Chapter 6
Chaos in Discontinuous Differential Equations

This chapter is devoted to proving chaos for periodically perturbed piecewise smooth ODEs. We study two cases: firstly, when the homoclinic orbit of the unperturbed piecewise smooth ODE transversally crosses the discontinuity surface, and secondly, when a part of homoclinic orbit is sliding on the discontinuity surface.

6.1 Transversal Homoclinic Bifurcation

6.1.1 Discontinuous Differential Equations

DDEs occur in several situations such as in mechanical systems with dry frictions or with impacts or in control theory, electronics, economics, medicine and biology [1–8]. Recently attempts have been made to extend the theory of chaos to differential equations with discontinuous right-hand sides. For examples, planar discontinuous differential equations are investigated in [9, 10], piecewise linear three-dimensional discontinuous differential equations are investigated in [11, 12] and weakly discontinuous systems are studied in [13–15]. Melnikov type analysis is also presented for DDEs in [16–21]. An overview of some aspects of chaotic dynamics in hybrid systems is given in [22]. A survey of controlling chaotic differential equations is presented in [23]. The switchability of flows of general DDEs is discussed in [24–26]. Planar discontinuous differential equations are investigated in [27, 28] using analytic and numeric approaches. Periodic and almost periodic solutions of DDEs are considered in [29–33].

In [34] bifurcations of bounded solutions from homoclinic orbits are investigated for time perturbed discontinuous differential equations in any finite dimensional space. We anticipated that under the conditions of [34] not only the existence of bounded solutions on $\mathbb{R}$, but also chaotic solutions could occur. The purpose of this section is to justify this conjecture about the existence of chaotic solutions. To handle this kind of problem one has to face the new problem that stable and unstable

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manifolds may only be Lipschitz in the state variable, even if they are possibly smooth with respect to parameters. So it is not clear what the notion of transverse intersection of invariant manifolds would be.

6.1.2 Setting of the Problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set in $\mathbb{R}^n$ and $G(z)$ be a C^r-function on $\bar{\Omega}$, with $r \geq 2$. We set

$$\Omega_{\pm} = \{z \in \Omega \mid \pm G(z) > 0\}, \quad \Omega_0 := \{z \in \Omega \mid G(z) = 0\}.$$

Let $f_{\pm}(z) \in C_b^r(\bar{\Omega}_{\pm})$ and $g \in C_b^r(\mathbb{R} \times \bar{\Omega} \times \mathbb{R})$, i.e. $f_{\pm}$ and g have uniformly bounded derivatives up to the r-th order on $\bar{\Omega}_{\pm}$ and $\mathbb{R} \times \bar{\Omega} \times \mathbb{R}$, respectively. We also assume that the r-th order derivatives of $f_{\pm}$ and g are uniformly continuous. Let $\varepsilon_0 \in (0,1)$. Throughout this section ε will denote a real parameter so that $|\varepsilon| \leq \varepsilon_0$. Particularly ε is bounded.

Remark 6.1.1. For technical purposes, we C_b^r-smoothly extend $f_{\pm}$ on $\mathbb{R}^n$, g on $\mathbb{R}^{n+2}$ and $\gamma_{\pm}$, γ_0 on $\mathbb{R}$ in such a way that

$$\sup\{|f_{\pm}(z)| \mid z \in \mathbb{R}^n\} \leq 2 \sup\{|f_{\pm}(z)| \mid z \in \bar{\Omega}_{\pm}\},$$
$$\sup\{|g(t,z,\varepsilon)| \mid (t,z,\varepsilon) \in \mathbb{R}^{n+2}\} \leq 2 \sup\{|g(t,z,\varepsilon)| \mid t \in \mathbb{R}, z \in \bar{\Omega}, |\varepsilon| \leq \varepsilon_0\}.$$

We also assume that up to the r-th order all the derivatives of the extended $f_{\pm}$ and g are uniformly continuous and continue to keep the same notations for extended mappings and functions.

We say that a function $z(t)$ is a solution of the equation

$$\dot{z} = f_{\pm}(z) + \varepsilon g(t,z,\varepsilon), \quad z \in \bar{\Omega}_{\pm}, \tag{6.1.1}$$

if it is continuous, piecewise C^1 satisfies Eq. (6.1.1) on $\Omega_{\pm}$ and, moreover, the following holds: if for some t_0 we have $z(t_0) \in \Omega_0$, then there exists $r > 0$ so that for any $t \in (t_0 - r, t_0 + r)$ with $t \neq t_0$, we have $z(t) \in \Omega_- \cup \Omega_+$. Moreover, if, for example, $z(t) \in \Omega_-$ for any $t \in (t_0 - r, t_0)$, then the left derivative of $z(t)$ at $t = t_0$ satisfies: $\dot{z}(t_0^-) = f_-(z(t_0)) + \varepsilon g(t_0, z(t_0), \varepsilon)$; similarly, if $z(t) \in \Omega_-$ for any $t \in (t_0, t_0 + r)$, then $\dot{z}(t_0^+) = f_-(z(t_0)) + \varepsilon g(t_0, z(t_0), \varepsilon)$. A similar meaning is assumed when $z(t) \in \Omega_+$ for either $t \in (t_0 - r, t_0)$ or $t \in (t_0, t_0 + r)$. Note that since $z(t) \notin \Omega_0$ for $t \in (t_0 - r, t_0 + r) \setminus \{t_0\}$ we have either $z(t) \in \Omega_-$ or $z(t) \in \Omega_+$ when $t \in (t_0 - r, t_0)$ or $t \in (t_0, t_0 + r)$.

We assume (Figure 6.1) that

(H1) For $\varepsilon = 0$ Eq. (6.1.1) has the hyperbolic equilibrium $x = 0 \in \Omega_-$ and a continuous (not necessarily C^1) solution $\gamma(t)$ which is homoclinic to $x = 0$ and consists of three branches

$$\gamma(t) = \begin{cases} \gamma_-(t), \text{ if } t \le -\bar{T}, \\ \gamma_0(t), \text{ if } -\bar{T} \le t \le \bar{T}, \\ \gamma_+(t), \text{ if } t \ge \bar{T}, \end{cases}$$

where $\gamma_\pm(t) \in \Omega_-$ for $|t| > \bar{T}$, $\gamma_0(t) \in \Omega_+$ for $|t| < \bar{T}$ and

$$\gamma_-(-\bar{T}) = \gamma_0(-\bar{T}) \in \Omega_0, \quad \gamma_+(\bar{T}) = \gamma_0(\bar{T}) \in \Omega_0.$$

(H2) It results: $G'(\gamma(-\bar{T}))f_\pm(\gamma(-\bar{T})) > 0$ and $G'(\gamma(\bar{T}))f_\pm(\gamma(\bar{T})) < 0$.

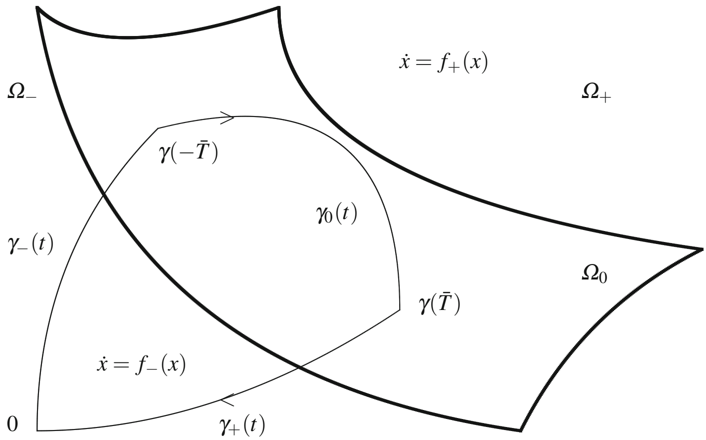


Fig. 6.1 Transversal homoclinic cycle $\gamma(t)$ of $\dot{x} = f_\pm(x)$.

According to (H1) and because of roughness of exponential dichotomies the linear systems $\dot{x} = f'_-(\gamma_-(t))x$ and $\dot{x} = f'_-(\gamma_+(t))x$ have exponential dichotomies on $(-\infty, -\bar{T}]$ and $[\bar{T}, \infty)$ respectively, that is, projections $P_\pm : \mathbb{R}^n \to \mathbb{R}^n$ and positive numbers $k \ge 1$ and $\delta > 0$ exist so that the following hold:

$$\begin{aligned} &\|X_-(t)P_-X_-^{-1}(s)\| \le k\mathrm{e}^{-\delta(t-s)}, && \text{if } s \le t \le -\bar{T}, \\ &\|X_-(t)(\mathbb{I}-P_-)X_-^{-1}(s)\| \le k\mathrm{e}^{\delta(t-s)}, && \text{if } t \le s \le -\bar{T}, \\ &\|X_+(t)P_+X_+^{-1}(s)\| \le k\mathrm{e}^{-\delta(t-s)}, && \text{if } \bar{T} \le s \le t, \\ &\|X_+(t)(\mathbb{I}-P_+)X_+^{-1}(s)\| \le k\mathrm{e}^{\delta(t-s)}, && \text{if } \bar{T} \le t \le s, \end{aligned} \tag{6.1.2}$$

where $X_-(t)$ and $X_+(t)$ are the fundamental matrices of the linear systems $\dot{x} = f'_-(\gamma_-(t))x$ and $\dot{x} = f'_-(\gamma_+(t))x$, respectively, so that $X_-(-\bar{T}) = X_+(\bar{T}) = \mathbb{I}$. Later in this section we will need to extend the validity of (6.1.2) to a larger set of values of s,t. So, let us take, for example, $u(t) = X_+(t)(\mathbb{I}-P_+)X_+^{-1}(s)$, with $\bar{T} \le s \le t \le s+2$. Then,

$$u(t) = u(s) + \int_s^t f'_-(\gamma_+(\tau))u(\tau)\, d\tau$$

and hence (using also $|u(s)| \le k$ (see (6.1.2))

$$|u(t)| \le k + K_- \int_s^t |u(\tau)|\, d\tau$$

where $K_- = \sup\{f'_-(\gamma_+(t)) \mid t \ge \bar{T}\}$. From Gronwall inequality (cf Section 2.5.1) we obtain:

$$|X_+(t)(\mathbb{I} - P_+)X_+^{-1}(s)| \le k\mathrm{e}^{K_-(t-s)} \le \hat{k}\,\mathrm{e}^{\delta(t-s)}, \quad \text{if } \bar{T} \le s \le t \le s+2,$$

where, for example, $\hat{k} = k\max\{1, \mathrm{e}^{2(K_- - \delta)}\}$. By similar arguments we prove that possibly replacing k with a larger value:

$$\begin{aligned}
&\|X_-(t)P_-X_-^{-1}(s)\| \le k\mathrm{e}^{-\delta(t-s)}, && \text{if } s-2 \le s,t \le -\bar{T},\\
&\|X_-(t)(\mathbb{I} - P_-)X_-^{-1}(s)\| \le k\mathrm{e}^{\delta(t-s)}, && \text{if } t-2 \le s,t \le -\bar{T},\\
&\|X_+(t)P_+X_+^{-1}(s)\| \le k\mathrm{e}^{-\delta(t-s)}, && \text{if } \bar{T} \le s,t \le t+2,\\
&\|X_+(t)(\mathbb{I} - P_+)X_+^{-1}(s)\| \le k\mathrm{e}^{\delta(t-s)}, && \text{if } \bar{T} \le s,t \le s+2.
\end{aligned} \tag{6.1.3}$$

We now state our third assumption. It is a kind of nondegeneracy condition of the homoclinic orbit $\gamma(t)$ with respect to $\dot{x} = f_\pm(x)$, that reduces to the known notion of nondegeneracy in the smooth case [35, 36]. This is discussed in more detail in Section 6.1.3.

Let $R_0 : \mathbb{R}^n \to \mathbb{R}^n$ be the projection onto $\mathscr{N}G'(\gamma(\bar{T}))$ along the direction of $\dot{\gamma}_0(\bar{T})$, i. e.

$$R_0 w = w - \frac{G'(\gamma(\bar{T}))w}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})}\dot{\gamma}_0(\bar{T})$$

and $X_0(t)$ be the fundamental solution of the linear system $\dot{z} = f'_+(\gamma_0(t))z$, $-\bar{T} \le t \le \bar{T}$, satisfying $X_0(-\bar{T}) = \mathbb{I}$. Then let

$$\mathscr{S}' = \mathscr{N}P_- \cap \mathscr{N}G'(\gamma(-\bar{T})) \quad \text{and} \quad \mathscr{S}'' = \mathscr{R}P_+ \cap \mathscr{N}G'(\gamma(\bar{T}))\,.$$

Since $\dot{\gamma}_-(-\bar{T}) \notin \mathscr{N}G'(\gamma(-\bar{T}))$, $\dim \mathscr{N}G'(\gamma(-\bar{T})) = n-1$ and $\dot{\gamma}_-(-\bar{T}) \in \mathscr{N}P_-$, we have $\dim[\mathscr{N}P_- + \mathscr{N}G'(\gamma(-\bar{T}))] = n$ and hence:

$$\begin{aligned}
\dim \mathscr{S}' &= \dim[\mathscr{N}P_- \cap \mathscr{N}G'(\gamma(-\bar{T}))]\\
&= \dim \mathscr{N}P_- + \dim \mathscr{N}G'(\gamma(-\bar{T})) - n = \dim \mathscr{N}P_- - 1\,.
\end{aligned}$$

Similarly, from $\dot{\gamma}_+(\bar{T}) \notin \mathscr{N}G'(\gamma(\bar{T}))$, $\dot{\gamma}_+(\bar{T}) \in \mathscr{R}P_+$ and $\dim \mathscr{N}G'(\gamma(\bar{T})) = n-1$, we see that

$$\begin{aligned}
\dim \mathscr{S}'' &= \dim[\mathscr{R}P_+ \cap \mathscr{N}G'(\gamma(\bar{T}))]\\
&= \dim \mathscr{R}P_+ + \dim \mathscr{N}G'(\gamma(\bar{T})) - n = \dim \mathscr{R}P_+ - 1\,.
\end{aligned}$$

We assume that the following condition holds:

(H3) $\mathscr{S}'' + R_0[X_0(\bar{T})\mathscr{S}']$ has codimension 1 in $\mathscr{R}R_0$.

Lemma 6.1.2. *From (H3), the linear subspaces $\mathscr{S}''$ and $\mathscr{S}''' = R_0[X_0(\bar{T})\mathscr{S}']$ intersect transversally in $\mathscr{R}R_0$. Moreover, we have* $\dim \mathscr{S}''' = \dim \mathscr{S}'$.

Proof. We have $\dim \mathscr{S}''' \le \dim \mathscr{S}' = \dim \mathscr{N}P_- - 1$. Moreover from (H3) we get $\dim[\mathscr{S}'' + \mathscr{S}'''] = n-2$, and then:

$$\begin{aligned}\dim\left[\mathscr{S}'' \cap \mathscr{S}'''\right] &= \dim \mathscr{S}'' + \dim \mathscr{S}''' - \dim\left[\mathscr{S}'' + \mathscr{S}'''\right]\\ &\le \dim \mathscr{R}P_+ - 1 + \dim \mathscr{N}P_- - 1 - (n-2) = \dim \mathscr{R}P_+ + \dim \mathscr{N}P_- - n = 0\,.\end{aligned}$$

So the inequality is an equality and $\dim \mathscr{S}''' = \dim \mathscr{S}'$. The proof is finished. □

According to Lemma 6.1.2, we have a unitary vector $\psi \in \mathscr{R}R_0$ so that

$$\mathbb{R}^n = \operatorname{span}\{\psi\} \oplus \mathscr{N}R_0 \oplus \mathscr{S}'' \oplus \mathscr{S}''' \tag{6.1.4}$$

and

$$\langle \psi, v\rangle = 0, \quad \text{for any } v \in \mathscr{S}'' \oplus \mathscr{S}'''. \tag{6.1.5}$$

The main result of this section is the following:

Theorem 6.1.3. *Assume that $f_\pm(z)$ and $g(t,z,\varepsilon)$ are C^2–functions with bounded derivatives and that their second order derivatives are uniformly continuous. Let conditions (H1), (H2) and (H3) hold. Then there exists a C^2-function $\mathscr{M}(\alpha)$ of the real variable α so that if $\mathscr{M}(\alpha^0) = 0$ and $\mathscr{M}'(\alpha^0) \neq 0$ for some $\alpha^0 \in \mathbb{R}$, then the following hold: there exist $\rho > 0$, $\widetilde{c}_1 > 0$ and $\widetilde{\varepsilon} > 0$ so that for any $0 \neq \varepsilon \in (-\widetilde{\varepsilon}, \widetilde{\varepsilon})$, there exists $\nu_\varepsilon \in (0, |\varepsilon|)$ (cf* (6.1.91)*) so that for any increasing sequence $\mathscr{T} = \{T_m\}_{m\in\mathbb{Z}}$ that satisfies*

$$T_{m+1} - T_m > \bar{T} + 1 - 2\delta^{-1}\ln|\varepsilon| \textit{ for any } m \in \mathbb{Z}$$

along with the following recurrence condition

$$|g(t+T_{2m}, z, 0) - g(t,z,0)| < \nu_\varepsilon \quad \textit{for any } (t,z,m) \in \mathbb{R}^{n+1}\times\mathbb{Z}, \tag{6.1.6}$$

there exist unique sequences $\hat{\alpha} = \{\hat{\alpha}_m\}_{m\in\mathbb{Z}}, \hat{\beta} = \{\hat{\beta}_m\}_{m\in\mathbb{Z}} \in \ell^\infty(\mathbb{R})$ (depending on $\mathscr{T}$ and ε, i.e. $\hat{\alpha} = \hat{\alpha}_{\mathscr{T}}(\varepsilon)$, $\hat{\beta} = \hat{\beta}_{\mathscr{T}}(\varepsilon)$) so that $\sup_{m\in\mathbb{Z}} |\hat{\alpha}_m - \alpha^0| < \widetilde{c}_1|\varepsilon|$, $\sup_{m\in\mathbb{Z}} |\hat{\beta}_m - \alpha^0| < \widetilde{c}_1|\varepsilon|$ *and a unique solution $z(t,\mathscr{T},\varepsilon)$ of Eq.* (6.1.1) *satisfying*

$$\begin{aligned}&\sup\nolimits_{t\in[T_{2m-1}+\hat{\beta}_{m-1}, T_{2m}-\bar{T}+\hat{\alpha}_m]} |z(t) - \gamma_-(t - T_{2m} - \hat{\alpha}_m)| < \rho\,,\\ &\sup\nolimits_{t\in[T_{2m}-\bar{T}+\hat{\alpha}_m, T_{2m}+\bar{T}+\hat{\beta}_m]} |z(t) - \gamma_0(t - T_{2m} - \hat{\alpha}_m)| < \rho\,,\\ &\sup\nolimits_{t\in[T_{2m}+\bar{T}+\hat{\beta}_m, T_{2m+1}+\hat{\beta}_m]} |z(t) - \gamma_+(t - T_{2m} - \hat{\beta}_m)| < \rho\,.\end{aligned} \tag{6.1.7}$$

We conclude this section with a remark on the projections of the dichotomies of the systems $\dot{x} = f'(\gamma_{\pm}(t))x$ on $[\bar{T},\infty)$ and $(-\infty,-\bar{T}]$:

$$P_{\pm}(t) = X_{\pm}(\pm t)P_{\pm}X_{\pm}^{-1}(\pm t). \tag{6.1.8}$$

Let P_0 be the projection of the dichotomy of the linear system $\dot{x} = f'(0)x$ on $\mathbb{R}$. We have (see Lemma 2.5.1) $\lim_{t\to\infty}\|P_{\pm}(t)-P_0\| = 0$. Thus $T > \bar{T}$ exists so that

$$\mathscr{N}P_{+}(t') \oplus \mathscr{R}P_{-}(t'') = \mathbb{R}^n \quad \text{for any } t',t'' \geq T. \tag{6.1.9}$$

We prove that a positive constant $\widetilde{c}$ exists so that

$$\max\{|x_+|,|x_-|\} \leq \widetilde{c}|x_+ + x_-| \quad \forall (x_+,x_-) \in \mathscr{N}P_{+}(t') \times \mathscr{R}P_{-}(t''). \tag{6.1.10}$$

Since it is clear that $|x_+ + x_-| \leq 2\max\{|x_+|,|x_-|\}$ we get, then, that the two norms $|x_+ + x_-|$ and $\max\{|x_+|,|x_-|\}$ are equivalent. To prove the statement (6.1.10) take $0 < \upsilon < 1/2$ and fix $T > \bar{T}$ so that for any $t',t'' \geq T > \bar{T}$ we have

$$\|P_0 - P_{+}(t')\| \leq \upsilon, \quad \|P_0 - P_{-}(t'')\| \leq \upsilon\,.$$

Next consider a linear mapping $A_\upsilon : \mathbb{R}^n \mapsto \mathbb{R}^n$ given by

$$A_\upsilon z := (\mathbb{I} - P_{+}(t'))z + P_{-}(t'')z\,.$$

Note that

$$A_\upsilon z = z - \left[(P_{+}(t') - P_0) + (P_0 - P_{-}(t'')\right] z\,.$$

Since $\|(P_{+}(t') - P_0) + (P_0 - P_{-}(t'')\| \leq 2\upsilon < 1$, A_υ is invertible and

$$\|A_\upsilon\| \leq 1 + 2\upsilon, \quad \|A_\upsilon^{-1}\| \leq 1/(1-2\upsilon)\,.$$

So for any $x \in \mathbb{R}^n$ there is a unique $z \in \mathbb{R}^n$ so that

$$x = A_\upsilon z = x_+ + x_-$$

where $x_+ = (\mathbb{I} - P_{+}(t'))z \in \mathscr{N}P_{+}(t')$ and $x_- = P_{-}(t'')z \in \mathscr{R}P_{-}(t'')$. Then

$$|x_+| \leq \|\mathbb{I} - P_{+}(t')\||z| \leq \|\mathbb{I} - P_{+}(t')\|\|A_\upsilon^{-1}\||x| \leq \frac{\|\mathbb{I} - P_0\| + \upsilon}{1-2\upsilon}|x|\,,$$
$$|x_-| \leq \|P_{-}(t'')\||z| \leq \|P_{-}(t'')\|\|A_\upsilon^{-1}\||x| \leq \frac{\|P_0\| + \upsilon}{1-2\upsilon}|x|\,.$$

This proves (6.1.10) with, for example,

$$\widetilde{c} = \frac{\max\{\|\mathbb{I} - P_0\| + \upsilon, \|P_0\| + \upsilon\|\}}{1-2\upsilon} \leq \frac{1 + \|P_0\| + \upsilon}{1-2\upsilon} \leq 2(1 + \|P_0\|)$$

for $\upsilon \leq \frac{1+\|P_0\|}{1+4(1+\|P_0\|)} < \frac{1}{2}$.

6.1.3 Geometric Interpretation of Nondegeneracy Condition

Now we present a geometric meaning of condition (H3). For any $x \in \Omega_0$ near $\gamma(-\bar{T})$ we consider the solution $\phi_-(t,x)$ of $\dot{x} = f_-(x)$ and the solution $\phi_0(t,x)$ of $\dot{x} = f_+(x)$ so that $\phi_-(-\bar{T},x) = \phi_0(-\bar{T},x) = x$, respectively. Similarly, for any $\widetilde{x} \in \Omega_0$ near $\gamma(\bar{T})$ we take a solution $\phi_+(t,\widetilde{x})$ of $\dot{x} = f_-(x)$ so that $\phi_+(\bar{T},\widetilde{x}) = \widetilde{x}$.

By the implicit function theorem, for any $x \in \Omega_0$ near $x_0 := \gamma(-\bar{T})$ there is a unique time $\tau(x)$ so that

$$G(\phi_0(\tau(x),x)) = 0, \quad \tau(x_0) = \bar{T}\,. \tag{6.1.11}$$

In summary, for any $x \in \Omega_0$ near x_0, we have constructed a solution $\phi(t,x)$ of $\dot{x} = f_\pm(x)$ defined as

$$\phi(t,x) = \begin{cases} \phi_-(t,x), & \text{for } t \le -\bar{T}\,, \\ \phi_0(t,x), & \text{for } -\bar{T} \le t \le \tau(x)\,, \\ \phi_+(t-\tau(x)+\bar{T}, \phi_0(\tau(x),x)), & \text{for } \tau(x) \le t\,. \end{cases}$$

We recall the following properties of the function $\phi(t,x)$:

$$\begin{aligned} &\phi_-(t,\gamma(-\bar{T})) = \gamma_-(t), \text{ for } \ t \le -\bar{T}, \\ &\phi_0(t,\gamma(-\bar{T})) = \gamma_0(t), \ \ \text{for } \ -\bar{T} \le t \le \bar{T}, \\ &\phi_+(t,\gamma(\bar{T})) = \gamma_+(t), \ \ \text{for } \ t \ge \bar{T}, \\ &\phi_0(\tau(x),x) \in \Omega_0, \qquad \text{for any } \ x \in \Omega_0 \text{ (near } \gamma(-\bar{T})) \end{aligned} \tag{6.1.12}$$

and note that from (6.1.12) we get, for any $\eta \in \mathcal{N}G'(\gamma(\bar{T}))$:

$$\left[\frac{\partial \phi_0}{\partial x}(\bar{T},x_0) + \dot{\phi}_0(\bar{T},x_0)\tau'(x_0)\right]\eta \in \mathcal{N}G'(\gamma(\bar{T})) = \mathscr{R}R_0\,. \tag{6.1.13}$$

We are interested in the linearization $\widetilde{\phi}(t) := \frac{\partial \phi}{\partial x}(t,x_0)\eta$ of $\phi(t,x)$ at $x = x_0$ along $\eta \in \mathcal{N}G'(\gamma(-\bar{T})) = T_{\gamma(-\bar{T})}\Omega_0$ that is using $\phi_\pm(\pm\bar{T},x) = x$, $\phi_0(-\bar{T},x) = x$ and (6.1.12):

$$\widetilde{\phi}(t) = \begin{cases} X_-(t)\eta, & t \le -\bar{T}\,, \\ X_0(t)\eta, & -\bar{T} \le t \le \bar{T}\,, \\ \tilde{X}_+(t)\eta, & \bar{T} < t\,, \end{cases}$$

where

$$\begin{aligned} \tilde{X}_+(t) =& \frac{\partial \phi_+}{\partial x}(t,\gamma(\bar{T}))\left[\dot{\phi}_0(\bar{T},x_0)\tau'(x_0) + \frac{\partial \phi_0}{\partial x}(\bar{T},x_0)\right] - \dot{\phi}_+(t,x_0)\tau'(x_0) \\ =& X_+(t)\left[(\dot{\gamma}_0(\bar{T}) - \dot{\gamma}_+(\bar{T}))\tau'(x_0) + X_0(\bar{T})\right]. \end{aligned} \tag{6.1.14}$$

Next, differentiating (6.1.11) we get $G'(\gamma(\bar{T}))\left[\frac{\partial\phi_0}{\partial x}(\bar{T},x_0)+\dot{\phi}_0(\bar{T},x_0)\tau'(x_0)\right]=0$, that is,

$$G'(\gamma(\bar{T}))\left[X_0(\bar{T})+\dot{\gamma}_0(\bar{T})\tau'(x_0)\right]=0.$$

As a consequence, we have, for any $\eta\in\mathscr{N}G'(\gamma(-\bar{T}))$:

$$\tau'(x_0)\eta=-\frac{G'(\gamma(\bar{T}))X_0(\bar{T})\eta}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})}.$$

Plugging everything together and using the definition of R_0, we finally arrive at:

$$\left[\frac{\partial\phi_0}{\partial x}(\bar{T},x_0)+\dot{\phi}_0(\bar{T},x_0)\tau'(x_0)\right]\eta=[X_0(\bar{T})+\dot{\gamma}_0(\bar{T})\tau'(x_0)]\eta=R_0X_0(\bar{T})\eta$$

and

$$\tilde{X}_+(t)\eta=X_+(t)[R_0X_0(\bar{T})\eta-\dot{\gamma}_+(\bar{T})\tau'(x_0)\eta].$$

Now, if $\widetilde{\phi}(t)$ is bounded on $\mathbb{R}$ we need $\eta\in\mathscr{N}P_-$ and hence, being $\eta\in\mathscr{N}G'(\gamma(-\bar{T}))$, we need $\eta\in\mathscr{S}'$. Moreover, since $\dot{\gamma}_+(\bar{T})\in\mathscr{R}P_+$ we see that $\tilde{X}_+(t)\eta$ is bounded on $\mathbb{R}_+$ if and only if so is $X_+(t)R_0X_0(\bar{T})\eta$, i.e. $R_0X_0(\bar{T})\eta\in\mathscr{R}P_+$. But $R_0X_0(\bar{T})\eta\in R_0X_0(\bar{T})\mathscr{S}'\subset\mathscr{R}R_0$. Hence assumption (H3) implies that $R_0X_0(\bar{T})\eta\in(\mathscr{R}R_0\cap\mathscr{R}P_+)\cap R_0X_0(\bar{T})\mathscr{S}'=\mathscr{S}''\cap\mathscr{S}'''=\{0\}$ as we proved in Lemma 6.1.2. In summary we derive the following result.

Theorem 6.1.4. *Condition $(H3)$ is equivalent to, say, that $\widetilde{\phi}(t)$ is bounded if and only if it is equal to zero. This corresponds to some nondegenerate condition on $\gamma(t)$ with respect to $\dot{x}=f_\pm(x)$.*

For the smooth case, i.e. when $f_-(x)=f_+(x)=f(x)\in C^r(\Omega)$, we have $\dot{\gamma}_0(\bar{T})=\dot{\gamma}_+(\bar{T})$ and hence $\widetilde{\phi}(t)=X(t)\eta$ where $X(t)$ is the fundamental matrix of the variational equation $\dot{x}=f'(\gamma(t))x$ along $\gamma(t)$ with $X(-\bar{T})=\mathbb{I}$. Note that $\eta\in T_{\gamma(-\bar{T})}\Omega_0$ and $T_{\gamma(-\bar{T})}\Omega_0$ is a transversal section to the homoclinic solution $\gamma(t)$ at $\gamma(-\bar{T})$. So in the smooth case, Theorem 6.1.4 states that condition (H3) is equivalent to the property that the only bounded solutions of the variational equation $\dot{x}=f'(\gamma(t))x$ are multiples of $\dot{\gamma}(t)$. Hence in the smooth case, condition (H3) is just the well-known *nondegeneracy condition* of $\gamma(t)$ (cf [35]).

Finally, we observe that (6.1.14) can be written as

$$\tilde{X}_+(t)=X_+(t)[\mathbb{I}+S]X_0(\bar{T})$$

where S is the so called *transition matrix* S [8, 13, 14, 19] and is given by

$$Sw:=\left(\dot{\gamma}_+(\bar{T})-\dot{\gamma}_0(\bar{T})\right)\frac{G'(\gamma(\bar{T}))w}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})}=\left(\dot{\gamma}_+(\bar{T})-\dot{\gamma}_0(\bar{T})\right)\frac{((R_0-\mathbb{I})w,\dot{\gamma}_0(\bar{T}))}{\|\dot{\gamma}_0(\bar{T})\|^2}$$

with the last equality following easily from the definition of R_0, where $(\cdot,\cdot)$ is a scalar product on $\mathbb{R}^n$ with the corresponding norm $\|\cdot\|$.

6.1.4 Orbits Close to the Lower Homoclinic Branches

Let $\rho > 0$ be sufficiently small, $\alpha, \beta \in \mathbb{R}$ so that $|\beta - \alpha| < \min\{1, 2\bar{T}\}$, and $\ell_T^\infty(\mathbb{R})$ be the space of doubly infinite sequences $\{T_m\}_{m\in\mathbb{Z}}$ so that $T_{m+1} - T_m \geq T + 1$ where T is chosen so that (6.1.9) holds. Note that $T_m - T_0 \geq mT$ if m is positive and $T_m - T_0 \leq mT$ if m is negative.

In this section we show how to construct solutions $z_m^-(t)$ and $z_m^+(t)$ of (6.1.1) in the intervals $[T_{2m-1} + \alpha - 1, T_{2m} - \bar{T} + \alpha]$ and $[T_{2m} + \bar{T} + \beta, T_{2m+1} + \beta + 1]$ respectively, in such a way that

$$
\begin{aligned}
&\sup_{t\in[T_{2m-1}-1, T_{2m}-\bar{T}]} |z_m^-(t+\alpha) - \gamma_-(t - T_{2m})| < \rho, \\
&\sup_{t\in[T_{2m}+\bar{T}, T_{2m+1}+1]} |z_m^+(t+\beta) - \gamma_+(t - T_{2m})| < \rho\,.
\end{aligned}
\tag{6.1.15}
$$

Note that $T_{2m-1} + \alpha - 1 < T_{2m} - \bar{T} + \alpha < T_{2m} + \bar{T} + \beta < T_{2m+1} + \beta + 1$. We show how to construct $z_m^-(t)$ for $t \in [T_{2m-1} + \alpha - 1, T_{2m} - \bar{T} + \alpha]$, the construction of $z_m^+(t)$ for $t \in [T_{2m} + \bar{T} + \beta, T_{2m+1} + \beta + 1]$ is similar. Let

$$
\begin{aligned}
I_m^- := [T_{2m-1} - 1, T_{2m} - \bar{T}], \quad I_m^+ &:= [T_{2m} + \bar{T}, T_{2m+1} + 1], \\
I_{m,\alpha}^- &:= [T_{2m-1} + \alpha - 1, T_{2m} - \bar{T} + \alpha], \\
I_{m,\beta}^+ &:= [T_{2m} + \bar{T} + \beta, T_{2m+1} + \beta + 1]
\end{aligned}
\tag{6.1.16}
$$

and set, for $t \in I_m^-$

$$x(t) = z_m^-(t+\alpha) - \gamma_-(t - T_{2m})$$

and

$$
\begin{aligned}
h_m^-(t, x, \alpha, \varepsilon) = {} & f_-(x + \gamma_-(t - T_{2m})) - f_-(\gamma_-(t - T_{2m})) \\
& - f'_-(\gamma_-(t - T_{2m}))x + \varepsilon g(t + \alpha, x + \gamma_-(t - T_{2m}), \varepsilon).
\end{aligned}
\tag{6.1.17}
$$

Then $z_m^-(t)$ satisfies Eq. (6.1.1) for $t \in I_{m,\alpha}^-$ together with (6.1.15) if and only if $x(t)$ is a solution, in I_m^-, of the equation

$$\dot{x} - f'_-(\gamma_-(t - T_{2m}))x = h_m^-(t, x, \alpha, \varepsilon), \tag{6.1.18}$$

so that $\sup_{t\in I_m^-} |x(t)| < \rho$.

Remark 6.1.5. According to Remark 6.1.1, we see that up to the r-th order all derivatives of $h_m^-(t, x, \alpha, \varepsilon)$ with respect to (x, α, ε) are bounded and uniformly continuous in (x, α, ε) uniformly with respect to $t \in I_m^-$ and $m \in \mathbb{Z}$. This statement easily follows from the fact that for $t \leq -\bar{T}$, one has $h_m^-(t + T_{2m}, x, \alpha, \varepsilon) = f_-(x + \gamma_-(t)) - f_-(\gamma_-(t)) - f'_-(\gamma_-(t))x + \varepsilon g(t + T_{2m} + \alpha, x + \gamma_-(t), \varepsilon)$ and the conclusion holds as far as $f(x)$ and $g(t + T_{2m} + \alpha, x + \gamma_-(t), \varepsilon)$ are concerned.

We will need the following Lemma [37, 38]:

Lemma 6.1.6. *Let the linear system $\dot{x} = A(t)x$ have an exponential dichotomy on $(-\infty, -\bar{T}]$ with projection P, and let $X(t)$ be its fundamental matrix so that $X(-\bar{T}) = \mathbb{I}$. Set $P(t) := X(t)PX^{-1}(t)$. Then for any continuous function $h(t) \in C^0([-T, -\bar{T}])$, $\xi_- \in \mathscr{N}P$ and $\varphi_- \in \mathscr{R}P(-T)$, the linear non homogeneous system*

$$\dot{x} = A(t)x + h(t) \tag{6.1.19}$$

has a unique solution $x(t)$ so that

$$(\mathbb{I} - P)x(-\bar{T}) = \xi_-, \quad P(-T)x(-T) = \varphi_- \tag{6.1.20}$$

and this solution satisfies

$$\begin{aligned} x(t) = X(t)\xi_- + X(t)PX^{-1}(-T)\varphi_- + \int_{-T}^{t} X(t)PX^{-1}(s)h(s)ds \\ - \int_{t}^{\bar{T}} X(t)(\mathbb{I} - P)X^{-1}(s)h(s)ds. \end{aligned} \tag{6.1.21}$$

Proof. We can directly verify that (6.1.21) solves (6.1.19) and it satisfies (6.1.20) as well. Next, if $h = 0$, $\xi_- = 0$ and $\varphi_- = 0$, then (6.1.19) implies $x(t) = X(t)x_0$ for some x_0, while (6.1.20) gives $(\mathbb{I} - P)x_0 = 0$ and $X(-T)Px_0 = 0$. Since $X(-T)$ is invertible, we obtain $x_0 = 0$, which yields to the uniqueness of $x(t)$. The proof is finished. □

Remark 6.1.7. From (6.1.2) and (6.1.21) we immediately obtain the following estimate for $|x(t)|$:

$$\sup_{-T \le t \le -\bar{T}} |x(t)| \le k \left[|\xi_-| + |\varphi_-| + 2\delta^{-1} \sup_{-T \le t \le -\bar{T}} |h(t)| \right]. \tag{6.1.22}$$

We apply Lemma 6.1.6 and Remark 6.1.7 with $A(t) = f'_-(\gamma_-(t - T_{2m}))$ in the interval I_m^- (instead of $[-T, -\bar{T}]$). Note that the fundamental matrix $X(t)$ and the projection P of the dichotomy on $(-\infty, T_{2m} - \bar{T}]$ of the linear system $\dot{x} = f'_-(\gamma_-(t - T_{2m}))x$ are $X_-(t - T_{2m})$ and P_-, respectively. Thus, in the notation of (6.1.8) and Lemma 6.1.6 we have

$$\begin{aligned} P_{-,m} :&= P(T_{2m-1} - 1) = X_-(T_{2m-1} - T_{2m} - 1)P_-X_-^{-1}(T_{2m-1} - T_{2m} - 1) \\ &= P_-(T_{2m} - T_{2m-1} + 1). \end{aligned}$$

Set:

$$\|x\|_{I_m^-} = \sup_{t \in I_m^-} |x(t)|.$$

Then a trivial application of Lemma 6.1.6 and (6.1.22) gives the following

Corollary 6.1.8. *Let $h(t) \in C^0(I_m^-)$, $\xi_- \in \mathscr{N}P_-$ and $\varphi_- \in \mathscr{R}P_{-,m}$. Then the linear nonhomogeneous system*

$$\dot{x} = f'_-(\gamma_-(t - T_{2m}))x + h(t)$$

has a unique solution $x(t) \in C^1(I_m^-)$ *so that*

$$(\mathbb{I} - P_-)x(T_{2m} - \bar{T}) = \xi_-, \quad P_{-,m}x(T_{2m-1} - 1) = \varphi_- . \tag{6.1.23}$$

Moreover this solution satisfies (*see* (6.1.22))

$$\|x(t)\|_{I_m^-} \le k\left[|\xi_-| + |\varphi_-| + 2\delta^{-1}\|h(t)\|_{I_m^-}\right] \tag{6.1.24}$$

and

$$\begin{aligned} x(t) = {} & X_-(t - T_{2m})\xi_- + X_-(t - T_{2m})P_-X_-^{-1}(T_{2m-1} - 1 - T_{2m})\varphi_- \\ & + \int_{T_{2m-1}-1}^{t} X_-(t - T_{2m})P_-X_-^{-1}(s - T_{2m})h(s)ds \\ & - \int_{t}^{T_{2m}-\bar{T}} X_-(t - T_{2m})(\mathbb{I} - P_-)X_-^{-1}(s - T_{2m})h(s)ds. \end{aligned} \tag{6.1.25}$$

Using Corollary 6.1.8 we define a map from $C^0(I_m^-) \times \mathscr{N}P_- \times \mathscr{R}P_{-,m} \times \mathbb{R}^2$ into $C^0(I_m^-)$ as

$$(x(t), \xi_-, \varphi_-, \alpha, \varepsilon) \mapsto \hat{x}(t) \tag{6.1.26}$$

where $y(t) = \hat{x}(t)$ is the unique solution given by Corollary 6.1.8 of the equation

$$\dot{y}(t) - f'_-(\gamma_-(t - T_{2m}))y(t) = h_m^-(t, x(t), \alpha, \varepsilon)$$

that satisfies conditions (6.1.23). We observe that the map

$$(x(t), \alpha, \varepsilon) \mapsto h_m^-(t, x(t), \alpha, \varepsilon)$$

is a C^r map from $C^0(I_m^-) \times \mathbb{R}^2$ into $C^0(I_m^-)$ [39] and hence, from (6.1.25) we see that so is the map (6.1.26) from $C^0(I_m^-) \times \mathscr{N}P_- \times \mathscr{R}P_{-,m} \times \mathbb{R}^2$ into $C^0(I_m^-)$. Next, from (6.1.17) we obtain immediately:

$$\|h_m^-(\cdot, x, \alpha, \varepsilon)\| \le \Delta_-(|x|)|x| + N|\varepsilon| \tag{6.1.27}$$

where

$$\Delta_-(r) = \sup\left\{\left|f'_-(x + \gamma_-(t)) - f'_-(\gamma_-(t))\right| \mid t \le -\bar{T}, |x| \le r\right\}$$

is an increasing function so that $\Delta_-(0) = 0$ and

$$N = \sup\left\{|g(t, z, \varepsilon)| \mid (t, z, \varepsilon) \in \mathbb{R}^{n+2}\right\}$$

and hence, using (6.1.24) we get:

$$\|\hat{x}\|_{I_m^-} \le k\left[|\xi_-| + |\varphi_-| + 2\delta^{-1}\Delta_-(\|x\|_{I_m^-})\|x\|_{I_m^-} + 2\delta^{-1}N|\varepsilon|\right]. \tag{6.1.28}$$

Similarly, for fixed $(\xi_-,\varphi_-,\alpha,\varepsilon)\in\mathscr{N}P_-\times\mathscr{R}P_{-,m}\times\mathbb{R}^2$ and $x_1(t),x_2(t)\in C^0(I_m^-)$ we see that

$$\|\hat{x}_2-\hat{x}_1\|_{I_m^-}\le 2k\delta^{-1}\left[\Delta_-(\bar{r})+N'|\varepsilon|\right]\|x_2-x_1\|_{I_m^-} \tag{6.1.29}$$

where $\bar{r}=\max\{\|x_1\|_{I_m^-},\|x_2\|_{I_m^-}\}$ and

$$N'=\sup\left\{\left|\frac{\partial g}{\partial x}(t,z,\varepsilon)\right| \mid (t,z,\varepsilon)\in\mathbb{R}^{n+2}\right\}.$$

Thus if $\rho>0$, $|\xi_-|$, $|\varphi_-|$ and $|\varepsilon|$ are sufficiently small, the map (6.1.26) is a C^r-contraction in the ball of center $x(t)=0$ and radius ρ in $C^0(I_m^-)$, which is uniform with respect to the other parameters $(\xi_-,\varphi_-,\alpha,\varepsilon)$ and $m\in\mathbb{Z}$. Hence we obtain the following:

Theorem 6.1.9. *Take on (H1), (H2) and let* $(\xi_-,\varphi_-,\alpha,\varepsilon)\in\mathscr{N}P_-\times\mathscr{R}P_{-,m}\times\mathbb{R}^2$, $\rho>0$ *be such that* $2k\left[|\xi_-|+|\varphi_-|+2\delta^{-1}N|\varepsilon|\right]\le\rho$ *and* $4k\delta^{-1}\left[\Delta_-(\rho)+N'|\varepsilon|\right]<1$. *Then, for* $t\in I_m^-$, *Eq.* (6.1.18) *has a unique bounded solution* $x_m^-(t)=x_m^-(t,\xi_-,\varphi_-,\alpha,\varepsilon)$ *which is* C^r *in the parameters* $(\xi_-,\varphi_-,\alpha,\varepsilon)$ *and* $m\in\mathbb{Z}$, *and satisfies*

$$\|x_m^-(\cdot,\xi_-,\varphi_-,\alpha,\varepsilon)\|_{I_m^-}\le 2k\left[|\xi_-|+|\varphi_-|+2\delta^{-1}N|\varepsilon|\right]\le\rho \tag{6.1.30}$$

together with

$$(\mathbb{I}-P_-)x_m^-(T_{2m}-\bar{T})=\xi_-,\qquad P_{-,m}x_m^-(T_{2m-1}-1)=\varphi_-.$$

Moreover the derivatives of $x_m^-(t,\xi_-,\varphi_-,\alpha,\varepsilon)$ *with respect to* $(\xi_-,\varphi_-,\alpha,\varepsilon)$ *are also bounded in* I_m^- *uniformly with respect to* $(\xi_-,\varphi_-,\alpha,\varepsilon)$ *and* $m\in\mathbb{Z}$ *and they are uniformly continuous in* $(\xi_-,\varphi_-,\alpha,\varepsilon)$ *uniformly with respect to* m *and* $t\in I_m^-$.

