minimizer is uniformly bounded if $1 < p^- \leqslant p^+ < \infty$. Thus we obtain the following corollary.

Corollary 13.1.7. *If $p \in \mathcal{P}(r,R)$ with $1 < p^- \leqslant p^+ < \infty$ then for every $a, b \in \mathbb{R}$, $a < b$, there exists a unique minimizer with these boundary values. The minimizer is given by*

$$u(x) := \int_r^x \left(\frac{m}{p(y)}\right)^{\frac{1}{p(y)-1}} dy + a,$$

for some constant $m > 0$.

The following example shows that the Dirichlet energy integral does not always have a minimizer.

Example 13.1.8. For $p(x) := 1 + x$ in $(0,1)$ the minimizer does not exists when the difference between the boundary values is large enough. Fix $a = 0$ and let $m > 1$. Then

$$\int_0^1 m^{\frac{1}{p(x)-1}}\, dx = \int_0^1 m^{\frac{1}{x}}\, dx \geqslant \max\{1, \log m\} \int_0^1 \frac{dx}{x} = \infty.$$

Since $p^{1/(1-p)}$ lies between $1/e$ and 1, the condition of Theorem 13.1.5 is not satisfied for $b > \int_0^1 p(x)^{\frac{1}{1-p(x)}}\, dx$.

Example 13.1.9. Using Theorem 13.1.5 we plot some minimizers of the energy integral for $p(x) := 1.1 + x$ in $(0,1)$ (Fig. 13.2). The left boundary value is 0 and the number on the right is again the second boundary value, b. It is easy to see that if we multiply a minimizer by a real number the result need not to be a minimizer.

Next we study regularity of minimizers.

Corollary 13.1.10. *If $p \in \mathcal{P}(r,R)$ with $1 < p^- \leqslant p^+ < \infty$, then the minimizers are bi-Lipschitz continuous.*

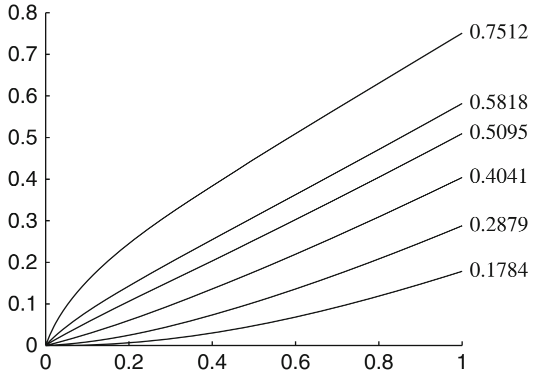

Fig. 13.2 Minimizers when $p(x) = 1.1 + x$

Proof. By Theorem 13.1.5, the minimizer has derivative $(m/p(x))^{1/(p(x)-1)}$ for some constant $m \geqslant 0$. Since

$$\left(\frac{m}{p(x)}\right)^{\frac{1}{p(x)-1}} \leqslant m^{\frac{1}{p(x)-1}} \leqslant \max\{m^{\frac{1}{p^--1}}, m^{\frac{1}{p^+-1}}\} < \infty$$

and

$$\left(\frac{m}{p(x)}\right)^{\frac{1}{p(x)-1}} \geqslant \left(\frac{m}{p^+}\right)^{\frac{1}{p(x)-1}} \geqslant \min\left\{\left(\frac{m}{p^+}\right)^{\frac{1}{p^--1}}, \left(\frac{m}{p^+}\right)^{\frac{1}{p^+-1}}\right\} > 0$$

for all $x \in (r, R)$, it follows from the mean-value theorem that the minimizer is bi-Lipschitz continuous. □

Corollary 13.1.11. *If $p \in \mathcal{P}(r, R)$ with $1 < p^- \leqslant p^+ < \infty$, then the derivative of the minimizer is α-Hölder continuous if and only if the exponent p is α-Hölder continuous.*

Proof. Let us denote $F(y) := (m/y)^{1/(y-1)}$. Then the derivative of the minimizer equals $F(p(x))$. Since F is differentiable on $(1, \infty)$ we obtain

$$|F(p(x)) - F(p(y))| = F'(\xi)|p(x) - p(y)|,$$

where $\xi \in (p(x), p(y))$, by the mean-value theorem. It is easy to see that F' is bounded and bounded away from 0 on $[p^-, p^+]$, so that $F(p(x))$ possesses the same degree of regularity as $p(x)$. □

The next result shows that if we relax the assumption $p^- > 1$ then we are a liable to lose a lot of the regularity of the minimizer.

Example 13.1.12. Let $p(x) := 1 + (\log(1/x))^{-1}$ in $(0, 1)$. Fix the left boundary value be 0 and let $b > 0$ be the right boundary value. We have

$$\int_0^1 m^{\frac{1}{p(x)-1}}\,dx = \int_0^1 x^{-\log m}\,dx = \frac{1}{1-\log m},$$

provided $m < \mathrm{e}$, so condition (13.1.6) is satisfied for any $b > 0$. Therefore the derivative of the minimizer is $(m/p(x))^{1/(p(x)-1)}$ for some $m > 0$, by Theorem 13.1.5. Thus for $0 < y < x < 1$ we have

$$|u(x) - u(y)| = \Big|\int_y^x x^{-\log m} p(x)^{-\frac{1}{p(x)-1}}\,dx\Big| \leqslant \frac{x^{1-\log m} - y^{1-\log m}}{1-\log m}$$
$$\leqslant \frac{(x-y)^{1-\log m}}{1-\log m}.$$

We see that u is $(1-\log m)$-Hölder continuous. Moreover, if b is such that $m > 1$, then the derivative is unbounded, hence not uniformly continuous.

Some minimizers are plotted in Fig. 13.3. The number on the right is again the second boundary value, b. The lower three curves are Lipschitz continuous, the following two are 0.738- and 0.530-Hölder continuous.

Theorem 13.1.13 (Harnack's inequality, [200]). *Let $p \in \mathcal{P}(r,R)$ with $1 < p^- \leqslant p^+ < \infty$. If $u \in W^{1,p(\cdot)}(r,R)$ is a minimizer with boundary values a and b, $0 \leqslant a < b$, then*

$$\sup_{y\in B(x,r')} u(y) \leqslant c \inf_{y\in B(x,r')} u(y)$$

for every $x \in (r,R)$ and every r' with $2B(x,r') \subset (r,R)$. The constant c depends only on p^-, p^+, $b-a$, and $R-r$.

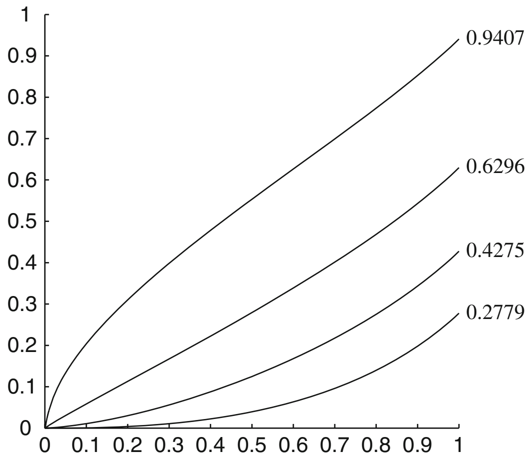

Fig. 13.3 Minimizers when $p(x) = 1 + (-\log(x))^{-1}$

Proof. We note that $t^{-1/(t-1)}$ lies between e^{-1} and 1 for all $t \geqslant 1$. By Theorem 13.1.5 we obtain

$$\sup_{B(x,r')} u = \int_r^{x+r'} \Big(\frac{m}{p(y)}\Big)^{\frac{1}{p(y)-1}}\,dy + a \leqslant \int_r^{x+r'} m^{\frac{1}{p(y)-1}}\,dy + a$$
$$\leqslant (x + r' - r) \max\Big\{m^{\frac{1}{p^+-1}}, m^{\frac{1}{p^--1}}\Big\} + a$$

and

$$\inf_{B(x,r')} u = \int_r^{x-r'} \Big(\frac{m}{p(y)}\Big)^{\frac{1}{p(y)-1}}\,dy + a \geqslant \mathrm{e}^{-1} \int_r^{x-r'} m^{\frac{1}{p(y)-1}}\,dy + a$$
$$\geqslant (x - r' - r) \min\Big\{m^{\frac{1}{p^+-1}}, m^{\frac{1}{p^--1}}\Big\} + a.$$

Since $2B(x, r') \subset (r, R)$, we deduce

$$\frac{x + r' - r}{x - r' - r} \leqslant 3.$$

Using this and the fact $\frac{c+a}{c'+a} \leqslant \max\{\frac{c}{c'}, 1\}$, the supremum and infimum estimates yield

$$\frac{\sup_{B(x,r')} u}{\inf_{B(x,r')} u} \leqslant 3\mathrm{e} \max\Big\{m^{-\frac{1}{p^--1}}, m^{\frac{1}{p^--1}}\Big\}.$$

Here the constant m is from Theorem 13.1.5 and it depends on the boundary values of u and p. Moreover m can be estimated in terms of p^-, p^+, $b - a$ and $R - r$. □

The following example shows that the constant in the Harnack inequality can not be independent of the minimizer even if the exponent is Lipschitz continuous. The example is from [208].

Example 13.1.14. We define

$$p(x) := \begin{cases} 3 & \text{for } 0 < x \leqslant \frac{1}{2}; \\ 3 - 2(x - \frac{1}{2}) & \text{for } \frac{1}{2} < x < 1. \end{cases}$$

Suppose that $u_b \in W^{1,p(\cdot)}(0, 1)$ is the minimizer of the Dirichlet energy integral for the boundary values 0 and $b > 0$ given by Theorem 13.1.5:

$$u_b(x) = \int_0^x \Big(\frac{m(b)}{p(y)}\Big)^{\frac{1}{p(y)-1}}\,dy.$$

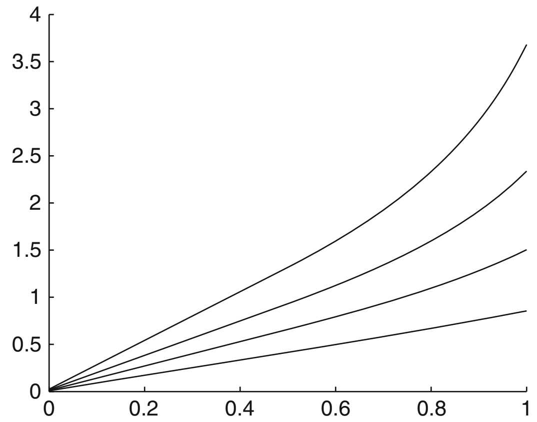

Fig. 13.4 Minimizers when p is Lipschitz continuous on the whole interval and constant on $(0, \frac{1}{2}]$

Note that if $b \to \infty$ then $m(b) \to \infty$. Three minimizers with $m(b) = 2, 4, 8$ are presented in Fig. 13.4.

