Chapter 5
Chaos in Partial Differential Equations

Functional analytical methods are presented in this chapter to predict chaos for periodically forced PDEs modeling vibrations of beams and depend on parameters.

5.1 Beams on Elastic Bearings

5.1.1 Weakly Nonlinear Beam Equation

This section deals with the beam equation (Figure 5.1)

$$\begin{aligned} & u_{tt} + u_{xxxx} + \varepsilon\delta u_t + \varepsilon\mu h(x, \sqrt{\varepsilon}t) = 0\,, \\ & u_{xx}(0,\cdot) = u_{xx}(\pi/4,\cdot) = 0\,, \\ & u_{xxx}(0,\cdot) = -\varepsilon f(u(0,\cdot)), \quad u_{xxx}(\pi/4,\cdot) = \varepsilon f(u(\pi/4,\cdot)) \end{aligned} \tag{5.1.1}$$

where $\varepsilon > 0$ and μ are sufficiently small parameters, $\delta > 0$ is a constant, $f \in C^2(\mathbb{R})$, $h \in C^2([0,\pi/4] \times \mathbb{R})$ and $h(x,t)$ is 1-periodic in t, provided an associated reduced equation has a homoclinic orbit (cf (5.1.9)). Equation (5.1.1) describes vibrations of a beam resting on two identical bearings with purely elastic responses which are determined by f. The length of the beam is $\pi/4$. Since $\varepsilon > 0$ is small, (5.1.1) is a semilinear, weakly damped, weakly forced and slowly varying problem.

Let us briefly recall some results related to Eq. (5.1.1). The undamped case ($\delta = 0$, $\mu = 0$ and $\varepsilon = 1$) was studied in [1,2] by using variational methods. In both papers, the problems studied are non-parametric.

The perturbation approach to the beam equation was earlier used in [3]. Recent results in this direction are given in [4,5]. We note that the problem (5.1.1) is more complicated than the one studied in [3–5], since in those papers the elastic response is distributed continuously along the beam, while in our case it is concentrated just at two end points of the beam. Moreover, the ε-smallness of the restoring force εf at the end points leads to a singularly perturbed problem in studying chaotic orbits

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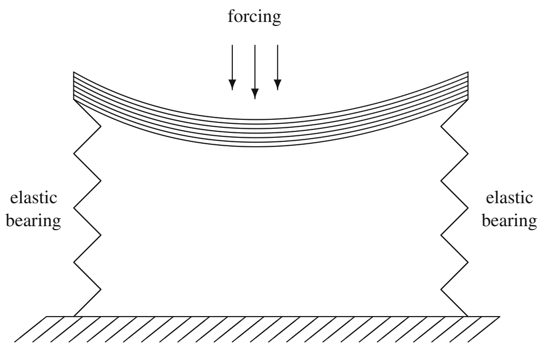

Fig. 5.1 The forced beam resting on two elastic bearings (5.1.1).

of (5.1.1). The existence of homoclinic and chaotic solutions has also been proved in [6–9] for different partial differential equations, with different methods compared with ours.

5.1.2 Setting of the Problem

First of all, we make the linear scale $t \leftrightarrow \sqrt{\varepsilon}t$ in (5.1.1), that is, we take $u(x,t) \leftrightarrow u(x,\sqrt{\varepsilon}t)$ to get the equivalent problem

$$\begin{aligned} &u_{tt} + \frac{1}{\varepsilon}u_{xxxx} + \sqrt{\varepsilon}\delta u_t + \mu h(x,t) = 0\,, \\ &u_{xx}(0,\cdot) = u_{xx}(\pi/4,\cdot) = 0\,, \\ &u_{xxx}(0,\cdot) = -\varepsilon f(u(0,\cdot)), \quad u_{xxx}(\pi/4,\cdot) = \varepsilon f(u(\pi/4,\cdot))\,. \end{aligned} \tag{5.1.2}$$

By a (weak) solution of (5.1.2), we mean any $u(x,t) \in C([0,\pi/4]\times\mathbb{R})$ satisfying the identity

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_0^{\pi/4} \Big\{u(x,t)\Big[v_{tt}(x,t) + \frac{1}{\varepsilon}v_{xxxx}(x,t) - \sqrt{\varepsilon}\delta v_t(x,t)\Big] + \mu h(x,t)v(x,t)\Big\}\,dx\,dt \\ &\quad + \int_{-\infty}^{\infty}\Big\{f(u(0,t))v(0,t) + f(u(\pi/4,t))v(\pi/4,t)\Big\}\,dt = 0 \end{aligned} \tag{5.1.3}$$

for any $v(x,t) \in C^\infty([0,\pi/4]\times\mathbb{R})$ so that $v(x,t)$ has a compact support and the following boundary value conditions hold

$$v_{xx}(0,\cdot) = v_{xx}(\pi/4,\cdot) = v_{xxx}(0,\cdot) = v_{xxx}(\pi/4,\cdot) = 0\,. \tag{5.1.4}$$

Now, it is well known [2] that there is an orthonormal system of eigenfunctions $\{w_i\}_{i=-1}^{\infty} \in L^2([0,\frac{\pi}{4}])$ of the eigenvalue problem

$$\begin{aligned}
&U^{(iv)}(x) = \kappa U(x),\\
&U''(0) = U''(\pi/4) = 0, \quad U'''(0) = U'''(\pi/4) = 0\,.
\end{aligned}$$

As a matter of fact (cf Section 5.1.5), the eigenfunctions $\{w_i\}_{i=-1}^{\infty}$ are uniformly bounded in $C^0([0,\frac{\pi}{4}])$, and setting $\kappa = \mu^4$, the eigenvalues of the above problem satisfy $\mu = \mu_k$, $k = -1,0,1,\dots$ with $\mu_{-1} = \mu_0 = 0$ and $\mu_k = 2(2k+1) + r(k)$, for any $k \in \mathbb{N}$, where $|r(k)| \le \bar{c}_1 \,\mathrm{e}^{-\bar{c}_2 k}$ for any $k \ge 1$, for some positive constants $\bar{c}_1$, $\bar{c}_2$. Furthermore, the eigenfunctions $w_{-1}(x)$ and $w_0(x)$ of the zero eigenvalue are:

$$w_{-1}(x) = \frac{2}{\sqrt{\pi}}, \quad w_0(x) = \frac{16}{\pi}\left(x - \frac{\pi}{8}\right)\sqrt{\frac{3}{\pi}}.$$

Thus we seek a solution $u(x,t)$ of (5.1.2) in the form

$$u(x,t) = y_1(t)w_{-1}(x) + y_2(t)w_0(x) + z(x,t)$$

where $z(x,t) \in C\left([0,\frac{\pi}{4}] \times \mathbb{R}\right)$ is orthogonal to the eigenfunctions $w_{-1}(x)$ and $w_0(x)$, satisfying

$$\int_0^{\pi/4} z(x,t)dx = \int_0^{\pi/4} xz(x,t)dx = 0. \tag{5.1.5}$$

To obtain the equations for $y_1(t)$, $y_2(t)$, and $z(x,t)$ we take $v(x,t) = \phi_1(t)w_{-1}(x) + \phi_2(t)w_0(x) + v_0(x,t)$ in (5.1.3) with $\phi_i \in C^\infty$, $v_0(x,t) \in C^\infty\left([0,\frac{\pi}{4}] \times \mathbb{R}\right)$ with compact supports so that $v_0(x,t)$ satisfies (5.1.4) and is orthogonal to $w_{-1}(x)$ and $w_0(x)$, i.e. it satisfies (5.1.5). Plugging the above expression for $v(x,t)$ into (5.1.3) and using the orthonormality, we arrive at the system of equations

$$\begin{aligned}
&\ddot{y}_1(t) + \sqrt{\varepsilon}\delta\dot{y}_1(t) + \frac{2}{\sqrt{\pi}}\mu\int_0^{\pi/4} h(x,t)\,dx\\
&+\frac{2}{\sqrt{\pi}}f\left(\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}y_2(t) + z(0,t)\right)\\
&+\frac{2}{\sqrt{\pi}}f\left(\frac{2}{\sqrt{\pi}}y_1(t) + 2\sqrt{\frac{3}{\pi}}y_2(t) + z(\pi/4,t)\right) = 0\,, \qquad (5.1.6)\\
&\ddot{y}_2(t) + \sqrt{\varepsilon}\delta\dot{y}_2(t) + \frac{16}{\pi}\sqrt{\frac{3}{\pi}}\mu\int_0^{\pi/4} h(x,t)\left(x - \frac{\pi}{8}\right)dx\\
&-2\sqrt{\frac{3}{\pi}}f\left(\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}y_2(t) + z(0,t)\right)
\end{aligned}$$

$$+2\sqrt{\frac{3}{\pi}}f\left(\frac{2}{\sqrt{\pi}}y_1(t)+2\sqrt{\frac{3}{\pi}}y_2(t)+z(\pi/4,t)\right)=0, \tag{5.1.7}$$

$$\int_{-\infty}^{\infty}\int_0^{\pi/4}\left\{z(x,t)\left[v_{tt}(x,t)+\frac{1}{\varepsilon}v_{xxxx}(x,t)-\sqrt{\varepsilon}\delta v_t(x,t)\right]+\mu h(x,t)v(x,t)\right\}dx\,dt$$
$$+\int_{-\infty}^{\infty}\left\{f(u(0,t))v(0,t)+f(u(\pi/4,t))v(\pi/4,t)\right\}dt=0 \tag{5.1.8}$$

where we write $v(x,t)$ instead $v_0(x,t)$. Thus, in Eq. (5.1.8), $v(x,t)$ is any function in $C^\infty([0,\frac{\pi}{4}]\times\mathbb{R})$ having compact support so that the conditions (5.1.4), (5.1.5) (with $v(x,t)$ instead of $z(x,t)$) hold. We remark that in this way we have split up the original equation into two parts. Equation (5.1.8) corresponds, in some sense, to Eq. (5.1.1) on a infinite dimensional center manifold, while Eqs. (5.1.6)–(5.1.8) are the equations on a hyperbolic manifold for the unperturbed equation. Since the center manifold is infinitely dimensional, the standard center manifold reduction method (cf Sections 2.5.4, 2.5.5 and [10]) fails for (5.1.1). We use instead a regular singular perturbation method. In fact, the above splitting of Eq. (5.1.1) has also the advantage that the singular part (in ε) is only in the z equation while Eqs. (5.1.6) and (5.1.8) look regular in $\sqrt{\varepsilon}$.

Now we assume that the following conditions hold:

(H1) $f(0)=0$, $f'(0)<0$ and the equation $\ddot{x}+f(x)=0$ has a homoclinic solution $\gamma(t)\neq 0$ that is a nontrivial bounded solution so that $\lim_{t\to\pm\infty}\gamma(t)=0$;

(H2) let $\gamma_1(t):=\frac{\sqrt{\pi}}{2}\gamma\Big(2\sqrt{\frac{2}{\pi}}t\Big)$. Then the linear equation $\ddot{v}+\frac{24}{\pi}f'\left(\frac{2}{\sqrt{\pi}}\gamma_1(t)\right)v=0$ has no nontrivial bounded solutions.

Without loss of generality we can also assume that $\ddot{\gamma}(0)\neq\dot{\gamma}(0)=0$. This implies that $\gamma(t)=\gamma(-t)$ (and then $\gamma_1(t)=\gamma_1(-t)$) since both satisfy the Cauchy problem $\ddot{x}+f(x)=0$, $x(0)=\gamma(0)$ and $\dot{x}(0)=0$. Note also that (H1) implies that the system

$$\begin{aligned}
&\ddot{y}_1+\frac{2}{\sqrt{\pi}}f\Big(\frac{2}{\sqrt{\pi}}y_1-2\sqrt{\frac{3}{\pi}}y_2\Big)+\frac{2}{\sqrt{\pi}}f\Big(\frac{2}{\sqrt{\pi}}y_1+2\sqrt{\frac{3}{\pi}}y_2\Big)=0,\\
&\ddot{y}_2-2\sqrt{\frac{3}{\pi}}f\Big(\frac{2}{\sqrt{\pi}}y_1-2\sqrt{\frac{3}{\pi}}y_2\Big)+2\sqrt{\frac{3}{\pi}}f\Big(\frac{2}{\sqrt{\pi}}y_1+2\sqrt{\frac{3}{\pi}}y_2\Big)=0
\end{aligned} \tag{5.1.9}$$

has a hyperbolic equilibrium $y_1=y_2=0$ with the homoclinic orbit $(\gamma_1(t),0)$ and that (H2) is equivalent to requiring that the space of bounded solutions of the linear, fourth order system

$$\ddot{y}_1+\frac{8}{\pi}f'\Big(\frac{2}{\sqrt{\pi}}\gamma_1(t)\Big)y_1=0,\quad \ddot{y}_2+\frac{24}{\pi}f'\Big(\frac{2}{\sqrt{\pi}}\gamma_1(t)\Big)y_2=0 \tag{5.1.10}$$

is one-dimensional and spanned by $(y_1(t),\dot{y}_1(t),y_2(t),\dot{y}_2(t))=(\dot{\gamma}_1(t),\ddot{\gamma}_1(t),0,0)$. We look for chaotic solutions of Equations (5.1.6)–(5.1.8) so that the sup-norm of $|y_2(t)|+|z(x,t)|$ on $[0,\frac{\pi}{4}]\times\mathbb{R}$ is small and $y_1(t)$ is orbitally near to $\gamma_1(t)$.

5.1.3 Preliminary Results

We begin our analysis by studying some linear problems associated with Eqs. (5.1.6)–(5.1.8). To start with, let us consider, for $i \in \mathbb{N}$, the following linear non-homogeneous equation

$$\ddot{z}_i(t) + \sqrt{\varepsilon}\delta \dot{z}_i(t) + \frac{1}{\varepsilon}\mu_i^4 z_i(t) = h_i(t)\,, \tag{5.1.11}$$

where $h_i(t)$ belongs to the Banach space $L^\infty(\mathbb{R})$ of bounded measurable functions on $\mathbb{R}$, with norm $\|h_i\|_\infty := \operatorname{ess\,sup}_{t\in\mathbb{R}} |h_i(t)| < \infty$. This equation comes from searching a solution of Eq. (5.1.17) of the form

$$z(x,t) = \sum_{i=1}^{\infty} z_i(t) w_i(x)$$

with $z_i(t) \in W^{2,\infty}(\mathbb{R})$. The only bounded solution of (5.1.11) for $0 < \varepsilon < 2\min_{i\geq 1}\{\frac{\mu_i^2}{\delta}\}$ is given by

$$z_i(t) = L_{i,\varepsilon}h_i := \frac{2\sqrt{\varepsilon}}{\omega_{i,\varepsilon}} \int_{-\infty}^{t} \mathrm{e}^{-\sqrt{\varepsilon}\delta(t-s)/2} \sin\left(\frac{\omega_{i,\varepsilon}}{2\sqrt{\varepsilon}}(t-s)\right) \times h_i(s)\,ds\,, \tag{5.1.12}$$

where $\omega_{i,\varepsilon} = \sqrt{4\mu_i^4 - \varepsilon^2\delta^2}$. Moreover it is easy to see that

$$\|z_i\|_\infty \leq \frac{4}{\delta\mu_i^2}\|h_i\|_\infty\,, \tag{5.1.13}$$

($\|z\|_\infty$ being the sup-norm of $z(t)$) and

$$\|\dot{z}_i\|_\infty \leq \left(\frac{2\sqrt{\varepsilon}}{\mu_i^2} + \frac{2}{\delta\sqrt{\varepsilon}}\right)\|h_i\|_\infty\,, \tag{5.1.14}$$

provided $0 < \varepsilon < \sqrt{3}\min_{i\geq 1}\{\frac{\mu_i^2}{\delta}\}$. Let $h = \{h_i(t)\}_{i=1}^\infty$, $h_i \in L^\infty(\mathbb{R})$ be a sequence of uniformly bounded measurable functions on $\mathbb{R}$, that is, satisfying $\|h\|_\infty := \sup_i \|h_i\|_\infty < \infty$. Consider the function

$$z(x,t) = \sum_{i=1}^{\infty} z_i(t) w_i(x) \tag{5.1.15}$$

where $z_i(t)$ are given by (5.1.12). We put

$$M_1 := \sup\left\{|w_i(x)| \,:\, x \in \left[0, \frac{\pi}{4}\right],\, i \in \mathbb{N}\right\}; \quad M_2 := 4M_1 \sum_{i=1}^{\infty} \frac{1}{\mu_i^2}\,, \tag{5.1.16}$$

with the last series being convergent because of the properties of μ_k, $k \in \mathbb{N}$.

Now, let $H_1(x,t) \in L^\infty([0,\pi/4]\times\mathbb{R})$, $H_2(t), H_3(t) \in L^\infty(\mathbb{R})$ be bounded measurable functions and consider the equation

$$\int_{-\infty}^{\infty}\int_0^{\pi/4}\Big\{z(x,t)\Big[v_{tt}(x,t)+\frac{1}{\varepsilon}v_{xxxx}(x,t)-\sqrt{\varepsilon}\delta v_t(x,t)\Big]+H_1(x,t)v(x,t)\Big]\,dx\,dt$$
$$+\int_{-\infty}^{\infty}\Big\{H_2(t)v(0,t)+H_3(t)v(\pi/4,t)\Big\}\,dt=0 \qquad (5.1.17)$$

for any $v(x,t)\in C^\infty([0,\pi/4]\times\mathbb{R})$ so that $v(x,t)$ has compact support and the boundary conditions (5.1.4), (5.1.5) hold. For $i\in\mathbb{N}$ let

$$h_i(t)=-\left(\int_0^{\pi/4}H_1(x,t)w_i(x)\,dx+H_2(t)w_i(0)+H_3(t)w_i(\pi/4)\right) \qquad (5.1.18)$$

and take $z_i(t)$, $z(x,t)$ as in (5.1.12), (5.1.15). Note that

$$|h_i(t)|\le M_1\left[\frac{\pi}{4}\|H_1(\cdot,t)\|_\infty+|H_2(t)|+|H_3(t)|\right] \qquad (5.1.19)$$

where $\|H_1(\cdot,t)\|_\infty=\sup_{0\le x\le\frac{\pi}{4}}|H_1(x,t)|$ and, similarly,

$$|\dot h_i(t)|\le M_1\left[\frac{\pi}{4}\|H_{1t}(\cdot,t)\|_\infty+|\dot H_2(t)|+|\dot H_3(t)|\right] \qquad (5.1.20)$$

provided $\dot H_2(t)$, $\dot H_2(t)$, and the partial derivative of $H_1(x,t)$ with respect to t, $H_{1t}(x,t)$, are bounded measurable functions. Then, we can prove as in [11] that $z(x,t)$ is a solution of Eq. (5.1.17).

Let $m\ge[\varepsilon^{-3/4}]+1$, with $[\varepsilon^{-3/4}]$ being the integer part of $\varepsilon^{-3/4}$. From now on we assume that $0<\varepsilon\le(1/2)^{4/3}$ so that $m\ge 3$. Then, for any $E=\{e_n\}_{n\in\mathbb{Z}}\in\mathscr{E}$, we put

$$\ell_E^\infty=\Big\{\alpha:=\{\alpha_j\}_{j\in\mathbb{Z}}\in\ell^\infty\mid\alpha_j\in\mathbb{R}\quad\text{and}\quad\alpha_j=0\quad\text{if}\quad e_j=0\Big\},$$

with ℓ^∞ being the Banach space of bounded, doubly infinity sequences of real numbers, endowed with the sup-norm. We will also consider a bounded subset of $\mathscr{E}\times\ell^\infty$:

$$X=\Big\{(E,\alpha)\in\mathscr{E}\times\ell^\infty\mid\alpha\in\ell_E^\infty\quad\text{and}\quad\|\alpha\|\le 2\Big\}.$$

Note that X is closed. In fact if $(E_n,\alpha_n)\to(E,\alpha)$ as $n\to\infty$, then, for any fixed $j\in\mathbb{Z}$, we have (with obvious meaning of symbols) $e_j^{(n)}=e_j$ for any $n\in\mathbb{N}$ sufficiently large. Hence $\alpha_j^{(n)}=0$ if $e_j=0$ and n is large enough. Thus $\alpha_j=0$ if $e_j=0$, that is, $(E,\alpha)\in X$.

For any $\xi=(E,\alpha)\in X$ we take the function $\gamma_\xi=\gamma_{(E,\alpha)}\in L^\infty(\mathbb{R})$ defined by

$$\gamma_\xi(t)=\begin{cases}\gamma_1(t-2jm-\alpha_j), & \text{if}\quad(2j-1)m<t\le(2j+1)m\quad\text{and}\quad e_j=1\\ 0, & \text{if}\quad(2j-1)m<t\le(2j+1)m\quad\text{and}\quad e_j=0.\end{cases}$$

For the sake of simplicity we will silently include, in the above definitions, also the end points of the intervals $[(2j-1)m,(2j+1)m]$, $j \in \mathbb{Z}$. We remark that $\gamma_\xi(t)$ has the following properties:

(i) $\gamma_\xi(t)$ is a bounded, piecewise C^2-function, with possible jumps at the points $(2j-1)m$, $j \in \mathbb{Z}$, and satisfies, in any of the intervals $((2j-1)m,(2j+1)m)$, the equation

$$\ddot{x} + \frac{4}{\sqrt{\pi}} f\left(\frac{2}{\sqrt{\pi}}x\right) = 0. \tag{5.1.21}$$

(ii) $\gamma_\xi(t)$, $\dot{\gamma}_\xi(t)$, $\ddot{\gamma}_\xi(t)$ belong to $L^\infty(\mathbb{R})$ and are bounded uniformly with respect to (ξ,m).

(iii) $\gamma_\xi(t)$, $\dot{\gamma}_\xi(t)$, $\ddot{\gamma}_\xi(t)$ are Lipschitz continuous function in $\alpha \in \ell^\infty_E$ uniformly with respect to (E,m). In fact, let $(E,\alpha'),(E,\alpha'') \in X$ and assume that $e_j = 1$ (if $e_j = 0$ there is nothing to prove). Then, for any $t \in ((2j-1)m,(2j+1)m]$ we have, for some $\theta \in \mathbb{R}$:

$$|\gamma_{\xi'}(t) - \gamma_{\xi''}(t)| \le |\dot{\gamma}_1(\theta)||\alpha'_j - \alpha''_j| \le \sqrt{2}\|\dot{\gamma}\|_\infty \, \|\alpha' - \alpha''\|. \tag{5.1.22}$$

A similar argument applies to $\dot{\gamma}_\xi(t)$, whereas we will use point (i) to reduce the study of the Lipschitz continuity of $\ddot{\gamma}_\xi(t)$ to that of $\gamma_\xi(t)$.

The following result deals with the solvability of Eq. (5.1.17).

Theorem 5.1.1. *For any given functions $H_1(x,t) \in L^\infty([0,\pi/4]\times\mathbb{R})$, $H_2(t),H_3(t) \in L^\infty(\mathbb{R})$ and for $0 < \varepsilon < \min_i\{\sqrt{3}\mu_i^2/\delta\}$, Equation* (5.1.17) *has a unique solution $z(x,t) \in C([0,\pi/4]\times\mathbb{R})$ of the form*

$$z(x,t) = \sum_{i=1}^{\infty} z_i(t)w_i(x)$$

with $z_i(t) \in W^{2,\infty}(\mathbb{R})$. Such a solution satisfies condition (5.1.5)*, moreover if $h_i(t)$ is defined as in* (5.1.18) *the following hold:*

(a) *Assume that there exist positive constants k_1, k_2, α_j and β so that*

$$|h_i(t)| \le k_1 + k_2\,\mathrm{e}^{-\beta|t-2jm-\alpha_j|}$$

for any $t \in ((2j-1)m,(2j+1)m]$ and $j \in \mathbb{Z}$. Then

$$\|z\|_\infty \le M_2\left[\frac{k_1}{\delta} + \left(\frac{1}{\delta^3} + \frac{2}{\beta}\right)k_2\sqrt{\varepsilon}\right].$$

(b) *Assume that for any $i,j \in \mathbb{Z}$, $h_i(t) \in W^{1,\infty}((2j-1)m,(2j+1)m)$ and that both $h_i(t)$ and $\dot{h}_i(t)$ satisfy the condition of point (a), then we have*

$$\|z\|_\infty \le M_2\left[5\varepsilon\left(\frac{1}{\delta^5} + 1 + \frac{1}{\beta}\right)(k_1+k_2) + \frac{2\sqrt{\varepsilon}}{\delta}k_1\right]$$

provided ε satisfies the further estimate $\sqrt{\varepsilon} < 2\delta^2$.

Proof. We only need to prove (a) and (b). Let $(2j-1)m < t \le (2j+1)m$ and $0 < \varepsilon < \min_i\{\sqrt{3}\mu_i^2\delta^{-1}\}$. We have

$$\left|\int_{-\infty}^{t} \mathrm{e}^{-\sqrt{\varepsilon}\delta(t-s)/2}\sin\frac{\omega_{i,\varepsilon}}{2\sqrt{\varepsilon}}(t-s)\,h_i(s)\,ds\right| \le \int_{-\infty}^{t}\mathrm{e}^{-\sqrt{\varepsilon}\delta(t-s)/2}[k_1+k_2\varphi(s)]\,ds,$$

where $\varphi(t) = \mathrm{e}^{-\beta|t-2jm-\alpha_j|}$ for $t \in ((2j-1)m,(2j+1)m]$. Then we have

$$\int_{-\infty}^{t}\mathrm{e}^{-\sqrt{\varepsilon}\delta(t-s)/2}\,ds \le \frac{2}{\sqrt{\varepsilon}\delta},$$

and similarly, using also $t > (2j-1)m$,

$$\int_{-\infty}^{(2j-3)m}\mathrm{e}^{-\sqrt{\varepsilon}\delta(t-s)/2}\,\varphi(s)\,ds \le \int_{-\infty}^{(2j-3)m}\mathrm{e}^{-\sqrt{\varepsilon}\delta(t-s)/2}\,ds \le \frac{2}{\sqrt{\varepsilon}\delta}\,\mathrm{e}^{-\sqrt{\varepsilon}\delta m} < \frac{2}{\delta^3},$$

since $m > \varepsilon^{-3/4}$ and $\theta^2\mathrm{e}^{-\theta} < 1$, when $\theta > 0$. Next,

$$\int_{(2j-3)m}^{(2j-1)m}\mathrm{e}^{-\sqrt{\varepsilon}\delta(t-s)/2}\,\varphi(s)\,ds \le \int_{-m-\alpha_{j-1}}^{m-\alpha_{j-1}}\mathrm{e}^{-\beta|s|}\,ds \le 2\int_0^\infty \mathrm{e}^{-\beta s}\,ds \le 2\beta^{-1},$$

and similarly

$$\int_{(2j-1)m}^{t}\mathrm{e}^{-\sqrt{\varepsilon}\delta(t-s)/2}\,\varphi(s)\,ds \le \int_{-\infty}^{\infty}\mathrm{e}^{-\beta|s|}\,ds \le 2\beta^{-1}.$$

Plugging everything together and using (5.1.12) and $\omega_{i,\varepsilon} \ge \mu_i^2$ since $\varepsilon\delta < \sqrt{3}\mu_i^2$, we obtain

$$\|z_i\|_\infty \le \frac{4}{\mu_i^2}\left[\frac{k_1}{\delta} + k_2\sqrt{\varepsilon}\left(\frac{1}{\delta^3}+\frac{2}{\beta}\right)\right].$$

Thus (a) follows from (5.1.15) and (5.1.16). Now we prove (b). For $(2j-1)m < t \le (2j+1)m$, write

$$\frac{\omega_{i,\varepsilon}}{2\sqrt{\varepsilon}}z_i(t) = \zeta_{i,j} + \tilde{z}_{i,j}(t) \tag{5.1.23}$$

with

$$\zeta_{i,j} = \int_{-\infty}^{(2j-3)m}\mathrm{e}^{-\sqrt{\varepsilon}\delta(t-s)/2}\sin\left(\frac{\omega_{i,\varepsilon}}{2\sqrt{\varepsilon}}(t-s)\right)h_i(s)ds,$$

$$\tilde{z}_{i,j}(t) = \int_{(2j-3)m}^{t}\mathrm{e}^{-\sqrt{\varepsilon}\delta(t-s)/2}\sin\left(\frac{\omega_{i,\varepsilon}}{2\sqrt{\varepsilon}}(t-s)\right)h_i(s)ds.$$

From the proof of point (a) we obtain:

$$|\zeta_{i,j}| \le \frac{2}{\sqrt{\varepsilon}\delta}\,\mathrm{e}^{-\sqrt{\varepsilon}\delta m}(k_1+k_2) \le \frac{10\sqrt{\varepsilon}}{\delta^5}(k_1+k_2) \tag{5.1.24}$$

since $\theta^4 \mathrm{e}^{-\theta} \le (4/\mathrm{e})^4 < 5$. On the other hand, by the same method in the above, we obtain

$$\begin{aligned} \left| \int_{(2j-3)m}^{t} \mathrm{e}^{-\sqrt{\varepsilon}\delta(t-s)/2} \cos\left(\frac{\omega_{i,\varepsilon}}{2\sqrt{\varepsilon}}(t-s)\right) \dot{h}_i(s)\, ds \right| &\le \frac{2}{\sqrt{\varepsilon}\delta} k_1 + \frac{4}{\beta} k_2, \\ \left| \int_{(2j-3)m}^{t} \mathrm{e}^{-\sqrt{\varepsilon}\delta(t-s)/2} \sin\left(\frac{\omega_{i,\varepsilon}}{2\sqrt{\varepsilon}}(t-s)\right) \dot{h}_i(s)\, ds \right| &\le \frac{2}{\sqrt{\varepsilon}\delta} k_1 + \frac{4}{\beta} k_2. \end{aligned} \tag{5.1.25}$$

Then, taking

$$\lambda = \frac{\sqrt{\varepsilon}\delta}{2}, \quad \omega = \frac{\omega_{i,\varepsilon}}{2\sqrt{\varepsilon}}$$

and integrating by parts the function of the s variable

$$\mathrm{e}^{-\lambda(t-s)} \sin(\omega(t-s)) h_i(s)$$

in the two intervals $[(2j-3)m, (2j-1)m]$, $[(2j-1)m, t]$ and adding the results we get, using also (5.1.25):

$$\left| \int_{(2j-3)m}^{t} \mathrm{e}^{-\lambda(t-s)} \sin(\omega(t-s)) h_i(s) ds \right| \le \frac{\omega}{\lambda^2+\omega^2} |h_i(t)|$$

$$+ \frac{\lambda+\omega}{\lambda^2+\omega^2} [|h_i((2j-1)m^+)| + |h_i((2j-1)m^-)| + \mathrm{e}^{-2\lambda m} |h_i((2j-3)m^+)|]$$

$$+ \frac{\lambda+\omega}{\lambda^2+\omega^2} \left[\frac{2}{\sqrt{\varepsilon}\delta} k_1 + \frac{4}{\beta} k_2\right] \le \frac{\lambda+\omega}{\lambda^2+\omega^2} \left[(3+\mathrm{e}^{-2\lambda m})(k_1+k_2) + \frac{2}{\sqrt{\varepsilon}\delta} k_1 + \frac{4}{\beta} k_2\right].$$

Finally, since

$$\frac{\varepsilon\delta + \omega_{i,\varepsilon}}{\omega_{i,\varepsilon}(\varepsilon^2\delta^2 + \omega_{i,\varepsilon}^2)} \le \frac{\sqrt{2}}{\omega_{i,\varepsilon}\sqrt{\varepsilon^2\delta^2+\omega_{i,\varepsilon}^2}} = \frac{\sqrt{2}}{2\mu_i^2 \omega_{i,\varepsilon}} \le \frac{\sqrt{2}}{2\mu_i^4} \le \frac{1}{\mu_i^2},$$

we obtain after some algebra:

$$\left| \frac{2\sqrt{\varepsilon}}{\omega_{i,\varepsilon}} \tilde{z}_{i,j}(t) \right| \le \frac{4\varepsilon}{\mu_i^2} \left[(3+\mathrm{e}^{-\sqrt{\varepsilon}\delta m})(k_1+k_2) + \frac{4}{\beta} k_2 + \frac{2}{\sqrt{\varepsilon}\delta} k_1\right].$$

Hence, using (5.1.23), (5.1.24), the assumption $\sqrt{\varepsilon} < 2\delta^2$ and the fact that $\mathrm{e}^{-\sqrt{\varepsilon}\delta m} \le \frac{1}{(\sqrt{\varepsilon}\delta m)^2} < \frac{\sqrt{\varepsilon}}{\delta^2}$:

$$\begin{aligned} \|z_i\|_\infty \le & \frac{4}{\mu_i^2} \left\{ \left[\frac{5}{\delta^5} + 3 + \frac{\sqrt{\varepsilon}}{\delta^2}\right] \varepsilon(k_1+k_2) + \frac{4\varepsilon}{\beta} k_2 + \frac{2\sqrt{\varepsilon}}{\delta} k_1 \right\} \\ \le & \frac{4}{\mu_i^2} \left\{ 5\varepsilon \left[\frac{1}{\delta^5} + 1 + \frac{1}{\beta}\right] (k_1+k_2) + \frac{2\sqrt{\varepsilon}}{\delta} k_1 \right\} \end{aligned}.$$

Again, the conclusion follows from (5.1.15) and (5.1.16). The proof is finished. □

In the following we denote by $L_\varepsilon(H_1,H_2,H_3)$ the unique bounded solution of the form (5.1.15) of Eq. (5.1.17) and note that L_ε is a bounded linear map from the space of bounded measurable functions to the space of bounded continuous functions, that is,

$$L_\varepsilon(H_1+\hat{H}_1,H_2+\hat{H}_2,H_3+\hat{H}_2)=L_\varepsilon(H_1,H_2,H_3)+L_\varepsilon(\hat{H}_1,\hat{H}_2,\hat{H}_3).$$

We now study the linear non-homogeneous equation

$$\begin{aligned} &\ddot{x}_1+\frac{8}{\pi}f'\Big(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\Big)x_1=h(t)\,,\\ &\dot{x}_1(2jm+\alpha_j)=0, \quad \text{for any } j\in\mathbb{Z} \text{ such that } e_j=1\,. \end{aligned} \tag{5.1.26}$$

Here $h\in L^\infty(\mathbb{R})$, and $x_1(t)$, $\dot{x}_1(t)$ are absolutely continuous functions so that (5.1.26) holds almost everywhere. Let us put

$$a=\sqrt{8|f'(0)|/\pi}\,.$$

Lemma 5.1.2. *There exist positive constants $A,B,C\in\mathbb{R}$ and $m_0\in\mathbb{N}$ so that for any $\xi=(E,\alpha)\in X$, $m\geq m_0$, and $j\in\mathbb{Z}$, there exist linear functionals $\mathscr{L}_{m,\xi,j}:L^\infty(\mathbb{R})\to\mathbb{R}$, so that $\|\mathscr{L}_{m,\xi,j}\|\leq Ae_j\mathrm{e}^{-am}$, with the property that if $h\in L^\infty(\mathbb{R})$ then* (5.1.26) *has a unique C^1 solution $x_1(t,\xi)$ bounded on $\mathbb{R}$ if and only if*

$$\mathscr{L}_{m,\xi,j}h+\int_{(2j-1)m}^{(2j+1)m}\dot{\gamma}_\xi(t)h(t)\,dt=0 \tag{5.1.27}$$

for any $j\in\mathbb{Z}$. Moreover, the following properties hold:

(i)
$$\|x_1(\cdot,\xi)\|_\infty\leq B\|h\|_\infty,\quad \|\dot{x}_1(\cdot,\xi)\|_\infty\leq B\|h\|_\infty. \tag{5.1.28}$$

(ii) *Let $x_p(t)$ be the unique bounded solution of equation $\ddot{x}_p+\frac{8}{\pi}f'(0)x_p=h(t)$, then*

$$|x_1(t,\xi)-x_p(t)|\leq C\big(\mathrm{e}^{-am/2}+\mathrm{e}^{-a|t-2jm-\alpha_j|/2}\big)\|h\|_\infty \tag{5.1.29}$$

for $(2j-1)m\leq t\leq(2j+1)m$ and any $j\in\mathbb{Z}$.

(iii) *Let $\xi'=(E,\alpha')$, $\xi''=(E,\alpha'')$ with $\alpha',\alpha''\in\ell_E^\infty$ and ξ be either ξ' or ξ''. Assume that $h(t,\xi)\in L^\infty(\mathbb{R})$ satisfies* (5.1.27). *Then there exists a constant, c_1, independent of ξ, so that the following holds:*

$$\begin{aligned} &\max\{\|x_1(\cdot,\xi')-x_1(\cdot,\xi'')\|_\infty,\|\dot{x}_1(\cdot,\xi')-\dot{x}_1(\cdot,\xi'')\|_\infty\}\\ &\leq B\|h(t,\xi')-h(t,\xi'')\|_\infty+c_1\|h(t,\xi'')\|_\infty\|\alpha'-\alpha''\|_\infty. \end{aligned} \tag{5.1.30}$$

Finally, for any $m\geq m_0$, the map $\mathscr{L}_m:X\times L^\infty(\mathbb{R})\to\ell^\infty(\mathbb{R})$ defined as $\mathscr{L}_m(\xi,h)=\{\mathscr{L}_{m,\xi,j}h\}_{j\in\mathbb{Z}}$ is Lipschitz in $\alpha\in\ell_E^\infty$ uniformly with respect to (E,m).