Proof. Only the last part of the statement needs to be proved. We know that $x_m^-(t,\xi_-,\varphi_-,\alpha,\varepsilon)$ is the unique fixed point of the map given by the right-hand side of Eq. (6.1.25) with $h_m^-(t,x(t),\alpha,\varepsilon)$ instead of $h(t)$. Since $\xi_-\in\mathscr{N}P_-$ we have $|X_-(t-T_{2m})\xi_-|=|X_-(t-T_{2m})(\mathbb{I}-P_-)X_-(-\bar{T})\xi_-|\le k\mathrm{e}^{\delta(t-T_{2m}-\bar{T})}|\xi_-|\le k|\xi_-|$ for any $t\in I_m^-$. A similar argument shows that $|X_-(t-T_{2m})P_-X_-^{-1}(T_{2m-1}-1-T_{2m})\varphi_-|\le k|\varphi_-|$ for any $t\in I_m^-$. As a consequence, the right-hand side of (6.1.25) consists of a bounded linear map in (ξ_-,φ_-), with bound independent of $m\in\mathbb{Z}$, and the nonlinear map from $C_b^0(I_m^-)\times\mathbb{R}\times\mathbb{R}$:

$$(x(\cdot),\alpha,\varepsilon)\mapsto\int_{T_{2m-1}-1}^{t}X_-(t-T_{2m})P_-X_-^{-1}(s-T_{2m})h_m^-(s,x(s),\alpha,\varepsilon)ds$$
$$-\int_t^{T_{2m}-\bar{T}}X_-(t-T_{2m})(\mathbb{I}-P_-)X_-^{-1}(s-T_{2m})h_m^-(s,x(s),\alpha,\varepsilon)ds$$

whose derivatives up to the r-th order are bounded and uniformly continuous in (x,α,ε) uniformly with respect to m because of the properties of $h_m^-(t,x,\alpha,\varepsilon)$ (see Remark 6.1.5 and 6.1.2). The proof is complete. □

We are now ready to prove the main result of this section:

Theorem 6.1.10. *Take on (H1), (H2) and let* $(\xi_-, \varphi_-, \alpha, \varepsilon) \in \mathscr{N}P_- \times \mathscr{R}P_{-,m} \times \mathbb{R}^2$, $\rho > 0$ *be such that* $2k\left[|\xi_-| + |\varphi_-| + 2\delta^{-1}N|\varepsilon|\right] \le \rho$ *and* $4k\delta^{-1}\left[\Delta_-(\rho) + N'|\varepsilon|\right] < 1$. *Then, for* $t \in I^-_{m,\alpha}$, *equation* $\dot{z} = f_-(z) + \varepsilon g(t,z,\varepsilon)$ *has a unique bounded solution* $z^-_m(t) = z^-_m(t, \xi_-, \varphi_-, \alpha, \varepsilon)$ *which is* C^r *in the parameters* $(\xi_-, \varphi_-, \alpha, \varepsilon)$ *and satisfies*

$$\|z^-_m(\cdot + \alpha, \xi_-, \varphi_-, \alpha, \varepsilon) - \gamma_-(\cdot - T_{2m})\|_{I^-_m} \le 2k\left[|\xi_-| + |\varphi_-| + 2\delta^{-1}N|\varepsilon|\right] \le \rho \tag{6.1.31}$$

together with

$$(\mathbb{I} - P_-)[z^-_m(T_{2m} - \bar{T} + \alpha) - \gamma_-(-\bar{T})] = \xi_-,$$
$$P_{-,m}[z^-_m(T_{2m-1} + \alpha - 1) - \gamma_-(T_{2m-1} - T_{2m} - 1)] = \varphi_-.$$

Moreover $x^-_m(t) := z^-_m(t + \alpha, \xi_-, \varphi_-, \alpha, \varepsilon) - \gamma_-(t - T_{2m})$ *is the unique fixed point of the map (6.1.25) and* $z^-_m(t, \xi_-, \varphi_-, \alpha, \varepsilon)$ *and its derivatives with respect to* $(\xi_-, \varphi_-, \alpha, \varepsilon)$ *are also bounded in* I^-_m *uniformly with respect to* $(\xi_-, \varphi_-, \alpha, \varepsilon)$ *and* $m \in \mathbb{Z}$, *uniformly continuous in* $(\xi_-, \varphi_-, \alpha, \varepsilon)$ *uniformly with respect to* (t, m) *with* $t \in I^-_m$, $m \in \mathbb{Z}$ *and satisfy:*

$$\begin{aligned}
&\frac{\partial z^-_m}{\partial \xi_-}(t + \alpha, 0, 0, \alpha, 0) = X_-(t - T_{2m})(\mathbb{I} - P_-),\\
&\frac{\partial z^-_m}{\partial \varphi_-}(t + \alpha, 0, 0, \alpha, 0)\varphi_- = X_-(t - T_{2m})P_-X_-^{-1}(T_{2m-1} - T_{2m} - 1)\varphi_-\\
&\frac{\partial z^-_m}{\partial \varepsilon}(t + \alpha, 0, 0, \alpha, 0)\\
&\quad = \int_{T_{2m-1}-1}^{t} X_-(t - T_{2m})P_-X_-^{-1}(s - T_{2m})g(s + \alpha, \gamma_-(s - T_{2m}), 0)ds\\
&\qquad - \int_t^{T_{2m}-\bar{T}} X_-(t - T_{2m})(\mathbb{I} - P_-)X_-^{-1}(s - T_{2m})g(s + \alpha, \gamma_-(s - T_{2m}), 0)ds
\end{aligned} \tag{6.1.32}$$

Proof. Setting $x(t) := z^-_m(t + \alpha) - \gamma_-(t - T_{2m})$ the existence of $z^-_m(t, \xi_-, \varphi_-, \alpha, \varepsilon)$ satisfying (6.1.31) follows from Theorem 6.1.9. Thus we only need to prove (6.1.32). From (6.1.28) we see that $x^-_m(t, 0, 0, \alpha, 0) = 0$ and then differentiating equation (6.1.25) with $x^-_m(t, \xi_-, \varphi_-, \alpha, \varepsilon)$ instead of $x(t)$ and $h^-_m(t, x^-_m(t, \xi_-, \varphi_-, \alpha, \varepsilon), \alpha, \varepsilon)$ instead of $h(t)$ we see that

$$\frac{\partial z^-_m}{\partial \xi_-}(t + \alpha, 0, 0, \alpha, 0)\xi_- = \frac{\partial x^-_m}{\partial \xi_-}(t, 0, 0, \alpha, 0)\xi_- = X_-(t - T_{2m})\xi_-.$$

Similarly we obtain the rest of (6.1.32). □

Remark 6.1.11. The function $z^-_m(t) = z^-_m(t, \xi_-, \varphi_-, \alpha, \varepsilon)$ is a bounded solution of Eq. (6.1.1) in the interval $I^-_{m,\alpha}$ as long as it remains in Ω_- for $t \in I^-_{m,\alpha}$, and satisfies (6.1.31). However in order that $z_m^-(t) \in \Omega_-$ for $t \in I_{m,\alpha}^-$ it is sufficient that $G(z_m^-(T_{2m} - \bar{T} + \alpha)) = 0$. This follows directly from (H2) and (6.1.31).

Next, let

$$
\begin{aligned}
\Delta_+(r) &:= \sup\left\{\left|f'_-(x+\gamma_+(t)) - f'_-(\gamma_+(t))\right| \mid \bar{T} \le t, |x| \le r\right\}, \\
P_{+,m} &:= P_+(T_{2m+1} - T_{2m} + 1) \\
&= X_+(T_{2m+1} - T_{2m} + 1)P_+X_+(T_{2m+1} - T_{2m} + 1)^{-1}, \qquad (6.1.33) \\
h_m^+(t,x,\beta,\varepsilon) &= f_-(x+\gamma_+(t-T_{2m})) - f_-(\gamma_+(t-T_{2m})) \\
&\quad - f'_-(\gamma_+(t-T_{2m}))x + \varepsilon g(t+\beta, x+\gamma_+(t-T_{2m}), \varepsilon).
\end{aligned}
$$

By an almost identical argument we show the following:

Theorem 6.1.12. *Take on* (H1), (H2) *and let* $(\xi_+, \varphi_+, \beta, \varepsilon) \in \mathscr{R}P_+ \times \mathscr{N}P_{+,m} \times \mathbb{R}^2$ *and* $\rho > 0$ *be such that* $2k\left[|\xi_+| + |\varphi_+| + 2\delta^{-1}N|\varepsilon|\right] \le \rho$ *and* $4k\delta^{-1}\left[\Delta_+(\rho) + N'|\varepsilon|\right] < 1$. *Then, for* $t \in I_{m,\beta}^+$, *equation* $\dot{z} = f_+(z) + \varepsilon g(t,z,\varepsilon)$ *has a unique bounded solution* $z_m^+(t) = z_m^+(t, \xi_+, \varphi_+, \beta, \varepsilon)$ *which is* C^r *in the parameters* $(\xi_+, \varphi_+, \beta, \varepsilon)$ *and satisfies*

$$
\|z_m^+(\cdot+\beta, \xi_+, \varphi_+, \beta, \varepsilon) - \gamma_+(\cdot - T_{2m})\|_{I_m^+} \le 2k\left[|\xi_+| + |\varphi_+| + 2\delta^{-1}N|\varepsilon|\right] \le \rho \qquad (6.1.34)
$$

together with

$$
\begin{gathered}
P_+[z_m^+(T_{2m} + \bar{T} + \beta) - \gamma_+(\bar{T})] = \xi_+, \\
(\mathbb{I} - P_{+,m})[z_m^+(T_{2m+1} + \beta + 1) - \gamma_+(T_{2m+1} - T_{2m} + 1)] = \varphi_+.
\end{gathered}
$$

Moreover $x_m^+(t) := z_m^+(t+\beta, \xi_+, \varphi_+, \beta, \varepsilon) - \gamma_+(t - T_{2m})$ *is the unique fixed point of the map*

$$
\begin{aligned}
&(x(t), \xi_+, \varphi_+, \beta, \varepsilon) \mapsto \\
&\qquad X_+(t-T_{2m})\xi_+ + X_+(t-T_{2m})(\mathbb{I} - P_+)X_+^{-1}(T_{2m+1} - T_{2m} + 1)\varphi_+ \\
&\qquad + \int_{T_{2m}+\bar{T}}^{t} X_+(t-T_{2m})P_+X_+^{-1}(s-T_{2m})h_m^+(s,x(s),\beta,\varepsilon)ds \\
&\qquad - \int_{t}^{T_{2m+1}+1} X_+(t-T_{2m})(\mathbb{I} - P_+)X_+^{-1}(s-T_{2m})h_m^+(s,x(s),\beta,\varepsilon)ds, \qquad (6.1.35)
\end{aligned}
$$

and $z_m^+(t, \xi_+, \varphi_+, \beta, \varepsilon)$ *and its derivatives with respect to* $(\xi_+, \varphi_+, \beta, \varepsilon)$ *are also bounded in* I_m^+ *uniformly with respect to* $(\xi_+, \varphi_+, \beta, \varepsilon)$ *and* $m \in \mathbb{Z}$, *uniformly continuous in* $(\xi_+, \varphi_+, \beta, \varepsilon)$ *uniformly with respect to* (t,m) *with* $t \in I_m^+$, $m \in \mathbb{Z}$ *and satisfy:*

$$
\frac{\partial z_m^+}{\partial \xi_+}(t+\beta, 0, 0, \beta, 0) = X_+(t - T_{2m})P_+
$$

$$\frac{\partial z_m^+}{\partial \varphi_+}(t+\beta,0,0,\beta,0)\varphi_+ = X_+(t-T_{2m})(\mathbb{I}-P_+)X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi_+$$

$$\begin{aligned}\frac{\partial z_m^+}{\partial \varepsilon}&(t+\beta,0,0,\beta,0)\\ &= \int_{T_{2m}+\bar{T}}^{t} X_+(t-T_{2m})P_+X_+^{-1}(s-T_{2m})g(s+\beta,\gamma_+(s-T_{2m}),0)ds\\ &\quad - \int_{t}^{T_{2m+1}+1} X_+(t-T_{2m})(\mathbb{I}-P_+)X_+^{-1}(s-T_{2m})g(s+\beta,\gamma_+(s-T_{2m}),0)ds.\end{aligned} \tag{6.1.36}$$

Remark 6.1.13. Note that $z_m^-(t,\xi_-,\varphi_-,\alpha,\varepsilon)$ (resp. $z_m^+(t,\xi_+,\varphi_+,\alpha,\varepsilon)$) depends on m by means of T_{2m-1} and T_{2m} (resp. T_{2m} and T_{2m+1}). Consequently, we may also write $x^-(t,\xi_-,\varphi_-,\alpha,\varepsilon,T_{2m},T_{2m-1})$, $x^+(t,\xi_+,\varphi_+,\alpha,\varepsilon,T_{2m},T_{2m+1})$ instead of $x_m^-(t,\xi_-,\varphi_-,\alpha,\varepsilon)$, $x_m^+(t,\xi_+,\varphi_+,\alpha,\varepsilon)$ and say that $x^-(t,\xi_-,\varphi_-,\alpha,\varepsilon,T_{2m},T_{2m-1})$, $x^+(t,\xi_+,\varphi_+,\alpha,\varepsilon,T_{2m},T_{2m+1})$, respectively, is uniformly continuous with respect to $(\xi_-,\varphi_-,\alpha,\varepsilon)$, resp. $(\xi_+,\varphi_+,\beta,\varepsilon)$, uniformly with respect to T_{2m},T_{2m-1}, resp. T_{2m},T_{2m+1}, and $t\in I_m^-$, (resp. $t\in I_m^+$).

6.1.5 Orbits Close to the Upper Homoclinic Branch

Theorem 6.1.14. *Take on (H1), (H2). Then there exist positive constants c, ε_0 and $\widetilde{\rho}_0$ so that for any $\alpha,\beta,\varepsilon\in\mathbb{R}$ and $\bar{\xi}\in\mathbb{R}^n$ so that $|\beta-\alpha|<\min\{1,2\bar{T}\}$, $|\varepsilon|\le\varepsilon_0$ and $|\bar{\xi}-\gamma_0(-\bar{T})|<\widetilde{\rho}_0$, there exists a unique solution $z_m^0(t)=z_m^0(t,\bar{\xi},\alpha,\beta,\varepsilon)$ of equation $\dot{z}=f_+(z)+\varepsilon g(t,z,\varepsilon)$, for $t\in[T_{2m}-\bar{T}+\alpha,T_{2m}+\bar{T}+\beta]$ so that*

$$z_m^0(T_{2m}-\bar{T}+\alpha)=\bar{\xi}$$

and

$$\|z_m^0(t)-\gamma_0(t-T_{2m}-\alpha)\|_{[T_{2m}-\bar{T}+\alpha,T_{2m}+\bar{T}+\beta]}\le c[|\bar{\xi}-\gamma_0(-\bar{T})|+2N\delta^{-1}|\varepsilon|]\,. \tag{6.1.37}$$

Moreover $z_m^0(t,\bar{\xi},\alpha,\beta,\varepsilon)$ and its derivatives with respect to $(\bar{\xi},\alpha,\beta,\varepsilon)$ are bounded in $[T_{2m}-\bar{T}+\alpha,T_{2m}+\bar{T}+\beta]$ uniformly with respect to $m\in\mathbb{Z}$, uniformly continuous in $(\bar{\xi},\alpha,\beta,\varepsilon)$, uniformly with respect to $t\in[T_{2m}-\bar{T}+\alpha,T_{2m}+\bar{T}+\beta]$, $m\in\mathbb{Z}$, and have the following properties:

(i) $x_m^0(t)=z_m^0(t+\alpha,\bar{\xi},\alpha,\beta,\varepsilon)-\gamma_0(t-T_{2m})$ *is a fixed point of the map*

$$\begin{aligned}x(t)\mapsto\ & X_0(t-T_{2m})\left[\bar{\xi}-\gamma_0(-\bar{T})\right]\\ &+\int_{T_{2m}-\bar{T}}^{t} X_0(t-T_{2m})X_0^{-1}(s-T_{2m})h_m^0(s,x(s),\alpha,\varepsilon)ds\end{aligned} \tag{6.1.38}$$

where

$$h_m^0(t,x,\alpha,\varepsilon) = f_+(x+\gamma_0(t-T_{2m})) - f_+(\gamma_0(t-T_{2m})) - f_+'(\gamma_0(t-T_{2m}))x$$
$$+\varepsilon g(t+\alpha, x+\gamma_0(t-T_{2m}),\varepsilon).$$

(ii) *The following equalities hold:*

$$\begin{aligned}
&\frac{\partial z_m^0}{\partial \alpha}(t,\gamma_0(\bar{T}),\alpha,\beta,0) = -\dot{\gamma}_0(t-T_{2m}-\alpha),\\
&\frac{\partial z_m^0}{\partial \beta}(t,\gamma_0(-\bar{T}),\alpha,\beta,0) = 0,\\
&\frac{\partial z_m^0}{\partial \bar{\xi}}(t,\gamma_0(-\bar{T}),\alpha,\beta,0) = X_0(t-T_{2m}-\alpha),\\
&\frac{\partial z_m^0}{\partial \varepsilon}(t+\alpha,\gamma_0(-\bar{T}),\alpha,\beta,0)\\
&\quad = \int_{T_{2m}-\bar{T}}^{t} X_0(t-T_{2m})X_0^{-1}(s-T_{2m})g(s+\alpha,\gamma_0(s-T_{2m}),0)\,ds.
\end{aligned} \tag{6.1.39}$$

Proof. The statement concerning the existence of the solution $z_m^0(t) = z_m^0(t,\xi,\alpha,\beta,\varepsilon)$ from which (6.1.37) holds, follows from the continuous dependence on the data. Moreover the fact that $x_m^0(t)$ is a fixed point of the map (6.1.38) follows from the variation of constants formula. The boundedness and continuity properties of $z_m^0(t,\xi,\alpha,\beta,\varepsilon)$ follow from the similar properties of $h_m^0(t,x,\alpha,\varepsilon)$ as in Theorems 6.1.10, 6.1.12. Then, because of uniqueness of fixed points we also get:

$$z_m^0(t,\gamma_0(-\bar{T}),\alpha,\beta,0) = \gamma_0(t-T_{2m}-\alpha)$$

from which the first two equalities of point (ii) easily follow. Differentiating (6.1.38) with respect to ξ, ε respectively and using the fact that $h_m^0(t,x,\alpha,0)$ is of the second order in x, we derive the other two equalities in (ii). □

Note that if

$$c[\widetilde{\rho}_0 + 2N\delta^{-1}\varepsilon_0] < \rho$$

from (6.1.37) we obtain:

$$\sup\{|z_m^0(t+\alpha) - \gamma_0(t-T_{2m})| \mid t \in [T_{2m}-\bar{T}, T_{2m}+\bar{T}+\beta-\alpha]\} < \rho. \tag{6.1.40}$$

Remark 6.1.15. Note that $z_m^0(t,\bar{\xi},\alpha,\beta,\varepsilon))$ depends on m by means of T_{2m}. Thus we may also write $z^0(t,\bar{\xi},\alpha,\beta,\varepsilon,T_{2m})$ instead of $z_m^0(t,\bar{\xi},\alpha,\beta,\varepsilon)$ and say that $z^0(t,\bar{\xi},\alpha,\beta,\varepsilon,T_{2m})$ is uniformly continuous in $(\bar{\xi},\alpha,\beta,\varepsilon)$ uniformly with respect to T_{2m} and $t \in [T_{2m}-\bar{T}+\alpha, T_{2m}+\bar{T}+\beta]$.

6.1.6 Bifurcation Equation

Let $\varepsilon_0 > 0$, $\widetilde{\rho}_0 > 0$ and $c > 0$ be constants as in Theorem 6.1.14, $C := \max\{c, 2k\}$, $\chi < 1$ a positive constant that will be specified and fixed below and $\rho_0 \le c\widetilde{\rho}_0$ be the largest positive number satisfying

$$4k\delta^{-1}\left[\Delta_{\pm}(\rho_0) + \frac{N'\delta}{2NC}\rho_0\right] \le 1.$$

Next, let $0 < \rho < \rho_0$ and $\varepsilon_\rho := \min\left\{\frac{\rho\delta}{2CN}, \varepsilon_0\right\}$. For any $\alpha = \{\alpha_m\}_{m\in\mathbb{Z}} \in \ell^\infty(\mathbb{R})$ and $\varepsilon \in (-\varepsilon_\rho, \varepsilon_\rho)$ we set

$$\begin{aligned}
&\ell^\infty_{\rho,\alpha,\varepsilon} := \Big\{\theta := \{(\varphi_m^-, \varphi_m^+, \xi_m^-, \xi_m^+, \bar{\xi}_m, \beta_m)\}_{m\in\mathbb{Z}} \in \ell^\infty(\mathbb{R}^{5n+1}) : \\
&(\varphi_m^-, \varphi_m^+, \xi_m^-, \xi_m^+, \bar{\xi}_m, \beta_m) \in \mathscr{R}P_{-,m} \times \mathscr{N}P_{+,m} \times \mathscr{N}P_- \times \mathscr{R}P_+ \times \mathbb{R}^{n+1}, \\
&2k\left[|\xi_m^\pm| + |\varphi_m^\pm| + 2\delta^{-1}N|\varepsilon|\right] < \rho, \quad c[|\bar{\xi}_m - \gamma_0(-\bar{T})| + 2N\delta^{-1}|\varepsilon|] < \rho, \\
&\sup_{m\in\mathbb{Z}} |\alpha_{m+1} - \beta_m| < \chi\Big\}
\end{aligned}$$

and

$$\ell^\infty_\rho = \left\{(\theta, \alpha, \varepsilon) \in \ell^\infty_{\rho,\alpha,\varepsilon} \times \ell^\infty(\mathbb{R}) \times (-\varepsilon_\rho, \varepsilon_\rho) : \alpha \in \ell^\infty_\chi\right\}$$

where

$$\ell^\infty_\chi = \left\{\alpha \in \ell^\infty(\mathbb{R}) : \sup_{m\in\mathbb{Z}} |\alpha_m - \alpha_{m-1}| < \chi\right\}.$$

Note that because of the choice of ρ, ε_ρ, $\ell^\infty_{\rho,\alpha,\varepsilon}$, ℓ^∞_ρ and ℓ^∞_χ are open nonempty subsets of

$$\ell^\infty(\mathscr{R}P_{-,m} \times \mathscr{N}P_{+,m} \times \mathscr{N}P_- \times \mathscr{R}P_+ \times \mathbb{R}^n \times \mathbb{R}),$$

$$\ell^\infty(\mathscr{R}P_{-,m} \times \mathscr{N}P_{+,m} \times \mathscr{N}P_- \times \mathscr{R}P_+ \times \mathbb{R}^n \times \mathbb{R}) \times \ell^\infty(\mathbb{R}) \times (-\varepsilon_\rho, \varepsilon_\rho)$$

and $\ell^\infty(\mathbb{R})$, respectively. In $\ell^\infty_{\rho,\alpha,\varepsilon}$ we take the norm

$$\begin{aligned}
\|\theta\| &= \left\|\{(\varphi_m^-, \varphi_m^+, \xi_m^-, \xi_m^+, \bar{\xi}_m, \beta_m)\}_{m\in\mathbb{Z}}\right\| \\
&= \sup_{m\in\mathbb{Z}} \max\left\{|\varphi_m^- + \varphi_m^+|, |\xi_m^-|, |\xi_m^+|, |\bar{\xi}_m|, |\beta_m|\right\}.
\end{aligned}$$

Let $\mathscr{T} = \{T_m\}_{m\in\mathbb{Z}}$ be given as in Section 6.1.4 and take $(\theta, \alpha, \varepsilon) \in \ell^\infty_\rho$. In this section we want to find such conditions that system (6.1.1) has a solution $z(t)$ defined on $\mathbb{R}$ so that any $m \in \mathbb{Z}$ satisfies:

$$\begin{aligned}
&\|z(t) - \gamma_-(t - T_{2m} - \alpha_m)\|_{\widetilde{I}_m^-} < \rho, \\
&\|z(t) - \gamma_0(t - T_{2m} - \alpha_m)\|_{\widetilde{I}_m^0} < \rho, \\
&\|z(t) - \gamma_+(t - T_{2m} - \beta_m)\|_{\widetilde{I}_m^+} < \rho
\end{aligned}$$

where $\widetilde{I}_m^- = [T_{2m-1}+\alpha_m-1, T_{2m}-\bar{T}+\alpha_m]$, $I_m^0 = [T_{2m}-\bar{T}+\alpha_m, T_{2m}+\bar{T}+\beta_m]$ and $\widetilde{I}_m^+ = [T_{2m}+\bar{T}+\beta_m, T_{2m+1}+\beta_m]$.

We note that for any $(\theta,\alpha,\varepsilon)\in\ell_\rho^\infty$ assumptions of Theorems 6.1.10, 6.1.12 and 6.1.14 are satisfied. Indeed we have

$$4k\delta^{-1}\left[\Delta_\pm(\rho)+N'|\varepsilon|\right] < 4k\delta^{-1}\left[\Delta_\pm(\rho)+N'\varepsilon_\rho\right] < 4k\delta^{-1}\left[\Delta_\pm(\rho_0)+\frac{N'\delta}{2NC}\rho_0\right]\le 1$$

along with $|\varepsilon|<\varepsilon_0$ and

$$|\bar{\xi}-\gamma_0(-\bar{T})| < \frac{\rho}{c} < \frac{\rho_0}{c} \le \widetilde{\rho}_0\,.$$

So according to the previous sections and because of uniqueness of the solutions $z_m^+(t,\xi_m^+,\varphi_m^+,\beta_m,\varepsilon)$, $z_m^-(t,\xi_m^-,\varphi_m^-,\alpha_m,\varepsilon)$ and $z_m^0(t,\bar{\xi}_m,\alpha_m,\beta_m,\varepsilon)$ we see that such a solution can be found if and only if we are able to solve the infinite set of equations $(m\in\mathbb{Z})$:

$$\begin{cases} z_m^+(T_{2m+1}+\beta_m,\xi_m^+,\varphi_m^+,\beta_m,\varepsilon)-z_{m+1}^-(T_{2m+1}+\beta_m,\xi_{m+1}^-,\varphi_{m+1}^-,\alpha_{m+1},\varepsilon)=0,\\ z_m^0(T_{2m}-\bar{T}+\alpha_m,\bar{\xi}_m,\alpha_m,\beta_m,\varepsilon)-z_m^-(T_{2m}-\bar{T}+\alpha_m,\xi_m^-,\varphi_m^-,\alpha_m,\varepsilon)=0,\\ z_m^0(T_{2m}+\bar{T}+\beta_m,\bar{\xi}_m,\alpha_m,\beta_m,\varepsilon)-z_m^+(T_{2m}+\bar{T}+\beta_m,\xi_m^+,\varphi_m^+,\beta_m,\varepsilon)=0,\\ G(z_m^-(T_{2m}-\bar{T}+\alpha_m,\xi_m^-,\varphi_m^-,\alpha_m,\varepsilon))=0,\\ G(z_m^0(T_{2m}+\bar{T}+\beta_m,\bar{\xi}_m,\alpha_m,\beta_m,\varepsilon))=0,\\ G(z_m^+(T_{2m}+\bar{T}+\beta_m,\xi_m^+,\varphi_m^+,\beta_m,\varepsilon))=0\,. \end{cases} \tag{6.1.41}$$

Since $T_{2m+1}+\alpha_{m+1}-1<T_{2m+1}+\beta_m$, system (6.1.41) is well posed. Note that from Theorem 6.1.14, the second of the above equations reads:

$$\bar{\xi}_m = z_m^-(T_{2m}-\bar{T}+\alpha_m,\xi_m^-,\varphi_m^-,\alpha_m,\varepsilon)$$

and gives the sequence $\{\bar{\xi}_m\}_{m\in\mathbb{Z}}$ in terms of the sequences $\{\xi_m^-\}_{m\in\mathbb{Z}}$, $\{\varphi_m^-\}_{m\in\mathbb{Z}}$, $\{\alpha_m\}_{m\in\mathbb{Z}}$, and ε. Moreover, if ρ is sufficiently small, $z_m^0(T_{2m}+\bar{T}+\beta_m,\bar{\xi}_m,\alpha_m,\beta_m,\varepsilon)$ is close to $\gamma_0(\bar{T}+\beta_m-\alpha_m)$, while $z_m^+(T_{2m}+\bar{T}+\beta_m,\xi_m^+,\varphi_m^+,\beta_m,\varepsilon)$ is close to $\gamma_+(\bar{T})=\gamma_0(\bar{T})$. So there is a positive constant $\chi<\min\{1,2\bar{T}\}$ so that the 5th and the 6th equations in (6.1.41) imply that the 3rd equation is equivalent to

$$R_0\left[z_m^0(T_{2m}+\bar{T}+\beta_m,\bar{\xi}_m,\alpha_m,\beta_m,\varepsilon)-z_m^+(T_{2m}+\bar{T}+\beta_m,\xi_m^+,\varphi_m^+,\beta_m,\varepsilon)\right]=0$$

where $R_0:\mathbb{R}^n\to\mathbb{R}^n$ is the projection defined in Section 6.1.2. From now on, we fix such a χ. Here we use the fact $|\beta_m-\alpha_m|<2\chi$ for any $m\in\mathbb{Z}$, so $\gamma_0(\bar{T}+\beta_m-\alpha_m)$ and $\gamma_0(\bar{T})$ are sufficiently close for χ is small enough uniformly for any $m\in\mathbb{Z}$.

Let

$$\ell_1^\infty=\ell^\infty(\mathbb{R}^n\times\mathbb{R}^n\times\mathscr{R}R_0\times\mathbb{R}\times\mathbb{R}\times\mathbb{R})$$

with the norm

$$\sup_{m\in\mathbb{Z}} \max\{|a_m|,|b_m|,|c_m|,|d_m|,|e_m|,|f_m|\}$$

for $\{(a_m,b_m,c_m,d_m,e_m,f_m)\}_{m\in\mathbb{Z}} \in \ell_1^\infty$. We define a map $\mathscr{G}_{\mathscr{T}} \in C^r\left(\ell_\rho^\infty, \ell_1^\infty\right)$ as

$$\mathscr{G}_{\mathscr{T}}(\theta,\alpha,\varepsilon) = \mathscr{G}_{\mathscr{T}}(\{(\varphi_m^-,\varphi_m^+,\xi_m^-,\xi_m^+,\bar{\xi}_m,\beta_m)\}_{m\in\mathbb{Z}},\{\alpha_m\}_{m\in\mathbb{Z}},\varepsilon) :=$$

$$\left\{\begin{array}{c} z_m^+(T_{2m+1}+\beta_m,\xi_m^+,\varphi_m^+,\beta_m,\varepsilon) - z_{m+1}^-(T_{2m+1}+\beta_m,\xi_{m+1}^-,\varphi_{m+1}^-,\alpha_{m+1},\varepsilon) \\ \bar{\xi}_m - z_m^-(T_{2m}-\bar{T}+\alpha_m,\xi_m^-,\varphi_m^-,\alpha_m,\varepsilon) \\ R_0[z_m^0(T_{2m}+\bar{T}+\beta_m,\bar{\xi}_m,\alpha_m,\beta_m,\varepsilon) - z_m^+(T_{2m}+\bar{T}+\beta_m,\xi_m^+,\varphi_m^+,\beta_m,\varepsilon)] \\ G(z_m^-(T_{2m}-\bar{T}+\alpha_m,\xi_m^-,\varphi_m^-,\alpha_m,\varepsilon)) \\ G(z_m^0(T_{2m}+\bar{T}+\beta_m,\bar{\xi}_m,\alpha_m,\beta_m,\varepsilon)) \\ G(z_m^+(T_{2m}+\bar{T}+\beta_m,\xi_m^+,\varphi_m^+,\beta_m,\varepsilon)) \end{array}\right\}_{m\in\mathbb{Z}}$$

so that Eq. (6.1.41) reads

$$\mathscr{G}_{\mathscr{T}}(\theta,\alpha,\varepsilon) = 0. \tag{6.1.42}$$

Before giving our main result we state few properties of the map $\mathscr{G}_{\mathscr{T}}$. First, from [39] it follows that $\mathscr{G}_{\mathscr{T}}$ is C^r and has bounded derivatives. More precisely, from the continuity properties of the solutions $z_m^+(t,\xi_m^+,\varphi_m^+,\beta_m,\varepsilon)$, $z_m^-(t,\xi_m^-,\varphi_m^-,\alpha_m,\varepsilon)$, and $z_m^0(t,\bar{\xi}_m,\alpha_m,\beta_m,\varepsilon)$ we see that $\mathscr{G}_{\mathscr{T}}(\theta,\alpha,\varepsilon)$ and its derivatives are bounded and uniformly continuous in $(\theta,\alpha,\varepsilon)$ uniformly with respect to $\mathscr{T} \in \ell_T^\infty(\mathbb{R})$. Next, for any $\alpha \in \ell_\chi^\infty$, we set:

$$\theta_\alpha = \{(0,0,0,0,\gamma_0(-\bar{T}),\alpha_m)\}_{m\in\mathbb{Z}}.$$

From (6.1.31), (6.1.34), (6.1.37), and $G(\gamma_\pm(\pm\bar{T}))=0$, $\gamma_\pm(\pm\bar{T})=\gamma_0(\pm\bar{T})$, we get

$$\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0) = \left\{\begin{array}{c} \gamma_+(T_{2m+1}-T_{2m}) - \gamma_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1}) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right\}_{m\in\mathbb{Z}}.$$

Now, for $t \geq T$ we have

$$|\gamma_+(t)| \leq \int_t^\infty |\dot{\gamma}_+(s)|ds \leq \int_t^\infty k\,\mathrm{e}^{-\delta(s-\bar{T})}\,|\dot{\gamma}_+(\bar{T})|ds = k\delta^{-1}\,\mathrm{e}^{-\delta(t-\bar{T})}\,|\dot{\gamma}_+(\bar{T})|$$

and similarly

$$|\gamma_-(t)| \leq k\delta^{-1}\,\mathrm{e}^{\delta(t+\bar{T})}\,|\dot{\gamma}_-(-\bar{T})|$$

for any $t \le -\bar{T}$. Thus

$$\begin{aligned}
&|\gamma_+(T_{2m+1}-T_{2m}) - \gamma_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})| \\
&\quad \le k\delta^{-1}\,\mathrm{e}^{-\delta(T_{2m+1}-T_{2m}-\bar{T})}\,|\dot{\gamma}_+(\bar{T})| + k\delta^{-1}\,\mathrm{e}^{\delta(T_{2m+1}-T_{2m+2}+\bar{T}+1)}\,|\dot{\gamma}_-(-\bar{T})| \\
&\quad \le 2k\delta^{-1}\,\mathrm{e}^{-\delta(T-\bar{T})}\max\{|\dot{\gamma}_-(-\bar{T})|, |\dot{\gamma}_+(\bar{T})|\},
\end{aligned}$$

that is,

$$\|\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\| \le 2k\delta^{-1}\,\mathrm{e}^{-\delta(T-\bar{T})}\max\{|\dot{\gamma}_-(-\bar{T})|, |\dot{\gamma}_+(\bar{T})|\}. \tag{6.1.43}$$

Similarly we get:

$$\frac{d}{d\alpha}[\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)]\widetilde{\alpha} = \left\{\begin{matrix} \dot{\gamma}_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})(\widetilde{\alpha}_{m+1}-\widetilde{\alpha}_m) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right\}_{m\in\mathbb{Z}}$$

and hence

$$\left\|\frac{d}{d\alpha}[\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)]\right\| \le 2k\delta^{-1}\,\mathrm{e}^{-\delta(T-\bar{T})}\,|\dot{\gamma}_-(-\bar{T})|. \tag{6.1.44}$$

Next, from Theorems 6.1.10, 6.1.12, 6.1.14, the equality $R_0\dot{\gamma}_0(\bar{T}) = 0$ and the identities

$$\begin{aligned}
P_-X_-^{-1}(T_{2m-1}-T_{2m}-1)\varphi_m^- &= X_-^{-1}(T_{2m-1}-T_{2m}-1)\varphi_m^-, \\
(\mathbb{I}-P_+)X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi_m^+ &= X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi_m^+
\end{aligned} \tag{6.1.45}$$

