Chapter 3
Chaos in Discrete Dynamical Systems

This chapter is devoted to functional analytical methods for showing chaos in discrete dynamical systems involving difference equations, diffeomorphisms, regular and singular ODEs with impulses, and inflated mappings as well.

3.1 Transversal Bounded Solutions

3.1.1 Difference Equations

In this section, we consider difference equations of the form

$$x_{k+1} = f(x_k) + h(x_k, \mu, k) \tag{3.1.1}$$

with $x_k \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$. We make the following assumptions of (3.1.1):

(i) f, h are C^3-smooth in all non-discrete arguments.
(ii) $f(0) = 0$ and $h(\cdot, 0, \cdot) = 0$.
(iii) The eigenvalues of $Df(0)$ are non-zero and all lie off the unit circle.
(iv) The unperturbed equation $x_{k+1} = f(x_k)$ has a homoclinic solution. That is, there exists a nonzero sequence $\gamma = \{\gamma_k\}_{k\in\mathbb{Z}}$ so that $\lim_{k\to\pm\infty} \gamma_k = 0$ and $\gamma_{k+1} = f(\gamma_k)$. Moreover, $Df(\gamma_k)$, $k \in \mathbb{Z}$ are nonsingular.

Our aim is to find a set of parameters μ for which (3.1.1) has a transversal bounded solution $\{\bar{x}_k\}_{k\in\mathbb{Z}}$ near $\{\gamma_k\}_{k\in\mathbb{Z}}$, i.e. the linearization of (3.1.1) along $\{\bar{x}_k\}_{k\in\mathbb{Z}}$ given by

$$v_{k+1} = \big(Df(\bar{x}_k) + D_x h(\bar{x}_k, \mu, k)\big) v_k, \quad k \in \mathbb{Z}$$

has the only bounded solution $v_k = 0$, $\forall\, k \in \mathbb{Z}$ (cf Lemma 2.5.2). When h is independent of k, i.e. (3.1.1) is a mapping, we know from Section 2.5.2 that the existence of such a bounded solution means the existence of a transversal homoclinic orbit and

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thus chaos. In general, (3.1.1) can be associated with quasiperiodically perturbed systems [1–3]. To derive these sets, higher dimensional Melnikov mappings are introduced. Simple zero points of those mappings give wedge-shaped regions in $\mathbb{R}^m$ for μ representing the desired sets.

We establish a complete analogy between the Melnikov theories for difference equations and ordinary differential equations (cf Section 4.1). Two-dimensional mappings are considered in [2, 4, 5]. Mappings in arbitrary finite dimensions are considered in [6–8] but the dimension is 1 in [8], which is released in this section, for the intersection of tangent spaces and stable and unstable manifolds along a homoclinic solution to a hyperbolic fixed point of the unperturbed mapping, and while the transversality is not proved in [6]. In this section, no restriction is given on the dimension of the phase space or on the dimension of intersection of stable and unstable manifolds. Other types of homoclinic bifurcations are given in [9].

3.1.2 Variational Equation

The norm and scalar product of $\mathbb{R}^n$ are denoted by $|\cdot|$, $\langle\cdot,\cdot\rangle$, respectively. Let us consider the unperturbed equation

$$x_{k+1} = f(x_k)\,. \tag{3.1.2}$$

For (3.1.2) we adopt the standard notation W^s, W^u for the local stable and local unstable manifolds, respectively, of the origin and $d_s = \dim W^s$, $d_u = \dim W^u$. Since $x = 0$ is a hyperbolic equilibrium, $\{\gamma_k\}_{k\in\mathbb{Z}}$ must approach the origin along W^s as $k \to +\infty$ and along W^u as $k \to -\infty$. By the *variational equation* of (3.1.2) along $\{\gamma_k\}_{k\in\mathbb{Z}}$ we mean the linear difference equation

$$u_{k+1} = Df(\gamma_k)u_k\,. \tag{3.1.3}$$

We note that as $k \to \pm\infty$, $Df(\gamma_k) \to Df(0)$, a hyperbolic matrix. Thus, the following result yields two solutions for (3.1.3), one for $k \in \mathbb{Z}_+$ and one for $k \in \mathbb{Z}_-$.