In $\left(0, \frac{1}{2}\right)$ the minimizer is linear, $u_b(x) = \sqrt{\frac{m(b)}{3}}x$. In $\left(\frac{1}{2}, \frac{3}{5}\right)$ the gradient of u_b increases from $\sqrt{\frac{m(b)}{3}}$ to $\left(\frac{5m(b)}{14}\right)^{\frac{5}{9}}$. At $\frac{11}{20}$, the midpoint of $\left(\frac{1}{2}, \frac{3}{5}\right)$, the gradient of u_b equals $\left(\frac{10m(b)}{29}\right)^{\frac{10}{19}}$. Hence

$$u_b(\tfrac{3}{5}) \geqslant \sqrt{\frac{m(b)}{3}}\frac{1}{2} + \frac{1}{20}\left(\frac{10m(b)}{29}\right)^{\frac{10}{19}}.$$

Let $B := B\left(\frac{1}{2}, \frac{1}{10}\right) = \left(\frac{2}{5}, \frac{3}{5}\right)$. Then

$$\frac{\sup_{x\in B}|u_b(x)|}{\inf_{x\in B}|u_b(x)|} \geqslant \frac{\sqrt{\frac{m(b)}{3}}\frac{1}{2} + \frac{1}{20}\left(\frac{10m(b)}{29}\right)^{\frac{10}{19}}}{\sqrt{\frac{m(b)}{3}}\frac{2}{5}} = \frac{5}{4} + \frac{1}{8\sqrt{3}}\left(\frac{10}{29}\right)^{\frac{10}{19}} m(b)^{\frac{1}{38}} \to \infty$$

as $b \to \infty$.

Fan and Fan [139] have considered more complicated one-dimensional variable exponent differential equations.

Theorem 13.1.15 (Theorem 1.1, [139]). *Let $I := [0,T] \subset \mathbb{R}$ and $g \in C(I \times \mathbb{R}^N, \mathbb{R}^N)$, and suppose that there exists $r > 0$ such that $zg(x,z) \geqslant 0$ for all $x \in I$ and $z \in \mathbb{R}^N$ with $|z| = r$. If $p \in C(I)$ and $p^- > 1$, then the equation*

$$\begin{cases} \big(|u'|^{p(x)-2}u'\big)' = g(x,u), \;\; x \in I, \\ u(0) - u(T) = u'(0) - u'(T) = 0, \end{cases} \tag{13.1.16}$$

has at least one weak solution $u \in C^1(I, \mathbb{R}^N)$ such that $|u(x)| \leqslant r$ for all $x \in I$.

We refer to the survey [201] for further variants of this result.

13.2 Minimizers

We start this section by discussing existence of minimizers for given boundary values. Then we move to regularity of minimizers and Harnack's inequality. These results are collected from many papers and for every theorem a reference is given.

Definition 13.2.1. A function $u \in W^{1,p(\cdot)}(\Omega)$ is a minimizer for a boundary value function $w \in W^{1,p(\cdot)}(\Omega)$ if $u - w \in W_0^{1,p(\cdot)}(\Omega)$ and

$$\int_\Omega |\nabla u|^{p(x)}\, dx \leqslant \int_\Omega |\nabla v|^{p(x)}\, dx$$

for every function v with $u - v \in W_0^{1,p(\cdot)}(\Omega)$.

Theorem 13.2.2 ([196]). *Let Ω be bounded and let $p \in \mathcal{P}(\Omega)$ with $1 < p^- \leqslant p^+ < \infty$ be such that the $p(\cdot)$-Poincaré inequality holds. Assume that $w \in W^{1,p(\cdot)}(\Omega)$. Then there exists a unique minimizer for the boundary value function w.*

Let $n \geqslant 3$, $q_1 \in (1, n/(n-1))$ and $q_2 \in (q_1^*, n)$. Hästö [213] constructed a bounded domain Ω, a continuous exponent $p \in \mathcal{P}(\Omega)$ with $p^- = q_1$ and $p^+ = q_2$, and a boundary value function $w \in W^{1,p(\cdot)}(\Omega)$ such that there does not exists a minimizer for the boundary value function w.

Note that if we had $q_2 \leqslant q_1^*$ in the previous theorem, then a minimizer would always exist, by Lemma 8.2.14 and Theorems 13.2.2, so in this sense Theorem 13.2.2 is the best possible.

Theorem 13.2.3 ([213]). *Let Ω be bounded and let $p \in \mathcal{P}(\Omega)$ with $1 < p^- \leqslant p^+ < \infty$. Suppose that $w \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$. Then there exists a unique minimizer for the boundary value function w.*

The proofs of Theorems 13.2.2 and 13.2.3 are based on a well known functional analysis result: in a reflexive Banach spaces there exists an element that minimizes every convex, lower semicontinuous and coercive operator. The space $W^{1,p(\cdot)}(\Omega)$ is a reflexive Banach space by Theorem 8.1.6. Convexity follows since $t \mapsto t^p$ is convex for every $1 < p < \infty$. Theorem 3.2.9 yields lower semicontinuity. Therefore, we need only worry about coercivity. In this setting coercivity means that $\|u\|_{p(\cdot)} \to \infty$ implies $\|\nabla u\|_{p(\cdot)} \to \infty$. Clearly this holds if the Poincaré inequality holds, see Sect. 8.2. If the boundary value function is bounded we may restrict our studied to uniformly bounded Sobolev functions and use the Poincaré inequality in the constant exponent case $p = 1$.

Assume that p is a bounded variable exponent with $p^- = 1$. For $\lambda > 1$ we set $p_\lambda := \max\{p, \lambda\}$; we can find Dirichlet $p_\lambda(\cdot)$-energy minimizers u_λ for the given bounded boundary value function $f \in W^{1,p_\delta(\cdot)}(\Omega)$ for some $\delta > 1$. The following result says that (u_λ) has a converging subsequence as $\lambda \to 1$. We denote $Y := \{x \in \Omega : p(x) = 1\}$.

Theorem 13.2.4 ([206]). *Let $p \in \mathcal{P}(\Omega)$ be bounded with $p^- = 1$ and let (λ_j) be a sequence decreasing to 1. Let (u_{λ_j}) be a sequence of Dirichlet $p_{\lambda_j}(\cdot)$-minimizers in Ω for a boundary value function $f \in W^{1,p_\delta(\cdot)}(\Omega) \cap L^\infty(\Omega)$, for some $\delta > 1$.*

Then there exists a subsequence (λ_j) and $u \in L^\infty(\Omega)$ such that:

(a) *$u_{\lambda_j} \to u$ in $L^{p_\delta(\cdot)}_{loc}(\Omega)$ for $\delta \in [1, \frac{n}{n-1})$;*
(b) *$u_{\lambda_j} \rightharpoonup u$ in $W^{1,p(\cdot)}_{loc}(\Omega \setminus Y)$;*
(c) *u is a weak solution of the $p(\cdot)$-Laplace equation in $\Omega \setminus Y$ (see the next section for the definition of weak solutions).*

If, in addition, p is log*-Hölder continuous and*

$$\lim_{x \to y} |p(x) - 1| \log \frac{1}{|x - y|} = 0$$

for every $y \in Y$, then the limit function u belongs to a variable exponent mixed BV-Sobolev space in Ω and it minimizes the BV-Sobolev energy among all functions with the same boundary values.

The limit function u form the previous theorem can be discontinuous as presented in Fig. 13.5 that is from [206].

At the opposite limit, when $p \to \infty$, we have the following result, where $p^\lambda := \min\{p, \lambda\}$, see also [270, 281].

Theorem 13.2.5 ([200]). *Let $p \in \mathcal{P}(\Omega)$ with $n < p^- \leqslant p^+ = \infty$. Assume that $f \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ with $\int_\Omega |\nabla f|^{p(x)} dx < \infty$. Let u_λ be the Dirichlet $p^\lambda(\cdot)$-energy minimizer for the boundary value function f. Then there exist a sequence (λ_i) converging to infinity and a function $u_\infty \in W^{1,p(\cdot)}(\Omega)$ such that (u_{λ_i}) converges locally uniformly to u_∞ in Ω. Moreover, $\int_\Omega |\nabla u_\infty|^{p(x)}\, dx$ is finite and $|\nabla u_\infty| \leqslant 1$ almost everywhere in $\{p = \infty\}$.*

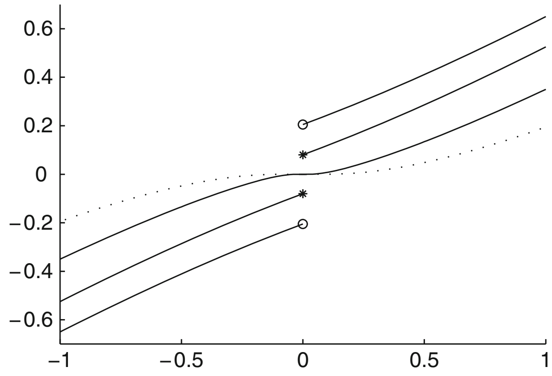

Fig. 13.5 Four BV-Sobolev-minimizers for $p(x) = 1 + |x|$ with different boundary values

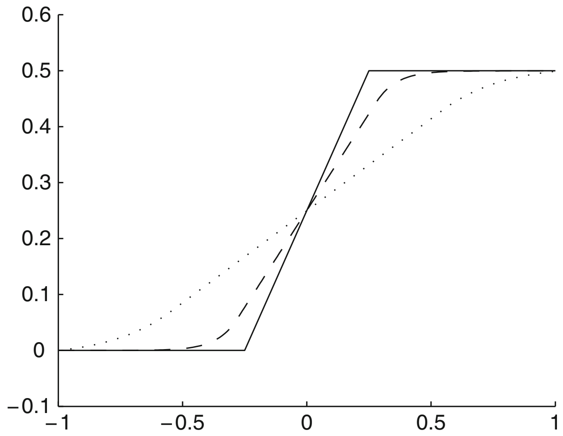

Fig. 13.6 A limit function and two solutions with $\lambda = 10, 100$

The next example shows that Harnack's inequality need not to hold for the limit function u_∞ in the form of Theorem 13.1.13. The example is from [200, Example 4.9].

Example 13.2.6. We define $p \in \mathcal{P}(0,1)$ by

$$p(x) = \begin{cases} \frac{3}{|x| - \frac{1}{4}}, & |x| > \frac{1}{4} \\ \infty, & |x| \leqslant \frac{1}{4} \end{cases}$$

and choose boundary values 0 and $\frac{1}{2}$. Figure 13.6 presents the limit function u_∞ (line) and $p^\lambda(\cdot)$-solutions with λ equal to 10 (dot) and 100 (dash). Note that the limit function u_∞ equals 0 on $(-1, -\frac{1}{4})$.

Next we study continuity of the minimizer and its gradient. A function $F : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a *Carathéodory function* if $x \mapsto F(x, z)$ is measurable for every $z \in \mathbb{R}^n$ and $z \mapsto F(x, z)$ is continuous for almost every $x \in \Omega$. Let $F : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function such that

$$c^{-1}|z|^{p(x)} \leqslant F(x, z) \leqslant c(1 + |z|^{p(x)})$$

for some $c \geqslant 1$.