(that follow from $\varphi_m^- \in \mathscr{R}P_{-,m}$, $\varphi_m^+ \in \mathscr{N}P_{+,m}$), we see that the derivative $D_1\mathscr{G}_{\mathscr{T}}$ of $\mathscr{G}_{\mathscr{T}}$ with respect to $\theta \in \ell^\infty_{\rho,\alpha,\varepsilon}$ at the point $(\theta_\alpha,\alpha,0)$ is given by

$$D_1\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\theta = \left\{\begin{matrix} \mathscr{L}_\alpha(\varphi_{m+1}^-, \varphi_m^+, \xi_{m+1}^-, \xi_m^+, \bar{\xi}_m, \beta_m) \\ \bar{\xi}_m - \xi_m^- - X_-^{-1}(T_{2m-1}-T_{2m}-1)\varphi_m^- \\ R_0[X_0(\bar{T})\bar{\xi}_m - \xi_m^+ - X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi_m^+] \\ G'(\gamma_0(-\bar{T}))\cdot[\xi_m^- + X_-^{-1}(T_{2m-1}-T_{2m}-1)\varphi_m^-] \\ G'(\gamma_0(\bar{T}))\cdot[X_0(\bar{T})\bar{\xi}_m + \dot{\gamma}_0(\bar{T})\beta_m] \\ G'(\gamma_+(\bar{T}))\cdot[\xi_m^+ + X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi_m^+] \end{matrix}\right\}_{m\in\mathbb{Z}}$$

where, we recall $\theta = \{(\varphi_m^-, \varphi_m^+, \xi_m^-, \xi_m^+, \bar{\xi}_m, \beta_m)\}_{m\in\mathbb{Z}}$, and

$$\begin{aligned}
&\mathscr{L}_\alpha(\varphi^-_{m+1},\varphi^+_m,\xi^-_{m+1},\xi^+_m,\bar{\xi}_m,\beta_m)\\
&\quad = X_+(T_{2m+1}-T_{2m})\xi^+_m\\
&\qquad -X_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})\xi^-_{m+1}-\dot{\gamma}_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})\beta_m\\
&\qquad +X_+(T_{2m+1}-T_{2m})(\mathbb{I}-P_+)X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi^+_m\\
&\qquad -X_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})P_-X_-^{-1}(T_{2m+1}-T_{2m+2}-1)\varphi^-_{m+1}\,.
\end{aligned}$$

Then, using again (6.1.45) we obtain:

$$\begin{aligned}
&\left|X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi^+_m\right| \le k\mathrm{e}^{-\delta(T_{2m+1}-T_{2m}-\bar{T}+1)}\,|\varphi^+_m| \le k\mathrm{e}^{-\delta(T-\bar{T}+2)}\,|\varphi^+_m|\\
&\left|X_-^{-1}(T_{2m-1}-T_{2m}-1)\varphi^-_m\right| \le k\mathrm{e}^{-\delta(T_{2m}-T_{2m-1}+1-\bar{T})}\,|\varphi^-_m| \le k\mathrm{e}^{-\delta(T-\bar{T}+2)}\,|\varphi^-_m|\,.
\end{aligned} \tag{6.1.46}$$

Moreover,

$$|X_+(T_{2m+1}-T_{2m})\xi^+_m| = |X_+(T_{2m+1}-T_{2m})P_+X_+^{-1}(\bar{T})\xi^+_m| \le k\mathrm{e}^{-\delta(T-\bar{T}+1)}\,|\xi^+_m| \tag{6.1.47}$$

and, since $|\alpha_m-\alpha_{m+1}|<1$ implies that $T_{2m+2}-T_{2m+1}-\alpha_m+\alpha_{m+1}\ge T>\bar{T}$:

$$\begin{aligned}
&|X_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})\xi^-_{m+1}|\\
&\quad = |X_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})(\mathbb{I}-P_-)X_-^{-1}(-\bar{T})\xi^-_{m+1}|\\
&\quad \le k\mathrm{e}^{-\delta(T-\bar{T})}\,|\xi^-_{m+1}|,\\
&\qquad |\dot{\gamma}_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})| \le k\mathrm{e}^{-\delta(T-\bar{T})}\,|\dot{\gamma}_-(-\bar{T})|
\end{aligned} \tag{6.1.48}$$

for any $m\in\mathbb{Z}$. Next,

$$\begin{aligned}
&X_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})P_-X_-^{-1}(T_{2m+1}-T_{2m+2}-1)\varphi^-_{m+1}\\
&\quad \in \mathscr{R}P_-(T_{2m+2}-T_{2m+1}-\alpha_m+\alpha_{m+1}),\\
&X_+(T_{2m+1}-T_{2m})(\mathbb{I}-P_+)X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi^+_m\\
&\quad \in \mathscr{N}P_+(T_{2m+1}-T_{2m}),
\end{aligned}$$

and (see (6.1.9))

$$\mathscr{N}P_+(T_{2m+1}-T_{2m})\oplus\mathscr{R}P_-(T_{2m+2}-T_{2m+1}-\alpha_m+\alpha_{m+1})=\mathbb{R}^n.$$

Hence the linear map

$$\begin{aligned}
\mathscr{L}_{\alpha,m} : (\varphi^-_{m+1},\varphi^+_m) \mapsto\ & X_+(T_{2m+1}-T_{2m})(\mathbb{I}-P_+)X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi^+_m\\
& -X_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})P_-X_-^{-1}(T_{2m+1}-T_{2m+2}-1)\varphi^-_{m+1}
\end{aligned}$$

is a linear isomorphism from $\mathscr{R}P_{-,m+1}\oplus\mathscr{N}P_{+,m}=\mathbb{R}^n$ into $\mathscr{N}P_+(T_{2m+1}-T_{2m})\oplus\mathscr{R}P_-(T_{2m+2}-T_{2m+1}-\alpha_m+\alpha_{m+1})=\mathbb{R}^n$ whose inverse is given by:

$$\begin{aligned}\mathscr{L}_{\alpha,m}^{-1} : (\tilde{\varphi}_{m+1}^-, \tilde{\varphi}_m^+) \mapsto\, & X_+(T_{2m+1} - T_{2m} + 1)(\mathbb{I} - P_+)X_+^{-1}(T_{2m+1} - T_{2m})\tilde{\varphi}_m^+ \\ & -X_-(T_{2m+1} - T_{2m+2} - 1)P_- X_-^{-1}(T_{2m+1} - T_{2m+2} + \alpha_m - \alpha_{m+1})\tilde{\varphi}_{m+1}^- .\end{aligned}$$

Note that (see (6.1.3)):

$$\begin{aligned} & |X_-(T_{2m+1} - T_{2m+2} - 1)P_- X_-^{-1}(T_{2m+1} - T_{2m+2} + \alpha_m - \alpha_{m+1})\tilde{\varphi}_{m+1}^-| \\ & \quad \le k\, \mathrm{e}^{\delta(1+\alpha_m - \alpha_{m+1})}\, |\tilde{\varphi}_{m+1}^-| \le k\, \mathrm{e}^{\delta(1+\chi)}\, |\tilde{\varphi}_{m+1}^-|; \\ & |X_+(T_{2m+1} - T_{2m} + 1)(\mathbb{I} - P_+)X_+^{-1}(T_{2m+1} - T_{2m})\tilde{\varphi}_{m+1}^-| \le k\, \mathrm{e}^{\delta}\, |\tilde{\varphi}_{m+1}^-| \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial \alpha}\mathscr{L}_{\alpha,m}(\varphi_{m+1}^-, \varphi_m^+)\alpha = -f_-'\left(\gamma_-(T_{2m+1} - T_{2m+2} + \alpha_m - \alpha_{m+1})\right) \cdot \\ X_-(T_{2m+1} - T_{2m+2} + \alpha_m - \alpha_{m+1})P_- X_-^{-1}(T_{2m+1} - T_{2m+2} - 1)\varphi_{m+1}^-(\alpha_m - \alpha_{m+1}) . \end{aligned}$$

Thus we obtain (see also (6.1.10)):

$$\begin{aligned} |\mathscr{L}_{\alpha,m}(\varphi_{m+1}^-, \varphi_m^+)| &\le k\, \mathrm{e}^{-\delta}\, |\varphi_m^+| + k\, \mathrm{e}^{-\delta(1-\chi)}\, |\varphi_{m+1}^-| \le k\tilde{c} |\varphi_m^+ + \varphi_{m+1}^-|, \\ |\mathscr{L}_{\alpha,m}^{-1}(\widetilde{\varphi}_{m+1}^-, \varphi_m^+)| &\le k\, \mathrm{e}^{\delta}\, |\varphi_m^+| + k\, \mathrm{e}^{\delta(1+\chi)}\, |\widetilde{\varphi}_{m+1}^-| \le k\tilde{c}\, \mathrm{e}^{2\delta}\, |\varphi_m^+ + \varphi_{m+1}^-|, \\ \left| \frac{\partial}{\partial \alpha}\mathscr{L}_{\alpha,m}(\varphi_{m+1}^-, \varphi_m^+)\right| &\le 2N_- k |\varphi_{m+1}^-| \end{aligned}$$

for $N_- := \sup_{x\in\mathbb{R}^n} |f_-(x)|$. So, using also $\frac{\partial}{\partial\alpha}\mathscr{L}_{\alpha,m}^{-1} = \mathscr{L}_{\alpha,m}^{-1} \circ \frac{\partial}{\partial\alpha}\mathscr{L}_{\alpha,m} \circ \mathscr{L}_{\alpha,m}^{-1}$:

$$\begin{gathered} \|\mathscr{L}_{\alpha,m}\| \le k\tilde{c} \quad \text{and} \quad \|\mathscr{L}_{\alpha,m}^{-1}\| \le k\tilde{c}\, \mathrm{e}^{2\delta}, \\ \left\| \tfrac{\partial}{\partial\alpha}\mathscr{L}_{\alpha,m} \right\| \le 2N_- k \quad \text{and} \quad \left\| \tfrac{\partial}{\partial\alpha}\mathscr{L}_{\alpha,m}^{-1} \right\| \le 2N_- k^3 \tilde{c}^2\, \mathrm{e}^{4\delta} . \end{gathered}$$

Next, using (6.1.47), (6.1.48):

$$\begin{aligned} & \left|\mathscr{L}_\alpha(\varphi_{m+1}^-, \varphi_m^+, \xi_{m+1}^-, \xi_m^+, \bar{\xi}_m, \beta_m) - \mathscr{L}_{\alpha,m}(\varphi_{m+1}^-, \varphi_m^+)\right| \\ & \quad \le k\, \mathrm{e}^{-\delta(T-\bar{T})}(2 + |\dot{\gamma}_-(-\bar{T})|)\|\theta\| \end{aligned} \tag{6.1.49}$$

(recall $\theta = \{(\varphi_m^-, \varphi_m^+, \xi_m^-, \xi_m^+, \bar{\xi}_m, \beta_m)\}_{m\in\mathbb{Z}}$). We define $\mathscr{H}_\alpha : \ell^\infty_{\rho,\alpha,\varepsilon} \to \ell^\infty_1$ as

$$\mathscr{H}_\alpha \theta = \left\{ \begin{array}{c} \mathscr{L}_{\alpha,m}(\varphi_{m+1}^-, \varphi_m^+) \\ \bar{\xi}_m - \xi_m^- \\ R_0[X_0(\bar{T})\bar{\xi}_m - \xi_m^+] \\ G'(\gamma_0(-\bar{T}))\xi_m^- \\ G'(\gamma_0(\bar{T})) \cdot [X_0(\bar{T})\bar{\xi}_m + \dot{\gamma}_0(\bar{T})\beta_m] \\ G'(\gamma_+(\bar{T})) \cdot \xi_m^+ \end{array} \right\}_{m\in\mathbb{Z}} .$$

Clearly

$$\frac{\partial}{\partial\alpha}\mathscr{H}_\alpha\theta = \begin{Bmatrix} \frac{\partial}{\partial\alpha}\mathscr{L}_{\alpha,m}(\varphi_{m+1}^-, \varphi_m^+) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}_{m\in\mathbb{Z}}$$

and so

$$\left\|\frac{\partial}{\partial\alpha}\mathscr{H}_\alpha\right\| \le 2N_-k\,. \tag{6.1.50}$$

Next, note that

$$[D_1\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0) - \mathscr{H}_\alpha]\theta$$
$$= \begin{Bmatrix} \mathscr{L}_\alpha(\varphi_{m+1}^-, \varphi_m^+, \xi_{m+1}^-, \xi_m^+, \bar{\xi}_m, \beta_m) - \mathscr{L}_{\alpha,m}(\varphi_{m+1}^-, \varphi_m^+) \\ -X_-^{-1}(T_{2m-1} - T_{2m} - 1)\varphi_m^- \\ -R_0X_+^{-1}(T_{2m+1} - T_{2m} + 1)\varphi_m^+ \\ G'(\gamma_0(-\bar{T}))X_-^{-1}(T_{2m-1} - T_{2m} - 1)\varphi_m^- \\ 0 \\ G'(\gamma_+(\bar{T}))X_+^{-1}(T_{2m+1} - T_{2m} + 1)\varphi_m^+ \end{Bmatrix}_{m\in\mathbb{Z}}\,. \tag{6.1.51}$$

Hence, from (6.1.46) and (6.1.49), we get

$$\|D_1\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0) - \mathscr{H}_\alpha\| \le \widetilde{c}_3 k\,\mathrm{e}^{-\delta(T-\bar{T})} \tag{6.1.52}$$

where

$$\widetilde{c}_3 := \max\left\{2 + |\dot{\gamma}_-(-\bar{T})|, \|R_0\|\,\mathrm{e}^{-2\delta}, |G'(\gamma_0(-\bar{T}))|\,\mathrm{e}^{-2\delta}, |G'(\gamma_+(\bar{T}))|\,\mathrm{e}^{-2\delta}\right\}.$$

Next, given $\{(a_m, b_m, c_m, d_m, e_m, f_m)\}_{m\in\mathbb{Z}} \in \ell_1^\infty$ we want to solve the linear equation

$$\mathscr{H}_\alpha\theta = \begin{Bmatrix} a_m \\ b_m \\ c_m \\ d_m \\ e_m \\ f_m \end{Bmatrix}_{m\in\mathbb{Z}} \tag{6.1.53}$$

that is the set of equations:

$$\begin{cases} \mathscr{L}_{\alpha,m}(\varphi_{m+1}^-, \varphi_m^+) = a_m, \\ \bar{\xi}_m - \xi_m^- = b_m, \\ R_0[X_0(\bar{T})\bar{\xi}_m - \xi_m^+] = c_m, \\ G'(\gamma_0(-\bar{T}))\xi_m^- = d_m, \\ G'(\gamma_0(\bar{T})) \cdot [X_0(\bar{T})\bar{\xi}_m + \dot{\gamma}_0(\bar{T})\beta_m] = e_m, \\ G'(\gamma_+(\bar{T})) \cdot \xi_m^+ = f_m. \end{cases} \tag{6.1.54}$$

To solve (6.1.54) we write:

$$\begin{gathered} \xi_m^- = \eta_m^\perp + \mu_m^- \dot{\gamma}_-(-\bar{T}), \\ \xi_m^+ = \zeta_m^\perp + \mu_m^+ \dot{\gamma}_+(\bar{T}), \quad m \in \mathbb{Z}, \\ \{\eta_m^\perp\}_{m\in\mathbb{Z}} \in \ell^\infty(\mathscr{S}'), \quad \{\zeta_m^\perp\}_{m\in\mathbb{Z}} \in \ell^\infty(\mathscr{S}''), \quad \{\mu_m^\pm\}_{m\in\mathbb{Z}} \in \ell^\infty(\mathbb{R}), \end{gathered} \tag{6.1.55}$$

and plug (6.1.55) into (6.1.54). We obtain

$$\begin{gathered} (\varphi_{m+1}^-, \varphi_m^+) = \mathscr{L}_{\alpha,m}^{-1} a_m, \\ \mu_m^- = \frac{d_m}{G'(\gamma_-(-\bar{T}))\dot{\gamma}_-(-\bar{T})}, \\ \mu_m^+ = \frac{f_m}{G'(\gamma_+(\bar{T}))\dot{\gamma}_+(\bar{T})}, \\ \bar{\xi}_m = \eta_m^\perp + \mu_m^- \dot{\gamma}_-(-\bar{T}) + b_m, \\ \beta_m = \frac{e_m - G'(\gamma_0(\bar{T}))X_0(\bar{T})\bar{\xi}_m}{G'(\gamma_0(\bar{T}))\dot{\gamma}_0(\bar{T})}, \\ R_0X_0(\bar{T})\eta_m^\perp - \zeta_m^\perp = c_m - \mu_m^- R_0X_0(\bar{T})\dot{\gamma}_-(-\bar{T}) - R_0X_0(\bar{T})b_m + \mu_m^+ R_0\dot{\gamma}_+(\bar{T}). \end{gathered} \tag{6.1.56}$$

Now we denote by $\Pi : \mathscr{R}R_0 \to \mathscr{S}'' \oplus \mathscr{S}''' \subset \mathscr{R}R_0$ the orthogonal projection onto $\mathscr{S}'' \oplus \mathscr{S}'''$ along span$\{\psi\}$ (recall that $\psi \in \mathscr{R}R_0 = \mathscr{N}G'(\gamma(T))$ is a unitary vector so that (6.1.4) and (6.1.5) hold). In other words:

$$(\mathbb{I} - \Pi)w = \langle \psi, w\rangle \psi \tag{6.1.57}$$

for any $w \in \mathscr{R}R_0$. Assumption (H3) implies that the linear mapping $\mathscr{S}'' \oplus \mathscr{S}' \mapsto \mathscr{S}'' \oplus \mathscr{S}''' = \mathscr{R}\Pi$ defined as $(\zeta^\perp, \eta^\perp) \to -\zeta^\perp + R_0X_0(\bar{T})\eta^\perp$ is invertible. So in order to solve (6.1.56), we need to suppose

$$\{(a_m, b_m, c_m, d_m, e_m, f_m)\}_{m\in\mathbb{Z}} \in \ell^\infty(\mathscr{S}^{iv}),$$

where

$$\mathscr{S}^{iv} = \Big\{(a,b,c,d,e,f) \in \mathbb{R}^{2n} \times \mathscr{R}R_0 \times \mathbb{R}^3 : (\mathbb{I} - \Pi)L(a,b,c,d,e,f) = 0\Big\}$$

and $L : \mathbb{R}^n \times \mathbb{R}^n \times \mathscr{R}R_0 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathscr{R}R_0$ is the linear map given by:

$$\begin{aligned} L(a,b,c,d,e,f) = c - \frac{d}{G'(\gamma_-(-\bar{T}))\dot{\gamma}_-(-\bar{T})} R_0 X_0(\bar{T})\dot{\gamma}_-(-\bar{T}) \\ -R_0 X_0(\bar{T})b + \frac{f}{G'(\gamma_+(\bar{T}))\dot{\gamma}_+(\bar{T})} R_0 \dot{\gamma}_+(\bar{T}) . \end{aligned} \tag{6.1.58}$$

Note that $\mathscr{S}^{iv}$ is a codimension 1 linear subspace of $\mathbb{R}^{2n} \times \mathscr{R}R_0 \times \mathbb{R}^3$. Hence $\widetilde{\psi} \in \mathbb{R}^{2n} \times \mathscr{R}R_0 \times \mathbb{R}^3$ exists so that

$$\operatorname{span}\{\widetilde{\psi}\} \oplus \mathscr{S}^{iv} = \mathbb{R}^{2n} \times \mathscr{R}R_0 \times \mathbb{R}^3 .$$

Of course, to be more precisely, we can take $\widetilde{\psi}$ so that $\langle \widetilde{\psi}, v\rangle = 0$ for any $v \in \mathscr{S}^{iv}$, where $\langle \cdot,\cdot \rangle$ is the usual scalar product on $\mathbb{R}^{3n+3}$. To construct such a $\widetilde{\psi}$ we note that from (6.1.57), it follows that $(\mathbb{I} - \Pi)Lv = \langle \psi, Lv\rangle \psi = \langle L^*\psi, v\rangle \psi$, where we take the natural restriction of $\langle \cdot,\cdot \rangle$ onto $\mathscr{R}R_0 \subset \mathbb{R}^n$. Thus $v = (a,b,c,d,e,f) \in \mathscr{S}^{iv}$ if and only if $\langle L^*\psi, v\rangle = 0$ or $v \in \{L^*\psi\}^{\perp}$ and we can take

$$\widetilde{\psi} = L^*\psi / |L^*\psi| .$$

Let $\widetilde{\Pi} : \mathbb{R}^{2n} \times \mathscr{R}R_0 \times \mathbb{R}^3 \to \mathscr{S}^{iv}$ be the orthogonal projection onto $\mathscr{S}^{iv}$ along $\operatorname{span}\{\widetilde{\psi}\}$. Then

$$(\mathbb{I} - \widetilde{\Pi})v = \langle \widetilde{\psi}, v\rangle \widetilde{\psi} = \frac{\langle L^*\psi, v\rangle}{|L^*\psi|} \widetilde{\psi} = \frac{\langle \psi, Lv\rangle}{|L^*\psi|} \widetilde{\psi} .$$

We set

$$\ell^\infty_\psi = \ell^\infty(\operatorname{span}\{\widetilde{\psi}\}) \subset \ell^\infty_1 .$$

Let $\Pi_\psi : \ell^\infty_1 \to \ell^\infty(\mathscr{S}^{iv})$ be the projection onto $\ell^\infty(\mathscr{S}^{iv})$ along ℓ^∞_ψ given by

$$\Pi_\psi \left(\{(a_m,b_m,c_m,d_m,e_m,f_m)\}_{m\in\mathbb{Z}} \right) = \left\{ \widetilde{\Pi}(a_m,b_m,c_m,d_m,e_m,f_m) \right\}_{m\in\mathbb{Z}} .$$

In summary, we see from (6.1.56) that there is a continuous inverse $\mathscr{H}_\alpha^{-1} : \ell^\infty(\mathscr{S}^{iv}) \mapsto \ell^\infty_2$, where

$$\begin{aligned} \ell^\infty_2 = \Big\{ \{(\varphi_m^-, \varphi_m^+, \xi_m^-, \xi_m^+, \bar{\xi}_m, \beta_m)\}_{m\in\mathbb{Z}} \in \ell^\infty\left(\mathbb{R}^{5n+1}\right) : \\ (\varphi_m^-, \varphi_m^+, \xi_m^-, \xi_m^+, \bar{\xi}_m, \beta_m) \in \mathscr{R}P_{-,m} \times \mathscr{N}P_{+,m} \times \mathscr{N}P_- \times \mathscr{R}P_+ \times \mathbb{R}^{n+1}, \forall m \in \mathbb{Z} \Big\} . \end{aligned}$$

Note that from (6.1.56) it easily follows that $\|\mathscr{H}_\alpha^{-1}\|$ and $\left\|\frac{\partial}{\partial\alpha}\mathscr{H}_\alpha^{-1}\right\| \le \left\|\frac{\partial}{\partial\alpha}\mathscr{H}_\alpha\right\| \|\mathscr{H}_\alpha^{-1}\|^2$ are uniformly bounded with respect to α.

Finally, we define projections onto $\mathscr{R}G'(\gamma(\bar{T}))$ and $\mathscr{R}G'(\gamma(-\bar{T}))$, respectively, as

$$
\begin{aligned}
(\mathbb{I}-R_+)w &= \frac{G'(\gamma(\bar{T}))w}{G'(\gamma(\bar{T}))\dot{\gamma}_+(\bar{T})}\dot{\gamma}_+(\bar{T}) \\
(\mathbb{I}-R_-)w &= \frac{G'(\gamma(-\bar{T}))w}{G'(\gamma(-\bar{T}))\dot{\gamma}_-(-\bar{T})}\dot{\gamma}_-(-\bar{T}).
\end{aligned} \tag{6.1.59}
$$

Note that R_+ is the projection onto $\mathscr{N}G'(\gamma(\bar{T}))$ along $\dot{\gamma}_+(\bar{T})$ whereas R_- is the projection onto $\mathscr{N}G'(\gamma(-\bar{T}))$ along $\dot{\gamma}_-(-\bar{T})$. First, we observe that for any $w \in \mathbb{R}^n$ we have $[\mathbb{I}-P_+]R_+P_+ = 0$, since $\dot{\gamma}_+(\bar{T}) \in \mathscr{R}P_+$. So $R_+P_+ = P_+R_+P_+$ and then for any $w \in \mathbb{R}^n$ we have $R_+P_+w \in \mathscr{R}P_+ \cap \mathscr{R}R_+ = \mathscr{S}''$. As a consequence, we see that $\psi^* R_+P_+w = 0$ for any $w \in \mathbb{R}^n$ (see (6.1.4)). Similarly we see that $P_-R_-[\mathbb{I}-P_-] = 0$, hence $R_-[\mathbb{I}-P_-]w \in \mathscr{N}P_- \cap \mathscr{R}R_- = \mathscr{N}P_- \cap \mathscr{N}G'(\gamma(-T)) = \mathscr{S}'$ for any $w \in \mathbb{R}^n$. As a consequence, we get $\psi^* R_0X_0(T)R_-[\mathbb{I}-P_-]w = 0$ for any $w \in \mathbb{R}^n$ since $R_0X_0(T)R_-(\mathbb{I}-P_+)w \in R_0X_0(T)\mathscr{S}'$. Consequently we arrive at

$$
P_+^* R_+^* \psi = 0, \quad \left(\mathbb{I}-P_-^*\right) R_-^* X_0(\bar{T})^* R_0^* \psi = 0. \tag{6.1.60}
$$

Next we set:

$$
\psi(t) = \begin{cases} X_-^{-1*}(t) R_-^* X_0(\bar{T})^* R_0^* \psi, & \text{if } t \le -\bar{T}, \\ X_0^{-1*}(t) X_0(\bar{T})^* R_0^* \psi, & \text{if } -\bar{T} < t \le \bar{T}, \\ X_+^{-1*}(t) R_+^* \psi, & \text{if } t > \bar{T}, \end{cases} \tag{6.1.61}
$$

and

$$
\mathscr{M}(\alpha) = \int_{-\infty}^{\infty} \psi^*(t) g(t+\alpha, \gamma(t), 0)\, dt. \tag{6.1.62}
$$

Using (6.1.60), we easily obtain:

$$
\begin{aligned}
&|\psi(t)| \le \|X_+^{-1*}(t)(\mathbb{I}-P_+^*)X_+^*(\bar{T})\| |R_+^*\psi| \le k\|R_+\|\, \mathrm{e}^{-\delta(t-\bar{T})} && \text{if } t \ge \bar{T}, \\
&|\psi(t)| \le k\|R_0X_0(\bar{T})R_-\|\, \mathrm{e}^{\delta(t+\bar{T})} && \text{if } t \le -\bar{T}.
\end{aligned} \tag{6.1.63}
$$

Thus $\mathscr{M}(\alpha)$ is a well defined C^2 function because of Lebesgue theorem. We are now ready to state the following result.

Theorem 6.1.16. *Assume that $f_\pm(z)$ and $g(t,z,\varepsilon)$ are C^r–functions with bounded derivatives and that their r-order derivatives are uniformly continuous. Assume, moreover, that conditions (H1), (H2) and (H3) hold.*

Then given $c_0 > 0$ there exist constants $\rho_0 > 0$, $\chi > 0$ and $c_1 > 0$ so that for any $0 < \rho < \rho_0$, there is $\bar{\varepsilon}_\rho > 0$ so that for any ε, $0 < |\varepsilon| < \bar{\varepsilon}_\rho$, for any increasing sequence $\mathscr{T} = \{T_m\}_{m\in\mathbb{Z}} \subset \mathbb{R}$ with $T_m - T_{m-1} > \bar{T} + 1 - 2\delta^{-1}\ln|\varepsilon|$ so that

$$
\mathscr{M}\left(T_{2m} + \alpha_m^0\right) = 0 \; \forall m \in \mathbb{Z} \text{ and } \inf_{m\in\mathbb{Z}} |\mathscr{M}'\left(T_{2m} + \alpha_m^0\right)| > c_0 \tag{6.1.64}
$$

for some $\alpha_0 = \{\alpha_m^0\}_{m\in\mathbb{Z}} \in \ell_\chi^\infty$, there exist unique sequences $\{\hat{\alpha}_m\}_{m\in\mathbb{Z}} = \left\{\hat{\alpha}_m(\mathscr{T}, \varepsilon)\right\}_{m\in\mathbb{Z}} \in \ell_\chi^\infty(\mathbb{R})$ and $\{\hat{\beta}_m\}_{m\in\mathbb{Z}} = \{\hat{\beta}_m(\mathscr{T},\varepsilon)\}_{m\in\mathbb{Z}} \in \ell^\infty(\mathbb{R})$ with $|\hat{\alpha}_m(\mathscr{T},\varepsilon) - \alpha_m^0| <$

$c_1|\varepsilon|$ *and* $|\hat\beta_m(\mathscr{T},\varepsilon)-\alpha_m^0|<c_1|\varepsilon|\ \forall m\in\mathbb{Z}$*, and a unique bounded solution* $z(t)=z(\mathscr{T},\varepsilon)(t)$ *of system* (6.1.1) *so that*

$$\begin{aligned}
&\sup_{t\in[T_{2m-1}+\hat\beta_{m-1},T_{2m}-\bar T+\hat\alpha_m]}|z(t)-\gamma_-(t-T_{2m}-\hat\alpha_m)|<\rho,\\
&\sup_{t\in[T_{2m}-\bar T+\hat\alpha_m,T_{2m}+\bar T+\hat\beta_m]}|z(t)-\gamma_0(t-T_{2m}-\hat\alpha_m)|<\rho,\\
&\sup_{t\in[T_{2m}+\bar T+\hat\beta_m,T_{2m+1}+\hat\beta_m]}|z(t)-\gamma_+(t-T_{2m}-\hat\beta_m)|<\rho
\end{aligned}$$

for any $m\in\mathbb{Z}$ *(cf* (6.1.7)*). Hence* $z(t)$ *is orbitally close to* $\gamma(t)$ *in the sense that* $\mathrm{dist}(z(t),\Gamma)<\rho$ *where* $\Gamma=\{\gamma(t)\mid t\in\mathbb{R}\}$ *is the orbit of* $\gamma(t)$*.*

Proof. If ρ and $\bar\varepsilon_\rho<\varepsilon_\rho$ are sufficiently small then, for $t\in I^-_{m,\alpha}$, the solution $z(t)$ we look for must satisfy $z(t)=z^-_m(t,\xi_-,\varphi_-,\alpha,\varepsilon)$ for some value of the parameters $(\xi_-,\varphi_-,\alpha,\varepsilon)$ and similarly in the other intervals $[T_{2m}-\bar T+\alpha,T_{2m}+\bar T+\beta]$ and $I^+_{m,\beta}$. So, we solve Eq. (6.1.42) for $(\theta,\alpha)\in\ell^\infty_{\rho,\alpha,\varepsilon}\times\ell^\infty_\chi$ in terms of $\mathscr{T}$ and $\varepsilon\in(-\bar\varepsilon_\rho,\bar\varepsilon_\rho)$. Set

$$\begin{aligned}
&\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon)\\
&=\mathscr{G}_{\mathscr{T}}(\theta,\alpha,\varepsilon)-\mathscr{H}_\alpha(\theta-\theta_\alpha)\\
&=\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\\
&\quad+[\mathscr{G}_{\mathscr{T}}(\theta,\alpha,0)-\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)-D_1\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)(\theta-\theta_\alpha)]\\
&\quad+(D_1\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)-\mathscr{H}_\alpha)(\theta-\theta_\alpha)+\varepsilon\int_0^1 D_3\mathscr{G}_{\mathscr{T}}(\theta,\alpha,\tau\varepsilon)\,d\tau
\end{aligned}$$

where $D_3\mathscr{G}_{\mathscr{T}}(\theta,\alpha,\varepsilon)$ denotes the derivative of $\mathscr{G}_{\mathscr{T}}$ with respect to ε. It is easy to see that

$$\begin{gathered}
\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)=\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon),\quad D_1\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon)=D_1\mathscr{G}_{\mathscr{T}}(\theta,\alpha,\varepsilon)-\mathscr{H}_\alpha,\\
D_1\mathscr{F}_{\mathscr{T}}(\theta_1,\alpha,\varepsilon)-D_1\mathscr{F}_{\mathscr{T}}(\theta_2,\alpha,\varepsilon)=D_1\mathscr{G}_{\mathscr{T}}(\theta_1,\alpha,\varepsilon)-D_1\mathscr{G}_{\mathscr{T}}(\theta_2,\alpha,\varepsilon),\\
D_2\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon)=D_2\mathscr{G}_{\mathscr{T}}(\theta,\alpha,\varepsilon)-\frac{\partial\mathscr{H}_\alpha}{\partial\alpha}(\theta-\theta_\alpha)-\mathscr{H}_\alpha\frac{\partial\theta_\alpha}{\partial\alpha}.
\end{gathered}\tag{6.1.65}$$

For simplicity we also set:

$$\mu=\mathrm{e}^{-\delta(T-\bar T)}.$$

From the definition of $\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon)$ we see that Eq. (6.1.42) has the form

$$\theta-\theta_\alpha+\mathscr{H}_\alpha^{-1}\Pi_\psi\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon)=0,\tag{6.1.66}$$

and

$$\big(\mathbb{I}-\Pi_\psi\big)\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon)=0.\tag{6.1.67}$$

We denote with $c^{(1)}_{\mathscr{G}}$, resp. $c^{(2)}_{\mathscr{G}}$, upper bounds for the norms of the first order, resp. second order, derivatives of $\mathscr{G}_{\mathscr{T}}(\theta,\alpha,\varepsilon)$, in ℓ^∞_ρ. Thus for example,

$$c_{\mathscr{G}}^{(1)} = \sup_{(\theta,\alpha,\varepsilon)\in\ell_\rho^\infty} \{\|D_1\mathscr{G}_{\mathscr{T}}(\theta,\alpha,\varepsilon)\|, \|D_2\mathscr{G}_{\mathscr{T}}(\theta,\alpha,\varepsilon)\|, \|D_3\mathscr{G}_{\mathscr{T}}(\theta,\alpha,\varepsilon)\|\}$$

and $c_{\mathscr{G}}^{(2)}$ is similar. Then

$$\begin{aligned}
&\mathscr{G}_{\mathscr{T}}(\theta,\alpha,0) - \mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0) - D_1\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)(\theta-\theta_\alpha)\\
&\quad = \int_0^1 (D_1\mathscr{G}_{\mathscr{T}}(\tau\theta + (1-\tau)\theta_\alpha,\alpha,0) - D_1\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0))\, d\tau(\theta-\theta_\alpha)\\
&\quad = \eta(\theta,\theta_\alpha,\alpha)(\theta-\theta_\alpha),
\end{aligned}$$

where

$$\|\eta(\theta,\theta_\alpha,\alpha)\| \le c_{\mathscr{G}}^{(2)}\|\theta-\theta_\alpha\|.$$

Hence, since

$$\begin{aligned}
\mathscr{F}_{\mathscr{T}}\ &(\theta,\alpha,\varepsilon) - \mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)\\
&= \int_0^1 [D_1\mathscr{F}_{\mathscr{T}}(\tau\theta+(1-\tau)\theta_\alpha,\alpha,\varepsilon)]\, d\tau(\theta-\theta_\alpha)\\
&= \int_0^1 [D_1\mathscr{F}_{\mathscr{T}}(\tau\theta+(1-\tau)\theta_\alpha,\alpha,\varepsilon) - D_1\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)] d\tau(\theta-\theta_\alpha)\\
&\quad + D_1\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)(\theta-\theta_\alpha)\\
&= \int_0^1 [D_1\mathscr{G}_{\mathscr{T}}(\tau\theta+(1-\tau)\theta_\alpha,\alpha,\varepsilon) - D_1\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)] d\tau(\theta-\theta_\alpha)\\
&\quad + [D_1\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon) - \mathscr{H}_\alpha](\theta-\theta_\alpha)
\end{aligned} \tag{6.1.68}$$

(see also (6.1.65)) we derive, using also (6.1.52) (recall $\mu = \mathrm{e}^{-\delta(T-\bar{T})}$)

$$\|\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon) - \mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)\| \le \frac{1}{2}c_{\mathscr{G}}^{(2)}\|\theta-\theta_\alpha\|^2 + (k\widetilde{c}_3\mu + c_{\mathscr{G}}^{(2)}|\varepsilon|)\|\theta-\theta_\alpha\| \tag{6.1.69}$$

and (see also (6.1.43), (6.1.65))

$$\|\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon)\| \le \frac{c_{\mathscr{G}}^{(2)}}{2}\|\theta-\theta_\alpha\|^2 + (k\widetilde{c}_3\mu + c_{\mathscr{G}}^{(2)}|\varepsilon|)\|\theta-\theta_\alpha\| + c_{\mathscr{G}}^{(1)}|\varepsilon| + c_\gamma\mu \tag{6.1.70}$$

where $c_\gamma = 2k\delta^{-1}\max\{|\dot{\gamma}_-(-\bar{T})|, |\dot{\gamma}_+(\bar{T})|\}$. Note that c_γ, $c_{\mathscr{G}}^{(1)}$, $c_{\mathscr{G}}^{(2)}$ and $\widetilde{c}_3$ do not depend on $(\alpha,\mathscr{T},\varepsilon)\in\ell_\chi^\infty\times\ell_{\mathscr{T}}^\infty(\mathbb{R})\times\mathbb{R}$. Next, from (6.1.50), (6.1.52) and (6.1.65) we get

$$\begin{aligned}
&\|D_1\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\| \le k\widetilde{c}_3\mu,\\
&\|D_1\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon) - D_1\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)\| \le c_{\mathscr{G}}^{(2)}\|\theta-\theta_\alpha\|,\\
&\|D_2\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon) - D_2\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)\| \le \left(c_{\mathscr{G}}^{(2)} + 2kN_-\right)\|\theta-\theta_\alpha\|.
\end{aligned} \tag{6.1.71}$$

From (6.1.70) and (6.1.71) we conclude that

$$\lim_{(\theta,\varepsilon,\mu)\to(\theta_\alpha,0,0)} \mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon)=0, \qquad \lim_{(\theta,\varepsilon,\mu)\to(\theta_\alpha,0,0)} D_1\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon)=0$$

uniformly with respect to α. Thus, if $\bar{\rho}_0>0$, $\mu_0>0$ and $0<\bar{\varepsilon}_0\le\varepsilon_\rho$ are sufficiently small and $0<\mu<\mu_0$, $|\varepsilon|<\bar{\varepsilon}_0$, from the implicit function theorem the existence follows of a unique solution $\theta=\theta_{\mathscr{T}}(\alpha,\varepsilon)$ of (6.1.66) which is defined for any $\alpha\in\ell^\infty_\chi$, $|\varepsilon|<\bar{\varepsilon}_0$, $0<\mu\le\mu_0$ and $\mathscr{T}=\{T_m\}_{m\in\mathbb{Z}}$ so that $T_{m+1}-T_m>T+1$ where $T-\bar{T}=-\delta^{-1}\ln\mu$. Moreover $\theta_{\mathscr{T}}(\alpha,\varepsilon)$ satisfies

$$\sup_{\alpha,\mathscr{T},\varepsilon}\|\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha\|<\bar{\rho}_0 \tag{6.1.72}$$

with the sup being taken over all α, $\mathscr{T}$ and ε satisfying the above conditions. Next, using (6.1.66) with $\theta_{\mathscr{T}}(\alpha,\varepsilon)$ instead of θ and (6.1.70), we see that:

$$\begin{aligned}&\|\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha\|\le\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|\|\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)\|\le\\ &\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|\left(\frac{c_{\mathscr{G}}^{(2)}}{2}\|\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha\|^2+(k\widetilde{c}_3\mu+c_{\mathscr{G}}^{(2)}|\varepsilon|)\|\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha\|\right.\\ &\qquad\qquad\left.+c_{\mathscr{G}}^{(1)}|\varepsilon|+c_\gamma\mu\right).\end{aligned}$$

Hence if $\bar{\rho}_0$, μ_0 and ε_0 are so small that

$$\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|[c_{\mathscr{G}}^{(2)}(\bar{\rho}_0+2\varepsilon_0)+2k\widetilde{c}_3\mu_0]<1 \tag{6.1.73}$$

we obtain:

$$\|\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha\|\le2\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|(c_\gamma\mu+c_{\mathscr{G}}^{(1)}|\varepsilon|). \tag{6.1.74}$$

Note that since $\widetilde{\Pi}$ is an orthogonal projection, it is enough to choose μ_0, ε_0 and $\bar{\rho}_0$ in such a way that $c_{\mathscr{G}}^{(2)}(\bar{\rho}_0+2\varepsilon_0)+2k\widetilde{c}_3\mu_0<\|\mathscr{H}_\alpha^{-1}\|^{-1}$. Moreover, plugging (6.1.74) into (6.1.69) we obtain

$$\begin{aligned}&\|\mathscr{F}_{\mathscr{T}}\ (\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)-\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)\|\\ &\quad\le2c_{\mathscr{G}}^{(2)}\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|^2(c_\gamma\mu+c_{\mathscr{G}}^{(1)}|\varepsilon|)^2\\ &\qquad+2(k\widetilde{c}_3\mu+c_{\mathscr{G}}^{(2)}|\varepsilon|)\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|(c_\gamma\mu+c_{\mathscr{G}}^{(1)}|\varepsilon|)\le\Lambda_1(\mu+|\varepsilon|)^2\end{aligned} \tag{6.1.75}$$

where $\Lambda_1>0$ is independent of $(\mathscr{T},\alpha,\mu,\varepsilon)$. For example:

$$\Lambda_1=2\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|\max\{c_\gamma,c_{\mathscr{G}}^{(1)},c_{\mathscr{G}}^{(2)},k\widetilde{c}_3\}^2\left[\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|c_{\mathscr{G}}^{(2)}+1\right].$$