Proof. The equation

$$\ddot{x}+\frac{8}{\pi}f'\left(\frac{2}{\sqrt{\pi}}\gamma_1(t)\right)x=0 \tag{5.1.31}$$

has a fundamental solution $u(t), v(t)$ with

$$u(0)=1, \quad \dot{u}(0)=0, \quad v(0)=0, \quad \dot{v}(0)=1\,.$$

Then v is bounded, odd and u is unbounded, even with asymptotic properties:

$$v(t), \dot{v}(t)\sim \mathrm{e}^{-a|t|}, u(t), \dot{u}(t)\sim \mathrm{e}^{a|t|} \text{ as } t\to\pm\infty\,.$$

Note that $\dot{\gamma}_1(t)$ is a solution of (5.1.31) so that $\dot{\gamma}_1(t)\sim \mathrm{e}^{-a|t|}$ and $\dot{\gamma}_1(0)=0$, $\ddot{\gamma}_1(0)\neq 0$, we get $v(t)=\frac{\dot{\gamma}_1(t)}{\ddot{\gamma}_1(0)}$. Let us pause for a moment to recall some of the properties of the functions $u(t)$, $v(t)$ that will be used later. Equation (5.1.31), or, as a system

$$\dot{u}_1=u_2, \quad \dot{u}_2=-\frac{8}{\pi}f'\left(\frac{2}{\sqrt{\pi}}\gamma_1(t)\right)u_1, \tag{5.1.32}$$

has an exponential dichotomy on $\mathbb{R}_+$ and $\mathbb{R}_-$ with exponent a (cf Section 2.5.1). Thus projections P_+, P_- exist so that $\operatorname{rank}P_+=\operatorname{rank}P_-=1$ and

$$\begin{aligned}
&\|X(t)P_+X^{-1}(s)\|\le k\mathrm{e}^{-a(t-s)}, &&\text{if}\quad 0\le s\le t,\\
&\|X(t)(\mathbb{I}-P_+)X^{-1}(s)\|\le k\mathrm{e}^{a(t-s)}, &&\text{if}\quad 0\le t\le s,\\
&\|X(t)P_-X^{-1}(s)\|\le k\mathrm{e}^{-a(t-s)}, &&\text{if}\quad s\le t\le 0,\\
&\|X(t)(\mathbb{I}-P_-)X^{-1}(s)\|\le k\mathrm{e}^{a(t-s)}, &&\text{if}\quad t\le s\le 0
\end{aligned} \tag{5.1.33}$$

where

$$X(t)=\begin{pmatrix} u(t) & v(t)\\ \dot{u}(t) & \dot{v}(t)\end{pmatrix}$$

is the fundamental matrix of (5.1.32) so that $X(0)=\mathbb{I}$. Although P_+ and P_- are not uniquely defined, $\mathscr{R}P_+$ and $\mathscr{N}P_-$ are precisely the one–dimensional vector spaces consisting of all initial conditions one has to assign to the linear system (5.1.32) to obtain solutions bounded on $\mathbb{R}_+$, $\mathbb{R}_-$ respectively. Moreover, any projection possessing $\mathscr{R}P_+$ as range (resp. $\mathscr{N}P_-$ as kernel) satisfies conditions (5.1.33). Now, since $v(t), \dot{v}(t)\to 0$, as $|t|\to\infty$, we see that we can take:

$$P_+\begin{pmatrix}x_1\\ x_2\end{pmatrix}=(\mathbb{I}-P_-)\begin{pmatrix}x_1\\ x_2\end{pmatrix}=x_2\begin{pmatrix}0\\ 1\end{pmatrix}.$$

Hence the matrix of P_+ and $\mathbb{I}-P_-$ with respect to the canonical basis of $\mathbb{R}^2$ is $\begin{pmatrix}0&0\\0&1\end{pmatrix}$. Then Eqs. (5.1.33) read:

$$|v(t)\dot{u}(s)|, |v(t)u(s)|, |\dot{v}(t)\dot{u}(s)|, |\dot{v}(t)u(s)|\le k\mathrm{e}^{-a|t-s|} \tag{5.1.34}$$

if $0 \le s \le t$ or $t \le s \le 0$, whereas

$$|u(t)\dot{v}(s)|, |u(t)v(s)|, |\dot{u}(t)\dot{v}(s)|, |\dot{u}(t)v(s)| \le k\mathrm{e}^{-a|t-s|} \tag{5.1.35}$$

if $0 \le t \le s$ or $s \le t \le 0$. Now, let us go back to the proof of the Lemma. We consider Eq. (5.1.26) on $[(2j-1)m, (2j+1)m]$ according to $e_j = 0$ or $e_j = 1$. When $e_j = 0$ (5.1.26) has the general solution

$$\begin{aligned} x_1(t) = & -\frac{1}{2a}\int_{(2j-1)m}^{t} \mathrm{e}^{-a(t-s)}\,h(s)\,ds - \frac{1}{2a}\int_{t}^{(2j+1)m} \mathrm{e}^{a(t-s)}\,h(s)\,ds \\ & + a_j\,\mathrm{e}^{a(t-(2j+1)m)} + b_j\,\mathrm{e}^{-a(t-(2j-1)m)} \end{aligned} \tag{5.1.36}$$

with $a_j, b_j \in \mathbb{R}$. When $e_j = 1$ we distinguish between $t \in [2jm + \alpha_j, (2j+1)m]$ and $t \in [(2j-1)m, 2jm + \alpha_j]$. If $t \in [2jm + \alpha_j, (2j+1)m]$ we write the general solution of Equation (5.1.26) with the condition $\dot{x}_1(2jm + \alpha_j) = 0$ as

$$\begin{aligned} x_1(t) = & \int_{2jm+\alpha_j}^{t} v(t-2jm-\alpha_j)u(s-2jm-\alpha_j)h(s)\,ds \\ & + \int_{t}^{(2j+1)m} u(t-2jm-\alpha_j)v(s-2jm-\alpha_j)h(s)\,ds \\ & + a_j^{+}u(t-2jm-\alpha_j)/u(m-\alpha_j) \end{aligned} \tag{5.1.37}$$

where $a_j^{+} \in \mathbb{R}$. If $t \in [(2j-1)m, 2jm + \alpha_j]$ we take

$$\begin{aligned} x_1(t) = & -\int_{t}^{2jm+\alpha_j} v(t-2jm-\alpha_j)u(s-2jm-\alpha_j)h(s)\,ds \\ & - \int_{(2j-1)m}^{t} u(t-2jm-\alpha_j)v(s-2jm-\alpha_j)h(s)\,ds \\ & + a_j^{-}u(t-2jm-\alpha_j)/u(-m-\alpha_j) \end{aligned} \tag{5.1.38}$$

where $a_j^{-} \in \mathbb{R}$. We note that $\dot{x}_1(2jm + \alpha_j) = 0$ in both (5.1.37) and (5.1.38). Thus to obtain a C^1 solution we only need that

$$x_1((2jm+\alpha_j)_-) = x_1((2jm+\alpha_j)_+), \quad \text{for any } j \in \mathbb{Z} \text{ such that } e_j = 1,$$

that is,

$$\int_{(2j-1)m}^{(2j+1)m} v(s-2jm-\alpha_j)h(s)\,ds = \frac{a_j^{-}}{u(-m-\alpha_j)} - \frac{a_j^{+}}{u(m-\alpha_j)}\,. \tag{5.1.39}$$

We note that from Eq. (5.1.36) we get, for any $j \in \mathbb{Z}$:

$$\sup_{(2j-1)m \le t \le (2j+1)m} |x_1(t)| \le |a_j| + |b_j| + \frac{1}{a^2}\operatorname{ess\,sup}_{(2j-1)m \le t \le (2j+1)m}|h(t)| \tag{5.1.40}$$

and

$$\sup_{(2j-1)m\le t\le(2j+1)m}|\dot{x}_1(t)|\le a(|a_j|+|b_j|)+\frac{1}{a}\operatorname{ess\,sup}_{(2j-1)m\le t\le(2j+1)m}|h(t)|. \tag{5.1.41}$$

A similar conclusion also follows (when $e_j=1$) from (5.1.37) and (5.1.38) using (5.1.34), (5.1.35). Equation (5.1.39) is the compatibility condition where the linear maps $\mathscr{L}_{m,\xi,j}$ come from. For the moment, we forget about these conditions and choose the constants a_j, b_j, a_j^+, a_j^- so that the equalities

$$\begin{aligned}
x_1(((2j+1)m)_-)&=x_1((2j+1)m)_+), \quad j\in\mathbb{Z}\\
\dot{x}_1(((2j+1)m)_-)&=\dot{x}_1((2j+1)m)_+), \quad j\in\mathbb{Z}
\end{aligned} \tag{5.1.42}$$

are satisfied. According to the values of e_j, e_{j+1} they read

$$\begin{aligned}
&a_j-b_{j+1}+b_j\,\mathrm{e}^{-2am}-a_{j+1}\,\mathrm{e}^{-2am}\\
&\quad=\frac{1}{2a}\int_{(2j-1)m}^{(2j+1)m}\mathrm{e}^{-a((2j+1)m-s)}\,h(s)\,ds-\frac{1}{2a}\int_{(2j+1)m}^{(2j+3)m}\mathrm{e}^{a((2j+1)m-s)}\,h(s)\,ds,\\
&a_j+b_{j+1}-b_j\,\mathrm{e}^{-2am}-a_{j+1}\,\mathrm{e}^{-2am}\\
&\quad=-\frac{1}{2a}\int_{(2j-1)m}^{(2j+1)m}\mathrm{e}^{-a((2j+1)m-s)}\,h(s)\,ds-\frac{1}{2a}\int_{(2j+1)m}^{(2j+3)m}\mathrm{e}^{a((2j+1)m-s)}\,h(s)\,ds,
\end{aligned} \tag{5.1.43}$$

if $e_j=e_{j+1}=0$, or

$$\begin{aligned}
&a_j-a_{j+1}^-+b_j\,\mathrm{e}^{-2am}\\
&\quad=\frac{1}{2a}\int_{(2j-1)m}^{(2j+1)m}\mathrm{e}^{-a((2j+1)m-s)}\,h(s)\,ds\\
&\qquad-\int_{(2j+1)m}^{2(j+1)m+\alpha_{j+1}}v(-m-\alpha_{j+1})u(s-2(j+1)m-\alpha_{j+1})h(s)\,ds,\\
&a_j-a_{j+1}^-\frac{\dot{u}(-m-\alpha_{j+1})}{au(-m-\alpha_{j+1})}-b_j\,\mathrm{e}^{-2am}\\
&\quad=-\frac{1}{2a}\int_{(2j-1)m}^{(2j+1)m}\mathrm{e}^{-a((2j+1)m-s)}\,h(s)\,ds\\
&\qquad-\frac{1}{a}\int_{(2j+1)m}^{2(j+1)m+\alpha_{j+1}}\dot{v}(-m-\alpha_{j+1})u(s-2(j+1)m-\alpha_{j+1})h(s)\,ds,
\end{aligned} \tag{5.1.44}$$

if $e_j=0$, $e_{j+1}=1$, or

$$
\begin{aligned}
&a_j^+ - b_{j+1} - a_{j+1}\,\mathrm{e}^{-2am} \\
&\quad = -\frac{1}{2a}\int_{(2j+1)m}^{(2j+3)m} \mathrm{e}^{a((2j+1)m-s)}\,h(s)\,ds \\
&\qquad - \int_{2jm+\alpha_j}^{(2j+1)m} v(m-\alpha_j)u(s-2jm-\alpha_j)h(s)\,ds, \\
&a_j^+ \frac{\dot u(m-\alpha_j)}{au(m-\alpha_j)} + b_{j+1} - a_{j+1}\,\mathrm{e}^{-2am} \\
&\quad = -\frac{1}{2a}\int_{(2j+1)m}^{(2j+3)m} \mathrm{e}^{a((2j+1)m-s)}\,h(s)\,ds \\
&\qquad - \frac{1}{a}\int_{2jm+\alpha_j}^{(2j+1)m} \dot v(m-\alpha_j)u(s-2jm-\alpha_j)h(s)\,ds,
\end{aligned}
\tag{5.1.45}
$$

if $e_j = 1$, $e_{j+1} = 0$, or

$$
\begin{aligned}
&a_j^+ - a_{j+1}^- \\
&\quad = -\int_{(2j+1)m}^{2(j+1)m+\alpha_{j+1}} v(-m-\alpha_{j+1})u(s-2(j+1)m-\alpha_{j+1})h(s)\,ds \\
&\qquad - \int_{2jm+\alpha_j}^{(2j+1)m} v(m-\alpha_j)u(s-2jm-\alpha_j)h(s)\,ds, \\
&a_j^+ \frac{\dot u(m-\alpha_j)}{au(m-\alpha_j)} - a_{j+1}^- \frac{\dot u(-m-\alpha_{j+1})}{au(-m-\alpha_{j+1})} \\
&\quad = -\frac{1}{a}\int_{(2j+1)m}^{2(j+1)m+\alpha_{j+1}} \dot v(-m-\alpha_{j+1})u(s-2(j+1)m-\alpha_{j+1})h(s)\,ds \\
&\qquad - \frac{1}{a}\int_{2jm+\alpha_j}^{(2j+1)m} \dot v(m-\alpha_j)u(s-2jm-\alpha_j)h(s)\,ds,
\end{aligned}
\tag{5.1.46}
$$

if $e_j = e_{j+1} = 1$. We note that when $\xi = (E,\alpha)$ is fixed, for any $j \in \mathbb{Z}$ only one among Equations (5.1.44)–(5.1.46) occurs. We consider these equations as a unique equation for the variable

$$\{(\tilde a_j, \tilde b_j)\}_{j\in\mathbb{Z}} \in \ell^\infty \times \ell^\infty$$

where $(\tilde a_j, \tilde b_j) = (a_j, b_j)$ if $e_j = 0$ whereas $(\tilde a_j, \tilde b_j) = (a_j^-, a_j^+)$ if $e_j = 1$. The left-hand sides of (5.1.44)–(5.1.46) define a linear bounded operator

$$
L_{m,\xi} : \ell^\infty \times \ell^\infty \to \ell^\infty \times \ell^\infty, \quad L_{m,\xi}\begin{pmatrix}\{\tilde a_j\} \\ \{\tilde b_j\}\end{pmatrix} = \begin{pmatrix}\{\hat a_j\} \\ \{\hat b_j\}\end{pmatrix}
\tag{5.1.47}
$$

where

$$\begin{aligned}
\hat{a}_j =& (1-e_j)\widetilde{a}_j - [e_{j+1} + (1-e_{j+1})\,\mathrm{e}^{-2am}]\widetilde{a}_{j+1} \\
&+ [e_j + (1-e_j)\,\mathrm{e}^{-2am}]\widetilde{b}_j - (1-e_{j+1})\widetilde{b}_{j+1}\,, \\
\hat{b}_j =& (1-e_j)\widetilde{a}_j - \left[\frac{\dot{u}(-m-\alpha_{j+1})}{au(-m-\alpha_{j+1})} e_{j+1} + (1-e_{j+1})\,\mathrm{e}^{-2am}\right]\widetilde{a}_{j+1} \\
&+ \left[\frac{\dot{u}(m-\alpha_j)}{au(m-\alpha_j)} e_j - (1-e_j)\,\mathrm{e}^{-2am}\right]\widetilde{b}_j + (1-e_{j+1})\widetilde{b}_{j+1}\,.
\end{aligned} \tag{5.1.48}$$

Now, since $0 \le 1-e_j \le 1$, $|\alpha_j| \le 2$, and

$$\lim_{t\to\pm\infty} \frac{\dot{u}(t)}{au(t)} = \pm 1 \tag{5.1.49}$$

we see that $m_0 \in \mathbb{N}$ exists so that for any $m \ge m_0$, $\xi \in X$ and $j \in \mathbb{Z}$, we have

$$|\hat{a}_j| < 3(\|\widetilde{a}\|_\infty + \|\widetilde{b}\|_\infty), \qquad |\hat{b}_j| < 3(\|\widetilde{a}\|_\infty + \|\widetilde{b}\|_\infty)$$

or $\|L_{m,\xi}\| < 6$. Now, we want to show that for m sufficiently large and any $\xi \in X$, the map $L_{m,\xi} : \ell^\infty \times \ell^\infty \to \ell^\infty \times \ell^\infty$ is invertible. To this end, we claim that when $m \to \infty$, the linear map $L_{m,\xi}$ tends to the map L_E defined as follows:

$$L_E \begin{pmatrix} \{\widetilde{a}_j\} \\ \{\widetilde{b}_j\} \end{pmatrix} = \begin{pmatrix} \{(1-e_j)\widetilde{a}_j - e_{j+1}\widetilde{a}_{j+1} + e_j\widetilde{b}_j - (1-e_{j+1})\widetilde{b}_{j+1}\} \\ \{(1-e_j)\widetilde{a}_j + e_{j+1}\widetilde{a}_{j+1} + e_j\widetilde{b}_j + (1-e_{j+1})\widetilde{b}_{j+1}\} \end{pmatrix}$$

in the sense that

$$\|L_{m,\xi} - L_E\| \to 0 \tag{5.1.50}$$

as $m \to \infty$ uniformly with respect to $\xi = (E,\alpha) \in X$. In fact,

$$\begin{aligned}
&(L_{m,\xi} - L_E)\begin{pmatrix} \{\widetilde{a}_j\} \\ \{\widetilde{b}_j\} \end{pmatrix} \\
&= \begin{pmatrix} \{(e_{j+1}-1)\,\mathrm{e}^{-2am}\,\widetilde{a}_{j+1} + (1-e_j)\,\mathrm{e}^{-2am}\,\widetilde{b}_j\} \\ \left\{ \begin{array}{c} \left[\left(\dfrac{\dot{u}(m-\alpha_j)}{au(m-\alpha_j)} - 1\right) e_j - (1-e_j)\,\mathrm{e}^{-2am}\right]\widetilde{b}_j \\ -\left[\left(\dfrac{\dot{u}(-m-\alpha_{j+1})}{au(-m-\alpha_{j+1})} + 1\right) e_{j+1} + (1-e_{j+1})\,\mathrm{e}^{-2am}\right]\widetilde{a}_{j+1} \end{array} \right\} \end{pmatrix}.
\end{aligned}$$

Thus (5.1.50) follows from (5.1.49) and $\|\alpha\| \le 2$. Next, the equation:

$$L_E \begin{pmatrix} \{\widetilde{a}_j\} \\ \{\widetilde{b}_j\} \end{pmatrix} = \begin{pmatrix} \{\bar{A}_j\} \\ \{\bar{B}_j\} \end{pmatrix}$$

is equivalent to the infinite dimensional system ($j \in \mathbb{Z}$):

$$\begin{cases} (1-e_j)\widetilde{a}_j + e_j\widetilde{b}_j = \dfrac{\bar{A}_j+\bar{B}_j}{2}, \\ e_{j+1}\widetilde{a}_{j+1} + (1-e_{j+1})\widetilde{b}_{j+1} = \dfrac{\bar{B}_j-\bar{A}_j}{2}. \end{cases}$$

Changing j with $j-1$ we obtain

$$\begin{cases} (1-e_{j-1})\widetilde{a}_{j-1} + e_{j-1}\widetilde{b}_{j-1} = \dfrac{\bar{A}_{j-1}+\bar{B}_{j-1}}{2}, \\ e_j\widetilde{a}_j + (1-e_j)\widetilde{b}_j = \dfrac{\bar{B}_{j-1}-\bar{A}_{j-1}}{2}. \end{cases}$$

Thus, for any $j \in \mathbb{Z}$, $(\widetilde{a}_j, \widetilde{b}_j)$ satisfies

$$\begin{cases} e_j\widetilde{a}_j + (1-e_j)\widetilde{b}_j = \dfrac{\bar{B}_{j-1}-\bar{A}_{j-1}}{2}, \\ (1-e_j)\widetilde{a}_j + e_j\widetilde{b}_j = \dfrac{\bar{A}_j+\bar{B}_j}{2}, \end{cases}$$

which is a linear system in the unknown $(\widetilde{a}_j, \widetilde{b}_j)$ having the solution

$$\widetilde{a}_j = \frac{1}{2}\frac{(1-e_j)(\bar{A}_j+\bar{B}_j)+e_j(\bar{A}_{j-1}-\bar{B}_{j-1})}{1-2e_j},$$
$$\widetilde{b}_j = \frac{1}{2}\frac{(1-e_j)(\bar{B}_{j-1}-\bar{A}_{j-1})-e_j(\bar{A}_j+\bar{B}_j)}{1-2e_j}.$$

Since e_j is either 0 or 1 we see that $|1-2e_j| = 1$ and then

$$|\widetilde{a}_j|, |\widetilde{b}_j| \le \frac{1}{2}(|\bar{A}_{j-1}|+|\bar{A}_j|) + \frac{1}{2}(|\bar{B}_{j-1}|+|\bar{B}_j|)$$

or

$$\|\widetilde{a}\|_\infty + \|\widetilde{b}\|_\infty \le 2(\|\bar{A}\|_\infty + \|\bar{B}\|_\infty).$$

That is, L_E^{-1} exists and $\|L_E^{-1}\| \le 2$. As a consequence, for any m sufficiently large and $\xi \in X$, $L_{m,\xi}$ has a bounded inverse $L_{m,\xi}^{-1}$ so that, say,

$$\|L_{m,\xi}^{-1}\| \le 3\,. \tag{5.1.51}$$

Thus we can uniquely solve Eqs. (5.1.44)–(5.1.46) for $\tilde{a}_j = \tilde{a}_j(h,\xi)$, $\tilde{b}_j = \tilde{b}_j(h,\xi)$ and a constant $\tilde{c}$ independent of $\xi \in X$ and $m \in \mathbb{N}$ (provided $m \ge m_0$, with m_0 sufficiently large) exists so that

$$|\tilde{a}_j(h,\xi)| \le \tilde{c}\|h\|_\infty, \quad |\tilde{b}_j(h,\xi)| \le \tilde{c}\|h\|_\infty \tag{5.1.52}$$

for any $j \in \mathbb{Z}$. Consequently, the compatibility condition (5.1.39) reads

$$\int_{(2j-1)m}^{(2j+1)m} \dot{\gamma}_\xi(s)h(s)\,ds = -\mathscr{L}_{m,\xi,j}(h) := \ddot{\gamma}_1(0)\left[\frac{a_j^-(h,\xi)}{u(-m-\alpha_j)} - \frac{a_j^+(h,\xi)}{u(m-\alpha_j)}\right]$$

for any $j \in \mathbb{Z}$ so that $e_j = 1$. Since we do not need any compatibility condition when $e_j = 0$, we set

$$\mathscr{L}_{m,\xi,j}(h) = 0 \quad \text{for any } j \in \mathbb{Z} \text{ such that } e_j = 0.$$

Clearly, the existence of a constant $B > 0$ so that Equation (5.1.28) holds, following from Eqs. (5.1.40), (5.1.41) and (5.1.52). Similarly the existence of the constant A as in the statement of the Lemma follows from (5.1.52) together with the fact that $|\alpha_j| \leq 2$ for any $j \in \mathbb{Z}$ and $u(t) \sim \mathrm{e}^{a|t|}$ as $|t| \to \infty$.

Now we estimate $\bar{v}(t) = x_1(t) - x_p(t)$, $x_p(t)$ being the unique bounded solution of the equation $\ddot{x} + \frac{8}{\pi}f'(0)x = h(t)$. Observe that $\bar{v}(t)$ is a C^1 solution, bounded on $\mathbb{R}$, of the differential equation:

$$\ddot{x} + \frac{8}{\pi}f'(0)x + w(t) = 0$$

where $w(t) = \frac{8}{\pi}\Big(f'\Big(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\Big) - f'(0)\Big)x_1(t)$. Thus

$$\bar{v}(t) = \frac{1}{2a}\int_{-\infty}^{t} \mathrm{e}^{-a(t-s)}\,w(s)\,ds + \frac{1}{2a}\int_{t}^{\infty} \mathrm{e}^{a(t-s)}\,w(s)\,ds\,.$$

Let $A_1 = 1 + \max_{t\in\mathbb{R}}|\gamma(t)|$ and $N = \max_{x\in[-A_1,A_1]}\{|f'(x)|,|f''(x)|\}$.Then

$$|w(s)| \leq \frac{16}{\pi\sqrt{\pi}}BN\|h\|_\infty|\gamma_\xi(s)|$$

and hence

$$|\bar{v}(t)| \leq \frac{16BN\|h\|_\infty}{2a\pi\sqrt{\pi}}\left\{\int_{-\infty}^{t} \mathrm{e}^{-a(t-s)}\,|\gamma_\xi(s)|\,ds + \int_{t}^{\infty} \mathrm{e}^{a(t-s)}\,|\gamma_\xi(s)|\,ds\right\}.$$

So, we consider the integrals

$$I(t,\xi) := \int_{-\infty}^{t} \mathrm{e}^{-a(t-s)}\,|\gamma_\xi(s)|\,ds, \quad J(t,\xi) := \int_{t}^{\infty} \mathrm{e}^{a(t-s)}\,|\gamma_\xi(s)|\,ds\,.$$

For any $\xi = (E,\alpha) \in X$, $E = \{e_j\}_{j\in\mathbb{Z}} \in \mathscr{E}$, $\alpha := \{\alpha_j\}_{j\in\mathbb{Z}} \in \ell_E^\infty$, let $\widetilde{\xi} = (\widetilde{E},\widetilde{\alpha}) \in X$ be defined as

$$\widetilde{E} := \{e_{-j}\}_{j\in\mathbb{Z}} \in \mathscr{E}, \qquad \widetilde{\alpha} := \{-\alpha_{-j}\}_{j\in\mathbb{Z}} \in \ell_{\widetilde{E}}^\infty.$$

From the definitions of $\gamma_\xi(t)$ and $\gamma_1(t) = \gamma_1(-t)$ we see that $\gamma_\xi(t) = \gamma_{\widetilde{\xi}}(-t)$ for any $t \in \mathbb{R}$, $t \neq (2j-1)m$, $j \in \mathbb{Z}$, and then

$$J(t,\xi) = \int_{-\infty}^{-t} \mathrm{e}^{a(t+s)} |\gamma_\xi(-s)|\, ds = \int_{-\infty}^{-t} \mathrm{e}^{-a(-t-s)} |\gamma_{\widetilde{\xi}}(s)|\, ds = I(-t,\widetilde{\xi}).$$

Thus we see that it is enough to estimate $I(t,\xi)$. Let $(2j-1)m < t \leq (2j+1)m$. We have

$$\int_{-\infty}^{(2j-3)m} \mathrm{e}^{-a(t-s)} |\gamma_\xi(s)|\, ds \leq \frac{A_1}{a}\, \mathrm{e}^{-2am} < \frac{A_1}{a}\, \mathrm{e}^{-am/2}\,.$$

Next, we estimate

$$\int_{(2j-3)m}^{(2j-1)m} \mathrm{e}^{-a(t-s)} |\gamma_\xi(s)|\, ds, \qquad \int_{(2j-1)m}^{t} \mathrm{e}^{-a(t-s)} |\gamma_\xi(s)|\, ds\,.$$

Since $\gamma_\xi(t) = 0$ if $(2i-1)m < t \leq (2i+1)m$ and $e_i = 0$ we see that we can assume that $e_{j-1} = e_j = 1$ and

$$\gamma_\xi(t) = \begin{cases} \gamma_1(t - 2(j-1)m - \alpha_{j-1}), & \text{if } (2j-3)m < t \leq (2j-1)m, \\ \gamma_1(t - 2jm - \alpha_j), & \text{if } (2j-1)m < t \leq (2j+1)m. \end{cases}$$

Now, let $A_2 > 0$ be such that

$$\max\Big\{ |\gamma_1(t)|, |\dot{\gamma}_1(t)|, |\ddot{\gamma}_1(t)| \Big\} \leq A_2\, \mathrm{e}^{-a|t|}\,. \tag{5.1.53}$$

Then

$$\int_{(2j-3)m}^{(2j-1)m} \mathrm{e}^{-a(t-s)} |\gamma_\xi(s)|\, ds \leq \int_{(2j-3)m}^{(2j-1)m} \mathrm{e}^{-a(t-s)} |\gamma_1(s - 2(j-1)m - \alpha_{j-1})| ds$$

$$\leq A_2 \int_{(2j-3)m}^{(2j-1)m} \mathrm{e}^{-a(t-s)}\, \mathrm{e}^{-a|s-2(j-1)m-\alpha_{j-1}|}\, ds$$

$$\leq A_2 \int_{2(j-1)m+\alpha_{j-1}}^{(2j-1)m} \mathrm{e}^{-a(t-s)}\, \mathrm{e}^{-a(s-2(j-1)m-\alpha_{j-1})}\, ds + A_2 \int_{(2j-3)m}^{2(j-1)m+\alpha_{j-1}} \mathrm{e}^{-a(t-s)}\, ds$$

$$\leq A_2\, \mathrm{e}^{-a(m-2)}(m+2) + \frac{A_2}{a}\, \mathrm{e}^{-a(m-2)} \leq \frac{A_2(\mathrm{e}^{4a}+1)}{a}\, \mathrm{e}^{-a(m-2)/2}\,.$$

Finally, if $(2j-1)m < t \leq 2jm + \alpha_j$ we have:

$$\int_{(2j-1)m}^{t} \mathrm{e}^{-a(t-s)} |\gamma_\xi(s)|\, ds \leq A_2 \int_{(2j-1)m}^{t} \mathrm{e}^{-a(t-s)}\, \mathrm{e}^{a(s-2jm-\alpha_j)}\, ds$$

$$\leq \frac{A_2}{2a}\, \mathrm{e}^{-a|t-2jm-\alpha_j|} \leq \frac{A_2}{2a}\, \mathrm{e}^{-a|t-2jm-\alpha_j|/2}$$

whereas if $2jm+\alpha_j < t \le (2j+1)m$

$$\int_{(2j-1)m}^{t} e^{-a(t-s)}\,|\gamma_\xi(s)|\,ds$$

$$\le A_2 \int_{(2j-1)m}^{t} e^{-a(t-s)}\, e^{-a|s-2jm-\alpha_j|}\, ds$$

$$\le A_2 \int_{(2j-1)m}^{2jm+\alpha_j} e^{-a(t-s)}\, e^{-a(2jm+\alpha_j-s)}\, ds + A_2 \int_{2jm+\alpha_j}^{t} e^{-a(t-s)}\, e^{-a(s-2jm-\alpha_j)}\, ds$$

$$\le \frac{A_2}{2a}\, e^{-a(t-2jm-\alpha_j)} + A_2\, e^{-a(t-2jm-\alpha_j)}(t-2jm-\alpha_j) \le \frac{3A_2}{2a}\, e^{-a(t-2jm-\alpha_j)/2},$$

since $a\theta\, e^{-a\theta} \le e^{-a\theta/2}$ for any $\theta \ge 0$. The fact that inequality (5.1.29) holds in the closed interval $[(2j-1)m,(2j+1)m]$ follows from continuity. We now prove (iii). Let $w(t) \in C^\infty(\mathbb{R})$ be a smooth function so that $\operatorname{supp} w \in (-1,1)$ and $w'(0) = 1$ and set

$$\hat{x}_1(t) = x_1(t,\xi') - x_1(t,\xi'') + e_j \dot{x}_1(2jm+\alpha_j',\xi'')w(t-2jm-\alpha_j')$$

if $(2j-1)m < t \le (2j+1)m$ and $j \in \mathbb{Z}$. Note that $\hat{x}_1(t)$ is a bounded C^1-function on $\mathbb{R}$ that satisfies, in any interval $((2j-1)m,(2j+1)m]$, the equation:

$$\ddot{x}_1 + \frac{8}{\pi} f'(\frac{2}{\sqrt{\pi}}\gamma_{\xi'}(t))x_1$$

$$= h(t,\xi') - h(t,\xi'')$$

$$+\frac{8}{\pi}\left[f'\left(\frac{2}{\sqrt{\pi}}\gamma_{\xi''}(t)\right) - f'\left(\frac{2}{\sqrt{\pi}}\gamma_{\xi'}(t)\right)\right] x_1(t,\xi'')$$

$$-e_j\dot{x}_1(2jm+\alpha_j',\xi'')\left[\ddot{w}(t-2jm-\alpha_j') + \frac{8}{\pi} f'\left(\frac{2}{\sqrt{\pi}}\gamma_{\xi'}(t)\right) w(t-2jm-\alpha_j')\right]$$

together with $\dot{x}_1(2jm+\alpha_j') = 0$ when $e_j = 1$. Thus, because of (i) and (5.1.22),

$$\max\{\|x_1(\cdot,\xi') - x_1(t,\xi'')\|_\infty, \|\dot{x}_1(\cdot,\xi') - \dot{x}_1(t,\xi'')\|_\infty\}$$

$$\le B\|h(\cdot,\xi') - h(\cdot,\xi'')\|_\infty$$

$$+\tilde{B}\sup_{j\in\mathbb{Z}} |e_j\dot{x}_1(2jm+\alpha_j',\xi'')| + \frac{16B^2N}{\pi\sqrt{\pi}} \|h(\cdot,\xi'')\|_\infty \|\gamma_{\xi'} - \gamma_{\xi''}\|_\infty$$

$$\le B\|h(\cdot,\xi') - h(\cdot,\xi'')\|_\infty + \tilde{B}\sup_{j\in\mathbb{Z}} |e_j\dot{x}_1(2jm+\alpha_j',\xi'')|$$

$$+B_1\|h(\cdot,\xi'')\|_\infty\, \|\alpha' - \alpha''\| \tag{5.1.54}$$

for some choice of the positive constants B_1 and $\tilde{B}$. On the other hand, when $e_j = 1$, we have, since $\dot{x}_1(2jm+\alpha_j'', \xi'') = 0$,

$$\dot{x}_1(2jm+\alpha_j', \xi'') = \int_{2jm+\alpha_j''}^{2jm+\alpha_j'} \ddot{x}_1(t,\xi'')dt$$

$$= \int_{2jm+\alpha_j''}^{2jm+\alpha_j'} \left(h(t,\xi'') - \frac{8}{\pi} f'\left(\frac{2}{\sqrt{\pi}}\gamma_{\xi''}(t)\right) x_1(t,\xi'')\right) dt$$

and hence

$$\begin{aligned} |\dot{x}_1(2jm+\alpha_j',\xi'')| &\le \left[1+\frac{8B}{\pi}|f'(0)|\right] \|h(\cdot,\xi'')\|_\infty |\alpha_j' - \alpha_j''| \\ &\quad + \frac{8B}{\pi}\|h(\cdot,\xi'')\|_\infty \int_0^{\alpha_j'-\alpha_j''} \left| f'\left(\frac{2}{\sqrt{\pi}}\gamma_1(t)\right) - f'(0)\right| dt \\ &\le \left\{1+\frac{8B}{\pi}[|f'(0)|+A_1 N]\right\} \|h(\cdot,\xi'')\|_\infty |\alpha_j' - \alpha_j''|. \end{aligned} \tag{5.1.55}$$

Then (iii) follows from (5.1.54), (5.1.55). Finally, the proof of Lipschitz continuity of the map $\mathscr{L}_m$ with respect to α is given in Section 5.1.6. □

Now we consider the equation

$$\ddot{x}_2 + \frac{24}{\pi} f'\left(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\right) x_2 = h \in L^\infty(\mathbb{R}) \tag{5.1.56}$$

and prove the following.

Lemma 5.1.3. *There exist positive constants $B_1, C_1 \in \mathbb{R}$ and $m_1 \in \mathbb{N}$, so that for any $\xi = (E,\alpha) \in X$ and $m \ge m_1$, Equation* (5.1.56) *has a unique C^1 solution $x_2(t,\xi)$ which is bounded on $\mathbb{R}$ and satisfies*

$$\|x_2(\cdot,\xi)\|_\infty \le B_1\|h\|_\infty, \quad \|\dot{x}_2(\cdot,\xi)\|_\infty \le B_1\|h\|_\infty. \tag{5.1.57}$$

Moreover the following properties hold:

(i) *Let $z_p(t)$ be the unique bounded solution of equation $\ddot{z}_p + \frac{24}{\pi} f'(0) z_p = h(t)$, then*

$$|x_2(t,\xi) - z_p(t)| \le C_1\left(\mathrm{e}^{-am/2} + \mathrm{e}^{-a|t-2jm-\alpha_j|/2}\right)\|h\|_\infty \tag{5.1.58}$$

for $(2j-1)m \le t \le (2j+1)m$ and any $j \in \mathbb{Z}$.

