Chapter 4
Chaos in Ordinary Differential Equations

Functional analytical methods are presented in this chapter to predict chaos for ODEs depending on parameters. Several types of ODEs are considered. We also study multivalued perturbations of ODEs, and coupled infinite-dimensional ODEs on the lattice $\mathbb{Z}$ as well. Moreover, the structure of bifurcation parameters for homoclinic orbits is investigated.

4.1 Higher Dimensional ODEs

4.1.1 Parameterized Higher Dimensional ODEs

In this section, we consider ODEs of the form

$$\dot{x} = f(x) + h(x, \mu, t) \tag{4.1.1}$$

with $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$. We make the following assumptions of (4.1.1):

(i) f and h are C^3 in all arguments.
(ii) $f(0) = 0$ and $h(\cdot, 0, \cdot) = 0$.
(iii) The eigenvalues of $Df(0)$ lie off the imaginary axis.
(iv) The unperturbed equation has a homoclinic solution, i.e. there is a nonzero differentiable function $\gamma(t)$ so that $\lim_{t\to\pm\infty} \gamma(t) = 0$ and $\dot{\gamma}(t) = f(\gamma(t))$.
(v) $h(x, \mu, t+1) = h(x, \mu, t)$ for $t \in \mathbb{R}$.

Let Ψ_μ be the period map of (4.1.1), i.e. $\Psi_\mu(x) = \phi_\mu(x, 1)$ where $\phi_\mu(x, t)$ is the solution of (4.1.1) with the initial condition $\phi_\mu(x, 0) = x$. The purpose of this section is to find a set of parameters μ for which the period map Ψ_μ of (4.1.1) has a transversal homoclinic orbit. For this reason, higher dimensional Melnikov mappings are introduced. Simple zero points of those mappings give wedge-shaped regions in $\mathbb{R}^m$ for μ where Ψ_μ possesses transversal homoclinic orbits. This result is a continuous version of Section 3.1, where difference equations are studied. Melnikov theory for

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ODEs is also given in a lot of work [1–7]. This method is usually applied when the unperturbed equation

$$\dot{x} = f(x) \tag{4.1.2}$$

is integrable [8].

4.1.2 Variational Equations

For (4.1.2) we adopt the standard notations W^s, W^u for the stable and unstable manifolds, respectively, of the origin and $d_s = \dim W^s$, $d_u = \dim W^u$. Since $x = 0$ is a hyperbolic equilibrium, γ lies on $W^s \cap W^u$. By the *variational equation* along γ we mean the linear differential equation

$$\dot{u} = Df(\gamma(t))u. \tag{4.1.3}$$

Now, we can repeat the arguments of Section 3.1.2 to (4.1.3), but since it is straightforward, we do not go into details, and we refer the readers to [3, Theorem 2] and [9, Theorem 3.1.2]. Consequently, the following results hold.

Theorem 4.1.1. *There exists a fundamental solution U for* (4.1.3) *along with constants $M > 0$, $K_0 > 0$ and four projections P_{ss}, P_{su}, P_{us}, P_{uu} so that $P_{ss} + P_{su} + P_{us} + P_{uu} = \mathbb{I}$ and the following hold:*

(i) $|U(t)(P_{ss} + P_{us})U(s)^{-1}| \leq K_0 e^{2M(s-t)}$, *for* $0 \leq s \leq t$,
(ii) $|U(t)(P_{su} + P_{uu})U(s)^{-1}| \leq K_0 e^{2M(t-s)}$, *for* $0 \leq t \leq s$,
(iii) $|U(t)(P_{ss} + P_{su})U(s)^{-1}| \leq K_0 e^{2M(t-s)}$, *for* $t \leq s \leq 0$,
(iv) $|U(t)(P_{us} + P_{uu})U(s)^{-1}| \leq K_0 e^{2M(s-t)}$, *for* $s \leq t \leq 0$.

Also $\operatorname{rank} P_{ss} = \operatorname{rank} P_{uu} = d$.

In the language of exponential dichotomies we see that Theorem 4.1.1 provides a two-sided exponential dichotomy. For $t \to -\infty$ an exponential dichotomy is given by the fundamental solution U and the projection $P_{us} + P_{uu}$ while for $t \to +\infty$ such an exponential dichotomy is given by U and $P_{ss} + P_{us}$.

Let u_j denote column j of U and assume that these are numbered so that

$$P_{uu} = \begin{pmatrix} \mathbb{I}_d & 0_d & 0 \\ 0_d & 0_d & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad P_{ss} = \begin{pmatrix} 0_d & 0_d & 0 \\ 0_d & \mathbb{I}_d & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here, $\mathbb{I}_d$ denotes the $d \times d$ identity matrix and 0_d denotes the $d \times d$ zero matrix.

For each $i = 1, \ldots, n$ we define $u_i^{\perp}(t)$ by $\langle u_i^{\perp}(t), u_j(t) \rangle = \delta_{ij}$, where $\langle \cdot, \cdot \rangle$ is the scalar product on $\mathbb{R}^n$. The vectors $u_i^{\perp}$ can be computed from the formula $U^{\perp *} = U^{-1}$ where $U^{\perp}$ denotes the matrix with $u_j^{\perp}$ as column j. Differentiating $UU^{\perp *} = \mathbb{I}$ we obtain $\dot{U}U^{\perp *} + U\dot{U}^{\perp *} = 0$ so that $\dot{U}^{\perp} = -(U^{-1}\dot{U}U^{\perp *})^* = -Df(\gamma)^* U^{\perp}$. Thus, $U^{\perp}$

is the adjoint of U. Note that $\{u_i^\perp(t) \mid i=1,2,\ldots,d\}$ is a basis of bounded solutions on $\mathbb{R}$ of the adjoint variational equation $\dot{w} = -Df(\gamma)^* w$. The function $\dot{\gamma}$ is always a solution to the variational equation (4.1.3) and we may assume that $u_{2d} = \dot{\gamma}$, since $\dot{\gamma}$ is a linear combination of columns u_{d+1} through u_{2d} of U and a linear transformation of these columns preserves the projections.

Now we define the following Banach spaces

$$Z = \left\{ z \in C\big((-\infty,\infty),\mathbb{R}^n\big) \mid \sup_{t\in\mathbb{R}} |z(t)| < \infty \right\},$$

$$Y = \left\{ z \in C^1\big((-\infty,\infty),\mathbb{R}^n\big) \mid z, \dot{z} \in Z \right\},$$

with the usual supremum norms.

Theorem 4.1.2. *The linear equation*

$$\dot{u} = Df(\gamma(t))u + z, \quad z \in Z.$$

has a solution $u = K(z)(t) \in Y$ *if and only if*

$$z \in \widetilde{Z} := \left\{ z \in Z \mid \int_{-\infty}^{\infty} P_{uu} U(s)^{-1} z(s)\, ds = 0 \right\}.$$

Moreover, if $z \in \widetilde{Z}$ *then we can take*

$$K(z)(t) = \begin{cases} U(t)\Big[\int_{-\infty}^{0} P_{su}U(s)^{-1}z(s)\,ds + \int_0^t (P_{ss}+P_{su})U(s)^{-1}z(s)\,ds \\ \quad - \int_t^{\infty} (P_{us}+P_{uu})U(s)^{-1}z(s)\,ds\Big], \quad \text{for } t \geq 0, \\ U(t)\Big[-\int_0^{\infty} P_{us}U(s)^{-1}z(s)\,ds + \int_0^t (P_{ss}+P_{us})U(s)^{-1}z(s)\,ds \\ \quad + \int_{-\infty}^{t} (P_{su}+P_{uu})U(s)^{-1}z(s)\,ds\Big], \quad \text{for } t \leq 0. \end{cases}$$

Note that $z \in \widetilde{Z} \Leftrightarrow \int\limits_{-\infty}^{\infty} \langle u_i^\perp(t), z(s) \rangle\, ds = 0$ for all $i = 1,2,\ldots,d$.

Theorem 4.1.3. *Define a projection* $\Pi : Z \to Z$ *by*

$$\Pi(z)(t) := \varphi(t) \int_{-\infty}^{\infty} U(t) P_{uu} U(s)^{-1} z(s)\, ds,$$

for a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ *satisfying* $\sup_t \left|\varphi(t)u_j(t)\right| < \infty$ *for all* j *and* $\int_{-\infty}^{\infty} \varphi(s)\, ds = 1$. *Then* $\mathscr{R}(\mathbb{I} - \Pi) = \widetilde{Z}$.

4.1.3 Melnikov Mappings

Without loss of generality, we can suppose that f and h as well as all their partial derivatives up to the order 3 are uniformly bounded on the whole spaces of definition. We study the equation (cf Theorem 2.2.4)

$$\begin{gathered} F_{\mu,\varepsilon,y}(x) = \dot{x} - f(x) - h(x,\mu,t) - \varepsilon|\mu|L(x-y) = 0, \\ F_{\mu,\varepsilon,y} : Y \to Z, \end{gathered} \tag{4.1.4}$$

where $L : Y \to Z$ is a linear continuous mapping so that $||L|| \leq 1$, $y \in Y$ and $\varepsilon \in \mathbb{R}$ is small. It is clear that solutions of (4.1.4) near γ with $\varepsilon = 0$ are homoclinic ones of (4.1.1). We make in (4.1.4) the change of variable

$$x(t) = \gamma(t-\alpha) + w(t), \quad \langle w(0), \dot{\gamma}^{\perp}(-\alpha) \rangle = 0, \tag{4.1.5}$$

where $\alpha \in \mathscr{I} \subset \mathbb{R}$ and $\mathscr{I}$ is a given bounded open interval. We note that (4.1.5) defines a tubular neighbourhood of the manifold $\{\gamma(t-\alpha)\}_{\alpha\in\mathscr{I}}$ in Y when w is sufficiently small (cf Section 2.4.3). Hence (4.1.4) has the form

$$\begin{aligned} G_{\alpha,\mu,\varepsilon,y}(w) &= \dot{w} - f(\gamma(t-\alpha)+w)) + f(\gamma(t-\alpha)) \\ &\qquad -h(\gamma(t-\alpha)+w,\mu,t) - \varepsilon|\mu|L\big(w+\gamma(t-\alpha)-y\big) = 0, \\ &\qquad G_{\alpha,\mu,\varepsilon,y} : Y \to Z. \end{aligned}$$

We have

$$D_w G_{\alpha,0,0,y}(0)u = \dot{u} - Df(\gamma(t-\alpha))u.$$

By putting

$$U_\alpha(t) = U(t-\alpha), \quad U_\alpha^{\perp}(t) = U^{\perp}(t-\alpha),$$

Theorem 4.1.1 is valid when U is replaced by U_α and (4.1.3) by

$$\dot{u} = Df(\gamma(t-\alpha))u,$$

respectively, but $K_0 > 0$ should be enlarged. Moreover, we put

$$\gamma_\alpha(t) = \gamma(t-\alpha), \quad u_{j,\alpha} = u_j(t-\alpha), \quad u_{j,\alpha}^{\perp} = u_j^{\perp}(t-\alpha).$$

Consequently, by taking

$$Q = \Big\{ y \in Y \mid \sup_{t\in\mathbb{R}} \big(|y(t)| + |\dot{y}(t)|\big) < \sup_{t\in\mathbb{R}} \big(|\gamma(t)| + |\dot{\gamma}(t)|\big) + 1 \Big\}$$

and by using the same approach as in [3], [5, p. 709] and Section 3.1.3 along with Theorems 4.1.2 and 4.1.3, there are open small neighborhoods $0 \in O \subset \mathbb{R}^{d-1}$, $0 \in V \subset \mathbb{R}, 0 \in W \subset \mathbb{R}^m$ and a mapping

$$G \in C^3\left(Y \times O \times \mathscr{I} \times W \times V \times Q, Z\right),$$

so that any solution of (4.1.4) near γ_α for $\mu \in W$, $\varepsilon \in V$, $y \in Q$ is determined by the equation $G(z,\beta,\alpha,\mu,\varepsilon,y) = 0$ and this solution has the form

$$x = \gamma_\alpha + z, \quad P_{ss}U_\alpha^{-1}(0)\big(z(0) - \sum_{j=1}^{d-1} \beta_j u_{j+d,\alpha}(0)\big) = 0, \tag{4.1.6}$$

where $\beta = (\beta_1, \ldots, \beta_{d-1})$. We remark that $\{u_{j,\alpha}(0)\}_{j=1}^n$ are linearly independent, $u_{2d,\alpha}(0) = \dot{\gamma}_\alpha(0) = \dot{\gamma}(-\alpha)$, as well as

$$\Big\{v \in \mathbb{R}^n \mid \langle v, \dot{\gamma}^\perp(-\alpha)\rangle = 0\Big\} = \operatorname{span}\Big\{\{u_{j,\alpha}(0)\}_{j=1}^n \setminus \{u_{2d,\alpha}(0)\}\Big\},$$

and

$$0 = P_{ss}U_\alpha^{-1}(0)w = P_{ss}U_\alpha^{\perp *}(0)w \Longleftrightarrow \langle u_{j+d,\alpha}^\perp(0), w\rangle = 0, \quad \forall j, 1 \le j \le d.$$

Hence (4.1.5) and (4.1.6) provide a suitable decomposition of any x in (4.1.4) near the manifold $\{\gamma(t-\alpha)\}_{\alpha\in\mathscr{I}}$. Now by using the Lyapunov-Schmidt procedure (see again [3, Theorem 8], [5, p. 709] and Section 3.1.3), the study of the equation $G(z,\beta,\alpha,\mu,\varepsilon,y) = 0$ can be expressed in the following theorem for z, μ, ε, β small, $y \in Q$ and $\alpha \in \mathscr{I}$.

Theorem 4.1.4. *U and d are the same as in Theorem 4.1.1. Then there exist small neighborhoods $0 \in O_1 \subset \mathbb{R}^{d-1}$, $0 \in W_1 \subset \mathbb{R}^m$, $0 \in V_1 \subset \mathbb{R}$ and a C^3 function $H : Q \times O_1 \times \mathscr{I} \times W_1 \times V_1 \to \mathbb{R}^d$ denoted $(y,\beta,\alpha,\mu,\varepsilon) \to H(y,\beta,\alpha,\mu,\varepsilon)$ with the following properties:*

(i) *The equation $H(y,\beta,\alpha,\mu,\varepsilon) = 0$ holds if and only if* (4.1.4) *has a solution near γ_α and each such $(y,\beta,\alpha,\mu,\varepsilon)$ determines only one solution of* (4.1.4),
(ii) $H(y,0,\alpha,0,0) = 0$,
(iii) $\frac{\partial H_i}{\partial \mu_j}(y,0,\alpha,0,0) = -\int_{-\infty}^{\infty} \Big\langle u_i^\perp(t), \frac{\partial h}{\partial \mu_j}(\gamma(t),0,t+\alpha)\Big\rangle\, dt$,
(iv) $\frac{\partial H_i}{\partial \beta_j}(y,0,\alpha,0,0) = 0$,
(v) $\frac{\partial^2 H_i}{\partial \beta_k \partial \beta_j}(y,0,\alpha,0,0) = -\int_{-\infty}^{\infty} \big\langle u_i^\perp(t), D^2 f(\gamma(t)) u_{d+j}(t) u_{d+k}(t)\big\rangle\, dt$.

We introduce the following notations:

$$a_{ij}(\alpha) = -\int_{-\infty}^{\infty} \Big\langle u_i^\perp(t), \frac{\partial h}{\partial \mu_j}(\gamma(t),0,t+\alpha)\Big\rangle\, dt,$$
$$b_{ijk} = -\int_{-\infty}^{\infty} \Big\langle u_i^\perp, D^2 f(\gamma) u_{d+j} u_{d+k}\Big\rangle\, dt.$$

Finally, we take the mapping $M_\mu : \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$\big(M_\mu(\alpha,\beta)\big)_i = \sum_{j=1}^{m} a_{ij}(\alpha)\mu_j + \frac{1}{2}\sum_{j,k=1}^{d-1} b_{ijk}\beta_j\beta_k\,.$$

Note that we can take any bases of bounded solutions of the adjoint and adjoint variational equations (with $u_{2d} = \dot{\gamma}$) for constructing the Melnikov function M_μ. Now we can state the main result of this section.

Theorem 4.1.5. *Let $d > 1$. If M_{μ_0} has a simple root (α_0,β_0), i.e. (α_0,β_0) satisfies $M_{\mu_0}(\alpha_0,\beta_0) = 0$ and $D_{(\alpha,\beta)}M_{\mu_0}(\alpha_0,\beta_0)$ is a regular matrix, then there is a wedge-shaped region in $\mathbb{R}^m$ for μ of the form*

$$\mathscr{R} = \Big\{ s^2\tilde{\mu} \,\Big|\, s \text{ is from a small open neighborhood of } 0 \in \mathbb{R} \text{ and } \tilde{\mu} \text{ is from a small open neighborhood of } \mu_0 \in \mathbb{R}^m \text{ satisfying } |\tilde{\mu}| = |\mu_0| \Big\},$$

so that for any $\mu \in \mathscr{R}\setminus\{0\}$, period map Ψ_μ of (4.1.1) possesses a transversal homoclinic orbit.

Proof. Let us take $\mathscr{I} = (\alpha_0 - 1, \alpha_0 + 1)$ and let us consider the mapping defined by

$$\Phi(y,\tilde{\beta},\alpha,\tilde{\mu},\tilde{\varepsilon},s) = \begin{cases} \dfrac{1}{s^2}H(y,s\tilde{\beta},\alpha,s^2\tilde{\mu},s^3\tilde{\varepsilon}), & \text{for} \quad s \neq 0, \\ M_{\tilde{\mu}}(\alpha,\tilde{\beta}), & \text{for} \quad s = 0\,. \end{cases}$$

According to (ii)–(v) of Theorem 4.1.4, the mapping Φ is C^1-smooth near

$$(y,\tilde{\beta},\alpha,\tilde{\mu},\tilde{\varepsilon},s) = (y,\beta_0,\alpha_0,\mu_0,0,0), \quad y \in Q$$

with respect to the variables $\tilde{\beta},\alpha$. Since

$$M_{\mu_0}(\alpha_0,\beta_0) = 0 \quad \text{and} \quad D_{(\alpha,\beta)}M_{\mu_0}(\alpha_0,\beta_0) \quad \text{is a regular matrix}\,,$$

we can apply the implicit function theorem to solving locally and uniquely the equation $\Phi = 0$ in the variables $\tilde{\beta},\alpha$, where $\tilde{\mu}$ is near $\tilde{\mu}_0$ satisfying $|\tilde{\mu}| = |\mu_0|$. This gives for $\varepsilon = 0$, by (i) of Theorem 4.1.4, the existence of $\mathscr{R}$ on which Ψ_μ has a homoclinic orbit. Moreover, we can suppose that the corresponding solutions of (4.1.4) lie in Q.

To prove the transversality of these homoclinic orbits, we fix $\mu \in \mathscr{R}\setminus\{0\}$ and take $y = \tilde{\gamma}$, where $\tilde{\gamma}$ is the solution of (4.1.4) for which the transversality of the corresponding homoclinic orbit of Ψ_μ should be proved. Then we vary $\varepsilon = s^3\tilde{\varepsilon}$ small. Note that $s \neq 0$ is also fixed due to $\mu = s^2\tilde{\mu}$ and $|\tilde{\mu}| = |\mu_0|$ as well. Since the local uniqueness of solutions of (4.1.4) near $\tilde{\gamma}$ is satisfied for any $\tilde{\varepsilon}$ sufficiently small according to the above application of the implicit function theorem, such equation (4.1.4) (with the fixed $\mu \in \mathscr{R}\setminus\{0\}$, $\varepsilon = s^3\tilde{\varepsilon}$ where $s \neq 0$ is also fixed and the special $y = \tilde{\gamma}$) has the only solution $x = \tilde{\gamma}$ near $\tilde{\gamma}$ for any $\tilde{\varepsilon}$ sufficiently small. Hence Theorem 2.2.4 gives the invertibility of $DF_{\mu,0,\tilde{\gamma}}(\tilde{\gamma})$, so the only bounded solution on $\mathbb{R}$ of the equation $\dot{v} = Df(\tilde{\gamma})v + D_xh(\tilde{\gamma},\mu,t)v$ is $v = 0$. Then Lemma 2.5.2 implies the transversality of these homoclinic orbits of Ψ_μ for $\mu \in \mathscr{R}\setminus\{0\}$. □

Remark 4.1.6. (a) If M_{μ_0} has a simple zero point (α_0, β_0), then $M_{r^2\mu_0}$ has also a simple zero point at $(\alpha_0, r\beta_0)$ for any $r \in \mathbb{R} \setminus \{0\}$.

(b) If $d = 1$ then we take the function $M_\mu(\alpha) = \sum_{j=1}^m a_{1j}(\alpha)\mu_j$, which is the usual Melnikov function. So for any simple zero α_0 of $M_{\mu_0}(\alpha) = 0$, when μ_0 is fixed, there is a two-sided wedge-shaped region in $\mathbb{R}^m$ for μ of the form

$$\mathscr{R} = \Big\{ s\tilde{\mu} \,\Big|\, s \text{ is from a small open neighborhood of } 0 \in \mathbb{R} \text{ and } \tilde{\mu} \text{ is from a small open neighborhood of } \mu_0 \in \mathbb{R}^m \text{ satisfying } |\tilde{\mu}| = |\mu_0| \Big\}$$

so that for any $\mu \in \mathscr{R} \setminus \{0\}$, the period map Ψ_μ of Eq. (4.1.1) possesses a transversal homoclinic orbit.

Remark 4.1.7. A standard perturbation theory [10–13], which can be verified by repeating the above arguments, implies the existence of a unique 1-periodic solution of (4.1.1) for any μ small, which is, in addition, hyperbolic. Then the transversal homoclinic solution of Theorem 4.1.5 is exponentially asymptotic to this periodic orbit.

Remark 4.1.8. Note that we can take any bases of bounded solutions of the adjoint variational and variational equations (with $u_{2d} = \dot{\gamma}$) for constructing the Melnikov function M_μ. Similar observations can be applied to detecting the other continuous Melnikov functions in this book.

Remark 4.1.9. The above results can be generalized to ODEs possessing heteroclinic orbits to *semi-hyperbolic equilibria* [14].

4.1.4 The Second Order Melnikov Function

When Melnikov function M_μ is identically zero then we need to compute the *second order Melnikov function*. Since in general computations are awkward, we consider the simplest case given by a C^3-equation

$$\ddot{x} = f(x) + \varepsilon q(t) \tag{4.1.7}$$

with 2π-periodic $q(t)$, and $\ddot{x} = f(x)$ has a homoclinic solution $p(t)$ to 0 with $f'(0) > 0$. We can suppose $\dot{p}(0) = 0$. We are looking for bounded solutions of (4.1.7) near $p(t)$. We briefly repeat the above arguments, so we shift $t \leftrightarrow t + \alpha$ and take $x = p + v$ in (4.1.7) with $v \in Y_0 := \{v \in Y \mid \dot{v}(0) = 0\}$ to obtain

$$\ddot{v} - f'(p)v = f(p+v) - f'(p)v - f(p) + \varepsilon q(t+\alpha)\,.$$

By introducing the projection $\Pi : X \to X$ as $\Pi z := \int_{-\infty}^{\infty} z(t)\dot{p}(t)\,dt / \int_{-\infty}^{\infty} \dot{p}^2(t)\,dt \cdot p$, the Lyapunov-Schmidt method splits (4.1.7) into two equations

$$\ddot{v} - f'(p)v = (\mathbb{I} - \Pi)\left\{f(p+v) - f'(p)v - f(p) + \varepsilon q(t+\alpha)\right\} \tag{4.1.8}$$

and

$$\int_{-\infty}^{\infty} \left\{f(p(t)+v(t)) - f'(p(t))v(t) - f(p(t)) + \varepsilon q(t+\alpha)\right\} \dot{p}(t)\,dt = 0\,. \tag{4.1.9}$$

By the implicit function theorem, we can uniquely solve (4.1.8) to get $v = v(\varepsilon,\alpha) \in Y_0$ with $v(0,\alpha) = 0$, so we put $v(\varepsilon,\alpha) = \varepsilon w(\varepsilon,\alpha)$, and inserting this into (4.1.9), we get the scalar bifurcation equation

$$B(\varepsilon,\alpha) := \int_{-\infty}^{\infty} \Big\{f(p(t)+\varepsilon w(\varepsilon,\alpha)(t)) - f'(p(t))\varepsilon w(\varepsilon,\alpha)(t)$$
$$-f(p(t)) + \varepsilon q(t+\alpha)\Big\}\dot{p}(t)\,dt = 0\,.$$

Clearly $B(0,\alpha) = 0$ and $B_\varepsilon(0,\alpha) = \int_{-\infty}^{\infty} q(t+\alpha)\dot{p}(t)\,dt = M(\alpha)$, where $M(\alpha)$ is the Melnikov function for (4.1.7). We have until now repeated arguments of Section 4.1.3 to (4.1.7). When $M(\alpha) = 0$, then we proceed further to derive

$$B_{\varepsilon\varepsilon}(0,\alpha) = \int_{-\infty}^{+\infty} \dot{p}(t) f''(p(t)) w(0,\alpha)^2(t)\,dt\,.$$

Note that by (4.1.8), $w(0,\alpha)$ solves $\ddot{w}(0,\alpha)(t) = f'(p(t))w(0,\alpha)(t) + q(t+\alpha)$. Summarizing the second order Melnikov function is given by

$$M_2(\alpha) := \int_{-\infty}^{+\infty} \dot{p}(t) f''(p(t)) v_\alpha^2(t)dt\,, \tag{4.1.10}$$

where $v_\alpha(t)$ is any fixed bounded solution of the equation

$$\ddot{x} = f'(p(t))x + q(t+\alpha).$$

This solution exists thanks to the fact that $M(\alpha) = 0$ (cf Theorem 4.1.2). Note that any two of these bounded solutions differ for a multiple of $\dot{p}(t)$, and hence $v_{\alpha+2\pi}(t) = v_\alpha(t) + \lambda\dot{p}(t)$, for some $\lambda \in \mathbb{R}$. On the other hand, $M_2(\alpha)$ does not depend on the particular solution $v_\alpha(t)$ we choose. This easily follows from that $\ddot{p}(t)$ is a bounded solution of the non homogeneous system

$$\ddot{x} = f'(p(t))x + f''(p(t))\dot{p}(t)^2$$

and $\dot{v}_\alpha(t)$ is a bounded solution of

$$\ddot{x} = f'(p(t))x + f''(p(t))\dot{p}(t)v_\alpha + \dot{q}(t+\alpha).$$

Hence:

$$\int_{-\infty}^{+\infty} \dot{p}(t) f''(p(t))\dot{p}(t)^2 dt = 0$$

and

$$\int_{-\infty}^{+\infty} \dot{p}(t) f''(p(t)) \dot{p}(t) v_\alpha(t) dt = -\int_{-\infty}^{+\infty} \dot{p}(t) \dot{q}(t+\alpha) = M'(\alpha) = 0.$$

Note that $M_2(\alpha)$ is 2π-periodic since the bifurcation function itself is 2π-periodic.

4.1.5 Application to Periodically Perturbed ODEs

We illustrate our theory on the following example. Consider the equation

$$\begin{aligned} \ddot{x} &= x - 2xz^2 + \dot{x}^2 + \mu_1 \cos \omega t - \mu_2 z, \\ \ddot{y} &= y - 2yz^2 + \dot{x}\dot{y}, \\ \ddot{z} &= z - 2z^3 + y\dot{y} + \mu_1 \cos \omega t + (\mu_2 - \mu_1)\dot{z}. \end{aligned} \tag{4.1.11}$$

This equation is studied in Example 1 of [3]. In the space $(x, \dot{x}, y, \dot{y}, z, \dot{z})$, the eigenvalues of $Df(0)$ are $\lambda_1 = \lambda_2 = \lambda_3 = -1$, $\lambda_4 = \lambda_5 = \lambda_6 = 1$. A homoclinic solution when $\mu = 0$ is given by $x = 0, y = 0, z = r$, i.e. $\gamma = (0,0,0,0,r,\dot{r})$ where $r(t) = \operatorname{sech} t$. Note that $\ddot{r} = r - r^3$ and $\ddot{z} = z - z^3$ is the familiar Duffing equation (cf Chapter 1). The linearization of (4.1.11) at γ has the form

$$\ddot{x} = \left(1 - 2\gamma^2\right) x, \quad \ddot{y} = \left(1 - 2\gamma^2\right) y, \quad \ddot{z} = \left(1 - 6\gamma^2\right) z.$$

Clearly $d = 3$ and by Remark 4.1.8, it is readily to find

$$u_4 = (r, \dot{r}, 0,0,0,0), \quad u_5 = (0,0,r,\dot{r},0,0), \quad u_6 = (0,0,0,0,\dot{r},\ddot{r})$$

$$u_1^\perp = (-\dot{r}, r, 0,0,0,0), \quad u_2^\perp = (0,0,-\dot{r},r,0,0), \quad u_3^\perp = (0,0,0,0,-\ddot{r},\dot{r}).$$

Using these results, we easily get

$$M_\mu(\alpha, \beta_1, \beta_2) = \begin{cases} a_{11}(\alpha)\mu_1 + 2\mu_2 - \dfrac{\pi}{8}\beta_1^2, \\ -\dfrac{\pi}{8}\beta_1\beta_2, \\ a_{31}(\alpha)\mu_1 - \dfrac{2}{3}\mu_2 - \dfrac{\pi}{8}\beta_2^2, \end{cases}$$

where

$$a_{11}(\alpha) = -\pi \cos \omega\alpha \operatorname{sech} \frac{\pi\omega}{2}, \quad a_{31}(\alpha) = \frac{2}{3} - \pi\omega \sin \omega\alpha \operatorname{sech} \frac{\pi\omega}{2}.$$

There are the following solutions of $M_\mu(\alpha, \beta) = 0$ (see Remark 4.1.6 (a))

$$\beta(\alpha) = \left(\sqrt{\frac{8}{\pi}(a_{11}+3a_{31})}, 0\right), \quad \mu(\alpha) = \left(1, \frac{3}{2}a_{31}\right) \tag{4.1.12}$$

$$\beta(\alpha) = \left(0, \sqrt{\frac{8}{3\pi}(a_{11}+3a_{31})}\right), \quad \mu(\alpha) = \left(1, -\frac{1}{2}a_{11}\right). \tag{4.1.13}$$

The linearization $D_{(\alpha,\beta)}M_\mu(\alpha,\beta)$ at the points (4.1.12) reads

$$\begin{pmatrix} a'_{11} & -\frac{\pi}{4}\sqrt{\frac{8}{\pi}(a_{11}+3a_{31})} & 0 \\ 0 & 0 & -\frac{\pi}{8}\sqrt{\frac{8}{\pi}(a_{11}+3a_{31})} \\ a'_{31} & 0 & 0 \end{pmatrix},$$

and at the points (4.1.13) it has the form

$$\begin{pmatrix} a'_{11} & 0 & 0 \\ 0 & -\frac{\pi}{8}\sqrt{\frac{8}{3\pi}(a_{11}+3a_{31})} & 0 \\ a'_{31} & 0 & -\frac{\pi}{4}\sqrt{\frac{8}{3\pi}(a_{11}+3a_{31})} \end{pmatrix}.$$

Next, we have $a_{11}(\alpha)+3a_{31}(\alpha) \geq 2-\pi(3\omega+1)\operatorname{sech}\frac{\pi\omega}{2} > 0$ for $\omega > \omega_0$, where $\omega_0 \doteq 1.95332$ is the only positive root of $\pi(3\omega_0+1)\operatorname{sech}\frac{\pi\omega_0}{2} = 2$. So for $\omega > \omega_0$ the points (4.1.12), involving (4.1.13), are simple zero points of $M_\mu(\alpha,\beta)$ when $\alpha \neq \frac{\pi(2k+1)}{2\omega}$, $\alpha \neq \frac{\pi k}{\omega}$, $k \in \mathbb{Z}$. Hence for $\omega > \omega_0$, there are two small open wedge-shaped regions in the μ_1–μ_2 plane with the limit slopes given by

$$1 \pm \frac{3}{2}\pi\omega \operatorname{sech}\frac{\pi\omega}{2} \quad \text{and} \quad \pm\frac{\pi}{2}\operatorname{sech}\frac{\pi\omega}{2}$$

containing parameters for which the period map of (4.1.11) possesses a transversal homoclinic orbit near γ. Since $1 \pm \frac{3}{2}\pi\omega \operatorname{sech}\frac{\pi\omega}{2} \sim 1 \pm 3\pi\omega\, e^{-\pi\omega/2}$ and $\pm\frac{\pi}{2}\operatorname{sech}\frac{\pi\omega}{2} \sim \pm\pi\, e^{-\pi\omega/2}$ for large values of ω, i.e. for rapid forcing, these wedge-shaped regions become very narrow as $\omega \to \infty$. For instance, if $\omega = 10$ then $\frac{3}{2}\pi\omega \operatorname{sech}\frac{\pi\omega}{2} \doteq 0.0000142033$ while $\frac{\pi}{2}\operatorname{sech}\frac{\pi\omega}{2} \doteq 4.73443 \times 10^{-7}$. Finally note that $1+\frac{3}{2}\pi\,\omega_0 \operatorname{sech}\frac{\pi\omega_0}{2} \doteq 1.85423$, $1-\frac{3}{2}\pi\omega_0 \operatorname{sech}\frac{\pi\omega_0}{2} = \frac{\pi}{2}\operatorname{sech}\frac{\pi\omega_0}{2} \doteq 0.145773$ and functions $\frac{3}{2}\pi\omega \operatorname{sech}\frac{\pi\omega}{2}$, $\frac{\pi}{2}\operatorname{sech}\frac{\pi\omega}{2}$ are rapidly decreasing on $[\omega_0,\infty)$.