Next, differentiating the equality

$$\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha+\mathscr{H}_\alpha^{-1}\Pi_\psi\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)=0$$

with respect to α we obtain:

$$\begin{aligned}\frac{\partial}{\partial\alpha}[\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha] &= -\mathscr{H}_\alpha^{-1}\Pi_\psi\tfrac{\partial}{\partial\alpha}\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)\\ &\quad-\left[\frac{\partial}{\partial\alpha}\mathscr{H}_\alpha^{-1}\Pi_\psi\right]\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)\\ &= -\mathscr{H}_\alpha^{-1}\Pi_\psi\Big\{\tfrac{\partial}{\partial\alpha}[\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)-\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)]\\ &\quad+\frac{\partial}{\partial\alpha}[\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)-\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,0)]+\frac{\partial}{\partial\alpha}\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\Big\}\\ &\quad-\left[\frac{\partial}{\partial\alpha}\mathscr{H}_\alpha^{-1}\Pi_\psi\right]\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon).\end{aligned} \tag{6.1.76}$$

Then note that

$$\begin{aligned}&\frac{\partial}{\partial\alpha}\;[\;\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)-\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)]\\ &\quad=\frac{\partial}{\partial\alpha}\int_0^1 D_1\mathscr{F}_{\mathscr{T}}(\tau\theta_{\mathscr{T}}(\alpha,\varepsilon)+(1-\tau)\theta_\alpha,\alpha,\varepsilon)\,d\tau(\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha)\\ &\quad=\Big\{\int_0^1 D_1^2\mathscr{F}_{\mathscr{T}}(\tau\theta_{\mathscr{T}}(\alpha,\varepsilon)+(1-\tau)\theta_\alpha,\alpha,\varepsilon)\frac{\partial}{\partial\alpha}[\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha]\tau\,d\tau\\ &\qquad+\int_0^1 D_1^2\mathscr{F}_{\mathscr{T}}(\tau\theta_{\mathscr{T}}(\alpha,\varepsilon)+(1-\tau)\theta_\alpha,\alpha,\varepsilon)\frac{d}{d\alpha}\theta_\alpha\,d\tau\\ &\qquad+\int_0^1 D_1D_2\mathscr{F}_{\mathscr{T}}(\tau\theta_{\mathscr{T}}(\alpha,\varepsilon)+(1-\tau)\theta_\alpha,\alpha,\varepsilon)\,d\tau\Big\}(\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha)\\ &\qquad+\int_0^1 D_1\mathscr{F}_{\mathscr{T}}(\tau\theta_{\mathscr{T}}(\alpha,\varepsilon)+(1-\tau)\theta_\alpha,\alpha,\varepsilon)\,d\tau\frac{\partial}{\partial\alpha}[\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha].\end{aligned} \tag{6.1.77}$$

First we derive

$$\begin{aligned}&\left\|\int_0^1 D_1^2\mathscr{F}_{\mathscr{T}}(\tau\theta_{\mathscr{T}}(\alpha,\varepsilon)+(1-\tau)\theta_\alpha,\alpha,\varepsilon)\frac{\partial}{\partial\alpha}[\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha]\tau\,d\tau\right\|\\ &\quad\le\int_0^1 c_{\mathscr{G}}^{(2)}\tau\,d\tau\left\|\frac{\partial}{\partial\alpha}[\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha]\right\|=\frac{1}{2}c_{\mathscr{G}}^{(2)}\left\|\frac{\partial}{\partial\alpha}[\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha]\right\|.\end{aligned}$$

Next, from (6.1.71) we obtain

$$\left\| \int_0^1 D_1 \mathscr{F}_{\mathscr{T}}(\tau\theta_{\mathscr{T}}(\alpha,\varepsilon)+(1-\tau)\theta_\alpha,\alpha,\varepsilon)\,d\tau \frac{\partial}{\partial\alpha}[\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha] \right\|$$

$$\leq \Big(\int_0^1 \|D_1\mathscr{F}_{\mathscr{T}}(\tau\theta_{\mathscr{T}}(\alpha,\varepsilon)+(1-\tau)\theta_\alpha,\alpha,\varepsilon)-D_1\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)]\|\,d\tau$$

$$+\|D_1\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)-D_1\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\|+\|D_1\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\|\Big)$$

$$\left\|\frac{\partial}{\partial\alpha}[\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha]\right\|$$

$$\leq \Big(\int_0^1 c_{\mathscr{G}}^{(2)}\|\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha\|\,\tau\,d\tau + c_{\mathscr{G}}^{(2)}|\varepsilon|+k\widetilde{c}_3\mu\Big)\left\|\frac{\partial}{\partial\alpha}[\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha]\right\|$$

$$\leq \left(c_{\mathscr{G}}^{(2)}\left(\frac{1}{2}\|\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha\|+|\varepsilon|\right)+k\widetilde{c}_3\mu\right)\left\|\frac{\partial}{\partial\alpha}[\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha]\right\|.$$

Finally, using (6.1.50), (6.1.72) and (6.1.74), the identity

$$\frac{d\theta_\alpha}{d\alpha}=(0,0,0,0,0,\mathbb{I}) \tag{6.1.78}$$

and $D_1D_2\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon)=D_1D_2\mathscr{G}_{\mathscr{T}}(\theta,\alpha,\varepsilon)-\frac{\partial\mathscr{H}_\alpha}{\partial\alpha}$, we conclude

$$\begin{aligned}&\left\|\frac{\partial}{\partial\alpha}[\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)-\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)]\right\|\\ &\leq [c_{\mathscr{G}}^{(2)}(\bar{\rho}_0+\varepsilon_0)+k\widetilde{c}_3\mu_0]\left\|\frac{\partial}{\partial\alpha}[\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha]\right\|\\ &\quad +4\left(c_{\mathscr{G}}^{(2)}+kN_-\right)\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|\left(c_\gamma\mu+c_{\mathscr{G}}^{(1)}|\varepsilon|\right).\end{aligned} \tag{6.1.79}$$

Similarly, we obtain

$$\begin{aligned}&\left\|\frac{\partial}{\partial\alpha}[\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)-\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,0)]\right\|=|\varepsilon|\\ &\left\|\frac{\partial}{\partial\alpha}\int_0^1 D_3\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\tau\varepsilon)\,d\tau\right\|\leq 2c_{\mathscr{G}}^{(2)}|\varepsilon|.\end{aligned} \tag{6.1.80}$$

Now, since

$$\left\|\frac{\partial}{\partial\alpha}\mathscr{H}_\alpha^{-1}\Pi_\psi\right\|\leq\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|^2\left\|\frac{\partial}{\partial\alpha}\mathscr{H}_\alpha\right\|\leq 2kN_-\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|^2,$$

we derive, using also (6.1.75), (6.1.43):

$$\begin{aligned}&\left\|\left[\frac{\partial}{\partial\alpha}\mathscr{H}_\alpha^{-1}\Pi_\psi\right]\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)\right\|\\&\le 2kN_-\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|^2\\&\quad\cdot\left\{\|\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)-\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)\|+\|\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)\|\right\}\\&\le 2kN_-\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|^2\left[\Lambda_1(\mu+|\varepsilon|)^2+c_\gamma\mu+c_{\mathscr{G}}^{(1)}|\varepsilon|\right].\end{aligned}\tag{6.1.81}$$

Plugging (6.1.79), (6.1.80), (6.1.81) into (6.1.76) and assuming, instead of (6.1.73), that

$$2\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|[c_{\mathscr{G}}^{(2)}(\bar\rho_0+\bar\varepsilon_0)+k\widetilde{c}_3\mu_0]\le 1$$

we obtain

$$\begin{aligned}&\left\|\frac{\partial}{\partial\alpha}[\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha]\right\|\\&\le 2\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|\Big\{4\left(c_{\mathscr{G}}^{(2)}+kN_-\right)\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|\left(c_\gamma\mu+c_{\mathscr{G}}^{(1)}|\varepsilon|\right)\\&\quad+2c_{\mathscr{G}}^{(2)}|\varepsilon|+c_\gamma\mu+2kN_-\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|\left[\Lambda_1(\mu+|\varepsilon|)^2+c_\gamma\mu+c_{\mathscr{G}}^{(1)}|\varepsilon|\right]\Big\}\\&\le\Lambda_2(\mu+|\varepsilon|),\end{aligned}\tag{6.1.82}$$

where Λ_2 is a positive constant that does not depend on $(\mathscr{T},\alpha,\mu,\varepsilon)$. We now take

$$\mu=\varepsilon^2$$

that is $T=\bar T-2\delta^{-1}\ln|\varepsilon|$. Note that from (6.1.74), we get:

$$\|\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha\|\le 2\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|(c_\gamma|\varepsilon|+c_{\mathscr{G}}^{(1)})|\varepsilon|.\tag{6.1.83}$$

Then, if we can solve the equation $(\mathbb{I}-\Pi_\psi)\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)=0$ for $\alpha=\alpha_{\mathscr{T}}(\varepsilon)=\{\alpha_{m,\mathscr{T}}(\varepsilon)\}_{m\in\mathbb{Z}}$ and define $z_{m,\mathscr{T}}^{\pm}(t,\varepsilon)$, $z_{m,\mathscr{T}}^{0}(t,\varepsilon)$ as $z_m^+(t,\xi_m^+,\varphi_m^+,\beta_m,\varepsilon)$, $z_m^-(t,\xi_m^-,\varphi_m^-,\alpha_m,\varepsilon)$ and $z_m^0(t,\bar\xi_m,\alpha_m,\beta_m,\varepsilon)$, with

$$\theta_{\mathscr{T}}(\varepsilon)=\theta_{\mathscr{T}}(\alpha_{\mathscr{T}}(\varepsilon),\varepsilon)$$

instead of $\theta=\left\{(\varphi_m^-,\varphi_m^+,\xi_m^-,\xi_m^+,\bar\xi_m,\beta_m)\right\}_{m\mathbb{Z}}$ and with $\mu=\varepsilon^2$, we see that condition (6.1.7) follows from (6.1.31), (6.1.34) and (6.1.40) provided $|\varepsilon|<\varepsilon_\rho$, taking ε_ρ smaller if necessary. Thus to complete the proof of Theorem 6.1.16 we only need to show that the equation

$$(\mathbb{I}-\Pi_\psi)\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)=0$$

can be solved for α in terms of $\varepsilon\in(-\varepsilon_\rho,\varepsilon_\rho)$ and $\mathscr{T}$ satisfying the conditions of Theorem 6.1.16. Now, from (6.1.83) we see that

$$\lim_{\varepsilon\to 0}(\mathbb{I}-\Pi_\psi)\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)=\lim_{\varepsilon\to 0}(\mathbb{I}-\Pi_\psi)\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)=0$$

uniformly with respect to $(\alpha,\mathscr{T})$ (recall that, see (6.1.43), $\|\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\|\le c_\gamma\mu = c_\gamma\varepsilon^2$). Hence we are led to prove that the bifurcation function

$$\frac{1}{\varepsilon}(\mathbb{I}-\Pi_\psi)\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)=0 \tag{6.1.84}$$

can be solved for α in terms of $\varepsilon\in(-\varepsilon_\rho,\varepsilon_\rho)$, $\varepsilon\neq 0$, and $\mathscr{T}$ satisfying the conditions of Theorem 6.1.16. We observe that, with $\mu=\varepsilon^2$, (6.1.75) reads:

$$\|\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)-\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)\|\le\Lambda_1(1+|\varepsilon|)^2\varepsilon^2.$$

Hence, using also (6.1.65) and (6.1.43) with $\mu=\mathrm{e}^{-\delta(T-\bar{T})}=\varepsilon^2$:

$$\begin{aligned}B_{\mathscr{T}}(\alpha,\varepsilon)&=\frac{1}{\varepsilon}\left(\mathbb{I}-\Pi_\psi\right)\left\{\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)+O(\varepsilon^2)\right\}\\&=\frac{1}{\varepsilon}\left(\mathbb{I}-\Pi_\psi\right)[\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)-\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)]+O(\varepsilon)\\&=\left(\mathbb{I}-\Pi_\psi\right)D_3\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)+O(\varepsilon)\end{aligned}$$

where $O(\varepsilon)$ is uniform with respect to $(\mathscr{T},\alpha)$. Now we look at:

$$D_1B_{\mathscr{T}}(\alpha,\varepsilon)=\frac{1}{\varepsilon}(\mathbb{I}-\Pi_\psi)\frac{\partial}{\partial\alpha}\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)\,. \tag{6.1.85}$$

Subtracting

$$\begin{aligned}&\left(D_1^2\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\frac{d\theta_\alpha}{d\alpha}+D_1D_2\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\right)(\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha)\\&\quad=\frac{d}{d\alpha}[D_1\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,0)](\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha)\end{aligned}$$

from both sides of (6.1.77) and using the uniform continuity of $D_1^2\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon)$, $D_1D_2\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon)$ in $(\theta,\alpha,\varepsilon)$, uniformly with respect to $\mathscr{T}$ we see that:

$$\begin{aligned}&\Big\|\frac{\partial}{\partial\alpha}\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)-\frac{\partial}{\partial\alpha}\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)\\&\quad-\left(D_1^2\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\frac{d\theta_\alpha}{d\alpha}+D_1D_2\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\right)(\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha)\Big\|\\&\le\left(\left(c_{\mathscr{G}}^{(2)}\|\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha\|+|\varepsilon|\right)+k\widetilde{c}_3\varepsilon^2\right)\left\|\frac{\partial}{\partial\alpha}(\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha)\right\|\\&\quad+\eta(\|\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha\|+|\varepsilon|)\|\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha\|\end{aligned}$$

where $\eta(r) \to 0$ as $r \to 0$, uniformly with respect to $(\mathscr{T}, \alpha, \varepsilon)$, So, using (6.1.83) and (6.1.82) with $\mu = \varepsilon^2$ we obtain:

$$\begin{aligned} &\frac{\partial}{\partial \alpha} \mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon) - \frac{\partial}{\partial \alpha}\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon) \\ &\quad - \frac{d}{d\alpha}[D_1\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,0)](\theta_{\mathscr{T}}(\alpha,\varepsilon) - \theta_\alpha) = o(\varepsilon) \end{aligned} \tag{6.1.86}$$

uniformly with respect to $(\alpha, \mathscr{T})$. So, plugging (6.1.86) into (6.1.85), using (6.1.65) and (6.1.44) with $\mu = \mathrm{e}^{-\delta(T-\bar{T})} = \varepsilon^2$, we obtain:

$$\begin{aligned} D_1 B_{\mathscr{T}}(\alpha,\varepsilon) &= (\mathbb{I} - \Pi_\psi)\frac{\partial}{\partial \alpha}\frac{\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon) - \mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,0)}{\varepsilon} \\ &\quad + (\mathbb{I} - \Pi_\psi)\left\{\varepsilon^{-1}\frac{d}{d\alpha}[D_1\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,0)][\theta_{\mathscr{T}}(\alpha,\varepsilon) - \theta_\alpha]\right\} + o(1) \\ &= \frac{d}{d\alpha}(\mathbb{I} - \Pi_\psi)D_3\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0) \\ &\quad + (\mathbb{I} - \Pi_\psi)\left\{\varepsilon^{-1}\frac{d}{d\alpha}[D_1\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0) - \mathscr{H}_\alpha][\theta_{\mathscr{T}}(\alpha,\varepsilon) - \theta_\alpha]\right\} \\ &\quad + o(1) \end{aligned}$$

with $o(1)$ being uniform with respect to α. But, differentiating (6.1.51) we see that

$$\frac{d}{d\alpha}(D_1\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0) - \mathscr{H}_\alpha) = \{(\mathscr{L}_m^\alpha,0,0,0,0,0)\}_{m\in\mathbb{Z}}$$

where

$$\begin{aligned} \mathscr{L}_m^\alpha(\widetilde{\alpha})(\theta) &= \mathscr{L}_m^\alpha(\widetilde{\alpha})(\varphi_{m+1}^-,\varphi_m^+,\xi_{m+1}^-,\xi_m^+,\bar{\xi}_m,\beta_m) \\ &= \left[\dot{X}_-(T_{2m+1} - T_{2m+2} + \alpha_m - \alpha_{m+1})(\widetilde{\alpha}_{m+1} - \widetilde{\alpha}_m)\right]\xi_{m+1}^- \\ &\quad + [\ddot{\gamma}_-(T_{2m+1} - T_{2m+2} + \alpha_m - \alpha_{m+1})(\widetilde{\alpha}_{m+1} - \widetilde{\alpha}_m)]\beta_m \\ &\le 2N_- k\delta^{-1}(\delta + |\dot{\gamma}_-(-\bar{T})|)\mu\|\theta\|\,\|\widetilde{\alpha}\| = O(\varepsilon^2)\|\theta\|\|\widetilde{\alpha}\| \end{aligned}$$

and hence
$$\left\|\frac{d}{d\alpha}[D_1\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0) - \mathscr{H}_\alpha]\right\| = O(\varepsilon^2).$$

In summary, we obtain:

$$D_1 B_{\mathscr{T}}(\alpha,\varepsilon) = \frac{d}{d\alpha}[(\mathbb{I} - \Pi_\psi)D_3\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)] + o(1) \tag{6.1.87}$$

uniformly with respect to α and $\mathscr{T}$. We have then

$$\lim_{\varepsilon\to 0} B_{\mathscr{T}}(\alpha,\varepsilon) = (\mathbb{I}-\Pi_\psi)D_3\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0) = \frac{1}{|L^*\psi|}\langle\psi, LD_3\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\rangle\widetilde{\psi},$$

$$\lim_{\varepsilon\to 0} D_1 B_{\mathscr{T}}(\alpha,\varepsilon) = \frac{d}{d\alpha}\frac{1}{|L^*\psi|}\langle\psi, LD_3\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\rangle\widetilde{\psi},$$

uniformly with respect to α and $\mathscr{T}$ (recall that L has been defined in (6.1.58)). To conclude the proof of Theorem 6.1.16 we evaluate $\langle\psi, LD_3\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\rangle$. We have:

$$D_3\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0) =$$

$$\left\{\begin{matrix} \frac{\partial z_m^+}{\partial\varepsilon}(T_{2m+1}+\alpha_m,0,0,\alpha_m,0) - \frac{\partial z_{m+1}^-}{\partial\varepsilon}(T_{2m+1}+\alpha_m,0,0,\alpha_{m+1},0) \\ -\frac{\partial z_m^-}{\partial\varepsilon}(T_{2m}-\bar{T}+\alpha_m,0,0,\alpha_m,0) \\ R_0\left[\frac{\partial z_m^0}{\partial\varepsilon}(T_{2m}+\bar{T}+\alpha_m,\gamma_0(-\bar{T}),\alpha_m,\alpha_m,0) - \frac{\partial z_m^+}{\partial\varepsilon}(T_{2m}+\bar{T}+\alpha_m,0,0,\alpha_m,0)\right] \\ G'(\gamma(-\bar{T}))\frac{\partial z_m^-}{\partial\varepsilon}(T_{2m}-\bar{T}+\alpha_m,0,0,\alpha_m,0) \\ G'(\gamma(\bar{T}))\frac{\partial z_m^0}{\partial\varepsilon}(T_{2m}+\bar{T}+\alpha_m,\gamma_0(-\bar{T}),\alpha_m,\alpha_m,0) \\ G'(\gamma(\bar{T}))\frac{\partial z_m^+}{\partial\varepsilon}(T_{2m}+\bar{T}+\alpha_m,0,0,\alpha_m,0) \end{matrix}\right\}_{m\in\mathbb{Z}}.$$

Thus:

$$\begin{aligned} LD\ {}_\varepsilon\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0) \\ = R_0\Big\{&\frac{\partial z_m^0}{\partial\varepsilon}(T_{2m}+\bar{T}+\alpha_m,\gamma_0(-\bar{T}),\alpha_m,\alpha_m,0) \\ &-\frac{\partial z_m^+}{\partial\varepsilon}(T_{2m}+\bar{T}+\alpha_m,0,0,\alpha_m,0) \\ &-\frac{G'(\gamma(-\bar{T}))\frac{\partial z_m^-}{\partial\varepsilon}(T_{2m}-\bar{T}+\alpha_m,0,0,\alpha_m,0)}{G'(\gamma(-\bar{T}))\dot{\gamma}_-(-\bar{T})}X_0(\bar{T})\dot{\gamma}_-(-\bar{T}) \\ &+X_0(\bar{T})\frac{\partial z_m^-}{\partial\varepsilon}(T_{2m}-\bar{T}+\alpha_m,0,0,\alpha_m,0) \\ &+\frac{G'(\gamma(\bar{T}))\frac{\partial z_m^+}{\partial\varepsilon}(T_{2m}+\bar{T}+\alpha_m,0,0,\alpha_m,0)}{G'(\gamma(\bar{T}))\dot{\gamma}_+(\bar{T})}\dot{\gamma}_+(\bar{T})\Big\} \end{aligned}$$

$$
\begin{aligned}
&= R_0\Big\{\frac{\partial z_m^0}{\partial \varepsilon}(T_{2m}+\bar{T}+\alpha_m,\gamma_0(-\bar{T}),\alpha_m,\alpha_m,0)\\
&\quad +X_0(\bar{T})R_-\frac{\partial z_m^-}{\partial \varepsilon}(T_{2m}-\bar{T}+\alpha_m,0,0,\alpha_m,0)\\
&\quad -R_+\frac{\partial z_m^+}{\partial \varepsilon}(T_{2m}+\bar{T}+\alpha_m,0,0,\alpha_m,0)\dot{\gamma}_+(\bar{T})\Big\}\\
&= R_0\Big\{\frac{\partial z_m^0}{\partial \varepsilon}(T_{2m}+\bar{T}+\alpha_m,\gamma_0(-\bar{T}),\alpha_m,\alpha_m,0)\\
&\quad +X_0(\bar{T})R_-\frac{\partial z_m^-}{\partial \varepsilon}(T_{2m}-\bar{T}+\alpha_m,0,0,\alpha_m,0)\Big\}\\
&\quad -R_+\frac{\partial z_m^+}{\partial \varepsilon}(T_{2m}+\bar{T}+\alpha_m,0,0,\alpha_m,0)\dot{\gamma}_+(\bar{T})
\end{aligned}
$$

since $\mathscr{R}R_+\subset\mathscr{R}R_0$. Next from Eqs. (6.1.32), (6.1.36), (6.1.39) we get:

$$
\begin{aligned}
&\frac{\partial z_m^0}{\partial \varepsilon}(T_{2m}+\bar{T}+\alpha_m,\gamma_0(-\bar{T}),\alpha_m,\alpha_m,0)\\
&\quad =\int_{-\bar{T}}^{\bar{T}} X_0(\bar{T})X_0^{-1}(t)g(t+T_{2m}+\alpha_m,\gamma_0(t),0)dt\,,\\
&\frac{\partial z_m^-}{\partial \varepsilon}(T_{2m}-\bar{T}+\alpha_m,0,0,\alpha_m,0)\\
&\quad =\int_{T_{2m-1}-T_{2m}-1}^{-\bar{T}} P_-X_-^{-1}(t)g(t+T_{2m}+\alpha_m,\gamma_-(t),0)dt\,,\\
&\frac{\partial z_m^+}{\partial \varepsilon}(T_{2m}+\bar{T}+\alpha_m,0,0,\alpha_m,0)\\
&\quad =-\int_{\bar{T}}^{T_{2m+1}-T_{2m}+1}(\mathbb{I}-P_+)X_+^{-1}(t)g(t+T_{2m}+\alpha_m,\gamma_+(t),0)dt\,.
\end{aligned}
\tag{6.1.88}
$$

As a consequence, using also (6.1.60), we get:

$$
\begin{aligned}
&\langle\psi,LD_3\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\rangle\\
&=\psi^*\Big[\int_{T_{2m-1}-T_{2m}-1}^{-\bar{T}} R_0X_0(\bar{T})R_-P_-X_-^{-1}(t)g(t+T_{2m}+\alpha_m,\gamma_-(t),0)dt\\
&\quad +\int_{-\bar{T}}^{\bar{T}} R_0X_0(\bar{T})X_0^{-1}(t)g(t+T_{2m}+\alpha_m,\gamma_0(t),0)dt\\
&\quad +\int_{\bar{T}}^{T_{2m+1}-T_{2m}+1} R_+(\mathbb{I}-P_+)X_+^{-1}(t)g(t+T_{2m}+\alpha_m,\gamma_+(t),0)dt\Big]\\
&=\int_{T_{2m-1}-T_{2m}-1}^{T_{2m+1}-T_{2m}+1}\psi^*(t)g(t+T_{2m}+\alpha_m,\gamma(t),0)dt\\
&=\int_{-\infty}^{\infty}\psi^*(t)g(t+T_{2m}+\alpha_m,\gamma(t),0)dt+O(\mathrm{e}^{-\delta(T+1)})\\
&=\int_{-\infty}^{\infty}\psi^*(t)g(t+T_{2m}+\alpha_m,\gamma(t),0)dt+O(\varepsilon^2)
\end{aligned}
\tag{6.1.89}
$$

where $\psi(t)$ has been defined in (6.1.61). Thus we prove that

$$B_{\mathscr{T}}(\alpha,\varepsilon)=\frac{1}{|L^*\psi|}\{\mathscr{M}(\alpha_m+T_{2m})\widetilde{\psi}\}_{m\in\mathbb{Z}}+O(\varepsilon),$$
$$D_1B_{\mathscr{T}}(\alpha,\varepsilon)=\frac{1}{|L^*\psi|}\{\mathscr{M}'(\alpha_m+T_{2m})\widetilde{\psi}\}_{m\in\mathbb{Z}}+o(1),$$

where $O(\varepsilon)$ and $o(1)$ are uniform with respect to α and $\mathscr{T}$. Now assume that $\mathscr{T}=\{T_m\}_{m\in\mathbb{Z}}$ and $\alpha_0=\{\alpha_m^0\}_{m\in\mathbb{Z}}$ satisfy the assumptions of Theorem 6.1.16. We have:

$$\lim_{\varepsilon\to 0}B_{\mathscr{T}}(\alpha_0,\varepsilon)=0,$$
$$\lim_{\varepsilon\to 0}D_1B_{\mathscr{T}}(\alpha_0,\varepsilon)=\frac{1}{|L^*\psi|}\{\mathscr{M}'(\alpha_m^0+T_{2m})\widetilde{\psi}\}_{m\in\mathbb{Z}}$$

uniformly with respect to $\mathscr{T}$. That is $\|D_1B_{\mathscr{T}}(\alpha_0,\varepsilon)\|>\frac{c_0}{2|L^*\psi|}$ provided $|\varepsilon|$ is sufficiently small. From the implicit function theorem we deduce the existence of $0<\bar{\varepsilon}_\rho<\varepsilon_0$ so that for any $0\neq\varepsilon\in(-\bar{\varepsilon}_\rho,\bar{\varepsilon}_\rho)$ and any sequence $\mathscr{T}=\{T_m\}_{m\in\mathbb{Z}}$ that satisfy the assumption of Theorem 6.1.16 there exists a unique sequence $\alpha(\mathscr{T},\varepsilon)=\{\alpha_m(\mathscr{T},\varepsilon)\}_{m\in\mathbb{Z}}\in\ell_\chi^\infty$ so that $\alpha(\mathscr{T},0)=\alpha_0$ and

$$B_{\mathscr{T}}(\alpha(\mathscr{T},\varepsilon),\varepsilon)=0.$$

Taking $\theta_{\mathscr{T}}(\varepsilon)=\theta_{\mathscr{T}}(\alpha(\mathscr{T},\varepsilon),\varepsilon)$ and

$$z(t)=\begin{cases} z_{m,\mathscr{T}}^-(t,\varepsilon), & \text{if } t\in[T_{2m-1}+\beta_{m-1,\mathscr{T}}(\varepsilon),T_{2m}-\bar{T}+\alpha_{m,\mathscr{T}}(\varepsilon)],\\ z_{m,\mathscr{T}}^0(t,\varepsilon), & \text{if } t\in[T_{2m}-\bar{T}+\alpha_{m,\mathscr{T}}(\varepsilon),T_{2m}+\bar{T}+\beta_{m,\mathscr{T}}(\varepsilon)],\\ z_{m,\mathscr{T}}^+(t,\varepsilon), & \text{if } t\in[T_{2m}+\bar{T}+\beta_{m,\mathscr{T}}(\varepsilon),T_{2m+1}+\beta_{m,\mathscr{T}}(\varepsilon)],\end{cases}$$

we see that $z(t)$ satisfies the conclusion of Theorem 6.1.16 with $\hat{\alpha}_m=\alpha_m(\mathscr{T},\varepsilon)$ and $\hat{\beta}_m=\beta_m(\alpha(\mathscr{T},\varepsilon),\varepsilon)$. The proof is complete. □

Remark 6.1.17. Functions $\mathscr{M},\mathscr{M}':\mathbb{R}\to\mathbb{R}$ are bounded.

Remark 6.1.18. Following the above arguments, we can consider also cases when $\bar{m}\in\mathbb{Z}$ exists so that either $T_j=-\infty\ \forall j\leq 2\bar{m}-1$ or $T_j=\infty\ \forall j\geq 2\bar{m}+1$. Then Theorem 6.1.16 is obviously modified (see (6.1.97), (6.1.98) and (6.1.99) below).

Remark 6.1.19. Here we emphasize that during the proof of Theorem 6.1.16, we only use the fact that f and g are C^2 with bounded and uniformly continuous derivatives. We should need higher derivatives if α_0 is a degenerate root of $\mathscr{M}_{\mathscr{T}}(\alpha)=\{\mathscr{M}(T_{2m}+\alpha_m)\}_{m\in\mathbb{Z}}$, when condition (6.1.64) fails.

We are now able to give the proof of Theorem 6.1.3. First we show the following preparatory results.

Lemma 6.1.20. *For any* $\varepsilon\neq 0$ *there exists* $|\varepsilon|>\nu_\varepsilon>0$ *so that if a sequence* $\mathscr{T}=\{T_m\}_{m\in\mathbb{Z}}$ *satisfies* (6.1.6) *then also it holds*

$$|D_1 g(t+T_{2m},z,0) - D_1 g(t,z,0)| < |\varepsilon| \tag{6.1.90}$$

for any $(t,z,m) \in \mathbb{R}^{n+1} \times \mathbb{Z}$.

Proof. Let $\varepsilon \neq 0$. Take $n_\varepsilon \in \mathbb{N}$ and $\nu_\varepsilon > 0$ as

$$n_\varepsilon = 2\left[\frac{\|D_{11}g\|}{|\varepsilon|}\right] + 1, \quad \nu_\varepsilon := \frac{|\varepsilon|}{4n_\varepsilon} \tag{6.1.91}$$

and let $\mathscr{T} = \{T_m\}_{m\in\mathbb{Z}}$ be a sequence satisfying (6.1.6). Then we derive [40]:

$$\begin{aligned}
&|D_1 g(t+T_{2m},z,0) - D_1 g(t,z,0)| \\
&\leq \left| D_1 g(t+T_{2m},z,0) - n_\varepsilon \left[g\left(t+T_{2m}+\frac{1}{n_\varepsilon},z,0\right) - g(t+T_{2m},z,0)\right]\right| \\
&\quad + \left| D_1 g(t,z,0) - n_\varepsilon \left[g\left(t+\frac{1}{n_\varepsilon},z,0\right) - g(t,z,0)\right]\right| \\
&\quad + n_\varepsilon \left| g\left(t+T_{2m}+\frac{1}{n_\varepsilon},z,0\right) - g\left(t+\frac{1}{n_\varepsilon},z,0\right)\right| \\
&\quad + n_\varepsilon \left| g\left(t+T_{2m},z,0\right) - g(t,z,0)\right| \\
&\leq n_\varepsilon \int_0^{1/n_\varepsilon} |D_1 g\left(t+T_{2m}+\eta,z,0\right) - D_1 g(t+T_{2m},z,0)|\, d\eta \\
&\quad + n_\varepsilon \int_0^{1/n_\varepsilon} |D_1 g\left(t+\eta,z,0\right) - D_1 g(t,z,0)|\, d\eta + 2n_\varepsilon \nu_\varepsilon \\
&\leq \frac{\|D_{11}g\|}{n_\varepsilon} + 2n_\varepsilon \nu_\varepsilon < |\varepsilon|.
\end{aligned}$$

The proof of Lemma 6.1.20 is complete. □

Lemma 6.1.21. *If* $\varepsilon \neq 0$ *is sufficiently small then for any given sequence* $\{T_m\}_{m\in\mathbb{Z}}$ *with the properties of Lemma* 6.1.20*, a sequence* $\{\alpha_m^0\}_{m\in\mathbb{Z}} \in \ell_\chi^\infty$ *exists satisfying* (6.1.64) *for some* $c_0 > 0$.

Proof. Let $|\mathscr{M}'(\alpha^0)| = 4c_0$. We have:

$$\mathscr{M}(T_{2m}+\alpha) = \mathscr{M}(\alpha) + \int_{-\infty}^{\infty} \psi^*(t)[g(t+T_{2m}+\alpha,\gamma(t),0) - g(t+\alpha,\gamma(t),0)]dt$$

and hence:

$$|\mathscr{M}(T_{2m}+\alpha) - \mathscr{M}(\alpha)| \leq |\varepsilon| \int_{-\infty}^{\infty} |\psi^*(t)|dt \leq 2K\delta^{-1}|\varepsilon|$$

since $|\psi^*(t)| \leq K\mathrm{e}^{-\delta|t|}$ for some $K \geq 1$ (see (6.1.63)). Similarly, from (6.1.90) we get

$$|\mathscr{M}'(T_{2m}+\alpha) - \mathscr{M}'(\alpha)| \leq 2K\delta^{-1}|\varepsilon|\,.$$

Let $\chi/2 > \delta_1 > 0$ be so small that $\mathscr{M}(\alpha^0 - \delta_1)\mathscr{M}(\alpha^0 + \delta_1) < 0$ and $|\mathscr{M}'(\alpha)| \geq 2c_0$ for $\alpha \in [\alpha^0 - \delta_1, \alpha^0 + \delta_1]$. Then, there is an $\widetilde{\varepsilon}_0 > 0$ so that for $0 < |\varepsilon| < \widetilde{\varepsilon}_0$ and for any $m \in \mathbb{Z}$ the equation $\mathscr{M}(T_{2m}+\alpha)=0$ has a unique solution $\alpha_m^0=\alpha(T_{2m}) \in (\alpha^0-\delta_1, \alpha^0+\delta_1)$ along with $|\mathscr{M}'(T_{2m}+\alpha)| \geq c_0$ for $\alpha \in [\alpha^0-\delta_1, \alpha^0+\delta_1]$. The proof of Lemma 6.1.20 is complete. □

Now we proceed with the proof of Theorem 6.1.3. Using Lemma 6.1.21, assumptions of Theorem 6.1.16 are verified and consequently, we obtain sequences $\{\hat{\alpha}_{m,\mathscr{T}}(\varepsilon)\}$, $\{\hat{\beta}_{m,\mathscr{T}}(\varepsilon)\}$, and a unique solution $z(t)$ of Eq. (6.1.1) that satisfies (6.1.7). To prove that $\sup_{m\in\mathbb{Z}}|\hat{\alpha}_{m,\mathscr{T}}(\varepsilon)-\alpha^0| < c_1|\varepsilon|$ and $\sup_{m\in\mathbb{Z}}|\hat{\beta}_{m,\mathscr{T}}(\varepsilon)-\alpha^0| < c_1|\varepsilon|$ assume for simplicity that $\mathscr{M}'(\alpha^0)=4c_0$ (a similar argument applies when $\mathscr{M}'(\alpha^0)=-4c_0$). Then we have, since $\mathscr{M}'(T_{2m}+\alpha) > c_0$ for any $\alpha \in [\alpha_0-\delta_1, \alpha_0+\delta_1]$:

$$2K\delta^{-1}|\varepsilon| \geq \left| \int_{\alpha^0}^{\alpha_m^0} \mathscr{M}'(T_{2m}+\tau)d\tau \right| \geq c_0|\alpha_m^0-\alpha^0|,$$

hence

$$|\hat{\alpha}_{m,\mathscr{T}}(\varepsilon)-\alpha^0| \leq |\hat{\alpha}_m(\mathscr{T},\varepsilon)-\alpha_m^0|+|\alpha_m^0-\alpha^0| \leq c_1|\varepsilon|+\frac{2K|\varepsilon|}{\delta c_0}=\widetilde{c}_1|\varepsilon|.$$

Similarly we get (possibly changing $\widetilde{c}_1$): $|\hat{\beta}_{m,\mathscr{T}}(\varepsilon)-\alpha^0| \leq \widetilde{c}_1|\varepsilon|$. The proof of Theorem 6.1.3 is complete.

Remark 6.1.22. By (6.1.91), we get $\nu_\varepsilon \sim \varepsilon^2$ in Theorem 6.1.3.

6.1.7 Chaotic Behaviour

Set (cf Section 2.5.2)

$$\hat{\mathscr{E}} := \{e \in \mathscr{E} \mid \inf\{m \in \mathbb{Z} \mid e_m = 1\} = -\infty, \sup\{m \in \mathbb{Z} \mid e_m = 1\} = \infty\},$$

$$\mathscr{E}_+ := \{e \in \mathscr{E} \mid \inf\{m \in \mathbb{Z} \mid e_m = 1\} > -\infty, \sup\{m \in \mathbb{Z} \mid e_m = 1\} = \infty\},$$

$$\mathscr{E}_- := \{e \in \mathscr{E} \mid \inf\{m \in \mathbb{Z} \mid e_m = 1\} = -\infty, \sup\{m \in \mathbb{Z} \mid e_m = 1\} < \infty\},$$

$$\mathscr{E}_0 := \{e \in \mathscr{E} \mid \inf\{m \in \mathbb{Z} \mid e_m = 1\} > -\infty, \sup\{m \in \mathbb{Z} \mid e_m = 1\} < \infty\}.$$

Note that $\hat{\mathscr{E}}$, $\mathscr{E}_-$, $\mathscr{E}_+$, $\mathscr{E}_0$ are invariant under the Bernoulli shift. In this section we suppose for simplicity that assumptions of Theorem 6.1.16 are satisfied with a technical condition $\|\alpha_0\| < \chi/2$, i.e the following holds:

(C) For any $\varepsilon \neq 0$ sufficiently small there is a sequence $\mathscr{T}=\{T_m\}_{m\in\mathbb{Z}}$ so that $T_{m+1}-T_m > \bar{T}+1-2\delta^{-1}\ln|\varepsilon|$ along with the existence of an $\alpha_0=\{\alpha_m^0\}_{m\in\mathbb{Z}} \in \ell_\chi^\infty$ with $\|\alpha_0\| < \chi/2$, satisfying (6.1.64).

Let $\mathscr{T}=\{T_m\}_{m\in\mathbb{Z}}$ be as in assumption (C). Assume, first, that $e \in \hat{\mathscr{E}}$. Let $\{n_m^e\}_{m\in\mathbb{Z}}$ be a fixed increasing doubly-infinite sequence of integers so that $e_k = 1$ if and only if $k = n^e_m$. We define sequences $\mathscr{T}^e = \{T^e_m\}_{m\in\mathbb{Z}}$ and $\alpha^e_0 = \{\alpha^{0e}_m\}_{m\in\mathbb{Z}}$ as

$$T^e_m := \begin{cases} T_{2n^e_k}, & \text{if } m = 2k, \\ T_{2n^e_k - 1}, & \text{if } m = 2k-1, \end{cases} \tag{6.1.92}$$

and similarly

$$\alpha^{0e}_m := \alpha^0_{n^e_m}. \tag{6.1.93}$$

Note that $T^e_{m+1} - T^e_m > \bar{T} + 1 - 2\delta^{-1}\ln|\varepsilon|$ for any $m \in \mathbb{Z}$ and $\mathscr{M}_{\mathscr{T}^e}(\alpha)$ has a simple zero α^e_0, i.e. (6.1.64) holds with exchanges $\mathscr{T}^e \leftrightarrow \mathscr{T}$ and $\alpha^e_0 \leftrightarrow \alpha_0$. Since $|\alpha^{0e}_{m+1} - \alpha^{0e}_m| < \chi$ for any $m \in \mathbb{Z}$, assumptions of Theorem 6.1.16 are satisfied by $\mathscr{M}_{\mathscr{T}^e}(\alpha)$, $\mathscr{T}^e$ and α^e_0. Let $z(t) = z(t, \mathscr{T}^e)$ be the corresponding solution of Eq. (6.1.1) whose existence is stated in Theorem 6.1.16. Then $z(t)$ satisfies

$$\begin{aligned} &\sup\nolimits_{t\in[T^e_{2m-1}+\beta^e_{m-1}, T^e_{2m}-\bar{T}+\alpha^e_m]} |z(t) - \gamma_-(t - T^e_{2m} - \alpha^e_m)| < \rho, \\ &\sup\nolimits_{t\in[T^e_{2m}-\bar{T}+\alpha^e_m, T^e_{2m}+\bar{T}+\beta^e_m]} |z(t) - \gamma_0(t - T^e_{2m} - \alpha^e_m)| < \rho, \\ &\sup\nolimits_{t\in[T^e_{2m}+\bar{T}+\beta^e_m, T^e_{2m+1}+\beta^e_m]} |z(t) - \gamma_+(t - T^e_{2m} - \beta^e_m)| < \rho, \end{aligned} \tag{6.1.94}$$

where the sequences $\alpha^e = \{\alpha^e_m\}_{m\in\mathbb{Z}}$ and $\beta^e = \{\beta^e_m\}_{m\in\mathbb{Z}}$ are determined as in Theorem 6.1.16 (note here we remove hats for notational simplicity).