(ii) *Let $\xi' = (E,\alpha')$, $\xi'' = (E,\alpha'')$ with $\alpha',\alpha'' \in \ell_E^\infty$ and ξ be either ξ' or ξ''. Assume that $h(t,\xi) \in L^\infty(\mathbb{R})$. Then there exists a constant, $\hat{c}_1$, independent of ξ, so that the following holds:*

$$\max\left\{\|x_2(\cdot,\xi')-x_2(\cdot,\xi'')\|_\infty,\|\dot{x}_2(\cdot,\xi')-\dot{x}_2(\cdot,\xi'')\|_\infty\right\}$$
$$\leq B_1\|h(t,\xi')-h(t,\xi'')\|_\infty+\hat{c}_1\|h(t,\xi'')\|_\infty\|\alpha'-\alpha''\|_\infty. \quad (5.1.59)$$

Proof. Since the proof is very similar to that of Lemma 5.1.2 (actually simpler) we only sketch it emphasizing the differences. Because of assumption (H2), the homogeneous equation associated with (5.1.56) has an exponential dichotomy on $\mathbb{R}$, that is, there exists a projection P of rank one so that the fundamental system $X(t)$ of (5.1.56) satisfies:

$$\|X(t)PX^{-1}(s)\|\leq k\mathrm{e}^{-b(t-s)}, \qquad \text{for any } s\leq t,$$
$$\|X(t)(\mathbb{I}-P)X^{-1}(s)\|\leq k\mathrm{e}^{-b(t-s)}, \qquad \text{for any } t\leq s$$

where $b=\sqrt{\frac{24}{\pi}|f'(0)|}$. Let $v_0\in\mathscr{R}P$, $u_0\in\mathscr{N}P$ be unitary vectors, and set

$$\begin{pmatrix}u(t)\\ \dot{u}(t)\end{pmatrix}:=X(t)u_0, \qquad \begin{pmatrix}v(t)\\ \dot{v}(t)\end{pmatrix}:=X(t)v_0.$$

Then it can be proved that (5.1.34) holds for any $t\leq s$ whereas (5.1.35) holds for any $s\leq t$. Now, when $e_j=0$ Equation (5.1.36), with b instead of a, gives the solution to (5.1.56) but now, since when $e_j=1$ we do not impose the condition $\dot{x}_2(2jm+\alpha_j)=0$, we do not need to split the interval $[(2j-1)m,2(j+1)m]$ into two parts and the general solution of (5.1.56) can be written as:

$$\begin{aligned}x_1(t)=&\int_{(2j-1)m}^{t}v(t-2jm)u(s-2jm)h(s)\,ds\\&+\int_{t}^{(2j+1)m}u(t-2jm)v(s-2jm)h(s)\,ds\\&+a_ju(t-2jm)/u(-m)+b_jv(t-2jm)/v(m).\end{aligned}$$

It is easy to see that $x_1(t)$ belongs to $L^\infty(\mathbb{R})$ and is C^1 in any open interval $((2j-1)m,(2j+1)m)$. Thus we obtain a unique bounded C^1 solution of Eq. (5.1.56) provided we show that Eq. (5.1.42) can be uniquely solved. This fact and the properties (i), (ii) are proved in the proof of Lemma 5.1.2 and so we omit it. □

In order to apply Lemma 5.1.2, we consider the set

$$\mathscr{S}_{m,\xi}:=\left\{h\in L^\infty(\mathbb{R})\mid \mathscr{L}_{m,\xi,j}h+\int_{(2j-1)m}^{(2j+1)m}\dot{\gamma}_\xi(t)h(t)\,dt=0 \quad \text{for any} \quad j\in\mathbb{Z}\right\}.$$

Note that if $\xi=0$ (i.e. $(E,\alpha)=(0,0)$) then $\mathscr{S}_{m,\xi}=L^\infty(\mathbb{R})$. Then we construct a projection $Q_{m,\xi}:L^\infty(\mathbb{R})\to\mathscr{S}_{m,\xi}$ as follows. If $\xi=0$ we set $Q_{m,\xi}=\mathbb{I}$, whereas if $\xi\neq0$ (and hence $E\neq0$) we proceed in the following way. For any $c=\{c_i\}_{i\in\mathbb{Z}}\in\ell_E^\infty$, we put

$$\gamma_c(t) = c_j \dot{\gamma}_\xi(t) \quad \text{for} \quad (2j-1)m < t \leq (2j+1)m.$$

We recall that $\ell_E^\infty := \Big\{ c = \{c_i\}_{i\in\mathbb{Z}} \in \ell^\infty \mid c_i = 0 \quad \text{for} \quad e_i = 0 \Big\}$. Hence $\gamma_c \in L^\infty(\mathbb{R})$ and

$$|\gamma_c(t)| \leq \|c\|_\infty |\dot{\gamma}_\xi(t)| \leq \|c\|_\infty \|\dot{\gamma}_1\|_\infty.$$

For any $h \in L^\infty(\mathbb{R})$ we take $h_c = h - \gamma_c$ and consider the system of equations

$$\mathscr{L}_{m,\xi,j} h_c + \int_{(2j-1)m}^{(2j+1)m} \dot{\gamma}_\xi(t) h_c(t)\, dt = 0, \quad j \in \mathbb{Z}. \tag{5.1.60}$$

Our purpose is to determine a solution $c \in \ell_E^\infty$ of the above system. Note that when $e_j = 0$, one has $\mathscr{L}_{m,\xi,j} = 0$, $\gamma_\xi(t) = 0$ and then the above equation is trivially satisfied regardless of the value of c_j. This is the reason why we take $c_j = 0$ when $e_j = 0$. On the other hand, since $\gamma_\xi(t) = 0$ in $((2j-1)m, (2j+1)m]$ when $e_j = 0$, the value of c_j does not matter to defining $\gamma_c(t)$ in this interval. We can write (5.1.60) as

$$[\mathscr{M}_{m,\xi} + \mathscr{L}_{m,\xi} G_{m,\xi}] c = [\mathscr{L}_{m,\xi} + N_{m,\xi}] h \tag{5.1.61}$$

where

$$\mathscr{L}_{m,\xi} h = \{\mathscr{L}_{m,\xi,j} h\}_{j\in\mathbb{Z}} \in \ell_E^\infty, \quad \mathscr{M}_{m,\xi} c = \left\{ c_j \int_{(2j-1)m}^{(2j+1)m} \dot{\gamma}_\xi^2(t) dt \right\}_{j\in\mathbb{Z}} \in \ell_E^\infty,$$

$$G_{m,\xi} c = \gamma_c(t) = \sum_{j\in\mathbb{Z}} c_j \dot{\gamma}_\xi(t) \chi_{((2j-1)m,(2j+1)m]}(t) \in L^\infty(\mathbb{R}),$$

$$N_{m,\xi} h = \left\{ \int_{(2j-1)m}^{(2j+1)m} \dot{\gamma}_\xi(t) h(t) dt \right\}_{j\in\mathbb{Z}} \in \ell_E^\infty.$$

Note that for any fixed $E \in \mathscr{E}$, both sides of Eq. (5.1.61) are elements of ℓ_E^∞.

Now, we have already observed that $\|G_{m,\xi} c\|_\infty \leq \|\dot{\gamma}_1\|_\infty \cdot \|c\|_\infty$, moreover, from Lemma 5.1.2 it follows that $\|\mathscr{L}_{m,\xi} h\|_\infty \leq A\,\mathrm{e}^{-am} \|h\|_\infty$. Hence

$$\|\mathscr{L}_{m,\xi} G_{m,\xi} c\|_\infty \leq A\,\mathrm{e}^{-am} \|\dot{\gamma}\|_\infty \cdot \|c\|_\infty. \tag{5.1.62}$$

Next, setting

$$\tilde{A}_1 = \int_{-\infty}^{\infty} |\dot{\gamma}_1(t)|\, dt > 0, \quad \tilde{A}_2 = \int_{-\infty}^{\infty} \dot{\gamma}_1(t)^2\, dt > 0$$

we have, for m sufficiently large, and any $j \in \mathbb{Z}$, with $e_j = 1$

$$\frac{\tilde{A}_2}{2} \leq \left| \int_{(2j-1)m}^{(2j+1)m} \dot{\gamma}_\xi(t)^2 dt \right| = \left| \int_{-m-\alpha_j}^{m-\alpha_j} \dot{\gamma}_1(t)^2 dt \right| \leq \tilde{A}_2$$

since $|\alpha_j| \leq 2$ for any $j \in \mathbb{Z}$. Thus $\mathscr{M}_{m,\xi} : \ell_E^\infty \to \ell_E^\infty$ is a bounded linear map ($\|\mathscr{M}_{m,\xi}\| \leq \tilde{A}_2$) which is invertible and it is easy to see that its inverse $\mathscr{M}_{m,\xi}^{-1}$ satisfies:

$$\frac{1}{\tilde{A}_2} \leq \|\mathscr{M}_{m,\xi}^{-1}\| \leq \frac{2}{\tilde{A}_2}$$

provided $m \in \mathbb{N}$ is sufficiently large. Thus, using also (5.1.62) we see that $[\mathscr{M}_{m,\xi} + \mathscr{L}_{m,\xi} G_{m,\xi}]^{-1}$ exists and is bounded uniformly with respect to (ξ, m) provided m is large enough. Finally:

$$\|N_{m,\xi} h\| = \sup_{j\in\mathbb{Z}} \left| e_j \int_{-m}^{m} \dot{\gamma}_1(t-\alpha_j) h(t+2jm)dt \right| \leq \tilde{A}_1 \|h\| \tag{5.1.63}$$

and hence Equation (5.1.61) has the unique solution, linear with h

$$c(m,\xi)h = \left[\mathscr{M}_{m,\xi} + \mathscr{L}_{m,\xi} G_{m,\xi}\right]^{-1} \left[\mathscr{L}_{m,\xi} + N_{m,\xi}\right] h \in \ell_E^\infty$$

and the linear map $h \mapsto c(m,\xi)h$ is a bounded linear map from $L^\infty(\mathbb{R})$ into ℓ_E^∞ with bound independent of (m,ξ) (of course with $m \geq \bar{m}$ sufficiently large). We set

$$P_{m,\xi} h = \gamma_{c(m,\xi)h}, \quad Q_{m,\xi} = \mathbb{I} - P_{m,\xi}.$$

Obviously we mean that $c(m,0) = 0$ for any $m \in \mathbb{N}$ so that $P_{m,0} = 0$ and $Q_{m,0} = \mathbb{I}$. We have the following:

Theorem 5.1.4. $P_{m,\xi} : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ *is a projection on* $L^\infty(\mathbb{R})$ *which is uniformly bounded with respect to* (m,ξ) *and Lipschitz in* $\alpha \in \ell_E^\infty$ *uniformly with respect to* (m,E)*. That is, a constant L, independent of* (m,E)*, exists such that* $\|P_{m,(E,\alpha)} - P_{m,(E,\alpha')}\| \leq L\|\alpha - \alpha'\|$ *for any* $m \geq \bar{m}$ *and* $(E,\alpha),(E,\alpha') \in X$*. Furthermore*

$$|[P_{m,\xi} h](t)| \leq |c(m,\xi)| \|h\|_\infty |\dot{\gamma}_\xi(t)| \tag{5.1.64}$$

and $P_{m,\xi} h = 0$ *if and only if*

$$[\mathscr{L}_{m,\xi} + N_{m,\xi}]h = 0. \tag{5.1.65}$$

Proof. Since there is nothing to prove when $\xi = 0$ we assume $\xi \neq 0$. The fact that $P_{m,\xi}$ is bounded uniformly with respect to (m,ξ) and actually satisfies (5.1.64) has already been proved. We now prove the last statement: the equation $P_{m,\xi} h = 0$ holds if and only if $\gamma_{c(m,\xi)h} = 0$, that is, if and only if $h = h_{c(m,\xi)h}$. Thus (5.1.65) follows because $c(m,\xi)h$ satisfies Eq. (5.1.60). On the contrary, if h satisfies (5.1.65), we have $c(m,\xi)h = 0$ because of uniqueness and then $P_{m,\xi} h = 0$. We can now prove that $P_{m,\xi}$ is a projection. In fact, we have $P_{m,\xi}[Q_{m,\xi} h] = P_{m,\xi}[h - P_{m,\xi} h] = 0$ because $h - P_{m,\xi} h = h - \gamma_{c(m,\xi)h}$ satisfies (5.1.65). Thus $P_{m,\xi} = P_{m,\xi}^2$. Finally we prove the Lipschitz continuity of $P_{m,\xi}$. First we prove that

$$(\xi, h) \mapsto N_{m,\xi} h = \left\{ e_j \int_{-m}^{m} \dot{\gamma}_1(t-\alpha_j) h(t+2jm)dt \right\}_{j\in\mathbb{Z}}$$

from $X \times L^\infty$ into ℓ_E^∞, is Lipschitz continuous function in α uniformly with respect to (m,E). In fact, for $\tau'', \tau' \in \mathbb{R}$ with $|\tau''|, |\tau'| \le 2$, we have, using $\ddot{\gamma}_1(t) = \frac{4}{\sqrt{\pi}} f(\frac{2}{\sqrt{\pi}}\gamma_1(t))$, $|f'(x)| \le N$ and (5.1.53):

$$\begin{aligned}
&\left|\int_{-m}^{m} [\dot{\gamma}_1(t-\tau'') - \dot{\gamma}_1(t-\tau')] h(t+2jm)dt\right| \\
&\le \int_{-m}^{m}\int_0^1 |\ddot{\gamma}_1(t-\theta\tau''-(1-\theta)\tau')| d\theta\, dt\, \|h\|_\infty |\tau''-\tau'| \\
&\le \frac{8N}{\pi}\int_{-m}^{m}\int_0^1 |\gamma_1(t-\theta\tau''-(1-\theta)\tau')| d\theta\, dt\, \|h\|_\infty |\tau''-\tau'| \\
&\le \frac{8N}{\pi}\int_{-m}^{m}\int_0^1 A_2\, \mathrm{e}^{-a|t-\theta\tau''-(1-\theta)\tau'|} d\theta\, dt\, \|h\|_\infty |\tau''-\tau'| \\
&\le \frac{8N}{\pi}\int_{-m}^{m}\int_0^1 A_2\, \mathrm{e}^{-a(|t|-2)} d\theta\, dt\, \|h\|_\infty |\tau''-\tau'| \le \frac{16NA_2\,\mathrm{e}^{2a}}{a\pi}\|h\|_\infty |\tau''-\tau'|\,.
\end{aligned}$$

Similarly we can prove that the bounded linear maps $\mathscr{M}_{m,\xi} : \ell_E^\infty \to \ell_E^\infty$ and $G_{m,\xi} : \ell_E^\infty \to L^\infty$ are Lipschitz continuous function in α uniformly with respect to (E,m). Then the inverse $[\mathscr{M}_{m,\xi} + \mathscr{L}_{m,\xi} G_{m,\xi}]^{-1}$ has the same property and the same holds for the solution $c(m,\xi)h$ of Eq. (5.1.61). Finally, let $\xi' = (E,\alpha'), \xi'' = (E,\alpha'') \in X$. Then for any $t \in ((2j-1)m, (2j+1)m]$ we have

$$[P_{m,\xi'}h - P_{m,\xi''}h](t) = \dot{\gamma}_{\xi''}(t)[c_j(m,\xi')h - c_j(m,\xi'')h] + [\dot{\gamma}_{\xi'}(t) - \dot{\gamma}_{\xi''}(t)]c_j(m,\xi')h$$

and hence $P_{m,\xi}$ is Lipschitz continuous function in α uniformly with respect to (E,m), so are $c(m,\xi)$ and $\dot{\gamma}_\xi(t)$ and both are bounded uniformly with respect to (ξ,m). The proof is complete. □

Remark 5.1.5. (a) Obviously $Q_{m,\xi}$ is also Lipschitz continuous function in α, uniformly with respect to (m,E) and, using $P_{m,\xi}Q_{m,\xi} = 0$, we see that the equation

$$\mathscr{L}_{m,\xi,j}Q_{m,\xi}h + \int_{(2j-1)m}^{(2j+1)m} \dot{\gamma}_\xi(t)[Q_{m,\xi}h](t)dt = 0$$

holds for any $j \in \mathbb{Z}$. That is, $Q_{m,\xi}$ is a projection from $L^\infty(\mathbb{R})$ onto $\mathscr{S}_{m,\xi}$ which is bounded uniformly with respect to (ξ,m), so is $P_{m,\xi}$.

(b) It follows from the arguments in Section 5.1.6 that $\mathscr{L}_{m,\xi}$ is not differentiable in α. Hence $P_{m,\xi}$ and $Q_{m,\xi}$ are also not differentiable in α. So the Lipschitz continuity of these maps is their best smoothness in α.

(c) If $h(t) = \dot{\gamma}_\xi(t)$ and $c_j = e_j$ for any $j \in \mathbb{Z}$, we have $h_c(t) = \dot{\gamma}_\xi(t) - \dot{\gamma}_\xi(t) = 0$ and then (5.1.60) is satisfied. Thus, because of uniqueness, $P_{m,\xi}\dot{\gamma}_\xi = \dot{\gamma}_\xi$ or

$$Q_{m,\xi}\dot{\gamma}_\xi = 0. \tag{5.1.66}$$

5.1.4 Chaotic Solutions

We look for solutions of Eqs. (5.1.6)–(5.1.8), for which, the sup-norms of $y_1(t) - \gamma_\xi(t)$, $y_2(t)$ and $z(x,t)$ are small. Since the function $\gamma_\xi(t)$ has small jumps at the points $t = (2j-1)m$, $j \in \mathbb{Z}$, we introduce a function $v_\xi(t) \in L^\infty(\mathbb{R})$ which has small norm, so that

$$\Gamma_\xi(t) = \gamma_\xi(t) + v_\xi(t)$$

is C^1. As an example, we can take the function:

$$v_\xi(t) = \frac{p_j}{4m^2}\big(t-(2j-1)m\big)^2 + \frac{q_j}{8m^3}\big(t-(2j-1)m\big)^3$$

if $(2j-1)m < t \le (2j+1)m$, $j \in \mathbb{Z}$, where

$$\begin{aligned} p_j &= 3\big(\gamma_\xi(((2j+1)m)_+) - \gamma_\xi(((2j+1)m)_-)\big) \\ &\quad + 2m\big(\dot\gamma_\xi(((2j+1)m)_-) - \dot\gamma_\xi(((2j+1)m)_+)\big)\,, \\ q_j &= 2m\big(\dot\gamma_\xi(((2j+1)m)_+) - \dot\gamma_\xi(((2j+1)m)_-)\big) \\ &\quad + 2\big(\gamma_\xi(((2j+1)m)_-) - \gamma_\xi(((2j+1)m)_+)\big)\,. \end{aligned}$$

Again, we will silently include, in the definition of $v_\xi(t)$ and $\Gamma_\xi(t)$, also the end points of the intervals $[(2j-1)m,(2j+1)m]$ as we did for the function $\gamma_\xi(t)$. Next, from (5.1.53) we obtain, for any $j \in \mathbb{Z}$:

$$\max\Big\{|\gamma_\xi(((2j+1)m)_\pm)|, |\dot\gamma_\xi(((2j+1)m)_\pm)|\Big\} \le A_2\,\mathrm{e}^{2a}\,\mathrm{e}^{-am} = \bar A_2\,\mathrm{e}^{-am}$$

where $\bar A_2 = A_2\,\mathrm{e}^{2a}$. As a consequence, we get

$$\begin{aligned} \|v_\xi\|_\infty &\le (10+8m)\bar A_2\,\mathrm{e}^{-am}, \\ \|\dot v_\xi\|_\infty &\le (12+10m)\bar A_2\,\mathrm{e}^{-am}/m, \\ \|\ddot v_\xi\|_\infty &\le (9+8m)\bar A_2\,\mathrm{e}^{-am}/m^2\,, \end{aligned} \tag{5.1.67}$$

or, since $0 < \varepsilon \le 2^{-4/3}$ (and hence $m > \varepsilon^{-3/4} \ge 2$):

$$\|v_\xi\|_\infty < \frac{12\bar A_2}{a^{7/3}}\varepsilon, \quad \|\dot v_\xi\|_\infty < \frac{6\bar A_2}{a^{4/3}}\varepsilon, \quad \|\ddot v_\xi\|_\infty < \frac{6\bar A_2}{a}\varepsilon^{3/2}\,. \tag{5.1.68}$$

Note that to obtain the inequalities (5.1.68) from (5.1.67) we have used the fact that for $\lambda > 0$, and $\theta > 0$ we have $\theta^\lambda\,\mathrm{e}^{-\theta} \le (\lambda/\mathrm{e})^\lambda$ and $\left(\frac{4}{3\mathrm{e}}\right)^{4/3} < \frac{2}{5}$, $\frac{1}{\mathrm{e}} < \frac{1}{2}$, $\left(\frac{7}{3\mathrm{e}}\right)^{7/3} < 1$. Let $\Lambda = \max\left\{\frac{12\,\mathrm{e}^{2a}}{a^{7/3}}, \frac{6\,\mathrm{e}^{2a}}{a^{4/3}}, \frac{6\,\mathrm{e}^{2a}}{a}, \mathrm{e}^{2a}\right\}$, then:

$$\|v_\xi\|_\infty \le \Lambda A_2\varepsilon, \quad \|\dot v_\xi\|_\infty \le \Lambda A_2\varepsilon, \quad \|\ddot v_\xi\|_\infty \le \Lambda A_2\varepsilon^{3/2}\,. \tag{5.1.69}$$

For reasons that will be clearer later, we now prove that the functions $v_\xi(t)$, $\dot{v}_\xi(t)$ and $\ddot{v}_\xi(t)$ are Lipschitz continuous funcfions in α, uniformly with respect to (E,m) and that the Lipschitz constant is of the order $O(\varepsilon)$ as $\varepsilon \to 0$, uniformly with respect to (E,m). So, let $\xi' = (E,\alpha'), \xi'' = (E,\alpha'') \in X$. For any $t \in ((2j-1),(2j+1)m]$ we have (with obvious meaning of symbols):

$$|v_{\xi'}(t) - v_{\xi''}(t)| \le |p'_j - p''_j| + |q'_j - q''_j|$$

$$|\dot{v}_{\xi'}(t) - \dot{v}_{\xi''}(t)| \le \frac{2|p'_j - p''_j| + 3|q'_j - q''_j|}{2m}$$

$$|\ddot{v}_{\xi'}(t) - \ddot{v}_{\xi''}(t)| \le \frac{|p'_j - p''_j| + 3|q'_j - q''_j|}{2m^2}\,.$$

Thus it is enough to estimate $|p'_j - p''_j|$ and $|q'_j - q''_j|$. Assume $e_j = 1$, then

$$\gamma_\xi(((2j+1)m)_-) = \gamma_1(m - \alpha_j)$$

and hence, using (5.1.53) and $|\alpha'_j|, |\alpha''_j| \le 2$ (recall that $\bar{A}_2 = A_2\,\mathrm{e}^{2a}$),

$$|\gamma_{\xi'}(((2j+1)m)_-) - \gamma_{\xi''}(((2j+1)m)_-)| \le \bar{A}_2\,\mathrm{e}^{-am}\,|\alpha'_j - \alpha''_j|.$$

Similarly, if $e_{j+1} = 1$,

$$|\gamma_{\xi'}(((2j+1)m)_+) - \gamma_{\xi''}(((2j+1)m)_+)| \le \bar{A}_2\,\mathrm{e}^{-am}\,|\alpha'_{j+1} - \alpha''_{j+1}|\,.$$

On the other hand, if, say, $e_j = 0$ then $\gamma_\xi(((2j+1)m)_-) = 0$, $\alpha'_j = \alpha''_j = 0$ and the same conclusion holds. Thus we get, for any $j \in \mathbb{Z}$ (recall that $m > 3$):

$$|p'_j - p''_j| \le (6+4m)\bar{A}_2\,\mathrm{e}^{-am}\,\|\alpha' - \alpha''\| < 6m\bar{A}_2\,\mathrm{e}^{-am}\,\|\alpha' - \alpha''\|$$

and similarly,

$$|q'_j - q''_j| \le (4+4m)\bar{A}_2\,\mathrm{e}^{-am}\,\|\alpha' - \alpha''\| < 6m\bar{A}_2\,\mathrm{e}^{-am}\,\|\alpha' - \alpha''\|.$$

Hence, like for (5.1.69), we see that the following holds:

$$\begin{aligned}
&\|v_{\xi'} - v_{\xi''}\|_\infty < A_2 \Lambda \varepsilon \|\alpha' - \alpha''\|,\\
&\|\dot{v}_{\xi'} - \dot{v}_{\xi''}\|_\infty < A_2 \Lambda \varepsilon \|\alpha' - \alpha''\|,\\
&\|\ddot{v}_{\xi'} - \ddot{v}_{\xi''}\|_\infty < A_2 \Lambda \varepsilon^{3/2} \|\alpha' - \alpha''\|
\end{aligned} \tag{5.1.70}$$

which is what we want to prove. Now we replace $y_1(t)$ with $y_1(t) + \Gamma_\xi(t)$ in (5.1.6)–(5.1.8) and project the right-hand side of the differential equation for the new $y_1(t)$ to $\mathscr{S}_{m,\xi}$. Since $\gamma_\xi(t)$ satisfies (5.1.21) and $Q_{m,\xi}\dot{\gamma}_\xi(t) = 0$ (see (5.1.66)), we obtain:

$$
\begin{aligned}
&\ddot{y}_1(t) + \frac{8}{\pi} f'\left(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\right) y_1(t) \\
&= -Q_{m,\xi}\Big\{\sqrt{\varepsilon}\delta\dot{y}_1(t) + \frac{2}{\sqrt{\pi}}\mu\int_0^{\pi/4} h(x,t)\,dx \\
&\quad + \frac{2}{\sqrt{\pi}} f\left(\frac{2}{\sqrt{\pi}}[y_1(t)+\Gamma_\xi(t)] - 2\sqrt{\frac{3}{\pi}}y_2(t) + z(0,t)\right) - \frac{4}{\sqrt{\pi}} f\left(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\right) \\
&\quad + \frac{2}{\sqrt{\pi}} f\left(\frac{2}{\sqrt{\pi}}[y_1(t)+\Gamma_\xi(t)] + 2\sqrt{\frac{3}{\pi}}y_2(t) + z(\frac{\pi}{4},t)\right) \\
&\quad - \frac{8}{\pi} f'\left(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\right) y_1(t) + \sqrt{\varepsilon}\delta\dot{v}_\xi(t) + \ddot{v}_\xi(t)\Big\}, \qquad (5.1.71)
\end{aligned}
$$

$$
\begin{aligned}
&\ddot{y}_2(t) + \frac{24}{\pi} f'\left(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\right) y_2(t) \\
&= -\Big\{\sqrt{\varepsilon}\delta\dot{y}_2(t) + \frac{16}{\pi}\sqrt{\frac{3}{\pi}}\mu\int_0^{\pi/4} h(x,t)\Big(x-\frac{\pi}{8}\Big)\,dx \\
&\quad -2\sqrt{\frac{3}{\pi}} f\Big(\frac{2}{\sqrt{\pi}}[y_1(t)+\Gamma_\xi(t)] - 2\sqrt{\frac{3}{\pi}}y_2(t) + z(0,t)\Big) \\
&\quad +2\sqrt{\frac{3}{\pi}} f\Big(\frac{2}{\sqrt{\pi}}[y_1(t)+\Gamma_\xi(t)] + 2\sqrt{\frac{3}{\pi}}y_2(t) + z(\pi/4,t)\Big) \\
&\quad -\frac{24}{\pi} f'(\frac{2}{\sqrt{\pi}}\gamma_\xi(t))y_2(t)\Big\}, \qquad (5.1.72)
\end{aligned}
$$

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_0^{\pi/4}\Big\{z(x,t)\Big[v_{tt}(x,t) + \frac{1}{\varepsilon}v_{xxxx}(x,t) - \sqrt{\varepsilon}\delta v_t(x,t)\Big] + \mu h(x,t)v(x,t)\Big\}\,dx\,dt \\
&+ \int_{-\infty}^{\infty}\Big\{f(u(0,t))v(0,t) + f(u(\pi/4,t))v(\pi/4,t)\Big\}\,dt = 0, \qquad (5.1.73)
\end{aligned}
$$

in (5.1.73) we write $u(x,t)$ for $\frac{2}{\sqrt{\pi}}[y_1(t)+\Gamma_\xi(t)] + y_2(t)w_0(x) + z(x,t)$.

Let $C_b^1(\mathbb{R})$ be the space of C^1 functions bounded together with their first derivative on $\mathbb{R}$. To make notations simpler we define the Banach spaces Y_1 and Y_2 as the space $C_b^1(\mathbb{R})$ endowed with the norms

$$
\|y_1\| = \frac{2}{\sqrt{\pi}}\sup_{t\in\mathbb{R}}\{|y_1(t)|, |\dot{y}_1(t)|\}, \quad \|y_2\| = 2\sqrt{\frac{3}{\pi}}\sup_{t\in\mathbb{R}}\{|y_2(t)|, |\dot{y}_2(t)|\},
$$

respectively. Unless otherwise specified, $y_1(t)$, $\hat{y}_1(t)$, resp. $y_2(t)$, $\hat{y}_2(t)$ will denote functions in Y_1, resp. Y_2 and the norm in $Y_1 \times Y_2$ will be $\|y_1\| + \|y_2\|$. Next, let $\rho > 0$ be a fixed positive number, $y_1(t) \in Y_1$, $y_2(t) \in Y_2$ and $z(x,t) \in C_b^0([0, \frac{\pi}{4}] \times \mathbb{R})$ be such that $\|y_1\| + \|y_2\| + \|z\|_\infty \le \rho$. For any fixed choice of such functions we set:

$$
\begin{aligned}
H_1(x,t) &= \mu h(x,t),\\
H_2(t,\xi) &= f\left(\frac{2}{\sqrt{\pi}}[y_1(t) + \Gamma_\xi(t)] - 2\sqrt{\frac{3}{\pi}}y_2(t) + z(0,t)\right) - f\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right)\\
&\quad - f'\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right)\left[\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}y_2(t) + z(0,t)\right],\\
H_3(t,\xi) &= f\left(\frac{2}{\sqrt{\pi}}\left[y_1(t) + \Gamma_\xi(t)\right] + 2\sqrt{\frac{3}{\pi}}y_2(t) + z\left(\frac{\pi}{4},t\right)\right) - f\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right)\\
&\quad - f'\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right)\left[\frac{2}{\sqrt{\pi}}y_1(t) + 2\sqrt{\frac{3}{\pi}}y_2(t) + z\left(\frac{\pi}{4},t\right)\right],\\
\hat{H}_2(t,\xi) &= f\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right)\\
&\quad + \left[f'\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right) - f'(0)\right]\left[\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}y_2(t) + z(0,t)\right],\\
\hat{H}_3(t,\xi) &= f\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right) \qquad\qquad (5.1.74)\\
&\quad + \left[f'\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right) - f'(0)\right]\left[\frac{2}{\sqrt{\pi}}y_1(t) + 2\sqrt{\frac{3}{\pi}}y_2(t) + z\left(\frac{\pi}{4},t\right)\right],\\
\tilde{H}_{21}(t) &= f'(0)\left[\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}y_2(t)\right], \qquad \tilde{H}_{22}(t) = f'(0)z(0,t),\\
\tilde{H}_{31}(t) &= f'(0)\left[\frac{2}{\sqrt{\pi}}y_1(t) + 2\sqrt{\frac{3}{\pi}}y_2(t)\right], \qquad \tilde{H}_{32}(t) = f'(0)z\left(\frac{\pi}{4},t\right),\\
\hat{H}_{20}(t,\xi) &= \hat{H}_{30}(t,\xi) = f\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right) - f(0),\\
\hat{H}_{21}(t,\xi) &= \left[f'\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right) - f'(0)\right]\left[\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}y_2(t) + z(0,t)\right],\\
\hat{H}_{31}(t,\xi) &= \left[f'\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right) - f'(0)\right]\left[\frac{2}{\sqrt{\pi}}y_1(t) + 2\sqrt{\frac{3}{\pi}}y_2(t) + z(\pi/4,t)\right].
\end{aligned}
$$

Let us continue to denote with N an upper bound for $f'(x)$ and $f''(x)$ in a neighborhood of $\gamma(t)$. We have the following result.

Lemma 5.1.6. *There exist positive constant k_3 and a function $\tilde{\Delta}(\rho) > 0$ with $\lim_{\rho\to 0}\tilde{\Delta}(\rho) = 0$, so that if $\|y_1\| + \|y_2\| + \|z\|_\infty \le \rho \le 1$, $E \in \mathscr{E}$ and $\alpha', \alpha'' \in \ell_E^\infty$ the following hold*

$$|H_k(t,\xi') - H_k(t,\xi'')| \le \rho\tilde{\Delta}(\rho)\left[\varepsilon + \mathrm{e}^{-a|t-2jm|}\right]\|\alpha' - \alpha''\|, \quad k = 2,3,$$
$$|\hat{H}_{k1}(t,\xi') - \hat{H}_{k1}(t,\xi'')| \le k_3\rho\left[\varepsilon + \mathrm{e}^{-a|t-2jm|}\right]\|\alpha' - \alpha''\|, \quad k = 2,3$$

where $\xi' = (E,\alpha')$ and $\xi'' = (E,\alpha'')$ and $t \in ((2j-1)m,(2j+1)m]$. Furthermore, $\hat{H}_{20}(t,\xi') - \hat{H}_{20}(t,\xi'') = \hat{H}_{30}(t,\xi') - \hat{H}_{30}(t,\xi'')$ can be written as the sum of two piecewise C^1-functions $H_{01}(t) + H_{02}(t)$, so that

$$|H_{01}(t)| \le k_3\varepsilon\|\alpha' - \alpha''\|,$$
$$|H_{02}(t)| \le k_3\,\mathrm{e}^{-a|t-2jm|}\|\alpha' - \alpha''\|,$$
$$|\dot{H}_{02}(t)| \le k_3\,\mathrm{e}^{-a|t-2jm|}\|\alpha' - \alpha''\|$$

where $\xi' = (E,\alpha')$, $\xi'' = (E,\alpha'')$ and $t \in ((2j-1)m,(2j+1)m]$.

Proof. Let $e_j = 1$. Then, for any $t \in ((2j-1)m,(2j+1)m]$, we have

$$\begin{aligned}|\Gamma_{\xi'}(t) - \Gamma_{\xi''}(t)| &\le \left[|\dot{\gamma}_1(t-2jm-\theta\alpha'_j - (1-\theta)\alpha''_j)| + A_2\Lambda\varepsilon\right]\|\alpha' - \alpha''\| \\ &\le \left[A_2\Lambda\varepsilon + \bar{A}_2\,\mathrm{e}^{-a|t-2jm|}\right]\|\alpha' - \alpha''\|.\end{aligned}$$

Obviously a similar conclusion holds when $e_j = 0$ since in this case we have $\Gamma_\xi(t) = v_\xi(t)$ for any $t \in ((2j-1)m,(2j+1)m]$. Next, for any $x \in \mathbb{R}$ we have $|x + \frac{2}{\sqrt{\pi}}\Gamma_\xi(t)| \le |x| + \frac{2}{\sqrt{\pi}}\|v_\xi\|_\infty + \|\gamma\|_\infty \le |x| + \frac{2}{\sqrt{\pi}}\Lambda A_2\varepsilon + A_1$. Thus, for any (y_1,y_2,z) $|y_1| + |y_2| + |z| \le \rho$ and $\xi \in X$, the functions $f^{(k)}(y_1 + \Gamma_\xi(t) + y_2 + z)$, $k = 0,1,2$ are bounded. Since

$$\begin{aligned}H_2(t,\xi') - H_2(t,\xi'') = &\int_{\frac{2}{\sqrt{\pi}}\Gamma_{\xi''}(t)}^{\frac{2}{\sqrt{\pi}}\Gamma_{\xi'}(t)} f'\left(\frac{2}{\sqrt{\pi}}y_1(t) + \theta - 2\sqrt{\frac{3}{\pi}}y_2(t) + z(0,t)\right) \\ &-f'(\theta) - f''(\theta)\left[\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}y_2(t) + z(0,t)\right]d\theta \\ =&\int_{\frac{2}{\sqrt{\pi}}\Gamma_{\xi''}(t)}^{\frac{2}{\sqrt{\pi}}\Gamma_{\xi'}(t)}\int_0^1 f''\left(\theta + \sigma\left[\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}y_2(t) + z(0,t)\right]\right) \\ &-f''(\theta)d\sigma d\theta\left[\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}y_2(t) + z(0,t)\right],\end{aligned}$$

we obtain:

$$
\begin{aligned}
|H_2(t,\xi')-H_2(t,\xi'')| &\leq \frac{2}{\sqrt{\pi}}\rho\Delta_0(\rho)|\Gamma_{\xi'}(t)-\Gamma_{\xi''}(t)| \\
&\leq \rho\widetilde{\Delta}(\rho)\left[\varepsilon+\mathrm{e}^{-a|t-2jm|}\right]\|\alpha'-\alpha''\|
\end{aligned}
$$

where $\Delta_0(\rho) := \sup_{\{|y|\leq\rho\ ;|x|\leq A_1\}}|f''(x+y)-f''(x)| \to 0$ as $\rho \to 0$ and $\widetilde{\Delta}(\rho) = \frac{2}{\sqrt{\pi}}A_2\Lambda\Delta_0(\rho)$. Similarly,

$$
|H_3(t,\xi')-H_3(t,\xi'')| \leq \rho\widetilde{\Delta}(\rho)\left[\varepsilon+\mathrm{e}^{-a|t-2jm|}\right]\|\alpha'-\alpha''\|
$$

whereas for $k=2,3$ we get:

$$
\begin{aligned}
|\hat{H}_{k1}(t,\xi')-\hat{H}_{k1}(t,\xi'')| &\leq \frac{2N}{\sqrt{\pi}}\rho|\Gamma_{\xi'}(t)-\Gamma_{\xi''}(t)| \\
&\leq \frac{2A_2\Lambda N}{\sqrt{\pi}}\rho\left[\varepsilon+\mathrm{e}^{-a|t-2jm|}\right]\|\alpha'-\alpha''\|.
\end{aligned}
$$

The first part of the Lemma then follows. For the second we write:

$$
\hat{H}_{20}(t,\xi')-\hat{H}_{20}(t,\xi'') = H_{01}(t)+H_{02}(t)
$$

where

$$
H_{01}(t) = f\left(\frac{2}{\sqrt{\pi}}\Gamma_{\xi'}(t)\right)-f\left(\frac{2}{\sqrt{\pi}}\gamma_{\xi'}(t)\right)-f\left(\frac{2}{\sqrt{\pi}}\Gamma_{\xi''}(t)\right)+f\left(\frac{2}{\sqrt{\pi}}\gamma_{\xi''}(t)\right),
$$

$$
H_{02}(t) = f\left(\frac{2}{\sqrt{\pi}}\gamma_{\xi'}(t)\right)-f\left(\frac{2}{\sqrt{\pi}}\gamma_{\xi''}(t)\right).
$$

Then, using (5.1.22) and (5.1.70), we have

$$
\begin{aligned}
&|H_{01}(t)| \\
&\leq \left|f\left(\frac{2}{\sqrt{\pi}}\Gamma_{\xi'}(t)\right)-f\left(\frac{2}{\sqrt{\pi}}[\gamma_{\xi'}(t)+v_{\xi''}(t)]\right)\right| \\
&\quad+\left|f\left(\frac{2}{\sqrt{\pi}}\left[\gamma_{\xi'}(t)+v_{\xi''}(t)\right]\right)-f\left(\frac{2}{\sqrt{\pi}}\gamma_{\xi'}(t)\right)-f\left(\frac{2}{\sqrt{\pi}}\left[\gamma_{\xi''}(t)+v_{\xi''}(t)\right]\right)\right. \\
&\quad\left.+f\left(\frac{2}{\sqrt{\pi}}\gamma_{\xi''}(t)\right)\right| \leq \frac{2}{\sqrt{\pi}}NA_2\Lambda\varepsilon\|\alpha'-\alpha''\| \\
&\quad+\int_0^{\frac{2}{\sqrt{\pi}}v_{\xi''}(t)}\left|f'\left(\frac{2}{\sqrt{\pi}}\gamma_{\xi'}(t)+\theta\right)-f'\left(\frac{2}{\sqrt{\pi}}\gamma_{\xi''}(t)+\theta\right)\right|d\theta \\
&\leq \frac{2}{\sqrt{\pi}}NA_2\Lambda\varepsilon\left(1+\frac{2\sqrt{2}}{\sqrt{\pi}}\|\dot{\gamma}\|_\infty\right)\|\alpha'-\alpha''\| \leq k_3\varepsilon\|\alpha'-\alpha''\|.
\end{aligned}
$$

Finally, for any $t\in((2j-1)m,(2j+1)m]$, $j\in\mathbb{Z}$, with $e_j=1$, we have

$$H_{02}(t) = f\left(\frac{2}{\sqrt{\pi}}\gamma_1(t-2jm-\alpha_j')\right) - f\left(\frac{2}{\sqrt{\pi}}\gamma_1(t-2jm-\alpha_j'')\right)$$
$$= \frac{2}{\sqrt{\pi}}\int_{\alpha_j'}^{\alpha_j''} f'\left(\frac{2}{\sqrt{\pi}}\gamma_1(t-2jm-\theta)\right)\dot{\gamma}_1(t-2jm-\theta)d\theta.$$

Thus
$$|H_{02}(t)| \leq \frac{2\bar{A}_2 N}{\sqrt{\pi}}\,\mathrm{e}^{-a|t-2jm|}\,|\alpha_j' - \alpha_j''| \leq k_3\,\mathrm{e}^{-a|t-2jm|}\,\|\alpha' - \alpha''\|$$

and similarly, differentiating with respect to t, we have

$$|\dot{H}_{02}(t)| \leq \frac{2\bar{A}_2 N}{\sqrt{\pi}}\left(1 + \frac{2}{\sqrt{\pi}}\bar{A}_2\right)\mathrm{e}^{-a|t-2jm|}\,|\alpha_j' - \alpha_j''| \leq k_3\,\mathrm{e}^{-a|t-2jm|}\,\|\alpha' - \alpha''\|.$$

The proof is complete. □

Now, consider the unique solution, whose existence is stated in Theorem 5.1.1, of Eq. (5.1.73) with $\hat{z}(x,t)$ instead of $z(x,t)$ and $\frac{2}{\sqrt{\pi}}[y_1(t) + \Gamma_\xi(t)] + y_2(t)w_0(x) + z(x,t)$ instead of $u(x,t)$:

$$\hat{z}(x,t) = F_1(z, y_1, y_2, \xi, \mu, \varepsilon) + L_{1\varepsilon}(y_1, y_2) + L_{2\varepsilon}(z)$$

where
$$F_1(z, y_1, y_2, \xi, \mu, \varepsilon) := L_\varepsilon(H_1, H_2, H_3) + L_\varepsilon(0, \hat{H}_2, \hat{H}_3)\,,$$
$$L_{1\varepsilon}(y_1, y_2) := L_\varepsilon(0, \tilde{H}_{21}, \tilde{H}_{31})\,,$$
$$L_{2\varepsilon}(z) := L_\varepsilon(0, \tilde{H}_{22}, \tilde{H}_{32}).$$

We are thinking of $F_1(z, y_1, y_2, \xi, \mu, \varepsilon)$ as a map from

$$C_b^0([0,\pi/4]\times\mathbb{R}) \times Y_1 \times Y_2 \times X \times \mathbb{R} \times \mathbb{R}_+ \to C_b^0([0,\pi/4]\times\mathbb{R})\,.$$

We will need the following result.