Definition 13.2.7. A function $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a local minimizer of F if $|\nabla u| \in L^{p(\cdot)}_{\mathrm{loc}}(\Omega)$ and

$$\int_{\mathrm{spt}\,\psi} F(x, \nabla u)\,dx \leqslant \int_{\mathrm{spt}\,\psi} F(x, \nabla u + \nabla\psi)\,dx$$

for every $\psi \in W^{1,1}(\Omega)$ with compact support in Ω.

Every minimizer from Definition 13.2.1 satisfies the conditions of Definition 13.2.7.

In the excellent series of paper, Acerbi, Coscia and Mingione proved the fundamental $C^{1,\alpha}$ regularity for the model equation [4, 79] and extended it to more general equations and systems [5,7,8]. We refer here only some their theorems and recommend reader to look the nice survey of Mingione [297].

Fan and Zhao showed [147] that if the exponent is continuous and $1 < p^- \leqslant p^+ < \infty$, then every local minimizer of F is locally bounded. Their proofs are based on De Giorgi's method.

Theorem 13.2.8 ([4, 147]). *Let Ω be bounded and $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p^- \leqslant p^+ < \infty$. Then every local minimizer of F is locally α-Hölder continuous for $\alpha \in (0, 1)$ depending the* log*-Hölder constant of p.*

The proof of Acerbi and Mingione [4] gives also a slightly different version of the previous theorem. Namely if for every $x \in \Omega$ we have

$$|p(x) - p(y)| \log |x - y| \to 0 \text{ as } y \to x,$$

then every local minimizer of F is locally α-Hölder continuous for every $0 < \alpha < 1$.

Using the higher integrability of the gradients of local minimizers of F, Coscia and Mingione show that in some cases the gradients are continuous.

Theorem 13.2.9 ([79]). *Let Ω be bounded and $p \in \mathcal{P}(\Omega)$ be α-Hölder continuous with $1 < p^- \leqslant p^+ < \infty$ and $0 < \alpha \leqslant 1$. Then every local minimizer of F has locally β-Hölder continuous derivatives for some $\beta < \alpha$.*

Corollary 13.1.11 shows that in Theorem 13.2.9 β can not be strictly larger than α.

Definition 13.2.10. A function $u \in W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)$ is called a quasiminimizer if there exists a constant $\kappa \geqslant 1$ such that

$$\int_{\{v\neq 0\}} |\nabla u|^{p(x)}\,dx \leqslant \kappa \int_{\{v\neq 0\}} |\nabla(u+v)|^{p(x)}\,dx$$

for every open set $D \subset\subset \Omega$ and for every $v \in W^{1,p(\cdot)}(D)$ with compact support in D.

In the previous section we noted that a constant multiple of a minimizers need not be a minimizer. If u is a quasiminimizer with a constant κ, then $-u$ is also a quasiminimizer with the same constant, and if $\alpha \in \mathbb{R}$ then αu is a quasiminimizer with a constant $\max\{\alpha^{p^+-p^-}\kappa, \alpha^{p^--p^+}\kappa\}$.

If $u \in W^{1,p(\cdot)}(\Omega)$ is a quasiminimizer and $\kappa = 1$, then u is a minimizer in a sense of Definition 13.2.1 for its own boundary values. Examples of the quasiminimizers:

- Local minimizers and minimizers with a given boundary value function of

$$\int_{\Omega} \frac{|\nabla u|^{p(\cdot)}}{p(x)}\,dx.$$

  These are quasiminimizers with constant $\frac{p^+}{p^-}$.
- Local minimizers of a Carathéodory function with a growing conditions

$$c^{-1}|z|^{p(x)} \leqslant F(x, \nabla u) \leqslant c|z|^{p(x)}.$$

  These are quasiminimizers with constant c^2.

Fan and Zhao studied quasiminimizers in [148]. They proved higher integrability for gradients and showed that each quasiminimizer is locally Hölder continuous. Their proofs are based on De Giorgi's method.

Theorem 13.2.11 ([148]). *Let Ω be bounded and $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p^- \leqslant p^+ < \infty$. Let u be a quasiminimizer. Then u is locally Hölder continuous and $|\nabla u| \in L^{p(\cdot)+\varepsilon}_{\mathrm{loc}}(\Omega)$ for some $\varepsilon > 0$.*

Harjulehto, Kuusi, Lukkari, Marola and Parviainen [210] extend works of Fan and Zhao [147, 148] and showed that De Giorgi's method can be fully adapted to the variable exponent case.

Theorem 13.2.12 (Harnack's inequality, [210]). *Let Ω be bounded and $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p^- \leqslant p^+ < \infty$. Let u be a nonnegative $p(\cdot)$-quasiminimizer in Ω. Further, we consider only cubes Q so small that $10Q \subset \Omega$,*

$$\int_Q |u|^{p(y)}\,dy \leqslant 1 \text{ and } \int_Q |\nabla u|^{p(y)}\,dy \leqslant 1.$$

Then there exists a constant c such that

$$\operatorname*{ess\,sup}_{y\in Q} u(y) \leqslant c\,(\operatorname*{ess\,inf}_{y\in Q} u(y) + \operatorname{diam}(Q)).$$

The constant c depends on n, $p(\cdot)$, q, $s > p^+_{10Q} - p^-_{10Q}$, the quasiminimizing constant and the $L^{ns}(10Q)$-norm of u.

Since p is uniformly continuous we may choose the radius of Q so that $ns \leqslant p^-$, and hence the $L^{ns}(10Q)$-norm of u is finite. Note that Harnack's inequality implies that u is continuous, for the proof see [18]. For bounded quasiminimizers the result can be write in a slightly different form:

$$\operatorname*{ess\,sup}_{y\in Q} u(y) \leqslant c\,(\operatorname*{ess\,inf}_{y\in Q} u(y) + \operatorname{diam}(Q)^\alpha)$$

for any $\alpha \geqslant 1$, where the constant c depends on n, $p(\cdot)$, q, α, the quasiminimizing constant and the L^∞-norm of u [210].

13.3 Harmonic and Superharmonic Functions

The Euler–Lagrange equation of the Dirichlet energy integral minimization problem is the $p(\cdot)$-*Laplace equation*

$$\operatorname{div}(p(\cdot)|\nabla u|^{p(\cdot)-2}\nabla u) = 0.$$

Next we discuss its weak solutions.

Definition 13.3.1. A function $u \in W^{1,p(\cdot)}_{loc}(\Omega)$ is a *(weak) $p(\cdot)$-supersolution* in Ω, if

$$\int_\Omega p(x)|\nabla u|^{p(x)-2}\nabla u \cdot \nabla\psi\,dx \geqslant 0$$

for every non-negative test function $\psi \in C_0^\infty(\Omega)$. A function u is a *subsolution* in Ω if $-u$ is a supersolution in Ω. A function u is a *(weak) $p(\cdot)$-solution* in Ω if u and $-u$ are supersolutions in Ω.

Note that in some paper, and also in our next section, the test functions are on from $W^{1,p(\cdot)}(\Omega)$ with a compact support in Ω. For a function $u \in W^{1,p(\cdot)}(\Omega)$ these two test classes as well $H_0^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ give the same result provided that smooth functions are dense in the Sobolev space.

If $u \in W^{1,p(\cdot)}(\Omega)$ is a minimizer of the Dirichlet energy integral for given boundary value function or if $u \in W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)$ is a local minimizer of $|\nabla u|^{p(\cdot)}$ then it is a solution. If u is a solution, then it is a local minimizer and if a solution belongs to $W^{1,p(\cdot)}(\Omega)$, then it is a minimizer of the Dirichlet energy integral for its own boundary values. Minimizers of the Dirichlet energy integral with an obstacle are supersolutions [205].

A more general equation which has also been considered is

$$\operatorname{div}(p(\cdot)|\nabla u|^{p(\cdot)-2}\nabla u) = B(x,u). \tag{13.3.2}$$

Weak solutions are defined analogously. There are several results on the existence of various functions B. As a preliminary result we mention that the equation with homogeneous Dirichlet boundary data has a unique weak solution if $B = B(x) \in L^{(p^*)'(1+\varepsilon)}(\Omega)$ is independent of u [144, Theorem 4.2]. Sanchón and Urbano [349] have shown that the same conclusion holds for entropy solutions even if only $B \in L^1(\Omega)$.

Theorem 13.3.3 (Theorem 4.3, [144]). *Suppose that $p \in C(\overline{\Omega})$ and $|B(x,u)| \leqslant c + c|u|^{p^- -\varepsilon-1}$. Then* (13.3.2) *has a weak solution for Dirichlet boundary values $g \in W^{1,p(\cdot)}(\Omega)$.*

To the best of our knowledge, this is the most general result which does not require a largeness assumption on B. Newer results, by contrast, place restrictions on the growth of B at the origin or at ∞; in particular, these results do not include as a special case $B = 0$.

Theorem 13.3.4 (Theorem 4.7, [144]). *Let $p \in C(\overline{\Omega})$ with $1 < p^- \leqslant p^+ < \infty$. Suppose that the following three conditions hold.*

(a) $|B(x,u)| \leqslant c + c|u|^{p^*(x)-1-\varepsilon}$ *for some* $\varepsilon > 0$.
(b) *There exist $R > 0$ and $\theta > p^+$ such that $0 < \theta \int_0^u B(x,v)\,dv \leqslant u\,B(x,u)$ for all $u \in \mathbb{R} \setminus (-R,R)$ and $x \in \Omega$.*
(c) $B(x,u) = o\big(|u|^{p^+-1}\big)$ *as $u \to 0$ uniformly in x.*

Then (13.3.2) *has a weak solution in $W^{1,p(\cdot)}_0(\Omega)$.*

In Theorem 4.8 of the same paper it is shown that there exist infinitely many solutions if the third condition is replaced by the assumption that B is odd in the second argument, see also [220]. A variant of this result was proved in [68]: there it is assumed that $B(x,u) = -\lambda(x)|u|^{p(x)-2}u + b(x,u)$, where $\lambda \approx 1$ and b satisfies the same conditions as B in the previously stated theorem.

We now return to the $p(\cdot)$-Laplace equation. Alkhutov showed in [18] that solutions are locally bounded, locally bounded supersolutions satisfy the weak Harnack inequality and locally bounded solutions satisfy Harnack's inequality, see also [391]. Harjulehto, Kinnunen and Lukkari extended his result to unbounded supersolutions. Moser's iteration are used in both papers. The key estimate in Moser's iteration is the Caccioppoli estimate. In the proof of Harnack's inequality the Caccioppoli estimate is used for the function $u+R$, where R is a radius of a ball. The extra term R is used to handle negative powers which comes for putting together the variable exponent Caccioppoli estimate and a constant exponent modular form Sobolev inequality. Several different versions of the Caccioppoli estimate can be found from the literature, see for example [18, Lemma 1.1], [20, Proposition 6.1] and [206, Lemma 5.3].