Now, consider the sequence $\widetilde{n}^e_m := n^e_{m+1}$ instead of n^e_m and denote with $\widetilde{\mathscr{T}}^e$, $\widetilde{\alpha}^e$, $\widetilde{\beta}^e$ and $\widetilde{\alpha}^e_0$ the corresponding sequences:

$$\widetilde{T}^e_m = T^e_{m+2}, \quad \widetilde{\alpha}^e_m = \alpha^e_{m+1}, \quad \widetilde{\beta}^e_m = \beta^e_{m+1}, \quad \widetilde{\alpha}^{0e}_m = \alpha^{0e}_{m+1}. \tag{6.1.95}$$

Then $\mathscr{M}_{\widetilde{\mathscr{T}}^e}(\alpha)$ has a simple zero $\widetilde{\alpha}^e_0$ and Theorem 6.1.16 is applicable. But clearly $\widetilde{z}(t) := z(t, \widetilde{\mathscr{T}}^e)$ satisfies the same set of estimates (6.1.94) and hence, because of uniqueness, $z(t, \widetilde{\mathscr{T}}^e) = z(t, \mathscr{T}^e)$ depends only on e and $\mathscr{T}$ (and not on the choice of n^e_m). So we will write $z(t, \mathscr{T}, e)$ instead of $z(t, \mathscr{T}^e)$.

Now, assume that $e_j = 1$. Then $j = n^e_m$ for some $m \in \mathbb{Z}$ and (6.1.94) gives, provided $|\varepsilon|$ is sufficiently small,

$$\begin{aligned} |z(T_{2j}) - \gamma_0(-\alpha^0_j)| &\le |z(T_{2j}) - \gamma_0(-\alpha^e_m)| + |\gamma_0(-\alpha^e_m) - \gamma_0(-\alpha^0_j)| \\ &< \rho + \sup_{t\in\mathbb{R}} |\dot{\gamma}_0(t)|\,|\alpha^e_m - \alpha^0_j| \\ &< \rho + c_1|\varepsilon| \sup_{t\in\mathbb{R}} |\dot{\gamma}_0(t)| < \frac{3}{2}\rho \end{aligned}$$

since $T^e_{2m} = T_{2j}$. On the other hand, if $e_j = 0$, let $m \in \mathbb{Z}$ be such that $n^e_m < j < n^e_{m+1}$. Then $n^e_{m+1} - 1 \ge j \ge n^e_m + 1$ and so

$$\begin{aligned} T_{2j} - T_{2n_m^e} - \bar{T} - \beta_m^e &\geq T_{2n_m^e+2} - T_{2n_m^e} - \bar{T} - \|\alpha_0\| - c_1|\varepsilon| \\ &\geq \bar{T} + 2 - 4\delta^{-1}\ln|\varepsilon| - \|\alpha_0\| - c_1|\varepsilon| \\ &> 0 \end{aligned}$$

and

$$\begin{aligned} T_{2m+1}^e + \beta_m^e - T_{2j} &\geq T_{2n_{m+1}^e-1} - T_{2n_{m+1}^e-2} - \|\alpha_0\| - c_1|\varepsilon| \\ &\geq \bar{T} + 1 - 2\delta^{-1}\ln|\varepsilon| - \|\alpha_0\| - c_1|\varepsilon| \\ &> 0 \end{aligned}$$

for $0 < |\varepsilon| \ll 1$. Consequently, we have $T_{2j} \in \left[T_{2m}^e + \bar{T} + \beta_m^e, T_{2m+1}^e + \beta_m^e\right]$, and using (6.1.94), we get

$$|z(T_{2j})| \leq |z(T_{2j}) - \gamma_+(T_{2j} - T_{2n_m^e} - \beta_m^e)| + |\gamma_+(T_{2j} - T_{2n_m^e} - \beta_m^e)| < \frac{3}{2}\rho$$

since $T_{2j} - T_{2n_m^e} - \beta_m^e \geq T_{2n_m^e+2} - T_{2n_m^e} - \|\alpha_0\| - c_1|\varepsilon| > 2\bar{T} + 2 - 4\delta^{-1}\ln|\varepsilon| - \|\alpha_0\| - c_1|\varepsilon| \gg 1$ for $0 < |\varepsilon| \ll 1$, and thus $\left|\gamma_+(T_{2j} - T_{2n_m^e} - \beta_m^e)\right| < \rho/2$. So $z(t, \mathscr{T}, e)$ has the following property

$$\begin{aligned} &|z(T_{2j}) - \gamma_0(-\alpha_j^0)| < \frac{3}{2}\rho, \text{ if } e_j = 1\,, \\ &|z(T_{2j})| < \frac{3}{2}\rho, \qquad\qquad \text{if } e_j = 0\,. \end{aligned} \tag{6.1.96}$$

Next, assume that $e \in \mathscr{E}_+$ and let again $\{n_m^e\}_{m\in\mathbb{Z}}$ be a fixed increasing sequence of integers so that $e_k = 1$ if and only if $k = n_m^e$ and $\lim_{m\to\infty} n_m^e = \infty$. Corresponding to this sequence, we define $\mathscr{T}^e$ as in (6.1.92) and then we obtain α^e and β^e as in (6.1.94) with the difference that $T_m^e = -\infty$ and $\alpha_m^e = \beta_m^e = 0$ for any $m < 2\bar{m}$ where $\bar{m}$ is such that $e_{n_{\bar{m}}^e} = 1$ and $e_j = 0$ for any $j < n_{\bar{m}}^e$. According to this choice, by Remark 6.1.18, we obtain a solution $z(t) = z(t, \mathscr{T}^e)$ of Eq. (6.1.1) that satisfies (6.1.94) when $m > \bar{m}$ whereas for $m = \bar{m}$ it satisfies:

$$\begin{aligned} &\sup_{t\in(-\infty, T_{2\bar{m}}^e - \bar{T} + \alpha_{\bar{m}}^e]} |z(t) - \gamma_-(t - T_{2\bar{m}}^e - \alpha_{\bar{m}}^e)| < \rho, \\ &\sup_{t\in[T_{2\bar{m}}^e - \bar{T} + \alpha_{\bar{m}}^e, T_{2\bar{m}}^e + \bar{T} + \beta_{\bar{m}}^e]} |z(t) - \gamma_0(t - T_{2\bar{m}}^e - \alpha_{\bar{m}}^e)| < \rho, \\ &\sup_{t\in[T_{2\bar{m}}^e + \bar{T} + \beta_{\bar{m}}^e, T_{2\bar{m}+1}^e + \beta_{\bar{m}}^e]} |z(t) - \gamma_+(t - T_{2\bar{m}}^e - \beta_{\bar{m}}^e)| < \rho. \end{aligned} \tag{6.1.97}$$

Note, then, that if we take, as in the previous case, $\widetilde{n}_m^e = n_{m+1}^e$ and $\widetilde{\mathscr{T}^e}$, $\widetilde{\alpha}^e$, $\widetilde{\beta}^e$ as in (6.1.95), then (6.1.94) holds with $\widetilde{\mathscr{T}^e}$ instead $\mathscr{T}^e$, provided $m > \bar{m} - 1$ whereas (6.1.97) holds with $\widetilde{T}_{2(\bar{m}-1)}^e$ and $\widetilde{T}_{2\bar{m}-1}^e$ instead of $T_{2\bar{m}}^e$ and $T_{2\bar{m}+1}^e$ respectively. So in this case we can also see that $z(t, \mathscr{T}^e) = z(t, \mathscr{T}, e)$ depends only on $(\mathscr{T}, e)$ and not on the choice of the sequence n_m^e. Moreover, (6.1.96) holds also in this case. In fact if either $e_j = 1$ or $e_j = 0$ and there exists $m \in \mathbb{Z}$ so that $n_m^e < j < n_{m+1}^e$ the same proof as before goes through. If, instead, $e_j = 0$ and $j < n_{\bar{m}}^e$, then the estimate

$|z(T_{2j})| < \frac{3}{2}\rho$ follows from the first estimate in (6.1.97) since $2j \leq 2n^e_{\bar{m}} - 2$ and then $T^e_{2j} - T^e_{2\bar{m}} - \alpha^e_{\bar{m}} \leq T^e_{2n^e_{\bar{m}}-2} - T_{2n^e_{\bar{m}}} + \|\alpha_0\| + c_1|\varepsilon| \leq -2\bar{T} - 2 - 4\delta^{-1}\ln|\varepsilon| + \|\alpha_0\| + c_1|\varepsilon| \ll 0$ for $0 < |\varepsilon| \ll 1$.

Similarly, if $e \in \mathscr{E}_-$ then by Remark 6.1.18, we obtain a solution $z(t) = z(t, \mathscr{T}^e)$ of Eq. (6.1.1) that satisfies (6.1.94) when $m < \bar{m}$ whereas for $m = \bar{m}$ it satisfies

$$\begin{aligned}
&\sup\nolimits_{t\in(T^e_{2\bar{m}-1},T^e_{2\bar{m}}-\bar{T}+\alpha^e_{\bar{m}}]} |z(t) - \gamma_-(t - T^e_{2\bar{m}} - \alpha^e_{\bar{m}})| < \rho,\\
&\sup\nolimits_{t\in[T^e_{2\bar{m}}-\bar{T}+\alpha^e_{\bar{m}},T^e_{2\bar{m}}+\bar{T}+\beta^e_{\bar{m}}]} |z(t) - \gamma_0(t - T^e_{2\bar{m}} - \alpha^e_{\bar{m}})| < \rho,\\
&\sup\nolimits_{t\in[T^e_{2\bar{m}}+\bar{T}+\beta^e_{\bar{m}},\infty)} |z(t) - \gamma_+(t - T^e_{2\bar{m}} - \beta^e_{\bar{m}})| < \rho.
\end{aligned} \tag{6.1.98}$$

From an argument similar to the previous one (in this case, we can take, for example, $\widetilde{n}^e_m = n^e_{m-1}$) we see that $z(t, \mathscr{T}^e) = z(t, \mathscr{T}, e)$ depends only on $(\mathscr{T}, e)$ and not on the choice of the sequence n^e_m and (6.1.96) holds.

Next, assume that $e \in \mathscr{E}_0$ with $e \neq 0$. Then there are $\bar{m}_- < \bar{m}_+$ so that $e_k = 0$ if either $k < n^e_{m_-}$ or $k > n^e_{m_+}$ and Eq. (6.1.1) has a unique solution $z(t, \mathscr{T}^e)$ so that (6.1.94) holds when $\bar{m}_- < m < \bar{m}_+$ whereas when either $m = m_-$ or $m = m_+$ it satisfies

$$\begin{aligned}
&\sup\nolimits_{t\in(-\infty,T^e_{2\bar{m}_-}-\bar{T}+\alpha^e_{\bar{m}_-}]} |z(t) - \gamma_-(t - T^e_{2\bar{m}_-} - \alpha^e_{\bar{m}_-})| < \rho,\\
&\sup\nolimits_{t\in[T^e_{2\bar{m}_-}-\bar{T}+\alpha^e_{\bar{m}_-},T^e_{2\bar{m}_-}+\bar{T}+\beta^e_{\bar{m}_-}]} |z(t) - \gamma_0(t - T^e_{2\bar{m}_-} - \alpha^e_{\bar{m}})| < \rho,\\
&\sup\nolimits_{t\in[T^e_{2\bar{m}_-}+\bar{T}+\beta^e_{\bar{m}_-},T^e_{2\bar{m}_-+1}+\beta^e_{\bar{m}_-}]} |z(t) - \gamma_+(t - T^e_{2\bar{m}_-} - \beta^e_{\bar{m}_-})| < \rho,\\
&\sup\nolimits_{t\in(T^e_{2\bar{m}_+-1},T^e_{2\bar{m}_+}-\bar{T}+\alpha^e_{\bar{m}_+}]} |z(t) - \gamma_-(t - T^e_{2\bar{m}_+} - \alpha^e_{\bar{m}_+})| < \rho,\\
&\sup\nolimits_{t\in[T^e_{2\bar{m}_+}-\bar{T}+\alpha^e_{\bar{m}_+},T^e_{2\bar{m}_+}+\bar{T}+\beta^e_{\bar{m}_+}]} |z(t) - \gamma_0(t - T^e_{2\bar{m}_+} - \alpha^e_{\bar{m}_+})| < \rho,\\
&\sup\nolimits_{t\in[T^e_{2\bar{m}_+}+\bar{T}+\beta^e_{\bar{m}_+},\infty)} |z(t) - \gamma_+(t - T^e_{2\bar{m}_+} - \beta^e_{\bar{m}_+})| < \rho.
\end{aligned} \tag{6.1.99}$$

Moreover $z(t, \mathscr{T}^e) = z(t, \mathscr{T}, e)$ depends only on $(\mathscr{T}, e)$ and not on the choice of n^e_m and (6.1.96) holds.

Finally, if $e = 0$, that is $e_k = 0$ for any $k \in \mathbb{Z}$, by we define $z(t, \mathscr{T}, 0) = u(t)$ as the unique bounded solution of (6.1.1) so that

$$\sup_{t\in\mathbb{R}} |u(t)| < \rho. \tag{6.1.100}$$

The existence and uniqueness of $u(t)$ follow from the standard regular perturbation theory (see [41–44], Remark 4.1.7). Now we are able to prove the following theorem:

Theorem 6.1.23. *Let assumptions (H1), (H2), (H3) and (C) be satisfied. Then there exists* $\bar{\rho} > 0$ *so that for any* $0 < \rho < \bar{\rho}$ *there exists* $\varepsilon_0 > 0$ *so that for any* $\varepsilon \neq 0$, $|\varepsilon| < \varepsilon_0$ *and for any* $e \in \mathscr{E}$, *Eq.* (6.1.1) *has a unique solution* $z(t, \mathscr{T}, e, \varepsilon)$ *that satisfies one among* (6.1.94), (6.1.97), (6.1.98) *or* (6.1.99) *and consequently* (6.1.96). *Moreover, setting* $\mathscr{T}^{(k)} := \{T_{n+2k}\}_{n\in\mathbb{Z}}$, *we have*

$$z(t,\mathscr{T}^{(k+1)},\sigma(e),\varepsilon)=z(t,\mathscr{T}^{(k)},e,\varepsilon) \tag{6.1.101}$$

for any $t\in\mathbb{R}$ and $e\in\mathscr{E}$.

Proof. We only need to prove that (6.1.101) holds. Since $z(t,\mathscr{T},e,\varepsilon)$ does not depend on the choice of $\{n_m^e\}_{m\in\mathbb{Z}}$ we see that we can take $n_m^{\sigma(e)}=n_m^e-1$ and then, setting $\mathscr{T}'=\{T_{m+2}\}_{m\in\mathbb{Z}}$, we have, if $m=2k$:

$$T_{2k}'^{\sigma(e)}=T_{2n_k^{\sigma(e)}+2}=T_{2n_k^e}=T_{2k}^e$$

and, if $m=2k-1$:

$$T_{2k-1}'^{\sigma(e)}=T_{2n_k^{\sigma(e)}+1}=T_{2n_k^e-1}=T_{2k-1}^e$$

that is

$$\mathscr{T}'^{\sigma(e)}=\mathscr{T}^e. \tag{6.1.102}$$

Hence we see that, for any $t\in\mathbb{R}$ and any $e\in\mathscr{E}$, the following holds

$$z(t,\mathscr{T}',\sigma(e),\varepsilon)=z(t,\mathscr{T},e,\varepsilon). \tag{6.1.103}$$

Now, from the definition of $\mathscr{T}^{(k)}$ we see that $\mathscr{T}^{(k+1)}=\mathscr{T}^{(k)\prime}$, thus (6.1.101) follows from (6.1.103). The proof is complete. □

Now, we define $F_k:\mathbb{R}^n\to\mathbb{R}^n$ so that $F_k(\xi)$ is the value at time $T_{2(k+1)}$ of the solution $z(t)$ of Eq. (6.1.1) so that $z(T_{2k})=\xi$:

$$\dot z=f_\pm(z)+\varepsilon g(t,z,\varepsilon),\quad z(T_{2k})=\xi \tag{6.1.104}$$

and let $\Phi_k(e):=z(T_{2k},\mathscr{T}^{(k)},e,\varepsilon)$. Then we have:

$$\begin{aligned}\Phi_{k+1}\circ\sigma(e)&=z(T_{2(k+1)},\mathscr{T}^{(k+1)},\sigma(e),\varepsilon)=z(T_{2(k+1)},\mathscr{T}^{(k)},e,\varepsilon)\\&=F_k(z(T_{2k},\mathscr{T}^{(k)},e,\varepsilon))=F_k\circ\Phi_k(e).\end{aligned} \tag{6.1.105}$$

Note that (6.1.105) can be stated in the following way. Let

$$\mathscr{S}_k=\left\{(z(T_{2k},\mathscr{T}^{(k)},e,\varepsilon)\mid e\in\mathscr{E}\right\},\quad k\in\mathbb{Z}.$$

Although F_k may not be defined in the whole $\mathbb{R}^n$, for sure it is defined in the set $\mathscr{S}_k$. It is standard to prove (see [36], Section 3.5) that $\mathscr{S}_k$ are compact in $\mathbb{R}^n$ and $\Phi_k:\mathscr{E}\mapsto\mathscr{S}_k$ are continuous and clearly onto. Moreover, by (6.1.105), all $F_k:\mathscr{S}_k\to\mathscr{S}_{k+1}$ are homeomorphisms.

Remark 6.1.24. Here we silently suppose that F_k are defined. We can do that since we can modify (6.1.1) outside of a neighbourhood of the homoclinic orbit.

Next, let $e,e'\in\mathscr{E}$ be two different sequences in $\mathscr{E}$. Then there exists $j\in\mathbb{Z}$ so that, for example, $e_j'=0$ and $e_j=1$. From $[-\chi/2,\chi/2]\subset[-\bar T,\bar T]$ and (6.1.96) we see that

$$\begin{aligned}&\left|z(T_{2j},\mathscr{T},e,\varepsilon)-z(T_{2j},\mathscr{T},e',\varepsilon)\right|\\&\quad\geq\left|\gamma_0(-\alpha_j^0)\right|-\left|z(T_{2j},\mathscr{T},e,\varepsilon)-\gamma_0(-\alpha_j^0)\right|-\left|z(T_{2j},\mathscr{T},e',\varepsilon)\right|\\&\quad\geq\left|\gamma_0(-\alpha_m^0)\right|-3\rho\geq\min_{t\in[-\bar{T},\bar{T}]}|\gamma_0(t)|-3\rho>0\end{aligned}$$

provided ρ is sufficiently small. As a consequence, $z(T_{2j},\mathscr{T},e,\varepsilon)\neq z(T_{2j},\mathscr{T},e',\varepsilon)$ and, since both are solutions of the same Eq. (6.1.1):

$$z(t,\mathscr{T},e,\varepsilon)\neq z(t,\mathscr{T},e',\varepsilon) \tag{6.1.106}$$

for any $t\in\mathbb{R}$. Thus we have proved that the map $e\mapsto z(t,\mathscr{T},e,\varepsilon)$ is one-to-one. Hence if $\Phi_k(e)=\Phi_k(e')$ then $e=e'$ since otherwise:

$$\Phi_k(e)=z(T_{2k},\mathscr{T}^{(k)},e,\varepsilon)\neq z(T_{2k},\mathscr{T}^{(k)},e',\varepsilon)=\Phi_k(e').$$

So $\Phi_k:\mathscr{E}\to\mathscr{S}_k$ is one-to-one and a homeomorphism for any $k\in\mathbb{Z}$. In summary, we get another result.

Theorem 6.1.25. *Assume that (H1), (H2), (H3) and (C) hold. Then for any $\varepsilon\neq 0$ sufficiently small, the following diagrams commute:*

$$\begin{array}{ccc}\mathscr{E} & \xrightarrow{\quad\sigma\quad} & \mathscr{E}\\ {\scriptstyle\Phi_k}\big\downarrow & & \big\downarrow{\scriptstyle\Phi_{k+1}}\\ \mathscr{S}_k & \xrightarrow[\quad F_k\quad]{} & \mathscr{S}_{k+1}\end{array}$$

for all $k\in\mathbb{Z}$. Moreover, all Φ_k are homeomorphisms.

Sequences of 2-dimensional maps are also studied in [45].

Remark 6.1.26. We improve (6.1.94) as follows. First, assume that $e_j=1$, and $e_{j+1}=0$. The cases $e_j=0$, $e_{j+1}=1$ and $e_j=e_{j+1}=1$ can be similarly handled. Then, if $j=n_k^e$, we have $n_{k+1}^e>n_k^e+1$ and then if

$$t\in[T_{2n_k^e+1}+\beta_k^e,T_{2n_{k+1}^e-1}+\beta_k^e]=\bigcup_{j=2n_k^e+1}^{2(n_{k+1}^e-1)}[T_j+\beta_k^e,T_{j+1}+\beta_k^e],$$

we have $t\in[T_{2k}^e+\bar{T}+\beta_k^e,T_{2k+1}^e+\beta_k^e]$ and

$$t-T_{2k}^e-\beta_k^e\in[T_{2n_k^e+1}-T_{2n_k^e},T_{2n_{k+1}^e-1}-T_{2n_k^e}]\subset(\bar{T}+1-2\delta^{-1}\ln|\varepsilon|,\infty)$$

and hence if ε is small enough that $|\gamma_-(t)|<\rho$ for any $t\geq\bar{T}+1-2\delta^{-1}\ln|\varepsilon|$, by (6.1.94) we get:

$$\sup_{t\in[T_j+\beta_k^e,T_{j+1}+\beta_k^e]}|z(t)-u(t)|<3\rho$$

for any $j \in \{2n_k^e+1,...,2(n_{k+1}^e-1)\}$. On the other hand,

$$\sup_{t\in[T_{2n_k^e-1}+\beta_{k-1}^e, T_{2n_k^e}+\bar{T}+\beta_k^e]} |z(t)-\gamma(t-T_{2n_k^e}-\alpha_k^e)| < \rho,$$
$$\sup_{t\in[T_{2n_k^e}+\bar{T}+\beta_k^e, T_{2n_k^e+1}+\beta_k^e]} |z(t)-\gamma(t-T_{2n_k^e}-\beta_k^e)| < \rho\,.$$

In summary, we can roughly state that for $t \in [T_{2j-1}, T_{2j+1}]$ the solution $z(t)$ is close either to the homoclinic orbit $\gamma(t)$ or to the bounded solution according to $e_j = 1$ or $e_j = 0$.

6.1.8 Almost and Quasiperiodic Cases

In this section we assume that $g(t,x,\varepsilon)$ is almost periodic in t uniformly in (x,ε), that is, the following holds:

(H4) For any $\nu > 0$ there exists $L_\nu > 0$ so that in any interval of a length greater than L_ν there exists T_ν which is an *almost period* for ν satisfying:

$$|g(t+T_\nu,x,\varepsilon)-g(t,x,\varepsilon)| < \nu$$

for any $(t,x,\varepsilon) \in \mathbb{R}^{n+2}$.

Note that under (H4), function $\mathscr{M}(\alpha)$ is almost periodic in α. In this section we suppose the existence of a simple zero α^0 of $\mathscr{M}(\alpha)$. Then following the arguments of the proof of Theorem 6.1.3 we see that for any $\varepsilon \neq 0$ sufficiently small there is a sequence $\mathscr{T}^\varepsilon = \{T_m^\varepsilon\}_{m\in\mathbb{Z}}$ so that $T_{m+1}^\varepsilon - T_m^\varepsilon > \bar{T}+1+4|\alpha^0|-2\delta^{-1}\ln|\varepsilon|$ along with the existence of $\alpha^\varepsilon = \{\alpha_m^\varepsilon\}_{m\in\mathbb{Z}} \in \ell^\infty$ with $\|\alpha^\varepsilon\| \leq 2|\alpha^0|$, satisfying $\mathscr{M}(T_{2m}^\varepsilon+\alpha_m^\varepsilon) = 0$ for any $m \in \mathbb{Z}$ and $\inf_{m\in\mathbb{Z}}|\mathscr{M}'(T_{2m}^\varepsilon+\alpha_m^\varepsilon)| > c_0$ for some $c_0 > 0$. Then taking $T_{2m} = T_{2m}^\varepsilon + \alpha_m^\varepsilon$, $T_{2m-1} = T_{2m-1}^\varepsilon$ and $\alpha_0 = 0$, assumption (C) is satisfied. So applying Theorem 6.1.25, system (6.1.1) is chaotic for any $\varepsilon \neq 0$ small. In summary we obtain the following theorem.

Theorem 6.1.27. *Assume that (H1)–(H4) hold and that the almost periodic Melnikov function $\mathscr{M}(\alpha)$ has a simple zero. Then system* (6.1.1) *is chaotic for any $\varepsilon \neq 0$ sufficiently small.*

Next, it is well known (see [41–44], Remark 4.1.7) that near the hyperbolic equilibrium $x = 0$ of the equation $\dot{x} = f_-(x)$ there exists a unique almost periodic solution of $\dot{x} = f_-(x)+\varepsilon g(t,x,\varepsilon)$. More precisely, given $\rho > 0$ there exists $\bar{\varepsilon} > 0$ so that for any $|\varepsilon| < \bar{\varepsilon}$ equation $\dot{x} = f_-(x)+\varepsilon g(t,x,\varepsilon)$ has a solution $u(t) = u(t,\varepsilon)$ so that $|u(t)| < \rho$ for any $t \in \mathbb{R}$ and it is almost periodic with common almost periods as $g(t,x,\varepsilon)$, i.e. assumption (H4) holds in addition with

$$|u(t+T_\nu)-u(t)| < \hat{c}_0\nu \quad \forall t \in \mathbb{R}$$

for a positive constant $\hat{c}_0$. Note that $u(t)$ is a bounded solution of $\dot{x} = f_-(x) + \varepsilon g(t,x,\varepsilon)$ mentioned in (6.1.100). Thus the conclusion of Remark 6.1.26 holds with the further property that $u(t)$ is almost periodic.

Results of this section generalize the deterministic chaos of [42–44, 46] to the discontinuous almost periodic system (6.1.1).

Finally, if $g(t,x,\varepsilon)$ is quasiperiodic in t the following holds:

(H5) $g(t,x,\varepsilon) = q(\omega_1 t,\dots,\omega_m t,x,\varepsilon)$ for $\omega_1,\dots,\omega_m \in \mathbb{R}$ with $q \in C_b^r(\mathbb{R}^{m+n+1},\mathbb{R}^n)$ and $q(\theta_1,\dots,\theta_m,x,\varepsilon)$ is 1-periodic in each θ_i, $i = 1,2,\dots,m$. Moreover, ω_i, $i = 1,2,\dots,m$ are linearly independent of $\mathbb{Z}$, i.e. if $\sum_{i=1}^m l_i\omega_i = 0$, $l_i \in \mathbb{Z}$, $i = 1,2,\dots,m$, then $l_i = 0$, $i = 1,2,\cdots,m$.

Then $g(t,x,\varepsilon)$ satisfies assumption (H4) [40, 42] and hence the conclusion of Theorem 6.1.27 holds.

6.1.9 Periodic Case

Here we assume that $g(t+p,z,\varepsilon) = g(t,z,\varepsilon)$ that is $g(t,z,\varepsilon)$ is p-periodic. Then $\mathscr{M}(\alpha)$ is also p-periodic. We suppose the existence of a simple zero α^0 of $\mathscr{M}(\alpha)$. Then Theorem 6.1.3 is applicable with $T_m = mT$ and $2T = rp$ for $r \gg 1$, $r \in \mathbb{N}$. So

$$T_m^e = \begin{cases} 2n_k^e T, & \text{if } m = 2k, \\ (2n_k^e - 1)T, & \text{if } m = 2k-1. \end{cases}$$

Since we can take $n_m^{\sigma(e)} = n_m^e - 1$ we see that

$$T_m^{\sigma(e)} = \left.\begin{cases} 2n_k^e T - 2T, & \text{if } m = 2k \\ (2n_k^e - 1)T - 2T, & \text{if } m = 2k-1 \end{cases}\right\} = T_m^e - 2T$$

for any $m \in \mathbb{Z}$. Again we denote with $z(t) = z(t,\mathscr{T},e)$ the solution of equation (6.1.1) corresponding to the sequence $\mathscr{T}^e$. Then $Z(t) := z(t+2T)$ satisfies the equation

$$\dot{z} = f_\pm(z) + \varepsilon g(t,z,\varepsilon)$$

together with the estimates:

$$\begin{aligned} &\sup\nolimits_{t\in[T_{2m-1}^{\sigma(e)}+\beta_{m-1}^e,\,T_{2m}^{\sigma(e)}-\bar{T}+\alpha_m^e]} |Z(t) - \gamma_-(t - T_{2m}^{\sigma(e)} - \alpha_m^e)| < \rho, \\ &\sup\nolimits_{t\in[T_{2m}^{\sigma(e)}-\bar{T}+\alpha_m^e,\,T_{2m}^{\sigma(e)}+\bar{T}+\beta_m^e]} |Z(t) - \gamma_0(t - T_{2m}^{\sigma(e)} - \alpha_m^e)| < \rho, \\ &\sup\nolimits_{t\in[T_{2m}^{\sigma(e)}+\bar{T}+\beta_m^e,\,T_{2m-1}^{\sigma(e)}+\beta_m^e]} |Z(t) - \gamma_+(t - T_{2m}^{\sigma(e)} - \beta_m^e)| < \rho. \end{aligned} \tag{6.1.107}$$

Thus, because of uniqueness:

$$\alpha(\mathscr{T}^e,\varepsilon) = \alpha(\mathscr{T}^{\sigma(e)},\varepsilon) \in \ell^\infty(\mathbb{R}), \quad \beta(\mathscr{T}^e,\varepsilon) = \beta(\mathscr{T}^{\sigma(e)},\varepsilon) \in \ell^\infty(\mathbb{R})$$

and $z(t+2T,\mathscr{T},e,\varepsilon)=z(t,\mathscr{T},\sigma(e),\varepsilon)$. Thus, using (6.1.101) and recalling that $T_k = kT$:

$$z(T_{2(k+1)},\mathscr{T}^{(k+1)},e,\varepsilon)=z(T_{2k},\mathscr{T}^{(k+1)},\sigma(e),\varepsilon)=z(T_{2k},\mathscr{T}^{(k)},e,\varepsilon),$$

that is, we see that

$$\Phi_k(e)=\Phi(e),\quad \mathscr{S}_k=\mathscr{S}$$

are independent of k. Similarly, because of uniqueness and periodicity, the value at the time $T_{2(k+1)}=2(k+1)T$ of the solution of (6.1.104) is the same as the value at time $2T$ of the solution of

$$\dot z=f_\pm(z)+\varepsilon g(t,z,\varepsilon),\quad z(0)=\xi,$$

that is, also $F_k(\xi)=F(\xi)$ are independent of k and we have:

$$\Phi\circ\sigma=F\circ\Phi.$$

In summary we arrive at the following result.

Theorem 6.1.28. *Assume that $g(t+p,z,\varepsilon)=g(t,z,\varepsilon)$, that is, $g(t,z,\varepsilon)$ is p-periodic. If $\varepsilon\neq 0$ is sufficiently small and there is a simple zero α^0 of $\mathscr{M}(\alpha)$ then the following diagram commutes:*

$$\begin{array}{ccc}
\mathscr{E} & \xrightarrow{\quad\sigma\quad} & \mathscr{E}\\
\Phi\downarrow & & \downarrow\Phi\\
\mathscr{S} & \xrightarrow[\quad F\quad]{} & \mathscr{S}
\end{array}$$

for any $\mathbb{N}\ni r\gg 1$. Here $F=\varphi_\varepsilon^r=\varphi_\varepsilon\circ\ldots\circ\varphi_\varepsilon$ (r times) is the rth iterate of the p-period map φ_ε of (6.1.1).

Theorem 6.1.28 generalizes the deterministic chaos of Section 2.5.2 [36, 47] to the discontinuous periodic system (6.1.1).

6.1.10 Piecewise Smooth Planar Systems

In this section we apply the theory developed in the previous parts to a two–dimensional system $(x,y\in\mathbb{R})$

$$\begin{aligned}\dot x&=P^\pm(x,y),\\ \dot y&=Q^\pm(x,y),\end{aligned}\tag{6.1.108}$$

where $+$ holds if $(x,y)\in\Omega_+=\{(x,y)\mid G(x,y)>0\}$ and $-$ when $(x,y)\in\Omega_-=\{(x,y)\mid G(x,y)<0\}$. We will construct an explicit expression for $\mathscr{M}(\alpha)$ showing that it extends to the discontinuous case, the usual Melnikov function, thus validating the name of Melnikov function we have given to $\mathscr{M}(\alpha)$. Let us write the homoclinic orbit

$$\gamma(t) = \begin{cases} \gamma_-(t), & \text{if } t \le -\bar{T}, \\ \gamma_0(t), & \text{if } -\bar{T} \le t \le \bar{T}, \\ \gamma_+(t), & \text{if } \bar{T} \le t, \end{cases}$$

as

$$\gamma_\pm(t) = \begin{pmatrix} u_\pm(t) \\ v_\pm(t) \end{pmatrix} \in \bar{\Omega}_-, \quad \gamma_0(t) = \begin{pmatrix} u_0(t) \\ v_0(t) \end{pmatrix} \in \bar{\Omega}_+.$$

Then let

$$\begin{aligned} a_\pm(t) &= P_x^-(u_\pm(t), v_\pm(t)) + Q_y^-(u_\pm(t), v_\pm(t)), \\ a_0(t) &= P_x^+(u_0(t), v_0(t)) + Q_y^+(u_0(t), v_0(t)) \end{aligned}$$

be the trace of the Jacobian matrix of the linearization of (6.1.108) along $(u_\pm(t), v_\pm(t))$ and $(u_0(t), v_0(t))$ respectively, and

$$a(t) := \begin{cases} a_-(t), & \text{if } t < -\bar{T}, \\ a_0(t), & \text{if } -\bar{T} \le t \le \bar{T}, \\ a_+(t), & \text{if } t > \bar{T}. \end{cases}$$

Then $\begin{pmatrix} \dot{v}_\pm(t) \\ -\dot{u}_\pm(t) \end{pmatrix} \mathrm{e}^{-\int_{\pm\bar{T}}^t a_\pm(\tau)d\tau}$ satisfy the adjoint variational system:

$$\begin{aligned} \dot{x} &= -P_x^-(\gamma_\pm(t))x - Q_x^-(\gamma_\pm(t))y, \\ \dot{y} &= -P_y^-(\gamma_\pm(t))x - Q_y^-(\gamma_\pm(t))y \end{aligned}$$

and similarly $\begin{pmatrix} \dot{v}_0(t) \\ -\dot{u}_0(t) \end{pmatrix} \mathrm{e}^{-\int_{-\bar{T}}^t a_0(\tau)d\tau}$ satisfies the adjoint system:

$$\begin{aligned} \dot{x} &= -P_x^+(\gamma_0(t))x - Q_x^+(\gamma_0(t))y, \\ \dot{y} &= -P_y^+(\gamma_0(t))x - Q_y^+(\gamma_0(t))y. \end{aligned}$$

As a consequence,

$$\begin{pmatrix} \dot{v}_\pm(t) \\ -\dot{u}_\pm(t) \end{pmatrix} \mathrm{e}^{-\int_{\pm\bar{T}}^t a_\pm(\tau)d\tau} = X_\pm^*(t)^{-1} \begin{pmatrix} \dot{v}_\pm(\pm\bar{T}) \\ -\dot{u}_\pm(\pm\bar{T}) \end{pmatrix}$$

and

$$\begin{pmatrix} \dot{v}_0(t) \\ -\dot{u}_0(t) \end{pmatrix} \mathrm{e}^{-\int_{-\bar{T}}^t a_0(\tau)d\tau} = X_0^*(t)^{-1} \begin{pmatrix} \dot{v}_0(-\bar{T}) \\ -\dot{u}_0(-\bar{T}) \end{pmatrix}.$$

Next, since the system is two-dimensional, we have

$$\operatorname{span}\{\psi\} = \mathscr{R}R_0 = \operatorname{span}\left\{\begin{pmatrix} G_y(\gamma(\bar{T})) \\ -G_x(\gamma(\bar{T})) \end{pmatrix}\right\}.$$

So we take:

$$\psi = \frac{1}{|G'(\gamma(\bar{T}))|}\begin{pmatrix} G_y(\gamma(\bar{T})) \\ -G_x(\gamma(\bar{T})) \end{pmatrix}.$$

Let $\{e_1, e_2\}$ be the canonical basis of $\mathbb{R}^2$. According to the definition of $R_\pm$, R_0 we have

$$\begin{aligned}
R_+e_1 &= e_1 - \frac{G_x(\gamma(\bar{T}))}{G'(\gamma(\bar{T}))\dot{\gamma}_+(\bar{T})}\dot{\gamma}_+(\bar{T}), \\
R_+e_2 &= e_2 - \frac{G_y(\gamma(\bar{T}))}{G'(\gamma(\bar{T}))\dot{\gamma}_+(\bar{T})}\dot{\gamma}_+(\bar{T}), \\
R_-e_1 &= e_1 - \frac{G_x(\gamma(-\bar{T}))}{G'(\gamma(-\bar{T}))\dot{\gamma}_-(-\bar{T})}\dot{\gamma}_-(-\bar{T}), \\
R_-e_2 &= e_2 - \frac{G_y(\gamma(-\bar{T}))}{G'(\gamma(-\bar{T}))\dot{\gamma}_-(-\bar{T})}\dot{\gamma}_-(-\bar{T}), \\
R_0e_1 &= e_1 - \frac{G_x(\gamma(\bar{T}))}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})}\dot{\gamma}_0(\bar{T}), \\
R_0e_2 &= e_2 - \frac{G_y(\gamma(\bar{T}))}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})}\dot{\gamma}_0(\bar{T})
\end{aligned}$$

and then (here $\mathscr{M}_L$ denotes the matrix of the linear map L with respect to the basis $\{e_1, e_2\}$ of $\mathbb{R}^2$)

$$\begin{aligned}
\mathscr{M}_{R_+^*} &= \frac{1}{G'(\gamma(\bar{T}))\dot{\gamma}_+(\bar{T})}\begin{pmatrix} \dot{v}_+(\bar{T}) \\ -\dot{u}_+(\bar{T}) \end{pmatrix}\left(G_y(\gamma(\bar{T}))\ -G_x(\gamma(\bar{T}))\right), \\
\mathscr{M}_{R_-^*} &= \frac{1}{G'(\gamma(-\bar{T}))\dot{\gamma}_-(-\bar{T})}\begin{pmatrix} \dot{v}_-(-\bar{T}) \\ -\dot{u}_-(-\bar{T}) \end{pmatrix}\left(G_y(\gamma(-\bar{T}))\ -G_x(\gamma(-\bar{T}))\right), \\
\mathscr{M}_{R_0^*} &= \frac{1}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})}\begin{pmatrix} \dot{v}_0(\bar{T}) \\ -\dot{u}_0(\bar{T}) \end{pmatrix}\left(G_y(\gamma(\bar{T}))\ -G_x(\gamma(\bar{T}))\right).
\end{aligned}$$

As a consequence,

$$\begin{aligned}
&X_-^*(t)^{-1}R_-^*X_0^*(\bar{T})R_0^*\psi \\
&= \frac{|G'(\gamma(\bar{T}))|}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})}X_-^*(t)^{-1}R_-^*X_0^*(\bar{T})\begin{pmatrix} \dot{v}_0(\bar{T}) \\ -\dot{u}_0(\bar{T}) \end{pmatrix} \\
&= \frac{|G'(\gamma(\bar{T}))|}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})}X_-^*(t)^{-1}R_-^*\begin{pmatrix} \dot{v}_0(-\bar{T}) \\ -\dot{u}_0(-\bar{T}) \end{pmatrix}\mathrm{e}^{\int_{-\bar{T}}^{\bar{T}} a_0(\tau)d\tau}
\end{aligned}$$

$$= \frac{|G'(\gamma(\bar{T}))|}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})} \frac{G'(\gamma(-\bar{T}))\dot{\gamma}_0(-\bar{T})}{G'(\gamma(-\bar{T}))\dot{\gamma}_-(-\bar{T})} \begin{pmatrix} \dot{v}_-(t) \\ -\dot{u}_-(t) \end{pmatrix} \mathrm{e}^{\int_t^{\bar{T}} a(\tau)d\tau}$$

for any $t \leq -\bar{T}$. Similarly, for $-\bar{T} \leq t \leq \bar{T}$ we have:

$$X_0^*(t)^{-1} X_0^*(\bar{T}) R_0^* \psi = \frac{|G'(\gamma(\bar{T}))|}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})} \begin{pmatrix} \dot{v}_0(t) \\ -\dot{u}_0(t) \end{pmatrix} \mathrm{e}^{\int_t^{\bar{T}} a_0(\tau)d\tau}$$

and

$$X_+^*(t)^{-1} R_+^* \psi = \frac{|G'(\gamma(\bar{T}))|}{G'(\gamma(\bar{T}))\dot{\gamma}_+(\bar{T})} \begin{pmatrix} \dot{v}_+(t) \\ -\dot{u}_+(t) \end{pmatrix} \mathrm{e}^{\int_t^{\bar{T}} a_+(\tau)d\tau}$$

for $t \geq \bar{T}$. Putting everything together we obtain

$$\begin{aligned} \mathscr{M}(\alpha) = {} & \frac{|G'(\gamma(\bar{T}))|}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})} \\ & \cdot \left\{ \frac{G'(\gamma(-\bar{T}))\dot{\gamma}_0(-\bar{T})}{G'(\gamma(-\bar{T}))\dot{\gamma}_-(-\bar{T})} \int_{-\infty}^{-\bar{T}} \begin{pmatrix} \dot{v}_-(t) \\ -\dot{u}_-(t) \end{pmatrix} g(t+\alpha, \gamma(t), 0)\, \mathrm{e}^{-\int_{\bar{T}}^t a(\tau)d\tau}\, dt \right. \\ & + \int_{-\bar{T}}^{\bar{T}} \begin{pmatrix} \dot{v}_0(t) \\ -\dot{u}_0(t) \end{pmatrix} g(t+\alpha, \gamma(t), 0)\, \mathrm{e}^{-\int_{\bar{T}}^t a_0(\tau)d\tau}\, dt \\ & \left. + \frac{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})}{G'(\gamma(\bar{T}))\dot{\gamma}_+(\bar{T})} \int_{\bar{T}}^{\infty} \begin{pmatrix} \dot{v}_+(t) \\ -\dot{u}_+(t) \end{pmatrix} g(t+\alpha, \gamma(t), 0)\, \mathrm{e}^{-\int_{\bar{T}}^t a_+(\tau)d\tau}\, dt \right\} \end{aligned}$$

that can be written as:

$$\begin{aligned} \mathscr{M}(\alpha) = {} & -\frac{|G'(\gamma(\bar{T}))|}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})} \\ & \cdot \left\{ \frac{G'(\gamma(-\bar{T}))\dot{\gamma}_0(-\bar{T})}{G'(\gamma(-\bar{T}))\dot{\gamma}_-(-\bar{T})} \int_{-\infty}^{-\bar{T}} f_-(\gamma(t)) \wedge g(t+\alpha, \gamma(t), 0)\, \mathrm{e}^{-\int_{\bar{T}}^t a(\tau)d\tau}\, dt \right. \\ & + \int_{-\bar{T}}^{\bar{T}} f_+(\gamma(t)) \wedge g(t+\alpha, \gamma(t), 0)\, \mathrm{e}^{-\int_{\bar{T}}^t a(\tau)d\tau}\, dt \qquad (6.1.109) \\ & \left. + \frac{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})}{G'(\gamma(\bar{T}))\dot{\gamma}_+(\bar{T})} \int_{\bar{T}}^{\infty} f_-(\gamma(t)) \wedge g(t+\alpha, \gamma(t), 0)\, \mathrm{e}^{-\int_{\bar{T}}^t a(\tau)d\tau}\, dt \right\}, \end{aligned}$$

where

$$f_{\pm}(x,y) = \begin{pmatrix} P^{\pm}(x,y) \\ Q^{\pm}(x,y) \end{pmatrix}.$$

Note that we can write:

$$\mathscr{M}(\alpha) = -\frac{|G'(\gamma(\bar{T}))|\,\mathrm{e}^{\int_0^{\bar{T}} a_0(\tau)d\tau}}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})} \left\{ \int_{-\infty}^{\infty} f(\gamma(t)) \wedge g(t+\alpha, \gamma(t), 0)\, \mathrm{e}^{-\int_0^t a(\tau)d\tau} dt + \delta_- + \delta_+ \right\}$$

where

$$\delta_- = \left(\frac{G'(\gamma(-\bar{T}))\dot{\gamma}_0(-\bar{T})}{G'(\gamma(-\bar{T}))\dot{\gamma}_-(-\bar{T})} - 1 \right) \int_{-\infty}^{-\bar{T}} f_-(\gamma(t)) \wedge g(t+\alpha, \gamma(t), 0)\, \mathrm{e}^{-\int_0^t a(\tau)d\tau} dt,$$

$$\delta_+ = \left(\frac{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})}{G'(\gamma(\bar{T}))\dot{\gamma}_+(\bar{T})} - 1 \right) \int_{\bar{T}}^{\infty} f_-(\gamma(t)) \wedge g(t+\alpha, \gamma(t), 0)\, \mathrm{e}^{-\int_0^t a(\tau)d\tau} dt.$$

Remark that the extra terms $\delta_\pm$ vanish in cases $\dot{\gamma}_0(-\bar{T}) = \dot{\gamma}_-(-\bar{T})$ and $\dot{\gamma}_0(\bar{T}) = \dot{\gamma}_+(\bar{T})$. Thus $\mathscr{M}(\alpha)$ extends the usual Melnikov function (cf Section 4.1) to the discontinuous case.