Lemma 3.1.1. *Let $k \to A(k)$ be a matrix valued function on $\mathbb{Z}_+$ and suppose there exists a constant nonsingular matrix, A_0, and a scalar $a > 0$ so that* $\sup_{k\in\mathbb{Z}_+} |A(k) - A_0|\,\mathrm{e}^{4ak} < \infty$. *Then there exists a fundamental solution, $X(k)$ for k large, to the difference equation $x_{k+1} = A(k)x_k$ so that $\lim_{k\to\infty} X(k)A_0^{-k} = \mathbb{I}$.*

Proof. The proof is very similar to [10, Lemma 3.1.1] and [11, 1. Lemma], but we present it here for the readers' convenience. Let P be a matrix so that $P^{-1}A_0P = J$, where J is the Jordan form with the block-diagonal form $J = \mathrm{diag}\,(J_1, J_2, \dots, J_r)$. Let k_i be the order of J_i and λ_i is the eigenvalue corresponding to J_i. We arrange the Jordan blocks so that $|\lambda_i| \le |\lambda_{i+1}|$. By putting $y = P^{-1}x$ and $B(k) = P^{-1}A(k)P$, the equation $x_{k+1} = A(k)x_k$ has the form

$$y_{k+1} = B(k)y_k = Jy_k + (B(k) - J)y_k\,. \tag{3.1.4}$$

We fix one block J_i and define $p_i = k_1 + k_2 + \cdots + k_{i-1}$. Similarly we define q_i satisfying $|\lambda_{q_{i-1}}| < |\lambda_i|$ and $|\lambda_{q_i}| = |\lambda_i|$. We split the matrix J^k into $U_1(k), U_2(k)$, where

$$U_1(k) = (J_1^k, J_2^k, \dots, J_{q_{i-1}}^k, 0, \dots, 0),$$
$$U_2(k) = (0, 0, \dots, 0, J_{q_i}^k, \dots, J_r^k).$$

Since the spectrum $\sigma(U_1(1))$ is contained inside the circle with the radius $|\lambda_{q_{i-1}}|$, we can assume by [12, 3.126 Lemma]

$$|U_1(1)| \le |\lambda_{q_{i-1}}| + b \le |\lambda_i| - b$$

for $b > 0$ sufficiently small. Consequently, we obtain for $k \ge 0$ that $|U_1(k)| \le |U_1(1)|^k \le (|\lambda_i| - b)^k$. Since $\sigma(U_2(-1)) = \big(\sigma(U_2(1))\big)^{-1}$, we similarly have

$$|U_2(k)| \le (|\lambda_i| - b)^k, \quad \forall k \in \mathbb{Z}_-$$

again for $b > 0$ sufficiently small. Let e_k be the k-th column of the $n \times n$ identity matrix. By fixing $k_0 \in \mathbb{N}$ sufficiently large, let us define a mapping T_j for $k = k_0, k_0 + 1, \dots$ and for $j \in \{1, 2, \dots, k_i\}$ as follows:

$$T_j(y)_k = J^k e_{p_i+j} + \sum_{j=k_0}^{k-1} U_1(k-1-j)(B(j)-J)y_j - \sum_{j=k}^{\infty} U_2(k-1-j)(B(j)-J)y_j\,. \tag{3.1.5}$$

We consider this mapping on the Banach space:

$$Y = \Big\{ \{y_j\}_{j=k_0}^{\infty} : y_j \in \mathbb{R}^n, \quad \sup_{j \ge k_0} \big|y_j\big| (|\lambda_i| + b)^{-j} < \infty \Big\}$$

with the norm $|||y||| = \sup_{k \ge k_0} |y_k|(|\lambda_i| + b)^{-k}$ for $y = \{y_j\}_{j=k_0}^{\infty}$. To show that T_j is well defined, we compute

$$\sup_k |J^k e_{p_i+j}|(|\lambda_i| + b)^{-k} < \infty,$$

since $|J_i^k| < c_1(|\lambda_i| + d)^k$ for a $0 < d < b$ and $c_1 > 0$. By taking $b > 0$ satisfying