Theorem 13.3.5 (The weak Harnack inequality, [208]). *Let Ω be bounded and $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p^- \leqslant p^+ < \infty$. Assume that u is a $p(\cdot)$-supersolution which is nonnegative in a ball $4B \subset \Omega$ and $s > p^+_{4B} - p^-_{4B}$. Then there exists q_0 such that*

$$\Bigg(\fint_{2B} u^{q_0}\, dx \Bigg)^{\frac{1}{q_0}} \leqslant c \Big(\operatorname*{ess\,inf}_B u(x) + \operatorname{diam}(B) \Big),$$

where c depend on n, $p(\cdot)$, q and $L^{ns}(4B)$-norm of u.

Since the exponent $p(\cdot)$ is uniformly continuous, we can take for example $ns = p^-_{4B}$ by choosing $4B$ small enough. Thus the constants in the estimates are finite for all supersolutions u on a scale that depends only on $p(\cdot)$.

Combining techniques from [18] and [208] we obtain the following result. Recently Harjulehto, Hästö and Latvala noted that it can be extended to the case $p^- = 1$ [197].

Theorem 13.3.6. *Let Ω be bounded and $p \in \mathcal{P}^{\log}(\Omega)$ with $1 \leqslant p^- \leqslant p^+ < \infty$. Let B be a ball such that $4B \subset\subset \Omega$ and let u be a $p(\cdot)$-solution in Ω. Assume that $s > p^+_{4B} - p^-_{4B}$. Then*

$$\operatorname*{ess\,sup}_B |u| \leqslant c \Bigg(\Bigg(\fint_{2B} |u|^t\, dx \Bigg)^{\frac{1}{t}} + \operatorname{diam}(B) \Bigg)$$

for every $t > 0$. The constant c depends only on n, p, t and $L^{ns}(4B)$-norm of u.

Theorems 13.3.5 and 13.3.6 yields the following full version of Harnack's inequality.

Theorem 13.3.7 (Harnack's inequality [18, 208]). *Let Ω be bounded and $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p^- \leqslant p^+ < \infty$. Assume that B is a ball such that $4B \subset\subset \Omega$, and assume that $s > p^+_{4B} - p^-_{4B}$. Let u be a solution which is nonnegative in $4B$. Then*

$$\sup_{x \in B} u(x) \leqslant c \Big(\inf_{x \in B} u(x) + \operatorname{diam}(B) \Big),$$

where the constant c depends on n, p and the $L^{ns}(4B)$-norm of u.

The solutions are locally bounded, and hence the dependence of the $L^{ns}(4B)$-norm of u can be replaced by dependence of the supremum of u, as has been done in [18].

This Harnack inequality implies, as pointed out in [18], that solutions are locally Hölder continuous. Since there is the extra diameter term on the right-hand side, the inequality does not imply the strong maximum principle; by the strong maximum principle we mean that a solution can attain neither its minimum nor its maximum. Fan, Zhao and Zhang showed the strong maximum principle for weak solutions of $\operatorname{div}(|\nabla u(x)|^{p(x)-2}\nabla u) = 0$ when $p \in C^1(\overline{\Omega})$ with $1 < p^- \leqslant p^+ < \infty$ [153]. Their proof is based on choosing a suitable test function.

Since the constant in Harnack's inequality depends on the norm of u, many results that follows from Harnack's inequality have slightly different forms than in the constant exponent case even if $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p^- \leqslant p^+ < \infty$. Assume, for example, that u_i is an increasing sequence of solutions and let u be its point-wise limit. If p is a constant, then u is solution provided it is finite at some point. If p is a variable exponent, then u is solution provided that $u \in L^t_{\mathrm{loc}}$ for some $t > 0$ [205].

Alkhutov and Krashennikova studied boundary regularity of solutions in [20]. They proved a Wiener type capacity condition for boundary regularity. Behavior of solutions up to the boundary have also been studied in [127,277].

Next we define superharmonic functions by the comparison principle.

Definition 13.3.8. We say that a function $u : \Omega \to (-\infty, \infty]$ is $p(\cdot)$-*superharmonic* in Ω if:

(a) u is lower semicontinuous;
(b) u is finite almost everywhere and;
(c) The comparison principle holds: if h is a solution in $D \subset\subset \Omega$, continuous in $\overline{D}$ and $u \geqslant h$ on ∂D, then $u \geqslant h$ in D.

Every $p(\cdot)$-supersolution in Ω which satisfies

$$u(x) = \operatorname{ess} \liminf_{y \to x} u(y)$$

for every $x \in \Omega$ is $p(\cdot)$-superharmonic in Ω. On the other hand if u is a $p(\cdot)$-superharmonic function, then $\min\{u, \lambda\}$ is a $p(\cdot)$-supersolution for every λ. For the proofs see [205]. Lukkari showed in [275] that the weak solutions of

$$-\operatorname{div}(p(x)|\nabla u(x)|^{p(x)-2}\nabla u) = \mu \quad (\mu \text{ is a finite Radon measure})$$

are $p(\cdot)$-superharmonic. See also [48] for similar equations in the case of a system. Next we list properties of $p(\cdot)$-superharmonic functions. We assume that Ω is bounded and $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p^- \leqslant p^+ < \infty$.

- Let u be a $p(\cdot)$-superharmonic in Ω and $D \subset\subset \Omega$. Then there exists an increasing sequence (u_i) of continuous $p(\cdot)$-supersolutions converging to u point-wise everywhere in D [205].
- Higher integrability properties of $p(\cdot)$-superharmonic functions and their point-wise defined "gradient" has been studied in [205].
- Superharmonic functions can be point-wise estimated by Wolff's potential [278].
- The balayage is a superharmonic function [263].
- Assume that $u \in L^t_{loc}(\Omega)$, for some $t > 0$. If u is a non-negative $p(\cdot)$-superharmonic function in Ω, then $u < \infty$ $p(\cdot)$-quasieverywhere in Ω [208]. Here the assumption $u \in L^t_{loc}(\Omega)$ is needed to adapt the weak Harnack inequality.
- If u is a $p(\cdot)$-superharmonic function, then it is $p(\cdot)$-quasicontinuous [211].

13.4 Harnack's Inequality for A-harmonic Functions

Harnack's inequality, as stated in Theorem 13.3.7 and in the existence literature, is formulated only for weak solutions of $p(\cdot)$-Laplace equation, although the method covers all elliptic equations with Laplace type structural conditions. Hence we prove, by Moser's iteration, Harnack's inequality here. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set. We study elliptic equation of the form

$$-\operatorname{div} \mathsf{A}(x, \nabla u) = 0,$$

where the operator $\mathsf{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following structural conditions for constants $c_1, c_2 > 0$:

$$x \mapsto \mathsf{A}(x,\xi) \text{ is measurable for all } \xi \in \mathbb{R}^n.$$
$$\xi \mapsto \mathsf{A}(x,\xi) \text{ is continuous.}$$
$$\mathsf{A}(x,-\xi) = -\mathsf{A}(x,\xi) \text{ for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^n.$$
$$\mathsf{A}(x,\xi)\cdot\xi \geqslant c_1|\xi|^{p(x)} \text{ for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^n.$$
$$|\mathsf{A}(x,\xi)| \leqslant c_2|\xi|^{p(x)-1} \text{ for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^n.$$
$$(\mathsf{A}(x,\eta) - \mathsf{A}(x,\xi))\cdot(\eta-\xi) > 0 \text{ for all } x \in \Omega \text{ and } \eta,\xi \in \mathbb{R}^n, \eta \neq \xi.$$

By choosing c_2 larger, if necessary, we may assume that $c_2 \geqslant c_1$. For example the equations

$$-\operatorname{div}(p(x)|\nabla u|^{p(x)-1}\nabla u) = 0 \text{ and } -\operatorname{div}(|\nabla u|^{p(x)-1}\nabla u) = 0$$

satisfy the above conditions.

Following Definition 13.3.1 we define weak solutions in this case as follows.

Definition 13.4.1. A function $u \in W^{1,p(\cdot)}_{loc}(\Omega)$ is a *(weak)* A*-supersolution* in Ω, if

$$\int_\Omega \mathsf{A}(x, \nabla u) \cdot \nabla \psi \, dx \geqslant 0$$

for every non-negative $\psi \in W^{1,p(\cdot)}(\Omega)$ with a compact support in Ω. A function u is a A*-subsolution* in Ω if $-u$ is a supersolution in Ω. A function u is a *(weak)* A*-solution* in Ω if u and $-u$ are A-supersolutions in Ω.

We start by the following technical lemma that is need later.

Lemma 13.4.2. *Let f be a positive measurable function and assume that the exponent $p \in \mathcal{P}^{\log}(\Omega)$ is bounded. Then*

$$\fint_B f^{p_B^+ - p_B^-} \, dx \leqslant c \, \|f\|_{L^s(B)}^{p_B^+ - p_B^-}$$

for any $s > p_B^+ - p_B^-$ and $B \subset \Omega$. Here the constant depends only on the dimension n and $c_{\log}(p)$.

Proof. Let $q := p_B^+ - p_B^-$ and let R be the radius of the ball B. Hölder's inequality implies that

$$\fint_B f^{p_B^+ - p_B^-} \, dx \leqslant \bigg(\fint_B f^s \, dx \bigg)^{\frac{q}{s}} = cR^{-\frac{nq}{s}} \, \|f\|_{L^s(B)}^q.$$

By log-Hölder continuity, $R^{-\frac{q}{s}} \leqslant R^{-q} < c < \infty$ and hence the claim follows. □

Later we apply Lemma 13.4.2 with $f = u^{q'}$. In this case the upper bound written in terms of u is

$$c\|u\|_{L^{q's}(B)}^{q'(p_B^+ - p_B^-)}.$$

First we show that A-subsolutions are locally bounded above. We fix a ball $B := B(z, R)$ such that $R \leqslant 1$ and $4B \subset\subset \Omega$. We write

$$v := \max\{u, 0\} + R,$$

where u is a A-subsolution; also,

$$\Phi(f, q, A) := \bigg(\fint_A f^q \, dx \bigg)^{1/q}$$

for a nonnegative measurable function f and $q \neq 0$.

Lemma 13.4.3. *Let $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p^- \leqslant p^+ < \infty$. Let $1 \leqslant \tau < \kappa \leqslant 3$. Then*

$$\left(\fint_{\tau B} v^{\beta n' p_{4B}^-} \, dx \right)^{\frac{1}{n'}} \leqslant c \beta^{p_{4B}^-} \left(\frac{\kappa}{\kappa - \tau} \right)^{p_{4B}^+} \fint_{\kappa B} v^{(\beta-1)p_{4B}^- + p(x)} \, dx$$

for $\beta \geqslant 1$. The constant c depends only on n, p^-, p^+, $c_{\log}(p)$ and the structural constants c_1 and c_2.

Proof. We choose $\eta \in C_0^\infty(\kappa B)$ such that $0 \leqslant \eta \leqslant 1$. Let G be a function on $[0, \infty)$ with $G'(t) = \beta t^{\beta-1}$. The function G_j is defined by the cut-off derivative, $G_j'(t) := \beta \min\{t, j\}^{\beta-1}$. Fixing the origin, we see that

$$G_j(t) = \begin{cases} t^\beta, & \text{for } 0 \leqslant t \leqslant j, \\ j^\beta + \beta j^{\beta-1}(t-j), & \text{for } t \geqslant j. \end{cases}$$

We further define

$$H_j(\xi) := \int_R^\xi G_j'(t)^{p_{4B}^-} \, dt$$

for $\xi \geqslant R$.