4.2 ODEs with Nonresonant Center Manifolds

4.2.1 Parameterized Coupled Oscillators

To illustrate the ideas of this section consider the equations

$$\ddot{x} = x - 2x(x^2+y^2) - 2\mu_2\dot{x} + \mu_1 \cos\omega t, \tag{4.2.1a}$$

$$\ddot{y} = (1-k)y - 2y(x^2+y^2) - 2\mu_2\dot{y} + \mu_1 \cos p\omega t, \tag{4.2.1b}$$

where $p \in \mathbb{N}$ and $\omega > 0$. This system consists of a radially symmetric Duffing oscillator with an additional spring of stiffness k in the y equation along with damping and external forces added as perturbation terms. Let us assume $k > 1$ in (4.2.1b). Then, for the unperturbed equation, i.e. when $\mu_1 = \mu_2 = 0$, the linear part of (4.2.1a) has a hyperbolic equilibrium and the linear part of (4.2.1b) has a center. Furthermore, for small μ_2, the eigenvalues of $\ddot{y} = (1-k)y - 2\mu_2\dot{y}$ are complex functions, $\lambda(\mu_2)$, with $\Re(\lambda(\mu_2)) = -\mu_2$ so that we have $\Re(\lambda(0)) = 0$ and $\Re(\lambda'(0)) = -1$. Thus, for small $\mu_2 \neq 0$, the equilibrium of (4.2.1b) is weakly hyperbolic.

If we set $y = 0$ in (4.2.1a) we get the standard forced, and damped Duffing equation

$$\ddot{x} = x - 2x^3 - 2\mu_2\dot{x} + \mu_1 \cos\omega t\,. \tag{4.2.2}$$

Using Melnikov theory of Section 4.1 one can show (see Example 4.2.6 below) that for small $\mu_1 \neq 0$ and for $\mu_2 \neq 0$, within a range

$$|\mu_2| < \frac{3\pi\omega}{4}|\mu_1|\,\mathrm{sech}\,\frac{\pi\omega}{2}, \tag{4.2.3}$$

Equation (4.2.2) has a transverse homoclinic orbit and hence exhibits chaos. The purpose of this section is to show that if $\mu_1 \neq 0$, $\mu_2 \neq 0$ are chosen to produce chaos in (4.2.1a) when $y = 0$ and if $p\omega \neq \sqrt{k-1}$ then, as a consequence of the weak hyperbolicity in the y equation, there exists chaos in the full Eq. (4.2.1) which, in some sense, shadows the chaos obtained in (4.2.1a) with $y = 0$. Condition $p\omega \neq \sqrt{k-1}$ means non-resonance in (4.2.1b). Resonant systems of ODEs are studied in Section 4.3.

As an abstract version of (4.2.1) we consider differential equations of the form

$$\dot{x} = f(x,y,\mu,t) = f_0(x,y) + \mu_1 f_1(x,y,\mu,t) + \mu_2 f_2(x,y,\mu,t), \tag{4.2.4a}$$

$$\dot{y} = g(x,y,\mu,t) = g_0(x,y) + \mu_1 g_1(x,y,\mu,t) + \mu_2 g_2(x,y,\mu), \tag{4.2.4b}$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\mu = (\mu_1,\mu_2) \in \mathbb{R}^2$. We make the following assumptions of (4.2.4):

(i) Each f_i, g_i is C^4-smooth in all arguments.
(ii) f_1, f_2 and g_1 are periodic in t with period T.
(iii) $D_2 f_0(x,0) = 0$.

(iv) The eigenvalues of $D_1 f_0(0,0)$ lie off the imaginary axis.
(v) The equation $\dot{x} = f_0(x,0)$ has a homoclinic solution γ.
(vi) $g_0(x,0) = g_2(x,0,\mu) = 0$, $D_{21}g_0(0,0) = 0$ and $D_{22}g_0(0,0) = 0$.
(vii) The eigenvalues of $D_2 g_0(0,0)$ lie on the imaginary axis.
(viii) If a function $\lambda(\mu_2)$ is an eigenvalue of the matrix $D_2 g_0(0,0) + \mu_2 D_2 g_2(0,0,0)$ then $\Re(\lambda'(0)) < 0$.
(ix) $D_2 g_1(0,0,0,t) = 0$.

Hypothesis (viii) is based on the examples for which the μ_2 perturbation represents damping which cases all the eigenvalues of (4.2.4b) to move to the left of the imaginary axis. In fact, it is sufficient to assume that $\Re(\lambda'(0)) \neq 0$. In other words, (4.2.4b) is weakly hyperbolic. This more general assumption requires a little more work since it is necessary to include a nontrivial projection in Lemma 4.2.4 below.

4.2.2 Chaotic Dynamics on the Hyperbolic Subspace

In this section we consider the equation

$$\dot{x} = f(x,0,\mu,t) = f_0(x,0) + \mu_1 f_1(x,0,\mu,t) + \mu_2 f_2(x,0,\mu,t) \tag{4.2.5}$$

obtained by setting $y = 0$ in (4.2.4a). Equation (4.2.5) will be called the *reduced equation* obtained from (4.2.4). We apply to this equation Melnikov theory from Section 4.1 which we summarize here for the readers' convenience. By hypothesis, the equation $\dot{x} = f_0(x,0)$ has a hyperbolic equilibrium and a homoclinic solution γ. Then (4.2.5) has a unique small hyperbolic T-periodic solution $p_\mu(t)$ for $|\mu|$ small (cf [11], Remark 4.1.7). Let $\{u_1,\ldots,u_d\}$ denote a basis for the vector space of bounded solutions to the variational equation $\dot{u} = D_1 f_0(\gamma,0)u$ with $u_d = \dot{\gamma}$ and let $\{v_1,\ldots,v_d\}$ denote a basis for the vector space of bounded solutions to the adjoint variational equation $\dot{v} = -D_1 f_0(\gamma,0)^t v$. Now define the functions $a_{ij} : \mathbb{R} \to \mathbb{R}$, constants b_{ijk} and function $M : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^{d-1} \to \mathbb{R}^d$ by

$$\begin{aligned}
a_{ij}(\alpha) &= \int_{-\infty}^{\infty} \langle v_i(t), f_j(\gamma(t),0,0,t+\alpha)\rangle\, dt, \quad \begin{cases} 1 \le i \le d, \\ 1 \le j \le 2; \end{cases} \\
b_{ijk} &= \int_{-\infty}^{\infty} \langle v_i, D_{11} f_0(\gamma,0) u_j u_k\rangle\, dt, \quad \begin{cases} 1 \le i \le d, \\ 1 \le j,k \le d-1; \end{cases} \\
M_i(\mu,\alpha,\beta) &= \sum_{j=1}^{2} a_{ij}(\alpha)\mu_j + \frac{1}{2}\sum_{j,k=1}^{d-1} b_{ijk}\beta_j\beta_k, \quad 1 \le i \le d.
\end{aligned} \tag{4.2.6}$$

The function M is our bifurcation function and is used in Theorem 4.2.1 below. The integer d has a geometric interpretation. Let $P = \gamma(0)$ and let W^s, W^u denote the stable, unstable manifolds respectively of the origin for the unperturbed equation from (4.2.5). Then the entire orbit of γ lies in $W^s \cap W^u$ so that $P \in W^s \cap W^u$ and $\dot{\gamma}(0) \in T_P W^s \cap T_P W^u$. The vectors $\{u_1(0), \ldots, u_d(0)\}$ are a basis for $T_P W^s \cap T_P W^u$ and $d = \dim(T_P W^s \cap T_P W^u)$.

Suppose that $W^s \cap W^u$ has a connected component which is a manifold of dimension d and contains the orbit of γ. Then in (4.2.6), all $b_{ijk} = 0$, the hypotheses of Theorem 4.2.1 below cannot be satisfied and an alternate bifurcation function is required. Let W^h denote a homoclinic d-manifold containing γ, let U_0 be an open neighborhood of the origin in $\mathbb{R}^{d-1}$, let $\eta : U_0 \to W^h$ be a differentiable function-denoted $\beta \to \eta(\beta)$ with $\eta(0) = P$, let $t \to \gamma_\beta(t)$ be the solution to the unperturbed equation (4.2.5) satisfying $\gamma_\beta(0) = \eta(\beta)$, and assume that η is constructed so that $(\beta, t) \to \gamma_\beta(t)$ establishes local coordinates on W^h. In other words, the original orbit γ is embedded in a $(d-1)$-parameter family of homoclinic orbits. We suppose that $\left\{\dot{\gamma}_\beta(t), \frac{\partial \gamma_\beta}{\partial \beta_i}(t), i = 1, \ldots, d-1\right\}$, $\beta = (\beta_1, \ldots, \beta_{d-1})$, is a basis of bounded solutions of the variational equation $\dot{v} = D_1 f_0(\gamma_\beta, 0)v$. For each fixed β we let $\{v_{\beta 1}, \ldots, v_{\beta d}\}$ denote a basis for the vector space of bounded solutions to the adjoint variational equation $\dot{v} = -D_1 f_0(\gamma_\beta, 0)^t v$. Without loss of generality we can assume that each $v_{\beta i}$ depends differentially on β. Now define functions $a_{ij} : \mathbb{R} \times U_0 \to \mathbb{R}$ and $M : \mathbb{R}^2 \times \mathbb{R} \times U_0 \to \mathbb{R}^d$ by

$$
\begin{aligned}
a_{ij}(\alpha, \beta) &= \int_{-\infty}^{\infty} \langle v_{\beta i}(t), f_j(\gamma_\beta(t), 0, 0, t+\alpha)\rangle \, dt, \quad \begin{cases} 1 \le i \le d, \\ 1 \le j \le 2, \end{cases} \\
M_i(\mu, \alpha, \beta) &= \sum_{j=1}^{2} a_{ij}(\alpha, \beta)\mu_j, \quad 1 \le i \le d.
\end{aligned}
\tag{4.2.7}
$$

This is our bifurcation function for the homoclinic manifold case. By combining results from Section 4.1 we now get the following result.

Theorem 4.2.1. *M is the same as in* (4.2.6) *or* (4.2.7) *and suppose $(\mu_0, \alpha_0, \beta_0)$ are such that $M(\mu_0, \alpha_0, \beta_0) = 0$ and $D_{(\alpha,\beta)} M(\mu_0, \alpha_0, \beta_0)$ is nonsingular. Then there exists $\xi_0 > 0$ so that if $0 < \xi < \xi_0$ the equation $\dot{x} = f(x, 0, \xi\mu_0, t)$ has a homoclinic solution γ_ξ to $p_{\xi\mu_0}$. Furthermore, $\gamma_\xi(t) \to p_{\xi\mu_0}$ at an exponential rate as $t \to \pm\infty$, γ_ξ depends continuously on ξ, $\lim_{\xi \to 0} \gamma_\xi(t) = \gamma(t - \alpha_0)$ (or $= \gamma_{\beta_0}(t - \alpha_0)$), uniformly in t and the variational equation along γ_ξ has an exponential dichotomy for the whole line when $\xi \neq 0$.*

Following Sections 2.5.2 and 2.5.3, Theorem 4.2.1 establishes chaos for the differential equation $\dot{x} = f(x, 0, \xi\mu_0, t)$.

We remark that the constant K_ξ of the exponential dichotomy for the variational equation $\dot{u} = D_1 f(\gamma_\xi, 0, \xi\mu_0, t)u$ along $\gamma_\xi(t)$ tends to infinity as $\xi \to 0$. Indeed, let a_ξ, P_ξ, U_ξ be the corresponding constant, projection and fundamental solution from the definition of exponential dichotomy from Section 2.5.1, respectively. The roughness result for exponential dichotomies (cf Lemma 2.5.1) implies that we can take $a_\xi = a_0 > 0$ for some constant a_0. If $\sup_{\xi > 0} K_\xi < \infty$, then there is a sequence $\{\xi_i\}_{i=1}^{\infty}$ so that $\xi_i \to 0$, $K_{\xi_i} \to K_0$, $P_{\xi_i} \to P_0$ and $U_{\xi_i}(t) \to U_0(t)$ pointwise. Clearly, P_0 is a projection and $U_0(t)$ is the fundamental solution of $\dot{u} = D_1 f_0(\gamma, 0)u$ creating an exponential dichotomy for this equation on the whole line $\mathbb{R}$ with constants (K_0, a_0). This contradicts the existence of a bounded solution $\dot{\gamma}$ for this equation. Consequently, $K_\xi \to \infty$ as $\xi \to 0$.

4.2.3 Chaos in the Full Equation

We construct the bifurcation function M from (4.2.6) or (4.2.7), as in the preceding section, from the reduced equation (4.2.5). If M satisfies the hypotheses for Theorem 4.2.1 we have a transverse homoclinic solution and hence chaos for (4.2.5) when $\mu = \xi\mu_0$, $0 < \xi < \xi_0$. We now establish a condition for chaos to exist in the full equation (4.2.4). Since the exponential constant K_ξ of $\dot{u} = D_1 f(\gamma_\xi, 0, \xi\mu_0, t)u$ tends to infinity as $\xi \to 0$, as we showed in previous section, we have to deal with the full system (4.2.4). For this we consider the modification of (4.2.4) in the form

$$\begin{aligned} \dot{x} &= f(x, \lambda y, \mu, t)\,, \\ \dot{y} &= g_0(x,y) + \lambda\mu_1 g_1(x,y,\mu,t) + \mu_2 g_2(x,y,\mu), \\ &\quad 0 \le \lambda \le 1\,. \end{aligned} \tag{4.2.8}$$

The changes $x = \gamma + \sum_{i=1}^{d-1} \xi\beta_i u_i + \xi^2 u$, $y = \xi^2 v$, $\mu = \xi^2\mu_0$ with $\mu_0 \neq 0$ into (4.2.8) yield

$$\begin{aligned} \dot{u} = {} & D_1 f_0(\gamma,0)u + \frac{1}{2}\sum_{i,j=1}^{d-1} D_{11} f_0(\gamma,0)\beta_i\beta_j u_i u_j \\ & + \mu_{0,1} f_1(\gamma,0,0,t+\alpha) + \mu_{0,2} f_2(\gamma,0,0,t+\alpha) + O(\xi), \end{aligned} \tag{4.2.9a}$$

$$\begin{aligned} \dot{v} = {} & \left[D_2 g_0(\gamma,0) + \xi^2\mu_{0,2} D_2 g_2(\gamma,0,0)\right] v \\ & + \Bigg[D_2 g_0\left(\gamma + \xi\sum_{i=1}^{d-1}\beta_i u_i + \xi^2 u, 0\right) - D_2 g_0(\gamma,0) \\ & + D_{22} g_0\left(\gamma + \xi\sum_{i=1}^{d-1}\beta_i u_i + \xi^2 u, 0\right)\xi^2 v + O(\xi^4 v^2)\Bigg] v + \lambda\mu_{0,1} g_1(0,0,0,t+\alpha) \\ & + \lambda\mu_{0,1}\left\{g_1\left(\gamma + \xi\sum_{i=1}^{d-1}\beta_i u_i + \xi^2 u, \xi^2 v, \xi^2\mu_0, t+\alpha\right) - g_1(0,0,0,t+\alpha)\right\} \\ & + \xi^2\mu_{0,2}\left\{D_2 g_2\left(\gamma + \xi\sum_{i=1}^{d-1}\beta_i u_i + \xi^2 u, 0, \xi^2\mu_0\right) - D_2 g_2(\gamma,0,0) + O(\xi^2 v)\right\} v\,. \end{aligned} \tag{4.2.9b}$$

We consider the Banach spaces

$$X_n = \left\{ x \in C(\mathbb{R}, \mathbb{R}^n) \Big| \sup_{t \in \mathbb{R}} |x| < \infty \right\},$$

$$Y_n = \left\{ y \in X_n \Big| \int_{-\infty}^{\infty} \langle y(t), v(t) \rangle \, dt \text{ for every solution } v \in X_n \text{ of } \dot{v} = -Df_0(\gamma, 0)^t v \right\}$$

with the supremum norm $||x|| = \sup_{t \in \mathbb{R}} |x(t)|$. To solve (4.2.9a), we recall Theorems 4.1.2 and 4.1.3.

Lemma 4.2.2. *Given $h \in Y_n$, the equation $\dot{u} = D_1 f_0(\gamma(t), 0)u + h$ has a unique solution $u \in X_n$ satisfying $\langle u(0), u_i(0) \rangle = 0$ for every $i = 1, 2, \ldots, d$.*

Lemma 4.2.3. *There exists a projection $\Pi : X_n \to X_n$ so that $\mathscr{R}(\mathbb{I} - \Pi) = Y_n$.*

We also need the following lemma.

Lemma 4.2.4. *There exist constants $b > 0$, $B > 0$ and $\widetilde{\xi}_0 > 0$ so that given $\mu_{0,2} > 0$, for any $0 < \xi \leq \widetilde{\xi}_0$ the variational equation*

$$\dot{v} = \left[D_2 g_0(\gamma(t), 0) + \xi^2 \mu_{0,2} D_2 g_2(\gamma(t), 0, 0) \right] v$$

has an exponential dichotomy on $\mathbb{R}$ with constants $(B, b\xi^2 \mu_{0,2})$.

Proof. Write the given equation in the form $\dot{v} = Rv + S(t)v$ where

$$R = D_2 g_0(0,0) + \xi^2 \mu_{0,2} D_2 g_2(0,0,0),$$
$$S(t) = D_2 g_0(\gamma(t), 0) - D_2 g_0(0,0) + \xi^2 \mu_{0,2} \left[D_2 g_2(\gamma(t), 0, 0) - D_2 g_2(0,0,0) \right].$$

Let V_ξ be the fundamental solution for $\dot{v} = Rv + S(t)v$ with $V_\xi(0) = \mathbb{I}$. Then for $s \leq t$ we have

$$V_\xi(t) = \mathrm{e}^{(t-s)R} V_\xi(s) + \int_s^t \mathrm{e}^{(t-\tau)R} S(\tau) V_\xi(\tau) \, d\tau.$$

Using (vii) and (viii) for (4.2.4) we can, for $\widetilde{\xi}_0$ sufficiently small, find $K_1, b > 0$ so that $|\,\mathrm{e}^{(t-s)R}\,| \leq K_1 \mathrm{e}^{b\xi^2 \mu_{0,2}(s-t)}$ when $0 < \xi \leq \widetilde{\xi}_0$ and $s \leq t$. Now define

$$x(t) = |V_\xi(t) V_\xi(s)^{-1}| \, \mathrm{e}^{b\xi^2 \mu_{0,2}(t-s)}.$$

Then from the preceding equation for V_ξ we get

$$x(t) \leq K_1 + \int_s^t K_1 |S(\tau)| x(\tau) \, d\tau.$$

Hence, from the Gronwall inequality (cf Section 2.5.1),

$$x(t) \leq K_1 \mathrm{e}^{K_1 \int_s^t |S(\tau)| \, d\tau} \leq B$$

for a constant $B > 0$. □

We define the linear map $\mathscr{K} : Y_n \to X_n$ by $\mathscr{K}h = u$ where h, u are as in Lemma 4.2.2. Using the projection Π and the exponential dichotomy V_ξ from Lemma 4.2.4, where we suppose $\mu_{0,2} > 0$ (the case $\mu_{0,2} < 0$ can be handled analogously), we can rewrite (4.2.9) as the fixed point problem

$$
\begin{aligned}
u = {} & \mathscr{K}(\mathbb{I}-\Pi)\Bigg(\frac{1}{2}\sum_{i,j=1}^{d-1} D_{11}f_0(\gamma,0)\beta_i\beta_j u_i u_j \\
& + \mu_{0,1}f_1(\gamma,0,0,,t+\alpha) + \mu_{0,2}f_2(\gamma,0,0,t+\alpha) + O(\xi)\Bigg), \qquad \text{(4.2.10a)}
\end{aligned}
$$

$$
\begin{aligned}
v(t) = {} & \int_{-\infty}^{t} V_\xi(t)V_\xi(s)^{-1}\Bigg\{\Bigg[D_2 g_0\left(\gamma(s)+\xi\sum_{i=1}^{d-1}\beta_i u_i(s)+\xi^2 u(s),0\right) \\
& + D_{22}g_0\left(\gamma(s)+\xi\sum_{i=1}^{d-1}\beta_i u_i(s)+\xi^2 u(s),0\right)\xi^2 v(s) \\
& - D_2 g_0(\gamma(s),0) + O(\xi^4 v(s)^2)\Bigg]v(s) + \lambda\mu_{0,1}g_1(0,0,0,s+\alpha) \\
& + \lambda\mu_{0,1}\Bigg\{g_1\left(\gamma(s)+\xi\sum_{i=1}^{d-1}\beta_i u_i(s)+\xi^2 u(s),\xi^2 v(s),\xi^2\mu_0,s+\alpha\right) \\
& - g_1(0,0,0,s+\alpha)\Bigg\} \\
& + \xi^2\mu_{0,2}\Bigg\{D_2 g_2\left(\gamma(s)+\xi\sum_{i=1}^{d-1}\beta_i u_i(s)+\xi^2 u(s),0,\xi^2\mu_0\right) \\
& - D_2 g_2(\gamma(s),0,0) + O(\xi^2 v)\Bigg\}v(s)\Bigg\}ds \qquad \text{(4.2.10b)}
\end{aligned}
$$

along with the system of bifurcation equations

$$
\begin{aligned}
& \int_{-\infty}^{\infty}\Bigg\langle v_i(t),\frac{1}{2}\sum_{i,j=1}^{d-1}D_{11}f_0(\gamma(t),0)\beta_i\beta_j u_i(t)u_j(t) + \mu_{0,1}f_1(\gamma(t),0,0,t+\alpha) \\
& \quad + \mu_{0,2}f_2(\gamma(t),0,0,t+\alpha) + O(\xi)\Bigg\rangle dt = 0, \quad i = 1,2,\ldots,d \qquad \text{(4.2.11)}
\end{aligned}
$$

where $\{v_1,\ldots,v_d\}$ is a basis for the space of bounded solutions to the adjoint equation. Using (ix) we have

$$
\begin{aligned}
& D_2 g_0\left(\gamma+\xi\sum_{i=1}^{d-1}\beta_i u_i+\xi^2 u,0\right) - D_2 g_0(\gamma,0) + D_{22}g_0\left(\gamma+\xi\sum_{i=1}^{d-1}\beta_i u_i+\xi^2 u,0\right)\xi^2 v \\
& \quad = O\left(\xi^2|\gamma||v|\right) + O\left(\xi^4|u||v|\right) + O\left(\xi^2|\gamma||u|\right) + O\left(\xi\sum_{i=1}^{d-1}\beta_i|u_i|\right),
\end{aligned}
$$

$$\begin{aligned}
& g_1\left(\gamma+\xi\sum_{i=1}^{d-1}\beta_i u_i+\xi^2 u,\xi^2 v,\xi^2\mu_0,t+\alpha\right)-g_1(0,0,0,t+\alpha)\\
&\quad = O\left(\xi^2|\gamma||v|\right)+O\left(\xi^4|u||v|\right)+O\left(\xi^2\right)+O\left(\xi^4|v|^2\right)\\
&\quad\quad + O\left(\xi^2|u|\right)+O(|\gamma|)+O\left(\xi\sum_{i=1}^{d-1}\beta_i|u_i|\right),\\
& D_2 g_2\left(\gamma+\xi\sum_{i=1}^{d-1}\beta_i u_i+\xi^2 u,0,\xi^2\mu_0\right)-D_2 g_2(\gamma,0,0)\\
&\quad = O\left(\xi^2\right)+O\left(\xi^2|u|\right)+O\left(\xi\sum_{i=1}^{d-1}\beta_i|u_i|\right).
\end{aligned}$$

We note that $|\gamma(t)|\le c\mathrm{e}^{-a|t|}$ and $|u_i(t)|\le c\mathrm{e}^{-a|t|}$, $i=1,2,\dots,d$ for constants $c>0$, $a>0$. Moreover, it holds that

$$\begin{aligned}
\int_{-\infty}^{t}\mathrm{e}^{-b\xi^2\mu_{0,2}(t-s)}\,ds &= \frac{1}{b\xi^2\mu_{0,2}},\\
\int_{-\infty}^{t}\mathrm{e}^{-b\xi^2\mu_{0,2}(t-s)-a|s|}\,ds &\le \int_{-\infty}^{\infty}\mathrm{e}^{-a|s|}\,ds = 2/a\,.
\end{aligned}$$

Consequently, if we assume that

$$\begin{aligned}
&\sup_{0\le\alpha\le T}\sup_{\xi>0}\int_{-\infty}^{t}\left|V_\xi(t)V_\xi(s)^{-1}g_1(0,0,0,s+\alpha)\,ds\right|<\infty,\\
&\sup_{0\le\alpha\le T}\sup_{\xi>0}\int_{-\infty}^{t}\left|V_\xi(t)V_\xi(s)^{-1}D_4 g_1(0,0,0,s+\alpha)ds\right|<\infty
\end{aligned}\tag{4.2.12}$$

then we can apply the Banach fixed point theorem 2.2.1 on a ball centered at 0 in the space $X_n\times X_m$ to solving (4.2.10) for $\xi>0$ sufficiently small. Substituting this solution into (4.2.11) yields a system of bifurcation equations of the form

$$M(\mu,\alpha,\beta)+O(\xi)=0,\tag{4.2.13}$$

where M is as in (4.2.6) or (4.2.7). The case for (4.2.7) can be handled like above.

The assumptions of Theorem 4.2.1 imply the solvability of (4.2.13). This gives a transverse homoclinic orbit $\Gamma(\lambda,\xi^2\mu_0)(t)=\left(\Gamma_1(\lambda,\xi^2\mu_0)(t),\Gamma_2(\lambda,\xi^2\mu_0)(t)\right)$ of (4.2.8) near γ so that $\Gamma_1(\lambda,\xi^2\mu_0)(t)=\gamma(t)+O(\xi)$. The transversality follows exactly as in Section 4.1.3, so we omit its proof. Moreover, we have $\Gamma(0,\xi^2\mu_0)=(\gamma_\xi,0)$ for γ_ξ from Theorem 4.2.1, and $\Gamma(1,\xi^2\mu_0)$ is a homoclinic solution for (4.2.4). The dichotomy constants of the linearized system of (4.2.8) along $\Gamma(\lambda,\xi^2\mu_0)(t)$ are uniform for $0\le\lambda\le1$ and fixed ξ. This follows from the roughness result of exponential dichotomies from Lemma 2.5.1. Now we can follow directly a construction of a Smale horseshoe of Section 3.5.2 [7] along $\Gamma(\lambda,\xi^2\mu_0)(t)$ for fixed small

ξ. Thus we have a continuous family Σ_λ of Smale horseshoes for (4.2.8). This gives us the lifting of the Smale horseshoe of the reduced system to the full one.

The conditions (4.2.12) are, in fact, ones of nonresonance. To see this consider the equations

$$\dot{v} = [D_2g_0(\gamma,0) + \xi^2\mu_{0,2}D_2g_2(\gamma,0,0)]v + h,$$
$$\dot{w} = [D_2g_0(0,0) + \xi^2\mu_{0,2}D_2g_2(0,0,0)]w + h,$$

where $v,w,h \in X_m$. Then we get

$$\frac{d}{dt}(v-w) = \left[D_2g_0(0,0) + \xi^2\mu_{0,2}D_2g_2(0,0,0)\right](v-w)$$
$$+\Big[D_2g_0(\gamma,0) - D_2g_0(0,0) + \xi^2\mu_{0,2}(D_2g_2(\gamma,0,0) - D_2g_2(\gamma,0,0))\Big]v\,.$$

This gives

$$|v(t)-w(t)| \leq ||v||K_1\int_{-\infty}^{t} \mathrm{e}^{-b\xi^2\mu_{0,2}(t-s)-a|s|}\,ds \leq 2||v||K_1/a$$

for constants $K_1 > 0$, $a > 0$. Hence there is a constant $K_2 > 0$ so that

$$||w-v|| \leq K_2||v||, \quad ||w-v|| \leq K_2||w||\,.$$

These inequalities imply that assumption (4.2.12) is equivalent to the condition that when $\xi > 0$ the only bounded solution, $v_{\alpha,\xi}$, of

$$\dot{v} = \left[D_2g_0(0,0) + \xi^2\mu_{0,2}D_2g_2(0,0,0)\right]v + g_1(0,0,0,t+\alpha) \tag{4.2.14}$$

satisfies $\sup_{0\leq\alpha\leq T}\sup_{\xi>0}||v_{\alpha,\xi}|| < \infty$. Then also $\sup_{0\leq\alpha\leq T}\sup_{\xi>0}||\dot{v}_{\alpha,\xi}|| < \infty$. Hence by the Arzelà-Ascoli theorem 2.1.3, there is a sequence $\{\xi_i\}_{i=1}^{\infty}$, $\xi_i > 0$, $\xi_i \to 0$ so that $v_{\alpha,\xi_i} \to v_0$ and $\dot{v}_{\alpha,\xi_i} \to \dot{v}_0$ uniformly in compact intervals. Consequently, we get

$$\dot{v}_0 = D_2g_0(0,0)v_0 + g_1(0,0,0,t+\alpha)\,. \tag{4.2.15}$$

We note that $v_{\alpha,\xi}$, v_0 are T-periodic. We know [11] that (4.2.15) has a T-periodic solution if and only if

$$\int_0^T \langle w_i(t), g_1(0,0,0,t)\rangle\,dt = 0, \quad i = 1,2,\ldots,d_1, \tag{4.2.16}$$

where $\{w_1,\ldots,w_{d_1}\}$ is a basis of T-periodic solutions of the adjoint variational equation $\dot{w} = -D_2g_0(0,0)^t w$. Hence assumption (4.2.12) implies the validity of (4.2.16).

Conversely, let (4.2.16) hold. Then (4.2.15) has a T-periodic solution and we put $v = v_0 + w$ into (4.2.14) to get

$$\dot{w} = [D_2g_0(0,0) + \xi^2\mu_{0,2}D_2g_2(0,0,0)]w + \xi^2\mu_{0,2}D_2g_2(0,0,0)v_0\,. \tag{4.2.17}$$

The above arguments and Lemma 4.2.4 give that the unique solution $w_{\alpha,\xi} \in X_m$ of (4.2.17) satisfies $\sup_{0\leq\alpha\leq T}\sup_{\xi>0}||w_{\alpha,\xi}|| < \infty$. In summary, we see that assumption (4.2.12) is equivalent to condition (4.2.16).

Now we can state our results in the form of the next theorem.

Theorem 4.2.5. *Let* (i)-(ix) *hold. Let* M *be the same as in* (4.2.6) *or* (4.2.7) *and suppose* $(\mu_0, \alpha_0, \beta_0)$ *are such that*

$$M(\mu_0, \alpha_0, \beta_0) = 0 \text{ and } D_{(\alpha,\beta)}M(\mu_0, \alpha_0, \beta_0) \text{ is nonsingular.}$$

If condition (4.2.16) *holds then there exist* $\bar{\xi}_0 > 0$, $K > 0$ *so that if* $0 < \xi \leq \bar{\xi}_0$ *and if the parameters in* (4.2.4) *are given by* $\mu = \xi\mu_0$, *then there exists a continuous map* $\phi : \mathscr{E} \times [0,1] \to \mathbb{R}^{n+m}$ *(cf Section 2.5.2) and* $m_0 \in \mathbb{N}$ *so that:*

(i) $\phi_\lambda = \phi(\cdot,\lambda) : \mathscr{E} \to \mathbb{R}^{n+m}$ *is a homeomorphism of* $\mathscr{E}$ *onto a compact subset of* $\mathbb{R}^{n+m}$ *on which the* m_0*th iterate* $F_\lambda^{m_0}$ *of the period map* F_λ *of* (4.2.8) *is invariant and satisfies* $F_\lambda^{2m_0} \circ \phi_\lambda = \phi_\lambda \circ \sigma$ *where* σ *is the Bernoulli shift on* $\mathscr{E}$.
(ii) $\phi_0 = \phi(\cdot,0) : \mathscr{E} \to \mathbb{R}^n \times \{0\}$ *and* $F_0 = (G_0, 0)$ *for the period map* G_0 *of the reduced equation* (4.2.5).
(iii) F_1 *is the period map of the full system* (4.2.4).
(iv) $|\phi(x,\lambda) - \phi(x,0)| \leq K\sqrt{\xi}$ *for any* $(x,\lambda) \in \mathscr{E} \times [0,1]$.

Theorem 4.2.5 roughly states that the Smale horseshoe of the reduced equation (4.2.5) can be shadowed and continued to the full system (4.2.4).

4.2.4 Applications to Nonlinear ODEs

We now illustrate the above theory with two examples. For convenience in our calculations let us denote $r(t) = \operatorname{sech} t$. Note that $\ddot{r} = r - 2r^3$ and $\dddot{r} = (1 - 6r^2)\dot{r}$.

Example 4.2.6. As our first example consider the equations (4.2.1) from the introduction. The reduced equation is

$$\ddot{x} = x - 2x^3 - 2\mu_2\dot{x} + \mu_1 \cos\omega t$$

which we consider as a first order system in the phase space $(x,\dot{x})$. Since this system is in $\mathbb{R}^2$ we necessarily have $d = 1$. A bounded solution to the adjoint equation is $v = (-\ddot{r}, \dot{r})$ and from this we compute

$$a_{11}(\alpha) = \int_{-\infty}^{\infty} \dot{r}\cos\omega(t+\alpha)\,dt = \pi\omega\operatorname{sech}\frac{\pi\omega}{2}\sin\omega\alpha,$$
$$a_{12} = \int_{-\infty}^{\infty} -2\dot{r}^2\,dt = -\frac{4}{3}.$$

The bifurcation equation obtained from (4.2.6) is

$$M(\alpha,\mu) = \left(\pi\omega \operatorname{sech}\frac{\pi\omega}{2}\sin\omega\alpha\right)\mu_1 - \frac{4}{3}\mu_2 = 0.$$

We can satisfy this equation by choosing $\alpha_0 \in \left[-\frac{\pi}{2\omega}, \frac{\pi}{2\omega}\right]$ and then taking $\mu_{0,1} \neq 0$ and

$$\frac{\mu_{0,2}}{\mu_{0,1}} = \frac{3\pi\omega}{4}\operatorname{sech}\frac{\pi\omega}{2}\sin\omega\alpha_0.$$

Since in (4.2.6), $d = 1$, the transversality condition is

$$D_\alpha M(\alpha_0,\mu_0) = \pi\omega^2\mu_{0,1}\operatorname{sech}\frac{\pi\omega}{2}\cos\omega\alpha_0 \neq 0$$

which is satisfied for $\alpha_0 \in \left(-\frac{\pi}{2\omega}, \frac{\pi}{2\omega}\right)$. Let $m_0 := (3\pi\omega/4)\operatorname{sech}\pi\omega/2$. By varying α_0 we see that the reduced equation exhibits chaos for all sufficiently small $|\mu_0|$ satisfying $-m_0 < \mu_{0,2}/\mu_{0,1} < m_0$. Theorem 4.2.5 gives another result.