$$\frac{|\lambda_i| + b}{|\lambda_i| - b} < \mathrm{e}^{4a},$$

we have for a constant $c > 0$

$$\sup_k \sum_{j=k_0}^{k-1} \big|U_1(k-1-j)(B(j)-J)y_j\big| (|\lambda_i| + b)^{-k}$$
$$\le c(|\lambda_i| - b)^{-1} |||y||| \sup_k \left(\frac{|\lambda_i| - b}{|\lambda_i| + b}\right)^k \sum_{j=k_0}^{k-1} \left(\frac{|\lambda_i| + b}{|\lambda_i| - b}\,\mathrm{e}^{-4a}\right)^j < \infty$$

and

$$\sup_k \sum_{j=k}^{\infty} \left|U_2(k-1-j)(B(j)-J)y_j\right| (|\lambda_i|+b)^{-k} \le$$
$$\le c \sup_k \sum_{j=k}^{\infty} (|\lambda_i|-b)^{k-1-j} |||y||| (|\lambda_i|+b)^j \mathrm{e}^{-4aj} (|\lambda_i|+b)^{-k}$$
$$\le c(|\lambda_i|-b)^{-1} |||y||| \sup_k \left(\frac{|\lambda_i|-b}{|\lambda_i|+b}\right)^k \sum_{j=k}^{\infty} \left(\frac{|\lambda_i|+b}{|\lambda_i|-b}\,\mathrm{e}^{-4a}\right)^j < \infty .$$

Consequently, we arrive at $|||T_j(y)||| < \infty$, so $T_j : Y \to Y$. Furthermore, we have

$$\forall \varepsilon > 0\, \exists n_0 > k_0 : \left(\frac{|\lambda_i|-b}{|\lambda_i|+b}\right)^k < \varepsilon\ \forall k > n_0 .$$

By using this property, the contraction of T_j follows the same arguments as the well defined T_j. Consequently by Banach fixed point theorem 2.2.1, T_j has a fixed point $y(j)$ satisfying by (3.1.5)

$$|y(j)_k - J^k e_{p_i+j}| \le K_0(|\lambda_i|-b)^k$$

for a constant $K_0 > 0$. By defining the matrix $Y_i(k)$ of the order $n \times k_i$ with $y(j)_k$ in column j, we obtain

$$|Y_i(k) - F_i(k)|(|\lambda_i|-b)^{-k} \le K_0 ,$$

where $F_i(k)$ is the $n \times k_i$-matrix with J_i^k in rows p_i+1 through p_i+k_i and all other rows zero. Let $\overline{G}_i$ be the identity matrix of order $k_i \times k_i$. Then $\lim_{k\to\infty} Y_i(k) J_i^{-k} = G_i$ and G_i is the matrix of order $n \times k_i$ with $\overline{G}_i$ in rows p_i+1 through p_i+k_i and all other rows zero. This construction is done for the block J_i. To get the result, we take the $n \times n$ matrix $Y(k)$ with $Y_i(k)$ in columns p_i+1 through p_i+k_i for $i = 1,2,\ldots,r$. So $\lim_{k\to\infty} Y(k)J^{-k} = \mathbb{I}$. Finally, by putting $X(k) = PY(k)P^{-1}$ we arrive at $X(k+1) = A(k)X(k)$ satisfying

$$X(k)A_0^{-k} \to \mathbb{I} \quad \text{as} \quad k \to \infty .$$

The proof is finished. □

Our next result matches at $k = 0$ the two solutions of (3.1.3) provided by the preceding lemma. The proof of the following theorem is a slight extension of [10, Theorem 3.1.2] and [11, Theorem. 2], so we omit the proof.

Theorem 3.1.2. *Let $d_s = \dim W^s$, $d_u = \dim W^u$ for* (3.1.3) *and let $\mathbb{I}_s$, $\mathbb{I}_u$ denote the identity matrices of order d_s, d_u respectively. There exists a fundamental solution $U(k)$, $k \in \mathbb{Z}$ for* (3.1.3) *along with constants $M > 1$, $K_0 > 0$ and four projections P_{ss}, P_{su}, P_{us}, P_{uu} so that $P_{ss}+P_{su}+P_{us}+P_{uu} = \mathbb{I}$ and the following hold:*

(i) $|U(t)(P_{ss}+P_{us})U(s)^{-1}| \le K_0 M^{(s-t)}$ *for* $0 \le s \le t$,
(ii) $|U(t)(P_{su}+P_{uu})U(s)^{-1}| \le K_0 M^{(t-s)}$ *for* $0 \le t \le s$,
(iii) $|U(t)(P_{ss}+P_{su})U(s)^{-1}| \le K_0 M^{(t-s)}$ *for* $t \le s \le 0$,

(iv) $|U(t)(P_{us}+P_{uu})U(s)^{-1}| \le K_0 M^{(s-t)}$ *for* $s \le t \le 0$.