Lemma 5.1.7. *For any fixed, small,* $\varepsilon > 0$, $L_{1\varepsilon} : Y_1 \times Y_2 \to C_b^0([0,\pi/4]\times\mathbb{R})$ *and* $L_{2\varepsilon} : C_b^0([0,\pi/4]\times\mathbb{R}) \to C_b^0([0,\pi/4]\times\mathbb{R})$ *are bounded linear maps whose norms satisfy:*

$$\|L_{1\varepsilon}\| \leq 2M_1M_2|f'(0)|\delta^{-1}\,, \quad \|L_{2\varepsilon}\| \leq 2M_1M_2|f'(0)|\delta^{-1}\,. \tag{5.1.75}$$

Moreover a function $\Delta(\rho) > 0$ *exists so that* $\lim_{\rho\to 0}\Delta(\rho) = 0$ *and for* $\|y_1\| + \|y_2\| + \|z\| \leq \rho$, $\|\tilde{y}_1\| + \|\tilde{y}_2\| + \|\tilde{z}\| \leq \rho$ *the following hold:*

(i)

$$\|\, F_1\ (z, y_1, y_2, \mu, \xi, \varepsilon)\|_\infty$$
$$\leq\ \frac{\pi}{2}M_1M_2\sqrt{\varepsilon}|\mu|(\sqrt{\varepsilon}\|h\|_\infty + \delta^{-1}\|h_t\|_\infty)$$

$$+\frac{4}{\sqrt{\pi}}A_2M_1M_2N\Lambda\left[5\left(\frac{1}{\delta^5}+1+\frac{1}{a}\right)\left(\frac{1}{\Lambda}+\varepsilon\right)+2\delta^{-1}\sqrt{\varepsilon}\right]\varepsilon$$
$$+2M_1M_2\Big[\delta^{-1}\Delta(\rho)+\frac{2}{\sqrt{\pi}}A_2N\sqrt{\varepsilon}\Big(\frac{1}{\delta^3}+\frac{2}{a}+\frac{\Lambda}{\delta}\sqrt{\varepsilon}\Big)\Big]\cdot$$
$$(\|y_1\|+\|y_2\|+\|z\|_\infty). \tag{5.1.76}$$

(ii) *for any* $\xi'=(E,\alpha'),\xi''=(E,\alpha'')\in X$, μ', μ'', *we have*

$$\|F_1(z,y_1,y_2,\mu',\xi',\varepsilon)-F_1(\tilde{z},\tilde{y}_1,\tilde{y}_2,\mu'',\xi'',\varepsilon)\|_\infty$$
$$\leq\frac{\pi}{4}M_1M_2\sqrt{\varepsilon}\left[5\sqrt{\varepsilon}\left(\frac{1}{\delta^5}+1+\frac{1}{a}\right)+2\delta^{-1}\right](\|h\|_\infty+\|h_t\|_\infty)|\mu'-\mu''|$$
$$+2M_1M_2\Big[\delta^{-1}\Delta(\rho)+\frac{2}{\sqrt{\pi}}A_2N\sqrt{\varepsilon}\Big(\frac{1}{\delta^3}+\frac{2}{a}+\frac{\Lambda}{\delta}\sqrt{\varepsilon}\Big)\Big]\cdot$$
$$(\|y_1-\tilde{y}_1\|+\|y_2-\tilde{y}_2\|+\|z-\tilde{z}\|_\infty)$$
$$+4k_3M_1M_2\left(\frac{\sqrt{\varepsilon}}{\delta}+\frac{1}{\delta^3}+\frac{2}{a}\right)\rho\sqrt{\varepsilon}\|\alpha'-\alpha''\|$$
$$+10k_3M_1M_2\varepsilon\left(\frac{1}{5\delta}+\frac{1}{\delta^5}+1+\frac{1}{a}\right)\|\alpha'-\alpha''\|, \tag{5.1.77}$$

with k_3 *being the positive constant of Lemma 5.1.6.*

Proof. By following the above estimates, it is easy to derive (5.1.75) along with the estimate

$$\|L_\varepsilon(H_1,H_2,H_3)\|_\infty\leq\frac{M_1M_2\pi}{2}\sqrt{\varepsilon}|\mu|(\sqrt{\varepsilon}\|h\|_\infty+\delta^{-1}\|h_t\|_\infty)$$
$$+2M_1M_2\delta^{-1}\Delta(\rho)(\|y_1\|+\|y_2\|+\|z\|_\infty) \tag{5.1.78}$$

where

$$\Delta(\rho)=\sup_{\substack{|y_1|+|y_2|+|z|\leq\rho\\ -\infty<t<\infty}}\left|f'\left(y_1+\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)+y_2+z\right)-f'\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right)\right|\to 0$$

as $\rho\to 0$ (cf [11, Lemma 2, Eq. (3.17), (3.20)] for more details). Since $f(0)=0$ we have $\hat{H}_2(t,\xi)=\hat{H}_{20}(t,\xi)+\hat{H}_{21}(t,\xi)$ and $\hat{H}_3(t,\xi)=\hat{H}_{30}(t,\xi)+\hat{H}_{31}(t,\xi)$, $\hat{H}_{ij}(t)$ defined in (5.1.75). Now, $\hat{H}_{20}(t,\xi)\in C_b^1(\mathbb{R})$ and the following inequalities hold (see also (5.1.53)):

$$|\hat{H}_{20}(t,\xi)|\leq\frac{2N}{\sqrt{\pi}}|\Gamma_\xi(t)|\leq\frac{2}{\sqrt{\pi}}A_2N\left[\Lambda\varepsilon+\mathrm{e}^{-a|t-2jm-\alpha_j|}\right],$$
$$|\dot{\hat{H}}_{20}(t,\xi)|\leq\frac{2N}{\sqrt{\pi}}|\dot{\Gamma}_\xi(t)|\leq\frac{2}{\sqrt{\pi}}A_2N\left[\Lambda\varepsilon+\mathrm{e}^{-a|t-2jm-\alpha_j|}\right]$$

for $(2j-1)m < t \leq (2j+1)m,\ j \in \mathbb{Z}$. Hence, from Theorem 5.1.1–(b), (5.1.19), (5.1.20) we get

$$\|L_\varepsilon(0,\hat{H}_{20},\hat{H}_{30})\|_\infty \leq \frac{4}{\sqrt{\pi}} A_2 M_1 M_2 N \Lambda \left[5\left(\frac{1}{\delta^5}+1+\frac{1}{a}\right)\left(\frac{1}{\Lambda}+\varepsilon\right)+\frac{2}{\delta}\sqrt{\varepsilon}\right]\varepsilon \tag{5.1.79}$$

Next,

$$\begin{aligned}|\hat{H}_{21}(t,\xi)| &\leq \frac{2N}{\sqrt{\pi}}|\Gamma_\xi(t)|[\|y_1\|+\|y_2\|+\|z\|_\infty] \\ &\leq \frac{2}{\sqrt{\pi}} A_2 N[\Lambda\varepsilon + \mathrm{e}^{-a|t-2jm-\alpha_j|}][\|y_1\|+\|y_2\|+\|z\|_\infty], \\ |\hat{H}_{31}(t,\xi)| &\leq \frac{2N}{\sqrt{\pi}}|\Gamma_\xi(t)|[\|y_1\|+\|y_2\|+\|z\|_\infty] \\ &\leq \frac{2}{\sqrt{\pi}} A_2 N[\Lambda\varepsilon + \mathrm{e}^{-a|t-2jm-\alpha_j|}][\|y_1\|+\|y_2\|+\|z\|_\infty].\end{aligned}$$

Thus, from Theorem 5.1.1(a) and (5.1.19) we obtain:

$$\|L_\varepsilon(0,\hat{H}_{21},\hat{H}_{31})\|_\infty \leq \frac{4}{\sqrt{\pi}} M_1 M_2 A_2 N\sqrt{\varepsilon}\left(\frac{1}{\delta^3}+\frac{2}{a}+\Lambda\frac{\sqrt{\varepsilon}}{\delta}\right)[\|y_1\|+\|y_2\|+\|z\|_\infty] \tag{5.1.80}$$

and (5.1.76) follows from (5.1.78), (5.1.79), and (5.1.80). Finally, we prove (5.1.77). Using arguments similar to the above we see that

$$\begin{aligned}&\| F_1(z,y_1,y_2,\mu'',\xi'',\varepsilon) - F_1(\tilde{z},\tilde{y}_1,\tilde{y}_2,\mu'',\xi'',\varepsilon)\|_\infty \\ &\quad \leq 2M_1M_2\left[\delta^{-1}\Delta(\rho)+\frac{2}{\sqrt{\pi}}A_2N\sqrt{\varepsilon}\left(\frac{1}{\delta^3}+\frac{2}{a}+\frac{\Lambda}{\delta}\sqrt{\varepsilon}\right)\right] \\ &\qquad \cdot [\|y_1-\tilde{y}_1\|+\|y_2-\tilde{y}_2\|+\|z-\tilde{z}\|_\infty].\end{aligned}$$

Next,

$$\begin{aligned}&F_1(z,y_1,y_2,\mu',\xi',\varepsilon) - F_1(z,y_1,y_2,\mu'',\xi'',\varepsilon) \\ &\quad = L_\varepsilon((\mu'-\mu'')h,0,0) + L_\varepsilon(0,H_2(\cdot,\xi')-H_2(\cdot,\xi''),H_3(\cdot,\xi')-H_3(\cdot,\xi'')) \\ &\qquad + L_\varepsilon(0,\hat{H}_{20}(\cdot,\xi')-\hat{H}_{20}(\cdot,\xi''),\hat{H}_{30}(\cdot,\xi')-\hat{H}_{30}(\cdot,\xi'')) \\ &\qquad + L_\varepsilon(0,\hat{H}_{21}(\cdot,\xi')-\hat{H}_{21}(\cdot,\xi''),\hat{H}_{31}(\cdot,\xi')-\hat{H}_{31}(\cdot,\xi''))\end{aligned}$$

and hence, from Lemma 5.1.6, Theorem 5.1.1, (5.1.19) and (5.1.20) we obtain:

$$\begin{aligned}&\| F_1(z,y_1,y_2,\mu',\xi',\varepsilon) - F_1(z,y_1,y_2,\mu'',\xi'',\varepsilon)\|_\infty \\ &\quad \leq 4k_3M_1M_2\left(\frac{\sqrt{\varepsilon}}{\delta}+\frac{1}{\delta^3}+\frac{2}{a}\right)\rho\sqrt{\varepsilon}\|\alpha'-\alpha''\|\end{aligned}$$

$$+\frac{\pi}{4}M_1M_2\sqrt{\varepsilon}\left[5\sqrt{\varepsilon}\left(\frac{1}{\delta^5}+1\right)+\frac{2}{\delta}\right](\|h\|_\infty+\|h_t\|_\infty)|\mu'-\mu''|$$
$$+10k_3M_1M_2\varepsilon\left(\frac{1}{5\delta}+\frac{1}{\delta^5}+1+\frac{1}{a}\right)\|\alpha'-\alpha''\|.$$

(5.1.77) then follows from the above two estimates. The proof is complete. □

Now, for given $(y_1(t), y_2(t), z(x,t)) \in Y_1 \times Y_2 \times C_b^0([0,\frac{\pi}{4}]\times\mathbb{R})$, we denote with $(\hat{y}_1(t), \hat{y}_2(t))$ the unique solution of

$$\begin{aligned}\ddot{\hat{y}}_1(t)+\frac{8}{\pi}f'\left(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\right)\hat{y}_1(t)&=g_1(t),\\ \ddot{\hat{y}}_2(t)+\frac{24}{\pi}f'\left(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\right)\hat{y}_2(t)&=g_2(t)\end{aligned}\tag{5.1.81}$$

where $g_1(t)$, $g_2(t)$ are the right-hand sides of Eqs. (5.1.71), (5.1.72), that satisfy $\dot{\hat{y}}_1(2jm+\alpha_j)=0$ for any $j\in\mathbb{Z}$ so that $e_j=1$. These solutions exist because of Lemmas 5.1.2 and 5.1.3, moreover

$$\|\hat{y}_1\|\le B\|g_1\|,\quad \|\hat{y}_2\|\le B_1\|g_2\|\tag{5.1.82}$$

where B and B_1 have been defined in Lemma 5.1.2 and Lemma 5.1.3. Note that in the above formulas the norm on the left is the norm in Y_1 (resp. Y_2), while $\|g_1\| = \frac{2}{\sqrt{\pi}}\sup_{t\in\mathbb{R}}|g_1(t)|$ and $\|g_2\|=2\sqrt{\frac{3}{\pi}}\sup_{t\in\mathbb{R}}|g_2(t)|$. Let

$$g_{11}(t)=g_1(t)+Q_{m,\xi}\left\{\frac{2}{\sqrt{\pi}}f'\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right)[z(0,t)+z(\pi/4,t)]\right\},$$
$$g_{21}(t)=g_2(t)+2\sqrt{\frac{3}{\pi}}f'\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right)[z(\pi/4,t)-z(0,t)].$$

Then $(\hat{y}_1(t), \hat{y}_2(t))$ can be written as

$$\hat{y}_1(t)=\hat{y}_{11}(t)+\hat{y}_{10}(t),\quad \hat{y}_2(t)=\hat{y}_{21}(t)+\hat{y}_{20}(t)$$

where $(\hat{y}_{11}(t), \hat{y}_{21}(t)) \in Y_1\times Y_2$ is the unique bounded solution of

$$\begin{aligned}\ddot{\hat{y}}_{11}(t)+\frac{8}{\pi}f'\left(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\right)\hat{y}_{11}(t)&=g_{11}(t),\\ \ddot{\hat{y}}_{21}(t)+\frac{24}{\pi}f'\left(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\right)\hat{y}_{21}(t)&=g_{21}(t)\end{aligned}\tag{5.1.83}$$

that satisfies $\dot{\hat{y}}_{11}(2jm+\alpha_j)=0$ for any $j\in\mathbb{Z}$ so that $e_j=1$, and $(\hat{y}_{10}(t),\hat{y}_{20}(t))\in Y_1\times Y_2$ is the unique bounded solution of

$$\ddot{\hat{y}}_{10}(t)+\frac{8}{\pi}f'\left(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\right)\hat{y}_{10}(t)=-Q_{m,\xi}\left[\frac{2}{\sqrt{\pi}}f'\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right)[z(0,t)+z(\pi/4,t)]\right],$$

$$\ddot{\hat{y}}_{20}(t)+\frac{24}{\pi}f'\left(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\right)\hat{y}_{20}(t)=-2\sqrt{\frac{3}{\pi}}f'\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right)[z(\pi/4,t)-z(0,t)] \tag{5.1.84}$$

that satisfies $\dot{\hat{y}}_{10}(2jm+\alpha_j)=0$ for any $j\in\mathbb{Z}$ so that $e_j=1$. We set

$$F_2(z,y_1,y_2,\xi,\mu,\varepsilon)=(\hat{y}_{11},\hat{y}_{21})\in Y_1\times Y_2,\quad Lz=(\hat{y}_{10},\hat{y}_{20}).$$

Then we have the following result:

Lemma 5.1.8. $L: C_b^0([0,\pi/4]\times\mathbb{R})\to Y_1\times Y_2$ *is a bounded linear map. Moreover, positive constants* k_6 *and* k_7 *and a function* $\overline{\Delta}(\rho,\varepsilon)>0$ *exist so that* $\lim_{(\rho,\varepsilon)\to(0,0)}\overline{\Delta}(\rho,\varepsilon)=0$ *and for* $\|y_1\|+\|y_2\|+\|z\|\le\rho$, $\|\widetilde{y}_1\|+\|\widetilde{y}_2\|+\|\widetilde{z}\|\le\rho$ *the following hold:*

(i)

$$\|F_2(z,y_1,y_2,\xi,\mu,\varepsilon)\|\le\bar{\Delta}(\rho,\varepsilon)[\|y_1\|+\|y_2\|+\|z\|_\infty]+k_6|\mu|+k_7\varepsilon. \tag{5.1.85}$$

(ii) *For any* $\xi=(E,\alpha)$, $\widetilde{\xi}=(E,\widetilde{\alpha})\in X$, μ, $\widetilde{\mu}$, *we have*

$$\begin{aligned}\| F_2\ &(z,y_1,y_2,\xi,\mu,\varepsilon)-F_2(\widetilde{z},\widetilde{y}_1,\widetilde{y}_2,\widetilde{\xi},\widetilde{\mu},\varepsilon)\|\\ &\le\bar{\Delta}(\rho,\varepsilon)[\|y_1-\widetilde{y}_1\|+\|y_2-\widetilde{y}_2\|+\|z-\widetilde{z}\|_\infty]\\ &\quad+[\rho\bar{\Delta}(\rho,\varepsilon)+k_6|\mu|+k_7\varepsilon]\|\alpha-\widetilde{\alpha}\|_\infty+k_6|\mu-\widetilde{\mu}|.\end{aligned} \tag{5.1.86}$$

Proof. First we note that from Remark 5.1.5 (a) the existence follows of a constant $A_4>0$ so that $\|Q_{m,\xi}\|\le A_4$ and $\|Q_{m,\xi'}-Q_{m,\xi''}\|\le A_4\|\alpha'-\alpha''\|$ for any m sufficiently large and any $\xi,\xi',\xi''\in X$ with $\xi'=(E,\alpha')$, $\xi''=(E,\alpha'')$. Then, L is obviously linear and from (5.1.82) it easily follows that

$$\|\hat{y}_{10}\|+\|\hat{y}_{20}\|\le\frac{8N(A_4B+3B_1)}{\pi}\|z\|_\infty,$$

that is, L is bounded and

$$\|L\|\le\frac{8N(A_4B+3B_1)}{\pi}.$$

Next, it is easy to see that

$$\begin{aligned}\|g_{11}\|\le A_4\Big\{&\sqrt{\varepsilon}\delta\|y_1\|+|\mu|\|h\|_\infty+\frac{2\Lambda A_2}{\sqrt{\pi}}(1+\delta)\varepsilon^{3/2}+\frac{16\Lambda A_2N}{\pi\sqrt{\pi}}\varepsilon\\ &+\frac{16\Lambda A_2N}{\pi\sqrt{\pi}}\varepsilon\|y_1\|+\frac{8}{\pi}\Delta(\rho)[\|y_1\|+\|y_2\|+\|z\|_\infty]\Big\}\\ \le\ &\frac{1}{2B}\{\bar{\Delta}(\rho,\varepsilon)[\|y_1\|+\|y_2\|+\|z\|_\infty]+k_6|\mu|+2k_7\varepsilon\}\end{aligned}$$

where $\bar{\Delta}(\rho,\varepsilon) \to 0$ as $\rho + |\varepsilon| \to 0$ and k_6, k_7 are suitably chosen. Similarly

$$\|g_{21}\| \le \frac{1}{2B_1}\{\bar{\Delta}(\rho,\varepsilon)[\|y_1\| + \|y_2\| + \|z\|_\infty] + k_6|\mu|\}.$$

Thus (5.1.85) follows from (5.1.81).

To prove (5.1.86), let $(z(x,t), y_1(t), y_2(t), \xi, \mu)$, $(\widetilde{z}(x,t), \widetilde{y}_1(t), \widetilde{y}_2(t), \widetilde{\xi}, \widetilde{\mu})$ be in the statement of the theorem and write $g_{11}(t, z(0,t), z(\frac{\pi}{4},t), y_1(t), y_2(t), \xi, \mu, \varepsilon)$ for $g_{11}(t)$ and $\widetilde{g}_{11}(t)$ for $g_{11}(t, \widetilde{z}(0,t), \widetilde{z}(\pi/4,t), \widetilde{y}_1(t), \widetilde{y}_2(t), \widetilde{\xi}, \widetilde{\mu}, \varepsilon)$. From Lemma 5.1.2-(iii) and Lemma 5.1.3-(ii) we know that

$$\begin{aligned} &\| F_2(z, y_1, y_2, \xi, \mu, \varepsilon) - F_2(\widetilde{z}, \widetilde{y}_1, \widetilde{y}_2, \widetilde{\xi}, \widetilde{\mu}, \varepsilon)\| \\ &\le B\|g_{11} - \widetilde{g}_{11}\| + B_1\|g_{21} - \widetilde{g}_{21}\| + [c_1\|g_{11}\| + \hat{c}_1\|g_{21}\|]\,\|\alpha - \widetilde{\alpha}\| \end{aligned}$$

where

$$\widetilde{g}_{21}(t) = g_{21}(t, \widetilde{z}(0,t), \widetilde{z}(\pi/4,t), \widetilde{y}_1(t), \widetilde{y}_2(t), \widetilde{\xi}, \widetilde{\mu}, \varepsilon).$$

Now we have

$$g_{11}(t) - \widetilde{g}_{11}(t) = G_{11}(t) + \widetilde{G}_{11}(t)$$

where

$$\begin{aligned} G_{11}(t) &= g_{11}(t, z(0,t), z(\pi/4,t), y_1(t), y_2(t), \xi, \mu, \varepsilon) \\ &\quad - g_{11}(t, z(0,t), z(\pi/4,t), y_1(t), y_2(t), \widetilde{\xi}, \widetilde{\mu}, \varepsilon), \\ \widetilde{G}_{11}(t) &= g_{11}(t, z(0,t), z(\pi/4,t), y_1(t), y_2(t), \widetilde{\xi}, \widetilde{\mu}, \varepsilon) \\ &\quad - g_{11}(t, \widetilde{z}(0,t), \widetilde{z}(\pi/4,t), \widetilde{y}_1(t), \widetilde{y}_2(t), \widetilde{\xi}, \widetilde{\mu}, \varepsilon). \end{aligned}$$

An argument similar to the above shows that

$$\|\widetilde{G}_{11}\| \le \frac{1}{2B}\bar{\Delta}(\rho,\varepsilon)[\|y_1 - \widetilde{y}_1\| + \|y_2 - \widetilde{y}_2\| + \|z - \widetilde{z}\|_\infty].$$

On the other hand, since

$$\begin{aligned} g_{11}(t) = -Q_{m,\xi}\Big\{ &\sqrt{\varepsilon}\delta\dot{y}_1(t) + \frac{2}{\sqrt{\pi}}\mu \textstyle\int_0^{\pi/4} h(x,t)\,dx + \frac{2}{\sqrt{\pi}}[H_2(t,\xi) + H_3(t,\xi)] \\ &- \frac{8}{\pi}\Big[f'\Big(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\Big) - f'\Big(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\Big)\Big]y_1(t) + \sqrt{\varepsilon}\delta\dot{v}_\xi(t) + \ddot{v}_\xi(t) \\ &+ \frac{4}{\sqrt{\pi}}\Big[f\Big(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\Big) - f\Big(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\Big)\Big]\Big\} \end{aligned}$$

we have, using also the estimate for $H_{01}(t)$ given in the proof of Lemma 5.1.6,

$$\|G_{11}\| \le \frac{1}{2A_4B}\|Q_{m,\xi} - Q_{m,\widetilde{\xi}}\|\Big\{\bar{\Delta}(\rho,\varepsilon)[\|y_1\| + \|y_2\| + \|z\|_\infty] + k_6|\mu| + 2k_7\varepsilon\Big\}$$

$$+\frac{1}{2B}\left\{k_6|\mu-\widetilde{\mu}|+k_8[\rho\widetilde{\Delta}(\rho)(1+\varepsilon)+\varepsilon(1+\rho)]\|\alpha-\widetilde{\alpha}\|\right\}$$

$$\leq \frac{1}{2B}\left[\rho\bar{\Delta}(\rho,\varepsilon)+k_6|\mu|+2k_7\varepsilon+k_8\left(\rho\widetilde{\Delta}(\rho)(1+\varepsilon)+\varepsilon(1+\rho)\right)\right]\|\alpha-\widetilde{\alpha}\|$$

$$+\frac{1}{2B}k_6|\mu-\widetilde{\mu}|$$

and then

$$\|g_{11}-\widetilde{g}_{11}\| \leq \frac{1}{2B}\Big\{\bar{\Delta}(\rho,\varepsilon)[\|y_1-\widetilde{y}_1\|+\|y_2-\widetilde{y}_2\|+\|z-\widetilde{z}\|_\infty]+k_6|\mu-\widetilde{\mu}|$$

$$+\left[\rho\bar{\Delta}(\rho,\varepsilon)+k_6|\mu|+2k_7\varepsilon+k_8\left(\rho\widetilde{\Delta}(\rho)(1+\varepsilon)+\varepsilon(1+\rho)\right)\right]\|\alpha-\widetilde{\alpha}\|\Big\}.$$

Similarly

$$\|g_{21}-\widetilde{g}_{21}\| \leq \frac{1}{2B_1}\Big\{\bar{\Delta}(\rho,\varepsilon)[\|y_1-\widetilde{y}_1\|+\|y_2-\widetilde{y}_2\|+\|z-\widetilde{z}\|_\infty]$$

$$+k_8\left(\rho\widetilde{\Delta}(\rho)(1+\varepsilon)+\varepsilon(1+\rho)\right)\|\alpha-\widetilde{\alpha}\|+k_6|\mu-\widetilde{\mu}|\Big\},$$

hence, (5.1.86) follows from (5.1.30), (5.1.59) and (5.1.81) provided $\varepsilon>0$ and $\rho>0$ are sufficiently small. The proof is complete. □

Our goal is to prove that the map $(z(x,t),y_1(t),y_2(t))\mapsto(\hat{z}(x,t),\hat{y}_1(t),\hat{y}_2(t))$ has a unique fixed point which is then a solution of Eqs. (5.1.71)–(5.1.73). To this end, we will make use of the following result, whose proof is omitted since it is a slight modification of Lemma 3 in [11].

Lemma 5.1.9. *Let Z, Y be Banach spaces, $B_{Z\times Y}(\rho)$ be the closed ball centered at zero and of radius ρ, S be a set of parameters, $M\subset S\times(0,\bar{\sigma}]$, and $F: B_{Z\times Y}(\rho)\times M\times[-\bar{\mu},\bar{\mu}]\times(0,\bar{\sigma}]\to Z\times Y$ be a map defined as:*

$$F(z,y,\kappa,\mu,\sigma)=\begin{pmatrix}F_1(z,y,\kappa,\mu,\sigma)+L_{1\sigma}y+L_{2\sigma}z\\ F_2(z,y,\kappa,\mu,\sigma)+Lz\end{pmatrix},$$

with $L_{1\sigma}:Y\to Z$, $L_{2\sigma}:Z\to Z$ and $L:Z\to Y$ being uniformly bounded linear maps for $\sigma>0$ small. Assume that a constant C and a function $\Delta(\rho,\mu,\sigma)$ exist so that $\lim_{(\rho,\mu,\sigma)\to(0,0,0)}\Delta(\rho,\mu,\sigma)=0$, and

$$\begin{aligned}
&\|F_1(z,y,\kappa,\mu,\sigma)\|\leq C(|\mu|+\sigma)\sigma+\Delta(\rho,\mu,\sigma)(\|z\|+\|y\|),\\
&\|F_2(z,y,\kappa,\mu,\sigma)\|\leq C|\mu|+\Delta(\rho,\mu,\sigma)(\|z\|+\|y\|),\\
&\|L_{1\sigma}F_2(z,y,\kappa,\mu,\sigma)\|\leq C(|\mu|+\sigma)\sigma+\Delta(\rho,\mu,\sigma)(\|z\|+\|y\|)\\
&\|F_i(z_2,y_2,\kappa,\mu,\sigma)-F_i(z_1,y_1,\kappa,\mu,\sigma)\|\leq\Delta(\rho,\mu,\sigma)(\|z_2-z_1\|+\|y_2-y_1\|)
\end{aligned}\tag{5.1.87}$$

when $\|z\|+\|y\|<\rho$, $\|z_1\|+\|y_1\|<\rho$, *and* $\|z_2\|+\|y_2\|<\rho$. *If there are* $0<\lambda<1$ *and* $\bar{\sigma}_0>0$ *so that*

$$\|L_{1\sigma}L+L_{2\sigma}\|<\lambda$$

for any $0<\sigma\le\bar{\sigma}_0$, *then there exist* $\mu_0>0$, $\sigma_0>0$, $\rho_1>0$ *and* $\rho_2>0$ *so that for* $|\mu|\le\mu_0$, $\kappa\in M$, *and* $0<\sigma\le\sigma_0$, F *has a unique fixed point* $(z(\mu,\sigma,\kappa),y(\mu,\sigma,\kappa))\in B_Z(\rho_1)\times B_Y(\rho_2)$. *Moreover,*

$$\|z(\mu,\sigma,\kappa)\|+\|y(\mu,\sigma,\kappa)\|\le C_1(|\mu|+\sigma) \tag{5.1.88}$$

for some positive constant C_1 *independent of* (μ,σ,κ), *and*

$$\|z(\mu,\sigma,\kappa)\|/(|\mu|+\sigma)\to 0$$

uniformly with respect to κ, *as* $(\mu,\sigma)\to(0,0)$, $\sigma>0$. *Finally,* $(z(\mu,\sigma,\kappa),y(\mu,\sigma,\kappa))$ *is* C^r, $r\ge 0$, *in* (μ,σ) *if* $F(z,y,\kappa,\mu,\sigma)$ *is* C^r *in* (z,y,μ,σ).

We apply Lemma 5.1.9 with $\sigma=\sqrt{\varepsilon}\le\bar{\sigma}=(1/2)^{2/3}$, $S=X\times\mathbb{N}$, $\kappa=(\xi,m,\sigma)\in M:=X\times\{(m,\sigma)\in\mathbb{N}\times(0,\bar{\sigma}):m\ge[\sigma^{-3/2}]+1\}$ and

$$\begin{aligned}
&F_1(z,y_1,y_2,\xi,\mu,\sigma)=L_\varepsilon(H_1,H_2,H_3)+L_\varepsilon(0,\hat{H}_2,\hat{H}_3),\\
&F_2(z,y_1,y_2,\xi,\mu,\varepsilon)=(\hat{y}_{11},\hat{y}_{21}),\\
&L_{1\sigma}(y_1,y_2):=L_{1\varepsilon}(y_1,y_2)=L_\varepsilon(0,\tilde{H}_{21},\tilde{H}_{31}),\\
&L_{2\sigma}z:=L_{2\varepsilon}z=L_\varepsilon(0,\tilde{H}_{22},\tilde{H}_{32}),\\
&Lz=(\hat{y}_{10},\hat{y}_{20})
\end{aligned}$$

where $H_i(t)$, $\hat{H}_i(t)$ and $\tilde{H}_{ij}(t)$ have been defined in (5.1.75). We get the following result.

Theorem 5.1.10. *Assume that the conditions* (H1)–(H2) *hold and that* $\delta>0$ *is a fixed positive number so that*

(H3) $2M_1M_2|f'(0)|<\delta$.