Theorem 4.2.7. *If* $p\omega \neq \sqrt{k-1}$ *then the full equation* (4.2.1) *exhibits chaos for all sufficiently small* $\mu_1 \neq 0, \mu_2$ *satisfying* (4.2.3).

Example 4.2.8. As a generalization of the preceding example consider the equations

$$\begin{aligned} \ddot{x} &= x - 2x(x^2+y^2+z^2) - \mu_2(\dot{x}+\dot{y}) + \mu_1\cos\omega t, \\ \ddot{y} &= y - 2y(x^2+y^2+z^2) - \mu_2(\dot{x}+\dot{y}), \\ \ddot{z} &= (1-k)z - 2z(x^2+y^2+z^2) - \mu_2\dot{z} + \mu_1\cos p\omega t \end{aligned} \tag{4.2.18}$$

where, as before, we assume that $k > 1$ and $p \in \mathbb{N}$. We consider these equations as a first order system in the phase space $(x,\dot{x},y,\dot{y},z,\dot{z})$. The reduced equations of (4.2.18) are

$$\begin{aligned} \ddot{x} &= x - 2x(x^2+y^2) - \mu_2(\dot{x}+\dot{y}) + \mu_1\cos\omega t, \\ \ddot{y} &= y - 2y(x^2+y^2) - \mu_2(\dot{x}+\dot{y}). \end{aligned} \tag{4.2.19}$$

The unperturbed motion of (4.2.19) has a homoclinic 2-manifold with a family of homoclinic orbits given by $x = r(t)\cos\beta$, $y = r(t)\sin\beta$ (cf [9, p. 133]). Writing out the adjoint equation in $\mathbb{R}^4$ we obtain as a basis for the space of bounded solutions

$$v_{\beta 1} = (-\ddot{r}\cos\beta, \dot{r}\cos\beta, -\ddot{r}\sin\beta, \dot{r}\sin\beta),$$

$$v_{\beta 2} = (-\dot{r}\sin\beta, r\sin\beta, \dot{r}\cos\beta, -r\cos\beta).$$

Next we compute

$$\begin{aligned} a_{11}(\alpha,\beta) &= \int_{-\infty}^{\infty} \dot{r}\cos\beta\cos\omega(t+\alpha)\,dt = \pi\omega\operatorname{sech}\frac{\pi\omega}{2}\sin\omega\alpha\cos\beta, \\ a_{12}(\alpha,\beta) &= \int_{-\infty}^{\infty} -\dot{r}\cos\beta(\dot{r}\cos\beta+\dot{r}\sin\beta) - \dot{r}\sin\beta(\dot{r}\cos\beta+\dot{r}\sin\beta)\,dt \\ &= -\frac{2}{3}(\cos\beta+\sin\beta)^2, \end{aligned}$$

$$a_{21}(\alpha,\beta) = \int_{-\infty}^{\infty} r\sin\beta\cos\omega(t+\alpha)\,dt = \pi\,\mathrm{sech}\,\frac{\pi\omega}{2}\cos\omega\alpha\sin\beta,$$

$$a_{22}(\alpha,\beta) = \int_{-\infty}^{\infty} -r\sin\beta(\dot r\cos\beta+\dot r\sin\beta)+r\cos\beta(\dot r\cos\beta+\dot r\sin\beta)\,dt = 0.$$

In (4.2.7), $d=2$, β is a scalar and the bifurcation equation $M(\alpha,\beta,\mu)=0$ takes the form

$$a_{11}(\alpha,\beta)\mu_1 + a_{12}(\alpha,\beta)\mu_2 = 0, \quad a_{21}(\alpha,\beta)\mu_1 = 0.$$

A sufficient condition for a nontrivial solution is $a_{21}=0$ which is satisfied by $\omega\alpha_0^{\pm} = \pm\pi/2$. We then have

$$\frac{\mu_2}{\mu_1} = -\frac{a_{11}(\alpha_0^{\pm},\beta_0)}{a_{12}(\alpha_0^{\pm},\beta_0)} = \pm\frac{3\pi\omega\,\mathrm{sech}\,\frac{\pi\omega}{2}\cos\beta_0}{2(\cos\beta_0+\sin\beta_0)^2}.$$

We see from Figure 4.1 that the range is $\mathbb{R}$ of the function $H(\beta) := \frac{\cos\beta}{(\cos\beta+\sin\beta)^2}$ as $\beta\in[0,2\pi]\setminus\left\{\frac{3\pi}{4},\frac{7\pi}{4}\right\}$.

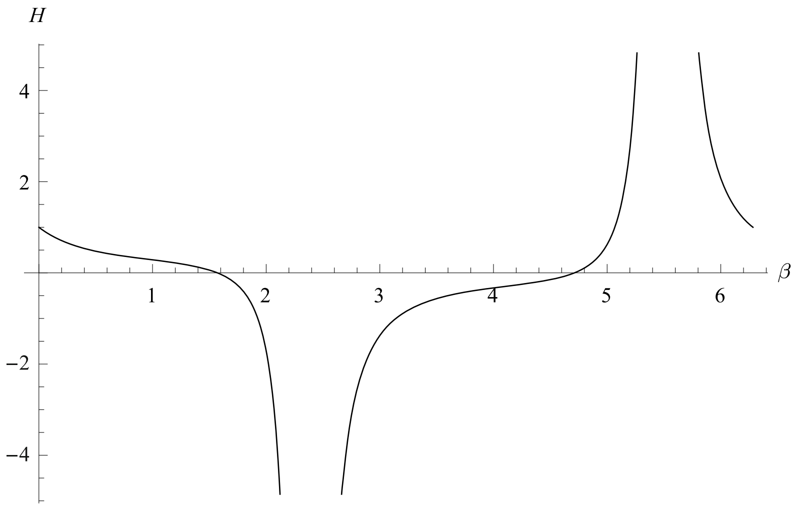

Fig. 4.1 The graph of the function $H(\beta)$ over $[0,2\pi]$.

It remains checking the transversality condition which takes the forms

$$\det D_{(\alpha,\beta)}M(\alpha_0^{+},\beta_0,\mu) = -\frac{\mu_1^2\pi^2\omega^2(\sin\beta_0+2\cos^3\beta_0)\sin\beta_0\,\mathrm{sech}^2\,\frac{\pi\omega}{2}}{(\cos\beta_0+\sin\beta_0)^2} \neq 0, \tag{4.2.20}$$

and

$$\det D_{(\alpha,\beta)} M(\alpha_0^-, \beta_0, \mu) = \frac{\mu_1^2 \pi^2 \omega^2 \operatorname{sech}^2 \frac{\pi\omega}{2} (2\cos 2\beta_0 - 2 - 2\sin 2\beta_0 + 3\sin 4\beta_0)}{4(\cos\beta_0 + \sin\beta_0)^2} \neq 0, \tag{4.2.21}$$

and (4.2.20) is satisfied for $\beta_0 \in [0,2\pi] \setminus \left\{0, \frac{3\pi}{4}, \pi, \frac{7\pi}{4}, 2\pi\right\}$, while (4.2.21) holds for

$$\beta_0 \in [0,2\pi] \setminus \left\{0, \frac{1}{2}\arccos\left(\frac{\sqrt{17}-1}{6}\right), -\frac{1}{2}\arccos\left(\frac{-\sqrt{17}-1}{6}\right) + \pi, \right.$$
$$\left. \frac{1}{2}\arccos\left(\frac{\sqrt{17}-1}{6}\right) + \pi, -\frac{1}{2}\arccos\left(\frac{-\sqrt{17}-1}{6}\right) + 2\pi, \frac{3\pi}{4}, \pi, \frac{7\pi}{4}, 2\pi \right\}.$$

Thus, the reduced equation exhibits chaos for all sufficiently small μ in the μ_1-μ_2 plane except along three lines of slopes $m = \pm m_0$, ∞, where $m_0 = \frac{3\pi\omega}{2} \operatorname{sech} \frac{\pi\omega}{2}$. From Theorem 4.2.5, if $p\omega \neq \sqrt{k-1}$ then the full equation exhibits chaos for all sufficiently small μ lying except along three lines of slopes $m = \pm m_0$, ∞. We obtain these transversal homoclinic orbits from (α_0^+, β_0). Moreover, we see from Figure 4.1 that the equation $H(\beta_0) = y$ has two solutions in $[0,2\pi)$ for any $y \in \mathbb{R}$. So we get two different transversal homoclinic orbits. Furthermore excluding also the next four lines of the slopes $\pm m_\pm$ with $m_\pm = \frac{3\pi\omega\sqrt{69\pm 3\sqrt{17}}}{32} \operatorname{sech} \frac{\pi\omega}{2}$ we can involve also the point (α_0^-, β_0), and consequently we get four different transversal homoclinic orbits. Note that $H(\beta + \pi) = -H(\beta)$, $H(0) = 1$ and $H\left(\mp\frac{1}{2}\arccos\left(\frac{\mp\sqrt{17}-1}{6}\right)\right) = \frac{\sqrt{69\pm 3\sqrt{17}}}{16}$.

4.3 ODEs with Resonant Center Manifolds

4.3.1 ODEs with Saddle-Center Parts

We consider differential equations of the form

$$\dot{x} = f(x,y,\mu,t) = f_0(x,y) + \mu_1 f_1(x,y,\mu,t) + \mu_2 f_2(x,y,\mu,t), \tag{4.3.1a}$$

$$\dot{y} = g(x,y,\mu,t) = g_0(x,y) + \mu_1 g_1(x,y,\mu,t) + \mu_2 g_2(x,y,\mu) \tag{4.3.1b}$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$. We make the following assumptions of (4.3.1):

(i) Each f_i, g_i are C^4-smooth in all arguments.
(ii) f_1, f_2 and g_1 are periodic in t with period T.
(iii) $D_2 f_0(x,0) = 0$.

(iv) The eigenvalues of $D_1 f_0(0,0)$ lie off the imaginary axis.
(v) The equation $\dot{x} = f_0(x,0)$ has a homoclinic solution γ.
(vi) $g_0(x,0) = g_2(x,0,\mu) = 0$, $D_{21}g_0(0,0) = 0$ and $D_{22}g_0(0,0) = 0$.
(vii) The eigenvalues of $D_2 g_0(0,0)$ lie on the imaginary axis.
(viii) If $\lambda(\mu_2)$ is an eigenvalue function of $D_2 g_0(0,0) + \mu_2 D_2 g_2(0,0,0)$ then $\Re(\lambda'(0)) < 0$.

In the hypothesis (viii), it is sufficient to assume that $\Re(\lambda'(0)) \neq 0$. In other words, (4.3.1b) is weakly hyperbolic with respect to μ_2. This more general assumption requires a little more work since it is necessary to include a nontrivial projection in Lemma 4.3.4 below. Consider the *reduced equation*

$$\dot{x} = f_0(x,0) + \mu_1 f_1(x,0,\mu,t) + \mu_2 f_2(x,0,\mu,t) \tag{4.3.2}$$

obtained by setting $y = 0$ in (4.3.1a). By hypothesis, the equation $\dot{x} = f_0(x,0)$ has a hyperbolic equilibrium and a homoclinic solution γ. Melnikov theory is used in Section 4.1 to obtain a transverse homoclinic solution in the reduced equation. The problem which naturally arises is showing that a transverse homoclinic solution for the reduced equation is shadowed by a transverse homoclinic solution for the full equation (4.3.1). This is done in Section 4.2 when the *center equation*

$$\dot{y} = g_0(0,y) + \mu_1 g_1(0,y,\mu,t) + \mu_2 g_2(0,y,\mu) \tag{4.3.3}$$

is not resonant at $y = 0$. The purpose of this section is to treat the resonant case and to detect a transverse homoclinic solution for the full system from a Melnikov function derived from the reduced and center equations. But the situation in this section is much more delicate than in Section 4.2.

Finally we note that a related problem is studied also in [15], where a three-dimensional ODE is considered with slowly varying one-dimensional variable. The approach in [15] is more geometrical than ours in this section.

4.3.2 Example of Coupled Oscillators at Resonance

We start with the equations

$$\begin{aligned} \ddot{x} &= x - 2x(x^2 + \xi y^2) - 2\mu_2 \delta \dot{x} + \mu_4 \cos(t+\alpha) + \mu_5 \sin(t+\alpha), \\ \ddot{y} &= -y - 2y(x^2 + y^2) - 2\mu_2 \dot{y} + \mu_1 \cos(t+\alpha) + \mu_3 \sin(t+\alpha). \end{aligned} \tag{4.3.4}$$

Here δ, ξ are positive constants and μ_i, $i = 1, \ldots, 5$ are small parameters. We put $\gamma(t) = \operatorname{sech} t$, $x = \gamma + \varepsilon^2 u$, $y = \varepsilon v$, $\mu_1 = \varepsilon^3 a_1$, $\mu_2 = \varepsilon^2$, $\mu_3 = \varepsilon^3 a_2$, $\mu_4 = \varepsilon^2 a_3$ and $\mu_5 = \varepsilon^2 a_4$, with $a_1, a_2, a_3, a_4 \in \mathbb{R}$ into (4.3.4) to get

$$\begin{aligned}\ddot{u} &= (1-6\gamma^2)u - 2\delta\dot{\gamma} - 2\xi\gamma v^2 + a_3\cos(t+\alpha) + a_4\sin(t+\alpha) + O(\varepsilon^2),\\ \ddot{v} &= -(1+2\gamma^2)v - 2\varepsilon^2\dot{v} - 4\varepsilon^2\gamma uv \\ &\quad -2\varepsilon^4u^2v - 2\varepsilon^2v^3 + \varepsilon^2a_1\cos(t+\alpha) + \varepsilon^2a_2\sin(t+\alpha)\,.\end{aligned} \tag{4.3.5}$$

First, we look for a 2π-periodic solution of the equation

$$\ddot{v}_{\varepsilon,\alpha,a} = -v_{\varepsilon,\alpha,a} - 2\varepsilon^2\dot{v}_{\varepsilon,\alpha,a} - 2\varepsilon^2v^3_{\varepsilon,\alpha,a} + \varepsilon^2a_1\cos(t+\alpha) + \varepsilon^2a_2\sin(t+\alpha)\,. \tag{4.3.6}$$

Clearly $v_{\varepsilon,\alpha,a}(t) = w_{\varepsilon,a}(t+\alpha)$ where $w_{\varepsilon,a}$ is a 2π-periodic solution of

$$\ddot{w}_{\varepsilon,a} = -w_{\varepsilon,a} - 2\varepsilon^2\dot{w}_{\varepsilon,a} - 2\varepsilon^2w^3_{\varepsilon,a} + \varepsilon^2a_1\cos t + \varepsilon^2a_2\sin t\,. \tag{4.3.7}$$

Consider the operator $L : C^2_{2\pi}(\mathbb{R}) \to C_{2\pi}(\mathbb{R})$ defined as $Lw = \ddot{w} + w$. Here $C^r_{2\pi}(\mathbb{R})$, $r \in \mathbb{Z}_+$, is the Banach space of C^r-smooth and 2π-periodic functions endowed with the maximum norm. We have

$$\mathscr{N}L = \operatorname{span}\{\cos t, \sin t\},$$
$$\mathscr{R}L = \left\{h \in C_{2\pi}(\mathbb{R}) \mid \int_0^{2\pi} h(t)\cos t\,dt = 0,\ \int_0^{2\pi} h(t)\sin t\,dt = 0\right\}.$$

Let $Q : C_{2\pi}(\mathbb{R}) \to \mathscr{R}L$ be the continuous projection

$$(Qw)(t) = w - \frac{1}{\pi}\cos t\int_0^{2\pi} w(t)\cos t\,dt - \frac{1}{\pi}\sin t\int_0^{2\pi} w(t)\sin t\,dt.$$

Equation (4.3.7) can now be split into a new differential equation

$$\ddot{w} + w = Q(-2\varepsilon^2\dot{w} - 2\varepsilon^2w^3 + \varepsilon^2a_1\cos t + \varepsilon^2a_2\sin t) = Q(-2\varepsilon^2\dot{w} - 2\varepsilon^2w^3)$$

and a bifurcation equation

$$\begin{aligned}&(\mathbb{I}-Q)(-2\varepsilon^2\dot{w} - 2\varepsilon^2w^3 + \varepsilon^2a_1\cos t + \varepsilon^2a_2\sin t)\\ &\quad = \varepsilon^2\left[a_1 - \frac{1}{\pi}\int_0^{2\pi}(2\dot{w}+2w^3)\cos t\,dt\right]\cos t\\ &\qquad + \varepsilon^2\left[a_2 - \frac{1}{\pi}\int_0^{2\pi}(2\dot{w}+2w^3)\sin t\,dt\right]\sin t = 0.\end{aligned}$$

The differential equation has a solution $w \in C^2_{2\pi}(\mathbb{R})$ of the form

$$w(t) = \varphi(\varepsilon, c_1, c_2)(t) + c_1\cos t + c_2\sin t$$

where c_1, c_2 are arbitrary and $\varphi = O(\varepsilon^2)$. Substituting this into the bifurcation equation gives

$$a_2 - \frac{1}{\pi}\int_0^{2\pi} \left[2(-c_1\sin t + c_2\cos t) + 2(c_1\cos t + c_2\sin t)^3\right]\sin t\,dt + O(\varepsilon^2) = 0,$$
$$a_1 - \frac{1}{\pi}\int_0^{2\pi} \left[2(-c_1\sin t + c_2\cos t) + 2(c_1\cos t + c_2\sin t)^3\right]\cos t\,dt + O(\varepsilon^2) = 0$$

or

$$\begin{aligned} 4c_1 - 3c_2^3 - 3c_1^2c_2 &= -2a_2 + O(\varepsilon^2), \\ 4c_2 + 3c_1^3 + 3c_2^2c_1 &= 2a_1 + O(\varepsilon^2). \end{aligned} \tag{4.3.8}$$

The determinant of the Jacobian of the left hand side of (4.3.8) is

$$16 + 27(c_1^2 + c_2^2)^2 \neq 0.$$

Now we have

$$|4c_1 - 3c_1^2c_2 - 3c_2^3| + |4c_2 + 3c_1^3 + 3c_1c_2^2| \geq (3(c_1^2 + c_2^2) - 4)(|c_1| + |c_2|)\,.$$

Hence the map

$$(c_1, c_2) \to (4c_1 - 3c_2^3 - 3c_1^2c_2, 4c_2 + 3c_1^3 + 3c_2^2c_1)$$

from $\mathbb{R}^2$ to $\mathbb{R}^2$ is proper and locally invertible and thus a diffeomorphism by the Banach-Mazur Theorem 2.2.6. Hence we can use the implicit function theorem to get solutions $c_1(a,\varepsilon)$ and $c_2(a,\varepsilon)$ to (4.3.8) for ε small and $a = (a_1, a_2) \in \mathbb{R}^2$ from bounded subsets. In summary, we have the following result:

Lemma 4.3.1. *For any $n \in \mathbb{N}$, there exist $\varepsilon_0 = \varepsilon_0(n) > 0$ and a differentiable function $c : (-n,n)^2 \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^2$ denoted $(a,\varepsilon) \to c(a,\varepsilon)$ so that* (4.3.6) *has a 2π-periodic solution of the form*:

$$v_{\varepsilon,\alpha,a}(t) = c_1(a,\varepsilon)\cos(t+\alpha) + c_2(a,\varepsilon)\sin(t+\alpha) + O(\varepsilon^2)\,. \tag{4.3.9}$$

We note that the function $c(a,\varepsilon)$ may also depend on n, but when $m > n$ these two functions $c(a,\varepsilon)$ from Lemma 4.3.1 coincide on the set $(-n,n)^2 \times (-\bar{\varepsilon}_0, \bar{\varepsilon}_0)$ with $\bar{\varepsilon}_0 = \min\{\varepsilon_0(n), \varepsilon_0(m)\}$.

We now substitute $v = w + v_{\varepsilon,\alpha,a}$ into (4.3.5) to get

$$\begin{aligned} \ddot{u} = {} & (1 - 6\gamma^2)u - 2\delta\dot{\gamma} - 2\xi\gamma\big(w + v_{\varepsilon,\alpha,a}(t)\big)^2 \\ & + a_3\cos(t+\alpha) + a_4\sin(t+\alpha) + O(\varepsilon), \end{aligned} \tag{4.3.10a}$$
$$\begin{aligned} \ddot{w} = {} & -(1 + 6\varepsilon^2 v_{\varepsilon,\alpha,a}^2)w - 2\varepsilon^2\dot{w} - 2\gamma^2 w \\ & - 2\gamma^2 v_{\varepsilon,\alpha,a} - 4\varepsilon^2\gamma u(w + v_{\varepsilon,\alpha,a}) - 2\varepsilon^4 u^2(w + v_{\varepsilon,\alpha,a}) \\ & - 6\varepsilon^2 w^2 v_{\varepsilon,\alpha,a} - 2\varepsilon^2 w^3. \end{aligned} \tag{4.3.10b}$$

To study (4.3.10) we must establish the existence of properties for an exponential dichotomy for the linear part of (4.3.10b) in three steps.

We first study the equation

$$\ddot{w} = -[1+\varepsilon^2\phi_\varepsilon(t)^2]w - 2\varepsilon^2\dot{w}, \tag{4.3.11}$$

where $\phi_\varepsilon(t) = \sqrt{6}v_{\varepsilon,\alpha,a}$.

Step 1. We put $w = \mathrm{e}^{-\varepsilon^2 t}z_1$ to get

$$\begin{aligned}\dot{z}_1 &= z_2,\\ \dot{z}_2 &= -[1+\varepsilon^2\phi_\varepsilon(t)^2 - \varepsilon^4]z_1.\end{aligned} \tag{4.3.12}$$

By Floquet theory [12,13] (4.3.12) has a solution, Z_ε, of the form $Z_\varepsilon = U_\varepsilon(t)\,\mathrm{e}^{tB_\varepsilon}$ where $U_\varepsilon(0) = \mathbb{I}$, $U_\varepsilon(t+2\pi) = U_\varepsilon(t)$ and

$$U_0(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

so that $\|U_0(t)\| = 1$. Stability is determined by the matrix B_ε and $Z_\varepsilon(2\pi) = \mathrm{e}^{2\pi B_\varepsilon}$, so we are interested in $Z_\varepsilon(2\pi)$. We have $Z_\varepsilon(t+2\pi) = Z_\varepsilon(t)Z_\varepsilon(2\pi)$ and from Liouville's formula (cf Section 2.5.1 and [12]) $\det Z_\varepsilon(2\pi) = 1$. Hence the eigenvalues of $Z_\varepsilon(2\pi)$ are a complex conjugate pair with norm 1 if and only if $|\operatorname{tr} Z_\varepsilon(2\pi)| < 2$. To compute an estimate for Z_ε we expand

$$\begin{aligned} z_1 &= u_0 + \varepsilon^2 u_1 + O(\varepsilon^4),\\ z_2 &= v_0 + \varepsilon^2 v_1 + O(\varepsilon^4),\\ \phi_\varepsilon &= \phi_0 + O(\varepsilon^2). \end{aligned}$$

Substituting these expansions into (4.3.12) we get

$$\dot{u}_0 = v_0, \quad \dot{v}_0 = -u_0, \qquad \dot{u}_1 = v_1, \quad \dot{v}_1 = -u_1 - \phi_0^2 u_0$$

and $Z_\varepsilon(0) = \mathbb{I}$ requires $u_1(0) = v_1(0) = 0$. By choosing either $u_0 = \cos t$, $v_0 = -\sin t$ or $u_0 = \sin t$, $v_0 = \cos t$, we find u_1, v_1 and then a computation shows that

$$Z_\varepsilon(2\pi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} \frac{1}{2}\int_0^{2\pi} \phi_0^2(s)\sin 2s\,ds & \int_0^{2\pi} \phi_0^2(s)\sin^2 s\,ds \\ -\int_0^{2\pi} \phi_0^2(s)\cos^2 s\,ds & -\frac{1}{2}\int_0^{2\pi} \phi_0^2(s)\sin 2s\,ds \end{pmatrix} + O(\varepsilon^4).$$

We have $\phi_0(t) = \sqrt{6}v_{0,\alpha,a}(t) = \sqrt{6}(c_1(a,0)\cos(t+\alpha) + c_2(a,0)\sin(t+\alpha))$. Thus, as long as $a \neq 0$ it follows from (4.3.8) that $c_1(a,0)^2 + c_2(a,0)^2 \neq 0$ and we can write

$$v_{0,a,\alpha} = c_5(a)\sin(t+\alpha+c_4(a))$$

where $c_5(a) = \sqrt{c_1(a,0)^2 + c_2(a,0)^2}$ and $c_4(a)$ is defined by the equality. Then $\phi_0(t) = c_3(a)\sin(t+\alpha+c_4(a))$ where $c_3(a) = \sqrt{6}c_5(a)$ and

$$\begin{aligned}
\int_0^{2\pi} \phi_0^2(s)\sin 2s\,ds &= c_3(a)^2\int_0^{2\pi} \sin 2s \sin^2(s+\alpha+c_4(a))\,ds\\
&= \frac{\pi}{2}c_3(a)^2\sin 2(\alpha+c_4(a)),\\
\int_0^{2\pi} \phi_0^2(s)\sin^2 s\,ds &= c_3(a)^2\int_0^{2\pi} \sin^2 s \sin^2(s+\alpha+c_4(a))\,ds\\
&= c_3(a)^2\Big(\frac{\pi}{2}+\frac{\pi}{4}\cos 2(\alpha+c_4(a))\Big),\\
\int_0^{2\pi} \phi_0^2(s)\cos^2 s\,ds &= c_3(a)^2\int_0^{2\pi} \cos^2 s \sin^2(s+\alpha+c_4(a))\,ds\\
&= c_3(a)^2\Big(\frac{\pi}{2}-\frac{\pi}{4}\cos 2(\alpha+c_4(a))\Big).
\end{aligned}$$

Hence

$$\begin{aligned}
Z_\varepsilon(2\pi) &= \mathbb{I}+\varepsilon^2 c_3(a)^2\frac{\pi}{4}\begin{pmatrix} \sin 2(\alpha+c_4(a)) & 2+\cos 2(\alpha+c_4(a))\\ -2+\cos 2(\alpha+c_4(a)) & -\sin 2(\alpha+c_4(a))\end{pmatrix}+O(\varepsilon^4)\\
&= \mathbb{I}+\varepsilon^2 A_\varepsilon
\end{aligned}$$

where the second equality defines the 2×2 matrix A_ε whose entries we denote are a_{ij}. If λ_A denotes an eigenvalue of A_ε then we can take

$$2\lambda_A = \mathrm{tr}A_\varepsilon+\sqrt{(\mathrm{tr}A_\varepsilon)^2-4\det A_\varepsilon}.$$

A direct computation shows $\det A_\varepsilon = \frac{3\pi^2}{16}c_3(a)^4+O(\varepsilon^2)$. Also, $\det Z_\varepsilon(2\pi)=1$ previously so that another calculation yields $\det Z_\varepsilon(2\pi)=1+\varepsilon^2\,\mathrm{tr}A_\varepsilon+\varepsilon^4\det A_\varepsilon=1$ and we get $\mathrm{tr}A_\varepsilon=-\varepsilon^2\det A_\varepsilon=-\varepsilon^2\frac{3\pi^2}{16}c_3(a)^4+O(\varepsilon^4)$. If we denote $\lambda_A=\varepsilon^2\lambda_A^R+i\lambda_A^I$ then

$$\begin{aligned}
\lambda_A^R &= \frac{1}{2\varepsilon^2}\,\mathrm{tr}A_\varepsilon = -\frac{3\pi^2}{32}c_3(a)^4+O(\varepsilon^2),\\
\lambda_A^I &= \sqrt{\det A_\varepsilon-\Big(\frac{1}{2}\,\mathrm{tr}A_\varepsilon\Big)^2} = \frac{\sqrt{3}\pi}{4}c_3(a)^2+O(\varepsilon).
\end{aligned}$$

Also, an eigenvalue, λ_Z, of $Z_\varepsilon(2\pi)$ is given by $\lambda_Z=1+\varepsilon^2\lambda_A$. The corresponding transformation matrix P_ε is

$$P_\varepsilon=\begin{pmatrix} a_{12} & 0\\ -a_{11}+\varepsilon^2\lambda_A^R & \lambda_A^I\end{pmatrix}\quad\text{with}\quad P_\varepsilon^{-1}=\begin{pmatrix} 1/a_{12} & 0\\ \dfrac{a_{11}-\varepsilon^2\lambda_A^R}{a_{12}\lambda_A^I} & \dfrac{1}{\lambda_A^I}\end{pmatrix}.$$

We have $\lambda_A^I>0$ for small ε, $\frac{\pi}{4}c_3(a)^2\le a_{12}\le\frac{3\pi}{4}c_3(a)^2$ and

$$P_\varepsilon^{-1} Z_\varepsilon(2\pi) P_\varepsilon = \begin{pmatrix} \Re\lambda_Z & \Im\lambda_Z \\ -\Im\lambda_Z & \Re\lambda_Z \end{pmatrix}.$$

Since $|\lambda_Z| = 1$ we can write

$$\begin{pmatrix} \Re\lambda_Z & \Im\lambda_Z \\ -\Im\lambda_Z & \Re\lambda_Z \end{pmatrix} = \mathrm{e}^{\Phi_\varepsilon} \quad \text{where} \quad \Phi_\varepsilon = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \quad \text{with} \quad \theta = \operatorname{Arg} \lambda_Z.$$

Now, we observe that the operator norm of a 2×2 square matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(that is the square root of the greatest eigenvalue of the symmetric matrix A^*A) is given by

$$\|A\|^2 = \frac{1}{2} \left[(a^2 + b^2 + c^2 + d^2) + \sqrt{[(a-d)^2 + (b+c)^2][(a+d)^2 + (b-c)^2]} \right]$$

and hence $\|A^{-1}\| = \frac{1}{|\det A|} \|A\|$ since

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Using these formulas we get

$$\begin{aligned} \|P_0\|^2 &= \frac{3\pi^2}{16} c_3(a)^4 \left[2 + \cos 2(\alpha + c_4(a))\right], \\ \det P_0 &= \frac{\sqrt{3}\pi^2}{16} c_3(a)^4 \left[2 + \cos 2(\alpha + c_4(a))\right], \\ \|P_0\| \|P_0^{-1}\| &= \sqrt{3}. \end{aligned}$$

We see that $\|P_\varepsilon\|$ and $\|P_\varepsilon^{-1}\|$ are both uniformly bounded for ε small and a bounded. Finally, we have

$$\begin{aligned} Z_\varepsilon(t) Z_\varepsilon(s)^{-1} &= U_\varepsilon(t)\, \mathrm{e}^{(t-s)B_\varepsilon}\, U_\varepsilon(s)^{-1} = U_\varepsilon(t) \exp\left(\frac{t-s}{2\pi} P_\varepsilon \Phi_\varepsilon P_\varepsilon^{-1} \right) U_\varepsilon(s)^{-1} \\ &= U_\varepsilon(t) P_\varepsilon\, \mathrm{e}^{\frac{t-s}{2\pi}\Phi_\varepsilon}\, P_\varepsilon^{-1} U_\varepsilon(s)^{-1} \\ &= U_\varepsilon(t) P_\varepsilon \begin{pmatrix} \cos\left(\frac{(t-s)\theta}{2\pi}\right) & \sin\left(\frac{(t-s)\theta}{2\pi}\right) \\ -\sin\left(\frac{(t-s)\theta}{2\pi}\right) & \cos\left(\frac{(t-s)\theta}{2\pi}\right) \end{pmatrix} P_\varepsilon^{-1} U_\varepsilon(s)^{-1}. \end{aligned}$$

Taking norms we get $\|Z_\varepsilon(t)Z_\varepsilon(s)^{-1}\| \le \sqrt{3}+\delta$ where $\delta \to 0$ as $\varepsilon \to 0$. This completes our study of (4.3.12).

Step 2. Next we write (4.3.11) as the system

$$\begin{aligned} \dot{w}_1 &= w_2, \\ \dot{w}_2 &= -w_1\left(1+\varepsilon^2\phi_\varepsilon(t)^2\right) - 2\varepsilon^2 w_2. \end{aligned} \tag{4.3.13}$$

Then the fundamental solution $\bar{W}_\varepsilon$ of (4.3.13) is given by

$$\bar{W}_\varepsilon(t) = \mathrm{e}^{-\varepsilon^2 t}\begin{pmatrix} 1 & 0 \\ -\varepsilon^2 & 1 \end{pmatrix} Z_\varepsilon(t).$$

This implies

$$\bar{W}_\varepsilon(t)\bar{W}_\varepsilon(s)^{-1} = \mathrm{e}^{-\varepsilon^2(t-s)}\begin{pmatrix} 1 & 0 \\ -\varepsilon^2 & 1 \end{pmatrix} Z_\varepsilon(t)Z_\varepsilon(s)^{-1}\begin{pmatrix} 1 & 0 \\ \varepsilon^2 & 1 \end{pmatrix}$$

and hence $\|\bar{W}_\varepsilon(t)\bar{W}_\varepsilon(s)^{-1}\| \le (\sqrt{3}+\delta)\,\mathrm{e}^{-\varepsilon^2(t-s)}$.