Also, $\operatorname{rank} P_{ss} = \operatorname{rank} P_{uu} = d$ *for some positive integer* d.

In the language of dichotomies (cf Section 2.5.1) we see that Theorem 3.1.2 provides a two-sided exponential dichotomy. For $k \to -\infty$ an exponential dichotomy is given by the fundamental solution $U(k)$ and the projection $P_{us}+P_{uu}$ while for $k \to +\infty$ such is given by $U(k)$ and $P_{ss}+P_{us}$.

Let $u_j(k)$ denote column j of $U(k)$ and assume that these are numbered so that

$$P_{uu} = \begin{pmatrix} \mathbb{I}_d & 0_d & 0 \\ 0_d & 0_d & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad P_{ss} = \begin{pmatrix} 0_d & 0_d & 0 \\ 0_d & \mathbb{I}_d & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here, $\mathbb{I}_d$ denotes the $d \times d$ identity matrix and 0_d denotes the $d \times d$ zero matrix.

For each $i = 1, \ldots, n$ we define $u_i^\perp(k)$ by $\langle u_i^\perp(k), u_j(k+1) \rangle = \delta_{ij}$. The vectors $u_i^\perp(k)$ can be computed from the formula $U(k)^{\perp *} = U(k+1)^{-1}$ where $U^\perp(k)$ denotes the matrix with $u_j^\perp(k)$ as column j. By using the identity $U(k+1)U(k)^{\perp *} = \mathbb{I}$ we obtain that $U(k+1)^\perp = \left(Df(\gamma_{k+1})^*\right)^{-1} U(k)^\perp$. Thus, $U^\perp(k)$ is the adjoint of $U(k)$. Note $\{u_i^\perp(k)\}_{k \in \mathbb{Z}}$, $i = 1, 2, \cdots, d$ is a basis of bounded solutions on $\mathbb{Z}$ to the adjoint variational equation $w_{k+1} = \left(Df(\gamma_{k+1})^*\right)^{-1} w_k$.

We take the Banach space

$$Z = \left\{ \{y_j\}_{j \in \mathbb{Z}} : y_j \in \mathbb{R}^n, \quad \sup_{j \in \mathbb{Z}} |y_j| < \infty \right\}$$

with the norm $||y|| = \sup_{k \in \mathbb{Z}} |y_k|$ for $y = \{y_j\}_{j \in \mathbb{Z}}$. Summation of the inequalities in Theorem 3.1.2 yields the following result.

Theorem 3.1.3. *Let* U *be the fundamental solution to* (3.1.3) *along with the projections* P_{ss}, P_{su}, P_{us}, P_{uu} *as in Theorem* 3.1.2*. Then there exists a constant* $K > 0$ *so that for any* $z \in Z$ *the following hold:*

(i) $\sum_{k=0}^{j} |U(j)(P_{ss}+P_{us})U(k)^{-1} z_k| \le K\|z\|$ *for* $j \ge 0$,
(ii) $\sum_{k=j}^{\infty} |U(j)(P_{su}+P_{uu})U(k)^{-1} z_k| \le K\|z\|$ *for* $j \ge 0$,
(iii) $\sum_{k=j}^{0} |U(j)(P_{ss}+P_{su})U(k)^{-1} z_k| \le K\|z\|$ *for* $j \le 0$,
(iv) $\sum_{k=-\infty}^{j} |U(j)(P_{us}+P_{uu})U(k)^{-1} z_k| \le K\|z\|$ *for* $j \le 0$.