First we show that $\psi := H_j(v) \eta^{p_{4B}^+}$ belongs to $W_0^{1,p(\cdot)}(\Omega)$. Since η has compact support in Ω, it suffices to show that $\psi \in W^{1,p(\cdot)}(\Omega)$. Note that ψ is non-negative, because η and v are. Since

$$|H_j(v)| \leqslant \frac{\beta^{p_{4B}^-}}{(\beta-1)p_{4B}^- + 1} j^{(\beta-1)p_{4B}^- + 1} + \beta^{p_{4B}^-} j^{(\beta-1)p_{4B}^-} v,$$

we find that $\psi \in L^{p(\cdot)}(\Omega)$. For the gradient we have

$$\begin{aligned} |\nabla \psi| &\leqslant p_{4B}^+ \eta^{p_{4B}^+ - 1} |\nabla \eta| H_j(v) + \eta^{p_{4B}^+} |G_j'(v)|^{p_{4B}^-} |\nabla v| \\ &\leqslant c(\eta) p_{4B}^+ H_j(v) + \eta^{p_{4B}^+} (\beta j^{\beta-1})^{p_{4B}^-} |\nabla v|, \end{aligned}$$

and hence $|\nabla \psi| \in L^{p(\cdot)}(\Omega)$.

Since u is a A-subsolution and ψ is an admissible test function, we have

$$\int_\Omega \mathsf{A}(x, -\nabla u) \cdot \nabla \psi \, dx \geqslant 0$$

and furthermore

$$\int_\Omega G_j'(v)^{p_{4B}^-}\eta^{p_{4B}^+}\mathsf{A}(x,\nabla u)\cdot\nabla v\,dx \leqslant \left|\int_\Omega p_{4B}^+\eta^{p_{4B}^+-1}H_j(v)\mathsf{A}(x,\nabla u)\cdot\nabla\eta\,dx\right|$$
$$\leqslant c_2\int_\Omega p_{4B}^+\eta^{p_{4B}^+-1}H_j(v)|\nabla u|^{p(x)-1}|\nabla\eta|\,dx.$$

Note that $\nabla v = 0$ and $H_j(v) = H_j(R) = 0$ whenever $u \leqslant 0$. If $u > 0$, then $\nabla v = \nabla u$. Hence we obtain

$$c_1\int_\Omega |\nabla v|^{p(x)}|G_j'(v)|^{p_{4B}^-}\eta^{p_B^+}\,dx \leqslant c_2\int_\Omega p_{4B}^+|\nabla v|^{p(x)-1}H_j(v)|\nabla\eta|\eta^{p_{4B}^+-1}\,dx.$$

We estimate the integrand on the right-hand side by Young's inequality,

$$ab \leqslant \Big(\frac{1}{\varepsilon}\Big)^{p-1}a^p + \varepsilon b^{p'},$$

for the exponents $p(x)$ and $p'(x)$. For $p(x) > 1$ this yields that

$$\begin{aligned}
&p_{4B}^+H_j(v)|\nabla v|^{p(x)-1}\eta^{p_{4B}^+-1}|\nabla\eta|\\
&\quad= p_{4B}^+|G_j'(v)|^{-\frac{p_{4B}^-}{p'(x)}}H_j(v)|\nabla\eta|\eta^{p_{4B}^+-\frac{p_{4B}^+}{p'(x)}-1}|G_j'(v)|^{\frac{p_{4B}^-}{p'(x)}}|\nabla v|^{p(x)-1}\eta^{\frac{p_{4B}^+}{p'(x)}}\\
&\quad\leqslant \varepsilon^{1-p(x)}{p_{4B}^+}^{p(x)}|G_j'(v)|^{-p_{4B}^-(p(x)-1)}H_j(v)^{p(x)}|\nabla\eta|^{p(x)}\eta^{p_{4B}^+-p(x)}\\
&\qquad+\varepsilon|G_j'(v)|^{p_{4B}^-}|\nabla v|^{p(x)}\eta^{p_{4B}^+}.
\end{aligned}$$

Combining this with the previous inequality and using $\eta \leqslant 1$ we obtain

$$\begin{aligned}
&c_1\int_\Omega |\nabla v|^{p(x)}|G_j'(v)|^{p_{4B}^-}\eta^{p_{4B}^+}\,dx\\
&\quad\leqslant c_2\varepsilon^{1-p_{4B}^+}\int_\Omega {p_{4B}^+}^{p(x)}|G_j'(v)|^{-p_{4B}^-(p(x)-1)}H_j(v)^{p(x)}|\nabla\eta|^{p(x)}\,dx\\
&\qquad+\varepsilon c_2\int_\Omega |G_j'(v)|^{p_{4B}^-}|\nabla v|^{p(x)}\eta^{p_{4B}^+}\,dx.
\end{aligned}$$

Next we choose ε so small that $\varepsilon c_2 = c_1/2$. Then we can absorb the second integral on the right-hand side into the left-hand side. Thus

$$\int_\Omega |\nabla v|^{p(x)}|G_j'(v)|^{p_{4B}^-}\eta^{p_{4B}^+}\,dx \leqslant c\int_\Omega |G_j'(v)|^{-p_{4B}^-(p(x)-1)}H_j(v)^{p(x)}|\nabla\eta|^{p(x)}\,dx.$$

Then we can use the trivial estimate $|\nabla v|^{p^-_{4B}} \leqslant 1 + |\nabla v|^{p(x)}$ and the previous estimate to derive

$$\begin{aligned}&\int_\Omega |\nabla v|^{p^-_{4B}} |G_j'(v)|^{p^-_{4B}} \eta^{p^+_{4B}}\, dx \\ &\quad \leqslant \int_\Omega |G'(v)|^{p^-_{4B}} \eta^{p^+_{4B}}\, dx + c\int_\Omega |G_j'(v)|^{-p^-_{4B}(p(x)-1)} H_j(v)^{p(x)} |\nabla \eta|^{p(x)}\, dx.\end{aligned}$$

Since $\eta \leqslant 1$ vanishes outside κB, we get

$$\begin{aligned}&\fint_{\kappa B} |\nabla (G_j(v)\eta^{p^+_{4B}})|^{p^-_{4B}}\, dx \\ &\quad = \fint_{\kappa B} |G_j(v) p^+_{4B} \eta^{p^+_{4B}-1}\nabla\eta + \eta^{p^+_{4B}} G_j'(v)\nabla v|^{p^-_{4B}}\, dx \\ &\quad \leqslant 2^{p^-_{4B}} {p^+_{4B}}^{p^-_{4B}} \fint_{\kappa B} |G_j(v)|^{p^-_{4B}} |\nabla\eta|^{p^-_{4B}}\, dx + 2^{p^-_{4B}} \fint_{\kappa B} |G_j'(v)|^{p^-_{4B}}\, dx \\ &\qquad + 2^{p^-_{4B}} c \fint_{\kappa B} |G_j'(v)|^{-p^-_{4B}(p(x)-1)} H_j(v)^{p(x)} |\nabla\eta|^{p(x)}\, dx.\end{aligned}$$

Next we use the constant exponent Sobolev-Poincaré inequality

$$\bigg(\fint_{\kappa B} \Big(\frac{|w|}{R}\Big)^{n' p^-_{4B}} dx\bigg)^{\frac{1}{n' p^-_{4B}}} \leqslant c\bigg(\fint_{\kappa B} |\nabla w|^{p^-_{4B}}\, dx\bigg)^{\frac{1}{p^-_{4B}}}$$

with the function $w = G_j(v)\eta^{p^+_{4B}} \in W^{1,p^-_{4B}}_0(\kappa B)$. We obtain that

$$\begin{aligned}&\bigg(\fint_{\kappa B} \Big(\frac{G_j(v)\eta^{p^+_{4B}}}{R}\Big)^{n' p^-_{4B}} dx\bigg)^{\frac{n-1}{n}} \leqslant c\fint_{\kappa B} |\nabla (G_j(v)\eta^{p^+_{4B}})|^{p^-_{4B}}\, dx \\ &\quad \leqslant c\fint_{\kappa B} |G_j(v)|^{p^-_{4B}} |\nabla\eta|^{p^-_{4B}}\, dx + c\fint_{\kappa B} |G_j'(v)|^{p^-_{4B}}\, dx \\ &\qquad + c\fint_{\kappa B} |G_j'(v)|^{-p^-_{4B}(p(x)-1)} H_j(v)^{p(x)} |\nabla\eta|^{p(x)}\, dx.\end{aligned}$$

Since $G_j \leqslant G$ and $G_j' \leqslant G'$, we may replace the functions G_j on the right-hand side by the function G. Then the right-hand side does not depend on j, and we may use monotone convergence on the left-hand side to conclude that

$$\bigg(\fint_{\kappa B}\Big(\frac{v^\beta\eta^{p^+_{4B}}}{R}\Big)^{n'p^-_{4B}}\,dx\bigg)^{\frac{1}{n'}}$$
$$\leqslant c\fint_{\kappa B} v^{\beta p^-_{4B}}|\nabla\eta|^{p^-_{4B}}\,dx + c\fint_{\kappa B}\beta^{p^-_{4B}}v^{-p^-_{4B}}v^{\beta p^-_{4B}}\,dx + I\,,$$

where I is given as

$$I = c\fint_{\kappa B}\beta^{(1-p(x))p^-_{4B}}v^{(\beta-1)(p^-_{4B}-p(x)p^-_{4B})}\times$$
$$\times\bigg(\frac{\beta p^-_{4B}}{(\beta-1)p^-_{4B}+1}\bigg)^{p(x)}v^{((\beta-1)p^-_{4B}+1)p(x)}|\nabla\eta|^{p(x)}\,dx.$$

We choose η so that $\eta = 1$ in τB and $|\nabla\eta| \leqslant \frac{c}{R(\kappa-\tau)} \leqslant \frac{c\kappa}{R(\kappa-\tau)}$. Since $v \geqslant R$, we obtain $v^{-p^-_{4B}} \leqslant R^{-p^-_{4B}}$ and

$$v^{\beta p^-_{4B}} = v^{(\beta-1)p^-_{4B}+p(x)}v^{p^-_{4B}-p(x)} \leqslant v^{(\beta-1)p^-_{4B}+p(x)}R^{p^-_{4B}-p(x)}.$$

By the log-Hölder continuity we have $R^{p^-_{4B}-p(x)} \leqslant c$. Thus there is a common integral average over $v^{(\beta-1)p^-_{4B}+p(x)}$ on the right-hand side. Since the measure of τB is comparable with the measure of κB, we can change the average on the left-hand side to the smaller ball. Multiplying both sides of the inequality by $R^{p^-_{4B}}$ now implies

$$\bigg(\fint_{\tau B} v^{\beta n' p^-_{4B}}\,dx\bigg)^{\frac{1}{n'}} \leqslant c\bigg[\Big(\frac{\kappa}{\kappa-\tau}\Big)^{p^-_{4B}} + \beta^{p^-_{4B}} + \beta^{p^-_{4B}}\Big(\frac{\kappa}{\kappa-\tau}\Big)^{p^+_{4B}}R^{p^-_{4B}-p^+_{4B}}\bigg]$$
$$\times\fint_{\kappa B} v^{(\beta-1)p^-_{4B}+p(x)}\,dx.$$

By the log-Hölder continuity of the exponent, the term $R^{p^-_{4B}-p^+_{4B}}$ is bounded by a constant and hence the claim follows. □

Lemma 13.4.4. *Let $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p^- \leqslant p^+ < \infty$. Let $1 \leqslant \tau < \kappa \leqslant 3$. Then*

$$\Phi(v, n'\beta, \tau B) \leqslant c^{\frac{1}{\beta}}\beta^{\frac{p^-_{4B}}{\beta}}\Big(\frac{r}{r-\varrho}\Big)^{\frac{p^+_{4B}}{\beta}}\Phi(v, q\beta, \kappa B)$$

for every $\beta \geqslant p^-_{4B}$, $1 < q < n'$ and $s > p^+_{4B} - p^-_{4B}$. The constant c depends only on n, p^-, p^+, $c_{\log}(p)$, and the $L^{q's}(4B)$-norm of v and the structural constants c_1 and c_2.