Let $\Gamma>0$ *be fixed. Then there exist positive numbers* $\rho_1>0$, $\rho_2>0$, $\varepsilon_0>0$, *and* $\mu_0>0$ *so that for any* $\xi\in X$, $0<\varepsilon<\varepsilon_0$, $|\mu|<\mu_0$, $m>\varepsilon^{-3/4}$ *and* $\varepsilon\le\Gamma|\mu|$, *the integro-differential system* (5.1.71)–(5.1.73) *has a unique bounded solution*

$$(z(x,t,\mu,\varepsilon,\delta,\xi,m),\quad y_1(t,\mu,\varepsilon,\delta,\xi,m),\quad y_2(t,\mu,\varepsilon,\delta,\xi,m))$$

so that

$$\|z(x,t,\mu,\varepsilon,\delta,\xi,m)\|_\infty<\rho_1,\quad \|y_1(t,\mu,\varepsilon,\delta,\xi,m)\|+\|y_2(t,\mu,\varepsilon,\delta,\xi,m)\|<\rho_2.$$

Moreover

$$\|z(\cdot,\cdot,\mu,\varepsilon,\delta,\xi,m)\|_\infty+\|y_1(\cdot,\mu,\varepsilon,\delta,\xi,m)\|+\|y_2(\cdot,\mu,\varepsilon,\delta,\xi,m)\|\le\tilde{C}_1(|\mu|+\sqrt{\varepsilon})$$

for some positive constant $\tilde{C}_1$ *independent of* (μ,ε,ξ)*, and*

$$\|z(\cdot,\cdot,\mu,\varepsilon,\delta,\xi,m)\|_\infty/(|\mu|+\sqrt{\varepsilon}) \to 0$$

uniformly with respect to (ξ,m)*, as* $(\mu,\varepsilon)\to(0,0)$*,* $\varepsilon>0$*. Finally,*

$$z(\cdot,\cdot,\mu,\varepsilon,\delta,\xi,m),\quad y_1(\cdot,\mu,\varepsilon,\delta,\xi,m),\quad y_2(\cdot,\mu,\varepsilon,\delta,\xi,m)$$

are Lipschitz in α *uniformly with respect to* (E,m) *and the Lipschitz constants are* $O(\sqrt{\varepsilon}+|\mu|)$ *for* y_1*,* y_2 *and* $o(\sqrt{\varepsilon}+|\mu|)$ *for* z*.*

Proof. We shall prove that the assumptions of Lemma 5.1.9 are satisfied. Of course, we take $Z=C_b^0([0,\pi/4]\times\mathbb{R})$, $Y=Y_1\times Y_2$ as Banach spaces, $S=X\times\mathbb{N}$ and $M=\{(\xi,m,\sigma)\mid \xi\in X, m\in\mathbb{N}, m>\sigma^{-3/2}\}$. The fact that $L_{1\sigma}=L_{1\varepsilon}$ and $L_{2\sigma}=L_{2\varepsilon}$ are bounded linear maps, as well as the fact that $\hat{z}=F_1(z,y_1,y_2,\xi,\mu,\varepsilon)$ satisfies the first and fourth conditions in (5.1.87) follow from Lemma 5.1.7. Similarly the facts that $L:Z\to Y$ is a bounded linear map and $F_2(z,y_1,y_2,\xi,\mu,\varepsilon)$ satisfies the second and fourth inequalities in (5.1.87) follow from Lemma 5.1.8 (see (5.1.85), (5.1.86)) and the assumption $\varepsilon\le\Gamma|\mu|$. Thus, in order to apply Lemma 5.1.9, we only need to prove that

$$\|L_{1\varepsilon}(\hat{y}_{11},\hat{y}_{21})\|_\infty \le C(|\mu|+\sqrt{\varepsilon})\sqrt{\varepsilon}+\Delta(\rho,\mu,\sqrt{\varepsilon})(\|z\|_\infty+\|y_1\|+\|y_2\|) \quad (5.1.89)$$

and that

$$\|(L_{1\varepsilon}L+L_{2\varepsilon})z\|_\infty \le \lambda\|z\|_\infty \quad (5.1.90)$$

for any $\varepsilon>0$ small enough and some $\lambda\in(0,1)$. First we prove (5.1.89). We have

$$L_{1\varepsilon}(\hat{y}_{11},\hat{y}_{21}) =$$

$$L_\varepsilon\left(0, f'(0)\left[\frac{2}{\sqrt{\pi}}\hat{y}_{11}(t)-2\sqrt{\frac{3}{\pi}}\hat{y}_{21}(t)\right], f'(0)\left[\frac{2}{\sqrt{\pi}}\hat{y}_{11}(t)+2\sqrt{\frac{3}{\pi}}\hat{y}_{21}(t)\right]\right).$$

Now, from (5.1.85), (5.1.86) and the definition of the norms in Y_1, Y_2, we see that $\hat{y}_{11}(t)$ and $\hat{y}_{21}(t)$ are bounded together with their first derivatives. Thus, using Theorem 5.1.1(b), (5.1.85), (5.1.86), and assumption (H3) we get:

$$\begin{aligned}\|L_{1\varepsilon}(\hat{y}_{11},\hat{y}_{21})\|_\infty &\le 2M_1M_2|f'(0)|\left[5\varepsilon\left(\frac{1}{\delta^5}+1\right)+\frac{2}{\delta}\sqrt{\varepsilon}\right]\cdot[\|\hat{y}_{11}\|+\|\hat{y}_{21}\|]\\ &\le \sqrt{\varepsilon}\left[5\sqrt{\varepsilon}\left(\frac{1}{\delta^4}+\delta\right)+2\right]\cdot[\|\hat{y}_{11}\|+\|\hat{y}_{21}\|]\\ &\le \tilde{c}_1\sqrt{\varepsilon}[\Delta(\rho)+\sqrt{\varepsilon}(\delta+\sqrt{\varepsilon})](\|y_1\|+\|y_2\|+\|z\|_\infty)\\ &\quad+\tilde{c}_2\sqrt{\varepsilon}(|\mu|\|h\|_\infty+\varepsilon(\delta\sqrt{\varepsilon}+2))\end{aligned}$$

for some suitable choice of the positive constants $\tilde{c}_1$ and $\tilde{c}_2$ (possibly dependent on δ). Thus (5.1.89) follows. Now, we look at $L_{1\varepsilon}Lz$. We have

$$L_{1\varepsilon}Lz = L_\varepsilon\left(0, f'(0)\left[\frac{2}{\sqrt{\pi}}\hat{y}_{10}(t) - 2\sqrt{\frac{3}{\pi}}\hat{y}_{20}(t)\right], f'(0)\left[\frac{2}{\sqrt{\pi}}\hat{y}_{10}(t) + 2\sqrt{\frac{3}{\pi}}\hat{y}_{20}(t)\right]\right)$$

where $(\hat{y}_{10}(t), \hat{y}_{20}(t)) \in Y_1 \times Y_2$ is the unique bounded solution of Equation (5.1.84) that satisfies $\dot{\hat{y}}_{10}(2jm + \alpha_j) = 0$ for any $j \in \mathbb{Z}$ so that $e_j = 1$. Let $(\hat{y}_{12}(t), \hat{y}_{22}(t)) \in Y_1 \times Y_2$ be the unique bounded solution of

$$\begin{aligned}
\ddot{\hat{y}}_{12}(t) + \frac{8}{\pi} f'(0)\hat{y}_{12}(t) &= -Q_{m,\xi}\left\{\frac{2}{\sqrt{\pi}} f'\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right)[z(0,t) + z(\pi/4,t)]\right\}, \\
\ddot{\hat{y}}_{22}(t) + \frac{24}{\pi} f'(0)\hat{y}_{22}(t) &= -2\sqrt{\frac{3}{\pi}} f'\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right)[z(\pi/4,t) - z(0,t)]
\end{aligned}$$

and $(\hat{y}_{13}(t), \hat{y}_{23}(t)) \in Y_1 \times Y_2$ be the unique bounded solution of

$$\begin{aligned}
\ddot{\hat{y}}_{13}(t) + \frac{8}{\pi} f'(0)\hat{y}_{13}(t) &= -\frac{2}{\sqrt{\pi}} f'(0)[z(0,t) + z(\pi/4,t)] \\
\ddot{\hat{y}}_{23}(t) + \frac{24}{\pi} f'(0)\hat{y}_{23}(t) &= -2\sqrt{\frac{3}{\pi}} f'(0)[z(\pi/4,t) - z(0,t)].
\end{aligned}$$

We set

$$\begin{aligned}
\bar{H}_{23}(t) &= f'(0)\left[\frac{2}{\sqrt{\pi}}(\hat{y}_{10}(t) - \hat{y}_{12}(t)) - 2\sqrt{\frac{3}{\pi}}(\hat{y}_{20}(t) - \hat{y}_{22}(t))\right], \\
\tilde{H}_{23}(t) &= f'(0)\left[\frac{2}{\sqrt{\pi}}(\hat{y}_{12}(t) - \hat{y}_{13}(t)) - 2\sqrt{\frac{3}{\pi}}(\hat{y}_{22}(t) - \hat{y}_{23}(t))\right], \\
\hat{H}_{23}(t) &= f'(0)\left[\frac{2}{\sqrt{\pi}}\hat{y}_{13}(t) - 2\sqrt{\frac{3}{\pi}}\hat{y}_{23}(t)\right], \\
\bar{H}_{33}(t) &= f'(0)\left[\frac{2}{\sqrt{\pi}}(\hat{y}_{10}(t) - \hat{y}_{12}(t)) + 2\sqrt{\frac{3}{\pi}}(\hat{y}_{20}(t) - \hat{y}_{22}(t))\right], \\
\tilde{H}_{33}(t) &= f'(0)\left[\frac{2}{\sqrt{\pi}}(\hat{y}_{12}(t) - \hat{y}_{13}(t)) + 2\sqrt{\frac{3}{\pi}}(\hat{y}_{22}(t) - \hat{y}_{23}(t))\right], \\
\hat{H}_{33}(t) &= f'(0)\left[\frac{2}{\sqrt{\pi}}\hat{y}_{13}(t) + 2\sqrt{\frac{3}{\pi}}\hat{y}_{23}(t)\right]
\end{aligned}$$

and note that

$$L_{1\varepsilon}Lz = L_\varepsilon\left(0, \bar{H}_{23}(t), \bar{H}_{33}(t)\right) + L_\varepsilon(0, \tilde{H}_{23}(t), \tilde{H}_{33}(t)) + L_\varepsilon(0, \hat{H}_{23}(t), \hat{H}_{33}(t)).$$

We know from [11, above equation (3.39)] that

$$\begin{aligned}
\|L_\varepsilon(0, \hat{H}_{23}(t), \hat{H}_{33}(t))\|_\infty &\leq 8M_1M_2|f'(0)|\sqrt{\varepsilon}\left(2a\delta^{-1} + \sqrt{\varepsilon}\right)\|z\|_\infty \\
&\leq 4\sqrt{\varepsilon}(2a + \delta\sqrt{\varepsilon})\|z\|_\infty.
\end{aligned}$$

Then from Lemma 5.1.2–(ii) and Lemma 5.1.3–(i) we obtain:

$$|\hat{y}_{10}(t)-\hat{y}_{12}(t)| \leq \frac{4A_4NC}{\sqrt{\pi}}\left(\mathrm{e}^{-am/2}+\mathrm{e}^{-a|t-2jm-\alpha_j|/2}\right)\|z\|_\infty,$$

$$|\hat{y}_{20}(t)-\hat{y}_{22}(t)| \leq 4NC_1\sqrt{\frac{3}{\pi}}\left(\mathrm{e}^{-am/2}+\mathrm{e}^{-a|t-2jm-\alpha_j|/2}\right)\|z\|_\infty$$

for $t\in((2j-1)m,(2j+1)m]$, whereas Lemma 5.1.2–(iii) and Lemma 5.1.3–(ii) with $E=\{0\}$ and $\alpha'=\alpha''=0$, give:

$$\|\hat{y}_{12}-\hat{y}_{13}\| \leq \frac{8B}{\pi}(A_4+1)N\|z\|_\infty\,, \quad \|\hat{y}_{22}-\hat{y}_{23}\| \leq \frac{48B_1}{\pi}N\|z\|_\infty\,,$$

with the norms of the left-hand sides being in Y_1 and Y_2 respectively. Thus, Theorem 5.1.1–(a) implies, after some algebra:

$$\|L_\varepsilon(0,\bar{H}_{23},\bar{H}_{33})\|_\infty \leq \frac{8N}{\pi}(A_4C+3C_1)\left(\frac{1}{(2a^2)^{1/3}}+\frac{1}{\delta^2}+\frac{4\delta}{a}\right)\sqrt{\varepsilon}\|z\|_\infty$$

using the inequality $\mathrm{e}^{-am/2}<\sqrt{\varepsilon}\left(\frac{1}{2a^2}\right)^{1/3}$ that follows from $\left(\frac{am}{2}\right)^{2/3}\mathrm{e}^{-\frac{am}{2}}<\frac{1}{2}$ and $m\geq\varepsilon^{-3/4}$. Next, applying again Theorem 5.1.1(b) with $k_2=0$ (and hence letting β tend to $+\infty$) gives:

$$\|L_\varepsilon(0,\tilde{H}_{23},\tilde{H}_{33})\|_\infty \leq \frac{8N}{\pi}(B(A_4+1)+6B_1)\sqrt{\varepsilon}\left[5\sqrt{\varepsilon}\left(\frac{1}{\delta^4}+\delta\right)+2\right]\|z\|_\infty.$$

Plugging everything together we obtain

$$\|L_{1\varepsilon}L\| \leq K\sqrt{\varepsilon}$$

where K is a positive constant depending only on δ. Thus, using (5.1.75) we get

$$\|L_{1\varepsilon}L+L_{2\varepsilon}\| \leq 2M_1M_2\delta^{-1}|f'(0)|+K\sqrt{\varepsilon}$$

and then, from assumption (H3), we see that $\varepsilon_0>0$ exists so that for any $\varepsilon\in(0,\varepsilon_0)$, (5.1.90) holds. Since the assumptions of Lemma 5.1.9 are satisfied we obtain a solution of Equations (5.1.71)–(5.1.73) provided $0<\varepsilon<\varepsilon_0$, $|\mu|<\mu_0$ and $\varepsilon\leq\Gamma|\mu|$.

Finally, we prove that this solution satisfies the Lipschitz condition in $\alpha\in\ell^\infty_E$ as stated in the Theorem. Let $\xi'=(E,\alpha')\in X$, $\xi''=(E,\alpha'')\in X$ and set

$$\begin{gathered}
y_1'(t)=y_1(t,\mu,\varepsilon,\delta,\xi',m), \quad y_1''(t)=y_1(t,\mu,\varepsilon,\delta,\xi'',m),\\
y_2'(t)=y_2(t,\mu,\varepsilon,\delta,\xi',m), \quad y_2''(t)=y_2(t,\mu,\varepsilon,\delta,\xi'',m),\\
z'(x,t)=z(x,t,\mu,\varepsilon,\delta,\xi',m),\ z''(x,t)=z(x,t,\mu,\varepsilon,\delta,\xi'',m).
\end{gathered}$$

Then $(z(x,t),y_1(t),y_2(t))=(z'(x,t)-z''(x,t),y_1'(t)-y_1''(t),y_2'(t)-y_2''(t))$ is a fixed point of the map

$$
\begin{aligned}
z(x,t) &= F_1\left(z(x,t)+z''(x,t), y_1(t)+y_1''(t), y_2(t)+y_1''(t), \xi', \mu, \varepsilon\right)\\
&\quad -F_1\left(z''(x,t), y_1''(t), y_2''(t), \xi'', \mu, \varepsilon\right) + L_{1\varepsilon}(y_1(t), y_2(t)) + L_{2\varepsilon} z(x,t,),\\
(y_1(t), y_2(t)) &= F_2(z(x,t)+z''(x,t), y_1(t)+y_1''(t), y_2(t)+y_1''(t), \xi', \mu, \varepsilon)\\
&\quad -F_2(z''(x,t), y_1''(t), y_2''(t), \xi'', \mu, \varepsilon) + Lz(x,t)\,.
\end{aligned}
\tag{5.1.91}
$$

From (5.1.86) we obtain

$$
\begin{aligned}
&\| \, F_2(z(x,t)+z''(x,t), y_1(t)+y_1''(t), y_2(t)+y_1''(t), \xi', \mu, \varepsilon)\\
&\quad -F_2(z''(x,t), y_1''(t), y_2''(t), \xi'', \mu, \varepsilon)\|\\
&\quad \le \bar{\Delta}(\rho,\varepsilon)(\|y_1\| + \|y_2\| + \|z\|_\infty) + k_4(|\mu| + \varepsilon + \rho\bar{\Delta}(\rho,\varepsilon))\|\alpha' - \alpha''\|
\end{aligned}
\tag{5.1.92}
$$

where $\bar{\Delta}(\rho,\varepsilon) \to 0$ as $\rho + \varepsilon \to 0^+$ and $k_4 > 0$ is a suitable constant. Thus, using Theorem 5.1.1(b) (with $k_2 = 0$ and $\beta = +\infty$) we see that a positive constant k_5 exists so that

$$
\begin{aligned}
&\| \, L_{1\varepsilon}(F_2(z(x,t)+z''(x,t), y_1(t)+y_1''(t), y_2(t)+y_1''(t), \xi', \mu, \varepsilon)\\
&\qquad - F_2(z''(x,t), y_1''(t), y_2''(t), \xi'', \mu, \varepsilon))\|_\infty\\
&\quad \le k_5\sqrt{\varepsilon}(|\mu| + \varepsilon + \rho\bar{\Delta}(\rho,\varepsilon))\|\alpha' - \alpha''\| + k_5\sqrt{\varepsilon}\bar{\Delta}(\rho,\varepsilon)(\|y\| + \|z\|_\infty)
\end{aligned}
\tag{5.1.93}
$$

for $\|y\| = \|y_1\| + \|y_2\|$. Now we replace $(y_1(t), y_2(t))$ in $L_{1\varepsilon}(y_1(t), y_2(t))$ in the first equation in (5.1.91) with the fixed point of the second equation in (5.1.91). Using Lemma 5.1.7, Lemma 5.1.8, (5.1.92) and (5.1.93), we get

$$
\begin{aligned}
&\|z\|_\infty \le \Delta_2(\rho,\varepsilon)(\|y\| + \|z\|_\infty) + k_9\sqrt{\varepsilon}(\sqrt{\varepsilon} + \rho + |\mu|)\|\alpha' - \alpha''\| + \lambda\|z\|_\infty\,,\\
&\|y\| \le \Delta_1(\rho,\varepsilon)(\|y\| + \|z\|_\infty) + k_4(\rho\bar{\Delta}(\rho,\varepsilon) + |\mu| + \varepsilon)\|\alpha' - \alpha''\| + \|L\|\|z\|_\infty
\end{aligned}
\tag{5.1.94}
$$

where $\Delta_1(\rho,\varepsilon), \Delta_2(\rho,\varepsilon) \to 0$ as $\rho + \varepsilon \to 0^+$ and k_9 is a positive constant. From (5.1.88) we know that $\rho = O(\sqrt{\varepsilon} + |\mu|)$. Thus, if ε is sufficiently small, we can solve the first inequality in (5.1.94) for $\|z\|_\infty$ and get:

$$
\|z\|_\infty \le \bar{\Delta}_2(\rho,\varepsilon)\|y\| + \sqrt{\varepsilon}O(|\mu| + \sqrt{\varepsilon})\|\alpha' - \alpha''\|
\tag{5.1.95}
$$

for $\bar{\Delta}_2(\rho,\varepsilon) \to 0$ as $\rho + \varepsilon \to 0^+$. Then we plug this estimate of $\|z\|_\infty$ into the second inequality in (5.1.94) and get:

$$
\|y\| \le O(|\mu| + \sqrt{\varepsilon})\|\alpha' - \alpha''\|\,.
$$

Finally, we plug again this estimate into (5.1.95) and obtain

$$
\|z\|_\infty \le o(\sqrt{\varepsilon} + |\mu|)\|\alpha' - \alpha''\|.
$$

The proof is complete. □

In order to find a bounded solution, near $\gamma_\xi(t)$, of Eqs. (5.1.6)–(5.1.8) we need to show that the equation

$$\begin{aligned}
&G(\xi,\varepsilon,\mu,\delta,m)\\
&:= P_{m,\xi}\Big\{\sqrt{\varepsilon}\delta\dot{y}_1(t,\mu,\varepsilon,\delta,\xi,m)+\frac{2}{\sqrt{\pi}}\mu\int_0^{\pi/4}h(x,t)\,dx\\
&\quad+\sqrt{\varepsilon}\delta\dot{\gamma}_\xi(t)+\frac{2}{\sqrt{\pi}}f\Big(\frac{2}{\sqrt{\pi}}[y_1(t,\mu,\varepsilon,\delta,\xi,m)+\Gamma_\xi(t)]\\
&\quad-2\sqrt{\frac{3}{\pi}}y_2(t,\mu,\varepsilon,\delta,\xi,m)+z(0,t,\mu,\varepsilon,\delta,\xi,m)\Big)\\
&\quad+\frac{2}{\sqrt{\pi}}f\Big(\frac{2}{\sqrt{\pi}}[y_1(t,\mu,\varepsilon,\delta,\xi,m)+\Gamma_\xi(t)]\\
&\quad+2\sqrt{\frac{3}{\pi}}y_2(t,\mu,\varepsilon,\delta,\xi,m)+z(\frac{\pi}{4},t,\mu,\varepsilon,\delta,\xi,m)\Big)\\
&\quad-\frac{4}{\sqrt{\pi}}f\left(\frac{2}{\sqrt{\pi}}\gamma_\xi(t))-\frac{8}{\pi}f'(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\right)y_1(t,\mu,\varepsilon,\delta,\xi,m)\\
&\quad+\sqrt{\varepsilon}\delta\dot{v}_\xi(t)+\ddot{v}_\xi(t)\Big\}=0
\end{aligned}$$

can be solved for some values of the parameters. From Theorem 5.1.10, we know that

$$\begin{aligned}
\|y_1(t,\mu,\varepsilon,\delta,\xi,m)\| &= O(|\mu|+\sqrt{\varepsilon}),\\
\|y_2(t,\mu,\varepsilon,\delta,\xi,m)\| &= O(|\mu|+\sqrt{\varepsilon}),\\
\|z(x,t,\mu,\varepsilon,\delta,\xi,m)\|_\infty &= o(|\mu|+\sqrt{\varepsilon}),\\
\|y_1(t,\mu,\varepsilon,\delta,\xi',m)-y_1(t,\mu,\varepsilon,\delta,\xi'',m)\| &\le O(|\mu|+\sqrt{\varepsilon})\|\alpha'-\alpha''\|,\\
\|y_2(t,\mu,\varepsilon,\delta,\xi',m)-y_2(t,\mu,\varepsilon,\delta,\xi'',m)\| &\le O(|\mu|+\sqrt{\varepsilon})\|\alpha'-\alpha''\|,\\
\|z(x,t,\mu,\varepsilon,\delta,\xi',m)-z(x,t,\mu,\varepsilon,\delta,\xi'',m)\|_\infty &\le o(|\mu|+\sqrt{\varepsilon})\|\alpha'-\alpha''\|
\end{aligned} \tag{5.1.96}$$

where $\xi=(E,\alpha)$, $\xi'=(E,\alpha')$, $\xi''=(E,\alpha'')$, and $O(|\mu|+\sqrt{\varepsilon})$, $o(|\mu|+\sqrt{\varepsilon})$ are uniform with respect to (ξ,m). Thus, we set $\mu=\sqrt{\varepsilon}\eta$, where η belongs to a compact subset of $\mathbb{R}\setminus\{0\}$ where the condition $\Gamma|\eta|\ge\varepsilon$ is satisfied (possibly taking ε smaller). By multiplying the equation $G(\xi,\varepsilon,\sqrt{\varepsilon}\eta,\delta,m)=0$ by $\varepsilon^{-1/2}$, we obtain the equation:

$$\widetilde{B}(\xi,\varepsilon,\eta,\delta,m):=P_{m,\xi}\left\{\delta\dot{\gamma}_\xi(t)+\frac{2}{\sqrt{\pi}}\eta\int_0^{\pi/4}h(x,t)\,dx+r(t,\xi,\varepsilon,\eta,\delta,m)\right\}=0 \tag{5.1.97}$$

where $\widetilde{B}(\xi,\varepsilon,\eta,\delta,m)=\varepsilon^{-1/2}G(\xi,\varepsilon,\sqrt{\varepsilon}\eta,\delta,m)$. Using (5.1.70) and (5.1.97) we see that

$$\begin{aligned} &\|r(t,\xi,\varepsilon,\eta,\delta,m)\|_\infty = o(1), \\ &\|r(t,\xi',\varepsilon,\eta,\delta,m) - r(t,\xi'',\varepsilon,\eta,\delta,m)\|_\infty \le o(1)\|\alpha' - \alpha''\| \end{aligned} \tag{5.1.98}$$

as $\varepsilon \to 0^+$ uniformly with respect to (ξ,η,m). Let

$$M_\eta(\alpha) = \delta \int_{-\infty}^{\infty} \dot{\gamma}_1(s)^2\,ds + \frac{2}{\sqrt{\pi}}\eta \int_{-\infty}^{\infty}\int_0^{\pi/4} \dot{\gamma}_1(s)h(x,s+\alpha)\,dx\,ds \tag{5.1.99}$$

and consider the space $\mathscr{C} = C^0\big([0,\pi/4]\times\mathbb{R},\mathbb{R}\big)$ endowed with the metric $d_{\mathscr{C}}$ given by

$$d_{\mathscr{C}}(u_1,u_2) = \sum_{n\in\mathbb{N}} 2^{-|n|} \max_{[0,\pi/4]\times[-n,n]} |u_1(x,t) - u_2(x,t)|\,.$$

Finally we define a (weak) solution of (5.1.1) to be any $u(x,t) \in C([0,\pi/4]\times\mathbb{R})$ satisfying the identity

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_0^{\pi/4} \Big\{u(x,t)\Big[v_{tt}(x,t) + v_{xxxx}(x,t) - \varepsilon\delta v_t(x,t)\Big] + \varepsilon\mu h(x,\sqrt{\varepsilon}t)v(x,t)\Big\}\,dx\,dt \\ &+\varepsilon\int_{-\infty}^{\infty}\Big\{f(u(0,t))v(0,t) + f(u(\pi/4,t))v(\pi/4,t)\Big\}\,dt = 0 \end{aligned} \tag{5.1.100}$$

for any $v(x,t) \in C^\infty([0,\pi/4]\times\mathbb{R})$ so that $v(x,t)$ has a compact support and satisfies boundary value conditions (5.1.4). Now we have the following result.

Theorem 5.1.11. *Let $f(x) \in C^2(\mathbb{R})$ and $h(x,t) = h(x,t+1) \in C^2([0,\pi/4]\times\mathbb{R})$ be so that (H1), (H2) hold. Let $\delta > 0$ be a fixed positive number that satisfies (H3). Then, if $\eta_0 \neq 0$ can be chosen in such a way that the equation $M_\eta(\alpha) = 0$ for $\eta = \eta_0$, has a simple root $\alpha_0 \in [0,1]$, there exist $\bar{\varepsilon} > 0$, $\bar{\eta} > 0$ so that for any $\varepsilon \in (0,\bar{\varepsilon}]$, $\mu = \sqrt{\varepsilon}\eta$ with $|\eta - \eta_0| \le \bar{\eta}$ and $m > \varepsilon^{-3/4}$, $m \in \mathbb{N}$, there is a continuous map $\Pi : \mathscr{E} \to C^0([0,\pi/4]\times\mathbb{R},\mathbb{R})$ so that $\Pi(E) = u_E(x,t)$ is a weak solution of Equation* (5.1.1). *Moreover, $\Pi : \mathscr{E} \to \Pi(\mathscr{E})$ is a homeomorphism satisfying*

$$\Pi(\sigma(E))(x,t) = \Pi(E)(x,t + (2m/\sqrt{\varepsilon}))$$

with $\sigma : \mathscr{E} \to \mathscr{E}$ being the Bernoulli shift. Consequently, the Smale horseshoe can be embedded into the dynamics of (5.1.1).

Proof. We will prove that Eq. (5.1.97) can be solved for any $\xi \in X$ and ε, μ and η as in the statement of the theorem. Of course, there is nothing to prove if $\xi = 0$ since $P_{m,0} = 0$. Thus we assume $E \neq 0$ and recall (see Theorem 5.1.4) that $P_{m,\xi}h = 0$ is equivalent to $[N_{m,\xi} + \mathscr{L}_{m,\xi}]h = 0$. So, we solve the equation

$$[N_{m,\xi} + \mathscr{L}_{m,\xi}]\left\{\delta\dot{\gamma}_\xi(t) + \frac{2}{\sqrt{\pi}}\eta\int_0^{\pi/4} h(x,t)\,dx + r(t,\xi,\varepsilon,\eta,\delta,m)\right\} = 0. \tag{5.1.101}$$

From (5.1.22) and (5.1.98) we know that the term in braces in the above equation is Lipschitz continuous function in $\alpha \in \ell_E^\infty$ uniformly with respect to (E,ε,η,m).

But $\|\mathscr{L}_{m,\xi}\| \le A\,\mathrm{e}^{-am} < \frac{2A}{5a^{4/3}}\varepsilon$ (having used again $\theta^{4/3}\mathrm{e}^{-\theta} < \frac{2}{5}$) and in Section 5.1.6 that follows, we will see that a positive constant $\tilde{A}$ exists so that $\|\mathscr{L}_{m,\xi'} - \mathscr{L}_{m,\xi''}\| \le \tilde{A}\,\mathrm{e}^{-am}\|\alpha' - \alpha''\| < \frac{2\tilde{A}}{5a^{4/3}}\varepsilon\|\alpha' - \alpha''\|$ for any $\xi' = (E,\alpha')$, $\xi'' = (E,\alpha'')$. As a consequence the function of ξ

$$\mathscr{L}_{m,\xi}\left\{\delta\dot{\gamma}_\xi(t) + \frac{2}{\sqrt{\pi}}\eta\int_0^{\pi/4} h(x,t)\,dx + r(t,\xi,\varepsilon,\eta,\delta,m)\right\}$$

is Lipschitz in $\alpha \in \ell_E^\infty$, with a $O(\varepsilon)$ Lipschitz constant which can be taken independently of (E,η,m). Next we consider

$$N_{m,\xi}\left\{\delta\dot{\gamma}_\xi(t) + \frac{2}{\sqrt{\pi}}\eta\int_0^{\pi/4} h(x,t)\,dx + r(t,\xi,\varepsilon,\eta,\delta,m)\right\}.$$

From the proof of Theorem 5.1.4 we know that $\xi \mapsto \|N_{m,\xi}\|$ is bounded uniformly with respect to (ξ,m) (see (5.1.63)) and Lipschitz continuous function in $\alpha \in \ell_E^\infty$ uniformly in (E,m) (actually we proved that $\|N_{m,\xi'} - N_{m,\xi''}\| \le \frac{16\bar{A}_2N}{a\pi}\|\alpha' - \alpha''\|$). So, using (5.1.98) we see that $N_{m,\xi}r(t,\xi,\varepsilon,\eta,\delta,m)$ is Lipschitz continuous function in $\alpha \in \ell_E^\infty$ uniformly in (E,m,η) and the Lipschitz constant tends to 0 as $\varepsilon \to 0$. Finally, we consider the map from ℓ_E^∞ into itself:

$$\alpha \mapsto N_{m,(E,\alpha)}\left\{\delta\dot{\gamma}_{(E,\alpha)}(t) + \frac{2}{\sqrt{\pi}}\eta\int_0^{\pi/4} h(x,t)\,dx\right\} - \widetilde{\mathscr{M}_\eta}(\alpha) \in \ell_E^\infty \tag{5.1.102}$$

where

$$\widetilde{\mathscr{M}_\eta}(\alpha) = \left\{e_j M_\eta(\alpha_j)\right\}_{j\in\mathbb{Z}}.$$

It is easy to see that the j–th component of the map (5.1.102) is given by the sum of the following two terms:

$$-e_j\int_{-\infty}^{-m-\alpha_j}\dot{\gamma}_1(t)\left[\delta\dot{\gamma}_1(t) + \frac{2}{\sqrt{\pi}}\eta\int_0^{\pi/4} h(x,t+\alpha_j)\,dx\right]dt,$$

$$-e_j\int_{m+\alpha_j}^{\infty}\dot{\gamma}_1(t)\left[\delta\dot{\gamma}_1(t) + \frac{2}{\sqrt{\pi}}\eta\int_0^{\pi/4} h(x,t+\alpha_j)\,dx\right]dt$$

and that the above functions are Lipschitz continuous function in α uniformly in (η,m,j) and with a $O(\varepsilon)$ Lipschitz constant, provided η belongs to a compact domain and ε is small. In fact, we have, for example, using also (5.1.53):

$$\left|\int_{-\infty}^{-m-\alpha_j'}\dot{\gamma}_1(t)\int_0^{\pi/4} h(x,t+\alpha_j')\,dx\,dt - \int_{-\infty}^{-m-\alpha_j''}\dot{\gamma}_1(t)\int_0^{\pi/4} h(x,t+\alpha_j'')\,dx\,dt\right|$$

$$\le \left|\int_{-m-\alpha_j''}^{-m-\alpha_j'}\dot{\gamma}_1(t)\int_0^{\pi/4} h(x,t+\alpha_j')\,dx\,dt\right|$$

$$
\begin{aligned}
&+\left|\int_{-\infty}^{-m-\alpha_j''}\dot\gamma_1(t)\int_0^{\pi/4}[h(x,t+\alpha_j')-h(x,t+\alpha_j'')]\,dx\,dt\right|\\
&\leq \frac{\bar A_2\pi}{4a}\mathrm{e}^{-am}[\|h\|_\infty|\mathrm{e}^{a\alpha_j'}-\mathrm{e}^{a\alpha_j''}|+\|h_t\|_\infty|\alpha_j'-\alpha_j''|]\\
&=O(\varepsilon)[\|h\|_\infty+\|h_t\|_\infty]\|\alpha'-\alpha''\|\,.
\end{aligned}
$$

A similar argument applies to the other quantities. Next, it is easy to see that the map $\widetilde{\mathscr{M}}_\eta:\ell_E^\infty\to\ell_E^\infty$ is C^1 in (α,η), and its derivative, with respect to α at the point $(\{e_j\alpha_0\}_{j\in\mathbb{Z}},\eta_0)\in\ell_E^\infty\times\mathbb{R}$, is given by:

$$
\alpha\mapsto\{M_{\eta_0}'(\alpha_0)\alpha_j\}_{j\in\mathbb{Z}}=\mathscr{M}_{\eta_0}'(\alpha_0)\alpha.
$$

As a matter of fact, we have:

$$
\tilde M_\eta(\alpha)-\tilde M_\eta(\alpha_0)-\tilde M_\eta'(\alpha_0)(\alpha-\alpha_0)=o(\|\alpha-\alpha_0\|)
$$

uniformly with respect to (η,E). So, we write (5.1.101) as a fixed point equation in ℓ_E^∞:

$$
\alpha=\alpha-\widetilde{\mathscr{M}}_{\eta_0}'(\alpha_0)^{-1}\widetilde{\mathscr{M}}_\eta(\alpha)-\widetilde{\mathscr{M}}_{\eta_0}'(\alpha_0)^{-1}R(\xi,\varepsilon,\eta,\delta)
$$

where $R(\xi,\varepsilon,\eta,\delta)$ is Lipschitz continuous function in $\alpha\in\ell_E^\infty$ with a $o(1)$ constant independent of (E,m,η). Moreover, the map $(\alpha,\eta)\mapsto\alpha-\widetilde{\mathscr{M}}_{\eta_0}'(\alpha_0)^{-1}\widetilde{\mathscr{M}}_\eta(\alpha)$ is C^1 and its $\alpha-$derivative vanishes at $\alpha=\alpha_0$ and $\eta=\eta_0$. Thus, from the uniform contraction principle 2.2.1 it follows the existence of $\bar\varepsilon>0$ and $\bar\eta>0$ so that for any $\varepsilon\in(0,\bar\varepsilon]$, $|\eta-\eta_0|\leq\bar\eta$ and $m>\varepsilon^{-3/4}$, $m\in\mathbb{N}$, the map

$$
\alpha\mapsto\alpha-\widetilde{\mathscr{M}}_{\eta_0}'(\alpha_0)^{-1}\widetilde{\mathscr{M}}_\eta(\alpha)-\widetilde{\mathscr{M}}_{\eta_0}'(\alpha_0)^{-1}R(\xi,\varepsilon,\eta,\delta)
$$

has a unique fixed point $\alpha=\alpha(E,m,\eta,\delta,\varepsilon)$ that tends to α_0 as $\varepsilon\to0$ and $\eta\to\eta_0$, uniformly with respect to (E,m). This implies that for any $\varepsilon\in(0,\varepsilon_0]$, $|\eta-\eta_0|\leq\bar\eta$ and $m>\varepsilon^{-3/4}$ the function

$$
\begin{aligned}
u_E(x,t):=&\left[y_1(\sqrt{\varepsilon}t,\sqrt{\varepsilon}\eta,\varepsilon,\delta,E,\alpha(E,m,\eta,\delta,\varepsilon))+\Gamma_\xi(t)\right]w_{-1}(x)\\
&+y_2(\sqrt{\varepsilon}t,\sqrt{\varepsilon}\eta,\varepsilon,\delta,E,\alpha(E,m,\eta,\delta,\varepsilon))w_0(x)\\
&+z(x,\sqrt{\varepsilon}t,\sqrt{\varepsilon}\eta,\varepsilon,\delta,E,\alpha(E,m,\eta,\delta,\varepsilon))
\end{aligned}
$$

is a solution of (5.1.101) near $\gamma_E(t)$ defined as

$$
\gamma_E(t)=\begin{cases}\gamma\Big(2\sqrt{\dfrac{2}{\pi}}\big(\sqrt{\varepsilon}t-2jm-\alpha_0\big)\Big), & \text{for}\quad (2j-1)m<\sqrt{\varepsilon}t\leq(2j+1)m\\ & \qquad\text{and}\quad e_j=1\,,\\[2ex] 0, & \text{for}\quad (2j-1)m<\sqrt{\varepsilon}t\leq(2j+1)m\\ & \qquad\text{and}\quad e_j=0\,.\end{cases}
$$

Since $u_E(x, 2jm\varepsilon^{-1/2})$ is near to $u = 0$ if $e_j = 0$ or to $u = \gamma\Big(-2\sqrt{\frac{2}{\pi}}\alpha_0\Big) \neq 0$ if $e_j = 1$, we see that for $\bar{\varepsilon}$ sufficiently small, the map $\Pi : E \to u_E$ is one-to-one and the choice of E determines the oscillatory properties of $u_E(x,t)$ near $\gamma(t)$. Moreover, $u_E(x,t)$ is the unique solution of (5.1.101) that satisfies the above oscillatory property and can be written as a totally convergent series:

$$u_E(x,t) = \sum_{i=-1}^{\infty} u_{i,E}(t) w_i(x).$$

Let $\sigma : \mathscr{E} \to \mathscr{E}$ be the shift map defined by $\sigma(\{e_j\}_{j\in\mathbb{Z}}) = \{e_{j+1}\}_{j\in\mathbb{Z}}$. Then $u_{\sigma(E)}(x,t)$ has the same oscillatory properties between $u = 0$ and $u = \gamma\Big(-2\sqrt{\frac{2}{\pi}}\alpha_0\Big) \neq 0$ as $u_E(x, t + 2m\varepsilon^{-1/2})$. But we have

$$u_E(x, t + 2m\varepsilon^{-1/2}) = \sum_{i=-1}^{\infty} u_{i,E}(t + 2m\varepsilon^{-1/2}) w_i(x)$$

and the series is again totally convergent. Thus, because of the uniqueness, we obtain:

$$u_{\sigma(E)}(x,t) = u_E(x, t + 2m/\sqrt{\varepsilon}).$$

We now prove the continuity of Π, with respect to the topologies on $\mathscr{E}$ and $\mathscr{C} = C\big([0,\pi/4] \times \mathbb{R}, \mathbb{R}\big)$ induced by the metrics $d_{\mathscr{E}}$ and $d_{\mathscr{C}}$. First, we observe that Theorem 5.1.1 implies the existence of a positive constant c_0 so that for any $E \in \mathscr{E}$, the components $u_{i,E}(t)$ of $u_E(x,t)$ satisfy:

$$\|u_{i,E}\|_\infty \leq c_0/(\mu_i^2 + 1), \quad \|\dot{u}_{i,E}\|_\infty \leq c_0 \tag{5.1.103}$$

with c_0 being a suitable constant (see (5.1.13), (5.1.14)). Hence, for any $R > 0$ there exists $n_0 \in \mathbb{N}$ so that, for any $E \in \mathscr{E}$, we have

$$\|u_E(x,t) - \sum_{i=-1}^{n_0} u_{i,E}(t) w_i(x)\|_\infty \leq 1/R.$$

Now, let $\{E_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathscr{E}$. From (5.1.103) and the Arzelà-Ascoli theorem 2.1.3 the existence follows of a subsequence $\{j_k^{(-1)}\}$ of $\{j_k^{(-2)} := k\}$ so that $u_{-1,E_{j_k^{(-1)}}}(t)$ converges uniformly in any interval $[-n,n]$. Then another application of the Arzelà-Ascoli theorem 2.1.3 implies the existence of a subsequence $\{j_k^{(0)}\}$ of $\{j_k^{(-1)}\}$ so that $u_{0,E_{j_k^{(0)}}}(t)$ converges uniformly in any interval $[-n,n]$. Proceeding in this way, for any $i = -1, 0, 1\ldots$, we construct a subsequence $\{j_k^{(i)}\}$ of $\{j_k^{(i-1)}\}$ so that $u_{i,E_{j_k^{(i)}}}(t)$ converges uniformly in any interval $[-n,n]$. Then, we use Cantor diagonal procedure to see that for any $i = -1, 0, 1\ldots$ the sequence $u_{i,E_{j_k^{(k)}}}(t)$ converges uniformly in any interval $[-n,n]$. Now, let E_{j_n} be a subsequence of E_j so that for any $i = -1, 0, \ldots$, $u_{i,E_{jn}}(t)$ converges to a continuous function $u_i(t)$ uniformly on any compact subset of $\mathbb{R}$. We have just proved that the set of such subsequences is not empty. From (5.1.103) we obtain $\|u_i\|_\infty \le c_0/(\mu_i^2+1)$ and hence the series $\sum_{i=-1}^{\infty} u_i(t)w_i(x)$ is totally convergent and defines a continuous function $u(x,t)$. Moreover, for $(x,t) \in [0, \frac{\pi}{4}] \times [-n,n]$ and any $R > 0$, we have

$$\left|u_{E_{j_k}}(x,t) - u(x,t)\right| \le \left|u_{E_{j_k}}(x,t) - \sum_{i=-1}^{n_0} u_{i,E_{j_k}}(t)w_i(x)\right| \\ + M_1 \sum_{i=-1}^{n_0} \left|u_{i,E_{j_k}}(t) - u_i(t)\right| + \left|u(x,t) - \sum_{i=-1}^{n_0} u_i(t)w_i(x)\right|.$$

So,

$$\overline{\lim}_{k\to\infty} |u_{E_{j_k}}(x,t) - u(x,t)| \le 2/R.$$

As a consequence, $u_{E_{jn}}(x,t) \to u(x,t)$ uniformly on compact sets. Thus the following statement holds:

for any given sequence $\{E_j\}_{j\in\mathbb{N}}$ in $\mathscr{E}$ there exists a subsequence $\{E_{j_k}\}_{k\in\mathbb{N}}$ so that $\{u_{E_{j_k}}(x,t)\}_{k\in\mathbb{N}}$ converges uniformly on compact sets to a continuous function

$$u(x,t) = \sum_{i=-1}^{\infty} u_{i,E}(t)w_i(x)$$

with the series being totally convergent and $u(x,t)$ being a weak solution of (5.1.1).