6.1.11 3D Quasiperiodic Piecewise Linear Systems

In in section, we consider the example

$$\dot{x} = \begin{cases} Ax + \varepsilon\,(g_1 \sin \omega_1 t + g_2 \sin \omega_2 t), & \text{for } \widetilde{a} \cdot x < d, \\ Ax + b + \varepsilon\,(g_1 \sin \omega_1 t + g_2 \sin \omega_2 t), & \text{for } \widetilde{a} \cdot x > d \end{cases} \tag{6.1.110}$$

of a quasiperiodically perturbed piecewise linear 3-dimensional differential equation. Here $d > 0$, $\omega_{1,2} > 0$, $\widetilde{a}, x, g_{1,2} \in \mathbb{R}^3$, $\widetilde{a} \cdot x$ is the scalar product in $\mathbb{R}^3$. Moreover, we consider system (6.1.110) under the following assumptions

(i) A is a 3×3-matrix with semi-simple eigenvalues, $\lambda_1, \lambda_2 > 0 > \lambda_3$ and with the corresponding eigenvectors, e_1, e_2, e_3.
(ii) Let $b = \sum_{i=1}^3 b_i e_i$ and $a_i := \widetilde{a} \cdot e_i$, $i = 1,2,3$. Then $a_1, b_3 \geq 0$, $a_2, a_3 > 0$ and $b_1, b_2 < 0$.

Remark 6.1.29. Certainly we can study more general systems

$$\dot{x} = \begin{cases} Ax + \varepsilon \sum_{k=1}^{m} g_k \sin \omega_k t, & \text{for } \widetilde{a} \cdot x < d, \\ Ax + b + \varepsilon \sum_{k=1}^{m} g_k \sin \omega_k t, & \text{for } \widetilde{a} \cdot x > d \end{cases}$$

but for simplicity we concentrate on (6.1.110) in this section.

If either $g_1 = 0$, $g_2 = 0$ or the ratio $\frac{\omega_1}{\omega_2}$ is rational, then we get the periodic case studied in [34]. Theorem 6.1.28, however, improves the result in [34] in the sense that here we obtain chaotic behaviour of the solutions. Thus, we focus here on the case

(iii) $g_1 \neq 0$, $g_2 \neq 0$ and ω_1/ω_2 is irrational.

Given the vectors in $\mathbb{R}^3$: $x = \sum_{i=1}^3 x_i e_i$ and $y = \sum_{i=1}^3 y_i e_i$ we define

$$\langle x, y\rangle = \sum_{i=1}^3 x_i y_i.$$

Then $\langle x, y\rangle$ is a scalar product in $\mathbb{R}^3$ that makes $\{e_1, e_2, e_3\}$ an orthonormal basis of $\mathbb{R}^3$. From now on we will write also (x_1, x_2, x_3) for the vector $x = \sum_{i=1}^3 x_i e_i$ and hence we identify e_1, e_2, e_3 with $(1,0,0), (0,1,0), (0,0,1)$ respectively.

Writing $x = \sum_{i=1}^3 x_i e_i$ and $g_j = \sum_{i=1}^3 g_{ji} e_i$, $j = 1,2$, (6.1.110) has the form

$$\dot{x}_i = \begin{cases} \lambda_i x_i + \varepsilon\,(g_{1i}\sin\omega_1 t + g_{2i}\sin\omega_2 t)\,, & \text{for} \quad \langle a, x\rangle < d\,, \\ \lambda_i x_i + b_i + \varepsilon\,(g_{1i}\sin\omega_1 t + g_{2i}\sin\omega_2 t)\,, & \text{for} \quad \langle a, x\rangle > d\,, \end{cases} \tag{6.1.111}$$

$i = 1,2,3$, where $a = \sum_{i=1}^3 a_i e_i$. Hence $G(x) = \langle a, x\rangle - d = \sum_{j=1}^3 a_j x_j - d$ and thus

$$\Omega_- = \Big\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \sum_{i=1}^3 a_i x_i < d\Big\},$$
$$\Omega_+ = \Big\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \sum_{i=1}^3 a_i x_i > d\Big\}.$$

Theorem 6.1.30. *If conditions* (i)–(ii) *and the next ones*

$$a_3 b_3 (\mathrm{e}^{2\lambda_3 \bar{T}} - 1) = d\lambda_3, \quad \sum_{j=1}^2 \frac{a_j b_j}{\lambda_j}(\mathrm{e}^{-2\lambda_j \bar{T}} - 1) = d \tag{6.1.112}$$

hold, then system

$$\dot{x}_i = \begin{cases} \lambda_i x_i, & \textit{for} \quad \langle a, x\rangle < d\,, \\ \lambda_i x_i + b_i, & \textit{for} \quad \langle a, x\rangle > d\,, \end{cases} \tag{6.1.113}$$

$i = 1,2,3$, *has a homoclinic orbit to* $x = 0$*:*

$$\gamma(t) = \begin{cases} \gamma_-(t), & \text{if } t \le -\bar{T}, \\ \gamma_0(t), & \text{if } -T \le t \le \bar{T}, \\ \gamma_+(t), & \text{if } t \ge \bar{T}, \end{cases}$$

where

$$\gamma_-(t) = \left(\mathrm{e}^{\lambda_1(t+\bar{T})}\left(\mathrm{e}^{-2\lambda_1\bar{T}} - 1\right)\frac{b_1}{\lambda_1}, \mathrm{e}^{\lambda_2(t+\bar{T})}\left(\mathrm{e}^{-2\lambda_2\bar{T}} - 1\right)\frac{b_2}{\lambda_2}, 0\right),$$

$$\gamma_0(t) = \left(\left(\mathrm{e}^{\lambda_1(t-\bar{T})} - 1\right)\frac{b_1}{\lambda_1}, \left(\mathrm{e}^{\lambda_2(t-\bar{T})} - 1\right)\frac{b_2}{\lambda_2}, \left(\mathrm{e}^{\lambda_3(t+\bar{T})} - 1\right)\frac{b_3}{\lambda_3}\right),$$

$$\gamma_+(t) = \left(0, 0, \frac{d}{a_3}\,\mathrm{e}^{\lambda_3(t-\bar{T})}\right),$$

and conditions (H1), (H2) and (H3) are satisfied.

Proof. With a view to constructing the homoclinic solution $\gamma(t)$ of system (6.1.113), we describe the local stable and unstable manifolds of the fixed point $(x_1,x_2,x_3) = (0,0,0) \in \Omega_-$: the local unstable manifold of the origin is

$$\mathscr{W}^s_{loc}(0) = \{(x_1,x_2,0) \mid x_1,x_2 \in \mathbb{R},\ a_1x_1 + a_2x_2 < d\}$$

and the local stable manifold is

$$\mathscr{W}^u_{loc}(0) = \{(0,0,x_3) \mid x_3 \in \mathbb{R},\ a_3x_3 < d\}.$$

Thus it must be:

$$\gamma_-(t) = \begin{pmatrix} c_1\,\mathrm{e}^{\lambda_1 t} \\ c_2\,\mathrm{e}^{\lambda_2 t} \\ 0 \end{pmatrix}, \quad \gamma_+(t) = \begin{pmatrix} 0 \\ 0 \\ c_3\,\mathrm{e}^{\lambda_3 t} \end{pmatrix}$$

as long as $\gamma_-(t), \gamma_+(t) \in \Omega_-$. Note that, if $c_1,c_2,c_3 \geq 0$ then, because of (ii), the scalar product $\langle a, \gamma_-(t)\rangle$ (resp. $\langle a, \gamma_+(t)\rangle$) is increasing (resp. decreasing) and hence $\gamma_-(t) \in \Omega_-$ for $t < -\bar{T}$ and $\gamma_+(t) \in \Omega_-$ for $t > \bar{T}$ together with $\gamma_-(-\bar{T}), \gamma_+(\bar{T}) \in \partial\Omega_-$ if and only if $\langle a, \gamma_-(-\bar{T})\rangle = \langle a, \gamma_+(\bar{T})\rangle = d$, that is, if the following conditions on the non-negative numbers $\bar{T}$, d, c_1, c_2, c_3 hold

$$a_1c_1\,\mathrm{e}^{-\lambda_1\bar{T}} + a_2c_2\,\mathrm{e}^{-\lambda_2\bar{T}} = d, \quad a_3c_3\,\mathrm{e}^{\lambda_3\bar{T}} = d. \tag{6.1.114}$$

Next we have to choose $c_1 \geq 0$, $c_2 \geq 0$ and $c_3 \geq 0$ in such a way that the solution $\gamma_0(t)$ of system (6.1.113) with $\gamma_0(-\bar{T}) = \gamma_-(-\bar{T})$ belongs to Ω_+ for $-\bar{T} < t < \bar{T}$ and satisfies $\gamma_0(\bar{T}) = \gamma_+(\bar{T})$. Now, it is easy to see that if the solution of (6.1.113) belongs to Ω_+ and satisfies $\gamma_0(-\bar{T}) = \gamma_-(-\bar{T})$, then it must be

$$\gamma_0(t) = \begin{pmatrix} \lambda_1^{-1}[\mathrm{e}^{\lambda_1 t}(b_1\,\mathrm{e}^{\lambda_1\bar{T}} + c_1\lambda_1) - b_1] \\ \lambda_2^{-1}[\mathrm{e}^{\lambda_2 t}(b_2\,\mathrm{e}^{\lambda_2\bar{T}} + c_2\lambda_2) - b_2] \\ b_3\lambda_3^{-1}(\mathrm{e}^{\lambda_3(t+\bar{T})} - 1) \end{pmatrix}.$$

Hence the condition $\gamma_0(\bar{T}) = \gamma_+(\bar{T})$ is equivalent to:

$$\begin{aligned} c_1\lambda_1 &= -2b_1\sinh(\lambda_1\bar{T}), \\ c_2\lambda_2 &= -2b_2\sinh(\lambda_2\bar{T}), \\ c_3\lambda_3 &= 2b_3\sinh(\lambda_3\bar{T}). \end{aligned} \tag{6.1.115}$$

Plugging these values of c_1,c_2,c_3 into (6.1.114) (note that $c_1,c_2,c_3 > 0$) we obtain (6.1.112) on $\bar{T}$, d. We assume that conditions (6.1.112) are satisfied and show that in this case, $\gamma_0(t) \in \Omega_+$ for any $t \in (-\bar{T}, \bar{T})$. To this end we consider the function:

$$\phi(t) := G(\gamma_0(t)) = \sum_{j=1}^{2} \frac{a_j b_j}{\lambda_j}(\mathrm{e}^{\lambda_j(t-\bar{T})} - 1) + \frac{a_3 b_3}{\lambda_3}(\mathrm{e}^{\lambda_3(t+\bar{T})} - 1) - d.$$

We derive

$$\phi''(t) = \sum_{j=1}^{2} a_j b_j \lambda_i \,\mathrm{e}^{\lambda_j(t-\bar{T})} + a_3 b_3 \lambda_3 \,\mathrm{e}^{\lambda_3(t+\bar{T})}.$$

From assumptions (ii) and (6.1.112) we see that

$$\phi(-\bar{T}) = \phi(\bar{T}) = 0, \quad \phi''(t) < 0 \quad \text{for any} \quad t \in \mathbb{R}.$$

Hence we obtain

$$\phi(t) > 0 \quad \text{on} \quad (-\bar{T}, \bar{T})$$

that gives $\gamma_0(t) \in \Omega_+$ for $-\bar{T} < t < \bar{T}$. Moreover, from $\phi(-\bar{T}) = 0$ and $\phi''(t) < 0$, we also get $\phi'(-\bar{T}) > 0$ and similarly $\phi'(\bar{T}) < 0$, that is,

$$\sum_{j=1}^{2} a_j b_j \,\mathrm{e}^{-2\lambda_j \bar{T}} + a_3 b_3 > 0, \quad \sum_{j=1}^{2} a_j b_j + a_3 b_3 \,\mathrm{e}^{2\lambda_3 \bar{T}} < 0. \tag{6.1.116}$$

Condition (H1) is verified. Now we verify (H2) by checking the inequalities:

$$G'(\gamma(-\bar{T})) f_\pm(\gamma(-\bar{T})) > 0 \quad \text{and} \quad G'(\gamma(\bar{T})) f_\pm(\gamma(\bar{T})) < 0$$

that in this case read:

$$\begin{aligned} &\sum_{j=1}^{2} a_j b_j (\mathrm{e}^{-2\lambda_j \bar{T}} - 1) > 0, && \sum_{j=1}^{2} a_j b_j \,\mathrm{e}^{-2\lambda_j \bar{T}} + a_3 b_3 > 0, \\ &\sum_{j=1}^{2} a_j b_j + a_3 b_3 \,\mathrm{e}^{2\lambda_3 \bar{T}} < 0, && d\lambda_3 < 0. \end{aligned} \tag{6.1.117}$$

The first and the fourth inequalities come immediately from assumptions (i)–(ii); the second and the third ones from (6.1.116). So (H2) also holds. Next we verify condition (H3). First we note that $\nabla G(x) = a$, for any $x \in \mathbb{R}^3$, and

$$P_+ = P_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.1.118}$$

Then, since $\mathscr{N}[G'(\gamma(\bar{T}))] = \{a\}^\perp$ and $a_3 > 0$, we get

$$\mathscr{S}'' = \mathscr{N}[G'(\gamma(\bar{T}))] \bigcap \mathscr{R}P_+ = \{0\}.$$

Similarly, since $\mathscr{N}P_- = \operatorname{span}\{e_1, e_2\}$ and $\mathscr{N}G'(\gamma(-\bar{T})) = \{a\}^\perp$, we obtain

$$\mathscr{S}' = \operatorname{span}\{(a_2, -a_1, 0)\}.$$

Next, from (6.1.113), we see that

$$X_0(t) = X_-(t) = \begin{pmatrix} e^{\lambda_1(t+\bar{T})} & 0 & 0 \\ 0 & e^{\lambda_2(t+\bar{T})} & 0 \\ 0 & 0 & e^{\lambda_3(t+\bar{T})} \end{pmatrix} \qquad (6.1.119)$$

and

$$X_+(t) = \begin{pmatrix} e^{\lambda_1(t-\bar{T})} & 0 & 0 \\ 0 & e^{\lambda_2(t-\bar{T})} & 0 \\ 0 & 0 & e^{\lambda_3(t-\bar{T})} \end{pmatrix}. \qquad (6.1.120)$$

Hence

$$X_0(\bar{T})\mathscr{S}' = \operatorname{span}\{w_0\} \quad \text{with} \quad w_0 := \begin{pmatrix} a_2\, e^{2\lambda_1 \bar{T}} \\ -a_1\, e^{2\lambda_2 \bar{T}} \\ 0 \end{pmatrix}. \qquad (6.1.121)$$

Since $\nabla G(x) = a$, we have:

$$\begin{aligned} R_0 w &= w - \frac{\langle a, w\rangle}{\langle a, \dot{\gamma}_0(\bar{T})\rangle}\dot{\gamma}_0(\bar{T}), \\ R_+ w &= w - \frac{\langle a, w\rangle}{\langle a, \dot{\gamma}_+(\bar{T})\rangle}\dot{\gamma}_+(\bar{T}), \qquad (6.1.122) \\ R_- w &= w - \frac{\langle a, w\rangle}{\langle a, \dot{\gamma}_-(-\bar{T})\rangle}\dot{\gamma}_-(-\bar{T}). \end{aligned}$$

As a consequence,

$$R_0 w_0 = w_0 - \frac{\langle a, w_0\rangle}{\langle a, \dot{\gamma}_0(\bar{T})\rangle}\dot{\gamma}_0(\bar{T}) \neq 0 \qquad (6.1.123)$$

since from (ii) it follows that w_0 is not parallel to $\dot{\gamma}_0(\bar{T}) = \begin{pmatrix} b_1 & b_2 & b_3\, e^{2\lambda_3 \bar{T}} \end{pmatrix}^*$. Thus we get $\mathscr{S}''' = R_0 X_0(\bar{T})\mathscr{S}' \neq \{0\}$ that is $\dim \mathscr{S}''' = 1$ and condition (H3) is satisfied. The proof is completed. □

We start with construction of $\psi(t)$: Since $\mathscr{S}'' = \{0\}$, we see that ψ is such that $\{\psi\}^\perp = \operatorname{span}\{a, R_0 w_0\}$. From (6.1.123) it is easy to see that $\langle a, R_0 w_0\rangle = 0$, hence we can take:

$$\psi = a \wedge R_0 w_0,$$

where $\wedge$ denotes the cross product.

First we construct $\psi(t)$ *for* $t \leq -\bar{T}$: Since: $(\mathbb{I} - P_-^*)R_-^* X_0^*(\bar{T})R_0^*\psi = 0$, we can compute $P_-^* R_-^* X_0^*(\bar{T})R_0^*\psi$ instead of $R_-^* X_0^*(\bar{T})R_0^*\psi$, with the first one being simpler. We recall that $R_0 w = w$ for any $w \in \{a\}^\perp$ and $R_0\dot{\gamma}_0(\bar{T}) = 0$. Thus the eigenvalues of R_0 are 0 (simple) and 1 (double). The same conclusion holds for $R_\pm$. As a consequence, we obtain:

$$\operatorname{trace} R_0 = \operatorname{trace} R_+ = \operatorname{trace} R_- = 2. \tag{6.1.124}$$

We also remark that

$$\dot\gamma_0(\bar T) = \begin{pmatrix} b_1 \\ b_2 \\ b_3\,\mathrm{e}^{2\lambda_3\bar T} \end{pmatrix}, \quad \dot\gamma_+(\bar T) = \begin{pmatrix} 0 \\ 0 \\ \dfrac{d\lambda_3}{a_3} \end{pmatrix}, \quad \dot\gamma_-(-\bar T) = \begin{pmatrix} b_1(\mathrm{e}^{-2\lambda_1\bar T}-1) \\ b_2(\mathrm{e}^{-2\lambda_2\bar T}-1) \\ 0 \end{pmatrix}$$

and, using (6.1.119)

$$X_0(\bar T) = \begin{pmatrix} \mathrm{e}^{2\lambda_1\bar T} & 0 & 0 \\ 0 & \mathrm{e}^{2\lambda_2\bar T} & 0 \\ 0 & 0 & \mathrm{e}^{2\lambda_3\bar T} \end{pmatrix}.$$

Hence we get:

$$P_-^* R_-^* X_0^*(\bar T) R_0^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{13} & A_{23} & A_{33} \end{pmatrix},$$

where

$$A_3 = (A_{13}, A_{23}, A_{33})^* = R_0 X_0(\bar T) R_- e_3 \tag{6.1.125}$$

is the third column of the matrix $R_0 X_0(\bar T) R_-$. Thus

$$P_-^* R_-^* X_0^*(\bar T) R_0^* \psi = \begin{pmatrix} 0 \\ 0 \\ \langle A_3, \psi\rangle \end{pmatrix}.$$

Since $\psi = a \wedge R_0 w_0$ we get, using (6.1.61) and (6.1.119):

$$\psi(t) = \mathrm{e}^{-\lambda_3(t+\bar T)} \langle A_3, a \wedge R_0 w_0\rangle e_3$$

for $t \le -\bar T$. Note that

$$\langle A_3, \psi\rangle = \det \begin{pmatrix} A_{13} & a_1 & (R_0w_0)_1 \\ A_{23} & a_2 & (R_0w_0)_2 \\ A_{33} & a_3 & (R_0w_0)_3 \end{pmatrix} = \det(A_3\ a\ R_0 w_0)$$

where $(R_0w_0)_j$ is the j-th component of R_0w_0 and that $A_3 = R_0[X_0(\bar T)R_- e_3] \in \mathscr{R}R_0 = \{a\}^\perp$ so both A_3 and R_0w_0 belong to span$\{a\}^\perp$, but of course this does not mean they are parallel. The computation of the vector A_3 is really messy even in an example as simple as this, so we don't proceed further with its computation now, but will do it later when we fix some particular values of the parameters.

Next, we look at the expression of $\psi(t)$ for $-\bar{T} < t \leq \bar{T}$: Since the linear system $\dot{x} = Ax$ is autonomous, and $X_0(-\bar{T}) = \mathbb{I}$, we have $X_0^{-1*}(t)X_0^*(\bar{T}) = X_0^*(-t)$. Next, to compute $R_0^*\psi$ we make use of the following identity.

Lemma 6.1.31. *For a given $3\times 3-$matrix M, it holds*

$$(Mu)\wedge v + u\wedge(Mv) - (\operatorname{trace} M)\, u\wedge v = -M^*(u\wedge v) \tag{6.1.126}$$

for any $u, v \in \mathbb{R}^3$.

Proof. Indeed, taking the scalar product with a vector $w \in \mathbb{R}^3$, (6.1.126) is equivalent to

$$\det(Mw,u,v) + \det(w,Mu,v) + \det(w,u,Mv) = (\operatorname{trace} \mathrm{M}) \det(w,u,v)\,. \tag{6.1.127}$$

To prove (6.1.127), we note that the map from $\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3$ to $\mathbb{R}$ given by $(w,u,v) \mapsto \det(Mw,u,v) + \det(w,Mu,v) + \det(w,u,Mv) \in \mathbb{R}$ is multilinear and alternating. Thus there exists $\kappa \in \mathbb{R}$ so that

$$\det(Mw,u,v) + \det(w,Mu,v) + \det(w,u,Mv) = \kappa \det(w,u,v).$$

Taking $w = e_1$, $u = e_2$ and $v = e_3$ we see that $\kappa = \operatorname{trace} M$ and (6.1.127) is proved. The proof of Lemma 6.1.31 is completed. □

We apply (6.1.126) with $M = R_0$, $u = a$ and $v = R_0w_0$. We get, since $\operatorname{trace} R_0 = 2$:

$$-R_0^*\psi = -R_0^*[a\wedge R_0w_0] = R_0a\wedge R_0w_0 + a\wedge[R_0R_0w_0] - 2a\wedge R_0w_0$$

$$= R_0a\wedge R_0w_0 - a\wedge R_0w_0 = -(\mathbb{I}-R_0)a\wedge R_0w_0 = -\frac{|a|^2}{\langle a,\dot{\gamma}_0(\bar{T})\rangle}\dot{\gamma}_0(\bar{T})\wedge R_0w_0$$

and then

$$\psi(t) = \frac{|a|^2}{\langle a,\dot{\gamma}_0(\bar{T})\rangle} X_0(-t)[\dot{\gamma}_0(\bar{T})\wedge R_0w_0]$$

for $-\bar{T} < t \leq \bar{T}$, since $X_0^*(t) = X_0(t)$.

Finally we compute $\psi(t)$ when $t > \bar{T}$: Applying again (6.1.126) with $M = R_+$, $u = a$ and $v = R_0w_0$. We get:

$$-R_+^*\psi = -R_+^*(a\wedge R_0w_0) = (R_+a)\wedge(R_0w_0) + a\wedge(R_+R_0w_0) - 2a\wedge R_0w_0$$

since $\operatorname{trace} R_+ = 2$. Now, we have:

$$(R_+a)\wedge(R_0w_0) = a\wedge R_0w_0 - \frac{|a|^2}{\langle a,\dot{\gamma}_+(\bar{T})\rangle}\dot{\gamma}_+(\bar{T})\wedge R_0w_0, \quad R_+R_0w_0 = R_0w_0$$

since $R_0w_0 \in \mathscr{R}Q = \mathscr{R}R_+$ and R_+ is a projection. Thus:

$$R_+^*\psi = R_+^*(a\wedge R_0w_0) = \frac{|a|^2}{\langle a,\dot{\gamma}_+(\bar{T})\rangle}\dot{\gamma}_+(\bar{T})\wedge R_0w_0 = \frac{|a|^2}{a_3} e_3\wedge R_0w_0$$

and

$$\psi(t) = \frac{|a|^2}{a_3} X_+^{-1}(t)[e_3 \wedge R_0 w_0]$$

for $t > \bar{T}$, since $X_+^*(t) = X_+(t)$. In summary, we conclude with the following result.

Theorem 6.1.32. *Let assumptions* (i)–(ii) *hold and suppose* (6.1.112) *is satisfied. Then the function* $\psi(t)$ *of* (6.1.61) *for the system* (6.1.113) *reads*

$$\psi(t) = \begin{cases} e^{-\lambda_3(t+\bar{T})} \langle A_3, a \wedge R_0 w_0 \rangle e_3, & \text{if } t \leq -\bar{T}, \\ \dfrac{|a|^2}{\langle a, \dot{\gamma}_0(\bar{T}) \rangle} X_0(-t)[\dot{\gamma}_0(\bar{T}) \wedge R_0 w_0], & \text{if } -\bar{T} < t \leq \bar{T}, \\ \dfrac{|a|^2}{a_3} X_+^{-1}(t)[e_3 \wedge R_0 w_0], & \text{if } t > \bar{T} \end{cases} \tag{6.1.128}$$

where $X_0(t)$, $X_+(t)$, w_0, R_0, A_3 *are given by* (6.1.119), (6.1.120), (6.1.121), (6.1.122), (6.1.125), *respectively.*

So we are in position to apply Theorem 6.1.16. Writing $g_j = (g_{j1}, g_{j2}, g_{j3})^*$, $j = 1, 2$, we get the Melnikov function (6.1.62)

$$\begin{aligned} \mathscr{M}(\alpha) &= \int_{-\infty}^{\infty} [\sin \omega_1 (t+\alpha) \psi^*(t) g_1 + \sin \omega_2 (t+\alpha) \psi^*(t) g_2] \, dt \\ &= \sin(\alpha\omega_1) \int_{-\infty}^{\infty} \cos(\omega_1 t) \psi^*(t) g_1 \, dt + \cos(\alpha\omega_1) \int_{-\infty}^{\infty} \sin(\omega_1 t) \psi^*(t) g_1 \, dt \\ &\quad + \sin(\alpha\omega_2) \int_{-\infty}^{\infty} \cos(\omega_2 t) \psi^*(t) g_2 \, dt + \cos(\alpha\omega_2) \int_{-\infty}^{\infty} \sin(\omega_2 t) \psi^*(t) g_2 \, dt \\ &= A_1(\omega_1) \sin(\omega_1 \alpha + \varpi_1(\omega_1)) + A_2(\omega_2) \sin(\omega_2 \alpha + \varpi_2(\omega_2)) \end{aligned}$$

where

$$A_i(\omega_i) := \sqrt{\left(\int_{-\infty}^{\infty} \cos \omega_i t \, \psi^*(t) g_i \, dt \right)^2 + \left(\int_{-\infty}^{\infty} \sin \omega_i t \, \psi^*(t) g_i \, dt \right)^2}$$

for $i = 1, 2$. Now we consider the following two possibilities:

1. Either $A_1(\omega_1) \neq 0, A_2(\omega_2) = 0$ or $A_1(\omega_1) = 0, A_2(\omega_2) \neq 0$. Then $\mathscr{M}(\alpha)$ has the simple zero $\alpha_0 = -\varpi_i(\omega_i)/\omega_i$, $i = 1, 2$, respectively.
2. $A_1(\omega_1) \neq 0$ and $A_2(\omega_2) \neq 0$. Let $s_i := \operatorname{sgn} A_i(\omega_i) \in \{-1, 1\}$, $i = 1, 2$. Then $s_1 \omega_1 A_1(\omega_1) + s_2 \omega_2 A_2(\omega_2) = \omega_1 |A_1(\omega_1)| + \omega_2 |A_2(\omega_2)| > 0$. Since $\cos \frac{1-s_i}{2} \pi = s_i$ and $\sin \frac{1-s_i}{2} \pi = 0$ for $i = 1, 2$, and ω_1 / ω_2 is irrational, from [40] the existence follows from α_0 (as a matter of fact infinitely many α_0 exists) so that $\omega_i \alpha_0 + \varpi_i(\omega_i)$ are near to $\frac{1-s_i}{2} \pi$ modulo 2π, $i = 1, 2$, and $\mathscr{M}(\alpha_0) = 0$ while

$$\mathscr{M}'(\alpha_0) \geq \frac{s_1 \omega_1 A_1(\omega_1) + s_2 \omega_2 A_2(\omega_2)}{2} > 0 \, .$$

Hence also in this case we have a simple zero of $\mathscr{M}(\alpha)$.

Consequently if $A_1(\omega_1)$ and $A_1(\omega_1)$ do not vanish simultaneously, Theorem 6.1.27 applies and we conclude that (6.1.110) behaves chaotically for any $\varepsilon \neq 0$ sufficiently small. Next, we note that $A_i(\omega_i) \neq 0$ if and only if

$$\Phi_i(\omega_i) := \int_{-\infty}^{\infty} \mathrm{e}^{-\omega_i t \imath}\, \psi^*(t) g_i\, dt \neq 0\,. \tag{6.1.129}$$

Since $\psi(t) \neq 0$, Plancherel Theorem (cf Section 2.1) ensures that

$$V(\omega) := \int_{-\infty}^{\infty} \mathrm{e}^{-\omega t \imath}\, \psi(t)\, dt \neq 0. \tag{6.1.130}$$

Note that $\Phi_i(\omega) = V(\omega)^* g_i$. So condition (6.1.129) is equivalent to the non-orthogonality of $V(\omega)$ to g_i. Furthermore, it is not difficult to observe that $\Phi_i(\omega)$ are analytic for $\omega > 0$. Indeed, we have $|\psi(t)| \leq k\mathrm{e}^{-\delta|t|}$, for some positive constants k and δ, and for $\omega, \eta \in \mathbb{R}$ we have: $\sin((\omega + \imath\eta)x) = \sin(\omega x)\mathrm{e}^{-\eta x} + \imath\,\mathrm{e}^{-\imath\omega x} \sinh \eta x$. Thus the function

$$\int_{-\infty}^{\infty} \sin(zt)\psi^*(t) g_i\, dt$$

is holomorphic in the strip $\{\omega + \imath\eta \in \mathbb{C} \mid |\eta| < \delta\}$. A similar argument works with $\cos(zt)$ instead of $\sin(zt)$. Consequently, when functions $\Phi_i(\omega)$ are not identically zero, they have at most countable many positive zeroes with possible accumulations at $+\infty$ (cf Section 2.6.5). In summary, we get the following result.

Theorem 6.1.33. *Let assumptions* (i)–(iii) *hold and suppose* (6.1.112) *holds. When both* $\Phi_1(\omega)$ *and* $\Phi_2(\omega)$ *are not identically zero, there is at most a countable set* $\{\widetilde{\omega}_j\} \subset (0,\infty)$ *with possible accumulating point at* $+\infty$ *so that if* $\omega_1, \omega_2 \in (0,\infty) \setminus \{\widetilde{\omega}_j\}$ *then system* (6.1.110) *is chaotic for any* $\varepsilon \neq 0$ *sufficiently small.*

Since in general, the above formulas are rather difficult to find the solution, now we consider the following concrete examples.

Example 6.1.34. We take

$$\begin{gathered} a_1 = 0, \quad a_2 = a_3 = 1, \quad \lambda_1 = 2, \quad \lambda_2 = 1, \quad \lambda_3 = -1, \\ b_1 = b_2 = -1, \quad b_3 = 1, \quad d = 3/4\,. \end{gathered} \tag{6.1.131}$$

Then (6.1.112) is satisfied with $\bar{T} = \ln 2$. With these parameters values we have:

$$R_0 w_0 = w_0 = 16 e_1, \quad \dot{\gamma}_0(\bar{T}) = -e_1 - e_2 + \frac{1}{4} e_3\,.$$

Thus,

$$\dot{\gamma}_0(\bar{T}) \wedge R_0 w_0 = 4e_2 + 16e_3$$

and we get

$$\psi(t) = \begin{cases} -\dfrac{64}{3}[\mathrm{e}^{-t} e_2 + \mathrm{e}^{t} e_3], & \text{for } -\ln 2 < t \leq \ln 2, \\ 64\,\mathrm{e}^{-t} e_2, & \text{for } t > \ln 2, \end{cases}$$

since

$$X_0(t) = X_-(t) = \begin{pmatrix} 4\,\mathrm{e}^{2t} & 0 & 0 \\ 0 & 2\,\mathrm{e}^{t} & 0 \\ 0 & 0 & \dfrac{1}{2}\,\mathrm{e}^{-t} \end{pmatrix}, \quad X_+(t) = \begin{pmatrix} \dfrac{1}{4}\,\mathrm{e}^{2t} & 0 & 0 \\ 0 & \dfrac{1}{2}\,\mathrm{e}^{t} & 0 \\ 0 & 0 & 2\,\mathrm{e}^{-t} \end{pmatrix}.$$

Finally we compute $\psi(t)$ for $t \leq -\ln 2$. First we need to know A_3 which is the third column of $R_0 X_0(\bar{T}) R_-$ that is

$$A_3 = R_0 X_0(\bar{T}) R_- e_3.$$

We have $R_- e_3 = -\frac{5}{4} e_1 - e_2 + e_3$, then $X_0(\bar{T}) R_- e_3 = -20 e_1 - 4 e_2 + \frac{1}{4} e_3$ and thus $A_3 = -15 e_1 + e_2 - e_3$. As a consequence,

$$\langle A_3, a \wedge R_0 w_0 \rangle = \det \begin{pmatrix} -15 & 0 & 16 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} = 32$$

and $\psi(t) = 64\,\mathrm{e}^{t} e_3$ for $t \leq -\ln 2$. We conclude that (see (6.1.128))

$$\psi(t) = \begin{cases} 64\,\mathrm{e}^{t} e_3, & \text{for } t \leq -\ln 2, \\ -\dfrac{64}{3}[\mathrm{e}^{-t} e_2 + \mathrm{e}^{t} e_3], & \text{for } -\ln 2 < t \leq \ln 2, \\ 64\,\mathrm{e}^{-t} e_2, & \text{for } t > \ln 2\,. \end{cases}$$

Putting this formula of $\psi(t)$ into (6.1.130), we finally obtain

$$V(\omega) = -\frac{256 \sin(\omega \ln 2)}{3(\omega^2 + 1)} \left[\omega(e_2 + e_3) + \imath(e_2 - e_3)\right].$$

Then from $\Phi_i(\omega) = V(\omega)^* g_i$, we have:

$$\Phi_i(\omega) = -\frac{256 \sin(\omega \ln 2)}{3(\omega^2 + 1)} \Big(\omega(g_{i2} + g_{i3}) + \imath(g_{i2} - g_{i3})\Big). \tag{6.1.132}$$

So for the parameters (6.1.131), $\Phi_i(\omega)$ is identically zero if and only if $g_{i2} = g_{i3} = 0$. Otherwise, it has only the simple positive zeroes $\widetilde{\omega}_j = \pi j / \ln 2$, $j \in \mathbb{N}$. In consequence of Theorem 6.1.33 we get the following.

Corollary 6.1.35. *Consider* (6.1.110) *with parameters* (6.1.131) *and* (iii) *holds. If either* $g_{i2} \neq 0$ *or* $g_{i3} \neq 0$ *for some* $i \in \{1, 2\}$ *and* $\omega_1, \omega_2 \neq \pi j / \ln 2$, $\forall j \in \mathbb{N}$ *then system* (6.1.110) *is chaotic for any* $\varepsilon \neq 0$ *small.*

Example 6.1.36. On the other hand, for the following set of parameters

$$\begin{aligned} &a_1 = a_2 = a_3 = 1, \quad b_1 = b_2 = -1, \quad b_3 = 13/8, \\ &\lambda_1 = 2, \quad \lambda_2 = 1, \quad \lambda_3 = -1, \quad d = 39/32, \end{aligned} \tag{6.1.133}$$

we get $\bar{T} = \ln 2$ and (see (6.1.128))

$$\psi(t) = \begin{cases} \frac{1344}{17} \mathrm{e}^t e_3, & \text{for} \quad t \leq -\ln 2, \\ -\frac{16}{17}(13\mathrm{e}^{-2t} e_1 + 26\mathrm{e}^{-t} e_2 + 20\mathrm{e}^t e_3), & \text{for} \quad -\ln 2 < t \leq \ln 2, \\ \frac{48}{17}(49\mathrm{e}^{-2t} e_1 + 18\mathrm{e}^{-t} e_2), & \text{for} \quad t > \ln 2. \end{cases}$$

Then

$$\Phi_i(\omega) = \frac{2^{6-\imath\omega}(13 \cdot 4^{\imath\omega} - 10)(g_{i1} + 2g_{i2} + g_{i1}\omega^2 + g_{i2}\omega^2 - 2g_{i3} + \omega^2 g_{i3} - \imath(g_{i2} + 3g_{i3})\omega)}{17(\omega - \imath)(\omega - 2\imath)(1 - \imath\omega)}$$

for $i = 1,2$. Clearly, for the parameters (6.1.133), $\Phi_i(\omega)$ is not identically zero. If $g_{i2} \neq -3g_{i3}$ then $\Phi_i(\omega)$ has no positive roots. If $g_{i2} = -3g_{i3}$ then $\Phi_i(\omega)$ has the only positive root $\omega_{i1} = \sqrt{\frac{g_{i1}-8g_{i3}}{2g_{i3}-g_{i1}}}$ provided $\frac{g_{i1}-8g_{i3}}{2g_{i3}-g_{i1}} > 0$. In consequence of Theorem 6.1.33 we obtain the following.