Step 3. Finally, we consider

$$\ddot{w} = -w(1+6\varepsilon^2 v_{\varepsilon,\alpha,a}^2(t)+2\gamma^2) - 2\varepsilon^2\dot{w}$$

which we write as

$$\begin{aligned} \dot{w}_1 &= w_2, \\ \dot{w}_2 &= -w_1(1+6\varepsilon^2 v_{\varepsilon,\alpha,a}^2(t)+2\gamma^2) - 2\varepsilon^2 w_2\,. \end{aligned} \tag{4.3.14}$$

Let W_ε be the fundamental solution of (4.3.14). We put

$$\Psi(t) = -2\gamma(t)^2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then for $t \ge s$ we get

$$W_\varepsilon(t)W_\varepsilon(s)^{-1} = \bar{W}_\varepsilon(t)\bar{W}_\varepsilon(s)^{-1} + \int_s^t \bar{W}_\varepsilon(t)\bar{W}_\varepsilon(z)^{-1}\Psi(z)W_\varepsilon(z)W_\varepsilon(s)^{-1}\,dz\,.$$

By putting $U(t) = W_\varepsilon(t)W_\varepsilon(s)^{-1}\,\mathrm{e}^{\varepsilon^2(t-s)}$ we obtain

$$\begin{aligned} \|U(t)\| &\le \left(\sqrt{3}+\delta\right) + \left(\sqrt{3}+\delta\right)\int_s^t \|\Psi(z)\|\|U(z)\|\,dz \\ &= \left(\sqrt{3}+\delta\right) + 2\left(\sqrt{3}+\delta\right)\int_s^t \gamma^2(z)\|U(z)\|\,dz \end{aligned}$$

which gives

$$\|U(t)\| \le \left(\sqrt{3}+\delta\right)\mathrm{e}^{2\left(\sqrt{3}+\delta\right)\int_s^t \gamma^2(z)\,dz}\,.$$

Now if either $t \geq s \gg 1$ or $s \leq t \ll -1$, then $\mathrm{e}^{2\sqrt{3}\int_s^t \gamma^2(z)\,dz}$ is about 1. So then we obtain

$$\|W_\varepsilon(t)W_\varepsilon(s)^{-1}\| \leq K_1\,\mathrm{e}^{-\varepsilon^2(t-s)}$$

with $K_1 \sim \sqrt{3}$ for $s \leq t \in (-\infty, -T_0] \cup [T_0, \infty)$ for $T_0 \gg 1$. Since W_0 satisfies

$$\begin{aligned}\dot{w}_1 &= w_2,\\ \dot{w}_2 &= -w_1(1+2\gamma^2).\end{aligned}$$

we see that

$$W_0(t) = C(t)\begin{pmatrix}1 & 0\\ 0 & 1/2\end{pmatrix},$$

where

$$C(t) = \begin{pmatrix}\cos t - \sin t\tanh t & \sin t + \cos t\tanh t\\ -\sin t - \cos t\tanh t - \sin t\,\mathrm{sech}^2 t & \cos t - \sin t\tanh t + \cos t\,\mathrm{sech}^2 t\end{pmatrix}.$$

Then we have

$$\|C(t)\|^2 = \frac{1}{2}\left(4 + \mathrm{sech}^4 t + \mathrm{sech}^2 t\sqrt{8 + \mathrm{sech}^4 t}\right) \leq 4, \quad \det C(t) = 2$$

which also imply $\|C(t)^{-1}\| \leq 1$. In summary, we arrive at

$$\|W_\varepsilon(t)W_\varepsilon(s)^{-1}\| \leq K_1\,\mathrm{e}^{-\varepsilon^2(t-s)}$$

with $K_1 \sim \sqrt{3} \times 2 \times \sqrt{3} = 6$ for $s \leq t \in \mathbb{R}$ and $\varepsilon > 0$ small. This is our exponential dichotomy for the linear part of (4.3.10b).

Remark 4.3.2. Note that in general the function $\tilde{f}_\varepsilon(t) = W_\varepsilon(t)\int_{-\infty}^t W_\varepsilon(s)^{-1} f(s)\,ds$ is $O(1/\varepsilon^2)$ for f bounded. But if $f \in L^1(\mathbb{R})$ such an expression is $O(1)$ and we can let $\varepsilon \to 0$. More precisely, set $\tilde{f}_0 := W_0(t)\int_{-\infty}^t W_0(s)^{-1} f(s)\,ds$ and let $\widetilde{T} > 0$ be large. Then $\tilde{f}_\varepsilon(t) = o(1)$ and $\tilde{f}_0(t) = o(1)$ uniformly for all $t \leq -\widetilde{T}$ and ε small. If $t \in [-\widetilde{T}, \widetilde{T}]$ then $\tilde{f}_\varepsilon(t) = W_\varepsilon(t)\int_{-\widetilde{T}}^t W_\varepsilon(s)^{-1} f(s)\,ds + W_\varepsilon(t)\int_{-\infty}^{-\widetilde{T}} W_\varepsilon(s)^{-1} f(s)\,ds$. Clearly $W_\varepsilon(t)\int_{-\infty}^{-\widetilde{T}} W_\varepsilon(s)^{-1} f(s)\,ds = o(1)$ and $W_0(t)\int_{-\infty}^{-\widetilde{T}} W_0(s)^{-1} f(s)\,ds = o(1)$ uniformly for all $t \in [-\widetilde{T}, \widetilde{T}]$ and ε small. Moreover

$$W_\varepsilon(t)\int_{-\widetilde{T}}^t W_\varepsilon(s)^{-1} f(s)\,ds \to W_0(t)\int_{-\widetilde{T}}^t W_0(s)^{-1} f(s)\,ds$$

uniformly for all $t \in [-\widetilde{T}, \widetilde{T}]$ as $\varepsilon \to 0$. Consequently, we obtain $\lim_{\varepsilon\to 0}\tilde{f}_\varepsilon(t) = \tilde{f}_0(t)$ uniformly in any interval $(-\infty, a]$ for $f \in L^1(\mathbb{R})$. If $t \geq \widetilde{T}$ then

$$\tilde{f}_\varepsilon(t) = W_\varepsilon(t)\int_{\widetilde{T}}^t W_\varepsilon(s)^{-1} f(s)\,ds + W_\varepsilon(t)\int_{-\infty}^{\widetilde{T}} W_\varepsilon(s)^{-1} f(s)\,ds\,.$$

We again deduce that $W_\varepsilon(t)\int_{\widetilde{T}}^t W_\varepsilon(s)^{-1}f(s)\,ds = o(1)$ and $W_0(t)\int_{\widetilde{T}}^t W_0(s)^{-1}f(s)\,ds = o(1)$ uniformly for all $t \geq \widetilde{T}$ and ε small. Next

$$W_\varepsilon(t)\int_{-\infty}^{\widetilde{T}} W_\varepsilon(s)^{-1}f(s)\,ds = W_\varepsilon(t)W_\varepsilon(\widetilde{T})^{-1}\tilde{f}_\varepsilon(\widetilde{T}).$$

In summary we obtain $\|\widetilde{f}_\varepsilon\| \leq \left(\sqrt{3}+o(1)\right)\|\widetilde{f}_0\|$. Moreover, when

$$\|f\|_{\tilde{a}} := \sup_{t\leq 0}|f(t)|\,\mathrm{e}^{-\tilde{a}t} < \infty$$

for $\tilde{a} > 0$, $\|\tilde{f}_\varepsilon\|_{\tilde{a}} \leq \frac{K_1}{\tilde{a}}\|f\|_{\tilde{a}}$. So if

$$X_{\tilde{a}} := \left\{f \in C(-\infty,0] \mid \|f\|_{\tilde{a}} < \infty\right\}$$

and $L_\varepsilon f := \tilde{f}_\varepsilon$, then $L_\varepsilon \in L(X_{\tilde{a}})$. Finally, we can check that $L_\varepsilon \to L_0$ as $\varepsilon \to 0$ in $L(X_{\tilde{a}})$ for $L_0 f = W_0(t)\int_{-\infty}^t W_0^{-1}(s)f(s)\,ds$.

Equation (4.3.10a) has the form

$$\ddot{u} = u(1-6\gamma^2(t)) + h(t), \qquad \dot{u}(0) = 0 \tag{4.3.15}$$

for $h(t) \in C_B(\mathbb{R})$ — the Banach space of bounded and continuous functions on $\mathbb{R}$ endowed with the supremum norm. For this we use the projection

$$\Pi h = \frac{\int_{-\infty}^{\infty} h(s)\dot{\gamma}(s)\,ds}{\int_{-\infty}^{\infty}\dot{\gamma}^2(s)\,ds}\,\dot{\gamma}(t).$$

From Section 4.1, (4.3.15) has a (unique) bounded solution $u = Kh$ if and only if $\Pi h = 0$. We write (4.3.10) in the form

$$\begin{aligned} u(t) &= K(\mathbb{I}-\Pi)\Big(-2\delta\dot{\gamma} - 2\xi\gamma\big[w+v_{\varepsilon,\alpha,a}(t)\big]^2 \\ &\qquad + a_3\cos(t+\alpha) + a_4\sin(t+\alpha)\Big) + O(\varepsilon), \end{aligned} \tag{4.3.16a}$$

$$\begin{aligned} w(t) &= \int_{-\infty}^{t} W_\varepsilon(t)W_\varepsilon(s)^{-1}\Big\{(0, -2\gamma^2 v_{\varepsilon,\alpha,a} - 4\varepsilon^2\gamma u(w+v_{\varepsilon,\alpha,a}) \\ &\qquad - 2\varepsilon^4 u^2(w+v_{\varepsilon,\alpha,a}) - 6\varepsilon^2 w^2 v_{\varepsilon,\alpha,a} - 2\varepsilon^2 w^3)\Big\}\,ds, \end{aligned} \tag{4.3.16b}$$

$$\begin{aligned} &\int_{-\infty}^{\infty}\Big(-2\delta\dot{\gamma} - 2\xi\gamma\big[w+v_{\varepsilon,\alpha,a}\big]^2 \\ &\qquad + a_3\cos(t+\alpha) + a_4\sin(t+\alpha)\Big)\dot{\gamma}(t)\,dt + O(\varepsilon) = 0 \end{aligned} \tag{4.3.16c}$$

for $w=(w_1,w_2)$. Since $v_{\varepsilon,\alpha,a}(t)=v_{0,\alpha,a}(t)+O(\varepsilon)$ by Lemma 4.3.1 and $\gamma\in L^1(\mathbb{R})$, we can consider, according to Remark 4.3.2, (4.3.16b) to be

$$\begin{aligned} w(t) = &\int_{-\infty}^{t} W_\varepsilon(t)W_\varepsilon(s)^{-1}(0,-2\gamma^2 v_{0,\alpha,a})\,ds \\ &-\int_{-\infty}^{t} W_\varepsilon(t)W_\varepsilon(s)^{-1}(0,-6\varepsilon^2 w^2 v_{\varepsilon,\alpha,a}-2\varepsilon^2 w^3)\,ds+o(1)\,. \end{aligned} \tag{4.3.17}$$

We note that

$$z_0(t)=(z_{01}(t),z_{02}(t))=\int_{-\infty}^{t} W_0(t)W_0(s)^{-1}(0,-2\gamma^2 v_{0,\alpha,a}(s))\,ds$$

solves

$$\begin{aligned} \dot z_{01} &= z_{02}, \\ \dot z_{02} &= -z_{01}-2\gamma^2(t)z_{01}-2\gamma^2(t)v_{0,\alpha,a}, \end{aligned} \qquad z_0(-\infty)=0$$

which is the limiting equation for $\varepsilon\to 0$ in (4.3.10b). Since $v_{0,\alpha,a}(t)=c_5(a)\sin(t+\alpha+c_4(a))$, we see that

$$z_{01}(t)=c_5(a)\,\mathrm{e}^{2t}\,\frac{\cos(t+\alpha+c_4(a))-\sin(t+\alpha+c_4(a))}{1+\mathrm{e}^{2t}}\,. \tag{4.3.18}$$

Then, with $s=\alpha+c_4(a)+\pi/4$, we have

$$\begin{aligned} \|z_0\|^2 &= \max_{t\in\mathbb{R}}\left(z_{01}(t)^2+z_{02}(t)^2\right) \\ &= \max_{t\in\mathbb{R}}\frac{2c_5(a)^2\,\mathrm{e}^{4t}}{(1+\mathrm{e}^{2t})^4}\big[1-2\sin 2(t+s)+4\cos^2(t+s) \\ &\quad +2\,\mathrm{e}^{2t}\left(1-\sin 2(t+s)\right)+\mathrm{e}^{4t}\big] \\ &\le \max_{t\in\mathbb{R}}\frac{2c_5(a)^2\,\mathrm{e}^{4t}}{(1+\mathrm{e}^{2t})^4}(7+4\,\mathrm{e}^{2t}+\mathrm{e}^{4t})=\frac{1029}{512}c_5(a)^2. \end{aligned}$$

Further, $\lim_{t\to\infty}\left(z_{01}(t)^2+z_{02}(t)^2\right)=2c_5(a)^2$ so that, finally,

$$\sqrt{2}c_5(a)\le\|z_0\|\le k_1c_5(a)$$

with $k_1=\sqrt{1029/512}\doteq 1.417662$. By using Remark 4.3.2 and (4.3.17), we have

$$\begin{aligned} \|w\| &\le \sqrt{3}\|z_0\|+6(6\|v_{\varepsilon,\alpha,a}\|\|w\|^2+2\|w\|^3)+o(1) \\ &\le \sqrt{3}k_1c_5(a)+36c_5(a)\|w\|^2+12\|w\|^3+o(1)\,. \end{aligned}$$

So if we choose $r_0>0$ so that

$$\begin{aligned} \sqrt{3}k_1c_5(a) + 36c_5(a)r_0^2 + 12r_0^3 &< r_0, \\ 72c_5(a)r_0 + 36r_0^2 &< 1, \end{aligned} \tag{4.3.19}$$

then for $\varepsilon > 0$ small and $\|w\| \le r_0$ we can uniquely solve (4.3.16) using the Banach fixed point theorem 2.2.1 on the ball

$$\left\{(u,w) \in C_B(\mathbb{R})^2 \mid \|u\| \le \tilde{K}, \quad \|w\| \le r_0\right\}$$

for a constant

$$\tilde{K} = \|K(\mathbb{I} - \Pi)\| \left(2\delta\|\dot{\gamma}\| + 2\xi\|\gamma\|[r_0 + c_5(a)]^2 + |a_3| + |a_4|\right) + 1\,.$$

To find the largest $c_5(a)$ in (4.3.19), we solve

$$\sqrt{3}k_1k_3 + 36k_2^2k_3 + 12k_2^3 = k_2\,, \quad 72k_2k_3 + 36k_2^2 = 1\,,$$

which has a solution

$$k_2 = \frac{\sqrt{-3 - 3\sqrt{3}k_1 + \sqrt{3}\sqrt{3 + 10\sqrt{3}k_1 + 9k_1^2}}}{6\sqrt{2}} \doteq 0.136179\,,$$

$$k_3 = \frac{5 + 3\sqrt{3}k_1 - \sqrt{9 + 30\sqrt{3}k_1 + 27k_1^2}}{12\sqrt{-6 - 6\sqrt{3}k_1 + 2\sqrt{9 + 30\sqrt{3}k_1 + 27k_1^2}}} \doteq 0.0339006\,.$$

So we take $r_0 = k_2$, $0 < c_5(a) < k_3$. Then (4.3.19) holds. Consequently, we have a bounded solution $w_{\alpha,a,\varepsilon} = (w_{1,\alpha,a,\varepsilon}, \dot{w}_{1,\alpha,a,\varepsilon})$ of (4.3.10b). Now we study the limit as $\varepsilon \to 0$. Let $\tilde{w}_{\alpha,a,\varepsilon}$, $\|\tilde{w}_{\alpha,a,\varepsilon}\| \le r_0$ solve

$$\begin{aligned} w(t) = &\int_{-\infty}^{t} W_\varepsilon(t)W_\varepsilon(s)^{-1}(0, -2\gamma^2 v_{0,\alpha,a})\,ds \\ &- \int_{-\infty}^{t} W_\varepsilon(t)W_\varepsilon(s)^{-1}(0, -6\varepsilon^2 w^2 v_{\varepsilon,\alpha,a} - 2\varepsilon^2 w^3)\,ds\,. \end{aligned} \tag{4.3.20}$$

We note that the right-hand side of (4.3.20), denoted $N_{\alpha,a,\varepsilon}(w)$, is a contraction on the ball $\{w \in C_B(\mathbb{R}) \mid \|w\| \le r_0\}$. So by the Banach fixed point theorem 2.2.1, $\tilde{w}_{\alpha,a,\varepsilon}$ exits and satisfies, according to (4.3.17), $\|\tilde{w}_{\alpha,a,\varepsilon} - w_{\alpha,a,\varepsilon}\| = o(1)$ as $\varepsilon \to 0$. Since $\gamma^2 \in X_{\tilde{a}}$ for some $\tilde{a} > 0$, and $N_{\alpha,a,\varepsilon} : X_{\tilde{a}} \to X_{\tilde{a}}$ is a contraction on any bounded subset, by Remark 4.3.2 we see that $\tilde{w}_{\alpha,a,\varepsilon} \to z_0$ as $\varepsilon \to 0$ in $X_{\tilde{a}}$. So $\tilde{w}_{\alpha,a,\varepsilon} \to z_0$ uniformly on $(-\infty, 0]$. Now let us fix an interval $[-n,n]$, $n \in \mathbb{N}$ and take a sequence $\{w_{\alpha,a,\varepsilon_i}\}_{i=0}^{\infty}$, $\varepsilon_i \to 0$. By the Arzelà-Ascoli theorem 2.1.3, we can suppose that $w_{\alpha,a,\varepsilon_i} \to \tilde{z}$ uniformly on $[-n,n]$. But we already know that $\tilde{z}(t) = z_0(t)$ on $[-n,0]$. Since $\tilde{z}(t)$ satisfies the same ODE on $[-n,n]$ as $z_0(t)$, we get $\tilde{z}(t) = z_0(t)$ also on $[0,n]$. These arguments imply that

$$w_{\alpha,a,\varepsilon}(t) \to z_0(t)$$

for $\varepsilon \to 0$ and uniformly in any compact interval on $\mathbb{R}$. Consequently, the limit bifurcation equation of (4.3.16c) is given by

$$M(\alpha) = \int_{-\infty}^{\infty} \Big(-2\delta\dot{\gamma}(t) - 2\xi\gamma(t)[z_{01}(t) + c_5(a)\sin(t+\alpha+c_4(a))]^2$$
$$+a_3\cos(t+\alpha) + a_4\sin(t+\alpha)\Big)\dot{\gamma}(t)\,dt = -\frac{4}{3}\delta + \pi(a_3\sin\alpha - a_4\cos\alpha)\operatorname{sech}\frac{\pi}{2} = 0\,.$$

The equation $M(\alpha) = 0$ has a simple root if and only if

$$4\delta < 3\pi\sqrt{a_3^2 + a_4^2}\operatorname{sech}\frac{\pi}{2}\,. \tag{4.3.21}$$

From (4.3.8) we derive

$$4(a_1^2 + a_2^2) = 9c_5(a)^6 + 16c_5(a)^2\,. \tag{4.3.22}$$

Since $c_5(a) < k_3$, we get

$$\sqrt{a_1^2 + a_2^2} < k_4 := \frac{\sqrt{9k_3^4 + 16}}{2}k_3 \doteq 0.0678013\,. \tag{4.3.23}$$

So if (4.3.21) holds, then we have a bounded solution for (4.3.4). Using the above method along with an approach from Section 4.1, we can show that it is a transverse homoclinic solution to a small periodic solution with appropriate shift-type dynamics. Finally, we obtain another result.

Theorem 4.3.3. *For any* $(a_1, a_2) \neq (0,0)$ *satisfying* (4.3.23) *there is a unique positive* $c_5(a)$ *solving* (4.3.22). *Then Eq.* (4.3.4) *has a transverse homoclinic solution for any* $\varepsilon > 0$ *sufficiently small with* $\mu_1 = \varepsilon^3 a_1$, $\mu_2 = \varepsilon^2$, $\mu_3 = \varepsilon^3 a_2$, $\mu_4 = \varepsilon^2 a_3$, $\mu_5 = \varepsilon^2 a_4$, *and* δ *satisfying condition* (4.3.21).

Note that if we suppose $\mu_4 = O(\varepsilon^3)$ and $\mu_5 = O(\varepsilon^3)$ in (4.3.4) then we get $M(\alpha) = \frac{4}{3}\delta$, so $M(\alpha) \neq 0$ and we do not get solutions of the desired form.

It is interesting to formulate the conditions in Theorem 4.3.3 in terms of the original parameters as they appear in (4.3.4). The equation $M(\alpha) = 0$, in place of (4.3.21), requires

$$0 < 2\delta\mu_2 < \frac{3}{2}\pi\sqrt{\mu_4^2 + \mu_5^2}\operatorname{sech}\frac{\pi}{2} \tag{4.3.24}$$

while (4.3.23) becomes

$$0 < \sqrt{\mu_1^2 + \mu_3^2} < k_4\mu_2^{3/2}\,. \tag{4.3.25}$$

The condition (4.3.24) is a restriction on the allowed damping relative to forcing in the first equation of (4.3.4). This result could be obtained by ignoring the center part of the problem, i.e. by setting $y = 0$ in the first equation of (4.3.4) and then applying classical Melnikov theory. The effect of the center manifold appears in condition (4.3.25) which imposes a limit on the magnitude of forcing relative to damping in the second equation of (4.3.4).

In this example, the hyperbolic and center parts of the analysis turn out to be separated but this is not always so. For example, if we replace $-2\xi xy^2$ in the first equation with $-2\xi \dot{x}y^2$, the Melnikov function $M(\alpha)$ acquires a contribution from the second equation. Indeed, it has now the form

$$M(\alpha) = -\frac{4}{3}\delta - \xi c_5(a)^2\left[\frac{8}{15} - \frac{2\pi}{3\sinh\pi}\sin 2(\alpha + c_4(a))\right] + \pi\,(a_3\sin\alpha - a_4\cos\alpha)\,\mathrm{sech}\,\frac{\pi}{2}.$$

By using (4.3.19) we study (4.3.16b) locally as a semilinear equation. In Section 4.3.4, we apply a global approach based on the averaging method [16] (cf Section 2.5.7) in order to study (4.3.5). This improves Theorem 4.3.3.

4.3.3 General Equations

To solve (4.3.1), we shift the time $t \longleftrightarrow t + \alpha$ and substitute

$$x = \gamma + \varepsilon\sum_{i=1}^{d-1}\beta_i u_i + \varepsilon^2 u, \quad y = \varepsilon v,$$
$$\mu_1 = \varepsilon^3\mu_{0,1}, \quad \mu_2 = \varepsilon^2\mu_{0,2}, \quad \mu_0 \neq 0$$

where $\{u_1,\dots,u_d\}$ is a basis for the vector space of bounded solutions for the linear system $\dot{u} = D_1 f_0(\gamma(t),0)u$ with $u_d = \dot{\gamma}$ and $\mu_0 = (\mu_{0,1},\mu_{0,2})$ is to be determined. We suppose $\mu_{0,2} > 0$. Introducing this change of variables into (4.3.1) yields

$$\dot{u} = D_1 f_0(\gamma,0)u + \frac{1}{2}\sum_{i,j=1}^{d-1} D_{11} f_0(\gamma,0)\beta_i\beta_j u_i u_j \tag{4.3.26a}$$
$$+\mu_{0,2} f_2(\gamma,0,0,t+\alpha) + \frac{1}{2}D_{22} f_0(\gamma,0)vv + O(\varepsilon),$$
$$\dot{v} = D_2 g_0(\gamma,0)v + \frac{\varepsilon^2}{6}D_{222} g_0(\gamma,0)v^3 \tag{4.3.26b}$$
$$+\varepsilon^2\mu_{0,1} g_1(0,0,0,t+\alpha) + \varepsilon^2\mu_{0,2} D_2 g_2(\gamma,0,0)v$$
$$+\phi_0(u,v,\varepsilon,t) + \varepsilon^2\mu_{0,1}\phi_1(u,v,\varepsilon,t) + \varepsilon^2\mu_{0,2}\phi_2(u,v,\varepsilon,t)$$

where

$$\phi_0(u,v,\varepsilon,t) = \frac{1}{\varepsilon} g_0\left(\gamma + \varepsilon \sum_{i=1}^{d-1} \beta_i u_i + \varepsilon^2 u, \varepsilon v\right) - D_2 g_0(\gamma,0)v - \frac{\varepsilon^2}{6} D_{222} g_0(\gamma,0)v^3,$$

$$\begin{aligned}\phi_1(u,v,\varepsilon,t) &= g_1\left(\gamma + \varepsilon \sum_{i=1}^{d-1} \beta_i u_i + \varepsilon^2 u, \varepsilon v, (\varepsilon^3 \mu_{0,1}, \varepsilon^2 \mu_{0,2}), t+\alpha\right) \\ &\quad - g_1(0,0,0,t+\alpha),\end{aligned}$$

$$\phi_2(u,v,\varepsilon,t) = \frac{1}{\varepsilon} g_2\left(\gamma + \varepsilon \sum_{i=1}^{d-1} \beta_i u_i + \varepsilon^2 u, \varepsilon v, (\varepsilon^3 \mu_{0,1}, \varepsilon^2 \mu_{0,2})\right) - D_2 g_2(\gamma,0,0)v.$$

We note that the functions γ and $u_i, i = 1,\ldots,d-1$ have a norm which is dominated by $\mathrm{e}^{-\tilde{a}|t|}$ for some $\tilde{a} > 0$. Using this fact and assumptions (i)–(viii) we have

$$\begin{aligned}\phi_0(u,v,\varepsilon,t) &= O(\varepsilon)\,\mathrm{e}^{-\tilde{a}|t|} + O(\varepsilon^3),\\ \phi_1(u,v,\varepsilon,t) &= O(1)\,\mathrm{e}^{-\tilde{a}|t|} + O(\varepsilon),\\ \phi_2(u,v,\varepsilon,t) &= O(\varepsilon).\end{aligned}$$

We consider the Banach spaces

$$\begin{aligned}X_n &= \left\{x \in C(\mathbb{R},\mathbb{R}^n) \;\middle|\; \sup_{t\in\mathbb{R}} |x(t)| < \infty\right\},\\ Y_n &= \Big\{y \in X_n \;\Big|\; \int_{-\infty}^{\infty} \langle y(t), v(t)\rangle\, dt = 0,\\ &\qquad \text{for every bounded solution } v \text{ to } \dot{v} = -D_1 f_0(\gamma,0)^t v\Big\}\end{aligned}$$

with the supremum norm $||x|| = \sup_{t\in\mathbb{R}} |x(t)|$. Now we recall the following results of Section 4.2.

Lemma 4.3.4. *There exist constants $b > 0$, $B > 0$ independent of ε so that given $\mu_{0,2} > 0$ the variational equation*

$$\dot{v} = \left[D_2 g_0(\gamma(t),0) + \varepsilon^2 \mu_{0,2} D_2 g_2(\gamma(t),0,0)\right] v$$

has an exponential dichotomy $(V_\varepsilon, \mathbb{I})$ on $\mathbb{R}$ with constants $(B, b\varepsilon^2 \mu_{0,2})$.

Lemma 4.3.5. *Given $h \in Y_n$, the equation $\dot{u} = D_1 f_0(\gamma(t),0)u + h$ has a unique solution $u \in X_n$ satisfying $\langle u(0), u_i(0)\rangle = 0$ for every $i = 1,2,\ldots,d$.*

Lemma 4.3.6. *There exists a continuous projection denoted $\Pi : X_n \to X_n$ so that $\mathscr{R}(\mathbb{I} - \Pi) = Y_n$.*

We define the linear map $\mathscr{K} : Y_n \to X_n$ by $\mathscr{K} h = u$ where h, u are the same as in Lemma 4.3.5. Now, we assume the following conditions:

(ix) For any $\varepsilon > 0$ small and $\alpha \in \mathbb{R}$, there is a $v_{\varepsilon,\alpha} \in X_m$ with $\dot{v}_{\varepsilon,\alpha} \in X_m$ satisfying

$$\begin{aligned}\dot{v}_{\varepsilon,\alpha}(t) = &\left(D_2 g_0(\gamma(t),0) + \varepsilon^2 \mu_{0,2} D_2 g_2(\gamma(t),0,0)\right) v_{\varepsilon,\alpha}(t)\\ &+ \frac{\varepsilon^2}{6} D_{222} g_0(0,0) v_{\varepsilon,\alpha}(t)^3 + \varepsilon^2 \mu_{0,1} g_1(0,0,0,t+\alpha)\end{aligned}$$

along with $\tilde{B} = \sup_{\varepsilon>0,\alpha} \|v_{\varepsilon,\alpha}\| < \infty$. Moreover, $v_{\varepsilon,\alpha}$ is C^1-smooth in $\varepsilon > 0$, α and $\sup_{\varepsilon>0,\alpha} \|\frac{\partial}{\partial\alpha} v_{\varepsilon,\alpha}\| < \infty$. Furthermore, there is a C^1-smooth $v_\alpha \in X_m$ so that $v_{\varepsilon,\alpha} \to v_\alpha$ and $\frac{\partial}{\partial\alpha} v_{\varepsilon,\alpha} \to \frac{\partial}{\partial\alpha} v_\alpha$ as $\varepsilon \to 0_+$ uniformly in any compact interval of $\mathbb{R}$ and uniformly for α as well.

(x) There are constants $\bar{B} > 0$, $\bar{b} > 0$ so that for any $\varepsilon > 0$ small and $\alpha \in \mathbb{R}$, the equation

$$\dot{w}(t) = \left(D_2 g_0(\gamma(t),0) + \varepsilon^2 \mu_{0,2} D_2 g_2(\gamma(t),0,0) + \frac{\varepsilon^2}{2} D_{222} g_0(0,0) v_{\varepsilon,\alpha}(t)^2\right) w(t)$$

has an exponential dichotomy $(W_\varepsilon, \mathbb{I})$ on $\mathbb{R}$ with constants $(\bar{B}, \bar{b}\varepsilon^2)$.