Now, assume that Π is not continuous. Then $E, E_j \in \mathscr{E}$, $j \in \mathbb{N}$ exist so that $d_{\mathscr{E}}(E_j, E) \to 0$, as $j \to \infty$ but $d_{\mathscr{C}}(u_{E_j}, u_E)$ is greater than a positive number for any $j \in \mathbb{N}$. Passing to a subsequence, if necessary, we can assume that $u_{E_j}(x,t)$ converges uniformly on compact sets to a weak solution $\hat{u}(x,t)$ of (5.1.1). Then, for any $(x,t) \in [0, \frac{\pi}{4}] \times \mathbb{R}$, we have

$$|\hat{u}(x,t) - \gamma_E(t)| \le |u_{E_{jn}}(x,t) - \hat{u}(x,t)| + |u_{E_{jn}}(x,t) - \gamma_{E_{jn}}(t)| + |\gamma_{E_{jn}}(t) - \gamma_E(t)|$$

and hence, passing to the limit for $n \to \infty$:

$$|\hat{u}(x,t) - \gamma_E(t)| \le \sup_n \|u_{E_{jn}} - \gamma_{E_{jn}}\|_\infty + \overline{\lim}_{n\to\infty} |\gamma_{E_{jn}}(t) - \gamma_E(t)|.$$

But, since $d_{\mathscr{E}}(E_j, E) \to 0$ we see that for $n > \overline{n}(\varepsilon, t)$ we have $\gamma_{E_{jn}}(t) = \gamma_E(t)$. So $\hat{u}(x,t)$ is orbitally close to $\gamma_E(t)$ and then, because of uniqueness,

$$\hat{u}(x,t) = u_E(x,t) = \Pi(E)$$

contradicting the assumption that Π was not continuous. The proof is complete. □

Remark 5.1.12. (a) If (H2) fails so that linear equation (5.1.10) has a two-dimensional space of bounded solutions on $\mathbb{R}$, then we can perform again the above procedure but we get a two-dimensional mapping like (5.1.99) of the form $M_\eta(\alpha,\beta)$, $(\alpha,\beta)\in\mathbb{R}^2$ (cf Section 4.1.3) and the existence of a simple root of function $M_\eta(\alpha,\beta)$ implies a result similar to Theorem 5.1.11.

(b) Assuming also that f is odd, i.e. $f(-y)=-f(y)$, then we get the additional homoclinic orbit $(0,\gamma_2(t)) := \left(0,\frac{1}{2}\sqrt{\frac{\pi}{3}}\gamma\left(2\sqrt{\frac{6}{\pi}}t\right)\right)$ for (5.1.9) and we can repeat the above approach by assuming the non-degeneracy of $\gamma_2(t)$ as in (H2). We get in this way another chaotic solutions of (5.1.1) when the corresponding mapping like (5.1.99) has a simple root. We do not perform here such computations.

(c) If we consider in (5.1.1) the time scale 1, i.e. we have $h(x,t)$ in (5.1.1), then (5.1.2) becomes a rapidly oscillating perturbed problem. So we should arrive at an exponentially small bifurcation problem [12, 13].

5.1.5 Useful Numerical Estimates

To get more information on condition (H3), we give in this section a numerical estimate of the constants M_1 and M_2 (see (5.1.16)). For this purpose, we recall [2]

$$w_k(x)=\frac{4}{\sqrt{\pi}W_k}\left[\cosh(\mu_k x)+\cos(\mu_k x)-\frac{\cosh\xi_k-\cos\xi_k}{\sinh\xi_k-\sin\xi_k}\big(\sinh(\mu_k x)+\sin(\mu_k x)\big)\right], \tag{5.1.104}$$

where $\xi_k=\mu_k\pi/4$ are determined by the equation $\cos\xi_k\cosh\xi_k=1$ and the constants W_k are given by the formula

$$W_k=\cosh\xi_k+\cos\xi_k-\frac{\cosh\xi_k-\cos\xi_k}{\sinh\xi_k-\sin\xi_k}\big(\sinh\xi_k+\sin\xi_k\big). \tag{5.1.105}$$

We first evaluate W_k. Numerically we find $\xi_1 \doteq 4.73004075$. Moreover, $0<\xi_1<\xi_2<\cdots$ and so $\cosh\xi_1<\cosh\xi_2<\cdots$. Since $\xi_k\sim\pi(2k+1)/2$ and $\cos(\pi(2k+1)/2)=0$, we get

$$|\sin\theta_k|\cdot|\xi_k-\pi(2k+1)/2|=|\cos\xi_k-\cos(\pi(2k+1)/2)|=\frac{1}{\cosh\xi_k}\le 2\,\mathrm{e}^{-\xi_k}$$

for a $\theta_k\in(\xi_k,\pi(2k+1)/2)$. But we have

$$1\ge|\sin\xi_k|=\sqrt{1-\cos^2\xi_k}\ge\sqrt{1-\cos^2\xi_1}\doteq 0.999844212\,,$$

since $0<\cos\xi_k=\operatorname{sech}\xi_k\le\operatorname{sech}\xi_1=\cos\xi_1$. Next, we can easily see that in fact $(4k-1)\pi/2<\xi_{2k-1}$, $\xi_{2k}<(4k+1)\pi/2$ and function $\cos x$ is positive in intervals $(\xi_k,\pi(2k+1)/2)$ for any $k\in\mathbb{N}$. So function $\sin x$ is increasing in these intervals, and it is positive on $[\xi_{2k},(4k+1)\pi/2]$ and negative on $[(4k-1)\pi/2,\xi_{2k-1}]$. Hence

$\sin\xi_{2k} = \sqrt{1-\cos^2\xi_{2k}}$. Using also $\cosh\xi_k = \frac{1}{\cos\xi_k}$ and $\sinh\xi_k = \sqrt{\cosh^2\xi_k - 1}$ form (5.1.105) we derive $W_{2k} = -2$. Similarly, from $\sin\xi_{2k-1} < 0$, $k \in \mathbb{N}$ we derive $\sin\xi_{2k-1} = -\sqrt{1-\cos^2\xi_{2k-1}}$ and then $W_{2k-1} = 2$. Consequently, $|W_k| = 2$ for any $n \in \mathbb{N}$. Next, (5.1.104) implies

$$\begin{aligned}
&|w_k(x)| \\
&\le \frac{2}{\sqrt{\pi}}\left(\left|\cosh(\mu_k x) - \frac{\cosh\xi_k \sinh(\mu_k x)}{\sinh\xi_k - \sin\xi_k}\right| + 1 + \cos\xi_k \frac{\sinh\xi_k}{\sinh\xi_k - 1} + \frac{\cosh\xi_k}{\sinh\xi_k - 1}\right) \\
&\le \frac{2}{\sqrt{\pi}}\left(\frac{\sinh\left(\mu_k\left(\frac{\pi}{4} - x\right)\right) + 2\cosh\xi_k + \cos\xi_k \sinh\xi_k}{\sinh\xi_k - 1} + 1\right) \\
&\le \frac{2}{\sqrt{\pi}}\left(\frac{\sinh\xi_1 + 2\cosh\xi_1 + \cos\xi_1 \sinh\xi_1}{\sinh\xi_1 - 1} + 1\right) \doteq 4.5949831827\,.
\end{aligned}$$

Hence $M_1 \le 4.594983183$. Now we estimate M_2. From the above arguments we deduce $|\sin\theta_k| \ge |\sin\xi_k| \ge |\sin\xi_1| \doteq 0.999844212$. This gives

$$|\xi_k - \pi(2k+1)/2| \le \frac{2}{|\sin\xi_1|}\,\mathrm{e}^{-\xi_1} \doteq 0.017654973\,.$$

So we obtain $\xi_k \ge \frac{\pi(2k+1)}{2} - 0.017654973 \ge \pi k$. Consequently, we arrive at

$$|\xi_k - \pi(2k+1)/2| \le \frac{2}{|\sin\xi_1|}\,\mathrm{e}^{-\xi_k} \le \frac{2}{|\sin\xi_1|}\,\mathrm{e}^{-\pi k} \le c\frac{\pi}{4}\,\mathrm{e}^{-\pi k} \qquad (5.1.106)$$

for $c \doteq 2.546875863$. Furthermore, since $\xi_k \ge \xi_1 > 4$, we have

$$\left|\frac{1}{\xi_k^2} - \frac{4}{\pi^2(2k+1)^2}\right| = 2\left|\frac{1}{\xi_k} + \frac{2}{\pi(2k+1)}\right| \cdot \left|\frac{\xi_k - \pi(2k+1)/2}{\xi_k \pi(2k+1)}\right| \le \frac{3}{16|\sin\xi_1|}\,\mathrm{e}^{-\pi k}\,.$$

Hence, we arrive at

$$\begin{aligned}
\sum_{k=7}^{\infty}\left|\frac{1}{\xi_k^2} - \frac{4}{\pi^2(2k+1)^2}\right| &\le \sum_{k=7}^{\infty}\frac{3}{16|\sin\xi_1|}\,\mathrm{e}^{-\pi k} = \frac{3}{16|\sin\xi_1|}\frac{\mathrm{e}^{-7\pi}}{1-\mathrm{e}^{-\pi}} \\
&\doteq 5.51594097 \cdot 10^{-11}\,.
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{k=1}^{\infty} 1/\xi_k^2 \\
&\quad\le \sum_{k=1}^{6} 1/\xi_k^2 + \sum_{k=7}^{\infty}\left|\frac{1}{\xi_k^2} - \frac{4}{\pi^2(2k+1)^2}\right| + \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} - \frac{4}{\pi^2}\sum_{k=0}^{6}\frac{1}{(2k+1)^2} \\
&\quad\le \sum_{k=1}^{6} 1/\xi_k^2 + \frac{3}{16|\sin\xi_1|}\frac{\mathrm{e}^{-7\pi}}{1-\mathrm{e}^{-\pi}} + \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{6}\frac{1}{(2k+1)^2} \doteq 0.09438295\,.
\end{aligned}$$

This implies $M_2 = \frac{\pi^2}{4} M_1 \sum_{k=1}^{\infty} 1/\xi_k^2 \leq 1.07008241$. In summariy, we see that condition (H3) holds if

$$9.8340213469 \cdot |f'(0)| < \delta\,.$$

Finally, we note that $w_k(x)$ and $w_k\left(\frac{\pi}{4} - x\right)$ solve the same eigenvalue problem

$$u_{xxxx}(x) = \mu_k u(x), \quad u_{xx}(0) = u_{xx}(\pi/4) = u_{xxx}(0) = u_{xxx}(\pi/4) = 0\,.$$

Since $\{w_k \mid k \in \mathbb{N}\}$ is an orthonormal system in $L^2([0,\pi/4])$, we see that $w_k(x) = \pm w_k\left(\frac{\pi}{4} - x\right)$. But $w_k(\pi/4) = 4/\sqrt{\pi}$ and $w_k(0) = 4/\sqrt{\pi}$ when k is odd, and $w_k(0) = -4/\sqrt{\pi}$ when k is even. So $w_{2k}\left(\frac{\pi}{4} - x\right) = -w_{2k}(x)$ and $w_{2k-1}\left(\frac{\pi}{4} - x\right) = w_{2k-1}(x)$, $\forall k \in \mathbb{N}$.

5.1.6 Lipschitz Continuity

Here we prove the Lipschitz continuity property of the linear map $\mathscr{L}_{m,\xi} : L^\infty(\mathbb{R}) \to \ell^\infty$ defined as

$$\mathscr{L}_{m,\xi}(h) = \{\mathscr{L}_{m,\xi,j}(h)\}_{j\in\mathbb{Z}}$$

with respect to α uniformly in $E \in \mathscr{E}$ and $m \geq m_0$. We start with the family of linear maps $L_{m,\xi} : \ell^\infty \times \ell^\infty \to \ell^\infty \times \ell^\infty$ defined as

$$L_{m,\xi}(\widetilde{a},\widetilde{b}) = \{L_{m,\xi,j}(\widetilde{a},\widetilde{b})\}_{j\in\mathbb{Z}}$$

where $\widetilde{a} = \{\widetilde{a}_j\}_{j\in\mathbb{Z}}$, $\widetilde{b} = \{\widetilde{b}_j\}_{j\in\mathbb{Z}}$ and prove that it is Lipschitz continuous function in α uniformly withy respect to (E,m), $E \in \mathscr{E}$ and $m \geq m_0$.

As in the proof of Lemma 5.1.2, $u(t)$ denotes the (unbounded) solution of $\ddot{x} + \frac{8}{\pi} f'\left(\frac{2}{\sqrt{\pi}}\gamma_1(t)\right) x = 0$ so that $u(0) = 1$ and $\dot{u}(0) = 0$. For simplicity we also set: $\hat{u}(t) = \frac{\dot{u}(t)}{au(t)}$ and note that $\hat{u}(t)$ is uniformly continuous in $\mathbb{R}$ since $\lim_{t\to\pm\infty} \hat{u}(t) = \pm 1$ (see (5.1.49)). Moreover we have

$$\frac{d}{dt}\left(\frac{\dot{u}(t)}{au(t)}\right) = \frac{\ddot{u}(t)}{au(t)} - \frac{1}{a}\left(\frac{\dot{u}(t)}{u(t)}\right)^2 = -\frac{8}{a\pi} f'\left(\frac{2}{\sqrt{\pi}}\gamma_1(t)\right) - a\left(\frac{\dot{u}(t)}{au(t)}\right)^2 \to 0$$

as $t \to \pm\infty$. Hence $\frac{d\hat{u}}{dt}(t)$ is also uniformly continuous in $\mathbb{R}$. As a matter of fact, $\hat{u}(t)$ is Lipschitz continuous function with constant, say, $\tilde{\Lambda}$, since $\frac{d\hat{u}}{dt}(t)$ is bounded on $\mathbb{R}$.

Now, let $\xi = (E,\alpha)$, $\xi' = (E,\alpha')$ be elements of X and consider the difference $L_{m,\xi} - L_{m,\xi'}$. From (5.1.47), (5.1.48) we see that for any $\widetilde{a} = \{\widetilde{a}_j\}_{j\in\mathbb{Z}}$, $\widetilde{b} = \{\widetilde{b}_j\}_{j\in\mathbb{Z}}$, we have

$$[L_{m,\xi} - L_{m,\xi'}]\begin{pmatrix}\widetilde{a}\\ \widetilde{b}\end{pmatrix} = \begin{pmatrix}0\\ \widetilde{B}\end{pmatrix} \tag{5.1.107}$$

with $\widetilde{B} = \{\widetilde{B}_j\}_{j\in\mathbb{Z}}$ and

$$\widetilde{B}_j = [\hat{u}(-m-\alpha'_{j+1}) - \hat{u}(-m-\alpha_{j+1})]e_{j+1}\widetilde{a}_{j+1} + [\hat{u}(m-\alpha_j) - \hat{u}(m-\alpha'_j)]e_j\widetilde{b}_j. \tag{5.1.108}$$

Then we have, using the Lipschitz continuity of $\hat{u}(t)$:

$$\begin{aligned}|\widetilde{B}_j| \leq& |\hat{u}(m-\alpha_j) - \hat{u}(m-\alpha'_j)||\widetilde{b}_j| + |\hat{u}(-m-\alpha'_{j+1}) - \hat{u}(-m-\alpha_{j+1})||\widetilde{a}_{j+1}|\\ \leq& \tilde{\Lambda}|\alpha_j - \alpha'_j||\widetilde{b}_j| + \tilde{\Lambda}|\alpha_{j+1} - \alpha'_{j+1}||\widetilde{a}_{j+1}| \leq \tilde{\Lambda}\|\alpha-\alpha'\|_\infty[\|\widetilde{a}\|_\infty + \|\widetilde{b}\|_\infty].\end{aligned}$$

As a consequence,

$$\|L_{m,\xi} - L_{m,\xi'}\|_\infty \leq \tilde{\Lambda}\|\alpha - \alpha'\|_\infty \tag{5.1.109}$$

uniformly with respect to (E,m), $E \in \mathscr{E}$ and $m \geq m_0$. Then the same conclusion holds for the inverse map $L^{-1}_{m,\xi}$. In fact, from $L^{-1}_{m,\xi} - L^{-1}_{m,\xi'} = L^{-1}_{m,\xi}[L_{m,\xi'} - L_{m,\xi}]L^{-1}_{m,\xi'}$ we obtain $\|L^{-1}_{m,\xi} - L^{-1}_{m,\xi'}\| \leq 9\widetilde{\Lambda}\|\alpha-\alpha'\|$, since $\|L^{-1}_{m,\xi}\| \leq 3$ (see (5.1.51)). Now,

$$\mathscr{L}_{m,\xi,j}(h) = -e_j\ddot{\gamma}_1(0)\left[\frac{\widetilde{a}_j}{u(-m-\alpha_j)} - \frac{\widetilde{b}_j}{u(m-\alpha_j)}\right]$$

where $(\widetilde{a},\widetilde{b})$ is obtained by solving the equation $L_{m,\xi}(\widetilde{a},\widetilde{b}) = (A_\xi h, B_\xi h)$ and $A_\xi h$, $B_\xi h$ are the linear (in $h \in L^\infty(\mathbb{R})$) maps defined by the right-hand sides of Equations (5.1.44)–(5.1.46):

$$\begin{aligned}A_\xi h &= \left\{(1-e_j)C_j - (1-e_{j+1})\hat{C}_{j+1} - e_jD_j(\alpha_j) - e_{j+1}\hat{D}_{j+1}(\alpha_{j+1})\right\}_{j\in\mathbb{Z}},\\ B_\xi h &= \left\{-(1-e_j)C_j - (1-e_{j+1})\hat{C}_{j+1} - e_jF_j(\alpha_j) - e_{j+1}\hat{F}_{j+1}(\alpha_{j+1})\right\}_{j\in\mathbb{Z}},\end{aligned}$$

where

$$C_j = \frac{1}{2a}\int_{(2j-1)m}^{(2j+1)m} \mathrm{e}^{-a((2j+1)m-s)}\,h(s)ds,$$

$$\hat{C}_j = \frac{1}{2a}\int_{(2j-1)m}^{(2j+1)m} \mathrm{e}^{a((2j-1)m-s)}\,h(s)ds,$$

$$D_j(\alpha) = \int_{2jm+\alpha}^{(2j+1)m} v(m-\alpha)u(s-2jm-\alpha)h(s)ds,$$

$$\hat{D}_j(\alpha) = \int_{(2j-1)m}^{2jm+\alpha} v(-m-\alpha)u(s-2jm-\alpha)h(s)ds,$$

$$F_j(\alpha) = \frac{1}{a}\int_{2jm+\alpha}^{(2j+1)m} \dot{v}(m-\alpha)u(s-2jm-\alpha)h(s)ds$$

$$\hat{F}_j(\alpha) = \frac{1}{a}\int_{(2j-1)m}^{2jm+\alpha} \dot{v}(-m-\alpha)u(s-2jm-\alpha)h(s)ds.$$

So, if we prove that the linear map $h \mapsto (A_\xi h, B_\xi h)$ is bounded uniformly with respect to $\xi \in X$ and Lipschitz continuous function in α uniformly with respect to

(E,m), we get that $\mathscr{L}_{m,\xi}(h)$ is Lipschitz continuous function in α uniformly with respect to (E,m) and that the Lipschitz constant is $O(\mathrm{e}^{-am}) = O(\varepsilon)$ as $\varepsilon \to 0$ uniformly with respect to (E,m). Now, the fact that $A_\xi h$, $B_\xi h$ are bounded uniformly with respect to $\xi \in X$ easily follows from

$$\begin{gathered} \max\left\{|C_j|,|\hat{C}_j|\right\} \le \tfrac{1}{2a^2}\|h\|_\infty, \\ \max\left\{|D_j(\alpha)|,|\hat{D}_j(\alpha)|,|F_j(\alpha)|,|\hat{F}_j(\alpha)|\right\} \le \tfrac{k}{a}\|h\|_\infty. \end{gathered} \tag{5.1.110}$$

Then it is enough to study the Lipschitz continuity of the maps

$$\begin{gathered} (\xi,h) \mapsto \left\{D_j(\alpha_j)e_j\right\}_{j\in\mathbb{Z}}, \ \ (\xi,h) \mapsto \left\{\hat{D}_j(\alpha_j)e_j)\right\}_{j\in\mathbb{Z}}, \\ (\xi,h) \mapsto \left\{F_j(\alpha_j)e_j\right\}_{j\in\mathbb{Z}}, \ \ (\xi,h) \mapsto \left\{\hat{F}_j(\alpha_j)e_j)\right\}_{j\in\mathbb{Z}}, \end{gathered} \tag{5.1.111}$$

with respect to α. Writing $D_j(\alpha,m)$, $\hat{D}_j(\alpha,m)$, etc. to emphasize dependence on m we see that

$$\hat{D}_j(\alpha,m) = -D_{-j}(\alpha,-m), \qquad \hat{F}_j(\alpha,m) = -F_{-j}(\alpha,-m).$$

Thus we only need to look at $D_j(\alpha)$ and $F_j(\alpha)$. We focus our attention on the map $(\xi,h) \mapsto \{D_j(\alpha_j)e_j\}$, $\xi = (E,\alpha)$, $F_j(\alpha)$ being handled similarly. First, we look at the difference $D_j(\tau'') - D_j(\tau')$, where $\tau',\tau'' \in \mathbb{R}$, $\tau'' \ge \tau'$ and $|\tau'|,|\tau''| \le 2$. We see that $D_j(\tau'') - D_j(\tau')$ equals:

$$\begin{aligned} &\int_{2jm+\tau''}^{(2j+1)m} \left[v(m-\tau'')u(s-2jm-\tau'') - v(m-\tau')u(s-2jm-\tau')\right] h(s)ds \\ &\quad - \int_{2jm+\tau'}^{2jm+\tau''} v(m-\tau')u(s-2jm-\tau')h(s)ds\,. \end{aligned}$$

Then (5.1.34) implies

$$\left|\int_{2jm+\tau'}^{2jm+\tau''} v(m-\tau')u(s-2jm-\tau')h(s)ds\right| \le k\|h\|_\infty|\tau''-\tau'|\,.$$

Similarly, we get

$$\begin{aligned} &\left|\int_{2jm+\tau''}^{(2j+1)m} \left[v(m-\tau'')u(s-2jm-\tau'') - v(m-\tau')u(s-2jm-\tau')\right] h(s)ds\right| \\ &= \left|\int_{2jm+\tau''}^{(2j+1)m} \left(\int_{\tau'}^{\tau''} \left[\dot{v}(m-\tau)u(s-2jm-\tau) - v(m-\tau)\dot{u}(s-2jm-\tau)\right] d\tau\right)\right. \\ &\qquad \left.\cdot h(s)ds\right| \\ &\le \frac{2k}{a}\|h\|_\infty|\tau''-\tau'|\,. \end{aligned}$$

Consequently, we obtain

$$|D_j(\tau'') - D_j(\tau')| \le \left(\frac{2k}{a} + k\right) \|h\|_\infty |\tau'' - \tau'| \,.$$

Thus $(\xi, h) \mapsto \{D_j(\alpha_j)e_j\}_{j\in\mathbb{Z}}$ is Lipschitz continuous function in α with the constant $\frac{2k}{a} + k$ independent of (E, m). Similarly we can prove the global Lipschitz continuity in α of $F_j(\alpha)$. This completes the proof of the uniform Lipschitz continuity in α of $\mathscr{L}_{m,\xi}(h)$. Note that when $h \in L^\infty$, the maps in (5.1.111) are not differentiable in α.

5.2 Infinite Dimensional Non-Resonant Systems

5.2.1 Buckled Elastic Beam

To motivate the ideas of this section consider the partial differential equation

$$\ddot{u} = -u'''' - P_0 u'' + \left[\int_0^\pi u'(s)^2 \, ds\right] u'' - 2\mu_2 \dot{u} + \mu_1 \cos \omega_0 t \tag{5.2.1}$$

where P_0, μ_1, μ_2, ω_0 are constants and u is a real valued function of two variables $t \in \mathbb{R}$, $x \in [0, \pi]$, subject to the boundary conditions

$$u(0,t) = u(\pi,t) = u''(0,t) = u''(\pi,t) = 0 \,.$$

In (5.2.1), a superior dot denotes differentiation with respect to t and prime differentiation with respect to x. This is a model for oscillations of an elastic beam with a compressive axial load P_0 (Figure 5.2). When P_0 is sufficiently large, (5.2.1) can exhibit chaotic behavior. The first work on this was done in [3]. Some more recent work on the full equation is in [4, 14]. An undamped buckled beam is investigated in [15] to show Arnold diffusion type motions. We will discuss some of them in more detail when we return to this problem in Section 5.2.6.

In (5.2.1) substitute $u(x,t) = \sum_{k=1}^{\infty} u_k(t) \sin kx$, multiply by $\sin nx$ and integrate from 0 to π. This yields the infinite set of ordinary differential equations

$$\ddot{u}_n = n^2(P_0 - n^2)u_n - \frac{\pi}{2} n^2 \left[\sum_{k=1}^{\infty} k^2 u_k^2\right] u_n - 2\mu_2 \dot{u}_n + 2\mu_1 \left[\frac{1 - (-1)^n}{\pi n}\right] \cos \omega_0 t, \quad n = 1, 2, \dots .$$

We see that the linear parts of these equations are uncoupled and the equations are divided into two types. The system of equations defined by $1 \le n^2 < P_0$ has a hyperbolic equilibrium in origin whereas for the system of equations satisfying $n^2 \ge P_0$, this equilibrium is a center. For simplicity let us assume $1 < P_0 < 4$. Then

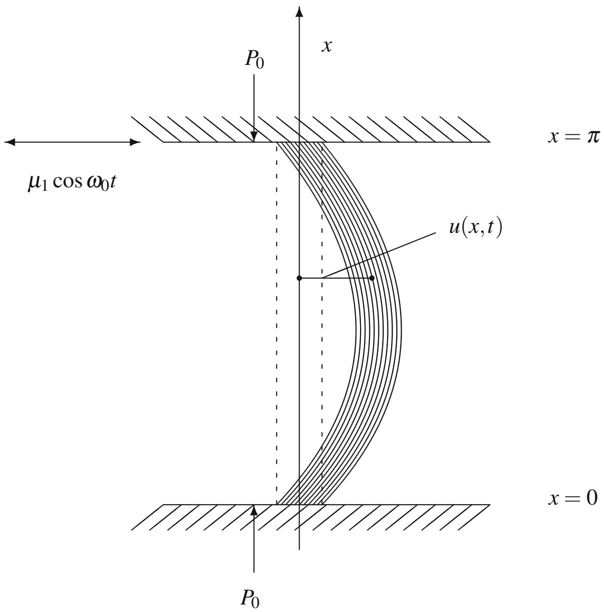

Fig. 5.2 The forced buckled beam (5.2.1).

only the equation with $n=1$ is hyperbolic while the system of remaining equations has a center. To emphasize this let us define $p=u_1$ and $q_n=u_{n+1}$, $n=1,2,\ldots$. The preceding equations now take the form

$$\ddot{p} = a^2 p - \frac{\pi}{2}\left[p^2 + \sum_{k=1}^{\infty}(k+1)^2 q_k^2\right] p - 2\mu_2 \dot{p} + \frac{4}{\pi}\mu_1 \cos\omega_0 t, \qquad (5.2.2a)$$

$$\begin{aligned} \ddot{q}_n &= -\omega_n^2 q_n - \frac{\pi}{2}(n+1)^2\left[p^2 + \sum_{k=1}^{\infty}(k+1)^2 q_k^2\right] q_n \\ &\quad -2\mu_2 \dot{q}_n + 2\mu_1\left[\frac{1-(-1)^{n+1}}{\pi(n+1)}\right]\cos\omega_0 t, \qquad (5.2.2b) \\ &\quad n = 1,2,\ldots \end{aligned}$$

where we have defined $a^2 = P_0 - 1$ and $\omega_n^2 = (n+1)^2\left[(n+1)^2 - P_0\right]$. In (5.2.2) we project onto the hyperbolic subspace by setting $q=0$ in (5.2.2a) to obtain what we shall call the *reduced equation*. In our example this is

$$\ddot{p} = a^2 p - \frac{\pi}{2}p^3 - 2\mu_2 \dot{p} + \frac{4}{\pi}\mu_1 \cos\omega_0 t. \qquad (5.2.3)$$

We see that this is the forced, damped Duffing equation with negative stiffness for which standard theory yields chaotic dynamics (cf Section 4.1). The purpose of this section is to show that the chaotic dynamics of (5.2.3) are, in some sense, shadowed in the dynamics of the full equation (5.2.2). To put our example in the first order form we define $x = (p, \dot{p})$ and

$$y = (q_1, \dot{q}_1/\omega_1, q_2, \dot{q}_2/\omega_2, \ldots).$$

Equations (5.2.2 a and b) now become

$$\begin{aligned}
\dot{x}_1 &= x_2\,, \\
\dot{x}_2 &= a^2 x_1 - \frac{\pi}{2}\left[x_1^2 + \sum_{k=1}^{\infty}(k+1)^2 y_{2k-1}^2\right]x_1 \\
&\qquad -2\mu_2 x_2 + \frac{4}{\pi}\mu_1 \cos\omega_0 t\,,
\end{aligned} \tag{5.2.4a}$$

$$\begin{aligned}
\dot{y}_{2n-1} &= \omega_n y_{2n}\,, \\
\dot{y}_{2n} &= -\omega_n y_{2n-1} - \frac{\pi}{2}\frac{(n+1)^2}{\omega_n}\left[x_1^2 + \sum_{k=1}^{\infty}(k+1)^2 y_{2k-1}^2\right] y_{2n-1} \\
&\qquad -2\mu_2 y_{2n} + 2\mu_1\left[\frac{1-(-1)^{n+1}}{\pi(n+1)\omega_n}\right]\cos\omega_0 t.
\end{aligned} \tag{5.2.4b}$$

For these equations we define the Hilbert space

$$\mathbb{Y} = \left\{ y = \{y_n\}_{n=1}^{\infty} \;\middle|\; y_n \in \mathbb{R}, \qquad \sum_{n=1}^{\infty} \omega_n^2 (y_{2n-1}^2 + y_{2n}^2) < \infty \right\}$$

with inner product $\langle u, v\rangle = \sum_{n=1}^{\infty} \omega_n^2 (u_{2n-1}v_{2n-1} + u_{2n}v_{2n})$. By a weak solution to (5.2.4) we mean a pair of functions $x_0 : \mathbb{R} \to \mathbb{R}^2$, $y_0 : \mathbb{R} \to \mathbb{Y}$ so that x_0 is differentiable and y_0 has a derivative $\dot{y}_0 \to \ell^2$, which satisfy (5.2.4a) pointwise in $\mathbb{R}^2$, (5.2.4b) pointwise in ℓ^2. Note that in this case we have

$$(u_1, u_2, \ldots) = (x, p_1, p_2, \ldots), \quad x^2 + \sum_{n=1}^{\infty} \omega_n^2 p_n^2 < \infty,$$

$$(\dot{u}_1, \dot{u}_2, \ldots) = (\dot{x}, \dot{p}_1, \dot{p}_2 \ldots) \in \ell^2$$

so that for the original differential equation (5.2.1), $u \in H^2(0,\pi) \cap H_0^1(0,\pi)$ and $\dot{u} \in L^2(0,\pi)$. This is discussed in [5]. In the next section we will formulate an abstract problem for which the hypotheses will consist of the essential features of (5.2.4). We have already mentioned one of them: when y is set equal to zero in (5.2.4a) the resulting equation is the transverse perturbation of an autonomous equation with a homoclinic solution. To see another important property we linearize (5.2.4b) in origin which yields the system of equations

$$\begin{aligned}
\dot{v}_{2n-1} &= \omega_n v_{2n}, \\
\dot{v}_{2n} &= -\omega_n v_{2n-1} - 2\mu_2 v_{2n}, \quad n \in \mathbb{N}.
\end{aligned} \tag{5.2.5}$$

Note that for each n we get a pair of equations uncoupled from the others and for $|\mu_2| < \omega_n$ we have a fundamental solution for (v_{2n-1}, v_{2n}) given by

$$V_n(t) = \begin{bmatrix} \cos\tilde{\omega}_n t + \frac{\mu_2}{\tilde{\omega}_n}\sin\tilde{\omega}_n t & \frac{\omega_n}{\tilde{\omega}_n}\sin\tilde{\omega}_n t \\ -\frac{\omega_n}{\tilde{\omega}_n}\sin\tilde{\omega}_n t & \cos\tilde{\omega}_n t - \frac{\mu_2}{\tilde{\omega}_n}\sin\tilde{\omega}_n t \end{bmatrix} \mathrm{e}^{-\mu_2 t}$$

where $\tilde{\omega}_n = \sqrt{\omega_n^2 - \mu_2^2}$. This solution has the properties $V_n(0) = \mathbb{I}$ and

$$|V_n(t)V_n(s)^{-1}| = |V_n(t)V_n(-s)| = |V_n(t-s)| \le K\,\mathrm{e}^{\mu_2(s-t)},$$

where $K > 0$ is independent of n. Using the sequence $\{V_n\}_{n=1}^{\infty}$ we can define a group $\{V_{\mu_2}(t)\}$ of bounded operators from $\mathbb{Y}$ to $\mathbb{Y}$ by

$$\begin{bmatrix} \left(V_{\mu_2}(t)y\right)_{2n-1} \\ \left(V_{\mu_2}(t)y\right)_{2n} \end{bmatrix} = V_n(t)\begin{bmatrix} y_{2n-1} \\ y_{2n} \end{bmatrix}.$$

Then $|V_{\mu_2}(t)V_{\mu_2}(s)^{-1}| \le K\,\mathrm{e}^{\mu_2(s-t)}$. For $y^0 \in \mathbb{Y}$, $y(t) = V_{\mu_2}(t)y^0$ is the weak solution to (5.2.5) satisfying $y(0) = y^0$. If we retain the forcing term from (5.2.4b) we obtain the system of nonhomogeneous variational equations

$$\begin{aligned} \dot{v}_{2n-1} &= \omega_n v_{2n}\,, \\ \dot{v}_{2n} &= -\omega_n v_{2n-1} - 2\mu_2 v_{2n} + \mu_1 \nu_n \cos\omega_0 t \end{aligned}$$

where $\nu_n = \dfrac{2[1-(-1)^{n+1}]}{\pi(n+1)\omega_n}$. Here we encounter the question of resonance. In the nonresonant case, i.e. $\omega_n \ne \omega_0$, the precedent has a particular solution in $\mathbb{Y}$ with components given by

$$\begin{bmatrix} v_{2n-1}(t) \\ v_{2n}(t) \end{bmatrix} = \frac{\mu_1\nu_n}{(\omega_n^2-\omega_0^2)^2 + 4\mu_2^2\omega_0^2}\begin{bmatrix} \omega_n(\omega_n^2-\omega_0^2)\cos\omega_0 t + 2\mu_2\omega_0\omega_n\sin\omega_0 t \\ -\omega_0(\omega_n^2-\omega_0^2)\sin\omega_0 t + 2\mu_2\omega_0^2\cos\omega_0 t \end{bmatrix}.$$

We make the existence of such a solution a separate hypothesis.

Finally, we mention other work on chaos in partial differential equations. For the complex Ginzburg-Landau equation in the near nonlinear Schrödinger regime, i.e. perturbed nonlinear Schrödinger equation, existence of homoclinic orbits is proved in [7, 16, 17], and existence of chaos is shown in [8, 18] under generic conditions. For perturbed sine-Gordon equation, existence of chaos and chaos cascade around a homoclinic tube was proved in [19–21]. For the reaction-diffusion equation, entropy study on the complexity of attractor is conducted in [22–24]. Chaotic oscillations of a linear wave equation with nonlinear boundary conditions are shown in [25]. The development of chaos and its controlling for PDEs is summarized in [26, 27].

5.2.2 Abstract Problem

Using the example in the preceding section as a model we now develop an abstract theory. Let $\mathbb{Y}$ and $\mathbb{H}$ be separable real Hilbert spaces with $\mathbb{Y} \subset \mathbb{H}$. We now consider differential equations of the form

$$\begin{aligned} \dot{x} &= f(x,y,\mu,t) = f_0(x,y) + \mu_1 f_1(x,y,\mu,t) + \mu_2 f_2(x,y,\mu,t)\,, \\ \dot{y} &= g(x,y,\mu,t) = Ay + g_0(x,y) + \mu_1 \nu \cos\omega_0 t + \mu_2 g_2(x,y,\mu) \end{aligned} \tag{5.2.6}$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{Y}$, $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$, $\nu \in \mathbb{Y}$. We make the following assumptions of (5.2.6):

(H1) $A \in L(\mathbb{Y}, \mathbb{H})$.
(H2) $f_0 \in C^4(\mathbb{R}^n \times \mathbb{Y}, \mathbb{R}^n)$, $f_1, f_2 \in C^4(\mathbb{R}^n \times \mathbb{Y} \times \mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^n)$, $g_0 \in C^4(\mathbb{R}^n \times \mathbb{Y}, \mathbb{Y})$ and $g_2 \in C^4(\mathbb{R}^n \times \mathbb{Y} \times \mathbb{R}^2, \mathbb{Y})$.
(H3) f_1 and f_2 are periodic in t with period $T = 2\pi/\omega_0$.
(H4) $f_0(0,0) = 0$ and $D_2 f_0(x,0) = 0$.
(H5) The eigenvalues of $D_1 f_0(0,0)$ lie off the imaginary axis.
(H6) The equation $\dot{x} = f_0(x,0)$ has a nontrivial solution homoclinic to $x = 0$.
(H7) $g_0(x,0) = g_2(x,0,\mu) = 0$, $D_{12} g_0(0,0) = 0$ and $D_{22} g_0(x,0) = 0$.
(H8) There are constants $K > 0$, $\delta > 0$ and $b > 0$ so that when $0 \le |\mu_2| \le \delta$ the variational equation $\dot{v} = \big(A + \mu_2 D_2 g_2(0,0,0)\big) v$ has a group $\{V_{\mu_2}(t)\}$ of bounded evolution operators from $\mathbb{Y}$ to $\mathbb{Y}$ satisfying $|V_{\mu_2}(t) V_{\mu_2}(s)^{-1}| \le K\, \mathrm{e}^{b\mu_2(s-t)}$.
(H9) There is a constant $K > 0$ so that the nonhomogeneous variational equation $\dot{v} = [A + \mu_2 D_2 g_2(0,0,0)]\, v + \mu_1 \nu \cos\omega_0 t$ has a particular solution $\psi : \mathbb{R} \to \mathbb{Y}$ satisfying $|\psi(t)| \le K|\mu_1||\nu|$.