Let us define a closed linear subspace of Z given by

$$Z_0 = \left\{ z \in Z : \sum_{k=-\infty}^{\infty} P_{uu} U(k+1)^{-1} z_k = 0 \right\}.$$

Note

$$0 = \sum_{k=-\infty}^{\infty} P_{uu} U(k+1)^{-1} z_k = \sum_{k=-\infty}^{\infty} P_{uu} U(k)^{\perp *} z_k \Leftrightarrow \sum_{k=-\infty}^{\infty} \langle u_j^\perp(k), z_k \rangle = 0$$

for all $j = 1, 2, \ldots, d$. We consider the difference equation:

$$z_{k+1} = Df(\gamma_k)z_k + w_k, \quad \{w_k\}_{k\in\mathbb{Z}} \in Z. \tag{3.1.6}$$

The following result is a Fredholm-like condition for (3.1.6).

Theorem 3.1.4. *Necessary and sufficient condition for the existence of a solution* $\{x_k\}_{k\in\mathbb{Z}} \in Z$ *of* (3.1.6) *is that* $\{w_k\}_{k\in\mathbb{Z}} \in Z_0$.

Proof. "$\Longrightarrow$"

Let $z = \{z_k\}_{k\in\mathbb{Z}}$ be a solution of (3.1.6). Denote $A(k) = Df(\gamma_k)$ and compute

$$P_{uu}U(k+1)^{-1}z_{k+1} = P_{uu}U(k+1)^{-1}A(k)z_k + P_{uu}U(k+1)^{-1}w_k\,.$$

Since $U(k+1) = A(k)U(k)$, $U(k+1)^{-1} = U(k)^{-1}A(k)^{-1}$, and hence

$$\sum_{k=-\infty}^{\infty} P_{uu}U(k+1)^{-1}z_{k+1} = \sum_{k=-\infty}^{\infty} P_{uu}U(k)^{-1}z_k + \sum_{k=-\infty}^{\infty} P_{uu}U(k+1)^{-1}w_k$$

which implies

$$\sum_{k=-\infty}^{\infty} P_{uu}U(k+1)^{-1}w_k = 0\,.$$

We note that Theorem 3.1.3 gives the convergence of these series.

"$\Longleftarrow$"

Let $w = \{w_k\}_{k=-\infty}^{\infty} \in Z_0$. We define the mapping $\mathscr{K}$ as follows:

$$\begin{aligned}\mathscr{K}(w)_k = U(k)\Big[&\sum_{j=-\infty}^{-1} P_{us}U(j+1)^{-1}w_j + \sum_{j=0}^{k-1}(P_{ss}+P_{us})U(j+1)^{-1}w_j \\ &-\sum_{j=k}^{\infty}(P_{su}+P_{uu})U(j+1)^{-1}w_j\Big],\end{aligned}$$

for $k \geq 0$,

$$\begin{aligned}\mathscr{K}(w)_k = U(k)\Big[&-\sum_{j=0}^{\infty} P_{su}U(j+1)^{-1}w_j + \sum_{j=-\infty}^{k-1}(P_{us}+P_{uu})U(j+1)^{-1}w_j \\ &-\sum_{j=k}^{-1}(P_{ss}+P_{su})U(j+1)^{-1}w_j\Big],\end{aligned}$$

for $k \leq 0$. Here we define $\sum_{j=0}^{-1} = 0$. Theorem 3.1.3 implies the well defined definition and continuity of $\mathscr{K} : Z_0 \to Z$ and by putting $z_k = \mathscr{K}(w)_k$, $\forall k \in \mathbb{Z}$ in (3.1.6), we easily verify that it is a solution. We note that the general solution of (3.1.6) has the form:

$$z = \sum_{j=1}^{d} \beta_j u_{j+d} + \mathscr{K}(w), \quad \beta_j \in \mathbb{R}\,.$$

The proof is finished. □

The next result provides an appropriate projection.

Theorem 3.1.5. *Let U be as in Theorem* 3.1.2 *and let Z_0 be as in Theorem* 3.1.4. *There exists a bounded projection $\Pi : Z \to Z$ so that $\mathscr{R}\,\Pi = Z_0$.*

Proof. We take Π in the form $\mathbb{I} - P$, where P is defined by

$$P(w)_k = \frac{U(k+1)}{a_{k+1}} P_{uu} \sum_{j=-\infty}^{\infty} U(j+1)^{-1} w_j\,,$$

and the sequence $\{a_k\}_{k\in\mathbb{Z}}$ satisfies

$$a_k > 0, \forall k \in \mathbb{Z}, \quad \sum_{k=-\infty}^{\infty} \frac{1}{a_{k+1}} = 1, \quad \sup_{k\in\mathbb{Z}} \frac{U(k+1)}{a_{k+1}} < \infty\,.$$