Let $\{v_1, v_2, \ldots, v_d\}$ be a basis of bounded solutions of $\dot{v} = -D_1 f_0(\gamma,0)^t v$. Using the projection Π and the exponential dichotomy W_ε from condition (x), we can rewrite (4.3.26), by changing $v = v_{\varepsilon,\alpha} + w$ in (4.3.26b), as the fixed point problem

$$\begin{aligned}u = &\mathscr{K}(\mathbb{I} - \Pi)\Big(\frac{1}{2}\sum_{i,j=1}^{d-1} D_{11} f_0(\gamma,0)\beta_i\beta_j u_i u_j + \mu_{0,2} f_2(\gamma,0,0,t+\alpha)\\ &+ \frac{1}{2} D_{22} f_0(\gamma,0)(v_{\varepsilon,\alpha}+w)(v_{\varepsilon,\alpha}+w) + O(\varepsilon)\Big),\end{aligned} \tag{4.3.27a}$$

$$\begin{aligned}w(t) = &\int_{-\infty}^{t} W_\varepsilon(t) W_\varepsilon(s)^{-1}\Big\{\frac{\varepsilon^2}{6}\left[D_{222} g_0(\gamma(s),0) - D_{222} g_0(0,0)\right] v_{\varepsilon,\alpha}(s)^3\\ &+ \frac{\varepsilon^2}{6} D_{222} g_0(\gamma(s),0)\left[3 v_{\varepsilon,\alpha}(s) w(s)^2 + w(s)^3\right]\\ &+ \frac{\varepsilon^2}{2}\left[D_{222} g_0(\gamma(s),0) - D_{222} g_0(0,0)\right] v_{\varepsilon,\alpha}(s)^2 w(s)\\ &+ \phi_0(u(s), v_{\varepsilon,\alpha}(s)+w(s),\varepsilon,s) + \varepsilon^2\mu_{0,1}\phi_1(u(s), v_{\varepsilon,\alpha}(s)+w(s),\varepsilon,s)\\ &+ \varepsilon^2\mu_{0,2}\phi_2(u(s), v_{\varepsilon,\alpha}(s)+w(s),\varepsilon,s)\Big\}\, ds,\end{aligned} \tag{4.3.27b}$$

$$\begin{aligned}&\int_{-\infty}^{\infty}\Big\langle v_i(t), \frac{1}{2}\sum_{i,j=1}^{d-1} D_{11} f_0(\gamma(t),0)\beta_i\beta_j u_i(t) u_j(t) + \mu_{0,2} f_2(\gamma(t),0,0,t+\alpha)\\ &\quad + \frac{1}{2} D_{22} f_0(\gamma(t),0)(v_{\varepsilon,\alpha}(t)+w(t))(v_{\varepsilon,\alpha}(t)+w(t)) + O(\varepsilon)\Big\rangle\, dt = 0,\\ &\qquad i = 1,2,\ldots,d.\end{aligned} \tag{4.3.28}$$

We note that $|\gamma(t)| \leq c\mathrm{e}^{-\tilde{a}|t|}$, $|u_i(t)| \leq c\mathrm{e}^{-\tilde{a}|t|}$, $|v_i(t)| \leq c\mathrm{e}^{-\tilde{a}|t|}$, $i = 1,2,\cdots,d$ for constants $c > 0$, $\tilde{a} > 0$. Moreover, it holds that

$$\int_{-\infty}^{t} \mathrm{e}^{-\bar{b}\varepsilon^2(t-s)}\,ds = \frac{1}{\bar{b}\varepsilon^2}, \qquad \int_{-\infty}^{t} \mathrm{e}^{-\bar{b}\varepsilon^2(t-s)-\tilde{a}|s|}\,ds \leq \int_{-\infty}^{\infty} \mathrm{e}^{-\tilde{a}|s|}\,ds = 2/\tilde{a}\,.$$

Using this we see that (4.3.27b) can be written as

$$w(t) = \frac{\varepsilon^2}{6}\int_{-\infty}^{t} W_\varepsilon(t)W_\varepsilon(s)^{-1}D_{222}g_0(0,0)\left(w(s)^3 + 3w(s)^2 v_{\varepsilon,\alpha}(s)\right)ds + O(\varepsilon)\,.$$

Using the above assumptions and the Banach fixed point theorem 2.2.1 on a ball in $X_n \times X_m$ centered at 0, (4.3.27) has a solution $(u,w) \in X_n \times X_m$ for any sufficiently small ε so that $w = O(\varepsilon)$. Substituting $w = O(\varepsilon)$ and using $v_{\varepsilon,\alpha} \to v_\alpha$, $\frac{\partial}{\partial\alpha}v_{\varepsilon,\alpha} \to \frac{\partial}{\partial\alpha}v_\alpha$ as $\varepsilon \to 0_+$ uniformly in any compact interval of $\mathbb{R}$ and uniformly for α as well we can write (4.3.28) as

$$M_i(\mu_0,\alpha,\beta) + o(1) = 0, \quad i = 1,2,\ldots,d, \tag{4.3.29}$$

where

$$M_i(\mu_0,\alpha,\beta) = \frac{1}{2}\sum_{j,k=1}^{d-1} b_{ijk}\beta_j\beta_k + a_i(\alpha)\mu_{0,2} + \frac{1}{2}\int_{-\infty}^{\infty}\langle v_i(t), D_{22}f_0(\gamma(t),0)v_\alpha(t)^2\rangle\,dt$$

and

$$a_i(\alpha) = \int_{-\infty}^{\infty}\langle v_i(t), f_2(\gamma(t),0,0,t+\alpha)\rangle\,dt, \quad 1 \leq i \leq d;$$

$$b_{ijk} = \int_{-\infty}^{\infty}\langle v_i, D_{11}f_0(\gamma,0)u_j u_k\rangle\,dt, \quad \begin{cases} 1 \leq i \leq d, \\ 1 \leq j,k \leq d-1. \end{cases}$$

We note that $v_\alpha(t)$ depends on μ_0. We put

$$M(\mu_0,\alpha,\beta) = (M_1(\mu_0,\alpha,\beta), M_2(\mu_0,\alpha,\beta),\ldots,M_d(\mu_0,\alpha,\beta))\,.$$

If we suppose (α_0,β_0) are such that $M(\mu_0,\alpha_0,\beta_0) = 0$ and $D_{(\alpha,\beta)}M(\mu_0,\alpha_0,\beta_0)$ is nonsingular then we can solve (4.3.29) by using the implicit function theorem. This gives a bounded solution of (4.3.1). As detected in Section 4.1, we can show that this solution is transversal, i.e. the linearization of (4.3.1) along that solution has an exponential dichotomy on the whole line $\mathbb{R}$. In summary, we get the following result:

Theorem 4.3.7. *Assume that conditions* (i)–(viii) *are satisfied and* (ix)–(x) *hold. If there are* (μ_0,α_0,β_0) *so that* $\mu_{0,2} > 0$, $M(\mu_0,\alpha_0,\beta_0) = 0$ *and* $D_{(\alpha,\beta)}M(\mu_0,\alpha_0,\beta_0)$ *is nonsingular, then for* $\mu_1 = \varepsilon^3\mu_{0,1}$, $\mu_2 = \varepsilon^2\mu_{0,2}$ *with* $\varepsilon > 0$ *small, Equation* (4.3.1) *has a transverse bounded solution with the appropriate shift-type irregular dynamics.*

For $t \geq s$, and using V_ε from Lemma 4.3.4, the equation in condition (x) can be rewritten as

$$w(t) = V_\varepsilon(t)V_\varepsilon(s)^{-1}w(s) + \frac{\varepsilon^2}{2}\int_s^t V_\varepsilon(t)V_\varepsilon(z)^{-1}D_{222}g_0\,(0,0)\,v_{\varepsilon,\alpha}(z)^2 w(z)\,dz.$$

This implies

$$|w(t)| \leq B\mathrm{e}^{-b\varepsilon^2\mu_{0,2}(t-s)}\,|w(s)| + \frac{\varepsilon^2}{2}B\tilde{B}^2\|D_{222}g_0(0,0)\|\int_s^t \mathrm{e}^{-b\varepsilon^2\mu_{0,2}(t-z)}\,|w(z)|\,dz$$

which implies

$$|w(t)|\,\mathrm{e}^{b\varepsilon^2\mu_{0,2}(t-s)} \leq B|w(s)| + \frac{\varepsilon^2}{2}B\tilde{B}^2\|D_{222}g_0(0,0)\|\int_s^t \mathrm{e}^{b\varepsilon^2\mu_{0,2}(z-s)}\,|w(z)|\,dz.$$

The Gronwall inequality (cf Section 2.5.1 and [11]) gives

$$|w(t)|\,\mathrm{e}^{b\varepsilon^2\mu_{0,2}(t-s)} \leq B\mathrm{e}^{\varepsilon^2 B\tilde{B}^2\|D_{222}g_0(0,0)\|(t-s)/2}\,|w(s)|.$$

Consequently, we obtain

$$|w(t)| \leq B\mathrm{e}^{\varepsilon^2(B\tilde{B}\|D_{222}g_0(0,0)\|/2 - b\mu_{0,2})(t-s)}\,|w(s)|.$$

Now we see that condition (x) holds provided that

$$\frac{B\tilde{B}^2}{2b}\|D_{222}g_0(0,0)\| < \mu_{0,2}.$$

As an application we return to (4.3.4) which we write in the form

$$\begin{aligned}\ddot{x} &= x - 2x(x^2+\xi y^2) - 2\mu_2\delta\dot{x} + a_3\mu_2\cos t + a_4\mu_2\sin t,\\ \ddot{y} &= -y - 2y(x^2+y^2) - 2\mu_2\dot{y} + a_1\mu_1\cos t + a_2\mu_1\sin t\end{aligned} \tag{4.3.30}$$

for which we use the usual first order form $x_1 = x$, $x_2 = \dot{x}$, $y_1 = y$, $y_2 = \dot{y}$. That is, we make, as at the beginning of Section 4.3.2, the substitutions $\mu_1 \to a_1\mu_1$, $\mu_2 \to \mu_2$, $\mu_3 \to a_2\mu_1$, $\mu_4 \to a_3\mu_2$, $\mu_5 \to a_4\mu_2$ for some parameters a_i, $i = 1,2,3,4$. Then (4.3.30) becomes

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= x_1 - 2x_1(x_1^2+\xi y_1^2) + \mu_2\Big(-2\delta x_2 + a_3\cos t + a_4\sin t\Big),\\ \dot{y}_1 &= y_2,\\ \dot{y}_2 &= -y_1 - 2y_1(x_1^2+y_1^2) - 2\mu_2 y_2 + \mu_1\Big(a_1\cos t + a_4\sin t\Big)\end{aligned}$$

which is clearly in the form of (4.3.1). We now check the hypotheses of Theorem 4.3.7 for (4.3.30). Conditions (i)–(viii) are easily verified.

In (ix) we write $v_{\varepsilon,\alpha} = (v, \dot v)$ and then obtain

$$\ddot v + (1+2\gamma^2)v + 2\varepsilon^2\mu_{0,2}\dot v + 2\varepsilon^2 v^3 = \varepsilon^2\mu_{0,1}[a_1\cos(t+\alpha) + a_2\sin(t+\alpha)]. \quad (4.3.31)$$

Note that this is the second equation in (4.3.5) when $u = 0$ and $\mu_{0,1} = \mu_{0,2} = 1$.

Setting $\mu_{0,1} = \mu_{0,2} = 1$ (since we already have parameters a_i), using the solution $v_{\varepsilon,\alpha,a}(t)$ of (4.3.6) and substituting $v(t) = w(t) + v_{\varepsilon,\alpha,a}(t)$ into (4.3.31), we get

$$\ddot w + (1+6\varepsilon^2 v_{\varepsilon,\alpha,a}^2 + 2\gamma^2)w + 2\varepsilon^2\dot w + 2\gamma^2 v_{\varepsilon,\alpha,a} + 6\varepsilon^2 w^2 v_{\varepsilon,\alpha,a} + 2\varepsilon^2 w^3 = 0 \quad (4.3.32)$$

which is (4.3.10b) when $u = 0$. Equation (4.3.32) can be rewritten as (4.3.16b) with $u = 0$ and then as (4.3.17). Taking $0 < c_5(a) < k_3$ the conditions of (4.3.19) are satisfied and we obtain the unique solvability of (4.3.32) with solution $w_{\varepsilon,\alpha,a}(t)$ satisfying $\|w_{\varepsilon,\alpha,a}\| \le r_0$. Consequently, condition (ix) is verified for (4.3.30) with $v_{\varepsilon,\alpha} = (v, \dot v)$ and $v_\alpha = (\tilde v, \dot{\tilde v})$ where

$$\begin{aligned} v(t) &= v_{\varepsilon,\alpha,a}(t) + w_{\varepsilon,\alpha,a}(t), \\ \tilde v(t) &= a_1\cos(t+\alpha) + a_2\sin(t+\alpha) + z_{01}(t). \end{aligned}$$

Concerning condition (x), we see that the equation from this condition has the form

$$\ddot w_1 + (1+2\gamma^2+6\varepsilon^2 v^2)w_1 + 2\varepsilon^2\mu_{0,2}\dot w_1 = 0$$

with $w_2 = \dot w_1$. Again using $\mu_{0,1} = \mu_{0,2} = 1$ and substituting for v we get

$$\ddot w_1 + (1+2\gamma^2+6\varepsilon^2 v_{\varepsilon,\alpha,a}^2)w_1 + 2\varepsilon^2\dot w_1 + 6\varepsilon^2(2v_{\varepsilon,\alpha,a}w_{\varepsilon,\alpha,a} + w_{\varepsilon,\alpha,a}^2)w_1 = 0,$$

which for $t \ge s$ has the form

$$\begin{aligned} w(t) = W_{\varepsilon,\alpha,a}(t)W_{\varepsilon,\alpha,a}(s)^{-1}w(s) - 6\varepsilon^2\int_{-\infty}^t W_{\varepsilon,\alpha,a}(t)W_{\varepsilon,\alpha,a}(s)^{-1}\cdot \\ \left\{\Big(0, (2v_{\varepsilon,\alpha,a}(z)w_{\varepsilon,\alpha,a}(z) + w_{\varepsilon,\alpha,a}(z)^2)w_1(z)\Big)\right\}dz. \end{aligned} \quad (4.3.33)$$

Since $\|v_{\varepsilon,\alpha,a}\| \le c_5(a) + O(\varepsilon)$ and $\|w_{\varepsilon,\alpha,a}\| \le r_0$, we get

$$\left|\Big(0, -6\varepsilon^2(2v_{\varepsilon,\alpha,a}(s)w_{\varepsilon,\alpha,a}(s) + w_{\varepsilon,\alpha,a}(s)^2)\Big)\right| \le \varepsilon^2\theta_\varepsilon,$$

for a constant

$$\theta_\varepsilon = 6(2c_5(a)r_0 + r_0^2) + O(\varepsilon).$$

From (4.3.33) we obtain

$$|w(t)| \le K_1 \mathrm{e}^{-\varepsilon^2(t-s)}|w(s)| + K_1\varepsilon^2\theta_\varepsilon\int_s^t \mathrm{e}^{-\varepsilon^2(t-z)}|w(z)|\,dz$$

which gives

$$|w(t)|\,\mathrm{e}^{\varepsilon^2(t-s)} \le K_1|w(s)| + K_1\varepsilon^2\theta_\varepsilon \int_s^t \mathrm{e}^{\varepsilon^2(z-s)}\,|w(z)|\,dz\,.$$

The Gronwall inequality again implies

$$|w(t)|\,\mathrm{e}^{\varepsilon^2(t-s)} \le K_1|w(s)|\,\mathrm{e}^{K_1\varepsilon^2\theta_\varepsilon(t-s)}\,.$$

Since $c_5(a) < k_3$, we see that $K_1\theta_0 < 1$ and then

$$|w(t)| \le K_1\,\mathrm{e}^{\varepsilon^2(K_1\theta_0-1)(t-s)/2}\,|w(s)|\,.$$

Hence we see that condition (x) is satisfied with $\bar{B} = K_1$ and $\bar{b} = (1-K_1\theta_0)/2$.

In summary, conditions (ix) and (x) are satisfied for (4.3.4).

Remark 4.3.8. The role of resonance is not clear in this section. But it is essential and it is hidden in assumptions (ix) and (x). For simplicity, we explain it again for example (4.3.4) by replacing the forcing terms $\cos t$, $\sin t$ with $\cos\pi t$, $\sin\pi t$, respectively. So we consider the equations

$$\begin{aligned}\ddot{x} &= x - 2x(x^2+\xi y^2) - 2\mu_2\delta\dot{x} + \mu_4\cos\pi(t+\alpha) + \mu_5\sin\pi(t+\alpha)\,,\\ \ddot{y} &= -y - 2y(x^2+y^2) - 2\mu_2\dot{y} + \mu_1\cos\pi(t+\alpha) + \mu_3\sin\pi(t+\alpha)\,.\end{aligned} \tag{4.3.34}$$

Certainly, the linear part of the second equation in (4.3.34) is nonresonant. Then in place of (4.3.6), we get

$$\ddot{\tilde{v}}_{\varepsilon,\alpha,a} = -\tilde{v}_{\varepsilon,\alpha,a} - 2\varepsilon^2\dot{\tilde{v}}_{\varepsilon,\alpha,a} - 2\varepsilon^2\tilde{v}^3_{\varepsilon,\alpha,a} + \varepsilon^2 a_1\cos\pi(t+\alpha) + \varepsilon^2 a_2\sin\pi(t+\alpha)\,.$$

Applying the method of Section 4.3.2, we obtain $\tilde{v}_{\varepsilon,\alpha,a}(t) = O(\varepsilon^2)$ and $\tilde{v}_{\varepsilon,\alpha,a}(t)$ is 2-period. Then (4.3.16b) gives $\tilde{w}_{\varepsilon,\alpha,a}(t) = O(\varepsilon^2)$ without any further restriction, i.e. a_1, a_2 are arbitrary nonzero. Consequently, the corresponding Melnikov function is independent of a_1, a_2. So the hyperbolic and center parts of (4.3.34) are always separated. This is consistent with the method in Section 4.2 for the nonresonant case. In summary, in the nonresonant case, the forcing terms in the center part do not affect the Melnikov function, while in the resonant case the forcing terms in center part do affect it in general.

4.3.4 Averaging Method

When Eq. (4.3.1) satisfies conditions (i)–(viii) the remaining task is to verify conditions (ix) and (x). We note that the equation in (x) is just the linearization of equation (ix) along $v_{\varepsilon,\alpha}(t)$. Consequently, we must study the equation of (ix) and its linearization. For this purpose, we can use also the method of averaging [16] (cf Section 2.5.7). As a concrete illustration of how this can be done we focus on (4.3.31). Using the matrix $C(t)$ from Section 4.3.2, we put

$$v(t) = c_1(t)v_1(t) + c_2(t)v_2(t),$$
$$v_1(t) = \cos t - \sin t \tanh t, \quad v_2(t) = \sin t + \cos t \tanh t$$

into (4.3.31) and set $\mu_{0,1} = \mu_{0,2} = 1$. We get the system

$$\begin{aligned} \dot{c}_1 &= \varepsilon^2 \Big[c_1\dot{v}_1(t) + c_2\dot{v}_2(t) + (c_1v_1(t) + c_2v_2(t))^3 \\ &\quad - \frac{a_1}{2}\cos(t+\alpha) - \frac{a_2}{2}\sin(t+\alpha) \Big] v_2(t), \\ \dot{c}_2 &= \varepsilon^2 \Big[-c_1\dot{v}_1(t) - c_2\dot{v}_2(t) - (c_1v_1(t) + c_2v_2(t))^3 \\ &\quad + \frac{a_1}{2}\cos(t+\alpha) + \frac{a_2}{2}\sin t + \alpha) \Big] v_1(t), \end{aligned} \tag{4.3.35}$$

where as usual we put $\dot{v}(t) = c_1(t)\dot{v}_1(t) + c_2(t)\dot{v}_2(t)$. Now we see that

$$v_i(t) \to v_{i,\pm}(t), \quad i = 1,2,$$

being exponentially fast as $t \to \pm\infty$ where

$$v_{1,\pm} = \cos t \mp \sin t, \quad v_{2,\pm}(t) = \sin t \pm \cos t\,.$$

Consequently, Equation (4.3.35) for $t \geq 0$ has the form

$$\begin{aligned} \dot{c}_1 &= \varepsilon^2 \Big\{ \Big(c_1\dot{v}_{1,+}(t) + c_2\dot{v}_{2,+}(t) + (c_1v_{1,+}(t) + c_2v_{2,+}(t))^3 \\ &\quad - \frac{a_1}{2}\cos(t+\alpha) - \frac{a_2}{2}\sin(t+\alpha) \Big) v_{2,+}(t) + h_+^1(c_1,c_2,\alpha,t) \Big\}, \\ \dot{c}_2 &= \varepsilon^2 \Big\{ \Big(- c_1\dot{v}_{1,+}(t) - c_2\dot{v}_{2,+}(t) - (c_1v_{1,+}(t) + c_2v_{2,+}(t))^3 \\ &\quad + \frac{a_1}{2}\cos(t+\alpha) + \frac{a_2}{2}\sin(t+\alpha) \Big) v_{1,+}(t) + h_+^2(c_1,c_2,\alpha,t) \Big\} \end{aligned} \tag{4.3.36}$$

while Eq. (4.3.35) for $t \leq 0$ has the form

$$\begin{aligned} \dot{c}_1 &= \varepsilon^2 \Big\{ \Big(c_1\dot{v}_{1,-}(t) + c_2\dot{v}_{2,-}(t) + (c_1v_{1,-}(t) + c_2v_{2,-}(t))^3 \\ &\quad - \frac{a_1}{2}\cos(t+\alpha) - \frac{a_2}{2}\sin(t+\alpha) \Big) v_{2,-}(t) + h_-^1(c_1,c_2,\alpha,t) \Big\}, \\ \dot{c}_2 &= \varepsilon^2 \Big\{ \Big(- c_1\dot{v}_{1,-}(t) - c_2\dot{v}_{2,-}(t) - (c_1v_{1,-}(t) + c_2v_{2,-}(t))^3 \\ &\quad + \frac{a_1}{2}\cos(t+\alpha) + \frac{a_2}{2}\sin(t+\alpha) \Big) v_{1,-}(t) + h_-^2(c_1,c_2,\alpha,t) \Big\} \end{aligned} \tag{4.3.37}$$

where $h_\pm^{1,2}(c_1,c_2,\alpha,t) \to 0$, being exponentially fast for $t \to \pm\infty$ and uniformly for $c_{1,2}$ on a bounded set. Now we average Eqs. (4.3.36) and (4.3.37) over $\mathbb{R}_\pm$, respectively, to get for $t \geq 0$ the system

$$
\begin{aligned}
\dot c_1 &= \frac{\varepsilon^2}{4}\Big(-4c_1+6c_1^2c_2+6c_2^3-(a_1+a_2)\cos\alpha+(a_1-a_2)\sin\alpha\Big),\\
\dot c_2 &= \frac{\varepsilon^2}{4}\Big(-4c_2-6c_1c_2^2-6c_1^3+(a_1-a_2)\cos\alpha+(a_1+a_2)\sin\alpha\Big),
\end{aligned}
\tag{4.3.38}
$$

while for $t \le 0$ we obtain the system

$$
\begin{aligned}
\dot c_1 &= \frac{\varepsilon^2}{4}\Big(-4c_1+6c_1^2c_2+6c_2^3+(a_1-a_2)\cos\alpha+(a_1+a_2)\sin\alpha\Big),\\
\dot c_2 &= \frac{\varepsilon^2}{4}\Big(-4c_2-6c_1c_2^2-6c_1^3+(a_1+a_2)\cos\alpha+(a_2-a_1)\sin\alpha\Big).
\end{aligned}
\tag{4.3.39}
$$

We put

$$
\begin{aligned}
A_{1,+} &= -(a_1+a_2)\cos\alpha+(a_1-a_2)\sin\alpha\,,\\
A_{2,+} &= (a_1-a_2)\cos\alpha+(a_1+a_2)\sin\alpha\,,\\
A_{1,-} &= (a_1-a_2)\cos\alpha+(a_1+a_2)\sin\alpha\,,\\
A_{2,-} &= (a_1+a_2)\cos\alpha+(a_2-a_1)\sin\alpha\,.
\end{aligned}
$$

The systems (4.3.38) and (4.3.39) form one system over $\mathbb{R}$ with a discontinuity at $t=0$. By using arguments of Section 4.3.2 (see (4.3.8)), we observe that the systems

$$
\begin{aligned}
-4c_1+6c_1^2c_2+6c_2^3+A_{1,\pm} &= 0\,,\\
-4c_2-6c_1c_2^2-6c_1^3+A_{2,\pm} &= 0
\end{aligned}
\tag{4.3.40}
$$

have unique solutions

$$
c_{a,\pm} = (c_{1,a,\pm}, c_{2,a,\pm})\,.
$$

Moreover, the eigenvalues of the linearization of (4.3.38), (4.3.39) at $c_{a,\pm}$ are

$$
[-4 \pm \imath 6\sqrt{3}(c_{1,a,\pm}^2+c_{2,a,\pm}^2)]\varepsilon^2/4\,.
$$

Consequently, we see that systems (4.3.38), (4.3.39) have unique weakly exponentially attracting equilibria $c_{a,\pm}$, respectively.

Note that for $a=0$ we get $c_{0,\pm}=0$ and then from (4.3.31) $v_{\varepsilon,\alpha}=0$ so the case $a=0$ is trivial. On the other hand, we need $v_{\varepsilon,\alpha}\neq 0$ for the influence of the center part to affect the Melnikov function. For this reason, we assume that $a \neq 0$.

Now if the point $c_{a,-}$ is in the basin of attraction of $c_{a,+}$, then we can construct a solution $c_a(t)$ of (4.3.38), (4.3.39) over $\mathbb{R}$ as follows:

$$
c_a(t) = \begin{cases} c_{a,-}, & \text{for} \quad t \le 0\,,\\ \text{the solution of (4.3.38) starting from } c_{a,-} & \text{for} \quad t \ge 0\,. \end{cases}
$$

This solution will generate, according to averaging theory [16] (cf Theorems 2.5.12, 2.5.13), a solution of (4.3.31) satisfying conditions (ix) and (x). We note that averaging theory can be applied to (4.3.36) and (4.3.37) since they are sums of periodic and exponentially fast decaying terms containing t variable. So (4.3.36) and (4.3.37) are KBM-vector fields.

To show that $c_{a,-}$ is in the basin of attraction of $c_{a,+}$ consider the function $H : \mathbb{R}^2 \to \mathbb{R}$ given by

$$H(c_1, c_2) = 3(c_1^2 + c_2^2)^2 - 2A_{2,+}c_1 + 2A_{1,+}c_2.$$

For further reference we note that

$$H(c_1, c_2) \leq 3(c_1^2 + c_2^2)^2 + 2\sqrt{A_{1,+}^2 + A_{2,+}^2}\sqrt{c_1^2 + c_2^2}, \tag{4.3.41}$$

and if $t \to (c_1(t), c_2(t))$ is a solution of (4.3.38),

$$\begin{aligned} \frac{d}{dt}H(c_1(t), c_2(t)) &= -2\varepsilon^2 \left[6(c_1^2 + c_2^2)^2 - A_{2,+}c_1 + A_{1,+}c_2\right] \\ &\leq -2\varepsilon^2 \sqrt{c_1^2 + c_2^2} \left[6(c_1^2 + c_2^2)^{3/2} - \sqrt{A_{1,+}^2 + A_{2,+}^2}\right]. \end{aligned} \tag{4.3.42}$$

We define two sets

$$\begin{aligned} D &= \left\{(c_1, c_2) \,\middle|\, c_1^2 + c_2^2 < (A_{1,+}^2 + A_{2,+}^2)^{1/3}\right\}, \\ U &= \left\{(c_1, c_2) \,\middle|\, H(c_1, c_2) < 5(A_{1,+}^2 + A_{2,+}^2)^{2/3}\right\}. \end{aligned}$$

Using (4.3.41) it is easy to verify that $D \subset U$. With (4.3.40) we obtain

$$\sqrt{A_{1,+}^2 + A_{2,+}^2}\sqrt{c_{1,a,+}^2 + c_{2,a,+}^2} \geq A_{2,+}c_{1,a,+} - A_{1,+}c_{2,a,+} = 6(c_{1,a,+}^2 + c_{2,a,+}^2)^2$$

from which it follows that $|c_{a,+}|^2 \leq (\frac{1}{6})^{2/3}(A_{1,+}^2 + A_{2,+}^2)^{1/3}$ so that $c_{a,+} \in U$.

If $t \to (c_1(t), c_2(t))$ is an orbit of (4.3.38) in the complement of $\bar{U}$ then

$$c_1(t)^2 + c_2(t)^2 \geq (A_{1,+}^2 + A_{2,+}^2)^{1/3}$$

and it follows from (4.3.42) that

$$\frac{d}{dt}H(c_1(t), c_2(t)) \leq -10\varepsilon^2(A_{1,+}^2 + A_{2,+}^2)^{2/3}.$$

Thus, $\bar{U}$ is an invariant global attractor. Since the divergence of (4.3.38) is $-2\varepsilon^2$, using Bendixson's criterion 2.5.10, we see that U contains no periodic orbits. Thus by the Poincarè-Bendixson theorem 2.5.9, U is in the basin of attraction for $c_{a,+}$, $c_{a,+}$ is a global attractor and, trivially, $c_{a,-}$ is in the basin of attraction of $c_{a,+}$.

In summary, we get the Melnikov function $M(\alpha)$ of Section 4.3.2 so that Theorem 4.3.3 holds for any $(a_1, a_2) \neq (0, 0)$ and we have the following improvement of Theorem 4.3.3.

Theorem 4.3.9. *Equation* (4.3.4) *has a transverse homoclinic solution for any* ξ, *and any small* μ_i, $i = 1,\dots,5$ *and* δ *satisfying condition* (4.3.24) *and* $(\mu_1,\mu_3) \neq (0,0)$.

Finally, we note that in spite of the fact that the results of Section 4.3.2 are improved in this section, that part is included here since it contains some useful derivations/computations such as the existence of periodic solutions and exponential dichotomies. We note that for general forms of coupled oscillators only local analysis as in Section 4.3.2 can be used to verify assumptions (ix) and (x). As our averaging technique uses the Poincarè-Bendixson theorem and Bendixson's criterion it cannot be used for higher-dimensional systems. In general, the situation depends on the form of the averaged equations.

4.4 Singularly Perturbed and Forced ODEs

4.4.1 Forced Singular ODEs

Consider a singular system of ODEs like

$$\begin{aligned} \varepsilon u' &= f(u,v) + \varepsilon h_1(t,u,v,\varepsilon), \quad u \in \mathbb{R}^n, \quad v \in \mathbb{R}^m, \\ v' &= g(u,v) + \varepsilon h_2(t,u,v,\varepsilon), \quad t \in \mathbb{R}, \quad \varepsilon \in \mathbb{R}, \end{aligned} \tag{4.4.1}$$

under the following conditions:

(a) f,g,h_1,h_2 are C_b^{r+1}-functions in their arguments, $r \geq 2$, defined for $(t,u,v,\varepsilon) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m\times(-\bar\varepsilon,\bar\varepsilon)$ and their $(r+1)$-derivatives are continuous in u uniformly with respect to (t,v,ε).
(b) $f(0,v) = 0$ for any $v \in \mathbb{R}^m$ and there exists $\delta > 0$ so that for any $v \in \mathbb{R}^m$ and $\lambda(v) \in \sigma(f_x(0,v))$ one has $|\Re\lambda(v)| > \delta > 0$.

Then setting $\varepsilon = 0$ in Eq. (4.4.1) we obtain the so-called *degenerate system*

$$v' = g(0,v), \quad v \in \mathbb{R}^m. \tag{4.4.2}$$

It was shown in [17] that given $T > 0$ the solutions of (4.4.1) are at a $O(\varepsilon)$-distance from the corresponding solutions of (4.4.2), for t in any compact subset of $(0,T]$. This result was improved in [18] leading to a condition similar to the above one about the eigenvalues of $f_u(0,v)$ [19]. Later, a geometric theory of singular systems was developed in [20]. This theory applies to the autonomous case and states, under certain hypotheses, the existence of a *center manifold* for (4.4.1) defined on compact subsets of $\mathbb{R}^m$ on which system (4.4.1) is a regular perturbation of the degenerate system (4.4.2). By means of this theory, a previous result given in [21] was improved in [20], concerning the existence of periodic solutions of (4.4.1). Afterwards geometric theory is used in [22, 23] to study the problem of bifurcation from a heteroclinic orbit of the degenerate system towards a heteroclinic orbit of the overall system (4.4.1). However, since the result of [20] holds in the autonomous case and with some roughness assumptions on system (4.4.2), conclusions in [22,23] are given just in the case of a *transverse heteroclinic orbit*. Later, using different methods, the non-autonomous case together with the homoclinic case have been handled in [24,25]. A result in [25], however, does not contain any conclusion of the smoothness of the bifurcating heteroclinic orbit with respect to the parameter ε, while four classes of differentiability (from C^{r+2} to C^{r-2}) are lost in [24]. Let us mention some related results in this direction. Attractive invariant manifolds of (4.4.1) are studied in [26] when h_1, h_2 are independent of t and $f_u(0,v)$ has all the eigenvalues with negative real parts. The same problem as in [26] is investigated in [27] when h_1, h_2 do depend on t.

4.4.2 Center Manifold Reduction

In this section we apply Theorem 2.5.8 to (4.4.1). Let $\tau = t/\varepsilon$ be the *fast time* and $\dot{}$ denote the derivative with respect to τ. Then (4.4.1) reads:

$$\begin{aligned} \dot{u} &= f(u,v) + \varepsilon h_1(t,u,v,\varepsilon), \\ \dot{v} &= \varepsilon\{g(u,v) + \varepsilon h_2(t,u,v,\varepsilon)\}, \\ \dot{t} &= \varepsilon. \end{aligned} \tag{4.4.3}$$

Take a C^∞-function $\phi : \mathbb{R} \to [0,\bar{\varepsilon}]$ so that $\phi(\varepsilon) = \bar{\varepsilon}$ for $\varepsilon \in (-\frac{\bar{\varepsilon}}{3}, \frac{\bar{\varepsilon}}{3})$, $|\frac{d\phi}{d\varepsilon}| < 2$ and $\operatorname{supp}\phi \subset [-\bar{\varepsilon},\bar{\varepsilon}]$. It is clear that $\phi \in C_b^{r+1}(\mathbb{R},\mathbb{R})$ since it has a compact support. Then, define $x = u, y = (v,t,\varepsilon\phi(\varepsilon))$ and consider, instead of (4.4.3), the following system

$$\begin{aligned} \dot{x} &= f_u(0,v)x + F(x,y) := A(y)x + F(x,y), \\ \dot{y} &= G(x,y), \end{aligned} \tag{4.4.4}$$

where

$$F(x,y) = F(x,(v,t,\varepsilon)) = f(x,v) - f_u(0,v)x + \varepsilon\phi(\varepsilon)h_1(t,x,v,\varepsilon\phi(\varepsilon)),$$

$$G(x,y) = G(x,(v,t,\varepsilon)) = \varepsilon\phi(\varepsilon)(g(x,v) + \varepsilon\phi(\varepsilon)h_2(t,x,v,\varepsilon\phi(\varepsilon)),1,0).$$

From the fact that the support of $\phi(\varepsilon)$ is a subset of $[-\bar{\varepsilon},\bar{\varepsilon}]$, it follows that $A(y)$, $F(x,y)$, $G(x,y)$ can be considered as C_b^r-functions in $(x,y) \in \mathbb{R}^n \times \mathbb{R}^{m+2}$ and that they satisfy the hypothesis (i) of Section 2.5.5. Moreover one has

$$|F(0,y)| + |F_x(0,y)| \le C|\varepsilon\phi(\varepsilon)| \le C\bar{\varepsilon}^2 < \sigma$$

provided $\bar{\varepsilon} \ll 1$. In the same way we see that $|G(x,y)|, |G_x(x,y)| < \sigma$. As regards the inequality $|G_y(x,y)| < \sigma$, this follows also from the fact that $\sup_{\varepsilon\in\mathbb{R}} |\frac{d}{d\varepsilon}[\varepsilon\phi(\varepsilon)]|$

$\leq \sup_{|\varepsilon|\leq\bar{\varepsilon}} |\varepsilon\phi'(\varepsilon)| + |\phi(\varepsilon)| \leq 3\bar{\varepsilon}$. All the hypotheses of Theorem 2.5.8 are then satisfied and hence the existence of a *global center manifold* for (4.4.4), satisfying the conclusions of Theorem 2.5.8, follows. This center manifold can be represented as:

$$\mathscr{C} = \Big\{(\xi,\eta,\alpha,\varepsilon) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times (-\bar{\varepsilon},\bar{\varepsilon}) \mid \xi = H(\eta,\alpha,\varepsilon)\Big\}$$

and is invariant under the flow given by (4.4.4). From $\frac{d\varepsilon}{d\tau} = 0$ we obtain that ε is constant, moreover, since any $\varepsilon \in (-\frac{\bar{\varepsilon}^2}{3}, \frac{\bar{\varepsilon}^2}{3})$ can be written as $\frac{\varepsilon}{\bar{\varepsilon}}\phi\left(\frac{\varepsilon}{\bar{\varepsilon}}\right)$, we see that for $|\varepsilon| < \varepsilon_0 = \frac{\bar{\varepsilon}^2}{3}$, such a manifold is invariant for (4.4.3). Any solution of (4.4.3) whose u-component is small must then satisfy (see property (P) of Theorem 2.5.8):

$$u(\tau) = H(y(\tau,\eta,\alpha,\varepsilon)),$$

where $y(\tau,\eta,\alpha,\varepsilon) = (v(\tau,\eta,\alpha,\varepsilon), \varepsilon\tau + \alpha, \varepsilon)$ and $v(\tau) = v(\tau,\eta,\alpha,\varepsilon)$ satisfies

$$\dot{v}(\tau) = \varepsilon\{g(H(v(\tau),\varepsilon\tau+\alpha,\varepsilon),v(\tau)) + \varepsilon h_2(\varepsilon\tau+\alpha, H(v(\tau),\varepsilon\tau+\alpha,\varepsilon),v(\tau),\varepsilon)\}$$

so that $\widetilde{v}(t) = v(t/\varepsilon)$ satisfying

$$\widetilde{v}'(t) = g(H(\widetilde{v}(t),t+\alpha,\varepsilon),\widetilde{v}(t)) + \varepsilon h_2(t+\alpha, H(\widetilde{v}(t),t+\alpha,\varepsilon),\widetilde{v}(t),\varepsilon). \quad (4.4.5)$$

Finally, note that $H(\eta,\alpha,0) = 0$ because of uniqueness. We have then shown the following.