By a weak solution to (5.2.6) we mean a pair of continuous functions $x_0 : \mathbb{R} \to \mathbb{R}^n$, $y_0 : \mathbb{R} \to \mathbb{Y}$ so that x_0 is differentiable and y_0 has a derivative $\dot{y}_0 : \mathbb{R} \to \mathbb{H}$, which satisfy (5.2.6) pointwise in $\mathbb{H}$. By (H8) we mean that $V_{\mu_2}(s)^{-1} = V_{\mu_2}(-s)$, $V_{\mu_2}(s) \circ V_{\mu_2}(t) = V_{\mu_2}(s+t)$, $V_{\mu_2}(0) = \mathbb{I}$ and that for $y_0 \in \mathbb{Y}$, $y(t) = V_{\mu_2}(t) y_0$ is the weak solution to $\dot{v} = [A + \mu_2 D_2 g_2(0,0,0)]\, v$ satisfying $y(0) = y_0$.

5.2.3 Chaos on the Hyperbolic Subspace

The reduced system of equations for (5.2.6) is

$$\dot{x} = f(x,0,\mu,t) = f_0(x,0) + \mu_1 f_1(x,0,\mu,t) + \mu_2 f_2(x,0,\mu,t) \tag{5.2.7}$$

with $x \in \mathbb{R}^n$. By (H6), (5.2.7) has a nontrivial homoclinic solution γ when $\mu = 0$. The variational equation along γ is the linear equation $\dot{u} = D_1 f_0(\gamma, 0) u$ and its adjoint variational equation

$$\dot{v} = -D_1 f_0(\gamma, 0)^* v\,. \tag{5.2.8}$$

By repeating arguments of Section 4.2.2, we have the following result (cf Theorem 4.2.1).

Theorem 5.2.1. *Let M be as in* (4.2.6) *or* (4.2.7) *and suppose μ_0, α_0, β_0 are such that $M(\mu_0,\alpha_0,\beta_0)=0$ and $D_{(\alpha,\beta)}M(\mu_0,\alpha_0,\beta_0)$ is nonsingular. Then there exists an interval $J=(0,\xi_0]$ so that for each $\xi\in J$ the equation $\dot{x}=f(x,0,\xi\mu_0,t)$ has a homoclinic solution γ_ξ to a small hyperbolic periodic solution. Furthermore, γ_ξ depends continuously on ξ, $\lim_{\xi\to 0}\gamma_\xi(t)=\gamma(t-\alpha_0)$ (or $=\gamma_{\beta_0}(t-\alpha_0)$, respectively) uniformly in t and the variational equation along γ_ξ has an exponential dichotomy on $\mathbb{R}$.*

Then we can show chaos for the differential equation $\dot{x}=f(x,0,\xi\mu_0,t)$. For this, first, for any $m\in\mathbb{N}$, $\xi\in J$ and $E=\{e_j\}_{j\in\mathbb{Z}}\in\mathscr{E}$ (cf Section 2.5.2) define the function $\gamma_{\xi,E,m}\in L^\infty(\mathbb{R},\mathbb{R}^n)$ by

$$\gamma_{\xi,E,m}(t)=\begin{cases}\gamma_\xi(t-2jmT), & \text{if}\quad (2j-1)mT<t\le(2j+1)mT \quad\text{and}\quad e_j=1,\\ \qquad 0, & \text{if}\quad (2j-1)mT<t\le(2j+1)mT \quad\text{and}\quad e_j=0.\end{cases}$$

Now following arguments of Sections 3.5.2 and 5.1.4, we obtain the following version of Smale-Birkhoff homoclinic theorem 2.5.4.

Theorem 5.2.2. (a) *Let μ_0, α_0, β_0, ξ_0 be as in Theorem 5.2.1. Fix $\xi\in(0,\xi_0]$ and let γ_ξ be obtained from Theorem 5.2.1. Then there exist an $\varepsilon_0>0$ and a function $\varepsilon\to M(\varepsilon)\in\mathbb{N}$ so that given ε with $0<\varepsilon\le\varepsilon_0$ and a positive integer $m\ge M(\varepsilon)$ the equation $\dot{x}=f(x,0,\xi\mu_0,t)$ has for each $E\in\mathscr{E}$ a unique solution $t\to x_E(t)$ satisfying*

$$|x_E(t)-\gamma_{\xi,E,m}(t)|\le\varepsilon,\quad \forall t\in\mathbb{R}.$$

(b) *x_E depends continuously on E and $x_E(t+2mT)=x_{\sigma(E)}(t)$ where σ is the Bernoulli shift on $\mathscr{E}$.*

(c) *The correspondence $\phi(E)=x_E(0)$ is a homeomorphism of $\mathscr{E}$ onto the compact subset Λ of $\mathbb{R}^n$ given by*

$$\Lambda:=\{x_E(0)\mid E\in\mathscr{E}\}$$

for which the $2m$th iterate F^{2m} of the period map F of (5.2.7) *is invariant and satisfies $F^{2m}\circ\phi=\phi\circ\sigma$.*

Theorem 5.2.2 asserts that the following diagram is commutative.

$$\begin{array}{ccc}\mathscr{E} & \xrightarrow{\quad\sigma\quad} & \mathscr{E}\\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\phi}\\ \Lambda & \xrightarrow[\quad F^{2m}\quad]{} & \Lambda\end{array}$$

This means that $F^{2m}:\Lambda\mapsto\Lambda$ has the same dynamics on Λ as the Bernoulli shift σ on $\mathscr{E}$. Consequently, F^{2m} is chaotic on Λ, so (5.2.7) is also chaotic. This construction is sometimes referred to as embedding a Smale horseshoe in the flow of the differential equation.

5.2.4 Chaos in the Full Equation

Since the homoclinic orbit γ_ξ obtained in Section 5.2.3 is hyperbolic the variational equation $\dot{u} = D_1 f(\gamma_\xi, 0, \xi\mu_0, t)u$ has an exponential dichotomy on $\mathbb{R}$ with constant K_ξ. Now, by Section 4.2.3, K_ξ tends to infinity as $\xi \to 0$. For this reason we consider the following modification of (5.2.6)

$$\begin{aligned} \dot{x} &= f(x,y,\mu,\lambda,t) := f(x,\lambda y,\mu,t)\,, \\ \dot{y} &= g(x,y,\mu,\lambda,t) := Ay + g_0(x,y) + \lambda\mu_1 v\cos\omega_0 t + \mu_2 g_2(x,y,\mu) \end{aligned} \tag{5.2.9}$$

for a parameter $\lambda \in [0,1]$. Now let $(\mu_0, \alpha_0, \beta_0)$ with $\mu_{0,2} \neq 0$ and γ_ξ be as in Theorem 5.2.1. Following the arguments of Section 4.2.3, we obtain a constant $\bar{\xi}_0$ and for each $\xi \in (0, \bar{\xi}_0]$ a homoclinic orbit

$$\Gamma(\lambda,\xi)(t) = \big(\Gamma_1(\lambda,\xi)(t), \Gamma_2(\lambda,\xi)(t)\big)$$

for (5.2.9) with $\mu = \xi\mu_0$ so that

$$\begin{aligned} \Gamma_1(\lambda,\xi)(t) \to \gamma(t-\alpha_0) \quad &(\text{or} \to \gamma_{\beta_0}(t-\alpha_0), \text{respectively})\,, \\ \text{and} \quad &\Gamma_2(\lambda,\xi)(t) \to 0 \end{aligned}$$

as $\xi \to 0$ uniformly for $\lambda \in [0,1]$. Moreover, we have $\Gamma(0,\xi) = (\gamma_\xi, 0)$ and $\Gamma(1,\xi)$ is a homoclinic solution for (5.2.6). The linearization of (5.2.9) with $\mu = \xi\mu_0$ along $\Gamma(\lambda,\xi)(t)$ has an exponential dichotomy on $\mathbb{R}$ with dichotomy constants uniformly with respect to $0 \le \lambda \le 1$ and fixed ξ. Analogous to the construction in Section 5.2.3, for each $E \in \mathscr{E}$, $\xi \in (0, \bar{\xi}_0]$ and $m \in \mathbb{N}$ we construct from $\Gamma(\lambda,\xi)$ a corresponding

$$\Gamma_E(\lambda,\xi,m) = (\Gamma_{1,E}(\lambda,\xi,m), \Gamma_{2,E}(\lambda,\xi,m))\,.$$

Similarly, from γ_ξ we obtain $\gamma_{\xi,E,m}$. Then we have $\Gamma_{1,E}(0,\xi,m) = \gamma_{\xi,E,m}$ and also $\Gamma_{2,E}(0,\xi,m) = 0$. Using the uniform exponential dichotomy, following Sections 3.5.2 and 5.1.4, we now obtain the following extension of Theorem 5.2.2.

Theorem 5.2.3. (a) *Let μ_0, α_0, β_0 be as in Theorem 5.2.1 with $\mu_{0,2} \neq 0$. Fix $\xi \in (0, \bar{\xi}_0]$ and let $\Gamma(\lambda,\xi,m)(t)$ be obtained above. Then there exist an $\bar{\varepsilon}_0 > 0$ and a function $\varepsilon \to \bar{M}(\varepsilon) \in \mathbb{N}$ so that given ε with $0 < \varepsilon \le \bar{\varepsilon}_0$ and a positive integer $m \ge \bar{M}(\varepsilon)$ Eq. (5.2.9) with $\mu = \xi\mu_0$ has for each $E \in \mathscr{E}$ a unique weak solution $t \to \big(x_{E,\lambda}(t), y_{E,\lambda}(t)\big)$ satisfying*

$$|x_{E,\lambda}(t) - \Gamma_{1,E}(\lambda,\xi,m)(t)| + |y_{E,\lambda}(t) - \Gamma_{2,E}(\lambda,\xi,m)(t)| \le \varepsilon \quad \forall t \in \mathbb{R}\,.$$

(b) *The functions* $\big(x_{E,\lambda}(t), y_{E,\lambda}(t)\big)$ *depend continuously on* E, λ *and we also have* $x_{E,\lambda}(t+2mT) = x_{\sigma(E),\lambda}(t)$, $y_{E,\lambda}(t+2mT) = y_{\sigma(E),\lambda}(t)$.

(c) *The correspondence* $\phi_\lambda(E) = \big(x_{E,\lambda}(0), y_{E,\lambda}(0)\big)$ *is a homeomorphism of* $\mathscr{E}$ *onto the compact subset* Λ_λ *of* $\mathbb{R}^n \times \mathbb{Y}$ *given by*

$$\Lambda_\lambda := \big\{ \big(x_{E,\lambda}(0), y_{E,\lambda}(0)\big) \mid E \in \mathscr{E} \big\}$$

for which the $2m$*th iterate* F_λ^{2m} *of the period map* F_λ *of* (5.2.9) *is invariant and satisfies* $F_\lambda^{2m} \circ \phi_\lambda = \phi_\lambda \circ \sigma$.

(d) $(x_{E,0}(t), y_{E,0}(t)) = (x_E(t), 0)$ *and* $\phi_0 = \phi$ *where* ϕ *is as in Theorem* 5.2.2.

In summary, we obtain the following main result.

Theorem 5.2.4. *Suppose (H1)–(H9) hold. Let* M *be as in* (4.2.6) *or* (4.2.7) *and suppose* $(\mu_0, \alpha_0, \beta_0)$ *are such that* $M(\mu_0, \alpha_0, \beta_0) = 0$ *and* $D_{(\alpha,\beta)}M(\mu_0, \alpha_0, \beta_0)$ *is nonsingular. Then there exists* $\bar{\xi}_0 > 0$ *so that if* $0 < \xi \leq \bar{\xi}_0$, *if the parameters in* (5.2.6) *are given by* $\mu = \xi\mu_0$, *and* $\mu_{0,2} \neq 0$ *then there exists a homeomorphism,* ϕ_1, *of* $\mathscr{E}$ *onto a compact subset of* $\mathbb{R}^n \times \mathbb{Y}$ *for which the* $2m$*th iterate,* F_1^{2m}, *of the period map* F_1 *of* (5.2.6) *is invariant and satisfies* $F_1^{2m} \circ \phi_1 = \phi_1 \circ \sigma$. *Here* $m \in \mathbb{N}$ *is sufficiently large.*

We might paraphrase Theorem 5.2.4, loosely, say, the Smale horseshoe embedded in the flow of the reduced equation (5.2.7) is shadowed by a horseshoe in the full equation (5.2.6).

5.2.5 Applications to Vibrating Elastic Beams

We now return to the example in Section 5.2.1 and apply our theory to the problem of vibrating elastic beams. We shall consider a number of different cases and generalizations. In each case our procedure will be:

(i) Use a Galerkin expansion to convert the partial differential equation to an infinite set of ordinary differential equations as (5.2.6).
(ii) Truncate the equation to get the finite problem(5.2.7).
(iii) Apply Theorem 5.2.2 to getting a Smale horseshoe for the finite problem. For this we must verify (H1) through (H6).
(iv) Use Theorem 5.2.4 to lift the horseshoe to the flow of the original partial differential differential equation. This requires (H7)–(H9).

5.2.6 Planer Motion with One Buckled Mode

The boundary value problem for planer deflections of an elastic beam with a compressive axial load P_0 and pinned ends is

$$\ddot{u} = -u'''' - P_0 u'' + \left[\int_0^\pi u'(s)^2\, ds\right] u'' - 2\mu_2 \dot{u} + \mu_1 \cos\omega_0 t,$$
$$u(0,t) = u(\pi,t) = u''(0,t) = u''(\pi,t) = 0$$

where $u(x,t)$ is the transverse deflection at a distance x from one end at time t. We consider the μ_i terms as perturbations. Our first step is to consider the linearized, unperturbed problem. We compute the eigenvalues in origin to be $\lambda_n = n^2(n^2 - P_0)$ with corresponding eigenfunctions $\varphi_n(x) = \sin nx$ for $n = 1, 2, \ldots$. For small P_0 the origin is a center. As P_0 is increased the first bifurcation occurs at $P_0 = 1$, the first *Euler buckling load*. The corresponding eigenfunction, $\varphi_1(x) = \sin x$, is referred to as the first *buckled mode*. The second bifurcation occurs at $P_0 = 4$. Thus, the simplest case, which we now consider, consists of $1 < P_0 < 4$. In the first equation we define

$$a^2 = \lambda_1 = P_0 - 1\,.$$

The eigenvalues for the center modes, or unbuckled modes, provide the frequencies used in (5.2.6) as we define

$$\omega_{n-1}^2 = \lambda_n = n^2[n^2 - P_0], \quad n = 2, 3, \ldots.$$

We now use the eigenfunctions for the Galerkin expansion $u(x,t) = \sum_{k=1}^{\infty} u_k(t)\sin kx$ and obtain the system of equations

$$\begin{aligned} \ddot{u}_n = n^2(P_0 - n^2)u_n - \frac{\pi}{2}n^2\left[\sum_{k=1}^{\infty} k^2 u_k^2\right] u_n \\ -2\mu_2\dot{u}_n + 2\mu_1\left[\frac{1-(-1)^n}{\pi n}\right]\cos\omega_0 t, \quad n = 1, 2, \ldots. \end{aligned} \tag{5.2.10}$$

To obtain a first order system as in (5.2.6) we define

$$x = (u_1, \dot{u}_1), \quad y = (u_2, \dot{u}_2/\omega_1, u_3, \dot{u}_3/\omega_2, \ldots).$$

The reduced equations are

$$\begin{aligned} \dot{x}_1 &= x_2\,, \\ \dot{x}_2 &= a^2 x_1 - \frac{\pi}{2}x_1^3 - 2\mu_2 x_2 + \frac{4}{\pi}\mu_1\cos\omega_0 t \end{aligned} \tag{5.2.11}$$

obtained by setting $y = 0$ in the hyperbolic part. When $\mu = 0$, (5.2.11) has a homoclinic solution given by $\gamma = (r, \dot{r})$ where $r(t) = (2a/\sqrt{\pi})\,\mathrm{sech}\, at$. Equation (5.2.8) becomes

$$\dot{v}_1 = -(a^2 - \frac{3\pi}{2}r^2)v_2, \qquad \dot{v}_2 = -v_1$$

with solution $(v_1, v_2) = (-\ddot{r}, \dot{r})$. We have $d = 1$ so the variable β does not appear, M is a scalar function, and the function $M = M_1$ becomes

$$M(\alpha) = \left[\frac{8\omega_0}{\sqrt{\pi}} \sin \omega_0 \alpha \operatorname{sech} \frac{\pi\omega_0}{2a}\right] \mu_1 - \left(\frac{16a^3}{3\pi}\right) \mu_2.$$

Thus, the conditions $M(\mu_0, \alpha_0) = 0$, $(\partial M/\partial \alpha)(\mu_0, \alpha_0) \neq 0$ are satisfied for all μ_0 so that $\left|\frac{\mu_{0,2}}{\mu_{0,1}}\right| < \frac{3\sqrt{\pi}\omega_0}{2a^3} \operatorname{sech} \frac{\pi\omega_0}{2a}$. Now we check condition (H9) which, for the present problem, requires us to consider the equation

$$\dot{v}_{2n-1} = \omega_n v_{2n},$$
$$\dot{v}_{2n} = -\omega_n v_{2n-1} - 2\mu_2 v_{2n} + \mu_1 \nu_n \cos \omega_0 t$$

where $\nu_n = \frac{2[1-(-1)^{n-1}]}{\pi(n+1)\omega_n}$. This system has a particular solution in $\mathbb{Y}$ with components given by

$$\begin{bmatrix} v_{2n-1}(t) \\ v_{2n}(t) \end{bmatrix} = \frac{\mu_1 \nu_n}{(\omega_n^2 - \omega_0^2)^2 + 4\mu_2^2\omega_0^2} \begin{bmatrix} \omega_n(\omega_n^2 - \omega_0^2)\cos \omega_0 t + 2\mu_2 \omega_0 \omega_n \sin \omega_0 t \\ -\omega_0(\omega_n^2 - \omega_0^2)\sin \omega_0 t + 2\mu_2 \omega_0^2 \cos \omega_0 t \end{bmatrix}.$$

From this we see that (H9) is satisfied whenever $\omega_0 \neq \omega_n$ for all n.

We note that while the conditions $M(\alpha) = 0$, $M'(\alpha) \neq 0$ can be satisfied with $\mu_2 = 0$, $\alpha = 0$ we require $\mu_2 \neq 0$ in Section 5.2.4 where we use a weak exponential dichotomy to lift the full equation. Thus, we obtain the following result using Theorem 5.2.4.

Theorem 5.2.5. *If $\omega_0 \neq \omega_n$ for all n then whenever μ_0 satisfies $\mu_{0,1} \neq 0$ and*

$$0 < \left|\frac{\mu_{0,2}}{\mu_{0,1}}\right| < \frac{3\sqrt{\pi}\omega_0}{2a^3} \operatorname{sech} \frac{\pi\omega_0}{2a}, \tag{5.2.12}$$

there exists a corresponding $\bar{\xi}_0 > 0$ so that if $0 < \xi \leq \bar{\xi}_0$, if the parameters in (5.2.10) *are given by $\mu = \xi\mu_0$ then there exists a compact subset of $\mathbb{R}^2 \times \mathbb{Y}$ on which the $2m$th iterate, F^{2m}, of the period map F of* (5.2.10) *is invariant and conjugate to the Bernoulli shift on $\mathcal{E}$. Here $m \in \mathbb{N}$ is sufficiently large.*

These results are stated in terms of the Galerkin equations (5.2.10) but they can be transferred back to the original partial differential equation. In this case we get a Bernoulli shift embedded in $\left[H_0^1(0,\pi) \cap H^2(0,\pi)\right] \times L^2(0,\pi)$. This is discussed in [5]. In the μ_1-μ_2 plane we get from the condition (5.2.12) four small open wedge-shaped regions of parameter values for which the partial differential equation exhibits chaos (Figure 5.3). These regions are bounded by the lines $\mu_1/\mu_2 = \pm\frac{3\sqrt{\pi}\omega_0}{2a^3} \operatorname{sech} \frac{\pi\omega_0}{2a}$ and $\mu_2 = 0$.

It is interesting to look at some history of this problem. The first work was done in [28] in which the author started with the PDE and carried out the Galerkin expansion but restricted his analysis to the reduced equation (5.2.11). The significance of that work is that it introduced the idea of Melnikov analysis. In subsequent work [3], the results are extended to infinite dimension but the Galerkin approach is abandoned in favor of nonlinear semigroup techniques directly in infinite dimensions. In our

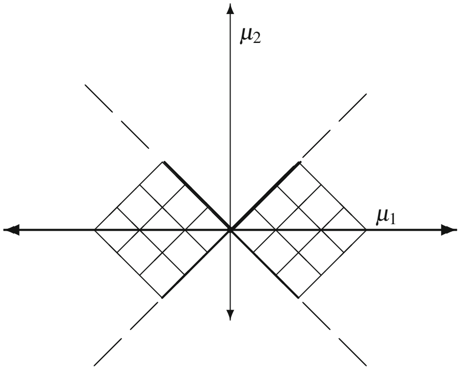

Fig. 5.3 The chaotic open wedge-shaped region of (5.2.10) in $\mathbb{R}^2$.

section we go back to the original, simpler analysis of the reduced equation and then show that the results apply to the original PDE. Some advantages of this are that the Galerkin projection is a technique familiar to many engineers and physicists and, also, we are able to utilize our general Melnikov results in Section 5.2.3. This is illustrated further in the generalizations to follow. We note that Equation (5.2.10) was treated also in [4].

5.2.7 Nonplaner Symmetric Beams

Let us consider a beam with symmetric cross section, pinned ends and compressive axial load P_0 and assume now that the beam is not constrained to defect in a plane. If $u(x,t)$ and $w(x,t)$ denote the transverse defections at position x and time t we obtain the following boundary value problem.

$$\begin{aligned}
\ddot{u} &= -u'''' - P_0 u'' + \left[\int_0^\pi \left(u'(s)^2 + w'(s)^2\right) ds\right] u'' \\
&\quad -2\mu_2 \dot{u}\cos\eta + \mu_1 \cos\zeta \cos\omega_0 t\,, \\
\ddot{w} &= -w'''' - P_0 w'' + \left[\int_0^\pi \left(u'(s)^2 + w'(s)^2\right) ds\right] w'' \\
&\quad -2\mu_2 \dot{w}\sin\eta + \mu_1 \sin\zeta \cos\omega_0 t\,, \\
u(0,t) &= u(\pi,t) = u''(0,t) = u''(\pi,t) = w(0,t) \\
&= w(\pi,t) = w''(0,t) = w''(\pi,t) = 0
\end{aligned}$$

where η, ζ are constants. The parameters μ_1, μ_2 represent the coefficients of, respectively, total transverse forcing and total viscous damping. These effects are distributed between the two directions of motion. The quantity $\tan\zeta$ represents the ratio of forcing in the u-direction to forcing in the w-direction while $\tan\eta$ plays the same role in the damping. We suppose $\eta, \zeta \in (0, \pi/2)$ in order to avoid certain degeneracies. In these equations we use the Galerkin expansions

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t)\sin kx, \quad w(x,t) = \sum_{k=1}^{\infty} w_k(t)\sin kx$$

and proceed as before. This yields the system of equations

$$\begin{aligned}
\ddot{u}_n &= n^2(P_0 - n^2)u_n - \frac{\pi}{2}n^2\left[\sum_{k=1}^{\infty} k^2(u_k^2 + w_k^2)\right]u_n \\
&\quad -2\mu_2\dot{u}_n\cos\eta + 2\mu_1\cos\zeta\left[\frac{1-(-1)^n}{\pi n}\right]\cos\omega_0 t, \\
\ddot{w}_n &= n^2(P_0 - n^2)w_n - \frac{\pi}{2}n^2\left[\sum_{k=1}^{\infty} k^2(u_k^2 + w_k^2)\right]w_n \\
&\quad -2\mu_2\dot{w}_n\sin\eta + 2\mu_1\sin\zeta\left[\frac{1-(-1)^n}{\pi n}\right]\cos\omega_0 t.
\end{aligned} \tag{5.2.13}$$

As before, we assume $1 < P_0 < 4$ and define $a^2 = P_0 - 1$, $\omega_{n-1}^2 = n(n^2 - P_0)$, $n = 2,3,\ldots$. Equations (5.2.13) take the form of (5.2.6) when we define $x = (u_1, \dot{u}_1, w_1, \dot{w}_1)$ and $y = (u_2, \dot{u}_2/\omega_1, w_2, \dot{w}_2/\omega_1, u_3, \dot{u}_3/\omega_2, w_3, \dot{w}_3/\omega_2, \ldots)$. The reduced equations are

$$\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= a^2x_1 - \frac{\pi}{2}(x_1^2 + x_3^2)x_1 - 2\mu_2 x_2\cos\eta + \frac{4}{\pi}\mu_1\cos\zeta\cos\omega_0 t, \\
\dot{x}_3 &= x_4, \\
\dot{x}_4 &= a^2x_3 - \frac{\pi}{2}(x_1^2 + x_3^2)x_3 - 2\mu_2 x_4\sin\eta + \frac{4}{\pi}\mu_1\sin\zeta\cos\omega_0 t.
\end{aligned}$$

When $\mu = 0$ we have a two-dimensional homoclinic manifold given by $\gamma_\beta = (r\cos\beta, \dot{r}\cos\beta, r\sin\beta, \dot{r}\sin\beta)$ where, as before, $r(t) = (2a/\sqrt{\pi})\,\mathrm{sech}\,at$ and β is a parameter. The adjoint equations (5.2.8) take the form

$$\begin{aligned}
\dot{v}_1 &= \left[-a^2 + \frac{\pi}{2}(3r^2\cos^2\beta + r^2\sin^2\beta)\right]v_2 + \left(\pi r^2\sin\beta\cos\beta\right)v_4, \\
\dot{v}_2 &= -v_1, \\
\dot{v}_3 &= \left(\pi r^2\sin\beta\cos\beta\right)v_2 + \left[-a^2 + \frac{\pi}{2}(r^2\cos^2\beta + 3r^2\sin^2\beta)\right]v_4, \\
\dot{v}_4 &= -v_3.
\end{aligned}$$

A one-parameter family of bounded solutions to these equations is given by

$$\begin{aligned} v_{\beta 1} &= (-\dot{r}\sin\beta, r\sin\beta, \dot{r}\cos\beta, -r\cos\beta)\,, \\ v_{\beta 2} &= (-\ddot{r}\cos\beta, \dot{r}\cos\beta, -\ddot{r}\sin\beta, \dot{r}\sin\beta) \end{aligned} \tag{5.2.14}$$

and the function, M, as in (4.2.7) becomes

$$M_1(\mu,\alpha,\beta) = \left[\frac{8}{\sqrt{\pi}}\sin(\beta-\zeta)\cos\omega_0\alpha\,\mathrm{sech}\,\frac{\pi\omega_0}{2a}\right]\mu_1\,,$$

$$\begin{aligned} M_2(\mu,\alpha,\beta) = {} & \left[\frac{8\omega_0}{\sqrt{\pi}}\cos(\beta-\zeta)\sin\omega_0\alpha\,\mathrm{sech}\,\frac{\pi\omega_0}{2a}\right]\mu_1 \\ & - \left[\frac{16a^3(\cos\eta\cos^2\beta+\sin\eta\sin^2\beta)}{3\pi}\right]\mu_2\,. \end{aligned}$$

Next, the conditions $M(\mu_0,\alpha_0,\beta_0)=0$, $D_{(\alpha,\beta)}M(\mu_0,\alpha_0,\beta_0)$ nonsingular are satisfied in two different cases. Of course, we suppose $\mu_{0,1}\neq 0$, $\mu_{0,2}\neq 0$ and then put $\lambda_0 = \frac{\mu_{0,2}}{\mu_{0,1}}$. We have the following two cases:

Case 1. We can choose either $\beta_0 = \zeta$ and then look for a simple root of the equation

$$\lambda_0 = m_1 \sin\omega_0\alpha\,, \tag{5.2.15}$$

or $\beta_0 = \zeta + \pi$ and look for a simple root of the equation

$$\lambda_0 = -m_1\sin\omega_0\alpha \tag{5.2.16}$$

for

$$m_1 = \frac{3\sqrt{\pi}\omega_0}{2a^2(\cos\eta\cos^2\zeta+\sin\eta\sin^2\zeta)}\,\mathrm{sech}\,\frac{\pi\omega_0}{2a}\,.$$

Supposing under the condition

$$0 < |\lambda_0| < m_1\,, \tag{5.2.17}$$

there is a simple root α_0 of (5.2.15). Similarly, (5.2.16) has also a simple root $-\alpha_0$. According to the formulas (5.2.14) for v_{β_1} and v_{β_2}, these simple roots (ζ,α_0) and $(\zeta+\pi,-\alpha_0)$ give two different solutions of (5.2.13).

Case 2. We begin from choosing $\omega_0\alpha_0 = (2k_0+1)\frac{\pi}{2}$ for $k_0\in\{0,1\}$ and then we look for a simple root $\beta_0\neq\zeta+k\pi$, $\forall k\in\mathbb{Z}$ of

$$\lambda_0 = (-1)^{k_0}\Phi(\beta) \tag{5.2.18}$$

where

$$\Phi(\beta) = \frac{3\omega_0\sqrt{\pi}}{2a^3}\,\frac{\cos(\beta-\zeta)}{\cos\eta\cos^2\beta+\sin\eta\sin^2\beta}\,\mathrm{sech}\,\frac{\pi\omega_0}{2a}.$$

Let $m_2 = \max_{\beta\in\mathbb{R}} \Phi(\beta)$. A computation of the constant m_2 is discussed in [29]. Since $\Phi(\beta+\pi) = -\Phi(\beta)$, the range of Φ is the closed interval $[-m_2, m_2]$. We now split this case into two parts:

Part 2A). For $\eta = \pi/4$ we get $\Phi(\beta) = m_1\cos(\beta-\zeta)$, along with $m_2 = m_1 = \frac{3\omega_0\sqrt{\pi}}{\sqrt{2}a^3}\operatorname{sech}\frac{\pi\omega_0}{2a}$. Equation (5.2.18) has now the form

$$(-1)^{k_0}\frac{3\omega_0\sqrt{\pi}}{\sqrt{2}a^3}\operatorname{sech}\frac{\pi\omega_0}{2a}\cos(\beta-\zeta) = \lambda_0\,,$$

so under condition (5.2.17), there is a simple root β_0 different from $\zeta + k\pi$, $\forall k\in\mathbb{Z}$. This holds for both cases $k_0\in\{0,1\}$ so we have two different solutions of (5.2.13). In addition, the results of Case 1 still apply here. Thus, in this situation, we have in the μ_1-μ_2 plane four wedged-shaped regions of parameter values bounded by $\mu_2/\mu_1 = \pm m_1$, $\mu_2 = 0$ for which the partial differential equation exhibits chaos. Particularly, (5.2.13) has four distinct homoclinic solutions, two from Case 1, two from Case 2A. These regions are labeled *II* in Figure 5.4. In this case there are no regions labeled *I*.

Part 2B). For $\eta \neq \pi/4$ we get $\Phi'(\zeta)\neq 0$, so $m_1 < m_2$. Certainly for the solvability of (5.2.18) we need $|\lambda_0|\le m_2$. Now we claim:

Lemma 5.2.6. *If*

$$\lambda_0 \in (-m_2, m_2)\setminus\{\pm m_1, 0\}\,, \tag{5.2.19}$$

then Eq. (5.2.18) *has a simple root* $\beta_0\in[0,2\pi]\setminus\{\zeta,\zeta+\pi\}$.

Proof. Assume to the contrary that (5.2.18) has no simple roots for a $\lambda_0\in(-m_2, m_2)\setminus\{\pm m_1, 0\}$. Then there are $0\le\beta_1<\beta_2\le 2\pi$ so that

$$\Phi(\beta_{1,2}) = (-1)^{k_0}\lambda_0,\quad \Phi'(\beta_{1,2}) = 0,\quad \Phi''(\beta_{1,2}) = 0\,. \tag{5.2.20}$$

Note that $\beta_{1,2}\neq\zeta+k\pi$ and $\beta_{1,2}\neq\zeta+\frac{2k+1}{2}\pi$, $\forall k\in\{0,1\}$. After some calculation we derive from (5.2.20) that $\cos 2\beta_{1,2}\neq 0$, $\sin 2\beta_{1,2}\neq 0$ and that (5.2.20) is equivalent to

$$\begin{aligned}\frac{\cos(\beta_{1,2}-\zeta)}{\cos\eta\cos^2\beta_{1,2}+\sin\eta\sin^2\beta_{1,2}} &= \frac{\sin(\beta_{1,2}-\zeta)}{(\cos\eta-\sin\eta)\sin 2\beta_{1,2}}\\ = \frac{\cos(\beta_{1,2}-\zeta)}{2(\cos\eta-\sin\eta)\cos 2\beta_{1,2}} &= (-1)^{k_0}\frac{2a^3}{3\omega_0\sqrt{\pi}}\cosh\frac{\pi\omega_0}{2a}\lambda_0\,.\end{aligned} \tag{5.2.21}$$

From (5.2.21) we derive

$$\cos 2\beta_{1,2} = \frac{\cos\eta+\sin\eta}{3(\cos\eta-\sin\eta)},\quad 2\tan(\beta_{1,2}-\zeta) = \tan 2\beta_{1,2}\,. \tag{5.2.22}$$

Hence

$$\beta_2 \in \{\pi - \beta_1, \pi + \beta_1, 2\pi - \beta_1\} .$$

If $\beta_2 = \pi - \beta_1$ then from $2\tan(\beta_2 - \zeta) = \tan 2\beta_2$ we get $2\tan(\beta_1 + \zeta) = \tan 2\beta_1$, but $2\tan(\beta_1 - \zeta) = \tan 2\beta_1$, so $\tan(\beta_1 + \zeta) = \tan(\beta_1 - \zeta)$, i.e. $\zeta = k\pi/2$, $k \in \{0,1\}$. This contradicts $\zeta \in (0, \pi/2)$. If $\beta_2 = \pi + \beta_1$ then

$$(-1)^{k_0}\lambda_0 = \Phi(\beta_2) = \Phi(\beta_1 + \pi) = -\Phi(\beta_1) = (-1)^{k_0+1}\lambda_0$$

which implies $\lambda_0 = 0$, a contradiction. If $\beta_2 = 2\pi - \beta_1$ then again we derive $\tan(\beta_1 + \zeta) = \tan(\beta_1 - \zeta)$, so that $\zeta = k\pi/2$, $k \in \{0,1\}$, a contradiction to $\zeta \in (0, \pi/2)$. The proof is finished. □

Note that $\beta_0 \in \{\zeta, \zeta + \pi\}$ for the Case 1, while $\beta_0 \in [0, 2\pi) \setminus \{\zeta, \zeta + \pi\}$ for the Case 2. Lemma 5.2.6 can be applied to both cases $\alpha_0 = \frac{\pi}{2\omega_0}(2k_0 + 1)$, $k_0 \in \{0,1\}$, so Part 2B yields, in the μ_1-μ_2 plane, four wedge-shaped regions of parameter values bounded by $\mu_2/\mu_1 = \pm m_2$, $\mu_2/\mu_1 = \pm m_1$, $\mu_2 = 0$ for which (5.2.13) has two different homoclinic solutions. These regions are labeled *I* in Figure 5.4. Note that we have four different solutions of (5.2.13) in regions labeled *II*, since there Case 1 can be also applied (see (5.2.15) and (5.2.16)). This completes the analysis of the Melnikov function. We now check about resonance. Because in the present problem all coupling terms are nonlinear, the linear equation in (H9) consists in two copies of the system of equations in the preceding example. This yields the following result obtained from Theorem 5.2.4.

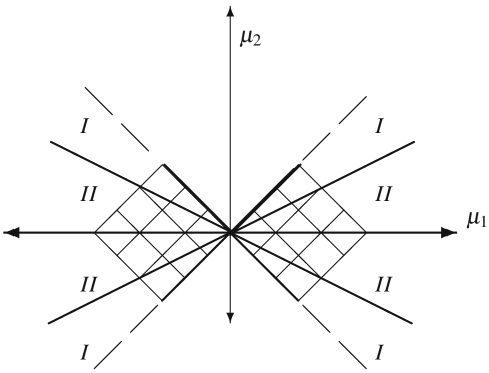

Fig. 5.4 The chaotic wedge-shaped regions of (5.2.13) in $\mathbb{R}^2$.

Theorem 5.2.7. *Suppose $\omega_0 \neq \omega_n$ for all n and let m_1, m_2 be as above.*

(a) *If $m_0 \neq 0$ satisfies one but not both of $|m_0| < m_i$ then if $\mu_{0,2}/\mu_{0,1} = m_0$ there exists a corresponding $\bar{\xi}_0 > 0$ so that if $0 < \xi \leq \bar{\xi}_0$, if the parameters in* (5.2.13) *are given by $\mu = \xi\mu_0$ then there exist two homoclinic orbits which can be used to construct a compact subset of $\mathbb{R}^4 \times \mathbb{Y}$ on which the $2m$th iterate, F^{2m}, of the period* *map F of* (5.2.13) *is invariant and conjugate to the Bernoulli shift on $\mathscr{E}$. Here $m \in \mathbb{N}$ is sufficiently large.*

(b) *If $m_0 \neq 0$ satisfies each of $|m_0| < m_i$ then there are four homoclinic orbits as in (i).*

In summary, we obtain eight open small wedge-shaped regions of parameter values in the μ_1-μ_2 plane bounded by the lines $\mu_2/\mu_1 = \pm m_1$, $\mu_2/\mu_1 = \pm m_2$ and $\mu_2 = 0$ with $m_1 \leq m_2$ for which the partial differential equation exhibits chaos (Figure 5.4). In the regions labeled I there are two homoclinics while in regions II there exist four. It is interesting to note that in this case, by adjusting the parameters η and ζ, it is possible to make the size of the wedge arbitrarily close to filling the μ_1-μ_2 plane.