Proof. Replacing β by β/p_{4B}^- in Lemma 13.4.3 we obtain

$$\left(\fint_{\tau B} \left(v^{\frac{\beta}{p_{4B}^-}} \right)^{n' p_{4B}^-} dx \right)^{\frac{1}{n'\beta}} \leqslant \left(c\beta^{p_{4B}^-} \left(\frac{\kappa}{\kappa - \tau} \right)^{p_{4B}^+} \fint_{\kappa B} v^{\left(\frac{\beta}{p_{4B}^-} - 1 \right) p_{4B}^- + p(x)} dx \right)^{\frac{1}{\beta}}.$$

This yields by Hölder's inequality and Lemma 13.4.2 that

$$\begin{aligned} &\left(\fint_{\tau B} v^{\beta n'} \, dx \right)^{\frac{n-1}{n\beta}} \\ &\quad \leqslant c^{\frac{1}{\beta}} \beta^{\frac{p_{4B}^-}{\beta}} \left(\frac{\kappa}{\kappa - \tau} \right)^{\frac{p_{4B}^+}{\beta}} \left(\fint_{\kappa B} v^{q'(p(x) - p_{4B}^-)} \, dx \right)^{\frac{1}{\beta q'}} \left(\fint_{\kappa B} v^{\beta q} \, dx \right)^{\frac{1}{\beta q}} \\ &\quad \leqslant c^{\frac{1}{\beta}} \beta^{\frac{p_{4B}^-}{\beta}} \left(\frac{\kappa}{\kappa - \tau} \right)^{\frac{p_{4B}^+}{\beta}} \left(1 + \|v\|_{L^{q's}(4B)}^{q'(p_{4B}^+ - p_{4B}^-)} \right)^{\frac{1}{\beta q'}} \left(\fint_{\kappa B} v^{\beta q} \, dx \right)^{\frac{1}{\beta q}}. \end{aligned}$$

To conclude the claim we include the term $\left(1 + \|v\|_{L^{q's}(4B)}^{q'(p_{4B}^+ - p_{4B}^-)} \right)^{\frac{1}{q'}} \leqslant 1 + \|v\|_{L^{q's}(4B)}^{q'(p_{4B}^+ - p_{4B}^-)}$ into the constant c. □

Theorem 13.4.5. *Let $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p^- \leqslant p^+ < \infty$. Let B be a ball with a radius $R \leqslant 1$ such that $4B \subset\subset \Omega$ and let u be a* A*-subsolution in Ω. Assume that $s > p_{4B}^+ - p_{4B}^-$. Then*

$$\operatorname*{ess\,sup}_B u \leqslant c \left(\left(\fint_{2B} |u|^t \, dx \right)^{\frac{1}{t}} + R \right)$$

for every $t > 0$. The constant c depends only on n, p^-, p^+, $c_{\log}(p)$, t, $L^{ns}(4B)$-norm of u and the structural constants c_1 and c_2.

Since the exponent p is uniformly continuous, we can take for example $ns = p_\Omega^-$ by choosing B small enough. Thus the constants in the estimates are finite for all solutions u in a scale that depends only on p.

Proof. By making s slightly smaller if necessary, we may assume that there exists $q \in (1, n')$ such that $\|u\|_{L^{q's}(4B)} < \infty$. Let $1 \leqslant \tau < \kappa \leqslant 3$. For $j = 0, 1, 2, \ldots$, we write $r_j := \tau + 2^{-j}(\kappa - \tau)$ and

$$\xi_j := \left(\frac{n'}{q} \right)^j q p_{4B}^-.$$

By Lemma 13.4.4 with $\beta = \left(\frac{n'}{q}\right)^j p^-_{4B}$ we obtain

$$\Phi(v, \xi_{j+1}, r_{j+1}B) \leqslant c^{\frac{q}{\xi_j}} \xi_j^{\frac{qp^-_{4B}}{\xi_j}} \left(\frac{r_j}{r_j - r_{j+1}}\right)^{\frac{qp^+_{4B}}{\xi_j}} \Phi(v, \xi_j, r_jB).$$

Iterating and letting $j \to \infty$ we find that

$$\begin{aligned}\operatorname*{ess\,sup}_{\tau B} |v| &\leqslant \prod_{j=0}^{\infty} c^{\frac{q}{\xi_j}} \xi_j^{\frac{qp^-_{4B}}{\xi_j}} \left(2^j \frac{\kappa}{\kappa - \tau}\right)^{\frac{qp^+_{4B}}{\xi_j}} \Phi(v, qp^-_{4B}, \kappa B) \\ &\leqslant c^{\frac{qn}{p^-_{4B}}} (n')^{\frac{qp^+_{4B}}{p^-_{4B}} \sum_{j=0}^{\infty} \frac{j}{(n')^j}} 2^{\frac{qp^+_{4B}}{p^-_{4B}} \sum_{j=0}^{\infty} \frac{j}{(n')^j}} \left(\frac{\kappa}{\kappa - \tau}\right)^{\frac{qnp^+_{4B}}{p^-_{4B}}} \Phi(v, qp^-_{4B}, \kappa B).\end{aligned}$$

By the root test the sums in the previous inequality are finite and hence

$$\operatorname*{ess\,sup}_{\tau B} |v| \leqslant c\big(1 - \tfrac{\tau}{\kappa}\big)^{-\frac{\lambda}{s}} \Phi(v, s, \kappa B), \tag{13.4.6}$$

where $\lambda := \frac{p^+_{B_{4R}} n' q}{(n' - q)}$ and $s := qp^-_{B_{4R}}$. By Hölder's inequality we see that $\Phi(v, s, \kappa B) \leqslant c\Phi(v, t, \kappa B)$ when $t \geqslant s$.

We then consider $t < s$. Let us show that

$$\operatorname*{ess\,sup}_{B} |v| \leqslant c\,\Phi(v, t, 2B),$$

for any $t \in (0, s)$. We adapt the argument of [280, Corollary 3.10]. Let $\sigma \in (\frac{1}{3}, 1)$. Denote

$$T(\sigma) := \operatorname*{ess\,sup}_{\sigma 2B} |v| \quad \text{and} \quad S(\sigma) := (1 - \sigma)^{\frac{\lambda}{t} - \frac{\lambda}{s}} \Phi(v, s, \sigma 2B).$$

Set $\sigma' := \frac{1+\sigma}{2}$. We rewrite the conclusion of the previous paragraph as

$$T(\sigma) \leqslant c\big(1 - \tfrac{\sigma}{\sigma'}\big)^{-\frac{\lambda}{s}} \Phi(v, s, \sigma' 2B) \approx (1 - \sigma)^{-\frac{\lambda}{s}} \Phi(v, s, \sigma' 2B).$$

Since $1 - \sigma' = \frac{1-\sigma}{2}$, we further obtain that

$$T(\sigma) \leqslant c(1 - \sigma)^{-\frac{\lambda}{t}} S(\sigma').$$

Using this in the second step, we estimate

$$\Big(\fint_{\sigma 2B} v^s\,dx\Big)^{\frac1s} \leqslant \Big(T(\sigma)^{s-t}\fint_{\sigma 2B} v^t\,dx\Big)^{\frac1s}$$
$$\leqslant c(1-\sigma)^{\frac{\lambda}{s}-\frac{\lambda}{t}} S(\sigma')^{1-\frac ts}\Big(\fint_{\sigma 2B} v^t\,dx\Big)^{\frac1s}.$$

Dividing both sides by $(1-\sigma)^{\frac{\lambda}{s}-\frac{\lambda}{t}}$, we obtain

$$S(\sigma) \leqslant cS(\sigma')^{1-\frac ts}\Big(\fint_{\sigma 2B} v^t\,dx\Big)^{\frac1s} \leqslant cS(\sigma')^{1-\frac ts}\Big(\fint_{2B} v^t\,dx\Big)^{\frac1s},$$

where we used $\sigma \approx 1$ in the second step. Iterating this inequality, we find that

$$S(\sigma) \leqslant c\Big(\fint_{2B} v^t\,dx\Big)^{\frac1s\sum_j(1-\frac ts)^j} = \Big(\fint_{2B} v^t\,dx\Big)^{\frac1t}.$$

We choose $\tau = 1$ and $\kappa = \frac32$ in (13.4.6) and $\sigma = \frac34$ in the above estimate. Combining these give the claim for the function v.

The same estimate holds also for $-\min\{u, 0\}$, since $-u$ is a solution. Thus the claim follows. □

If u is a solution then the above theorem holds also for $-\min\{u, 0\}$ and hence we obtain the following corollary by covering $D \subset\subset \Omega$ by a finitely many balls satisfying the conditions of the previous theorem.

Corollary 13.4.7. *Let $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p^- \leqslant p^+ < \infty$. Then every* A*-solution is locally bounded.*

Next we show that non-negative A-supersolutions satisfy the weak Harnack inequality. We write

$$v := u + R,$$

where u is a non-negative A-supersolution. Remember that $B = B(z, R)$ is fixed and $4B \subset\subset \Omega$.

We derive a suitable Caccioppoli type estimate with variable exponents.

Lemma 13.4.8 (Caccioppoli estimate). *Let $p \in \mathcal{P}(\Omega)$ with $1 < p^- \leqslant p^+ < \infty$. Let E be a measurable subset of $4B \subset\subset \Omega$ and $\eta \in C_0^\infty(4B)$ such that $0 \leqslant \eta \leqslant 1$. Then for every $\gamma_0 < 0$ there is a constant c depending only on p^+, $c_{log}(p)$, c_1, c_2 and γ_0 such that*

$$\int_E v^{\gamma-1}|\nabla u|^{p_E^-}\eta^{p_{4B}^+}\,dx \leqslant c\int_{4B}\big(\eta^{p_{4B}^+}v^{\gamma-1} + v^{\gamma+p(x)-1}|\nabla\eta|^{p(x)}\big)\,dx$$

for every $\gamma < \gamma_0 < 0$.