Theorem 4.4.1. *Consider system* (4.4.1) *and assume (a) and (b) hold. Then there exist* $\varepsilon_0, \rho > 0$ *and a* C^r*-function* $H : \mathbb{R}^m \times \mathbb{R} \times (-\varepsilon_0,\varepsilon_0) \to \mathbb{R}$ *so that the following properties hold:*

(i) $\sup_{(\eta,\alpha,\varepsilon)\in\mathbb{R}^m\times\mathbb{R}\times(-\varepsilon_0,\varepsilon_0)} |H(\eta,\alpha,\varepsilon)| \leq \rho.$

(ii) *For any* $\varepsilon \in (-\varepsilon_0,\varepsilon_0)$ *and* $\alpha \in \mathbb{R}$ *the manifold*

$$\mathscr{C}_{\alpha,\varepsilon} = \Big\{(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^m \mid \xi = H(\eta,\alpha,\varepsilon)\Big\}$$

is invariant for the flow of system (4.4.1)*, with* $t+\alpha$ *instead of* t*, in the sense that if* $(u(\alpha),v(\alpha)) \in \mathscr{C}_{\alpha,\varepsilon}$ *then* $(u(t),v(t)) \in \mathscr{C}_{\alpha,\varepsilon}$ *for any* $t \in \mathbb{R}$.

(iii) *Any solution* $(u(t),v(t))$ *of* (4.4.1)*, with* $t+\alpha$ *instead of* t*, showing that* $||u||_\infty < \rho$*, belongs to* $\mathscr{C}_{\alpha,\varepsilon}$.

As an example of application of this result assume that

(c) The degenerate system (4.4.2) has an orbit $\gamma(t)$ homoclinic to a hyperbolic equilibrium, and the variational system $\dot{v} = g_v(0,\gamma(t))v$ has the unique bounded solution $\dot{\gamma}(t)$ (up to a multiplicative constant).

Then the following theorem holds:

Theorem 4.4.2. *Assume (a), (b), (c) and define*

$$\Delta(\alpha) = \int_{-\infty}^{+\infty} \psi^*(t)\left\{ h_2(t+\alpha,0,\gamma(t),0) - g_u(0,\gamma(t))f_u(0,\gamma(t))^{-1}h_1(t+\alpha,0,\gamma(t),0)\right\} dt$$

with $\psi^*(t)$ *being the unique (up to a multiplicative constant) bounded solution to the adjoint variational system* $\dot{v} = -g_v(0,\gamma(t))^* v$. *Then, if* $\Delta(\alpha)$ *has a simple zero at* $\alpha = \alpha_0$, *there exist* $\rho > 0$, $\varepsilon_0 > 0$ *so that for* $|\varepsilon| < \varepsilon_0$, *system* (4.4.1) *has a unique solution* $(u(t,\varepsilon), v(t,\varepsilon))$ *which is* C^{r-1} *with respect to* ε, *bounded together with its derivatives (in* ε*), and satisfying also*:

$$|u(t,\varepsilon)| < \rho \text{ and } \sup_{t\in\mathbb{R}} |u(t,\varepsilon)| + |v(t,\varepsilon) - \gamma(t-\alpha_0)| \to 0 \text{ as } \varepsilon \to 0. \tag{4.4.6}$$

Proof. A solution satisfying (4.4.6) must lie in a manifold $\mathscr{C}_{\alpha,\varepsilon}$ owing to property (iii) of Theorem 4.4.1, hence its v-component must satisfy (4.4.5). The unperturbed system of (4.4.5) is the degenerate system (4.4.2). From regular perturbation theory (see Section 4.1) we obtain the Melnikov function

$$M(\alpha) = \int_{-\infty}^{+\infty} \psi^*(t)\left\{h_2(t+\alpha,0,\gamma(t),0) + g_u(0,\gamma(t))H_\varepsilon(\gamma(t),t+\alpha,0)\right\} dt.$$

Taking the derivative with respect to ε at $\varepsilon = 0$ of

$$\begin{aligned}\varepsilon\frac{\partial}{\partial t}H\ &(v(t,\eta_0,\alpha,\varepsilon),t+\alpha,\varepsilon)\\ &= f(H(v(t,\eta_0,\alpha,\varepsilon),t+\alpha,\varepsilon),v(t,\eta_0,\alpha,\varepsilon))\\ &\quad +\varepsilon h_1(t+\alpha,H(v(t,\eta_0,\alpha,\varepsilon),t+\alpha,\varepsilon),v(t,\eta_0,\alpha,\varepsilon),\varepsilon),\end{aligned}$$

we get (recall $H(\eta,\alpha,0) = 0$)

$$f_u(0,v(t,\eta_0,\alpha,0))H_\varepsilon(v(t,\eta_0,\alpha,0),t+\alpha,0) + h_1(t+\alpha,0,v(t,\eta_0,\alpha,0),0) = 0. \tag{4.4.7}$$

Now $v(t,\gamma(\alpha),\alpha,0)$ solves (4.4.2) with the condition $v(0) = \gamma(\alpha)$, as a consequence $v(t,\gamma(\alpha),\alpha,0) = \gamma(t)$ and using (4.4.7) we obtain:

$$H_\varepsilon(\gamma(t),t+\alpha,0) = -f_u(0,\gamma(t))^{-1}h_1(t+\alpha,0,\gamma(t),0)\} dt$$

and hence $M(\alpha) = \Delta(\alpha)$. □

Remark 4.4.3. From regular perturbation theory, it follows that the solution, whose existence is stated in Theorem 4.4.2, is C^{r-1} in ε. This improves previous results [24, 25].

As another application of Theorem 4.4.1, the degenerate system (4.4.2) has an orbit heteroclinic to semi-hyperbolic equilibria, but we do not go into details and we refer the readers to [28].

4.4.3 ODEs with Normal and Slow Variables

Only for the reader information, we note in this part an opposite case to (4.4.1) by considering a system

$$\begin{aligned} \dot{x} &= f(x,y) + \varepsilon h(x,y,t,\varepsilon)\,, \\ \dot{y} &= \varepsilon\,(Ay + g(y) + p(x,y,t,\varepsilon) + \varepsilon q(y,t,\varepsilon))\,, \end{aligned} \tag{4.4.8}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\varepsilon > 0$ is sufficiently small, A is an $m \times m$ matrix, and all mappings are smooth, 1-periodic in the time variable $t \in \mathbb{R}$ so that

(i) $f(0,0) = 0$, $g(0) = 0$, $g_x(0) = 0$, $p(0,\cdot,\cdot,\cdot) = 0$.
(ii) The eigenvalues of A and $f_x(0,0)$ lie off the imaginary axis.
(iii) There is a homoclinic solution $\gamma \neq 0$ so that $\lim_{t\to\pm\infty} \gamma(t) = 0$ and $\dot{\gamma}(t) = f(\gamma(t),0)$.

Here g_x, f_x mean derivatives of g and f with respect to x, respectively. The second equation of (4.4.8) has the usual canonical form of the averaging theory (cf Section 2.5.7) in the variable y with $x = 0$, and it is assumed [29] that its averaged equation with $x = 0$ possesses a hyperbolic equilibrium. Hence the homoclinic dynamics of the first equation of (4.4.8) is combined with the dynamics near the slow hyperbolic equilibrium of the averaged second equation of (4.4.8) when $x = 0$. Moreover, the transversality of bounded solutions on $\mathbb{R}$ of (4.4.8) is studied for the sufficiently small parameter $\varepsilon > 0$. Consequently, as a by-product chaotic behavior of (4.4.8) is shown for such ε in [29]. Systems of ODEs with normal and slow variables are investigated also in [30, 31].

Systems like (4.4.8) occur in certain weakly coupled systems. More general ODEs are studied in [32–37], and we refers the readers for further details to these papers.

4.4.4 Homoclinic Hopf Bifurcation

Finally we note that the method of Section 4.4.3 can be applied to systems of ODEs representing an interaction of the homoclinic and Hopf bifurcation, which are given by

$$\begin{aligned} \dot{x} &= f_1(x) + h_1(x,y,\lambda)\,, \\ \dot{y} &= f_2(y,\lambda) + \lambda h_2(x,y,\lambda) + h_3(x,y)\,, \end{aligned} \tag{4.4.9}$$

where $f_1 : \mathbb{R}^n \to \mathbb{R}^n$, $f_2 : \mathbb{R}^3 \to \mathbb{R}^2$, $h_1 : \mathbb{R}^{n+3} \to \mathbb{R}^n$, $h_2 : \mathbb{R}^{n+3} \to \mathbb{R}^2$, $h_3 : \mathbb{R}^{n+2} \to \mathbb{R}^2$ are smooth so that

(i) $f_2(0,\cdot) = 0$, $Df_2(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
(ii) $f_1(0) = 0$ and the eigenvalues of $Df_1(0)$ lie off the imaginary axis.
(iii) There is a homoclinic solution $\gamma \neq 0$ so that $\lim_{t\to\pm\infty} \gamma(t) = 0$ and $\dot{\gamma}(t) = f_1(\gamma(t))$.

(iv) $h_1(\cdot,0,0) = 0, h_2(0,\cdot,\cdot) = 0, h_3(0,\cdot) = 0, h_3(\cdot,0) = 0$.

The system (4.4.9) is an autoparametric system [38–40] consisting of two subsystems: Oscillator and Excited System.The Oscillator which is vibrating according to its nature is given by the second equation of (4.4.9) in the variable y possessing the *Hopf singularity* at $y = 0$ for $\lambda = 0, x = 0$ [41]. The Excited System is determined by the first equation of (4.4.9) in the variable x exhibiting a homoclinic structure to the equilibrium $x = 0$ for $\lambda = 0, y = 0$. (4.4.9) has for $\lambda = 0$ a semi–trivial solution $x = \gamma, y = 0$. Either chaotic or at least periodic dynamics of (4.4.9) near $\gamma \times \{0\}$ for $\lambda \neq 0$ sufficiently small is studied in [42], and we refer the readers to this paper for more details. We note that $x = 0, y = 0$ is a nonhyperbolic equilibrium of (4.4.9) for $\lambda = 0$ possessing a homoclinic loop $x = \gamma, y = 0$. Related research work is presented in [32, 34, 37, 43].

4.5 Bifurcation from Degenerate Homoclinics

4.5.1 Periodically Forced ODEs with Degenerate Homoclinics

In this section, we consider ODEs of the form

$$\dot{x} = f(x) + h(x,\mu,t), \quad x \in \mathbb{R}^n, \quad \mu \in \mathbb{R}^m \tag{4.5.1}$$

satisfying the following assumptions:

(i) f and h are C^∞ in all arguments.
(ii) $f(0) = 0$ and $h(\cdot,0,\cdot) = 0$.
(iii) The eigenvalues of $Df(0)$ lie off the imaginary axis.
(iv) The unperturbed equation has a homoclinic solution $\gamma \neq 0$ so that $\lim_{t\to\pm\infty} \gamma(t) = 0$ and $\dot{\gamma}(t) = f(\gamma(t))$.
(v) $h(x,\mu,t+1) = h(x,\mu,t)$ for any $t \in \mathbb{R}$.
(vi) The variational linear differential equation

$$\dot{u}(t) = Df(\gamma(t))u(t) \tag{4.5.2}$$

has precisely d, $d \geq 2$ linearly independent solutions bounded on $\mathbb{R}$.

For the unperturbed equation

$$\dot{x} = f(x), \tag{4.5.3}$$

we adopt the standard notation W^s, W^u for the stable and unstable manifolds, respectively, of the origin and $d_s = \dim W^s$, $d_u = \dim W^u$. Since $x = 0$ is a hyperbolic equilibrium, γ must approach the origin along W^s as $t \to +\infty$ and along W^u as $t \to -\infty$. Thus, γ lies on $W^s \cap W^u$. The condition (vi) means that the tangent spaces of W^s and W^u along γ have a d–dimensional intersection.

The case when h is independent of t, $m=3, d=2$ is studied in [44] and it is shown that the set of small parameters, for which homoclinics of (4.5.1) exist near γ, forms a Whitney umbrella (cf [45] and Figure 4.2).

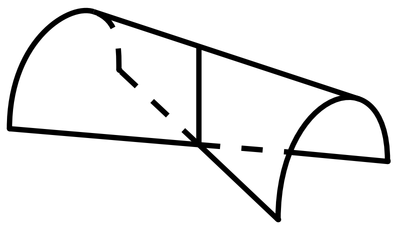

Fig. 4.2 The Whitney umbrella.

Equation (4.5.1) is considered in [46] with $d=2$ and

$$h(x,\mu,t)=h_1(x,\lambda)+\varepsilon h_2(x,\mu,t), \quad \mu=(\lambda,\varepsilon)\in\mathbb{R}^3\times\mathbb{R},$$

and it is shown that the set of small parameters, for which homoclinic points of (4.5.1) exist in a small section transverse to γ, is foliated by Whitney umbrellas. Bifurcation results for (4.5.1) are derived from [47] with $m=1$ and $d=2$. Bifurcation results in this direction are also established in [1,3–5].

Instead of (4.5.1), we consider

$$\dot{x}=f(x)+h(x,\mu,t+\alpha), \quad x\in\mathbb{R}^n, \quad \mu\in\mathbb{R}^m, \tag{4.5.4}$$

where $\alpha\in S^1=\mathbb{R}/\mathbb{Z}$ is considered as another global parameter. Here S^1 is the circle.

In this section, we always mean "generically" in the sense that certain transversality (nondegenerate) conditions are satisfied for the studied problems. Those conditions usually are rather involved formulas and their verification is tedious for a concrete example. On the other hand, if one of those transversality conditions fails then we are led to a higher–order degenerate singularity of the studied bifurcation equation with a vague normal form.

We also remark that we focus our attention in this section on describing the set of all small parameters of the above types of (4.5.1) for which homoclinics exist near γ. We do not investigate neither the numbers of those homoclinics nor which kind of bifurcations takes place. But more careful analysis of the bifurcation equations could lead to some results in that direction as [48]. However, their description is outside the scope of this section.

4.5.2 Bifurcation Equation

The bifurcation equation for finding homoclinics of (4.5.4) near γ is derived from Section 4.1.3, so we only recall its form:

$$H(\beta,\alpha,\mu)=(H_1(\beta,\alpha,\mu),\dots,H_d(\beta,\alpha,\mu))=0\,, \tag{4.5.5}$$

where $H:O_1\times\mathscr{I}\times W_1\to\mathbb{R}^d$ is smooth for small neighborhoods $0\in O_1\subset\mathbb{R}^{d-1}$, $0\in W_1\subset\mathbb{R}^m$, a bounded open interval $\mathscr{I}\subset\mathbb{R}$, and

$$H_i(\beta,\alpha,\mu)=\sum_{j=1}^{m}a_{ij}(\alpha)\mu_j+\frac{1}{2}\sum_{j,k=1}^{d-1}b_{ijk}\beta_j\beta_k+\quad \text{h.o.t}\,,$$

$$a_{ij}(\alpha)=-\int_{-\infty}^{\infty}\langle u_i^{\perp}(t),\frac{\partial h}{\partial\mu_j}(\gamma(t),0,t+\alpha)\rangle\,dt\,,$$

$$b_{ijk}=-\int_{-\infty}^{\infty}\langle u_i^{\perp},D^2f(\gamma)u_{d+j}u_{d+k}\rangle\,dt\,.$$

4.5.3 Bifurcation for 2-Parametric Systems

We investigate (4.5.1) in this section for $m=2$ and the condition (vi) holds with $d=2$. Then the bifurcation equation (4.5.5) has the form

$$\begin{aligned}a_{11}(\alpha)\mu_1+a_{12}(\alpha)\mu_2+b_1\beta^2+\quad \text{h.o.t.}=0\\ a_{21}(\alpha)\mu_1+a_{22}(\alpha)\mu_2+b_2\beta^2+\quad \text{h.o.t.}=0\,.\end{aligned} \tag{4.5.6}$$

Since the codimension is 1 of the set of all noninvertible 2×2–matrices in the space of 2×2–matrices (cf Theorem 2.6.2), generically we assume that there is a finite number of $\alpha_1,\dots,\alpha_{l_1}\in S^1$ so that

$$A(\alpha)=\begin{pmatrix}a_{11}(\alpha) & a_{12}(\alpha)\\ a_{21}(\alpha) & a_{22}(\alpha)\end{pmatrix}$$

is noninvertible only for $\alpha=\alpha_1,\dots,\alpha_{l_1}$.

A1. First of all, we study (4.5.6) for α near $\alpha_0\notin\{\alpha_1,\dots,\alpha_{l_1}\}$. Then by applying the implicit function theorem, we obtain from (4.5.6)

$$\mu_1=\mu_1(\alpha,\beta),\quad \mu_2=\mu_2(\alpha,\beta)$$

for α near α_0 and β small. Moreover, (4.5.6) implies

$$\mu_i(\alpha,\beta)=\beta^2\big(\mu_{i1}(\alpha)+\beta d_i(\alpha,\beta)\big),\quad i=1,2\,,$$

where μ_{i1}, d_i, $i=1,2$ are C^∞–smooth. Generically, we have the following possibilities:

A1.1. $\mu_{11}(\alpha_0)\neq0,\quad \mu_{21}(\alpha_0)\neq0.$

Theorem 4.5.1. *Generically in the case A1.1, the set of parameters* (α, μ_1, μ_2) *near* $(\alpha_0, 0, 0)$*, for which* (4.5.4) *has a homoclinic near* γ*, is diffeomorphically foliated along the* α*–axis by two curves*

$$\left(\alpha, \tau^2 + \tau^3 e_1(\alpha, \tau), \tau^2\right),$$

where $e_1 \in C^\infty$ *satisfies* $e_1(\alpha_0, 0) \neq 0$ *and* $\tau \in \mathbb{R}$ *is small (Figure* 4.3*).*

Proof. We take

$$\tau = \beta \sqrt{|\mu_{21}(\alpha) + \beta d_2(\alpha, \beta)|}.$$

Then our set has the form

$$\left(\alpha, \tau^2 \mu_{13}(\alpha) + \tau^3 d_3(\alpha, \tau), \tau^2 \operatorname{sgn} \mu_{21}(\alpha_0)\right),$$

where $\mu_{13}, d_3 \in C^\infty$, $\mu_{13}(\alpha_0) \neq 0$ and generically $d_3(\alpha_0, 0) \neq 0$. This set is diffeomorphic to

$$\left(\alpha, \tau^2 + \tau^3 d_3(\alpha, \tau)/\mu_{13}(\alpha), \tau^2\right).$$

The proof is finished. □

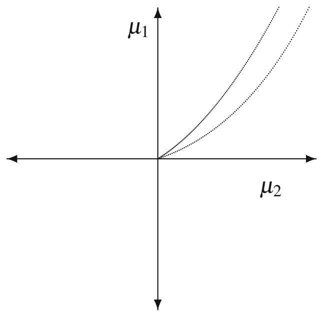

Fig. 4.3 $\mu_{11}(\alpha_0) > 0$, $\mu_{21}(\alpha_0) > 0$.

We note that generically we cannot avoid in the case A1.1 the following situation:

A1.1.1. $\quad \mu_{11}(\alpha_0) \neq 0, \quad \mu_{21}(\alpha_0) \neq 0, \quad e_1(\alpha_0, 0) = 0.$

We note that this case generically occurs only in a finite number of $\alpha_0 \notin \{\alpha_1, \ldots, \alpha_{l_1}\}$.

Theorem 4.5.2. *Generically in the case* A1.1.1*, the set of parameters* (α, μ_1, μ_2) *near* $(\alpha_0, 0, 0)$*, for which* (4.5.4) *has a homoclinic near* γ*, is diffeomorphically foliated along the* α*–axis by two curves*

$$\left(\alpha, \tau^2 + \tau^3(\alpha - \alpha_0) d_4(\alpha, \tau) + d_5(\alpha)\tau^4 + \tau^5 d_6(\alpha, \tau), \tau^2\right), \tag{4.5.7}$$

where $d_4, d_5, d_6 \in C^\infty$ *satisfy* $d_4(\alpha_0, 0) \neq 0$, $d_6(\alpha_0, 0) \neq 0$ *and* $\tau \in \mathbb{R}$ *is small* (*Figure* 4.4).

Proof. The statement of theorem is trivial, since by $e_1(\alpha_0, 0) = 0$, we have

$$e_1(\alpha, \tau) = (\alpha - \alpha_0) d_4(\alpha, \tau) + d_5(\alpha)\tau + \tau^2 d_6(\alpha, \tau).$$

To show the situation in Figure 4.4, we study the intersection of two curves (4.5.7) by solving for small $\tau > 0$ the equation

$$\begin{aligned} &\tau^2 + \tau^3(\alpha - \alpha_0) d_4(\alpha, \tau) + d_5(\alpha)\tau^4 + \tau^5 d_6(\alpha, \tau) \\ = &\ \tau^2 - \tau^3(\alpha - \alpha_0) d_4(\alpha, -\tau) + d_5(\alpha)\tau^4 - \tau^5 d_6(\alpha, -\tau). \\ &(\alpha - \alpha_0)\big(d_4(\alpha, \tau) + d_4(\alpha, -\tau)\big) = -\tau^2\big(d_6(\alpha, \tau) + d_6(\alpha, -\tau)\big). \end{aligned} \tag{4.5.8}$$

By the Whitney theorem 2.6.9, we have

$$d_4(\alpha, \tau) + d_4(\alpha, -\tau) = \tilde{d}_4(\alpha, \tau^2), \quad \tilde{d}_4 \in C^\infty,$$

$$d_6(\alpha, \tau) + d_6(\alpha, -\tau) = \tilde{d}_6(\alpha, \tau^2), \quad \tilde{d}_4 \in C^\infty.$$

Hence (4.5.8) is equivalent to

$$(\alpha - \alpha_0)\tilde{d}_4(\alpha, \tau^2) = -\tau^2 \tilde{d}_6(\alpha, \tau^2). \tag{4.5.9}$$

We can solve τ^2 from (4.5.9) to obtain

$$\tau^2 = \tau_1(\alpha), \quad \tau_1(\alpha_0) = 0, \quad \tau_1'(\alpha_0) \neq 0.$$

Now the situation in Figure 4.4 is clear. □

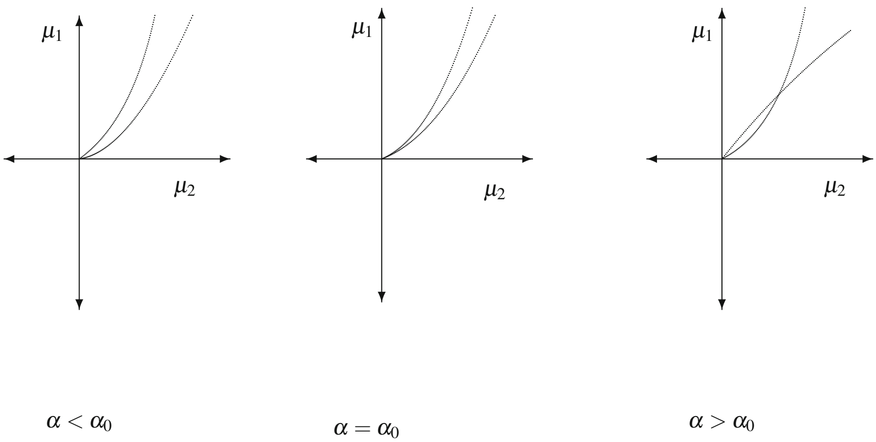

Fig. 4.4 $\mu_{11}(\alpha_0) > 0$, $\mu_{21}(\alpha_0) > 0$, $\tau_1'(\alpha_0) > 0$.

A1.2. $\mu_{11}(\alpha_0) = 0, \quad \mu_{11}'(\alpha_0) \neq 0, \quad \mu_{21}(\alpha_0) \neq 0.$

We note that this case generically occurs only in a finite number of $\alpha_0 \notin \{\alpha_1, \ldots, \alpha_{l_1}\}$.

Theorem 4.5.3. *Generically in the case A1.2, the set of parameters* (α, μ_1, μ_2) *near* $(\alpha_0, 0, 0)$*, for which* (4.5.4) *has a homoclinic near* γ*, is diffeomorphically foliated along the* α*–axis by two curves*

$$\left(\alpha, \tau^2(\alpha - \alpha_0) + \tau^3 e_2(\alpha, \tau), \tau^2\right),$$

where $e_2 \in C^\infty$ *satisfies* $e_2(\alpha_0, 0) \neq 0$ *and* $\tau \in \mathbb{R}$ *is small (Figure* 4.5*).*

Proof. Like in the above proof, our set is equivalent to

$$\left(\alpha, \tau^2 \mu_{13}(\alpha) + \tau^3 d_3(\alpha, \tau), \tau^2\right),$$

where $\mu_{13}, d_3 \in C^\infty$, $\mu_{13}(\alpha_0) = 0$, $\mu_{13}'(\alpha_0) \neq 0$, $d_3(\alpha_0, 0) \neq 0$. Hence we have

$$\left(\alpha, \tau^2(\alpha - \alpha_0)\mu_{14}(\alpha) + \tau^3 d_3(\alpha, \tau), \tau^2\right),$$

where $\mu_{14} \in C^\infty$, $\mu_{14}(\alpha_0) \neq 0$. Consequently, the set is diffeomorphic to

$$\left(\alpha, \tau^2(\alpha - \alpha_0) + \tau^3 d_3(\alpha, \tau)/\mu_{14}(\alpha), \tau^2\right).$$

The proof is finished. □

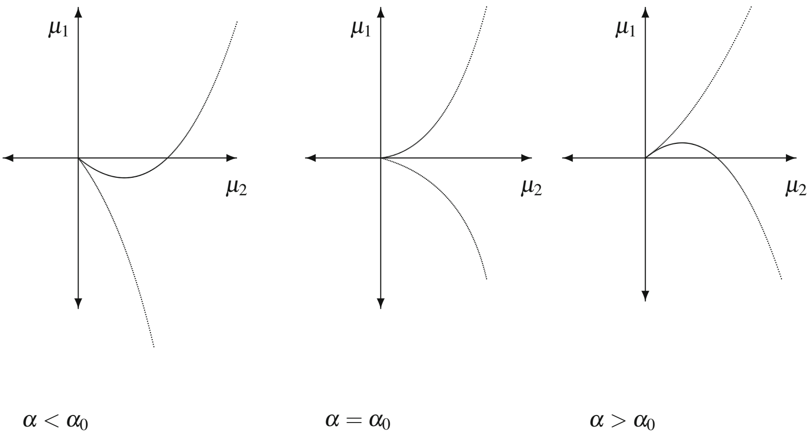

Fig. 4.5 $\mu_{11}'(\alpha_0) > 0, \mu_{21}(\alpha_0) > 0.$

A1.3. $\mu_{11}(\alpha_0) \neq 0, \quad \mu_{21}(\alpha_0) = 0, \quad \mu_{21}'(\alpha_0) \neq 0.$

It is clear that this case is the same as A1.2.

A2. The second case is when α is near $\alpha_0 \in \{\alpha_1, \ldots, \alpha_{l_1}\}$. So $A(\alpha_0)$ is noninvertible. We can assume

$$a_{11}(\alpha_0) \neq 0, \quad a_{21}(\alpha_0) = 0, \quad a'_{21}(\alpha_0) \neq 0, \quad a_{22}(\alpha_0) = 0, \quad a'_{22}(\alpha_0) \neq 0.$$

Then we solve

$$\mu_1 = \mu_1(\alpha, \beta, \mu_2)$$

from the first equation of (4.5.6) for α near α_0 and β, μ_2 small. Consequently, by inserting this solution into the second equation of (4.5.6), the bifurcation equation now is reduced to

$$\begin{aligned} Q(\alpha, \beta, \mu_2) = &(\alpha - \alpha_0)\tilde{a}_{21}(\alpha)\mu_1(\alpha, \beta, \mu_2) \\ &+(\alpha - \alpha_0)\tilde{a}_{22}(\alpha)\mu_2 + b_2\beta^2 + \quad \text{h.o.t.} = 0. \end{aligned} \tag{4.5.10}$$

We note

$$\mu_1(\alpha, \beta, 0) = O(\beta^2), \quad Q(\alpha, \beta, 0) = O(\beta^2), \quad Q(\alpha_0, 0, \mu_2) = O(\mu_2^2).$$

By using the Malgrange Preparation Theorem 2.6.8, generically (4.5.10) is equivalent to

$$Q_1(\alpha, \beta, \mu_2) = \beta^2 A(\alpha, \beta) + B(\alpha, \beta)\mu_2 + \mu_2^2 = 0, \tag{4.5.11}$$

where $A, B \in C^\infty$ satisfy

$$A(\alpha_0, 0) \neq 0, \quad B(\alpha, \beta) = (\alpha - \alpha_0)B_1(\alpha, \beta) + \beta B_2(\beta),$$

$$B_1, B_2 \in C^\infty, \quad B_1(\alpha_0, 0) \neq 0.$$

We take

$$\tau = \beta\sqrt{|A(\alpha, \beta)|}, \quad \eta = B(\alpha, \beta). \tag{4.5.12}$$

Then (4.5.11) is equivalent to

$$\tau^2 \operatorname{sgn} A(\alpha_0, 0) + \eta\mu_2 + \mu_2^2 = 0. \tag{4.5.13}$$

The discriminant of (4.5.13) is as follows:

$$D(\eta, \tau) = \eta^2 - 4\tau^2 \operatorname{sgn} A(\alpha_0, 0).$$

We note that

$$\mu_1 = \tilde{\mu}_1(\eta, \tau, \mu_2) = \mu_2 E(\eta, \tau, \mu_2) + \tau^2 F(\eta, \tau),$$

where $E, F \in C^\infty$ generically satisfy $E(0,0,0) \neq 0$ and $\frac{\partial E}{\partial \tau}(0,0,0) \neq 0$. Consequently, our set of parameters in the space (η, μ_1, μ_2) near $(0,0,0)$ has the form

$$\big(\eta, \mu_2 E(\eta, \tau, \mu_2) + \tau^2 F(\eta, \tau), \mu_2\big),$$

$$\tau^2 \operatorname{sgn} A(\alpha_0, 0) + \eta\mu_2 + \mu_2^2 = 0,$$

where $\tau \in \mathbb{R}$ is small. We consider the following two possibilities.

A2.1. $\operatorname{sgn} A(\alpha_0, 0) = -1$.

In this case, (4.5.13) has the form

$$\tau^2 = \mu_2(\mu_2 + \eta).$$

Hence

$$\tau = \pm\sqrt{\mu_2(\mu_2 + \eta)},$$

where either $\eta \geq 0$, $\mu_2 \geq 0$, $\mu_2 \leq -\eta$ or $\eta \leq 0$, $\mu_2 \leq 0$, $\mu_2 \geq -\eta$. Then

$$\begin{aligned}\mu_2 E(\eta, \tau, \mu_2) + \tau^2 F(\eta, \tau) &= \mu_2 \Big(E\big(\eta, \pm\sqrt{\mu_2(\mu_2+\eta)}, \mu_2\big) \\ &\quad + (\mu_2 + \eta) F\big(\eta, \pm\sqrt{\mu_2(\mu_2+\eta)}\big)\Big) \\ &= H_\pm(\eta, \mu_2).\end{aligned}$$

We compute

$$\begin{aligned}\big(H_+(\eta, \mu_2) - H_-(\eta, \mu_2)\big)/\mu_2 &= E\big(\eta, \sqrt{\mu_2(\mu_2+\eta)}, \mu_2\big) \\ &\quad + (\mu_2+\eta) F\big(\eta, \sqrt{\mu_2(\mu_2+\eta)}\big) \\ &\quad - E\big(\eta, -\sqrt{\mu_2(\mu_2+\eta)}, \mu_2\big) \\ &\quad - (\mu_2+\eta) F\big(\eta, -\sqrt{\mu_2(\mu_2+\eta)}\big) \\ &= \Big(\frac{\partial E}{\partial \tau}(\eta, \theta, \mu_2) + (\mu_2+\eta)\frac{\partial F}{\partial \tau}(\eta, \theta)\Big). \\ &\quad 2\sqrt{\mu_2(\mu_2+\eta)} \neq 0\end{aligned}$$

for any sufficiently small η and $\mu_2 \neq 0$, $\mu_2 \neq -\eta$. We also note that $H_\pm(\eta, \mu_2) = 0$ for sufficiently small μ_2, η only if $\mu_2 = 0$.

In summary, we obtain the following result.

Theorem 4.5.4. *Generically in the case* A2.1*, the set of parameters* (α, μ_1, μ_2) *near* $(\alpha_0, 0, 0)$*, for which* (4.5.4) *has a homoclinic near* γ *(see* (4.5.12)*), is diffeomorphically foliated along the* η*–axis by four curves*

$$\big(\eta, H_\pm(\eta, \mu_2), \mu_2\big)$$

where either $\eta \geq 0$, $\mu_2 \geq 0$, $\mu_2 \leq -\eta$ *or* $\eta \leq 0$, $\mu_2 \leq 0$, $\mu_2 \geq -\eta$ *(Figure* 4.6*).*

A2.2. $\operatorname{sgn} A(\alpha_0, 0) = 1$.

In this case, (4.5.13) has the form

$$\tau^2 + \eta\mu_2 + \mu_2^2 = 0.$$

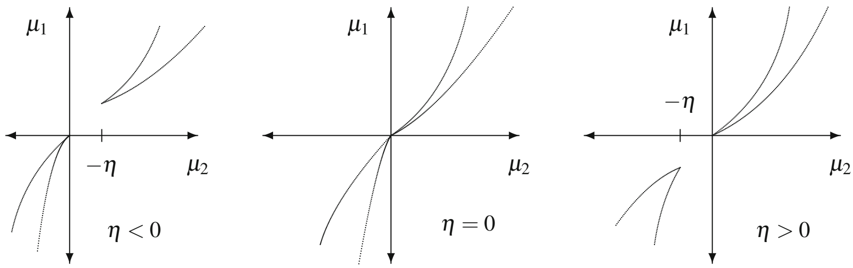

Fig. 4.6 $E(0,0) > 0$.