5.2.8 Nonplaner Nonsymmetric Beams

For the case of a nonsymmetric beam with nonplaner motion we have the boundary value problem

$$
\begin{aligned}
\ddot{u} = &-u'''' - P_0 u'' + \left[\int_0^\pi \left(u'(s)^2 + w'(s)^2\right) ds\right] u'' \\
&-2\mu_2 \dot{u} \cos\eta + \mu_1 \cos\zeta \cos\omega_0 t\,, \\
\ddot{w} = &-R^2 w'''' - P_0 w'' + \left[\int_0^\pi \left(u'(s)^2 + w'(s)^2\right) ds\right] w'' \\
&-2\mu_2 \dot{w} \sin\eta + \mu_1 \sin\zeta \cos\omega_0 t\,,
\end{aligned}
$$

$$
\begin{aligned}
u\ (0,t) &= u(\pi,t) = u''(0,t) = u''(\pi,t) \\
&= w(0,t) = w(\pi,t) = w''(0,t) = w''(\pi,t) = 0
\end{aligned}
$$

where R^2 is constant representing the stiffness ratio for the two directions. We assume that $R > 1$ which amounts to choosing w as the direction with stiffer cross-section. Note that $R = 1$ reduces to Section 5.2.7. As before we assume that η, $\zeta \in (0, \pi/2)$. The Galerkin expansion becomes

$$
\begin{aligned}
\ddot{u}_n = &\ n^2(P_0 - n^2)u_n - \frac{\pi}{2} n^2 \left[\sum_{k=1}^{\infty} k^2(u_k^2 + w_k^2)\right] u_n \\
&- 2\mu_2 \dot{u}_n \cos\eta + 2\mu_1 \cos\zeta \left[\frac{1-(-1)^n}{\pi n}\right] \cos\omega_0 t\,, \\
\ddot{w}_n = &\ n^2(P_0 - n^2 R^2)w_n - \frac{\pi}{2} n^2 \left[\sum_{k=1}^{\infty} k^2(u_k^2 + w_k^2)\right] w_n \\
&- 2\mu_2 \dot{w}_n \sin\eta + 2\mu_1 \sin\zeta \left[\frac{1-(-1)^n}{\pi n}\right] \cos\omega_0 t\,.
\end{aligned}
\tag{5.2.23}
$$

If P_0 is increased only enough to give one buckled mode, necessarily in the u direction, the problem reduces to Section 5.2.6. We shall assume here the next simplest case consisting of one buckled mode in each direction which occurs when $1 < P_0 < 4$ and $R^2 < P_0 < 4R^2$. Note that this requires $R < 2$ and we assume that $R^2 < P_0 < 4$. If the stiffness ratio is too high there will be multiple buckled in the u (soft) direction before occurrence of the first buckled mode in the w (stiff) direction. We define

$$a_1^2 = P_0 - 1, \qquad \omega_{n-1,1}^2 = n^2[(n^2 - P_0], \quad n = 2,3,\dots;$$
$$a_2^2 = P_0 - R^2, \qquad \omega_{n-1,2}^2 = n^2[n^2R^2 - P_0], \quad n = 2,3,\dots.$$

We put (5.2.23) in the form of (5.2.6) by defining

$$x = (u_1, \dot{u}_1, w_1, \dot{w}_1),$$

$$y = (u_2, \dot{u}_2/\omega_{1,1}, w_2, \dot{w}_2/\omega_{1,2}, u_3, \dot{u}_3/\omega_{2,1}, w_3, \dot{w}_3/\omega_{2,2}, \dots).$$

The reduced equations are

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = a_1^2 x_1 - \frac{\pi}{2}(x_1^2 + x_3^2)x_1 - 2\mu_2 x_2 \cos\eta + \frac{4}{\pi}\mu_1 \cos\zeta \cos\omega_0 t,$$
$$\dot{x}_3 = x_4,$$
$$\dot{x}_4 = a_2^2 x_3 - \frac{\pi}{2}(x_1^2 + x_3^2)x_3 - 2\mu_2 x_4 \sin\eta + \frac{4}{\pi}\mu_1 \sin\zeta \cos\omega_0 t.$$

For the unperturbed equations we have two homoclinic solutions given by

$$\gamma_1 = (r_1, \dot{r}_1, 0, 0), \qquad \gamma_2 = (0, 0, r_2, \dot{r}_2)$$

where $r_1(t) = (2a_1/\sqrt{\pi})\operatorname{sech} a_1 t$ and $r_2(t) = (2a_2/\sqrt{\pi})\operatorname{sech} a_2 t$. Using γ_1 the adjoint equations (5.2.8) become

$$\dot{v}_1 = \left(-a_1^2 + \frac{3\pi}{2} r_1^2\right) v_2, \quad \dot{v}_2 = -v_1,$$
$$\dot{v}_3 = \left(-a_2^2 + \frac{\pi}{2} r_1^2\right) v_4, \quad \dot{v}_4 = -v_3.$$

The essential issue here is to determine the space of bounded solutions to these equations. We can write these in the form

$$\ddot{v}_2 = \left(a_1^2 - \frac{3\pi}{2} r_1^2\right) v_2, \qquad \ddot{v}_4 = \left(a_2^2 - \frac{\pi}{2} r_1^2\right) v_4.$$

The v_2 equation has a one-dimensional space of bounded solutions spanned by the solution $v_2 = \dot{r}_1$, obtained from $\dot{\gamma}_1$. For the v_4 equation we have the following result.

Lemma 5.2.8. *Let* $\kappa > 0$. *The equation*

$$\ddot{v} + (-\lambda + \kappa \operatorname{sech}^2 t)v = 0$$

has a bounded solution if and only if there exists an integer M so that

$$\lambda = \frac{1}{4}\left(\sqrt{4\kappa+1} - 4M - 1\right)^2 \quad \textit{for} \quad 0 \le M < \frac{1}{4}\left(\sqrt{4\kappa+1} - 1\right)$$

$$\textit{or} \quad \lambda = \frac{1}{4}\left(\sqrt{4\kappa+1} - 4M - 3\right)^2 \quad \textit{for} \quad 0 \le M < \frac{1}{4}\left(\sqrt{4\kappa+1} - 3\right).$$

The idea for the proof of this lemma is to express the solution as the product of a power of $\operatorname{sech} t$ and a hypergeometric function with argument $-\sinh^2 t$. The condition for the existence of a bounded solution is that the hypergeometric series terminate and the resulting polynomial is of sufficiently small degree. The details for this have been worked out in Appendix of [30]. See also Sections 23, 25 of [31].

Applying Lemma 5.2.8 to the equation for v_4 we find that the condition for a bounded solution is $a_1 = a_2$ which is ruled out by the assumption of $R > 1$. Hence, the system of equations for v has a one-dimensional space of bounded solutions spanned by $v = (-\ddot{r}_1, \dot{r}, 0, 0)$ and the Melnikov function (4.2.6) is

$$M(\alpha) = \left[\frac{8\omega_0 \cos\zeta}{\sqrt{\pi}} \sin\omega_0\alpha \operatorname{sech}\frac{\pi\omega_0}{2a_1}\right]\mu_1 - \left(\frac{16a_1^3\cos\eta}{3\pi}\right)\mu_2.$$

The non-resonance hypothesis follows as in the previous examples which leads, in the present case, to the following result obtained from Theorem 5.2.4.

Theorem 5.2.9. *If* $\omega_0 \neq \omega_{n,i}$ *for all n and for* $i = 1, 2$, *then whenever* μ_0 *satisfies* $\mu_{0,1} \neq 0$ *and*

$$0 < \left|\frac{\mu_{0,2}}{\mu_{0,1}}\right| < \frac{3\sqrt{\pi}\,\omega_0\cos\zeta}{2a_1^3\cos\eta}\operatorname{sech}\frac{\pi\omega_0}{2a_1}$$

there exists a corresponding $\bar{\xi}_0 > 0$ *so that if* $0 < \xi \le \bar{\xi}_0$, *if the parameters in* (5.2.23) *are given by* $\mu = \xi\mu_0$ *then there exists a compact subset of* $\mathbb{R}^4 \times \mathbb{Y}$ *on which the* $2m$*th iterate,* F^{2m}, *of the period map F of* (5.2.23) *is invariant and conjugate to the Bernoulli shift on* $\mathscr{E}$. *Here* $m \in \mathbb{N}$ *is sufficiently large.*

Replacing γ_1 with γ_2 yields the following analogous result.

Theorem 5.2.10. *If* $\omega_0 \neq \omega_{n,i}$ *for all n and for* $i = 1, 2$, *then whenever* μ_0 *satisfies* $\mu_{0,1} \neq 0$ *and*

$$0 < \left|\frac{\mu_{0,2}}{\mu_{0,1}}\right| < \frac{3\sqrt{\pi}\,\omega_0\sin\zeta}{2a_2^3\sin\eta}\operatorname{sech}\frac{\pi\omega_0}{2a_2}$$

there exists a corresponding $\bar{\xi}_0 > 0$ *so that if* $0 < \xi \le \bar{\xi}_0$, *if the parameters in* (5.2.23) *are given by* $\mu = \xi\mu_0$ *then there exists a compact subset of* $\mathbb{R}^4 \times \mathbb{Y}$ *on which the* $2m$*th iterate,* F^{2m}, *of the period map F of* (5.2.23) *is invariant and conjugate to the Bernoulli shift on* $\mathscr{E}$. *Here* $m \in \mathbb{N}$ *is sufficiently large.*

In the μ_1-μ_2 plane in this case we get a diagram as in Figure 5.4. For parameter values in the regions labeled *I* there is one homoclinic orbit while for those in *II* there are two.

5.2.9 Multiple Buckled Modes

One has to consider the situation where the axial load, P_0, is increased sufficiently to produce multiple buckled modes. We will look at the case of a beam constrained to planer motion. The calculations for the non-planer case are similar. We return to the boundary value problem of Section 5.2.6 and use the same Galerkin equations

$$\begin{aligned}\ddot{u}_n = n^2(P_0 - n^2)u_n - \tfrac{\pi}{2}n^2\left[\sum_{k=1}^{\infty} k^2 u_k^2\right] u_n \\ -2\mu_2\dot{u}_n + 2\mu_1\left[\frac{1-(-1)^n}{\pi n}\right]\cos\omega_0 t, \quad n = 1, 2, \ldots .\end{aligned} \tag{5.2.24}$$

In the present case we assume that there exists an integer N so that $N^2 < P_0 < (N+1)^2$. We then define

$$\begin{aligned} a_n^2 &= n^2(P_0 - n^2), \quad \text{for } n = 1, 2, \ldots, N; \\ \omega_{n-N}^2 &= n^2(n^2 - P_0), \quad \text{for } n = N+1, N+2, \ldots \end{aligned}$$

and put (5.2.24) in the form of (5.2.6) by defining

$$\begin{aligned} x &= (u_1, \dot{u}_1, u_2, \dot{u}_2, \ldots, u_N, \dot{u}_N), \\ y &= (u_{N+1}, \dot{u}_{N+1}/\omega_1, u_{N+2}, \dot{u}_{N+2}/\omega_2, \ldots). \end{aligned}$$

A truncated version of the resulting equations with $N = 2$ was studied in [30]. The reduced equations are

$$\left.\begin{aligned} \dot{x}_{2n-1} &= x_{2n} \\ \dot{x}_{2n} &= a_n^2 x_{2n-1} - \frac{\pi n^2}{2}\left(\sum_{k=1}^{N} k^2 x_{2k-1}^2\right) x_{2n-1} \\ &\quad -2\mu_2 x_{2n} + 2\mu_1\left[\tfrac{1-(-1)^n}{\pi n}\right]\cos\omega_0 t \end{aligned}\right\} n = 1, 2, \ldots, N.$$

When $\mu = 0$ we have N homoclinic solutions given by

$$\gamma_m = (0, \ldots, 0, \underbrace{r_m, \dot{r}_m}_{2m-1, 2m}, 0, \ldots, 0), \quad m = 1, 2, \ldots, N$$

where $r_m(t) = (2a_m/m^2\sqrt{\pi})\,\mathrm{sech}\, a_m t$ and the adjoint equation (5.2.8) along γ_m is

$$\left.\begin{aligned} \dot{v}_{2n-1} &= \left(-a_n^2 + \frac{\pi m^2 n^2}{2} r_m^2\right) v_{2n}\,, \\ \dot{v}_{2n} &= -v_{2n-1}\,, \end{aligned}\right\} n \neq m$$

$$\begin{aligned} \dot{v}_{2m-1} &= \left(-a_m^2 + \frac{3\pi m^4}{2} r_m^2\right) v_{2m}\,, \\ \dot{v}_{2m} &= -v_{2m-1}\,. \end{aligned}$$

For the distinguished equation we have the bounded solution $v_{2m-1} = -\ddot{r}_m$, $v_{2m} = \dot{r}_m$ while for the equations with $n \neq m$ we must solve

$$\frac{d^2 v_{2n}}{dx^2} = \left(\frac{a_n^2}{a_m^2} - \frac{2n^2}{m^2}\operatorname{sech}^2 x\right) v_{2n}.$$

Using Lemma 5.2.8 we find that this last equation has a bounded solution if and only if there is an integer M so that one of the following conditions holds:

$$\frac{n^2(P_0 - n^2)}{m^2(P_0 - m^2)} = \frac{1}{4}\left[\sqrt{\frac{8n^2}{m^2} + 1} - 4M - 1\right]^2 \tag{5.2.25a}$$

$$\text{for } 0 \leq M < \frac{1}{4}\left(\sqrt{\frac{8n^2}{m^2} + 1} - 1\right),$$

$$\frac{n^2(P_0 - n^2)}{m^2(P_0 - m^2)} = \frac{1}{4}\left[\sqrt{\frac{8n^2}{m^2} + 1} - 4M - 3\right]^2 \tag{5.2.25b}$$

$$\text{for } 0 \leq M < \frac{1}{4}\left(\sqrt{\frac{8n^2}{m^2} + 1} - 3\right).$$

If, for some fixed m, none of the equations in (5.2.25 a and b) is satisfied for $n \neq m$ we can proceed much as in Section 5.2.6 since then the adjoint equation obtained from γ_m has a one-dimensional space of bounded solutions spanned by

$$v = (0, \ldots, 0, \underbrace{-\ddot{r}_m, \dot{r}_m}_{2m-1,2m}, 0, \ldots, 0).$$

One complication has been introduced by our assumption in the original partial differential equation that the transverse-applied load is uniform in x. This assumption causes the μ_1 terms to drop out in (5.2.24) for n even which prohibits nonsingular solutions of $M(\alpha) = 0$ as can be seen by examining Section 5.2.6. For this reason, we must choose m odd. Theorem 5.2.4 now yields the following result.

Theorem 5.2.11. *Let m be an odd integer,* $1 \leq m \leq N$, *and suppose* P_0 *is chosen so that none of the equations in* (5.2.25 a and b) *is satisfied. If* $\omega_0 \neq \omega_n$ *for all n, then* *whenever* μ_0 *satisfies* $\mu_{0,1} \neq 0$ *and*

$$0 < \left| \frac{\mu_{0,2}}{\mu_{0,1}} \right| < \frac{3m\sqrt{\pi}\,\omega_0}{2a_m^3} \operatorname{sech} \frac{\pi\omega_0}{2a_m}$$

there exists a corresponding $\bar{\xi}_0 > 0$ *so that if* $0 < \xi \leq \bar{\xi}_0$*, if the parameters in* (5.2.24) *are given by* $\mu = \xi\mu_0$ *then there exists a compact subset of* $\mathbb{R}^{2N} \times \mathbb{Y}$ *on which the* $2k$*th iterate,* F^{2k}*, of the period map* F *of* (5.2.24) *is invariant and conjugate to the Bernoulli shift on* $\mathscr{E}$*. Here* $k \in \mathbb{N}$ *is sufficiently large.*

We can simplify the preceding results by finding cases where the equations in (5.2.25) can never have a solution. The following is a helpful result along these lines.

Lemma 5.2.12. *The equations in* (5.2.25) *can never be satisfied for* $n < m \leq N$.

Proof. For (5.2.25a) we have $\frac{1}{4}\left(\sqrt{8n^2/m^2+1}-1\right) < \frac{1}{2}$ so we have only one equation to consider with $M = 0$. But then we have, first, $\frac{n^2(P_0-n^2)}{m^2(P_0-m^2)} > \frac{n^2}{m^2}$, and also

$$\frac{1}{4}\left[\sqrt{\frac{8n^2}{m^2}+1}-1\right]^2 - \frac{n^2}{m^2} = \frac{2\frac{n^2}{m^2}\left(\frac{n^2}{m^2}-1\right)}{2\frac{n^2}{m^2}+1+\sqrt{\frac{8n^2}{m^2}+1}} < 0$$

so that Equation (5.2.25a) has no solution for any P_0. Next we note that when $n < m$, we have $\frac{1}{4}\left(\sqrt{8n^2/m^2+1}-3\right) < 0$ so that there are no equations for (5.2.25b). □

When $m = N$ the preceding result will eliminate any restriction, obtained from (5.2.25), on P_0. This fact was shown with a different technique in [4] where they used a more general transverse forcing term which allowed for the possibility of a μ_2 term for each n in (5.2.24) and, hence, also for each n in the reduced equation. They then take $m = N$. Since, for our specific form of loading, we must have m odd we have the following result.

Theorem 5.2.13. *Let* N *and* P_0 *be as for* (5.2.24) *and suppose one of the following holds:*

(i) N *is odd and* $m = N$.
(ii) N *is even,* $N \geq 4$, $m = N-1$ *and*

$$P_0 \neq \frac{4N^2-(N-1)^2\left[\sqrt{9N^2-2N+1}-3(N-1)\right]^2}{4N^2-\left[\sqrt{9N^2-2N+1}-3(N-1)\right]^2}.$$

(iii) $N = 2$, $m = 1$ *and*

$$P_0 \neq \frac{37+5\sqrt{33}}{16}, \quad P_0 \neq \frac{55+9\sqrt{33}}{16}.$$

Suppose in addition that $\omega_n \neq \omega_0$ *for all n. Then whenever* μ_0 *satisfies* $\mu_{0,1} \neq 0$ *and*

$$0 < \left|\frac{\mu_{0,2}}{\mu_{0,1}}\right| < \frac{3m\sqrt{\pi}\,\omega_0}{2a_m^3} \operatorname{sech} \frac{\pi\omega_0}{2a_m}$$

there exists a corresponding $\bar{\xi}_0 > 0$ *so that if* $0 < \xi \leq \bar{\xi}_0$*, if the parameters in* (5.2.24) *are given by* $\mu = \xi\mu_0$ *then there exists a compact subset of* $\mathbb{R}^{2N} \times \mathbb{Y}$ *on which the* $2k$*th iterate,* F^{2k}*, of the period map* F *of* (5.2.24) *is invariant and conjugate to the Bernoulli shift on* $\mathscr{E}$*. Here* $k \in \mathbb{N}$ *is sufficiently large.*

Proof. The result is obtained by using γ_m and proceeding as in Section 5.2.6. This is valid as long as Equations (5.2.25) have no solutions for $n \neq m$ so it remains to show that this is true in each case. If (i) holds we can use Lemma 5.2.12.

If $m = N-1$ then, using Lemma 5.2.12, we need check only $n = N$. Define

$$f_a(N) = \frac{1}{4}\left(\sqrt{\frac{8N^2}{(N-1)^2}+1} - 1\right),$$

$$f_b(N) = \frac{1}{4}\left(\sqrt{\frac{8N^2}{(N-1)^2}+1} - 3\right).$$

Then (5.2.25a) must be checked for integers $M \in [0, f_a(N))$ and (5.2.25b) for integers $M \in [0, f_b(N))$.

In case (ii) we have $N \geq 4$ which implies $1/2 < f_a(N) \leq (\sqrt{137}-3)/12 < 1$ so we need consider only $M = 0$. In this case we solve

$$\frac{N^2(P_0 - N^2)}{(N-1)^2[P_0 - (N-1)^2]} = 4f_a(N)^2$$

for P_0 to get

$$P_0 = \frac{N^4 - 4f_a(N)^2(N-1)^4}{N^2 - 4f_a(N)^2(N-1)^2} = \frac{(N-1)^2}{2}\left[1 - 2\frac{N^2}{(N-1)^2} - \sqrt{\frac{8N^2}{(N-1)^2}+1}\right].$$

But this value is negative and can be discarded. Similarly, we have, for $N \geq 4$, $0 < f_b(N) \leq (\sqrt{137}-9)/12 < 1$, so in (5.2.25b) we need also consider only $M = 0$. Here we get

$$P_0 = \frac{N^4 - 4f_b(N)^2(N-1)^4}{N^2 - 4f_b(N)^2(N-1)^2} = \frac{4N^4 - (N-1)^2\left[\sqrt{9N^2-2N+1} - 3(N-1)\right]^2}{4N^2 - \left[\sqrt{9N^2-2N+1} - 3(N-1)\right]^2}.$$

Next, we consider (iii) where $N = 2$, $m = 1$. Since $2 > f_a(2) = (\sqrt{33} - 1)/4 > 1$ we must consider $M = 0$ and $M = 1$ in (5.2.25a). When $M = 0$ we get the value $P_0 = -(7 + \sqrt{33})/2 < 0$ which can be discarded while for $M = 1$ we have $P_0 = (37 + 5\sqrt{33})/16$. Finally, $0 < f_b(2) = (\sqrt{33} - 3)/4 < 1$, so only $M = 0$ must be considered in (5.2.25b) and this yields $P_0 = (55 + 9\sqrt{33})/16$. □

5.3 Periodically Forced Compressed Beam

5.3.1 Resonant Compressed Equation

This section is a continuation of Section 5.2, and it is devoted to the study of a system modelling a compressed beam with friction subjected to a small periodic forcing. Particularly we are interested in the existence of chaotic patterns. The model is described by the following PDE

$$u_{tt} + u_{xxxx} + \gamma u_{xx} - \kappa u_{xx} f\left(\int_0^\pi u_x^2(\xi,t)d\xi\right) = \varepsilon(\nu h(x,\sqrt{\varepsilon}t) - \delta u_t), \qquad (5.3.1)$$

$$u(0,t) = u(\pi,t) = 0 = u_{xx}(0,t) = u_{xx}(\pi,t) \qquad (5.3.2)$$

where $u(x,t) \in \mathbb{R}$ is the transverse deflection of the axis of the beam; $\gamma > 0$ is an external load, $\kappa > 0$ is a ratio indicating the external rigidity and $\delta > 0$ is the damping, ε and ν are small parameters, the function $h(x,t)$ represents the periodic (in time) forcing distributed along the whole beam. We assume that $h \in L^\infty\left(\mathbb{R}, L^2([0,\pi])\right)$ is a 1-periodic function of t with $\left\|\int_0^\pi h(x,\cdot)^2\,dx\right\|_\infty = 1$. Therefore $\varepsilon\nu$ represents the strength of the forcing.

Section 5.2 discusses Equation (5.3.1) when the external load γ is not resonant and $\kappa \in \mathbb{R}$ is fixed. Here we discuss the complementary case. Precisely we assume that γ is slightly larger than the i-th eigenvalue of the unperturbed problem: $\gamma = i^2 + \varepsilon\sigma^2$, where $i \in \mathbb{N}$ is fixed, $\varepsilon > 0$ and $\sigma \in (0,1]$. Therefore we will also assume that $\kappa = \varepsilon k$, so that the contribution given from the stress due to the external rigidity, does not drive the system too far away from the resonance.

5.3.2 Formulation of Weak Solutions

It is easily observed that the unperturbed problem

$$u_{xxxx} + \gamma u_{xx} = 0,$$

$$u(0,t) = u(\pi,t) = 0 = u_{xx}(0,t) = u_{xx}(\pi,t)$$

admits $\{j^2 \mid j \in \mathbb{N}\}$ as set of eigenvalues and that the corresponding eigenfunctions $\sqrt{\frac{2}{\pi}}\sin(jx)$, where $j \in \mathbb{N}$, form an orthonormal system in $L^2([0,\pi])$ which generates the space $H_0^2([0,\pi])$. First of all we make the linear scale $t \leftrightarrow \sqrt{\varepsilon}t$. Then Eqs. (5.3.1), (5.3.2) read:

$$u_{tt} + \frac{1}{\varepsilon}[u_{xxxx} + (i^2 + \varepsilon\sigma^2)u_{xx}] - kf\left(\int_0^\pi u_x^2(\xi,t)d\xi\right)u_{xx} = \nu h(x,t) - \sqrt{\varepsilon}\delta u_t,$$

$$u(0,t) = u(\pi,t) = 0 = u_{xx}(0,t) = u_{xx}(\pi,t). \tag{5.3.3}$$

We want to solve (5.3.3) in a *weak* form, that is, we look for a function $u \in L^\infty\left(\mathbb{R}, H_0^2([0,\pi])\right) \subset L^\infty([0,\pi]\times\mathbb{R})$ so that

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_0^\pi \Bigg\{u(x,t)\Bigg(\Psi_{tt} + \frac{1}{\varepsilon}\left[\Psi_{xxxx} + (i^2+\varepsilon\sigma^2)\Psi_{xx}\right]\\ -kf\left(\int_0^\pi u_x^2(\xi,t)d\xi\right)\Psi_{xx} - \sqrt{\varepsilon}\delta\Psi_t\Bigg) - \nu\Psi(x,t)h(x,t)\Bigg\}dxdt = 0\end{aligned} \tag{5.3.4}$$

for any $\Psi(x,t) \in C^\infty([0,\pi]\times\mathbb{R})$ with compact support so that

$$\Psi(0,t) = \Psi(\pi,t) = \Psi_{xx}(0,t) = \Psi_{xx}(\pi,t) = 0.$$

5.3.3 Chaotic Solutions

In this section, the existence of chaotic solutions is studied for (5.3.1). To start with, note that we can expand the function $u(x,t) \in L^\infty\left(\mathbb{R}, H_0^2([0,\pi])\right)$ as follows:

$$u(x,t) = \sqrt{\frac{2}{\pi}}\left[\sum_{0<l<i}\phi_l(t)\sin(lx) + y(t)\sin(ix) + \sum_{j>i} z_j(t)\sin(jx)\right],$$

where $\phi_l(t), y(t), z_j(t) \in L^\infty(\mathbb{R})$, the expansion holding in $H_0^2([0,\pi])$. Similarly we write:

$$\Psi(x,t) = \sqrt{\frac{2}{\pi}}\left[\sum_{l=1}^{i-1}\psi_l(t)\sin(lx) + \psi_i(t)\sin(ix) + \sum_{j=i+1}^{\infty}\psi_j(t)\sin(jx)\right],$$

where, for any $k \geq 1$, $\psi_k(t) \in C_0^\infty(\mathbb{R})$, the space of C^∞-functions on $\mathbb{R}$ having compact supports. Plugging the above expression for $u(x,t)$ and $\Psi(x,t)$ into (5.3.4) and using the orthonormality, we arrive at the system of equations for the components $(\phi_l(t), y(t), z_j(t))$ of $u(x,t)$

$$\ddot{\phi}_l(t) - \frac{i^2 - l^2 + \varepsilon\sigma^2}{\varepsilon} l^2 \phi_l(t) + kl^2 f\Big(\sum_{0<l<i} l^2 \phi_l(t)^2 + i^2 y^2(t) + \sum_{j>i} j^2 z_j(t)^2\Big)\phi_l(t)$$

$$+\sqrt{\varepsilon}\delta\dot{\phi}_l(t) - \nu\sqrt{\frac{2}{\pi}}\int_0^{\pi} h(x,t)\sin(lx)\,dx = 0, \tag{5.3.5}$$

$$\ddot{y}(t) - \sigma^2 i^2 y(t) + ki^2 f\Big(\sum_{0<l<i} l^2 \phi_l^2(t) + i^2 y^2(t) + \sum_{j>i} j^2 z_j^2(t)\Big) y(t) + \sqrt{\varepsilon}\delta\dot{y}(t)$$

$$-\nu\sqrt{\frac{2}{\pi}}\int_0^{\pi} h(x,t)\sin(ix)\,dx = 0 \tag{5.3.6}$$

$$\ddot{z}_j(t) + \frac{j^2 - i^2 - \varepsilon\sigma^2}{\varepsilon} j^2 z_j(t) + kj^2 f\Big(\sum_{0<l<i} l^2 \phi_l(t)^2 + i^2 y^2(t) + \sum_{j>i} j^2 z_j(t)^2\Big) z_j(t)$$

$$+\sqrt{\varepsilon}\delta\dot{z}_j(t) - \nu\sqrt{\frac{2}{\pi}}\int_0^{\pi} h(x,t)\sin(jx)\,dx = 0 \tag{5.3.7}$$

where $0 < l < i < j$. In this way we have decomposed the problem along three submanifolds: a strongly hyperbolic second order problem in $\mathbb{R}^{i-1}$, a hyperbolic second order problem in $\mathbb{R}$, and a second order problem in an infinite dimensional center manifold. We assume that $f(x)$ satisfies the following hypotheses:

(F1) The function $f \in C([0,\infty),[0,\infty)) \cap C^2((0,\infty),[0,\infty))$. Moreover we assume that the following conditions hold:

$$f(0) = 0, \quad \limsup_{x\to 0^+} |xf'(x^2)| < \infty, \quad \limsup_{x\to 0^+} |x^3 f''(x^2)| < \infty.$$

(F2) The equation

$$\ddot{y} - \sigma^2 y + kf(y^2)y = 0 \tag{5.3.8}$$

has a positive homoclinic solution that is a C^2-solution $\gamma(t) > 0$ so that $\lim_{|t|\to\infty} \gamma(t) = \lim_{|t|\to\infty} \dot{\gamma}(t) = 0$.

Remark 5.3.1. (a) Observe that $\gamma_i(t) = \gamma(it)/i$ solves the equation

$$\ddot{y} - i^2\sigma^2 y + ki^2 f(i^2 y^2)y = 0 \tag{5.3.9}$$

for any $i \in \mathbb{N}\setminus\{0\}$. That is, $\gamma_i(t)$ is a solution of the equation obtained from (5.3.6) taking $\phi_l(t) = 0$, $z_j(t) = 0$ and $\varepsilon = \nu = 0$. We will refer to Eq. (5.3.9) as the *unperturbed problem*.

(b) Equation (5.3.8) has the energy function

$$E(y,\dot{y}) = \dot{y}^2 + \int_0^{y^2} \left(kf(s) - \sigma^2\right) ds$$

which is even in both y and $\dot{y}$. Since $\lim_{t\to\infty} \gamma(t) = 0$, we see that $\dot{\gamma}(t) = 0$ has a solution t_0. It is easy to prove [32] that this solution is unique. Hence we can assume that $t_0 = 0$ and then $\gamma(t) = \gamma(-t)$ because of uniqueness. Thus $\gamma(t)$ has either a positive maximum or a negative minimum at the point $t = 0$. Since $-\gamma(t)$ satisfies Eq. (5.3.8) as $\gamma(t)$ does, we see that the assumption $\gamma(t) > 0$ is not restrictive. Then, $\gamma(t)$ is increasing on $(-\infty, 0]$ and decreasing on $[0,\infty)$. As a consequence, $0 \le \gamma(t) \le M := \gamma(0)$. Since the energy function $E(y,\dot{y})$ is constant along $(\gamma(t),\dot{\gamma}(t))$ and $\dot{\gamma}(0) = 0$ we get

$$\int_0^{M^2} \left(kf(s) - \sigma^2\right) ds = 0$$

(note that $\lim_{t\to\infty} E(\gamma(t),\dot{\gamma}(t)) = E(0,0) = 0$) and

$$\int_0^{x^2} \left(kf(s) - \sigma^2\right) ds < 0$$

for $0 < x < M$. Finally $kf(M^2) \neq \sigma^2$, since, otherwise $x = M$ would be a fixed point of Equation (5.3.8). As a matter of fact, we have $kf(M^2) > \sigma^2$, since the function $\int_0^{x^2} \left(kf(s) - \sigma^2\right) ds$ passes from negative values to 0 when $x \to M^-$ and then its derivative at $x = M$ must be nonnegative. As a consequence, assumption (F2) implies that the following condition holds:

(F2') There exists $M > 0$ so that $\int_0^{x^2} \left[kf(s) - \sigma^2\right] ds < 0$ for any $0 < x < M$ and $\int_0^{M^2} \left[kf(s) - \sigma^2\right] ds = 0$. Moreover $kf(M^2) > \sigma^2$.

On the other hand, if condition (F2') holds then the solution $\gamma(t)$ of (5.3.8), $\gamma(0) = M$ and $\dot{\gamma}(0) = 0$, satisfies $0 < \gamma(t) < M$ for any $t \neq 0$, and is homoclinic to the (hyperbolic) fixed point $x = 0$, $\dot{x} = 0$ of (5.3.8). Thus the two conditions (F2) and (F2') are equivalent. Finally we observe that the curve $(\gamma(t),\dot{\gamma}(t))$ is contained in the sector $\{(y,\dot{y}) \mid y \ge 0 \text{ and } |\dot{y}| \le \sigma y\}$, that is, $|\dot{\gamma}(t)| \le \sigma\gamma(t)$ for any $t \in \mathbb{R}$.

(c) Since we look for solutions close to the homoclinic orbit, in fact, it is enough that f is defined just for $0 \le x \le M^2 + 1$.

(b) Assumption (F1) is satisfied in particular if we take any function $f(x)$ of the form $f(x) = g(x^\alpha)$, where $\alpha \ge \frac{1}{2}$ and $g(x) \in C^2\left([0,\infty),[0,\infty)\right)$ is a positive function so that $g(0) = 0$.

We see that (5.3.5), (5.3.6) and (5.3.7) are similar to (5.1.6), (5.1.8) and (5.1.8). So we can repeat arguments of Section 5.1, i.e. we can apply a Lyapunov-Schmidt reduction method like for the system of (5.1.6), (5.1.8) and (5.1.8) to deriving a Melnikov function for (5.3.1), (5.3.2). We do not go into details, and we refer the readers to [33], we only here recall the following notations (cf Section 5.1.3). For any $E = \{e_j\}_{j\in\mathbb{Z}} \in \mathscr{E}$, we put

$$\ell_E^\infty = \Big\{ \alpha := \{\alpha_j\}_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{R}) \mid \alpha_j \in \mathbb{R} \text{ and } \alpha_j = 0 \text{ if } e_j = 0 \Big\},$$

with $\ell^\infty(\mathbb{R})$ being the Banach space of bounded, doubly infinity sequences of real numbers, endowed with the sup-norm. For any $(E,\alpha) \in \mathscr{E} \times \ell_E^\infty$ we take the function $\gamma_{(E,\alpha)} \in L^\infty(\mathbb{R})$ defined as

$$\gamma_{(E,\alpha)}(t) = \begin{cases} \gamma(t-2jm-\alpha_j) & \text{if } (2j-1)m < t \leq (2j+1)m \text{ and } e_j = 1 \\ 0 & \text{if } (2j-1)m < t \leq (2j+1)m \text{ and } e_j = 0. \end{cases}$$

Now we can state the following main result proved in [33].

Theorem 5.3.2. *Assume that the conditions (F1) and (F2) are satisfied, and that* $h \in L^\infty(\mathbb{R}, L^2([0,\pi]))$ *is* 1*-periodic with respect to* t *and* $\left\| \int_0^\pi h(x,\cdot)^2\,dx \right\|_\infty = 1$*. Assume, further, that* $\mu_0 \in \mathbb{R}$ *exists so that the function*

$$\bar{M}(\tau) := \delta \int_{-\infty}^{\infty} \dot{\gamma}(t)^2 dt - \mu_0 \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \int_0^\pi \dot{\gamma}(t) h(x,(t+\tau)/i) \sin(ix)\,dx\,dt$$

has a simple zero at $\tau = \tau_0 \in [0,1]$*, that is,* $\bar{M}(\tau_0) = 0$ *and* $\bar{M}'(\tau_0) \neq 0$*. Then there exist* $\bar{\rho} > 0$*,* $\bar{\varepsilon} > 0$ *and* $\bar{\mu} > 0$ *so that for any* $0 < \varepsilon < \bar{\varepsilon}$*,* $|\mu - \mu_0| \leq \bar{\mu}$ *and* $m > \varepsilon^{-3/4}$*, with* $m = ki$ *and* $k \in \mathbb{N}$*, there is a continuous function* $\alpha_{\varepsilon,\mu,m} : \mathscr{E} \to \ell^\infty(\mathbb{R})$ *so that* $\alpha_{\varepsilon,\mu,m}(E) \in \ell_E^\infty$ *and a continuous map* $\Pi_{\varepsilon,\mu,m} : \mathscr{E} \to L^\infty\left(\mathbb{R}, H_0^2([0,\pi])\right)$ *so that*

$$u_E(x,t,\varepsilon) := i^{-1} \Pi_{\varepsilon,\mu,m}(E)(x, i\sqrt{\varepsilon}t)$$

is a weak solution of (5.3.1) with $\nu = \sqrt{\varepsilon}\mu$ *that satisfies*

$$\text{ess sup}_{t\in\mathbb{R}} \left\| iu_E(x,t,\varepsilon) - \sqrt{\frac{2}{\pi}} \gamma_{(E,\alpha_{\varepsilon,\mu,m}(E))}(i\sqrt{\varepsilon}t) \sin(ix) \right\|_{H_0^2([0,\pi])} \leq \bar{\rho}$$

where $\|\cdot\|_{H_0^2([0,\pi])}$ *is the norm in* $H_0^2([0,\pi])$*. Moreover, the map* $\Pi_{\varepsilon,\mu,m} : \mathscr{E} \to \Pi(\mathscr{E})$ *is a homeomorphism satisfying*

$$\Pi_{\varepsilon,\mu,m}(\sigma(E))(x,t) = \Pi_{\varepsilon,\mu,m}(E)\,(x,t+2m).$$

Hence $u_{\sigma(E)}(x,t,\varepsilon) = u_E(x,t+2k/\sqrt{\varepsilon},\varepsilon)$*.*

Finally we note that from (F1) it follows that:

$$\lim_{x\to 0^+} xf'(x) = \lim_{x\to 0} x^2 f'(x^2) = 0, \quad \lim_{x\to 0^+} x^2 f''(x) = \lim_{x\to 0} x^4 f''(x^2) = 0.$$

Hence the function $xf(x^2)$ is C^1 on $\mathbb{R}$ and its second derivative is bounded on $K \setminus \{0\}$, with K being any fixed compact subset of $\mathbb{R}$. In fact, for $x \neq 0$, we have

$$\frac{d}{dx}[xf(x^2)] = 2x^2 f'(x^2) + f(x^2) \to 0 = \frac{d}{dx}[xf(x^2)]_{|x=0}$$

as $x \to 0$. Thus $\frac{d}{dx}[xf(x^2)]$ is continuous in $\mathbb{R}$. Next

$$\frac{d^2}{dx^2}[xf(x^2)] = 6xf'(x^2) + 4x^3 f''(x^2)$$

is bounded on $K \setminus \{0\}$ for any given compact subset K of $\mathbb{R}$ because of assumption (F1).

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