Corollary 6.1.37. *Consider* (6.1.110) *with parameters* (6.1.133) *and* (iii) *holds. If one of the following conditions is satisfied*

- $g_{i2} \neq -3g_{i3}$,
- $g_{i2} = -3g_{i3}$, $g_{i1} = 2g_{i3} \neq 0$,
- $g_{i2} = -3g_{i3}$, $g_{i1} \neq 2g_{i3}$, $\dfrac{g_{i1} - 8g_{i3}}{2g_{i3} - g_{i1}} < 0$,
- $g_{i2} = -3g_{i3}$, $g_{i1} \neq 2g_{i3}$, $\dfrac{g_{i1} - 8g_{i3}}{2g_{i3} - g_{i1}} > 0$ *and* $\omega_i \neq \sqrt{\dfrac{g_{i1} - 8g_{i3}}{2g_{i3} - g_{i1}}}$,

for some $i \in \{1,2\}$ *then system* (6.1.110) *is chaotic for any* $\varepsilon \neq 0$ *small.*

Remark 6.1.38. Parameters (6.1.131) and (6.1.133) give Examples 6.1.34 and 6.1.36 for which $\Phi_i(\omega)$ is either identically zero, or has infinitely many positive roots, or has no positive roots, or has finitely many positive roots.

Remark 6.1.39. If $\Phi_1(\omega_1) = 0$ and $\Phi_2(\omega_2) = 0$ then $\mathscr{M}(\alpha)$ is identically zero and a *second-order Melnikov function* must be derived as in Section 4.1.4. But those computations should be very awkward for (6.1.110), so we omit them.

Finally when $g_1 \neq 0$, $g_2 \neq 0$ and ω_1/ω_2 is rational, we get a different situation. For instance, consider Example 6.1.34 with $\omega_1 = 1$, $\omega_2 = 3$ and $g_{i2} = g_{i3}$, $i = 1,2$. Thus (6.1.110) is 2π-periodic and

$$\mathscr{M}(\alpha) = \Phi_1(1)\sin\alpha + \Phi_2(3)\sin 3\alpha = \sin\alpha - \frac{1}{3}\sin 3\alpha = \frac{4}{3}\sin^3\alpha$$

provided $\Phi_1(1) = 1$ and $\Phi_2(3) = -\frac{1}{3}$. From (6.1.132) we derive

$$g_{12} + g_{13} = -\frac{3}{128\sin(\ln 2)}, \quad g_{22} + g_{23} = \frac{5}{384\sin(3\ln 2)}.$$

Then the Melnikov function is $\mathscr{M}(\alpha) = \frac{4}{3}\sin^3\alpha$ and it has only the zero $\alpha_0 = 0$ in $[-\pi,\pi]$ which is not simple but has Brouwer index 1 (cf Section 2.2.4). So Theorem 6.1.27 is not applicable, but we still get a chaos for (6.1.110) with $\varepsilon \neq 0$ small as in Remark 3.1.9 [15].

6.1.12 Multiple Transversal Crossings

The above results can be extended to cases when homoclinics are transversally passing through several discontinuity manifolds. More precisely, let $\Omega \subset \mathbb{R}^n$ be a bounded open set in $\mathbb{R}^n$ and $G_j(z)$, $j = 1,\dots,p$ be C^r-functions on $\bar{\Omega}$, with $r \geq 2$. We set $S_j = \{z \in \Omega \mid G_j(z) = 0\}$, and

$$\Omega \setminus \bigcup_{j=1}^{p} S_j := \bigcup_{i=0}^{q} \Omega_i$$

with Ω_i being the connected components of $\Omega \setminus \cup_{j=1}^{p} S_j$. Let $f_i(z) \in C_b^r(\mathbb{R}^n)$ and $g_i(t,z,\varepsilon) \in C_b^r(\mathbb{R}^{n+2})$, i.e. $f_i(z)$ and $g_i(t,z,\varepsilon)$ have uniformly bounded derivatives up to the r-th order on $\mathbb{R}^n$ and $\mathbb{R}^{n+2}$, respectively. We also assume that the r-th order derivatives of $f_i(z)$ and $g_i(t,z,\varepsilon)$ are uniformly continuous. We set

$$f(z) := f_i(z), \qquad g(t,z,\varepsilon) := g_i(t,z,\varepsilon) \quad \text{if } z \in \Omega_i$$

and

$$G(z) := \Pi_{j=1}^{p} G_j(z).$$

Definition 6.1.40. We say that a piecewise C^1-function $z(t)$ is a solution of the equation

$$\dot{z} = f(z) + \varepsilon g(t,z,\varepsilon), \quad z \in \bar{\Omega}, \tag{6.1.134}$$

if it satisfies Eq. (6.1.134) when $z(t) \in \Omega_i$, and moreover, the following holds: if for some t_* we have $z(t_*) \in S_j$, then $z(t_*) \notin S_l$ for any $l \neq j$ and there exists $r > 0$ so that for any $t \in (t_* - r, t_* + r)$ with $t \neq t_*$, we have $z(t) \in \cup_{i=0}^{q} \Omega_i$. Moreover, if, for example, $z(t) \in \Omega_i$ for any $t \in (t_* - r, t_*)$, then the left derivative of $z(t)$ at $t = t_*$ satisfies: $\dot{z}(t_*^-) = f_i(z(t_*)) + \varepsilon g_i(t_*, z(t_*), \varepsilon)$; similarly, if $z(t) \in \Omega_i$ for any $t \in (t_*, t_* + r)$, then $\dot{z}(t_*^+) = f_i(z(t_*)) + \varepsilon g_i(t_*, z(t_*), \varepsilon)$.

Remark 6.1.41. Since $z(t) \in \cup_{i=0}^{q}\Omega_i$ for any $t \in (t_* - r, t_* + r) \setminus \{t_*\}$ there exist two indices $i = i'_j, i''_j$ so that $z(t) \in \Omega_{i'_j}$ when $t \in (t_* - r, t_*)$ and $z(t) \in \Omega_{i''_j}$ for $t \in (t_*, t_* + r)$. Moreover, since $z(t) \notin \cup_{j=1}^{p} S_j$, for any $t \in (t_* - r, t_*) \cup (t_*, t_* + r)$, $z(t) \in \cup_{j=1}^{p} S_j$ only for t in a discrete increasing subset $\{t_j\}$ of $\mathbb{R}$ with possible accumulation points at $\pm\infty$. Moreover $z(t) \in C^{r+1}(\mathbb{R} \setminus \{t_j\})$.

We assume (Figure 6.2) that:

(H1) For $\varepsilon = 0$ Eq. (6.1.134) has the hyperbolic equilibrium $x = 0 \in \Omega_0$ and a continuous, piecewise C^1-solution $\gamma(t) \in \Omega$ which is homoclinic to $x = 0$ and consists of three branches

$$\gamma(t) = \begin{cases} \gamma_-(t), & \text{if } t \le -\bar{T}, \\ \gamma_0(t), & \text{if } -\bar{T} \le t \le \bar{T}, \\ \gamma_+(t), & \text{if } t \ge \bar{T}, \end{cases}$$

where $\gamma_\pm(t) \in \Omega_0$ for $|t| > \bar{T}$, $\gamma_0(t) \in \Omega$ for $|t| < \bar{T}$ and

$$\gamma_-(-\bar{T}) = \gamma_0(-\bar{T}) \in \partial\Omega_0, \quad \gamma_+(\bar{T}) = \gamma_0(\bar{T}) \in \partial\Omega_0 .$$

(H2) At any point $t_* \in \mathbb{R}$ so that $\gamma(t_*) \in S_j$, we have

$$G'(\gamma(t_*))f_{i'_j}(\gamma(t_*)) \cdot G'(\gamma(t_*))f_{i''_j}(\gamma(t_*)) > 0 ,$$

where i'_j, i''_j are the two indices defined in Remark 6.1.41.

Let t_* be such that $\gamma(t_*) \in S_j$ for some j. Then (H2) means that both $\dot{\gamma}(t_*^+)$ and $\dot{\gamma}(t_*^-)$ are transverse to S_j at the point $\gamma(t_*)$. Next, since $\gamma(t) \in \Omega_0$ for $|t| \ge \bar{T}$, it follows that $\gamma_0(t)$ intersect $\cup_{i=1}^{p} S_i$ only a finite number of times denoted by $-\bar{T} = t_0 < t_1 < \cdots < t_{N-1} < t_N = \bar{T}$. In summary $\gamma(t) \in \cup_{i=1}^{p} S_i$ if and only if $t \in \{-\bar{T} = t_0 < t_1 < \cdots < t_{N-1} < t_N = \bar{T}\}$ and $\gamma(t)$ is continuous, piecewise C^1 in $\mathbb{R}$ and has left and right derivatives at the points $t = t_i$, $i = 0, \ldots, N$. Next for $l = 0, \ldots, N-1$, we define i_l, j_l so that $\gamma_0(t) \in \Omega_{i_l}$ for any $t \in (t_l, t_{l+1})$ and $\gamma_0(t_l) \in S_{j_l}$, $\gamma_0(t_N) \in S_{j_N}$. Thus, with reference to the notation of Remark 6.1.41, we have $i'_{j_l} = i_{l-1}$ and $i''_{j_l} = i_l$.

For $l = 1, \ldots, N$ let $X_l(t)$, $t \in [t_{l-1}, t_l]$ be the fundamental matrix solution of $\dot{x} = f'_{i_{l-1}}(\gamma_0(t))x$ with $X_l(t_{l-1}) = \mathbb{I}$. The transition matrix $\mathscr{S}_l : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$\mathscr{S}_l w := w + \left[\dot{\gamma}_0(t_l^+) - \dot{\gamma}_0(t_l^-)\right] \frac{G'(\gamma_0(t_l))w}{G'(\gamma_0(t_l))\dot{\gamma}_0(t_l^-)} \tag{6.1.135}$$

for $l = 1, \ldots, N-1$. It is easy to see that all $\mathscr{S}_l$ are invertible. Finally we define the fundamental matrix solution of the variational equation of (6.1.1) along $\gamma_0(t)$ at $\varepsilon = 0$ as follows:

$$X_0(t) := X_l(t)\mathscr{S}_{l-1}X_{l-1}(t_{l-1})\mathscr{S}_{l-2} \cdots \mathscr{S}_1 X_1(t_1) \quad \text{for} \quad t \in [t_{l-1}, t_l)$$

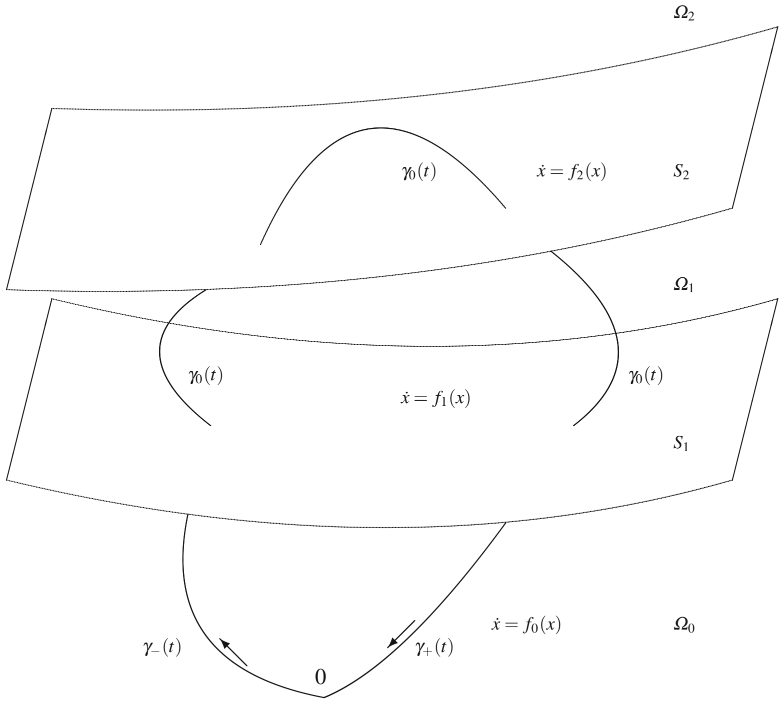


Fig. 6.2 Homoclinic orbit $\gamma(t)$ transversally crosses discontinuity manifolds S_1 and S_2. It may cross $S_{1,2}$ several but finite times before eventually getting in Ω_0.

and $l = 2, \ldots, N$, where we have $X_0(t) = X_1(t)$ on $[t_0, t_1)$. Note that $X_0(t)$ solves the following impulsive linear matrix differential Cauchy problem

$$\dot{X}_0(t) = Df(\gamma_0(t))X_0(t),$$

$$X_0(t_l^+) = \mathscr{S}_l X_0(t_l^-), \quad l = 1, \ldots, N-1, \quad X_0(-\bar{T}) = \mathbb{I}$$

for $t \in [-\bar{T}, \bar{T}]$. Now we can repeat the above arguments over (6.1.134) by introducing (6.1.61), (6.1.62) and then restate Theorem 6.1.16 and the other above results [48].

6.2 Sliding Homoclinic Bifurcation

6.2.1 Higher Dimensional Sliding Homoclinics

In [34] the problem of bifurcations from homoclinic orbits is studied whereas in Section 6.1 chaotic behaviour of solutions is proved for time perturbed discontinuous differential equations in a finite dimensional space, when the homoclinic orbits of the unperturbed problem crosses transversally the discontinuity manifold. Thus, it is natural to argue if a similar behaviour arises also when sliding homoclinic orbits are concerned. The purpose of this section is to give an affirmative answer to this question. It has been observed in Section 6.1 that one of the problems we have to face studying discontinuous differential equations, is the loss of smoothness of invariant manifolds, a problem persisting also in the sliding case. Moreover in the sliding case the additional problem arises, that is, during the *sliding time* the system should be considered only on the discontinuity manifold, thus reducing the dimension of the system. However, we show in this section that the method used in Section 6.1 to prove chaotic behavior can be arranged to handle the case of sliding homoclinic orbits, leading to a similar conclusion.

Typical examples of sliding motions are in relay controllers, impact oscillators and stick-slip friction systems where the stick motion corresponds to sliding. Many non-smooth models can be found in [6,7,11,14,27,28,49–54]. Sliding homoclinic solutions to pseudo-saddles (saddles lying on discontinuity curves/lines) of planar DDEs are studied in [6,51] both numerically and analytically. A theoretical discussion on sliding homoclinic solutions to saddles of planar DDEs is presented in [6]. However, we have not found any concrete example in literature with a sliding homoclinic orbit to a saddle, except in [28] where an example is given with two discontinuity lines. In our opinion the reason why it is so difficult to find examples is because when the discontinuity manifold is linear, the DDE must be nonlinear in the open subset the equilibrium point belongs to and this makes computations harder. Of course, one can imagine a linear system of ODE with a sliding homoclinic orbit to a nonlinear discontinuity manifold. But one can reduce to the linear discontinuity manifold (and then to a nonlinear equation) by a simple change of variables, and for computational reasons, it is better to work with linear discontinuity manifolds. For this reason we investigate examples of DDEs exhibiting sliding chaotic behaviour in consequence of Theorem 6.2.5 in Sections 6.2.2 and 6.2.3.

Now we go into details. Let $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$ with corresponding projections $P_z : \mathbb{R}^n \to \mathbb{R}$ and $P_y : \mathbb{R}^n \to \mathbb{R}^{n-1}$. For $x \in \mathbb{R}^n$ we write $x = (z,y) \in \mathbb{R} \times \mathbb{R}^{n-1}$. Consider a discontinuous system in $\mathbb{R}^n$ with a small parameter such as:

$$\dot{x} = f(x) + \varepsilon g(t,x,\varepsilon)\,, \tag{6.2.1}$$

where

$$f(x) = \begin{cases} f_+(z,y) & \text{for,} \quad z > 0\,, \\ f_-(z,y) & \text{for,} \quad z < 0\,, \end{cases}$$

with $f_\pm : \Omega \to \mathbb{R}^n$, $f_\pm \in C_b^r(\Omega)$ and $g : \mathbb{R} \times \Omega \times \mathbb{R} \to \mathbb{R}^n$, $g \in C_b^r(\mathbb{R} \times \Omega \times \mathbb{R})$, with Ω being a bounded open subset of $\mathbb{R}^n$ that has nonempty intersection with the hyperplane $z = 0$. Note that we allow the possibility that $f_+(0,y) \neq f_-(0,y)$. We also assume that the r-th order derivatives of $f_\pm(x)$ and $g(t,x,\varepsilon)$ are uniformly continuous. We set

$$\Omega_\pm = \{x = (z,y) \in \Omega \mid \pm z > 0\}, \quad \Omega_0 = \{x = (z,y) \in \Omega \mid z = 0\}.$$

By putting

$$f_\pm = (h_\pm(z,y), k_\pm(z,y)),$$

where $h_\pm : \Omega \to \mathbb{R}$ and $k_\pm : \Omega \to \mathbb{R}^{n-1}$, we assume that

(H1) For any $(0,y) \in \Omega_0$ it results:

$$h_-(0,y) - h_+(0,y) > 0. \tag{6.2.2}$$

Then we set (see [8, Eq. (2.12)]))

$$H(y) := V(y)\frac{k_+(0,y) - k_-(0,y)}{2} + \frac{k_+(0,y) + k_-(0,y)}{2},$$

where

$$V(y) := \frac{h_+(0,y) + h_-(0,y)}{h_-(0,y) - h_+(0,y)},$$

and for $(0,y) \in \Omega_0$, we consider the equation

$$\dot{y} = H(y). \tag{6.2.3}$$

Note that $H(y)$ has the following symmetric form with respect to indices $\pm$:

$$H(y) = \frac{h_-(0,y)k_+(0,y) - h_+(0,y)k_-(0,y)}{h_-(0,y) - h_+(0,y)}.$$

We suppose that

(H2) The unperturbed equation $\dot{x} = f_-(x)$ has a hyperbolic fixed point $x_0 \in \Omega_-$ and two solutions $\gamma_\pm(t)$, defined respectively for $t \geq \bar{T}$ and $t \leq -\bar{T}$, so that $\lim_{t\to\pm\infty} \gamma_\pm(t) = x_0$ and $\gamma_\pm(\pm\bar{T}) \in \Omega_0$.

(H3) Equation (6.2.3) has a solution $y_0(t)$, $(0, y_0(t)) \in \Omega_0$ for $-\bar{T} \leq t \leq \bar{T}$ so that

$$\gamma_-(-\bar{T}) = \gamma_0(-\bar{T}), \quad \gamma_+(\bar{T}) = \gamma_0(\bar{T})$$

where $\gamma_0(t) = (0, y_0(t))$, and the following hold:
$h_+(\gamma_0(t)) < 0$ for any $t \in [-\bar{T}, \bar{T}]$;
$h_-(\gamma_0(t)) > 0$ for any $t \in [-\bar{T}, \bar{T})$;
$h_-(\gamma_0(\bar{T})) = 0$ and $k_-(\gamma_0(\bar{T}))$ is not orthogonal to $\nabla_y h_-(\gamma_0(\bar{T})) \neq 0$. Here $\nabla_y h_-(\gamma_0(\bar{T}))$ is the gradient of $h_-(0,y)$ at the point $\gamma_0(\bar{T}) \in \Omega_0$.

Remark 6.2.1. 1. Note that the assumption that system (6.2.1) has a discontinuity on the hyperplane $z = 0$ is made only for sake of simplicity. We could have assumed, instead, that the singularity was at a hypersurface $x_1 = \varphi(x_2, \ldots, x_n)$ since we can reduce to our hypothesis by the simple change of variables:

$$y = (x_2, \ldots, x_n), \quad z = x_1 - \varphi(x_2, \ldots, x_n).$$

2. It will result from the argument given in the next sections that we may as well consider the case

$$g(x) = \begin{cases} g_+(t,z,y), & \text{for} \quad z > 0\,, \\ g_-(t,z,y), & \text{for} \quad z < 0\,, \end{cases}$$

with $g_\pm : \mathbb{R} \times \Omega \times \mathbb{R} \to \mathbb{R}^n$, $g_\pm \in C_b^r(\mathbb{R} \times \Omega \times \mathbb{R})$. However, for simplicity, we will continue to assume that $g \in C_b^r(\mathbb{R} \times \Omega \times \mathbb{R})$.

Remark 6.2.2. From (H3) it follows that $h_-^{-1}(0)$ is a submanifold $\mathscr{K}$ of Ω_0 of codimension 1 near the point $\gamma_0(\bar{T})$ (here we consider the restriction $h_- : \Omega_0 \to \mathbb{R}$). Moreover, since $V(y_0(\bar{T})) = -1$, we get

$$H(y_0(\bar{T})) = k_-(\gamma_0(\bar{T})),$$

so $\dot{\gamma}_0(\bar{T}) = (0, H(y_0(\bar{T}))) = (0, k_-(\gamma_0(\bar{T}))) = f_-(\gamma_0(\bar{T}))$. Thus condition (H3) means that $\dot{\gamma}_0(\bar{T})$ is transverse to $\mathscr{K}$ in Ω_0. Next, from (H3) it follows immediately that

$$\nabla_y h_-(0, y_0(\bar{T}))\dot{y}_0(\bar{T}) < 0.$$

Note that $\nabla_y h_-(0, y_0(t))\dot{y}_0(t) = h'_-(\gamma_0(t))\dot{\gamma}_0(t)$ for $t \in [-\bar{T}, \bar{T}]$. Finally, for the validity of the results of this section, it is enough that condition (H1) holds for y in a neigbourhood of $y_0(t)$, $-\bar{T} \le t \le \bar{T}$.

We set:

$$\gamma(t) = \begin{cases} \gamma_-(t), & \text{if } t \le -\bar{T}, \\ \gamma_0(t), & \text{if } -\bar{T} \le t \le \bar{T}, \\ \gamma_+(t), & \text{if } t \ge \bar{T} \end{cases}$$

and will refer to $\gamma(t)$ as the *sliding homoclinic* solution of (6.2.1) when $\varepsilon = 0$ (Figure 6.3).

We note that $\gamma(t)$ is C^1-smooth also at $t = \bar{T}$. In fact from $h_-(0, y_0(\bar{T})) = h_-(\gamma(\bar{T})) = 0$ we obtain $V(y_0(\bar{T})) = -1$ and then:

$$\dot{\gamma}_+(\bar{T}) = f_-(\gamma(\bar{T})) = \begin{pmatrix} h_-(\gamma(\bar{T})) \\ k_-(\gamma(\bar{T})) \end{pmatrix} = \begin{pmatrix} 0 \\ k_-(\gamma(\bar{T})) \end{pmatrix} = \begin{pmatrix} 0 \\ H(y_0(\bar{T})) \end{pmatrix} = \dot{\gamma}_0(\bar{T}).$$

Recalling $x = (z, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we set

$$f_\pm(x) + \varepsilon g(t, x, \varepsilon) = (h_\pm(t, z, y, \varepsilon), k_\pm(t, z, y, \varepsilon)).$$

and

$$H(t, y, \varepsilon) := \frac{h_-(t, 0, y, \varepsilon)k_+(t, 0, y, \varepsilon) - h_+(t, 0, y, \varepsilon)k_-(t, 0, y, \varepsilon)}{h_-(t, 0, y, \varepsilon) - h_+(t, 0, y, \varepsilon)}\,.$$

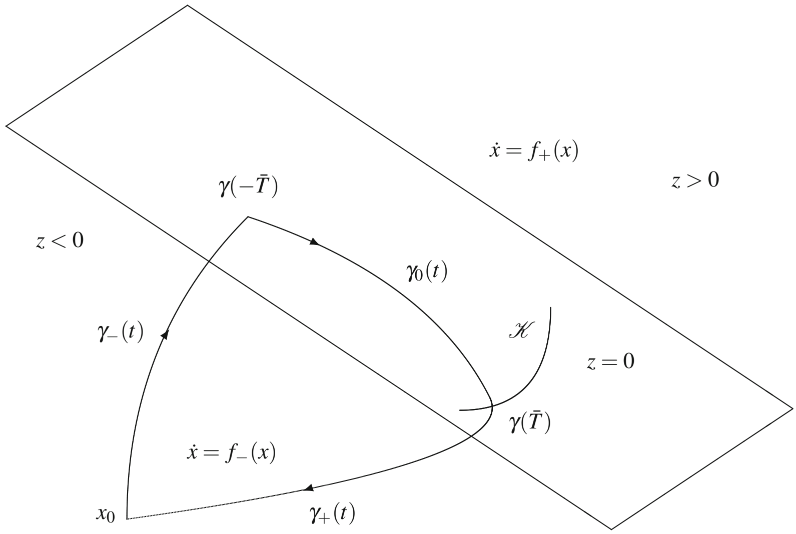


Fig. 6.3 A homoclinic sliding orbit $\gamma(t)$ of (6.2.1) with $\varepsilon = 0$ to the hyperbolic equilibrium $x = x_0$.

Note that $h_-(t,0,y,\varepsilon) - h_+(t,0,y,\varepsilon) = h_-(0,y) - h_+(0,y) > 0$ for any $y \in \Omega_0$ by (6.2.2). So $H(t,y,\varepsilon)$ is well defined. We are interested in the chaotic dynamics of (6.2.1) near $\gamma(t)$ for $\varepsilon \neq 0$ small.

Definition 6.2.3. By a *sliding* solution $x(t)$ of (6.2.1) we mean a function $x : \mathbb{R} \to \mathbb{R}^n$ for which the following hold:

There exists an increasing sequence $\{\widetilde{T}_m\}$ (possibly finite or with $m \leq m_0 \in \mathbb{Z}$, or $m \geq m_0 \in \mathbb{Z}$, with $m_0 \in \mathbb{Z}$, or $m \in \mathbb{Z}$) so that $x(t)$ is C^1-smooth for any $t \in \mathbb{R} \setminus \{\widetilde{T}_{2m}\}$ and possesses right and left derivatives at $t = \widetilde{T}_{2m}$. If $t \in (\widetilde{T}_{2m-1}, \widetilde{T}_{2m})$ then $x(t) \in \Omega_-$ and satisfies the equation $\dot{x} = f_-(x) + \varepsilon g(t,x,\varepsilon)$. If $t \in (\widetilde{T}_{2m}, \widetilde{T}_{2m+1})$ then $x(t) = (0, y(t)) \in \Omega_0$ and $y(t)$ satisfies the equation $\dot{y} = H(t,y,\varepsilon)$. At $t = \widetilde{T}_{2m+1}$ the equation $h_-(\widetilde{T}_{2m+1}, 0, y(\widetilde{T}_{2m+1}), \varepsilon) = 0$ is satisfied.

Since x_0 is a hyperbolic fixed point of $\dot{x} = f_-(x)$, the linear system $\dot{x} = f'_-(\gamma_+(t))x$ has an exponential dichotomy on $[\bar{T}, \infty)$ with projection P_+, and denotes by $X_+(t)$ its fundamental matrix with $X_+(\bar{T}) = \mathbb{I}$. Similarly the equation $\dot{x} = f'_-(\gamma_-(t))x$ has an exponential dichotomy on $(-\infty, -\bar{T}]$ with projection P_-, and denotes by $X_-(t)$ its fundamental matrix with $X_-(-\bar{T}) = \mathbb{I}$. Let

$$\mathscr{S}' := \mathscr{N}P_- \bigcap P_y(\mathbb{R}^n) = \{y \in \mathbb{R}^{n-1} \mid (0,y) \in \mathscr{N}P_-\} \subset \mathbb{R}^{n-1}.$$

Note that $\dim \mathscr{S}' = \dim \mathscr{N}P_- - 1$. Next we define projections Q and R as follows:

$Q : \mathbb{R}^n \to \mathbb{R}^n$ is the projection on $\mathbb{R}^n$ with $\mathscr{R}Q = \{0\} \times \mathbb{R}^{n-1}$ and $\mathscr{N}Q = \operatorname{span}\{\dot{\gamma}_-(-\bar{T})\}$,

$R : \mathscr{R}P_y \to \mathscr{R}P_y$ is the projection on $\mathscr{R}P_y$ so that $\mathscr{R}R = \mathscr{N}\nabla_y h_-(0, y_0(\bar{T}))$ and $\mathscr{N}R = \operatorname{span}\{\dot{y}_0(\bar{T})\}$.

Let $Y_0(t)$ be the fundamental solution of $\dot{y} = H'(y_0(t))y$, with $Y_0(-\bar{T}) = \mathbb{I}$. Since $\dim \mathscr{S}' = \dim \mathscr{N}P_- - 1$, it is obvious that $\dim \begin{pmatrix} 0 \\ RY_0(\bar{T})\mathscr{S}' \end{pmatrix} \le \dim \mathscr{N}P_- - 1$. Then

$$\begin{aligned} 0 &\le \dim \left[\begin{pmatrix} 0 \\ RY_0(\bar{T})\mathscr{S}' \end{pmatrix} \cap \mathscr{R}P_+ \right] \\ &= \dim \begin{pmatrix} 0 \\ RY_0(\bar{T})\mathscr{S}' \end{pmatrix} + \dim \mathscr{R}P_+ - \dim \left[\begin{pmatrix} 0 \\ RY_0(\bar{T})\mathscr{S}' \end{pmatrix} + \mathscr{R}P_+ \right] \\ &\le \dim \mathscr{N}P_- - 1 + \dim \mathscr{R}P_+ - \dim \left[\begin{pmatrix} 0 \\ RY_0(\bar{T})\mathscr{S}' \end{pmatrix} + \mathscr{R}P_+ \right] \\ &= n - 1 - \dim \left[\begin{pmatrix} 0 \\ RY_0(\bar{T})\mathscr{S}' \end{pmatrix} + \mathscr{R}P_+ \right] \end{aligned} \tag{6.2.4}$$

since $\dim \mathscr{R}P_+ + \dim \mathscr{N}P_- = n$. As a consequence,

$$\dim \left[\begin{pmatrix} 0 \\ RY_0(\bar{T})\mathscr{S}' \end{pmatrix} + \mathscr{R}P_+ \right] \le n - 1.$$

Our next assumption is as follows:

(H4) $\begin{pmatrix} 0 \\ RY_0(\bar{T})\mathscr{S}' \end{pmatrix} + \mathscr{R}P_+$ has codimension 1 in $\mathbb{R}^n$.

It follows from (H4) that a unitary vector $\psi \in \mathbb{R}^n$ exists so that

$$\{\psi\}^\perp = \begin{pmatrix} 0 \\ RY_0(\bar{T})(\mathscr{S}') \end{pmatrix} + \mathscr{R}P_+.$$

Using this vector we define the function

$$\psi(t) = \begin{cases} X_-^{-1}(t)^* P_-^* Q^* P_y^* Y_0(\bar{T})^* R^* P_y \psi, & \text{for} t \le -\bar{T}, \\ P_y^* Y_0^{-1}(t)^* Y_0(\bar{T})^* P_y \psi, & \\ -\dfrac{k_+(0, y_0(t)) + k_-(0, y_0(t))}{h_+(0, y_0(t)) - h_-(0, y_0(t))} P_z^* Y_0^{-1}(t)^* Y_0(\bar{T})^* P_y \psi, & \text{for } -\bar{T} < t \le \bar{T}, \\ X_+^{-1}(t)^* (\mathbb{I} - P_+^*) \psi, & \text{for } t > \bar{T}. \end{cases}$$

Set

$$\mathscr{M}(\alpha) := \int_{-\infty}^{\infty} \psi^*(t) g(t + \alpha, \gamma(t), 0) dt.$$

Remark 6.2.4. (i) Since $\dot{y}_0(\bar{T}) = Y_0(\bar{T})\dot{y}_0(-\bar{T})$, we get $RY_0(\bar{T})\dot{y}_0(-\bar{T}) = 0$. But (H4) and (6.2.4) imply that $\dim RY_0(\bar{T})\mathscr{S}' = \dim \mathscr{N}P_- - 1 = \dim Y_0(\bar{T})\mathscr{S}' = \dim \mathscr{S}'$. Then $RY_0(\bar{T}) : \mathscr{S}' \to RY_0(\bar{T})\mathscr{S}'$ is an isomorphism and hence $\dot{y}_0(-\bar{T}) \notin \mathscr{S}'$. This means that $\dot{\gamma}_0(-\bar{T})$ transversally crosses the unstable manifold W_0^u of $\dot{x} = f_-(x)$ at $\gamma_0(-\bar{T})$. Consequently recalling also (6.2.4), assumption (H4) is a kind of nondegeneracy and transversality condition as well.
(ii) If $\dim \mathscr{N}P_- = n-1$ and $\dot{y}_0(-\bar{T}) \notin \mathscr{S}'$, then $\mathscr{R}P_+ = \mathrm{span}\{\dot{\gamma}_+(\bar{T})\} = \mathrm{span}\{\dot{\gamma}_0(\bar{T})\}$ and $RY_0(\bar{T}) : \mathscr{S}' \to RY_0(\bar{T})\mathscr{S}'$ is 1 : 1. As a consequence, $\begin{pmatrix} 0 \\ RY_0(\bar{T})\mathscr{S}' \end{pmatrix} \cap \mathscr{R}P_+ = \{0\}$ and all the inequalities in (6.2.4) are equalities. Consequently, if $\dim \mathscr{N}P_- = n-1$ then $\dot{y}_0(-\bar{T}) \notin \mathscr{S}'$ if and only if (H4) holds. Moreover, we get $\psi = e_1 = (1,0,...,0)$ and $P_y\psi = 0$. Hence

$$\psi(t) = \begin{cases} 0, & \text{for } t \le \bar{T}, \\ X_+^{-1}(t)^*(\mathbb{I} - P_+^*)\psi, & \text{for } t > \bar{T} \end{cases} \tag{6.2.5}$$

and

$$\mathscr{M}(\alpha) = \int_{\bar{T}}^{\infty} \psi^*(t) g(t+\alpha, \gamma(t), 0) dt. \tag{6.2.6}$$

Formula (6.2.6) corresponds to formula [27, (2.45)] for the planar case, that is, the Melnikov function contains only the $\gamma_+(t)$ part of $\gamma(t)$.

We recall that $g(t,x,\varepsilon)$ is quasiperiodic in t, if hypothesis (H5) of Section 6.1.8 holds. Now we can directly follow the method of Section 6.1 so we omit details and we refer the readers to [55]. Here we state the following result:

Theorem 6.2.5. *Assume that* (*H*1)–(*H*4) *and* (*H*5) *of Section 6.1.8 hold. If* $\mathscr{M}$ *has a simple zero* α_0, *i.e.* $\mathscr{M}(\alpha_0) = 0$ *and* $\mathscr{M}'(\alpha_0) \neq 0$, *then for any* $\varepsilon \neq 0$ *sufficiently small, there are sequences* $\{T_k^\varepsilon\}_{k\in\mathbb{Z}} \subset \mathbb{R}$, $\{\mathscr{S}_k^\varepsilon\}_{k\in\mathbb{Z}}$, $\{\Phi_k^\varepsilon\}_{k\in\mathbb{Z}}$ *so that*

(a) $\inf_{k\in\mathbb{Z}}(T_{k+1}^\varepsilon - T_k^\varepsilon) \to \infty$ *as* $\varepsilon \to 0$,
(b) $\mathscr{S}_k^\varepsilon \subset \mathbb{R}^n$ *are compact,*
(c) $\Phi_k^\varepsilon : \mathscr{E} \mapsto \mathscr{S}_k^\varepsilon$ *are homeomorphisms,*
(d) *Let* $F_k^\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ *be defined so that* $F_k^\varepsilon(\xi)$ *is the value at time* $T_{2(k+1)}^\varepsilon$ *of the solution of Eq.* (6.2.1) *so that* $z(T_{2k}^\varepsilon) = \xi$. *Then the following diagrams commute:*

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{\quad\sigma\quad} & \mathscr{E} \\ \Phi_k^\varepsilon \big\downarrow & & \big\downarrow \Phi_{k+1}^\varepsilon \\ \mathscr{S}_k^\varepsilon & \xrightarrow[\quad F_k^\varepsilon \quad]{} & \mathscr{S}_{k+1}^\varepsilon \end{array}$$

for all $k \in \mathbb{Z}$. *If, in addition,* $g(t,z,\varepsilon)$ *is p-periodic in t then* $F^\varepsilon = \varphi_\varepsilon^{r_\varepsilon} = \varphi_\varepsilon \circ \ldots \circ \varphi_\varepsilon = F_k^\varepsilon$ *(*r_ε *times) is the* r_ε*th iterate of the p-period map* φ_ε *of* (6.2.1) *for some large* $r_\varepsilon \in \mathbb{N}$, $\mathscr{S}^\varepsilon = \mathscr{S}_k^\varepsilon$ *and* $\Phi^\varepsilon = \Phi_k^\varepsilon$, *that is, in the periodic case the above diagram is independent of k.*

Here we recall Remark 6.1.24. Finally, Theorem 6.2.5 generalizes results of [43, 44, 46, 47] to the DDE (6.2.1) (cf Section 4.1).

6.2.2 Planar Sliding Homoclinics

First, we apply our theory to the planar discontinuous system

$$\begin{aligned} \dot{z} &= f_+(z) + \varepsilon g(z,t,\varepsilon) \quad \text{for} \quad y > 1\,, \\ \dot{z} &= f_-(z) + \varepsilon g(z,t,\varepsilon) \quad \text{for} \quad y < 1 \end{aligned} \tag{6.2.7}$$

where $z = (x,y) \in \mathbb{R}^2$, $f_\pm$, g are C^3-smooth and g is 1-periodic in t. Here we set

$$q_\pm(z,t,\varepsilon) = f_\pm(z) + \varepsilon g(z,t,\varepsilon)\,.$$

On $y = 1$ (cf (6.2.3)), we consider the system

$$\begin{aligned} \dot{x} = & \frac{q_{+2}(x,1,t,\varepsilon)}{q_{+2}(x,1,t,\varepsilon) - q_{-2}(x,1,t,\varepsilon)} q_{+1}(x,1,t,\varepsilon) \\ & + \frac{q_{-2}(x,1,t,\varepsilon)}{q_{-2}(x,1,t,\varepsilon) - q_{+2}(x,1,t,\varepsilon)} q_{-1}(x,1,t,\varepsilon)\,, \end{aligned}$$

where $q_\pm = (q_{\pm 1}, q_{\pm 2})$. We suppose the following conditions hold:

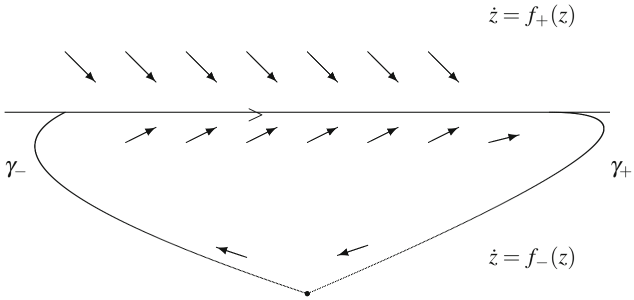


Fig. 6.4 A planar homoclinic sliding on the line of discontinuity.

(i) $f_-(0) = 0$ and $Df_-(0)$ has no eigenvalues on the imaginary axis.
(ii) There are two solutions $\gamma_-(s)$, $\gamma_+(s)$ of $\dot{z} = f_-(z)$, $y \leq 1$ defined on $\mathbb{R}_- = (-\infty, 0]$, $\mathbb{R}_+ = [0, +\infty)$, respectively, so that $\lim_{s\to\pm\infty} \gamma_\pm(s) = 0$ and $\gamma_\pm(s) = (x_\pm(s), y_\pm(s))$ with $y_\pm(0) = 1$, $x_-(0) < x_+(0)$. Moreover, $f_\pm(z) = (f_{\pm 1}(z), f_{\pm 2}(z))$ with $f_{\pm 1}(x,1) > 0$, $f_{+2}(x,1) < 0$ for $x_-(0) \leq x \leq x_+(0)$. Furthermore, $f_{-2}(x,1) > 0$ for $x_-(0) \leq x < x_+(0)$, $f_{-2}(x_+(0),1) = 0$ and $\partial_x f_{-2}(x_+(0),1) < 0$.

Assumptions (i) and (ii) mean that (6.2.7) for $\varepsilon = 0$ has a sliding homoclinic solution γ, created by $\gamma_\pm$, to a hyperbolic equilibrium 0 (Figure 6.4). Now we have a case of Remark 6.2.4-(ii), so we can use the formulas (6.2.5)–(6.2.6) to derive:

$$\mathscr{M}(\alpha) = \int_0^{+\infty} \psi(s)^* g(\gamma_+(s), \alpha + s, 0)\, ds \tag{6.2.8}$$

where $\psi(t)$ is a basis of a space of bounded solutions on $\mathbb{R}_+$ of the adjoint variational system $\dot{w} = -Df_-^*(\gamma_+(s))w$. By Theorem 6.2.5, we arrive at the following result.