Proof. We want to test with the function $\psi := v^\gamma \eta^{p^+_{4B}}$. Next we show that $\psi \in W_0^{1,p(\cdot)}(4B)$. Since η has compact support in $4B$ it suffuces to show that $\psi \in W^{1,p(\cdot)}(\Omega)$. We observe that $\psi \in L^{p(\cdot)}(\Omega)$ since $|v^\gamma|\eta^{p^+_{4B}} \leqslant R^\gamma$. Furthermore, we have

$$|\nabla\psi| \leqslant |\gamma v^{\gamma-1}\eta^{p^+_{4B}}\nabla u + v^\gamma p^+_{4B}\eta^{p^+_{4B}-1}\nabla\eta| \leqslant |\gamma|R^{\gamma-1}|\nabla u| + p^+_{4B}R^\gamma|\nabla\eta|,$$

from which we conclude that $|\nabla\psi| \in L^{p(\cdot)}(\Omega)$.

Using the fact that u is a A-supersolution and ψ is a nonnegative test function we find that

$$\begin{aligned} 0 &\leqslant \int\limits_{4B} \mathsf{A}(x,\nabla u)\cdot\nabla\psi(x)\,dx \\ &= \int\limits_{4B} \gamma\eta^{p^+_{4B}}v^{\gamma-1}\mathsf{A}(x,\nabla u)\cdot\nabla u\,dx + \int\limits_{4B} p^+_{4B}v^\gamma\eta^{p^+_{4B}-1}\mathsf{A}(x,\nabla u)\cdot\nabla\eta\,dx. \end{aligned}$$

Since γ is a negative number this implies by the structural conditions that

$$|\gamma_0|c_1\int\limits_{4B}\eta^{p^+_{4B}}v^{\gamma-1}|\nabla u|^{p(x)}\,dx \leqslant p^+_{4B}c_2\int\limits_{4B} v^\gamma\eta^{p^+_{4B}-1}|\nabla u|^{p(x)-1}\,|\nabla\eta|\,dx.$$

We denote the right-hand side of the previous inequality by I. Using Young's inequality, $0<\varepsilon\leqslant 1$, we obtain

$$\begin{aligned} I &\leqslant p^+_{4B}c_2\int\limits_{4B}\Big(\frac{1}{\varepsilon}\Big)^{p(x)-1}\Big(v^{\frac{\gamma+p(x)-1}{p(x)}}|\nabla\eta|\eta^{p^+_{4B}-\frac{p^+_{4B}}{p'(x)}-1}\Big)^{p(x)} \\ &\qquad + \varepsilon\Big(|\nabla u|^{p(x)-1}\eta^{\frac{p^+_{4B}}{p'(x)}}v^{\gamma-\frac{\gamma+p(x)-1}{p(x)}}\Big)^{p'(x)}\,dx \\ &\leqslant p^+_{4B}c_2\Big(\frac{1}{\varepsilon}\Big)^{p^+_{4B}-1}\int\limits_{4B} v^{\gamma+p(x)-1}|\nabla\eta|^{p(x)}\eta^{p^+_{4B}-p(x)}\,dx \\ &\qquad + p^+_{4B}c_2\varepsilon\int\limits_{4B}|\nabla u|^{p(x)}\eta^{p^+_{4B}}v^{\gamma-1}\,dx. \end{aligned}$$

By combining these inequalities we arrive at

$$\begin{aligned} &|\gamma_0|c_1\int\limits_{4B}|\nabla u|^{p(x)}\eta^{p^+_{4B}}v^{\gamma-1}\,dx \\ &\leqslant p^+_{4B}c_2\Big(\frac{1}{\varepsilon}\Big)^{p^+_{4B}-1}\int\limits_{4B} v^{\gamma+p(x)-1}|\nabla\eta|^{p(x)}\eta^{p^+_{4B}-p(x)}\,dx \\ &\qquad + p^+_{4B}c_2\varepsilon\int\limits_{4B}|\nabla u|^{p(x)}\eta^{p^+_{4B}}v^{\gamma-1}\,dx. \end{aligned}$$

By choosing

$$\varepsilon = \min\left\{1, \frac{|\gamma_0| c_1}{2 p_{4B}^+ c_2}\right\}$$

we can absorb the last term to the left-hand side and obtain

$$\int_{4B} |\nabla u|^{p(x)} \eta^{p_{4B}^+} v^{\gamma-1}\,dx \leqslant p_{4B}^+ c_2 \left(\frac{2p_{4B}^+ c_2}{|\gamma_0| c_1} + 1\right)^{p_{4B}^+ - 1} \frac{2}{|\gamma_0| c_1} \int_{4B} v^{\gamma+p(x)-1} |\nabla \eta|^{p(x)}\,dx.$$

Taking $f = v^{\gamma-1}\eta^{p_{4B}^+}$ and $g = |\nabla u|$ in the point-wise inequality

$$f(x) g(x)^{p_E^-} \leqslant f(x) + f(x) g(x)^{p(x)}$$

and using the previous inequality we obtain the claim. □

Lemma 13.4.9. *Let $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p^- \leqslant p^+ < \infty$. Assume that u is a nonnegative* A*-supersolution in $4B$ and let $1 \leqslant \tau < \kappa \leqslant 3$. Then*

$$\Phi(v, q\beta, \kappa B) \leqslant c^{\frac{1}{|\beta|}} (1 + |\beta|)^{\frac{p_{4B}^+}{|\beta|}} \left(\frac{\kappa}{\kappa - \tau}\right)^{\frac{p_{4B}^+}{|\beta|}} \Phi(v, n'\beta, \tau B)$$

for every $\beta < 0$ and $1 < q < n'$. The constant c depends on n, p^-, p^+, $c_{\log}(p)$, the $L^{q's}(4B)$-norm of u with $s > p_{4B}^+ - p_{4B}^-$ and the structural constants c_1 and c_2.

Proof. In the Caccioppoli estimate, Lemma 13.4.8, we take $E = 4B$ and $\gamma = \beta - p_{4B}^- + 1$. Then $\gamma < 1 - p_{4B}^-$ and thus

$$\int_{4B} v^{\beta - p_{4B}^-} |\nabla u|^{p_{4B}^-} \eta^{p_{4B}^+}\,dx \leqslant c \int_{4B} \left(\eta^{p_{4R}^+} v^{\beta - p_{4B}^-} + v^{\beta - p_{4B}^- + p(x)} |\nabla \eta|^{p(x)}\right) dx.$$

Next we take a cutoff function $\eta \in C_0^\infty(\kappa B)$ with $0 \leqslant \eta \leqslant 1$, $\eta = 1$ in τB and

$$|\nabla \eta| \leqslant \frac{c}{R(\kappa - \tau)} \leqslant \frac{c\kappa}{R(\kappa - \tau)}.$$

By the log-Hölder continuity of p we have

$$|\nabla \eta|^{-p(x)} \leqslant c R^{-p(x)} \left(\frac{\kappa}{\kappa - \tau}\right)^{p_{4B}^+} \leqslant c R^{-p_{4B}^-} \left(\frac{\kappa}{\kappa - \tau}\right)^{p_{4B}^+}. \tag{13.4.10}$$

With this choice of η we have

$$
\begin{aligned}
&\fint_{\kappa B} \Big|\nabla\Big(v^{\frac{\beta}{p_{4B}^-}}\eta^{\frac{p_{4B}^+}{p_{4B}^-}}\Big)\Big|^{p_{4B}^-}\,dx\\
&\quad\leqslant c\fint_{\kappa B} |\beta|^{p_{4B}^-}v^{\beta-p_{4B}^-}|\nabla u|^{p_{4B}^-}\eta^{p_{4B}^+}\,dx + c\fint_{\kappa B} v^{\beta}\eta^{p_{4B}^+-p_{4B}^-}|\nabla\eta|^{p_{4B}^-}\,dx\\
&\quad\leqslant c|\beta|^{p_{4B}^-}\fint_{\kappa B}\big(\eta^{p_{4B}^+}v^{\beta-p_{4B}^-} + v^{\beta-p_{4B}^-+p(x)}|\nabla\eta|^{p(x)}\big)\,dx + c\fint_{\kappa B} v^{\beta}|\nabla\eta|^{p_{4B}^-}\,dx\\
&\quad\leqslant c(1+|\beta|)^{p_{4B}^+}\bigg[\fint_{\kappa B} v^{\beta-p_{4B}^-}\,dx + \fint_{\kappa B} v^{\beta-p_{4B}^-+p(x)}|\nabla\eta|^{p(x)}\,dx + \fint_{\kappa B} v^{\beta}|\nabla\eta|^{p_{4B}^-}\,dx\bigg].
\end{aligned}
$$

Now the goal is to estimate each integrals in the brackets by

$$
\Big(\fint_{\kappa B} v^{q\beta}\,dx\Big)^{1/q}.
$$

The first integral can be estimated with Hölder's inequality. Since $v^{-p_{4B}^-}\leqslant R^{-p_{4B}^-}$, we have

$$
\fint_{\kappa B} v^{\beta-p_{4B}^-}\,dx \leqslant R^{-p_{4B}^-}\Big(\fint_{\kappa B} v^{q\beta}\,dx\Big)^{1/q}.
$$

By (13.4.10), Hölder's inequality and Lemma 13.4.2 for the second integral we have

$$
\begin{aligned}
&\fint_{\kappa B} v^{\beta-p_{4B}^-+p(x)}|\nabla\eta|^{p(x)}\,dx\\
&\quad\leqslant cR^{-p_{4B}^-}\Big(\frac{\kappa}{\kappa-\tau}\Big)^{p_{4B}^+}\fint_{\kappa B} v^{\beta-p_{4B}^-+p(x)}\,dx\\
&\quad\leqslant cR^{-p_{4B}^-}\Big(\frac{\kappa}{\kappa-\tau}\Big)^{p_{4B}^+}\Big(\fint_{\kappa B} v^{q'(p(x)-p_{4B}^-)}\,dx\Big)^{1/q'}\Big(\fint_{\kappa B} v^{q\beta}\,dx\Big)^{1/q}\\
&\quad\leqslant cR^{-p_{4B}^-}\Big(\frac{\kappa}{\kappa-\tau}\Big)^{p_{4B}^+}\Big(1+\|v\|_{L^{q's}(4B)}^{q'(p_{4B}^+-p_{4B}^-)}\Big)^{1/q'}\Big(\fint_{\kappa B} v^{q\beta}\,dx\Big)^{1/q}.
\end{aligned}
$$

Finally, for the third integral we obtain the estimate by Hölder's inequality.