Hence

$$\tau = \pm\sqrt{-\mu_2(\mu_2+\eta)}$$

where either $\eta \geq 0$, $-\eta \leq \mu_2 \leq 0$ or $\eta \leq 0$, $0 \leq \mu_2 \leq -\eta$. Then

$$\begin{aligned}\mu_2 E(\eta,\tau,\mu_2) + \tau^2 F(\eta,\tau) &= \mu_2\Big(E\big(\eta,\pm\sqrt{-\mu_2(\mu_2+\eta)},\mu_2\big) \\ &\quad -(\mu_2+\eta)F\big(\eta,\pm\sqrt{-\mu_2(\mu_2+\eta)}\big)\Big) \\ &= G_\pm(\eta,\mu_2)\,.\end{aligned}$$

Similarly like the above, we see that $G_+(\eta,\mu_2) \neq G_-(\eta,\mu_2)$ for any sufficiently small η and $\mu_2 \neq 0$, $\mu_2 \neq -\eta$. We also have that $G_\pm(\eta,\mu_2) = 0$ for sufficiently small μ_2, η only if $\mu_2 = 0$. We achieve the following result.

Theorem 4.5.5. *Generically in the case A2.2, the set of parameters* (α,μ_1,μ_2) *near* $(\alpha_0,0,0)$*, for which* (4.5.4) *has a homoclinic near* γ *(see* (4.5.12)*), is diffeomorphically foliated along the* η*–axis by a closed loop*

$$(\eta, H_\pm(\eta,\mu_2),\mu_2)$$

where either $\eta \geq 0$, $-\eta \leq \mu_2 \leq 0$ *or* $\eta \leq 0$, $0 \leq \mu_2 \leq -\eta$. *We note that for* $\eta = 0$ *this is just the point* $(0,0)$ (*Figure* 4.7).

4.5.4 Bifurcation for 4-Parametric Systems

In this section, we consider the case $m = 4$ and the condition (vi) holds with $d = 2$. Then the bifurcation equation (4.5.5) has the form

$$\begin{aligned}a_{11}(\alpha)\mu_1 + a_{12}(\alpha)\mu_2 + a_{13}(\alpha)\mu_3 + a_{14}(\alpha)\mu_4 + b_1\beta^2 + \quad \text{h.o.t.} = 0,\\ a_{21}(\alpha)\mu_1 + a_{22}(\alpha)\mu_2 + a_{13}(\alpha)\mu_3 + a_{24}(\alpha)\mu_4 + b_2\beta^2 + \quad \text{h.o.t.} = 0\,.\end{aligned} \tag{4.5.14}$$

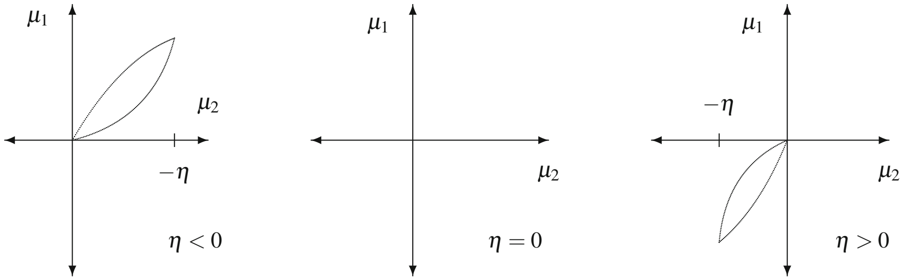

Fig. 4.7 $E(0,0) > 0$.

Since the codimension is 3 of the set of all 2×4–matrices with corank 1 in the space of 2×4–matrices (cf Theorem 2.6.2), generically we may assume the invertibility of the matrix $A(\alpha)$ for any $\alpha \in S^1$. Then by applying the implicit function theorem, we obtain from (4.5.14)

$$\mu_1 = \mu_1(\alpha,\beta,\mu_3,\mu_4), \quad \mu_2 = \mu_2(\alpha,\beta,\mu_3,\mu_4)$$

for $\alpha \in S^1$ and β, μ_3, μ_4 small. Moreover, (4.5.14) implies

$$\mu_i(\alpha,\beta,0,0) = O(\beta^2), \quad i = 1,2\,.$$

Generically we may assume

$$\Big(\frac{\partial^2\mu_1}{\partial^2\beta}(\alpha,0,0,0)\Big)^2 + \Big(\frac{\partial^2\mu_2}{\partial^2\beta}(\alpha,0,0,0)\Big)^2 \neq 0 \quad \forall,\alpha \in S^1\,.$$

We take the change of parameters

$$\mu_1 \leftrightarrow A_1(\alpha)\mu_1 + A_2(\alpha)\mu_2, \quad \mu_2 \leftrightarrow -A_2(\alpha)\mu_1 + A_1(\alpha)\mu_2\,,$$

where

$$A_1(\alpha) = \frac{\partial^2\mu_1}{\partial^2\beta}(\alpha,0,0,0)\Big/\Big(\Big(\frac{\partial^2\mu_1}{\partial^2\beta}(\alpha,0,0,0)\Big)^2 + \Big(\frac{\partial^2\mu_2}{\partial^2\beta}(\alpha,0,0,0)\Big)^2\Big),$$

$$A_2(\alpha) = \frac{\partial^2\mu_2}{\partial^2\beta}(\alpha,0,0,0)\Big/\Big(\Big(\frac{\partial^2\mu_1}{\partial^2\beta}(\alpha,0,0,0)\Big)^2 + \Big(\frac{\partial^2\mu_2}{\partial^2\beta}(\alpha,0,0,0)\Big)^2\Big).$$

For these new parameters, we have

$$\frac{\partial^2\mu_1}{\partial^2\beta}(\alpha,0,0,0) \neq 0\,.$$

Then we solve for β small the equation

$$\frac{\partial \mu_1}{\partial \beta}(\alpha, \beta, \mu_3, \mu_4) = 0$$

to obtain $\beta = \tilde{\beta}(\alpha, \mu_3, \mu_4)$, and by replacing β with $\beta + \tilde{\beta}(\alpha, \mu_3, \mu_4)$, we may assume that

$$\mu_1(\alpha, \beta, \mu_3, \mu_4) = \bar{\mu}_1(\alpha, \mu_3, \mu_4) + \beta^2 \tilde{\mu}_1(\alpha, \beta, \mu_3, \mu_4)$$

where $\tilde{\mu}_1(\alpha, 0, 0, 0) \neq 0$. Replacing β with $\beta\sqrt{|\tilde{\mu}_1(\alpha, \beta, \mu_3, \mu_4)|}$, we obtain

$$\mu_1(\alpha, \beta, \mu_3, \mu_4) = \bar{\mu}_1(\alpha, \mu_3, \mu_4) \pm \beta^2 .$$

Now we take the change of parameters

$$\mu_1 \leftrightarrow \pm\big(\mu_1 - \bar{\mu}_1(\alpha, \mu_3, \mu_4)\big), \quad \mu_2 \leftrightarrow \mu_2 - \mu_2(\alpha, 0, \mu_3, \mu_4) .$$

In this way, we arrive at

$$\mu_1(\alpha, \beta, \mu_3, \mu_4) = \beta^2, \quad \mu_2(\alpha, \beta, \mu_3, \mu_4) = \beta\rho(\alpha, \beta, \mu_3, \mu_4)$$

where $\rho \in C^\infty$ satisfies $\rho(\cdot, 0, 0, 0) = 0$. All the above changes of parameters give a local diffeomorphism

$$\Gamma_1 : S^1 \times \mathscr{O}_1 \to S^1 \times \mathbb{R}^4$$

foliated along S^1, where $\mathscr{O}_1$ is an open neighbourhood of $0 \in \mathbb{R}^4$. Generically we may assume that

$$\Big(\frac{\partial \rho}{\partial \mu_3}(\alpha, 0, 0, 0)\Big)^2 + \Big(\frac{\partial \rho}{\partial \mu_4}(\alpha, 0, 0, 0)\Big)^2 \neq 0, \quad \forall \alpha \in S^1 .$$

We take the change of parameters

$$\mu_3 \leftrightarrow D_1(\alpha)\mu_3 - D_2(\alpha)\mu_4, \quad \mu_4 \leftrightarrow D_2(\alpha)\mu_3 + D_1(\alpha)\mu_4 ,$$

where

$$D_1(\alpha) = \frac{\partial \rho}{\partial \mu_3}(\alpha, 0, 0, 0) \Big/ \Big(\Big(\frac{\partial \rho}{\partial \mu_3}(\alpha, 0, 0, 0)\Big)^2 + \Big(\frac{\partial \rho}{\partial \mu_4}(\alpha, 0, 0, 0)\Big)^2\Big),$$

$$D_2(\alpha) = \frac{\partial \rho}{\partial \mu_4}(\alpha, 0, 0, 0) \Big/ \Big(\Big(\frac{\partial \rho}{\partial \mu_3}(\alpha, 0, 0, 0)\Big)^2 + \Big(\frac{\partial \rho}{\partial \mu_4}(\alpha, 0, 0, 0)\Big)^2\Big).$$

For these new parameters, we have

$$\frac{\partial \rho}{\partial \mu_3}(\alpha, 0, 0, 0) \neq 0 .$$

Now we split

$$\big(\rho(\alpha,\beta,\mu_3,\mu_4)-\rho(\alpha,0,\mu_3,\mu_4)\big)/\beta=\rho_1(\alpha,\beta^2,\mu_3,\mu_4)+\beta\rho_2(\alpha,\beta^2,\mu_3,\mu_4)\,,$$

where $\rho_i\in C^\infty$, $i=1,2$. For an open neighbourhood $\mathscr{O}_2$ of $0\in\mathbb{R}^4$, we take a local diffeomorphism

$$\Gamma_2:S^1\times\mathscr{O}_2\to S^1\times\mathbb{R}^4$$

given by

$$\begin{aligned}&\Gamma_2(\lambda_5,\lambda_1,\lambda_2,\lambda_3,\lambda_4)\\&=\Big(\lambda_5,\lambda_1,\lambda_2-\lambda_1\rho_1(\lambda_5,\lambda_1,\lambda_3,\lambda_4),\rho(\lambda_5,0,\lambda_3,\lambda_4)+\lambda_1\rho_2(\lambda_5,\lambda_1,\lambda_3,\lambda_4),\lambda_4\Big)\,,\end{aligned}$$

which is foliated along S^1. In summary, we arrive at the following theorem.

Theorem 4.5.6. *Let* $d=2$, $m=4$ *in* (4.5.1). *Then generically the set of parameters* $(\alpha,\mu_1,\mu_2,\mu_3,\mu_4)$ *near* $(\alpha,0,0,0,0)$, $\alpha\in S^1$, *for which* (4.5.4) *has a homoclinic near* γ, *is diffeomorphically foliated along the* α*–axis by a surface of the Morin singularity* [49] *given as follows:*

$$(x_1,x_2,x_3)\to(x_1^2,x_1x_2,x_2,x_3)\,. \tag{4.5.15}$$

Proof. It is enough to take the composition of all the above changes of parameters [44, p. 221]. □

Remark 4.5.7. We note that singularity (4.5.15) is just the foliated Whitney umbrella of [46]. Moreover, the foliation along the α–axis is nontrivial. In each α–section, the diffeomorphism between the Morin singularity and the set of small parameters $\mu\in\mathbb{R}^4$ for which (4.5.4) has a homoclinic solution near γ, does depend smoothly on α. This is the main difference between our result and [46]. We do not restrict the existence of homoclinic solutions of (4.5.1) near γ by supposing that they cross a transverse section of γ at $t=0$. We really investigate all possible homoclinic solutions of (4.5.1) geometrically near γ. A similar nontrivial foliation along the α–axis holds for the result of Section 4.5.3. Furthermore, the result of Section 4.5.3 does not follow directly from Section 4.5.4. It is more delicate even for $m=1$, $d=2$ [47]. It seems that the case $m=3$, $d=2$ is more sophisticated than the case of Section 4.5.3. Finally, the result of Section 4.5.4 persists under further perturbations, that is, generically we get the same result for $m\geq 4$ with $d=2$.

4.5.5 Autonomous Perturbations

In this section, we study the case $d\geq 3$ of (4.5.1) with h independent of t. Then the bifurcation equation (4.5.5) is independent of α, so we put $\alpha=0$ in (4.5.5). Moreover, we assume that (4.5.3) is decoupled

$$\begin{gathered}\dot z_j=f_{1,j}(z_j),\quad \dot y=f_2(y),\\ j=1,2,\dots,d-2,\quad x=(z_1,z_2,\dots,z_{d-2},y)\,.\end{gathered} \tag{4.5.16}$$

Hence

$$\gamma = (\gamma_{1,1}, \gamma_{1,2}, \ldots, \gamma_{1,d-2}, \gamma_2),$$

and (4.5.2) has the form

$$\dot{u}_j = Df_{1,j}(\gamma_{1,j})u_j, \quad j = 1, 2, \ldots, d-2, \tag{4.5.17}$$

$$\dot{v} = Df_2(\gamma_2)v. \tag{4.5.18}$$

We suppose the following assumptions:

(H) The variational equations (4.5.17) with $j = 1, 2, \ldots, d-2$, respectively (4.5.18), have precisely 1, respectively 2, linearly independent solutions bounded on $\mathbb{R}$.

Let

$$W_{ss} = \times_{j=1}^{d-2} \{\gamma_{1,j}(t) \mid t \in \mathbb{R}\} \times \{\gamma_2(t) \mid t \in \mathbb{R}\}$$

be a homoclinic manifold. Theorem 4.1.1 is applicable separately to (4.5.17) and (4.5.18). Then a small transverse section Ψ at $\gamma(0)$ to W_{ss} in $\mathbb{R}^n$ is given, and we study the existence of homoclinic solutions of (4.5.1) crossing Ψ. This leads us to the bifurcation equation (4.5.5) possessing now the form

$$\Omega\mu + \beta^2\omega^* + \quad \text{h.o.t.} = 0, \tag{4.5.19}$$

where $\beta \in \mathbb{R}$ is small, $\omega \in \mathbb{R}^d$ is given and $\Omega : \mathbb{R}^m \to \mathbb{R}^d$ is a matrix. We suppose that $m \geq 2d-1$. Since the codimension is $m-d+1$ of the set of all $d \times m$–matrices with corank 1 in the space of $d \times m$–matrices (cf Theorem 2.6.2), generically we may assume that $\operatorname{rank}\Omega = d$ and so by applying the implicit function theorem to (4.5.19), we obtain

$$\mu_1 = \mu_1(\beta, \mu_2), \quad \mu_2 \in \mathbb{R}^{m-d} \quad \text{is small},$$

where $\mu_1 \in C^\infty$ satisfies $\mu_1(\beta, 0) = O(\beta^2)$. Consequently our set has the form

$$\left\{ \left(\mu_1(\beta, \mu_2), \mu_2\right) \mid \beta \in \mathbb{R}, \quad \mu_2 \in \mathbb{R}^{m-d} \quad \text{are small} \right\}.$$

We introduce a mapping $M : \mathscr{O} \to \mathbb{R}^m$ given by

$$M(\beta, \mu_2) = \left(\mu_1(\beta, \mu_2), \mu_2\right),$$

where $\mathscr{O}$ is an open neighbourhood of $0 \in \mathbb{R}^{m-d+1}$. The linearization $DM(0)$ has corank 1. Let $J^1\left(\mathbb{R}^{m-d+1}, \mathbb{R}^m\right)$ be the 1–jet bundle (cf Section 2.6), and let S_1 be a submanifold of $J^1\left(\mathbb{R}^{m-d+1}, \mathbb{R}^m\right)$ defined by

$$S_1 = \left\{\sigma \in J^1\left(\mathbb{R}^{m-d+1}, \mathbb{R}^m\right) \mid \operatorname{corank}\sigma = 1\right\}.$$

Since $m-d+1 \geq d$ and according to Theorem 2.6.3, the codimension is d of the set S_1 in $J^1\left(\mathbb{R}^{m-d+1}, \mathbb{R}^m\right)$, by recalling Theorems 2.6.6 and 2.6.7, we can assume that

$$j^1M \quad \text{intersects transversally} \quad S_1 \quad \text{at} \quad 0\,, \tag{4.5.20}$$

where

$$j^1M : \mathscr{O} \to J^1\left(\mathbb{R}^{m-d+1}, \mathbb{R}^m\right)$$

is the 1–jet mapping. By applying a result of [49] (see also a proof of [45, Theorem 4.6 on p. 179]), we immediately obtain the following theorem.

Theorem 4.5.8. *Let* $d \geq 3, m \geq 2d-1$ *in* (4.5.1) *when h is independent of t. Suppose* (4.5.16) *and that the assumption (H) holds for* (4.5.17), (4.5.18). *Then generically, when* $\operatorname{rank}\Omega = d$ *and* (4.5.20) *holds, the set of small parameters* $\mu \in \mathbb{R}^m$ *for which* (4.5.1) *has a homoclinic solution crossing* Ψ *is diffeomorphic to a surface of the Morin singularity given by*

$$\left(x_1, x_2, \dots, x_{m-d+1}\right) \to \left(x_1^2, x_1x_2, x_1x_3, \dots, x_1x_d, x_2, x_3, \dots, x_{m-d+1}\right).$$

Remark 4.5.9. 1. Theorem 4.5.8 is valid also for $d = 2$, but then we recover the result of [44] for $m = 3$. We note that the condition $m \geq 2d-1$ is a principal and not a technical restriction. Decoupling of (4.5.3) into (4.5.16) is motivated by examples of [1, 10, 50]: When several oscillators are weakly coupled then (4.5.16) is naturally satisfied. On the other hand, we are not able to find a reasonable result for the case $d \geq 3$ in general (4.5.1) without assuming the decoupling condition (4.5.16).

2. We have a cross-cap singularity [45, p. 179] in Theorem 4.5.8 with $m = 2d-1$.

3. The transversality condition (4.5.20) is the condition on the 2–jet of M at 0 [45, p. 179].

4. Under the assumptions of Theorem 4.5.8, there is a family $\Psi_{\gamma(t)}$ of small transverse sections to W_{ss} at $\gamma(t)$ for any t sufficiently small so that $\Psi_{\gamma(0)} = \Psi$, the family $\Psi_{\gamma(\cdot)}$ represents a tubular neighbourhood of W_{ss} in $\mathbb{R}^n$ near $\gamma(0)$ and the statement of Theorem 4.5.8 holds also for any $\Psi_{\gamma(t)}$.

Finally, we can study more degenerate Morin singularities of M. Let

$$J^k\left(\mathbb{R}^{m-d+1}, \mathbb{R}^m\right), \quad 2 \leq k \in \mathbb{N}$$

be the k–jet bundle, and let

$$j^kM : \mathscr{O} \to J^k\left(\mathbb{R}^{m-d+1}, \mathbb{R}^m\right)$$

be the k–jet mapping. Let S_{1_k} be the contact class in $J^k\left(\mathbb{R}^{m-d+1}, \mathbb{R}^m\right)$ [45, p. 174]. We know by [49] that S_{1_k} is a submanifold of $J^k\left(\mathbb{R}^{m-d+1}, \mathbb{R}^m\right)$ with codimension kd. Let us suppose that $j^kM(0) \in S_{1_k}$. Again by recalling Theorems 2.6.6 and 2.6.7, we can assume that

$$j^kM \quad \text{intersects transversally} \quad S_{1_k} \quad \text{at} \quad 0\,, \tag{4.5.21}$$

provided that $m-d+1 \geq kd$, i.e. $m \geq d(k+1)-1$. Results of [49] give the following theorem.

Theorem 4.5.10. *Let* $d \geq 3$, $m \geq d(k+1)-1$, $2 \leq k \in \mathbb{N}$ *in* (4.5.1) *when* h *is independent of* t. *Suppose* (4.5.16) *and that the assumption (H) holds for* (4.5.17), (4.5.18). *If* $\operatorname{rank}\Omega = d$ *and* $j^k M(0) \in S_{1_k}$ *holds with* (4.5.21) *as well, then the set of small parameters* $\mu \in \mathbb{R}^m$ *for which* (4.5.1) *has a homoclinic solution crossing* Ψ *is diffeomorphic to a surface of the Morin singularity given by*

$$\begin{aligned} y_j &= x_j, \quad 1 \leq j \leq m-d \\ y_{m-d+j} &= \sum_{r=1}^{k} x_{(j-1)k+r} x_{m-d+1}^r, \quad 1 \leq j \leq d-1 \\ y_m &= \sum_{r=1}^{k-1} x_{(d-1)k+r} x_{m-d+1}^r + x_{m-d+1}^{k+1}. \end{aligned}$$

The proof of Theorem 4.5.10 is outside the scope of this book.

4.6 Inflated ODEs

4.6.1 Inflated Carathéodory Type ODEs

Similar to Section 3.5, when we consider an orbit $x(t)$, $t \in \mathbb{R}$ of an ε-inflation of a differential equation $\dot{x} = f(t,x)$, then we deal with a differential inclusion

$$\begin{aligned} &\dot{x}(t) \in f(t,x(t)) + \varepsilon \mathscr{B}_{\mathbb{R}^n} \quad \text{for almost each (f.a.e.) } t \in \mathbb{R}, \\ &x(0) = x_0 . \end{aligned} \tag{4.6.1}$$

Here we suppose that $f : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ satisfies Carathéodory type conditions and it is globally Lipschitz continuous function in x (cf [51–53] and Section 2.5.8). We are again not interested in the existence of one solution of (4.6.1), but in the set of all trajectories of (4.6.1). So we consider a single-valued differential equation

$$\begin{aligned} &\dot{x}(t) = f(t,x(t)) + \varepsilon h(t), \quad h(t) \in \mathscr{B}_{\mathbb{R}^n} \text{ f.a.e. } t \in \mathbb{R}, \\ &x(0) = x_0 , \end{aligned} \tag{4.6.2}$$

where $h \in L^\infty(\mathbb{R},\mathbb{R}^n)$ is considered as a parameter. This orbit of (4.6.2) is denoted by $x(h)$. Since f is globally Lipschitz continuous function, this orbit is unique and continuously depends on h. Next, we define an ε-inflated orbit of (4.6.1) given by

$$\mathbf{x}^\varepsilon(x_0)(t) = \left\{ x(h)(t) \mid h \in L^\infty(\mathbb{R},\mathbb{R}^n),\, h(t) \in \mathscr{B}_{\mathbb{R}^n} \text{ f.a.e. } t \in \mathbb{R} \right\}.$$

Sets of $\mathbf{x}^\varepsilon(x_0)(t)$ are contractible into themselves to $x_0(t) = \mathbf{x}^0(x_0)(t)$ – the solution of $\dot{x}(t) = f(t,x(t))$ f.a.e. $t \in \mathbb{R}$, $x(0) = x_0$. For $t \neq 0$, the point $x_0(t)$ is in the interior of $\mathbf{x}^\varepsilon(x_0)(t)$. Moreover, $\mathbf{x}^\varepsilon(x_0)(t)$ are compact.

This approach of considering parameterized differential equations (4.6.2) instead of differential inclusions (4.6.1) is used in [53] for investigation of an ε-inflated dynamics near to a hyperbolic equilibrium of a differential equation. More precisely, we construct analogues of the stable and unstable manifolds, which are typical of a single-valued hyperbolic dynamics; moreover, we construct the maximal weakly invariant bounded set and prove that all such sets are graphs of Lipschitz maps.

4.6.2 Inflated Periodic ODEs

In this section we extend the results of Section 3.5.2 to continuous time case, i.e. we start from ODE

$$\dot{x} = h(t,x), \tag{4.6.3}$$

where $h \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ satisfies the following hypotheses:

(H1) h is 1-periodic in $t \in \mathbb{R}$. Moreover, (4.6.3) possesses a nonconstant hyperbolic 1-periodic solution $\gamma_0(t)$ along with a homoclinic one $\gamma(t)$ so that $\lim_{t\to\pm\infty} |\gamma(t) - \gamma_0(t)| = 0$. Furthermore, the variational equation $\dot{v} = Dh(t,\gamma(t))v$ has an exponential dichotomy on $\mathbb{R}$.

Let $\phi(t,x)$, $\phi(0,x) = x$ be the evolution operator of (4.6.3). By introducing the Poincarè map $f(x) = \phi(1,x)$ of (4.6.3), diffeomorphism f has a hyperbolic fixed point $x_0 = \gamma_0(0)$ along with a transversal homoclinic orbit $\{x_k^0\}_{k\in\mathbb{Z}}$, $x_k^0 = \gamma(k)$. So Theorem 2.5.4 can be applied to (4.6.3).

Next, we consider a differential inclusion in $\mathbb{R}^n$ of the form

$$\dot{x} \in h(t,x) + q(t,x,\mathscr{B}_{\mathbb{R}^n}), \tag{4.6.4}$$

where $q \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ is a 1-periodic mapping in $t \in \mathbb{R}$, satisfying the following hypotheses:

(H2) There are positive constants λ, Λ so that

$$|q(t,x,p) - q(t,\tilde{x},\tilde{p})| \leq \lambda|x-\tilde{x}| + \Lambda|p-\tilde{p}| \quad \text{and} \quad q(t,x,0) = 0$$

for all $t \in \mathbb{R}, x, \tilde{x} \in \mathbb{R}^n, p, \tilde{p} \in \mathscr{B}_{\mathbb{R}^n}$.

We put $\mathscr{L} = L^\infty(\mathbb{R},\mathbb{R}^n)$ with usual supremum norm $\|u\|_\infty = \operatorname{ess\,sup}_{t\in\mathbb{R}} |u(t)|$ and take $u \in \mathscr{B} := \{u \in \mathscr{L} \mid \|u\|_\infty \leq 1\}$. We remark (see Section 3.5.2) that (4.6.4) is equivalent, i.e. it has the same solution set, to the family of ODE

$$\dot{x} = h(t,x) + q(t,x,u(t)), \qquad u \in \mathscr{B}. \tag{4.6.5}$$

Now we can repeat the arguments of Section 3.5.2. We sketch main steps for the readers' convenience. First we note that (4.6.5) is a continuous time analogy of

(3.5.6). Then we fix $\omega \in \mathbb{N}$ large and for any $\xi \in \mathscr{E}$, $\xi = \{e_j\}_{j\in\mathbb{Z}}$ define a pseudo-orbit $x^\xi \in \mathscr{L}$ as follows for $t \in [2j\omega, \dots, 2(j+1)\omega)$, $j \in \mathbb{Z}$:

$$x^\xi(t) := \begin{cases} \gamma(t-(2j+1)\omega), & \text{for } e_j = 1, \\ \gamma_0(t-(2j+1)\omega), & \text{for } e_j = 0. \end{cases}$$

Following the proof of Lemma 3.5.1 (cf Theorem 4.1.2), we have another result.

Lemma 4.6.1. *There exist $\omega_0 \in \mathbb{N}$ and a constant $c > 0$ so that for any $\xi \in \mathscr{E}$, $u \in \mathscr{L}$, there is a unique solution $w \in W^{1,\infty}(\mathbb{R},\mathbb{R}^n)$ of the linear system*

$$\dot{w} = D_x h(t, x^\xi(t))w + u.$$

Moreover, w is linear in u and it holds $\|w\|_\infty \le c\|u\|_\infty$.

Following Theorems 3.5.2 and 3.5.3, we get

Theorem 4.6.2. *Assume λ and Λ are sufficiently small. Then there are $\omega_1 > \omega_0$, $\rho_0 > 0$ and $\widetilde{L} > 0$ so that for any $\mathbb{N} \ni \omega \ge \omega_1$ but fixed and for any $\xi \in \mathscr{E}$, $u \in \mathscr{B}$, there is a unique solution $x(u,\xi) \in \mathscr{L}$ of* (4.6.5) *so that $\|x(u,\xi) - x^\xi\|_\infty \le \rho_0$. Moreover, $\|x(u_1,\xi) - x(u_2,\xi)\|_\infty \le \widetilde{L}\|u_1 - u_2\|_\infty$ for any $\xi \in \mathscr{E}$ and $u_1, u_2 \in \mathscr{B}$. Furthermore, mapping $x : \mathscr{B} \times \mathscr{E} \to L^\infty_{loc}(\mathbb{R},\mathbb{R}^n)$ is continuous, where $L^\infty_{loc}(\mathbb{R},\mathbb{R}^n)$ is the usual topological vector space endowed with a metric*

$$d(u_1,u_2) := \sum_{k\in\mathbb{N}} \frac{\|u_1 - u_2\|_{k,\infty}}{2^{|k|+1}(1 + \|u_1 - u_2\|_{k,\infty})},$$

where $\|\cdot\|_{k,\infty}$ are the supremum norms on $[-k,k]$, $k \in \mathbb{N}$.

Next, it is easy to verify

$$x^{\sigma(\xi)}(t) = x^\xi(t + 2\omega).$$

Then by the 1-periodicity of (4.6.4) in t and the uniqueness of $x(u,\xi)$, from Theorem 4.6.2, we get

$$x(\widetilde{u}, \sigma(\xi))(t) = x(u,\xi)(t + 2\omega), \quad \forall t \in \mathbb{R}$$

for $\widetilde{u}(t) := u(t + 2\omega)$, i.e. it holds

$$x(u,\xi)(2k\omega) = x\left(\widetilde{\sigma}^k(u), \sigma^k(\xi)\right)(0), \quad \forall k \in \mathbb{Z} \tag{4.6.6}$$

for a shift homeomorphism $\widetilde{\sigma} : \mathscr{B} \to \mathscr{B}$ defined as $\widetilde{\sigma}(u) := \widetilde{u}$.

Let $\varphi_u(t,s,y)$ be the evolution operator of (4.6.5) for $t, s \in \mathbb{R}$, $y \in \mathbb{R}^n$. Here for simplicity we suppose a technical condition that h is also globally Lipschitz continuous function in x. Then clearly

$$x(u,\xi)(2(k+1)\omega) = \varphi_u(2(k+1)\omega, 2k\omega, x(u,\xi)(2k\omega)), \quad \forall k \in \mathbb{Z}. \tag{4.6.7}$$

So (4.6.6) and (4.6.7) yield

$$x\left(\widetilde{\sigma}^{k+1}(u), \sigma^{k+1}(\xi)\right)(0) = \varphi_u\left(2(k+1)\omega, 2k\omega, x\left(\widetilde{\sigma}^{k}(u), \sigma^{k}(\xi)\right)(0)\right), \quad \forall k \in \mathbb{Z},$$

that is,

$$x\left(\widetilde{\sigma}^{k+1}(u), \sigma(\xi)\right)(0) = \varphi_u\left(2(k+1)\omega, 2k\omega, x\left(\widetilde{\sigma}^{k}(u), \xi\right)(0)\right), \quad \forall k \in \mathbb{Z}. \tag{4.6.8}$$

Now, introducing the following mappings

$$\Sigma : \mathscr{B} \times \mathscr{E} \times \mathbb{Z} \mapsto \mathscr{B} \times \mathscr{E} \times \mathbb{Z}$$

$$\Sigma(u, \xi, k) := (u, \sigma(\xi), k+1),$$

$$\Phi : \mathscr{B} \times \mathscr{E} \times \mathbb{Z} \mapsto \mathscr{B} \times \mathbb{R}^n \times \mathbb{Z}$$

$$\Phi(u, \xi, k) := \left(u, x\left(\widetilde{\sigma}^{k}(u), \xi\right)(0), k\right),$$

$$F^{2\omega} : \mathscr{B} \times \mathbb{R}^n \times \mathbb{Z} \mapsto \mathscr{B} \times \mathbb{R}^n \times \mathbb{Z}$$

$$F^{2\omega}(u, x, k) := (u, \varphi_u(2(k+1)\omega, 2k\omega, x), k+1),$$

and using (4.6.8), we obtain the following analogy of Theorem 3.5.5.

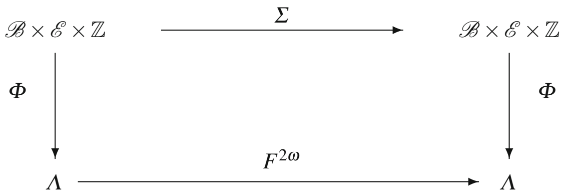

Fig. 4.8 Commutative diagram of inflated deterministic chaos.

Theorem 4.6.3. *The diagram of Figure* 4.8 *is commutative for the set*

$$\Lambda := \Phi(\mathscr{B} \times \mathscr{E} \times \mathbb{Z}).$$

Moreover, mappings Σ and Φ are homeomorphisms.

For $u = 0$, diagram of Figure 4.8 is again reduced to diagram of Figure 2.1 in Section 2.5.2 with $f(x) = \varphi_0(1, 0, x)$ for the 1-time, Poincarè map of (4.6.3). Finally, we can extend very similarly Theorem 3.5.6 to (4.6.4), but we do not write it since that extension is almost identical to Theorem 3.5.6.

4.6.3 Inflated Autonomous ODEs

In general, the situation is different when (4.6.3) is autonomous. Let us consider an ODE

$$\dot{x} = h(x), \tag{4.6.9}$$

where $h \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ satisfies the following assumption:

(A1) (4.6.9) possesses a solution $\gamma(t)$ homoclinic to a hyperbolic equilibrium 0. Moreover, the variational equation $\dot{v} = Dh(\gamma(t))v$ has the only bounded solution $\dot{\gamma}(t)$ on $\mathbb{R}$ up to constant multiplies.

Assumption (A1) means that γ is nondegenerate in the sense that the stable and unstable manifolds of 0 transversally intersect along γ (cf Section 2.5.4 and [7,54]). Moreover, we know from Section 4.1.2 that (A1) implies that the adjoint variational equation $\dot{v} = -Dh(\gamma(t))^* v$ has the only bounded solution $\psi(t)$ on $\mathbb{R}$ up to constant multiplies.