Theorem 6.2.6. *If there is a simple root of $\mathscr{M}$ given by* (6.2.8)*, then* (6.2.7) *is chaotic with $\varepsilon \neq 0$ small.*

As a concrete example we consider

$$\begin{aligned} &\dot{y} = z, \quad \dot{z} = y - \frac{1}{2}y^3 + yz, && \text{for } z < \mathrm{e}^{-\frac{4\sqrt{3}\pi}{9}}, \\ &\dot{y} = z, \quad \dot{z} = y - \frac{1}{2}y^3 + (y-q)z && \text{for } z > \mathrm{e}^{-\frac{4\sqrt{3}\pi}{9}} \end{aligned} \tag{6.2.9}$$

that have a sliding homoclinic orbit to a saddle $(0,0)$ for any $q \geq 6.947$. Indeed, we start from (6.2.12) with $\beta = 1/2$. Note the phase portrait of (6.2.9) looks like Figure 6.4. Then we get $\tau = \sqrt{3}/2$ (cf (6.2.16)), $\Omega_\tau = \mathrm{e}^{-\frac{4\sqrt{3}\pi}{9}}$ (cf (6.2.18)) and $y_+(\bar{T}) = \sqrt{2 + 2\mathrm{e}^{-\frac{4\sqrt{3}\pi}{9}}}$ (cf (6.2.19)). The segment

$$\left\{ \left(y, \mathrm{e}^{-\frac{4\sqrt{3}\pi}{9}} \right) \in \mathbb{R}^2 \mid 0 \leq y \leq y_+(\bar{T}) \right\}$$

is attractive from above for (6.2.9), if

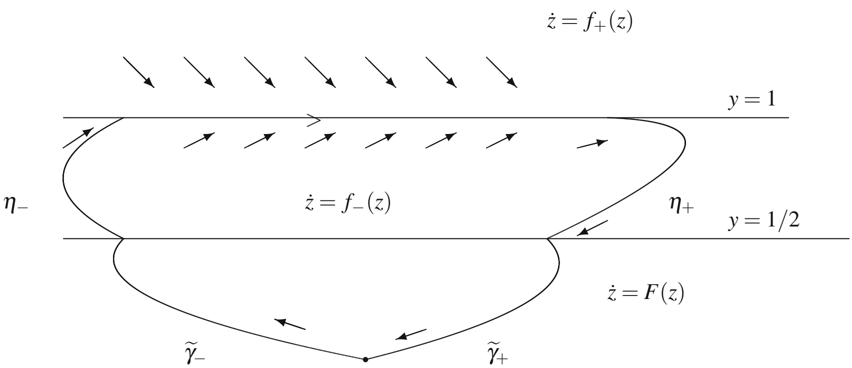


Fig. 6.5 A planar homoclinic sliding on the line of discontinuity with transversal crossing of another discontinuity line.

$$q > \max_{y\in[0,y_+(\bar{T})]} \frac{1}{\Omega_\tau}\left(y - \frac{1}{2}y^3 + \Omega_\tau y\right) = \frac{2\sqrt{6}}{9\Omega_\tau}(1+\Omega_\tau)^{3/2} \doteq 6.94609\,.$$

Hence we could take $q \geq 6.947$. Next we may also add its periodic perturbation

$$\begin{aligned} &\dot{y} = z, \quad \dot{z} = y - \frac{1}{2}y^3 + yz + \varepsilon\cos\omega t, && \text{for } z < \mathrm{e}^{-\frac{4\sqrt{3}\pi}{9}}\,, \\ &\dot{y} = z, \quad \dot{z} = y - \frac{1}{2}y^3 + (y-q)z + \varepsilon\cos\omega t && \text{for } z > \mathrm{e}^{-\frac{4\sqrt{3}\pi}{9}}\,. \end{aligned} \tag{6.2.10}$$

Then the Melnikov function is the same as in Section 6.2.3, and we could apply Theorem 6.2.8 with $F(1/2) \doteq 0.00228$ and $D(1/2) \doteq 25.3974$. Consequently, if either $0 < \omega < 0.0022$ or $\omega > 25.3975$ then (6.2.10) is chaotic.

The above approach to (6.2.7) can be generalized [28, 48] to cases when homoclinic orbit $\gamma(s)$ transversally crosses another curves of discontinuity. For simplicity, we suppose that such a discontinuity in (6.2.7) occurs at the level $y = 1/2$, i.e. we deal with the system

$$\begin{aligned} &\dot{z} = f_+(z) + \varepsilon g(z,t,\varepsilon), && \text{for } y > 1\,, \\ &\dot{z} = f_-(z) + \varepsilon g(z,t,\varepsilon), && \text{for } 1/2 < y < 1\,, \\ &\dot{z} = F(z) + \varepsilon g(z,t,\varepsilon), && \text{for } y < 1/2 \end{aligned} \tag{6.2.11}$$

where $z = (x,y) \in \mathbb{R}^2$, $f_\pm$, F, g are C^3-smooth and g is 1-periodic in t. We suppose the following conditions hold:

(a) $F(0) = 0$ and $DF(0)$ has no eigenvalues on the imaginary axis.

(b) There are two solutions η_-, η_+ of $\dot{z} = f_-(z)$, $1/2 \leq y \leq 1$ defined on $[a_-,0]$, $[0,a_+]$, $a_- < 0 < a_+$, respectively, so that $\eta_\pm(s) = (\widetilde{x}_\pm(s), \widetilde{y}_\pm(s))$ with $\widetilde{y}_\pm(0) = 1$, $\widetilde{y}_\pm(a_\pm) = 1/2$, $\widetilde{x}_-(0) < \widetilde{x}_+(0)$, $\widetilde{x}_-(a_-) < \widetilde{x}_+(a_+)$. Moreover, $f_\pm(z) = (f_{\pm1}(z), f_{\pm2}(z))$ with $f_{\pm1}(x,1) > 0$, $f_{+2}(x,1) < 0$ for $\widetilde{x}_-(0) \leq x \leq \widetilde{x}_+(0)$. Furthermore, $f_{-2}(x,1) > 0$ for $\widetilde{x}_-(0) \leq x < \widetilde{x}_+(0)$, $f_{-2}(\widetilde{x}_+(0),1) = 0$ and $\partial_x f_{-2}(\widetilde{x}_+(0),1) < 0$. Finally, we suppose that $f_{-2}(\eta_-(a_-)) > 0$ and $f_{-2}(\eta_+(a_+)) < 0$.

(c) There are two solutions $\widetilde{\gamma}_-(s)$, $\widetilde{\gamma}_+(s)$ of $\dot{z} = F(z)$, $y \leq 1/2$ defined on $\mathbb{R}_- = (-\infty,0]$, $\mathbb{R}_+ = [0,+\infty)$, respectively, so that $\lim_{s\to\pm\infty} \widetilde{\gamma}_\pm(s) = 0$ and $\widetilde{\gamma}_\pm(0) = \eta_\pm(a_\pm)$. Moreover, $F(z) = (F_1(z), F_2(z))$ with $F_2(\widetilde{\gamma}_-(0)) > 0$ and $F_2(\widetilde{\gamma}_+(0)) < 0$.

Again, assumptions (a), (b) and (c) imply that (6.2.11) for $\varepsilon = 0$ has a sliding homoclinic solution $\widetilde{\gamma}$, created by $\eta_\pm$ and $\widetilde{\gamma}_\pm$, to a hyperbolic equilibrium 0 (Figure 6.5). We do not make further computations for (6.2.11), instead, we refer to [28, 48] for more details.

6.2.3 Three-Dimensional Sliding Homoclinics

This section is devoted to a construction of a concrete example (cf (6.2.20), (6.2.21), (6.2.23)) of (6.2.1) to which the above theory is applied. Then we proceed with a more particular perturbation (cf Theorems 6.2.8, 6.2.9). In order to construct our example, we start from [56]

$$\begin{aligned}\dot{z} &= y - \beta y^3 + yz,\\ \dot{y} &= z\end{aligned} \tag{6.2.12}$$

for $\beta > 1/8$. Then $(0,0)$ is hyperbolic and $(1/\sqrt{\beta},0)$ is an unstable focus. Since $(0,0)$ is hyperbolic it has one-dimensional stable and unstable manifolds. In the following we first show that these two manifolds have the structure depicted in Figure 6.6 where the stable manifold is tangent (and the unstable manifold is transverse) to the horizontal straight line. Performing the transformation $u = 1 - \beta y^2$, $y > 0$, $v = z$ we get

$$\begin{aligned}\dot{u} &= -2\beta v \frac{\sqrt{1-u}}{\sqrt{\beta}}, \quad z < 1,\\ \dot{v} &= (u+v)\frac{\sqrt{1-u}}{\sqrt{\beta}}.\end{aligned} \tag{6.2.13}$$

Note that $(0,0)$ corresponds to $(1,0)$ and $(1/\sqrt{\beta},0)$ to $(0,0)$. Let $' = \frac{d}{d\theta}$ and consider the linear system

$$\begin{aligned}u' &= -2\beta v,\\ v' &= u + v,\\ u(0) &= 1, \quad v(0) = 0\end{aligned} \tag{6.2.14}$$

whose solution has the form

$$\begin{aligned}u_\tau(\theta) &= \mathrm{e}^{\theta/2}\cos(\tau\theta) - \frac{1}{2\tau}\mathrm{e}^{\theta/2}\sin(\tau\theta),\\ v_\tau(\theta) &= \frac{1}{\tau}\mathrm{e}^{\theta/2}\sin(\tau\theta)\end{aligned} \tag{6.2.15}$$

with

$$\tau = \frac{\sqrt{8\beta - 1}}{2}, \tag{6.2.16}$$

and so $\beta = \frac{4\tau^2+1}{8}$. Note that

$$u_\tau'(\theta) = -2\beta v_\tau(\theta) = -2\beta\frac{1}{\tau}\mathrm{e}^{\theta/2}\sin(\tau\theta)$$

has the opposite sign to $\sin(\tau\theta)$ thus $u_\tau(\theta) \le u_\tau(0) = 1$ for any $\theta \in \left(-\frac{\pi}{\tau}, \frac{\pi}{\tau}\right)$. On the other hand, if $\tau\theta \le -\pi$ we have

$$u_\tau(\theta) \le \mathrm{e}^{-\frac{\pi}{2\tau}}\frac{\sqrt{4\tau^2+1}}{2\tau} = \mathrm{e}^{-\frac{\pi}{2\tau}}\sqrt{1 + \left(\frac{1}{2\tau}\right)^2} < 1$$

since $e^{\pi s} > \sqrt{1+s^2}$ for any $s > 0$. As a consequence, $(u_\tau(\theta), v_\tau(\theta))$ is tangent to the line $u = 1$ from the left at $\theta = 0$ for $\theta \in (-\infty, \theta_\tau^+)$. Here $\theta_\tau^+ > 0$ is the least positive value so that $u_\tau(\theta) = 1$. Next, let θ_τ^- be the greatest negative value for which $v_\tau'(\theta) = 0$ and $v_\tau(\theta) > 0$. Then θ_τ^- solves the following system:

$$\cos(\tau\theta_\tau^-) + \frac{1}{2\tau}\sin(\tau\theta_\tau^-) = 0, \quad \sin(\tau\theta_\tau^-) > 0$$

so

$$\tau\theta_\tau^- = -\arctan 2\tau - \pi .$$

Given $\bar{T} > 0$ (we will fix it later) we consider the solution $\theta^-(t)$ of the equation:

$$\dot{\theta} = \sqrt{\frac{1 - u_\tau(\theta)}{\beta}}, \quad \theta(\bar{T}) = \theta_\tau^- . \tag{6.2.17}$$

Separating variables we see that

$$\int_{\theta_\tau^-}^{\theta^-(t)} \frac{d\theta}{\sqrt{1 - u_\tau(\theta)}} = \frac{t - \bar{T}}{\sqrt{\beta}}$$

or

$$\theta^-(t) = \Theta_-^{-1}\left(\frac{t - \bar{T}}{\sqrt{\beta}}\right), \quad \Theta_-(\theta) = \int_{\theta_\tau^-}^{\theta} \frac{d\theta}{\sqrt{1 - u_\tau(\theta)}} .$$

From (6.2.14) we easily see that

$$1 - u_\tau(\theta) = \beta\theta^2 + o(\theta^2)$$

as $\theta \to 0$. As a consequence, $\Theta_-(\theta)$ is an increasing function that tends to $+\infty$ as $\theta \to 0$. Thus $\theta^-(t)$ is increasing and tends to 0 as $t \to \infty$. Moreover, since $u_\tau(\theta) < 1$ for $\theta < 0$ we also see that $\theta(t) \to -\infty$ as $t \to -\infty$. Summarizing $\theta^-(t)$ is an increasing function defined on $(-\infty, \infty)$, taking values on $(-\infty, 0)$, $\theta^-(\bar{T}) = \theta_\tau^-$. Setting

$$y_+(t) = \sqrt{\frac{1 - u_\tau(\theta^-(t))}{\beta}}, \quad z_+(t) = v_\tau(\theta^-(t))$$

we see that $(z_+(t), y_+(t))$ is a solution of Eq. (6.2.12) so that

$$\lim_{t\to\infty}(z_+(t), y_+(t)) = \lim_{\theta\to 0}\left(v_\tau(\theta), \sqrt{\frac{1 - u_\tau(\theta)}{\beta}}\right) = (0,0),$$

$$\lim_{t\to-\infty}(z_+(t), y_+(t)) = \lim_{\theta\to-\infty}\left(v_\tau(\theta), \sqrt{\frac{1 - u_\tau(\theta)}{\beta}}\right) = \left(0, \sqrt{\frac{1}{\beta}}\right),$$

that is, $(z_+(t), y_+(t))$ is a heteroclinic connection from $\left(0, \sqrt{\frac{1}{\beta}}\right)$ to $(0,0)$. Next, we know that θ_τ^- is the greatest negative value so that $v'(\theta) = 0$ and $v(\theta) > 0$. This means that at $t = \bar{T}$ we have

$$z_+(\bar{T}) = v_\tau(\theta_\tau^-) := \Omega_\tau > 0, \quad \dot{z}_+(\bar{T}) = 0$$

and these two conditions are not satisfied when $t > \bar{T}$. Note that:

$$\Omega_\tau = \frac{1}{\tau} e^{\theta_\tau^-/2} \sin(\tau\theta_\tau^-) = 2 e^{\theta_\tau^-/2} \sqrt{\frac{1}{1+4\tau^2}} = e^{\theta_\tau^-/2} \sqrt{\frac{1}{2\beta}}, \tag{6.2.18}$$

moreover:

$$y_+(\bar{T}) = \sqrt{\frac{1 - u_\tau(\theta_\tau^-)}{\beta}} = \sqrt{\frac{1 + v_\tau(\theta_\tau^-)}{\beta}} = \sqrt{\frac{1 + \Omega_\tau}{\beta}}. \tag{6.2.19}$$

Now we consider the solution $(z_-(t), y_-(t))$ of Eq. (6.2.12) that belongs to the unstable manifold of the saddle $(0,0)$. Since $(z_-(t), y_-(t)) \to (0,0)$ as $t \to -\infty$ it follows that we have to look for a solution $(u(t), v(t))$ of (6.2.13) so that $(u(t), v(t)) \to (1,0)$ as $t \to -\infty$. Thus we consider again Eq. (6.2.14) with $\theta \in (0, \theta_\tau^+)$. Thus $\theta = \theta^+(t)$ is again a solution of

$$\dot{\theta} = \sqrt{\frac{1 - u_\tau(\theta)}{\beta}}$$

with the initial condition $\theta(0) = \theta_\tau^+$. So we obtain:

$$\int_{\theta_\tau^+}^{\theta^+(t)} \frac{d\theta}{\sqrt{1 - u_\tau(\theta)}} = \frac{t}{\sqrt{\beta}}$$

that is

$$\theta^+(t) = \Theta_+^{-1}\left(\frac{t}{\sqrt{\beta}}\right), \quad \Theta_+(\theta) = \int_{\theta_\tau^+}^{\theta} \frac{d\theta}{\sqrt{1 - u_\tau(\theta)}}.$$

Obviously $\Theta_+(\theta)$ is an increasing function and since $\theta \in (0, \theta_\tau^+)$, $\Theta_+(\theta) < 0$ for $0 \le \theta < \theta_\tau^+$. Arguing as before we see that $\lim_{\theta \to 0} \Theta_+(\theta) = -\infty$ and hence $\lim_{t \to -\infty} \theta^+(t) = 0$. For $t \in (-\infty, 0]$ (and hence $\theta^+(t) \in (0, \theta_\tau^+]$) we set:

$$y_-(t) = \sqrt{\frac{1 - u_\tau(\theta^+(t))}{\beta}}, \quad z_-(t) = v_\tau(\theta^+(t))$$

and note that the following hold:

$$y_-(0) = 0, \quad z_-(0) = v_\tau(\theta_\tau^+),$$

$$\lim_{t\to-\infty}(z_-(t),y_-(t)) = \lim_{\theta\to 0}\left(v_\tau(\theta),\sqrt{\frac{1-u_\tau(\theta)}{\beta}}\right) = (0,0)\,.$$

Now, since $u_\tau(\theta) < 1$ for $\theta \in (0,\theta_\tau^+)$, we see that $u'_\tau(\theta_\tau^+) \geq 0$ and then $z_-(0) = v_\tau(\theta_\tau^+) \leq 0$. But it must be $z_-(0) = v_\tau(\theta_\tau^+) < 0$ otherwise $(z_-(0),y_-(0)) = (0,0)$ because of uniqueness. Next $(z_-(t),y_-(t))$ belongs to the unstable manifold of the equilibrium $(0,0)$ and $y_-(t) > 0$ for any $t \in (-\infty,0)$, thus

$$\frac{(\dot{z}_-(t),\dot{y}_-(t))}{\sqrt{\dot{z}_-(t)^2+\dot{y}_-(t)^2}} \to v_-$$

as $t \to -\infty$, with v_- being the eigenvector of the positive eigenvalue of the linearization of Equation (6.2.12) at $(0,0)$, i.e.

$$\begin{aligned}\dot{z} &= y\,,\\ \dot{y} &= z\end{aligned}$$

having a positive second component. Hence $z_-(t) = \dot{y}_-(t)$ is eventually positive for $t \to -\infty$. Thus the curve $(z_-(t),y_-(t))$ has to pass from the first quadrant to the fourth one and this can be realized only by passing above the line $z = z_+(\bar{T})$ because otherwise it would intersect the curve $(z_+(t),y_+(t))$. As a consequence, $t_0 < 0$ must exist so that $z_-(t_0) = z_+(\bar{T}) = \Omega_\tau$ and $z_-(t) < \Omega_\tau$ for any $t < t_0$. We set

$$\bar{y}_\tau := \sqrt{\frac{1-u_\tau(\theta_\tau^+(t_0))}{\beta}}\,.$$

Shifting time we can suppose without loss of generality that $t_0 = -\bar{T}$. Thus we have found solutions $\tilde{\gamma}_\pm(t) = (z_\pm(t),y_\pm(t))$ of (6.2.12) so that

$$\begin{aligned}&\tilde{\gamma}_-(t) \to (0,0), && \text{as } t \to -\infty,\\ &\tilde{\gamma}_+(t) \to (0,0), && \text{as } t \to +\infty,\\ &z_-(-\bar{T}) = z_+(\bar{T}) = \Omega_\tau\ .\end{aligned}$$

The graphs of the above-mentioned invariant manifolds of (6.2.12) and the line $z = \Omega_\tau$ in the right half-plane for $\beta = 37/8$, i.e. $\tau = 3$, are given in Figure 6.6.

We are now able to construct our example. We take

$$\begin{aligned}\dot{z} &= y_1 - \beta y_1^3 + z y_1 + y_2^2,\\ \dot{y}_1 &= z, \qquad\qquad (6.2.20)\\ \dot{y}_2 &= y_2(1+z)\end{aligned}$$

for $z < \Omega_\tau$ and

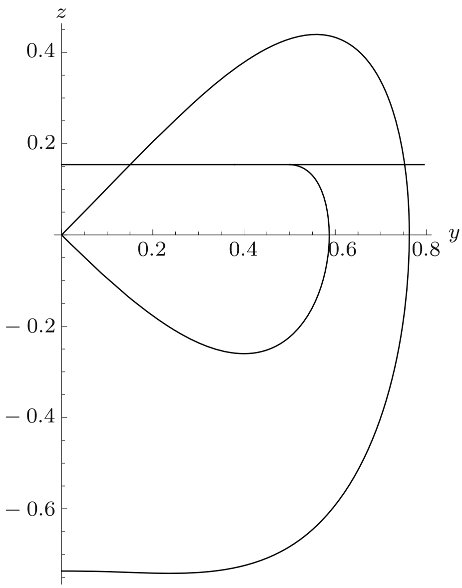


Fig. 6.6 The stable and unstable manifolds of system (6.2.12).

$$\begin{aligned}\dot{z} &= -z,\\ \dot{y}_1 &= 0,\\ \dot{y}_2 &= 0\end{aligned} \tag{6.2.21}$$

when $z > \Omega_\tau$, that is, we take:

$$\begin{aligned} f_+(z,y_1,y_2) &= \begin{pmatrix} -z \\ 0 \\ 0 \end{pmatrix} \quad \text{for } z > \Omega_\tau,\\ f_-(z,y_1,y_2) &= \begin{pmatrix} y_1 - \beta y_1^3 + zy_1 + y_2^2 \\ z \\ y_2(1+z) \end{pmatrix} \quad \text{for } z < \Omega_\tau\,. \end{aligned} \tag{6.2.22}$$

Then

$$h_-(\Omega_\tau,y_1,y_2) = y_1 - \beta y_1^3 + \Omega_\tau y_1 + y_2^2, \quad h_+(\Omega_\tau,y_1,y_2) = -\Omega_\tau$$

and

$$H(y_1,y_2) = \frac{\Omega_\tau}{y_1 - \beta y_1^3 + \Omega_\tau(y_1+1) + y_2^2} \begin{pmatrix} \Omega_\tau \\ y_2(1+\Omega_\tau) \end{pmatrix}.$$

We note that

$$h_-(\Omega_\tau,y_1,y_2) - h_+(\Omega_\tau,y_1,y_2) = y_1(1 - \beta y_1^2 + \Omega_\tau) + y_2^2 + \Omega_\tau > 0$$

if $0 \leq y_1 \leq \sqrt{\frac{1+\Omega_\tau}{\beta}}$. Then we take the solution $\tilde{y}_1(t)$ of

$$\dot{y}_1 = \frac{\Omega_\tau^2}{y_1 - \beta y_1^3 + \Omega_\tau(y_1+1)}$$

so that $y_1(0) = \bar{y}_\tau$ and let $\tilde{T}$ be such that $y_1(\tilde{T}) = \sqrt{\frac{1+\Omega_\tau}{\beta}}$. Note that according to the previous remark, $h_-(\Omega_\tau, y_1, y_2) - h_+(\Omega_\tau, y_1, y_2) > 0$ in a neighborhood of $\tilde{y}_1(t)$, $0 \leq t \leq \tilde{T}$. Thus we are in position to apply Remark 6.2.2. Now we define $\bar{T} = \frac{\tilde{T}}{2}$ and set

$$\gamma_0(t) = (\Omega_\tau, \tilde{y}_1(t+\bar{T}), 0), \quad \gamma_-(t) = (\tilde{\gamma}_-(t), 0), \quad \gamma_+(t) = (\tilde{\gamma}_+(t), 0)$$

and

$$\gamma(t) = \begin{cases} \gamma_-(t), & \text{if } t \leq -\bar{T}, \\ \gamma_0(t), & \text{if } -\bar{T} \leq t \leq \bar{T}, \\ \gamma_+(t), & \text{if } t \geq \bar{T} \end{cases}$$

is a sliding homoclinic orbit for the system (6.2.20), (6.2.21).

For concrete values of $\tau > 0$, we take $\beta = \frac{1}{8} + \frac{\tau^2}{2}$, compute Ω_τ and we solve (6.2.12) with initial values $z_s(\bar{T}) = \sqrt{\frac{1+\Omega_\tau}{\beta}}$, $y_s(\bar{T}) = \Omega_\tau$ to get $\widetilde{\gamma}_+(t)$ and $\gamma_+(t)$.

We now verify that system (6.2.20), (6.2.21) and $\gamma(t)$ satisfy conditions (H1)–(H4) of this section. We have already seen that (H1) is satisfied (see also Remark 6.2.2). Condition (H2) is also satisfied with $x_0 = (z^0, y_1^0, y_2^0) = (0,0,0)$. Note that in this example the discontinuity level is at $z = \Omega_\tau$ and not at $z = 0$ but we have observed that this fact does not make any difference. Now we verify (H3). It is trivial to verify that $h_+(\gamma(t)) < 0$ for $-\bar{T} \leq t \leq \bar{T}$, $h_-(\gamma(t)) > 0$ for $-\bar{T} \leq t < \bar{T}$ and $h_-(\gamma(\bar{T})) = y_+(\bar{T})(1 - \beta y_+(\bar{T})^2 + \Omega_\tau) = 0$. So we check the last condition in (H3). We have:

$$\nabla_y h_-(\gamma_0(\bar{T})) = -2\begin{pmatrix} 1+\Omega_\tau \\ 0 \end{pmatrix} \quad \text{and} \quad k_-(\gamma_0(\bar{T})) = \begin{pmatrix} \Omega_\tau \\ 0 \end{pmatrix}$$

from which we obtain

$$\nabla_y h_-(\gamma_0(\bar{T}) k_-(\gamma_0(\bar{T}))^* = -2\Omega_\tau(1+\Omega_\tau) \neq 0.$$

Finally, we check (H4). By Remark 6.2.4 it is enough to prove that $(\dot{y}_1(-\bar{T}), 0)^* \notin \mathscr{S}'$ or, equivalently, that $(1,0)^* \notin \mathscr{S}'$. Now, the variational system of (6.2.20) along $\gamma_-(t)$ is given by:

$$\begin{aligned} \dot{z} &= y_-(t)z + (1 - 3\beta y_-(t)^2 + z_-(t))y_1, \\ \dot{y}_1 &= z, \\ \dot{y}_2 &= (1 + z_-(t))y_2. \end{aligned}$$

Since this system has the bounded solution at $-\infty$: $\left(0,0,e^{t+y_-(t)}\right)$, and $\dim \mathscr{S}' = 1$ it follows that $\mathscr{S}' = \operatorname{span}\{(0,1)\}$ and hence $(1,0) \notin \mathscr{S}'$. Thus (H4) holds.

Finally we add a perturbation

$$\varepsilon g(t) = \varepsilon \begin{pmatrix} q(\omega t) \\ 0 \\ q_1(\omega_1 t) \end{pmatrix} \tag{6.2.23}$$

to (6.2.20), (6.2.21) and compute the Melnikov function. Here ω, ω_1 are positive constants and q_1, q_2 are almost periodic C^2-functions with bounded derivatives and their second order derivatives are uniformly continuous. To this end, we need to compute the solution $\psi(t)$ of the adjoint variational system:

$$\begin{aligned} \dot{z} &= -y_+(t)z - y_1 \\ \dot{y}_1 &= -(1-3\beta y_+(t)^2 + z_+(t))z \\ \dot{y}_2 &= -(1+z_+(t))y_2 \end{aligned} \tag{6.2.24}$$

with $\psi(0) = (1,0,0)$ (see (6.2.5)). Since $y_2 = 0$ is invariant for system (6.2.24) we get $\psi(t) = (\psi_1(t), \psi_2(t), 0)$ where $(z,y) = (\psi_1(t), \psi_2(t))$ is a bounded (at $+\infty$) solution of

$$\begin{aligned} \dot{z} &= -y_+(t)z - y_1, \\ \dot{y}_1 &= -(1-3\beta y_+(t)^2 + z_+(t))z \end{aligned} \tag{6.2.25}$$

that is

$$\psi(t) = \begin{pmatrix} \dot{y}_+(t) \\ -\dot{z}_+(t) \\ 0 \end{pmatrix} e^{-\int_{\bar{T}}^t y_+(s)ds}$$

and the Melnikov function is

$$\mathscr{M}(\alpha) = \int_{\bar{T}}^{\infty} \dot{y}_+(t)\, e^{-\int_{\bar{T}}^t y_+(s)ds} q(\omega t + \alpha)dt\,.$$

Since

$$\begin{aligned} \lim_{\omega\to 0} \mathscr{M}(\alpha) &= q(\alpha)\int_{\bar{T}}^{\infty} \dot{y}_+(t)\, e^{-\int_{\bar{T}}^t y_+(s)ds}\, dt, \\ \lim_{\omega\to 0} \mathscr{M}'(\alpha) &= q'(\alpha)\int_{\bar{T}}^{\infty} \dot{y}_+(t)\, e^{-\int_{\bar{T}}^t y_+(s)ds}\, dt, \end{aligned}$$

we see that if $q(\alpha)$ has a simple zero at some $\alpha = \alpha_0$ and

$$\int_{\bar{T}}^{\infty} \dot{y}_+(t)\, e^{-\int_{\bar{T}}^t y_+(s)ds}\, dt \neq 0, \tag{6.2.26}$$

then $\mathscr{M}(\alpha)$ will have a simple zero at some α near to α_0 for $\omega > 0$ small. To check condition (6.2.26) we recall that

$$y_+(t) = \sqrt{\frac{1 - u_\tau(\theta^-(t))}{\beta}} = \dot{\theta}^-(t)$$

so,

$$\int_{\bar{T}}^{t} y_+(s)ds = \theta^-(t) - \theta^-(\bar{T}) = \theta^-(t) - \theta_\tau^- . \tag{6.2.27}$$

Now, let $Y(\theta) = \sqrt{\frac{1-u_\tau(\theta)}{\beta}}$. Then:

$$y_+(t) = Y(\theta^-(t))$$

and

$$\dot{y}_+(t) = Y'(\theta^-(t))\dot{\theta}^-(t) . \tag{6.2.28}$$

Plugging (6.2.27), (6.2.28) into (6.2.26) we obtain:

$$\begin{aligned}
&\int_{\bar{T}}^{\infty} \dot{y}_+(t)\,\mathrm{e}^{-\int_{\bar{T}}^{t} y_+(s)ds}\, dt = \mathrm{e}^{\theta_\tau^-} \int_{\bar{T}}^{\infty} \mathrm{e}^{-\theta^-(t)}\, Y'(\theta^-(t))\dot{\theta}^-(t)dt \\
&= \mathrm{e}^{\theta_\tau^-} \int_{\theta_\tau^-}^{0} \mathrm{e}^{-\theta}\, Y'(\theta)d\theta = \mathrm{e}^{\theta_\tau^-} \left[Y(\theta)\,\mathrm{e}^{-\theta}\Big|_{\theta_\tau^-}^{0} + \int_{\theta_\tau^-}^{0} \mathrm{e}^{-\theta}\, Y(\theta)d\theta \right] \\
&= \mathrm{e}^{\theta_\tau^-} \int_{\theta_\tau^-}^{0} \mathrm{e}^{-\theta}\, Y(\theta)d\theta - Y(\theta_\tau^-) \\
&= \int_{\theta_\tau^-}^{0} \mathrm{e}^{\theta_\tau^- - \theta}\, Y(\theta)d\theta - \sqrt{\frac{1+\Omega_\tau}{\beta}} \\
&= \frac{1}{\sqrt{\beta}} \left(\int_{\theta_\tau^-}^{0} \mathrm{e}^{\theta_\tau^- - \theta} \sqrt{1-u_\tau(\theta)}d\theta - \sqrt{1+\Omega_\tau} \right).
\end{aligned} \tag{6.2.29}$$

We prove now that the expression (6.2.29) is negative for any $\tau > 0$. Using Cauchy-Schwarz-Bunyakovsky inequality we get

$$\int_{\theta_\tau^-}^{0} \mathrm{e}^{\theta_\tau^- - \theta} \sqrt{1-u_\tau(\theta)}d\theta \le \sqrt{\int_{\theta_\tau^-}^{0} \mathrm{e}^{\theta_\tau^- - \theta}\, d\theta} \sqrt{\int_{\theta_\tau^-}^{0} \mathrm{e}^{\theta_\tau^- - \theta}(1-u_\tau(\theta))d\theta}.$$

Next, we integrate

$$\int_{\theta_\tau^-}^{0} \mathrm{e}^{\theta_\tau^- - \theta}\, d\theta = 1 - \mathrm{e}^{\theta_\tau^-}$$

and

$$\begin{aligned}
\int_{\theta_\tau^-}^{0} \mathrm{e}^{\theta_\tau^- - \theta}(1-u_\tau(\theta))d\theta &= \int_{\theta_\tau^-}^{0} \mathrm{e}^{\theta_\tau^- - \theta} \left(1 - \mathrm{e}^{\theta/2}\cos(\tau\theta) + \frac{1}{2\tau}\,\mathrm{e}^{\theta/2}\sin(\tau\theta) \right) d\theta \\
&= 1 - \mathrm{e}^{\theta_\tau^-} + 2\,\mathrm{e}^{\theta_\tau^-/2} \sqrt{\frac{1}{1+4\tau^2}} = 1 - \mathrm{e}^{\theta_\tau^-} + \Omega_\tau.
\end{aligned}$$

Consequently:

$$\int_{\theta_\tau^-}^0 e^{\theta_\tau^- - \theta} \sqrt{1-u_\tau(\theta)}d\theta \le \sqrt{1-e^{\theta_\tau^-}}\sqrt{1-e^{\theta_\tau^-}+\Omega_\tau} < \sqrt{1+\Omega_\tau},$$

hence the expression (6.2.29) is negative for any value of $\tau > 0$. In summary, we obtain the following result.

Theorem 6.2.7. *Let $q(t)$ have a simple zero. Then there exist $\omega_0 > 0$ and $\varepsilon_0 > 0$ so that for $0 < |\omega| < \omega_0$ and $0 < |\varepsilon| < \varepsilon_0$, system*

$$\dot{x} = f_\pm(x) + \varepsilon g(t), \quad x \in \Omega_\pm \tag{6.2.30}$$

where $x = (z, y_1, y_2) \in \mathbb{R}^3$, $f_\pm(x)$ is as in (6.2.22) and $g(t)$ as in (6.2.23), is chaotic.

For example, if $q(t) = \cos t$ we get

$$\mathscr{M}(\alpha) = \int_{\bar{T}}^{\infty} \dot{y}_+(t)\, e^{-\int_{\bar{T}}^t y_+(s)ds} \cos(\omega t + \alpha)dt$$

and then

$$\mathscr{M}(\alpha) - \imath\mathscr{M}'(\alpha) = e^{\imath\alpha}\Psi_\tau(\omega), \quad \Psi_\tau(\omega) := \int_{\bar{T}}^{\infty} \dot{y}_+(t)\, e^{-\int_{\bar{T}}^t y_+(s)ds} e^{\imath\omega t}\, dt\,.$$

As a consequence if $\Psi_\tau(\omega) \ne 0$ then $\mathscr{M}(\alpha)$ has a simple zero. Since $\Psi_\tau(0) \ne 0$, $\Psi_\tau(\omega)$ is a nonzero analytical function. From Theorem 6.2.7 we know that (6.2.30) behaves chaotically for $|\omega| < \omega_0$ (and $|\varepsilon| < \varepsilon_0$) sufficiently small. However, for this particular example ($q(t) = \cos t$), (6.2.30) behaves chaotically also when ω is large. As a matter of fact, we have the following:

Theorem 6.2.8. *There exist continuous functions $F(\beta), D(\beta) : \left(\frac{1}{8}, \infty\right) \to (0, \infty)$ so that for any given constants $\beta > 1/8$, $\omega_1 > 0$, $\omega \in (0,\infty) \setminus [F(\beta), D(\beta)]$ and an almost periodic C^2–function $q_1(t)$ with bounded derivatives so that its second order derivative is uniformly continuous, there exists $\varepsilon_0 = \varepsilon_0(\beta, \omega, \omega_1, q_1(\cdot))$ so that for $0 < |\varepsilon| < \varepsilon_0$ and*

$$g(t) = \begin{pmatrix} \cos(\omega t) \\ 0 \\ q_1(\omega_1 t) \end{pmatrix}$$

system (6.2.20), (6.2.21) is chaotic. Moreover, it holds

$$\lim_{\tau \to 1/8_+} F(\beta) = 0, \quad \lim_{\beta\to\infty} F(\beta) = \frac{2\sqrt{2}}{\pi(2\sqrt{2}+1)} \doteq 0.235166\,,$$

$$\lim_{\beta \to 1/8_+} D(\beta) = \infty, \quad \lim_{\beta\to\infty} D(\beta) = \frac{3\sqrt{2}\pi}{2} + 4 - \sqrt{2} \doteq 9.25011\,.$$

Proof. We omit the proof of this theorem, since it is rather technical and refer the readers to [55] for more details. □

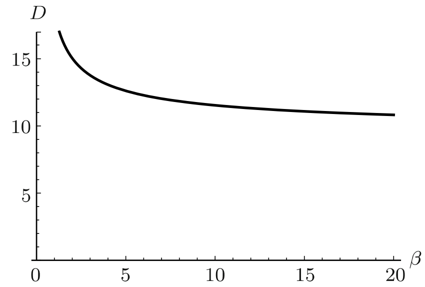
Here we only mention that

$$D(\beta) := B\left(\sqrt{8\beta-1}/2\right),$$

where

$$B(\tau) := \sqrt{\frac{8(1+\tau^{-1}\mathrm{e}^{-\frac{\pi}{2\tau}})}{4\tau^2+1}} + \frac{\sqrt{4+\tau^{-2}}}{2}\left(\frac{3\sqrt{2}\pi}{8}(\tau^{-2}+4\tau^{-1}+4)+(2+\tau^{-1})\left(2-\sqrt{1/2}\right)\right)$$

and the graph of $D(\beta)$ in interval $(1/8, 20]$ looks like



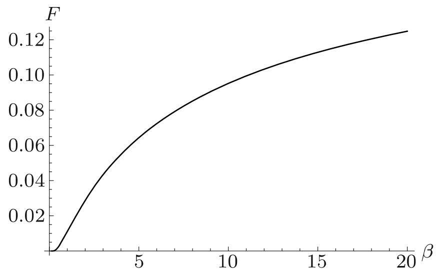
Furthermore, we have

$$F(\beta) := C\left(\sqrt{8\beta-1}/2\right)$$

where

$$C(\tau) := \frac{2\sqrt{2}\tau}{\sqrt{4\tau^2+1}} \frac{\sqrt{1+\Omega_\tau}-\sqrt{1-\mathrm{e}^{\theta_\tau^-}}\sqrt{1-\mathrm{e}^{\theta_\tau^-}+\Omega_\tau}}{2\arctan 2\tau\sqrt{\dfrac{1+\mathrm{e}^{-\frac{\pi}{2\tau}}}{1+\Omega_\tau}}+\pi-\arctan 2\tau}$$

and the graph of $F(\beta)$ in interval $(1/8, 20]$ looks like



For instance, a numerical evaluation shows that for $\beta = 25$: $D(25) \doteq 0.1337$ and $F(25) \doteq 10.6489$, so for $\omega \in (0,\infty) \setminus [0.13, 10.65]$, system (6.2.20), (6.2.21) is chaotic for $\varepsilon \neq 0$.

Furthermore, since $\Psi_\tau(\omega)$ is analytical (cf Section 2.6.5), there is at most a finite number of $\omega_1, \ldots, \omega_{n_\beta} \in [F(\beta), D(\beta)]$ so that for any $\omega > 0$ and $\omega \notin \{\omega_1, \ldots, \omega_{n_\beta}\}$, there is a chaos like in Theorem 6.2.8. An open problem remains to estimate n_β. On the other hand, the statement of Theorem 6.2.8 can be extended as follows.

Theorem 6.2.9. *There exists a continuous function* $G(\omega) : (0,\infty) \to \left[\frac{1}{8}, \infty\right)$ *so that for any given constants* $\omega \in (0,\infty)$, $\beta > G(\omega)$, $\omega_1 > 0$ *and an almost periodic* C^2-*function* $q_1(t)$ *with bounded derivatives so that its second order derivative is uniformly continuous, there exists* $\varepsilon_0 = \varepsilon_0(\beta, \omega, \omega_1, q_1(\cdot))$ *so that for* $0 < |\varepsilon| < \varepsilon_0$ *and*

$$g(t) = \begin{pmatrix} \cos(\omega t) \\ 0 \\ q_1(\omega_1 t) \end{pmatrix},$$

system (6.2.20), (6.2.21) *is chaotic.*

We again refer the readers to [55] for more details. A lower bound $G(\omega)$ for β could be numerically estimated, but we do not carry out these awkward computations in this section. By Theorem 6.2.8, it would be enough to estimate $G(\omega)$ in the interval $[0.2, 9.3]$.

6.3 Outlook

The above results could be extended to other types of discontinuous homoclinics. First we could study impact systems like in [17, 18, 20, 21, 49]. Second we could develop Melnikov theory for grazing homoclinics which has not yet been done. Discontinuous systems with grazing orbits are investigated in [6, 22, 24, 57, 58].

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