Now we have arrived at the inequality

$$\fint_{\kappa B} \Big|\nabla\Big(v^{\frac{\beta}{p^-_{4B}}}\eta^{\frac{p^+_{4B}}{p^-_{4B}}}\Big)\Big|^{p^-_{4B}}\,dx \leqslant c(1+|\beta|)^{p^+_{4B}} R^{-p^-_{4B}}\Big(\frac{\kappa}{\kappa-\tau}\Big)^{p^+_{4B}}\Big(\fint_{\kappa B} v^{q\beta}\,dx\Big)^{1/q},$$

where the term $1+\|v\|^{q'(p^+_{4B}-p^-_{4B})}_{L^{q's}(4B)}$ is inside the constant c.

By the constant exponent Sobolev inequality

$$\Big(\fint_{\kappa B}|u|^{n'p^-_{4B}}\,dx\Big)^{\frac{1}{n'p^-_{4B}}} \leqslant cR\Big(\fint_{\kappa B}|\nabla u|^{p^-_{4B}}\,dx\Big)^{\frac{1}{p^-_{4B}}},$$

where $u\in W^{1,p^-_{4B}}_0(\kappa B)$, we obtain

$$\begin{aligned}\Big(\fint_{\tau B} v^{\beta n'}\,dx\Big)^{\frac{n-1}{n}} &\leqslant \Big(c\fint_{\kappa B}\Big(v^{\frac{\beta}{p^-_{4B}}}\eta^{\frac{p^+_{4B}}{p^-_{4B}}}\Big)^{n'p^-_{4B}}\,dx\Big)^{\frac{n-1}{n}}\\ &\leqslant cR^{p^-_{4B}}\fint_{\kappa B}\Big|\nabla\Big(v^{\frac{\beta}{p^-_{4B}}}\eta^{\frac{p^+_{4B}}{p^-_{4B}}}\Big)\Big|^{p^-_{4B}}\,dx\\ &\leqslant c(1+|\beta|)^{p^+_{4B}}\Big(\frac{\kappa}{\kappa-\tau}\Big)^{p^+_{4B}}\Big(\fint_{\kappa B} v^{q\beta}\,dx\Big)^{1/q},\end{aligned}$$

where the term $1+\|v\|^{q'(p^+_{4B}-p^-_{4B})}_{L^{q's}(4B)}$ is inside the constant c. The claim follows from this since β is a negative number. □

The next lemma is the crucial passage from positive exponents to negative exponents in the Moser iteration scheme.

Lemma 13.4.11. *Let $p\in\mathcal{P}^{\log}(\Omega)$ with $1<p^-\leqslant p^+<\infty$. Assume that u is a nonnegative supersolution in $4B\subset\subset\Omega$ and $s>p^+_{4B}-p^-_{4B}$. Then there exist constants $q_0>0$ and c depending on n, p^-, p^+, $c_{\log}(p)$ and $L^s(4B)$-norm of u such that*

$$\Phi(v,q_0,2B)\leqslant c\Phi(v,-q_0,2B).$$

Proof. Choose a ball B' with a diameter r such that $2B'\subset 4B$ and a cutoff function $\eta\in C^\infty_0(2B')$ such that $\eta=1$ in B' and $|\nabla\eta|\leqslant c/r$. Taking $E=B'$ and $\gamma=1-p^-_{B'}$ in Caccioppoli estimate, Lemma 13.4.8, we have

$$\fint_{B'}|\nabla\log v|^{p^-_{B'}}\,dx\leqslant c\Big(\fint_{2B'} v^{-p^-_{B'}}+\fint_{2B'} v^{p(x)-p^-_{B'}}r^{-p(x)}\,dx\Big).$$

Using the estimate $v^{-p^-_{B'}} \leqslant R^{-p^-_{B'}} \leqslant cr^{-p^-_{B'}}$, the log-Hölder continuity of p and Lemma 13.4.2 we find that

$$\fint_{B'} |\nabla \log v|^{p^-_{B'}}\,dx \leqslant c\Big(r^{-p^-_{B'}} + r^{-p^-_{2B'}} \fint_{2B'} v^{p(x)-p^-_{B'}}\,dx\Big)$$
$$\leqslant c\big(r^{-p^-_{B'}} + r^{-p^-_{2B'}}(1 + \|v\|^{p^+_{4B}-p^-_{4B}}_{L^s(4B)}\big).$$

Let $f := \log v$. By the constant exponent Poincaré inequality, Hölder's inequality and the above estimate we obtain

$$\begin{aligned}\fint_{B'} |f - f_{B'}|\,dx &\leqslant \Big(r^{p^-_{B'}} \fint_{B'} |\nabla f|^{p^-_{B'}}\,dx\Big)^{1/p^-_{B'}} \\ &\leqslant c\big(1 + r^{p^-_{B'}-p^-_{2B'}}(1 + \|v\|^{p^+_{4B}-p^-_{4B}}_{L^s(4B)}\big)^{1/p^-_{B'}}.\end{aligned} \tag{13.4.12}$$

Note that $p^-_{B'} \geqslant p^-_{2B'}$ since $B' \subset 2B'$, so that the right-hand side is bounded (and $f \in \mathrm{BMO}(2B)$).

Since the BMO-estimate (13.4.12) holds for all balls $B' \subset 4B$, the measure theoretic John-Nirenberg lemma (see for example [219, Corollary 19.10, p. 371 in Dover's edition] or [280, Theorem 1.66, p. 40]) implies that there exist positive constants c_3 and c_4 depending on the right-hand side of (13.4.12) such that

$$\fint_{2B} e^{c_3|f-f_{2B}|}\,dx \leqslant c_4.$$

Using this we can conclude that

$$\Big(\fint_{2B} e^{c_3 f}\,dx\Big)\Big(\fint_{2B} e^{-c_3 f}\,dx\Big) = \Big(\fint_{2B} e^{c_3(f-f_{2B})}\,dx\Big)\Big(\fint_{2B} e^{-c_3(f-f_{2B})}\,dx\Big)$$
$$\leqslant \Big(\fint_{2B} e^{c_3|f-f_{2B}|}\,dx\Big)^2 \leqslant c_4^2,$$

which implies that

$$\Big(\fint_{2B} v^{c_3}\,dx\Big)^{1/c_3} = \Big(\fint_{2B} e^{c_3 f}\,dx\Big)^{1/c_3} \leqslant c_4^{2/c_3}\Big(\fint_{2B} e^{-c_3 f}\,dx\Big)^{-1/c_3}$$
$$= c_4^{2/c_3}\Big(\fint_{2B} v^{-c_3}\,dx\Big)^{-1/c_3},$$

so that we can take $q_0 = c_3$. □

Note that the exponent q_0 in Lemma 13.4.11 also depends on the $L^s(4B)$-norm of u. More precisely, the constant c_3 obtained from the John-Nirenberg lemma is a universal constant divided by the right-hand side of (13.4.12). Thus

$$q_0 = \frac{c}{c' + \|u\|_{L^s(4B)}^{p_{4B}^+ - p_{4B}^-}}.$$

Theorem 13.4.13 (The weak Harnack inequality). *Let $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p^- \leqslant p^+ < \infty$. Let B be a ball with a radius $R \leqslant 1$ such that $4B \subset\subset \Omega$. Assume that u is a nonnegative* A*-supersolution in $4B \subset\subset \Omega$ and $s > p_{4B}^+ - p_{4B}^-$. Then*

$$\Big(\fint_{B_{2R}} u^{q_0}\, dx \Big)^{1/q_0} \leqslant c\, (\operatorname*{ess\,inf}_{B_R} u(x) + R),$$

where q_0 is the exponent from Lemma 13.4.11 and c depends on n, p^-, p^+, $c_{\log}(p)$, q, $L^{ns}(4B)$-norm of u and the structural constants c_1 and c_2.

Since the exponent p is uniformly continuous, we can take for example $ns = p_\Omega^-$ by choosing R small enough. Thus the constants in the estimates are finite for all supersolutions u in a scale that depends only on p.

Proof. By making s slightly smaller if necessary, we may assume that there exists $q \in (1, n')$ such that $\|u\|_{L^{q's}(4B)} < \infty$. Let q_0 be as in the previous lemma, and assume without loss of generality that $q_0 < 1$.

Let $1 \leqslant \tau < \kappa \leqslant 3$, $r_j := \tau + 2^{-j}(\kappa - \tau)$ and

$$\xi_j := -\Big(\frac{n'}{q}\Big)^j q_0$$

for $j = 0, 1, 2, \ldots$ By Lemma 13.4.9 with $\beta = \frac{\xi_j}{q}$, we have

$$\Phi(v, \xi_j, r_j B) \leqslant c^{\frac{q}{|\xi_j|}} (1 + |\xi_j|)^{\frac{q p_{4B}^+}{|\xi_j|}} \Big(\frac{r_j}{r_j - r_{j+1}}\Big)^{\frac{q p_{4B}^+}{|\xi_j|}} \Phi(v, \xi_{j+1}, r_{j+1} B).$$

Iterating this inequality, and observing that $1 + |\xi_j| \leqslant 2(\frac{n'}{q})^j$ since $q_0 \leqslant 1$, we obtain

$$\begin{aligned} \Phi(v, -q_0, \kappa B) &\leqslant \prod_{j=0}^{\infty} c^{\frac{q}{|\xi_j|}} (1 + |\xi_j|)^{\frac{q p_{4B}^+}{|\xi_j|}} \Big(\frac{r_j}{r_j - r_{j+1}}\Big)^{\frac{q p_{4B}^+}{|\xi_j|}} \operatorname*{ess\,inf}_{x \in \tau B} v(x) \\ &\leqslant c^{\sum_{j=0}^\infty \frac{q}{|\xi_j|}} \Big(\frac{2n'}{q}\Big)^{\sum_{j=0}^\infty j \frac{q p_{4B}^+}{|\xi_j|}} \Big(\frac{\kappa}{\kappa - \tau}\Big)^{\sum_{j=0}^\infty \frac{q p_{4B}^+}{|\xi_j|}} \operatorname*{ess\,inf}_{x \in \tau B} v(x). \end{aligned}$$

All the series in the sums converge by the root test, so

$$\Phi(v, -q_0, \kappa B) \leqslant c \operatorname*{ess\,inf}_{x \in \tau B} v(x).$$

Next we choose $\tau := 1$ and $\kappa := 2$ and use Lemma 13.4.11 to get the claim. □

Combining Theorems 13.4.5 and 13.4.13 we obtain the following theorem.

Theorem 13.4.14 (Harnack's inequality). *Let $p \in \mathcal{P}^{\log}(\Omega)$ with $1 < p^- \leqslant p^+ < \infty$. Let B be a ball with a radius $R \leqslant 1$ such that $4B \subset\subset \Omega$. Let u be a non-negative A-solution in $4B$ and $s > p^+_{4B} - p^-_{4B}$. Then*

$$\operatorname*{ess\,sup}_{x \in B} u(x) \leqslant c \left(\operatorname*{ess\,inf}_{x \in B} u(x) + R \right),$$

where the constant c depends on n, p^-, p^+, $c_{\log}(p)$, the $L^{ns}(4B)$-norm of u and the structural constants c_1 and c_2.