Next, we consider a differential inclusion in $\mathbb{R}^n$ of the form

$$\dot{x} \in h(x) + \varepsilon q(x, \mathscr{B}_{\mathbb{R}^n}) \tag{4.6.10}$$

where $0 \neq \varepsilon \in \mathbb{R}$ is small and $q \in C(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ satisfies the following assumption:

(A2) There are positive constants λ, μ so that

$$|q(x,p) - q(\tilde{x}, \tilde{p})| \leq \lambda |x - \tilde{x}| + \mu |p - \tilde{p}|$$

for all $x, \tilde{x} \in \mathbb{R}^n, p, \tilde{p} \in \mathscr{B}_{\mathbb{R}^n}$.

Again (4.6.10) is equivalent to the family of ODEs

$$\dot{x} = h(x) + \varepsilon q(x, u(t)), \qquad u \in \mathscr{B}. \tag{4.6.11}$$

For any fixed $u \in \mathscr{B}$, (4.6.11) is the standard bifurcation problem studied in Section 4.1.3. Consequently, we can state the following result.

Theorem 4.6.4. *There is an $\varepsilon^0 > 0$ so that for any $|\varepsilon| < \varepsilon^0$ and $u \in \mathscr{B}$ there is a unique bounded solution x_u of* (4.6.11) *with a small amplitude. Next, let us set*

$$M_u(\alpha) := \int_{-\infty}^{\infty} \psi^*(t+\alpha) q(\gamma(t+\alpha), u(t))\, dt\,. \tag{4.6.12}$$

Then there is an $\varepsilon^0 \geq \varepsilon_0 = \varepsilon_0(u) > 0$ so that for any $0 < |\varepsilon| < \varepsilon_0$ it holds

(i) *If there is an $\alpha_0 \in \mathbb{R}$ so that $M_u(\alpha_0) = 0$ and M_u is strictly monotone at α_0, then there is a unique bounded solution x of* (4.6.11) *so that*

$$\|x - \gamma(\cdot + \alpha_0)\|_\infty \to 0$$

as $\varepsilon \to 0$, and x is asymptotic to x_u as $|t| \to \infty$. Moreover there is a Smale horseshoe type chaos when u is almost periodic.

(ii) *If M_u is changing the sign over $\mathbb{R}$, then there is a bounded solution x of* (4.6.11) *orbitally near to γ and x is asymptotic to x_u as $|t| \to \infty$. Moreover there is a Smale semi-horseshoe type chaos when u is almost periodic.*

(iii) *If $\inf_{\mathbb{R}} |M_u| > 0$ then there is no bounded solution of* (4.6.11) *near γ and asymptotic to x_u as $|t| \to \infty$.*

Remark 4.6.5. $\mathscr{B}$ contains two disjoint (possible empty) open subsets $\mathscr{B}_1$ and $\mathscr{B}_2$ which are satisfied either of (ii) or (iii) of Theorem 4.6.4.

Example 4.6.6. Let us consider an ε-inflated weakly damped Duffing equation

$$\ddot{x} \in x - 2x^3 + \varepsilon(-\delta\dot{x} + [-1,1])$$

for a $\delta > 0$. Then $\gamma(t) = (\gamma(t), \dot{\gamma}(t))$, $\gamma = \operatorname{sech} t$, $\psi(t) = (-\ddot{\gamma}(t), \dot{\gamma}(t))$, and thus (4.6.12) has the form

$$M_u(\alpha) = \int_{-\infty}^{\infty} \dot{\gamma}(t+\alpha)\left(-\delta\dot{\gamma}(t+\alpha) + u(t)\right) dt = -\frac{2}{3}\delta + \int_{-\infty}^{\infty} \dot{\gamma}(t+\alpha) u(t)\, dt.$$

Using

$$|M_u(\alpha)| \geq \frac{2}{3}\delta - \|u\|_\infty \int_{-\infty}^{\infty} |\dot{\gamma}(t+\alpha)|\, dt = \frac{2}{3}\delta - 2\|u\|_\infty,$$

we see that if $\|u\|_\infty < \min\left\{\frac{\delta}{3}, 1\right\}$ then $u \in \mathscr{B}_2$. Particularly, for $\delta > 3$ we get $\mathscr{B} = \mathscr{B}_2$. If $0 < \delta \leq 3$, then we take $u(t) = -\operatorname{sgn} t$. Hence

$$M_{-\operatorname{sgn}}(\alpha) = -\frac{2}{3}\delta - \int_{-\infty}^{\infty} \dot{\gamma}(t+\alpha) \operatorname{sgn} t\, dt = -\frac{2}{3}\delta + 2\operatorname{sech}\alpha.$$

We see that if $\delta = 3$ then $-\operatorname{sgn} t \in \mathscr{B} \setminus (\mathscr{B}_1 \cup \mathscr{B}_2)$ and if $0 < \delta < 3$ then $-\operatorname{sgn} t \in \mathscr{B}_1$. Finally we take $u(t) = \theta \cos t$ for $0 \leq \theta \leq 1$. Hence

$$M_{\theta\cos}(\alpha) = -\frac{2}{3}\delta + \theta \int_{-\infty}^{\infty} \dot{\gamma}(t+\alpha)\cos t\, dt = -\frac{2}{3}\delta - \pi\theta \operatorname{sech}\frac{\pi}{2}\sin\alpha.$$

If $0 < \delta < \frac{3}{2}\pi \operatorname{sech}\frac{\pi}{2} \doteq 1.87806$ then $\theta\cos t \in \mathscr{B}_2$ for $0 \leq \theta < \frac{2\delta}{3\pi}\cosh\frac{\pi}{2}$, $\theta\cos t \in \mathscr{B}_1$ for $1 \geq \theta > \frac{2\delta}{3\pi}\cosh\frac{\pi}{2}$ and $\frac{2\delta}{3\pi}\cosh\frac{\pi}{2}\cos t \in \mathscr{B} \setminus (\mathscr{B}_1 \cup \mathscr{B}_2)$. If $\delta = \frac{3}{2}\pi \operatorname{sech}\frac{\pi}{2}$ then $\theta\cos t \in \mathscr{B}_2$ for $0 \leq \theta < 1$ and $\cos t \in \mathscr{B} \setminus (\mathscr{B}_1 \cup \mathscr{B}_2)$. If $\delta > \frac{3}{2}\pi \operatorname{sech}\frac{\pi}{2}$ then $\theta\cos t \in \mathscr{B}_2$ for $0 \leq \theta \leq 1$. These inequalities are balance between the damping and forcing to either get chaos, or exclude it near the homoclinic solution.

Finally we remark that the inflated chaos could be extended also to the autonomous case (4.6.10) under the assumption

(A3) (4.6.9) possesses a hyperbolic nonconstant periodic solution $x_0(t)$ with a transversal homoclinic point $z \in W^s(x_0) \cap W^u(x_0)$, i.e. $T_z W^s(x_0) \cap T_z W^u(x_0) = \operatorname{span}\{h(z)\}$.

The method of [55] could be used together with our parameterized approach but this is outside scope of this book.

4.7 Nonlinear Diatomic Lattices

4.7.1 Forced and Coupled Nonlinear Lattices

We end this chapter with infinite dimensional ODEs [56, 57]. Let us consider a model of two one-dimensional interacting sublattices of harmonically coupled protons and heavy ions [58–61]. It represents the Bernal-Flower filaments in ice or more complex biological macromolecules in membranes, in which only the degrees of freedom that contribute predominantly to proton mobility have been conserved. In these systems, each proton lies between a pair of "oxygens". The proton part of the Hamiltonian is

$$H_p = \sum_n \frac{1}{2} m\dot{u}_n^2 + U(u_n) + \frac{1}{2} k_1 (u_{n+1} - u_n)^2 \,,$$

where u_n denotes the displacement of the nth proton with respect to the center of the oxygen pair and k_1 is the coupling between neighboring protons. Furthermore, $U(u) = \xi_0(1 - u^2/d_0^2)^2$ is the double-well potential with the potential barrier ξ_0, and $2d_0$ is the distance between its two minima. Finally, m is the mass of protons.

Similarly, the oxygen part of the Hamiltonian is

$$H_O = \sum_n \frac{1}{2} M\dot{\rho}_n^2 + \frac{1}{2} M\Omega_0^2 \rho_n^2 + \frac{1}{2} K_1 (\rho_{n+1} - \rho_n)^2 \,,$$

where ρ_n is the displacement between two oxygens, M is the mass of oxygens, Ω_0 is the frequency of the optical mode and K_1 is the harmonic coupling between neighboring oxygens.

The last part in the Hamiltonian of the model arises from the dynamical interaction between two sublattices and it is given by

$$H_{int} = \sum_n \chi \rho_n (u_n^2 - d_0^2) \,,$$

where χ measures the strength of the coupling. The Hamiltonian of the model is the sum of these three contributions $H = H_p + H_O + H_{int}$.

We are also interested in the influence of external field and damping. For the model studied here, since a spatially homogeneous field is not coupled to the optical motion ρ_n of the oxygens, a force term has to be considered only in the equation of motion of the protons.

In summary, we consider in this section the following coupled infinite chain of oscillators

$$\begin{aligned}\ddot{u}_n+\Gamma_1\dot{u}_n&=\frac{k_1}{m}\big(u_{n+1}-2u_n+u_{n-1}\big)+\frac{4\xi_0}{md_0^2}u_n\Big(1-\frac{u_n^2}{d_0^2}\Big)-2\frac{\chi}{m}\rho_n u_n+\frac{F}{m}\,,\\ \ddot{\rho}_n+\Gamma_2\dot{\rho}_n&=\frac{K_1}{M}\big(\rho_{n+1}-2\rho_n+\rho_{n-1}\big)-\Omega_0^2\rho_n-\frac{\chi}{M}(u_n^2-d_0^2)\,,\end{aligned} \tag{4.7.1}$$

where F is the external force on the protons and Γ_1,Γ_2 are the damping coefficients for the proton and oxygen motions.

We are interested in the existence of homoclinic and chaotic *spatially localized solutions* of (4.7.1). The existence of time periodic spatially localized solutions, the so-called *breathers* are studied in [62–68].

4.7.2 Spatially Localized Chaos

We assume in this section that $\Gamma_1=\varepsilon\delta_1$, $\Gamma_2=\varepsilon\delta_2$, $F/m=\varepsilon f(t)$, $k_1/m=\varepsilon\mu_1$, $K_1/M=\varepsilon\mu_2$, $-2\chi/m=\varepsilon\mu_3$, $-\chi/M=\varepsilon\mu_4$ for a small parameter $\varepsilon>0$, constants $\delta_1\geq 0$, $\delta_2>0$, μ_i, $i=1,2,3,4$ and a C^1-smooth T-periodic function $f(t)$. Putting

$$a^2:=\frac{4\xi_0}{md_0^4}\,,$$

(4.7.1) has the form

$$\begin{aligned}&\ddot{u}_n+\varepsilon\delta_1\dot{u}_n+a^2u_n\big(u_n^2-d_0^2\big)=\varepsilon\mu_1\big(u_{n+1}-2u_n+u_{n-1}\big)+\varepsilon\mu_3\rho_n u_n+\varepsilon f(t)\,,\\ &\ddot{\rho}_n+\varepsilon\delta_2\dot{\rho}_n+\Omega_0^2\rho_n=\varepsilon\mu_2\big(\rho_{n+1}-2\rho_n+\rho_{n-1}\big)+\varepsilon\mu_4(u_n^2-d_0^2)\,.\end{aligned} \tag{4.7.2}$$

We first consider the system

$$\begin{aligned}\ddot{u}+\varepsilon\delta_1\dot{u}+a^2u\big(u^2-d_0^2\big)&=\varepsilon\mu_3\rho u+\varepsilon f(t)\,,\\ \ddot{\rho}+\varepsilon\delta_2\dot{\rho}+\Omega_0^2\rho&=\varepsilon\mu_4(u^2-d_0^2)\,.\end{aligned} \tag{4.7.3}$$

The equation

$$\dot{u}=v,\quad \dot{v}=a^2(d_0^2-u^2)u$$

has a hyperbolic equilibrium $u=v=0$ and centers $u=\pm d_0$, $v=0$ [35]. Furthermore, there are two symmetric homoclinic solutions $(\gamma(t),\dot{\gamma}(t))$ and $(-\gamma(t),-\dot{\gamma}(t))$ for $\gamma(t)=\sqrt{2}d_0\,\mathrm{sech}\,ad_0t$. Now we make the change of variable $\rho\leftrightarrow\rho-\frac{\varepsilon\mu_4 d_0^2}{\Omega_0^2}$ in (4.7.3) to get

$$\begin{aligned}\ddot{u}+\varepsilon\delta_1\dot{u}+a^2u\big(u^2-d_0^2\big)&=\varepsilon\mu_3\left(\rho-\frac{\varepsilon\mu_4d_0^2}{\Omega_0^2}\right)u+\varepsilon f(t)\,,\\ \ddot{\rho}+\varepsilon\delta_2\dot{\rho}+\Omega_0^2\rho&=\varepsilon\mu_4u^2\,.\end{aligned}$$

To study a small T-periodic solution of the above system, we take its equivalent form

$$\ddot{u}+\varepsilon\delta_1\dot{u}+a^2u\big(u^2-d_0^2\big)= \\ \varepsilon\mu_3\left(\frac{\varepsilon\mu_4}{\Omega_\varepsilon}\int_{-\infty}^{t}\mathrm{e}^{-\varepsilon\delta_2(t-s)/2}\sin\Omega_\varepsilon(t-s)u^2(s)\,ds-\frac{\varepsilon\mu_4 d_0^2}{\Omega_0^2}\right)u+\varepsilon f(t) \tag{4.7.4}$$

where $\Omega_\varepsilon=\sqrt{\Omega_0^2-\frac{\varepsilon^2\delta_2^2}{4}}$ and $0<\varepsilon<2\Omega_0/\delta_2$. Now it is not difficult to prove for (4.7.4) by using the implicit function theorem the existence of a unique small T-periodic solution $u_\varepsilon(t)=O(\varepsilon)$, $\rho_\varepsilon(t)=O(\varepsilon)$ of (4.7.3) . Then we make in (4.7.2) the change of variables $u_n\leftrightarrow u_n+u_\varepsilon$, $\rho_n\leftrightarrow\rho_n+\rho_\varepsilon$ to get the chain

$$\begin{gathered}\dot{u}_n=v_n,\\ \dot{v}_n+\varepsilon\delta_1v_n-a^2u_nd_0^2+a^2u_n^3+3a^2u_n^2u_\varepsilon+3a^2u_nu_\varepsilon^2\\ =\varepsilon\mu_1\big(u_{n+1}-2u_n+u_{n-1}\big)+\varepsilon\mu_3\big(\rho_nu_n+\rho_nu_\varepsilon+\rho_\varepsilon u_n\big);\\ \dot{\rho}_n=\psi_n,\\ \dot{\psi}_n+\varepsilon\delta_2\psi_n+\Omega_0^2\rho_n=\varepsilon\mu_2\big(\rho_{n+1}-2\rho_n+\rho_{n-1}\big)+\varepsilon\mu_4(u_n^2+2u_\varepsilon u_n)\,.\end{gathered} \tag{4.7.5}$$

We consider (4.7.5) as an ODE on the Hilbert space

$$H:=\left\{z=\{(u_n,v_n,\rho_n,\psi_n)\}_{n\in\mathbb{Z}}\mid\sum_{n\in\mathbb{Z}}\big(u_n^2+v_n^2+\rho_n^2+\psi_n^2\big)<\infty\right\}$$

with the norm $\|z\|=\sqrt{\sum_{n\in\mathbb{Z}}(u_n^2+v_n^2+\rho_n^2+\psi_n^2)}$. The non-homogeneous linearization of (4.7.5) at $z=0$ has the form

$$\begin{gathered}\dot{u}_n=v_n+h_{n1}(t),\\ \dot{v}_n+\varepsilon\delta_1v_n+u_n(3a^2u_\varepsilon^2-a^2d_0^2-\varepsilon\mu_3\rho_\varepsilon),\\ -\varepsilon\mu_1\big(u_{n+1}-2u_n+u_{n-1}\big),-\varepsilon\mu_3\rho_nu_\varepsilon=h_{n2}(t);\\ \dot{\rho}_n=\psi_n+g_{n1}(t),\\ \dot{\psi}_n+\varepsilon\delta_2\psi_n+\Omega_0^2\rho_n-\varepsilon\mu_2\big(\rho_{n+1}-2\rho_n+\rho_{n-1}\big)-2\varepsilon\mu_4u_\varepsilon u_n=g_{n2}(t),\end{gathered} \tag{4.7.6}$$

with $w(t)=\big\{(h_{n1}(t),h_{n2}(t),g_{n1}(t),g_{n2}(t))\big\}_{n\in\mathbb{Z}}\in C_b(\mathbb{R},H)$ – the Banach space of all bounded continuous functions from $\mathbb{R}$ to H with the norm $|w|=\sup_{\mathbb{R}}\|w(t)\|$. We look for a solution $z\in C_b(\mathbb{R},H)$ of (4.7.5) for $\varepsilon>0$ small. For this reason, we consider the Hilbert spaces $H_2 := H_1 \times H_1$ and

$$H_1 := \left\{ \{u_n\}_{n\in\mathbb{Z}} \mid \sum_{n\in\mathbb{Z}} u_n^2 < \infty \right\}$$

with the corresponding standard norms and scalar products. We first study the equation

$$\dot{\rho} = \psi + g_1, \quad \dot{\psi} + \varepsilon\delta_2\psi + A_\varepsilon\rho = g_2 \tag{4.7.7}$$

on H_2 for $(g_1, g_2) \in C_b(\mathbb{R}, H_2)$ and

$$A_\varepsilon\rho = \left\{ \Omega_0^2\rho_n - \varepsilon\mu_2(\rho_{n+1} - 2\rho_n + \rho_{n-1}) \right\}_{n\in\mathbb{Z}}.$$

Clearly $A_\varepsilon : H_1 \to H_1$ is symmetrically and positively definite for ε small. Then for any small ε, there is a symmetrically and positively definite $B_\varepsilon : H_1 \to H_1$ so that

$$B_\varepsilon^2 = A_\varepsilon - \frac{\varepsilon^2\delta_2^2}{4}\mathbb{I}.$$

We take the operators $\cos B_\varepsilon t$ and $\sin B_\varepsilon t$ from H_1 to H_1. For any $\rho \in H_1$, we consider the function

$$\phi(t) := |\cos B_\varepsilon t\rho|^2 + |\sin B_\varepsilon t\rho|^2.$$

Then we have

$$\dot{\phi}(t) = -2\langle \cos B_\varepsilon t\rho, B_\varepsilon \sin B_\varepsilon t\rho\rangle + 2\langle \sin B_\varepsilon t\rho, B_\varepsilon \cos B_\varepsilon t\rho\rangle = 0.$$

Hence

$$|\cos B_\varepsilon t\rho|^2 + |\sin B_\varepsilon t\rho|^2 = \rho,$$

and then $\|\cos B_\varepsilon t\| \leq 1$ and $\|\sin B_\varepsilon t\| \leq 1$. Now, the equation

$$\dot{\rho} = \psi, \quad \dot{\psi} + \varepsilon\delta_2\psi + A_\varepsilon\rho = 0 \tag{4.7.8}$$

has the form $\ddot{\rho} + \varepsilon\delta_2\dot{\rho} + A_\varepsilon\rho = 0$ which has the general solution

$$\mathrm{e}^{-\varepsilon\delta_2 t/2}\left[\cos B_\varepsilon t\rho_1 + \sin B_\varepsilon t\rho_2\right]$$

for $\rho_{1,2} \in H_1$. Consequently, the fundamental solution of (4.7.8) has the form

$$V_\varepsilon(t) = \mathrm{e}^{-\varepsilon\delta_2 t/2}W_\varepsilon(t)$$

with uniformly bounded $W_\varepsilon(t)$ for $\varepsilon > 0$ small. Thus, the only bounded solution of (4.7.7) has the form

$$(\rho(t), \psi(t)) = \int_{-\infty}^{t} \mathrm{e}^{-\varepsilon\delta_2(t-s)/2}W_\varepsilon(t-s)\,(g_1(s), g_2(s))\,ds. \tag{4.7.9}$$

Hence

$$|(\rho,\psi)| \leq K_1|(g_1,g_2)|/\varepsilon$$

for a constant $K_1 > 0$ independent of $\varepsilon > 0$ small. Furthermore, it is not difficult to see that the linear system

$$\dot{u}_n = v_n + h_{n1}(t), \quad \dot{v}_n + \varepsilon\delta v_n - a^2 d_0^2 u_n = h_{n2}(t)$$

has a unique solution $\{(u_n(t),v_n(t))\}_{n\in\mathbb{Z}} \in C_b(\mathbb{R},H_2)$ so that

$$\left|\{(u_n(t),v_n(t))\}_{n\in\mathbb{Z}}\right| \leq K_2\left|\{(h_{n1}(t),h_{n2}(t))\}_{n\in\mathbb{Z}}\right|$$

for a constant $K_2 > 0$ independent of $\varepsilon > 0$ small. Now we turn back to (4.7.6). Summarizing the above arguments, we see, by using the Banach contraction mapping principle 2.2.1 for $\varepsilon > 0$ small, that (4.7.6) has for any $w(t) \in C_b(\mathbb{R},H)$ a unique solution $z \in C_b(\mathbb{R},H)$ so that $|z| \leq K_3|w|/\varepsilon$ for a constant $K_3 > 0$ independent of $\varepsilon > 0$ small. Since the system (4.7.6) is T-periodic, we get from Lemma 2.5.5 that (4.7.6) has an exponential dichotomy on $\mathbb{R}$ in the space H for any $\varepsilon > 0$ sufficiently small. Consequently, we get another result.

Theorem 4.7.1. *The T-periodic solution $u_n(t) = u_\varepsilon(t)$, $\rho_n(t) = \rho_\varepsilon(t)$ $\forall n \in \mathbb{Z}$ of* (4.7.2) *is hyperbolic in H for any $\varepsilon > 0$ sufficiently small, i.e. the zero equilibrium of* (4.7.5) *in H is hyperbolic.*

Now we look for more complicated solutions of (4.7.2). For this reason, we shift in (4.7.5) the time $t \leftrightarrow t+\alpha$ to get the system

$$\begin{aligned}
&\dot{u}_n = v_n\\
&\dot{v}_n + \varepsilon\delta_1 v_n - a^2 u_n d_0^2 + a^2 u_n^3 + 3a^2 u_n^2 u_\varepsilon(t+\alpha) + 3a^2 u_n u_\varepsilon^2(t+\alpha)\\
&\quad = \varepsilon\mu_1\left(u_{n+1} - 2u_n + u_{n-1}\right) + \varepsilon\mu_3\left(\rho_n u_n + \rho_n u_\varepsilon(t+\alpha) + \rho_\varepsilon(t+\alpha)u_n\right),\\
&\dot{\rho}_n = \psi_n\\
&\dot{\psi}_n + \varepsilon\delta_2\psi_n + \Omega_0^2\rho_n = \varepsilon\mu_2\left(\rho_{n+1} - 2\rho_n + \rho_{n-1}\right)\\
&\qquad + \varepsilon\mu_4(u_n^2 + 2u_\varepsilon(t+\alpha)u_n).
\end{aligned} \tag{4.7.10}$$

We look for a solution of (4.7.10) for $\varepsilon > 0$ small so that $u_n \sim 0$, $v_n \sim 0$ for $n \neq 0$ and $u_0 \sim \gamma$, $v_0 \sim \dot{\gamma}$. Let $(\rho_0,\psi_0) = \{(\rho_n^0,\psi_n^0)\}_{n\in\mathbb{Z}}$ be the unique bounded solution of (4.7.7) for $g_1 = 0$ and $g_2 = \{g_{n2}\}_{n\in\mathbb{Z}}$ with $g_{n2} = 0$ for $n \neq 0$ and $g_{02} = \varepsilon\mu_4(\gamma^2 + 2u_\varepsilon(t+\alpha)\gamma)$. Let us put $u_n^0 = v_n^0 = 0$ for $n \neq 0$ and $u_0^0 = \gamma$, $v_0^0 = \dot{\gamma}$. Now we make in (4.7.10) the change of variables $u_n \leftrightarrow u_n + u_n^0$, $v_n \leftrightarrow v_n + v_n^0$, $\rho_n \leftrightarrow \rho_n + \rho_n^0$, $\psi_n \leftrightarrow \psi_n + \psi_n^0$ to get for $n \neq 0$ the system

$$\begin{aligned}
&\dot{u}_n = v_n,\\
&\dot{v}_n + \varepsilon\delta_1 v_n - a^2 u_n d_0^2 + a^2 u_n^3 + 3a^2 u_n^2 u_\varepsilon(t+\alpha) + 3a^2 u_n u_\varepsilon^2(t+\alpha)\\
&= \varepsilon\mu_1\left(u_{n+1} + u_{n+1}^0 - 2u_n + u_{n-1} + u_{n-1}^0\right)\\
&\quad + \varepsilon\mu_3\left((\rho_n + \rho_n^0)u_n + (\rho_n + \rho_n^0)u_\varepsilon(t+\alpha) + \rho_\varepsilon(t+\alpha)u_n\right);
\end{aligned} \tag{4.7.11}$$

$$\begin{aligned}\dot{\rho}_n &= \psi_n,\\ \dot{\psi}_n + \varepsilon\delta_2\psi_n + \Omega_0^2\rho_n &= \varepsilon\mu_2\big(\rho_{n+1} - 2\rho_n + \rho_{n-1}\big)\\ &\quad + \varepsilon\mu_4(u_n^2 + 2u_\varepsilon(t+\alpha)u_n)\,.\end{aligned}$$

For the mode $n = 0$, we first note that the system

$$\dot{u}_0 = v_0, \quad \dot{v}_0 + a^2(3\gamma^2 - d_0^2)u_0 = h(t)$$

for $h(t) \in C_b(\mathbb{R},\mathbb{R})$ has a solution $(u_0, v_0) \in C_b(\mathbb{R},\mathbb{R}^2)$ (see Section 4.1) if and only if $\int_{-\infty}^{\infty} h(t)\dot{\gamma}(t)\,dt = 0$ and such a solution is unique if $\int_{-\infty}^{\infty} u_0(t)\dot{\gamma}(t)\,dt = 0$. Consequently, for the mode $n = 0$ we get from (4.7.10) the equations

$$\begin{gathered}\dot{u}_0 = v_0,\\ \dot{v}_0 + a^2(3\gamma^2 - d_0^2)u_0 = h(t) - \dot{\gamma}(t)\int_{-\infty}^{\infty} h(t)\dot{\gamma}(t)\,dt \Big/ \int_{-\infty}^{\infty}\dot{\gamma}(t)^2\,dt\,,\\ \int_{-\infty}^{\infty} u_0(t)\dot{\gamma}(t)\,dt = 0;\\ \dot{\rho}_0 = \psi_0\\ \dot{\psi}_0 + \varepsilon\delta_2\psi_0 + \Omega_0^2\rho_0 = \varepsilon\mu_2\big(\rho_1 - 2\rho_0 + \rho_{-1}\big)\\ + \varepsilon\mu_4(u_0^2 + 2u_0\gamma + 2u_\varepsilon(t+\alpha)u_0)\,,\end{gathered} \tag{4.7.12}$$

and

$$\int_{-\infty}^{\infty} h(t)\dot{\gamma}(t)\,dt = 0 \tag{4.7.13}$$

for

$$\begin{aligned}h(t) = &-a^2(u_0^3 + 3u_0^2\gamma) - \varepsilon\delta_1\dot{\gamma} - 3a^2(u_0+\gamma)^2u_\varepsilon(t+\alpha) - \varepsilon\delta_1 v_0\\ &- 3a^2(u_0+\gamma)u_\varepsilon^2(t+\alpha) + \varepsilon\mu_1\big(u_1 - 2(u_0+\gamma) + u_{-1}\big)\\ &+ \varepsilon\mu_3\big((\rho_0+\rho_0^0)(u_0+\gamma) + (\rho_0+\rho_0^0)u_\varepsilon(t+\alpha) + \rho_\varepsilon(t+\alpha)(u_0+\gamma)\big)\,.\end{aligned} \tag{4.7.14}$$

Now for $\varepsilon > 0$ small, we can solve (4.7.12) and (4.7.12) to get the solution

$$z = \Big\{\Big(u_n(t), v_n(t), \rho_n(t), \psi_n(t)\Big)\Big\}_{n\in\mathbb{Z}} \in C_b(\mathbb{R}, H),$$

so that $z = O(\varepsilon)$. Then we put this z into (4.7.15) to get the function $h_{\varepsilon,\alpha} \in C_b(\mathbb{R},\mathbb{R})$. We note $h_{\varepsilon,\alpha}(t) = O(\varepsilon)$ uniformly for $\varepsilon > 0$ small and $\alpha, t \in \mathbb{R}$. Clearly $h_{\varepsilon,\alpha}(t)$ is T-periodic in α. Then from (4.7.13) we get the bifurcation equation

$$Q(\varepsilon,\alpha) := \frac{1}{\varepsilon}\int_{-\infty}^{\infty} h_{\varepsilon,\alpha}(t)\dot{\gamma}(t)\,dt = 0\,.$$

If we put

$$\lim_{\varepsilon\to 0} u_\varepsilon(t)/\varepsilon = w(t), \quad \lim_{\varepsilon\to 0} \rho_\varepsilon(t)/\varepsilon = \zeta(t),$$

then from (4.7.3) we get

$$\ddot{w} - a^2 d_0^2 w = f(t), \quad \ddot{\zeta} + \Omega_0^2 \zeta = -\mu_4 d_0^2 .$$

Hence $\zeta = -\mu_4 d_0^2/\Omega_0^2$ and

$$w(t) = -\frac{1}{2ad_0}\int_{-\infty}^{t} \mathrm{e}^{-ad_0(t-s)} f(s)\,ds - \frac{1}{2ad_0}\int_{t}^{\infty} \mathrm{e}^{ad_0(t-s)} f(s)\,ds . \tag{4.7.15}$$

Clearly $w(t)$ is T-periodic. Furthermore, since $\gamma(t) \to 0$ as $t \to \pm\infty$ exponentially, from formula (4.7.9) we see that $\lim_{\varepsilon\to 0}(\rho_0, \psi_0)/\varepsilon = \{(\rho_{0n}, \psi_{0n})\}_{n\in\mathbb{Z}}$ with $\rho_{0n} = \psi_{0n} = 0$ for $n \neq 0$ and

$$\ddot{\rho}_{00} + \Omega_0^2 \rho_{00} = \mu_4 \gamma(t)^2 ,$$

i.e. $\rho_{00}(t) = \dfrac{\mu_4}{\Omega_0}\int_{-\infty}^{t} \sin\Omega_0(t-s)\gamma(s)^2\,ds$. In summary, from (4.7.15) we get

$$\begin{aligned} M(\alpha) := Q(0,\alpha) &= \int_{-\infty}^{\infty}\Big[-\delta_1\dot{\gamma}(t) - 3a^2\gamma(t)^2 w(t+\alpha) - 2\mu_1\gamma(t)\Big]\dot{\gamma}(t)\,dt \\ &= -\frac{4}{3}\delta_1 a d_0^3 + a^2\int_{-\infty}^{\infty}\gamma(t)^3\dot{w}(t+\alpha)\,dt . \end{aligned} \tag{4.7.16}$$

Clearly $M(\alpha)$ is T-periodic. We note that similarly we can prove that

$$\lim_{\varepsilon\to 0}\frac{\partial}{\partial\alpha}Q(\varepsilon,\alpha)/\varepsilon = M'(\alpha)$$

uniformly for $\alpha \in \mathbb{R}$. In summary, we get another result.

Theorem 4.7.2. *Let M be given by* (4.7.16). *If there is a simple zero α_0 of M, i.e. $M(\alpha_0) = 0$ and $M'(\alpha_0) \neq 0$, then* (4.7.2) *has for any $\varepsilon > 0$ small a bounded solution $z(t)$ with small u_n, ρ_n for $n \neq 0$ and (u_0, ρ_0) near $(\gamma(t-\alpha_0), 0)$.*

Now, it is not difficult to prove like in the finite-dimensional case (cf Section 4.1) that

$$\Big(z(t) - \Big\{(u_\varepsilon(t), \dot{u}_\varepsilon(t), \rho_\varepsilon(t), \dot{\rho}_\varepsilon(t))\Big\}_{n\in\mathbb{Z}}\Big) \to 0$$

is exponentially fast as $t \to \pm\infty$ in H. Moreover, near $z(t)$ we can construct the Smale horseshoe. Consequently, we get in this case the chaos in (4.7.2) with corresponding infinitely many periodic orbits with arbitrarily large periods. This Smale horseshoe of (4.7.2) is spatially localized but not exponentially like in breathers.

To be more concrete, we take

$$f(t) = \Upsilon \cos\omega t$$

for $\Upsilon > 0$. Then (4.7.15) gives

$$w(t) = -\frac{\Upsilon}{\omega^2 + a^2 d_0^2} \cos \omega t \,,$$

and the formula (4.7.16) has now the form

$$M(\alpha) = -\frac{4}{3}\delta_1 a d_0^3 + \frac{\omega \Upsilon \pi \sqrt{2}}{a} \operatorname{sech} \frac{\omega \pi}{2 a d_0} \sin \omega \alpha \,.$$

Consequently, if

$$8\sqrt{2}\delta_1 \xi_0 < 3m\omega \Upsilon \pi d_0 \operatorname{sech} \frac{\omega d_0 \pi \sqrt{m}}{4\sqrt{\xi_0}} \,, \tag{4.7.17}$$

then $M(\alpha)$ has a simple zero, so (4.7.2) is chaotic for any $\varepsilon > 0$ small. We note that the inequality (4.7.17) gives sufficient conditions between the magnitude of the forcing Υ and the damping δ_1 in order to get chaos in (4.7.2) for $\varepsilon > 0$ small. So chaos is generated by the proton part of (4.7.2). If $\delta_1 = 0$ then (4.7.2) is always chaotic for $f(t) = \Upsilon \cos \omega t$. Furthermore, if $\Gamma_1 > 0, \Gamma_2 > 0$ and $F = 0$, i.e. there is no forcing but damping then it is not difficult to prove that (4.7.1) has no nonconstant periodic solutions in the space H.

Finally, we note that similarly we can study the case when more than one modes are excited. We do not carry out here such computations [64].

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