We verify that this P is a projection, i.e. $P^2 = P$:

$$P(P(w))_k = P\left(\left\{\frac{U(s+1)}{a_{s+1}} P_{uu} \sum_{j=-\infty}^{\infty} U(j+1)^{-1} w_j\right\}_{s\in\mathbb{Z}}\right)$$
$$= \frac{U(k+1)}{a_{k+1}} P_{uu} \sum_{l=-\infty}^{\infty} U(l+1)^{-1} \left(\frac{U(l+1)}{a_{l+1}} P_{uu} \sum_{j=-\infty}^{\infty} U(j+1)^{-1} w_j\right) = P(w)_k\,.$$

Hence P is a projection. Now we verify that $\Pi = \mathbb{I} - P$ is such that $\Pi w \in Z_0$:

$$\sum_{k=-\infty}^{\infty} P_{uu} U(k+1)^{-1} \Pi(w)_k = \sum_{k=-\infty}^{\infty} P_{uu} U(k+1)^{-1} (\mathbb{I} - P)(w)_k$$
$$= \sum_{k=-\infty}^{\infty} P_{uu} U(k+1)^{-1} \left(w_k - \frac{U(k+1)}{a_{k+1}} P_{uu} \sum_{j=-\infty}^{\infty} U(j+1)^{-1} w_j\right)$$
$$= \sum_{k=-\infty}^{\infty} P_{uu} U(k+1)^{-1} w_k$$
$$- \sum_{k=-\infty}^{\infty} P_{uu} U(k+1)^{-1} \left(\frac{U(k+1)}{a_{k+1}} P_{uu} \sum_{j=-\infty}^{\infty} U(j+1)^{-1} w_j\right) = 0\,.$$

Consequently, Π has the desired properties. □

3.1.3 Perturbation Theory

We study the equation (cf Theorem 2.2.4):

$$\begin{aligned} F_{\mu,\varepsilon,y}(x)_k &= x_{k+1} - f(x_k) - h(x_k,\mu,k) - \varepsilon|\mu|\mathscr{L}(x-y-\gamma) = 0 \\ F_{\mu,\varepsilon,y} &: Z \to Z\,, \end{aligned} \tag{3.1.7}$$

where $\mathscr{L} : Z \to Z$ is a linear continuous mapping so that $||\mathscr{L}|| \le 1$, $y \in Z$, and $\varepsilon \in \mathbb{R}$ is small. It is clear that solutions of (3.1.7) with $\varepsilon = 0$ are bounded solutions of (3.1.1). We define mappings $L : Z \to Z$ and $G : Z \times \mathbb{R}^m \times \mathbb{R} \times Z \to Z$ as follows:

$$\begin{aligned} L(z)_k &= z_{k+1} - Df(\gamma_k)z_k, \\ G(z,\mu,\varepsilon,y)_k &= f(z_k+\gamma_k) - f(\gamma_k) - Df(\gamma_k)z_k + h(z_k+\gamma_k,\mu,k) + \varepsilon|\mu|\mathscr{L}(z-y)\,. \end{aligned}$$

By putting $x = z + \gamma$ in (3.1.7), this equation has the form:

$$L(z) = G(z,\mu,\varepsilon,y)\,. \tag{3.1.8}$$

We decompose (3.1.8) in the following way

$$L(z) = \Pi G(z,\mu,\varepsilon,y)\,, \quad 0 = (\mathbb{I} - \Pi)G(z,\mu,\varepsilon,y)\,.$$

By using Theorem 3.1.4, the above pair of equations is equivalent to

$$z = \sum_{j=1}^{d} \beta_j u_{j+d} + \mathscr{K}\big(\Pi G(z,\mu,\varepsilon,y)\big), \quad \beta_j \in \mathbb{R} \tag{3.1.9}$$

and

$$0 = (\mathbb{I} - \Pi)G(z,\mu,\varepsilon,y)\,. \tag{3.1.10}$$

Moreover by using the Lyapunov-Schmidt procedure from Section 2.2.3 like in [11, Theorem 8], the study of Eqs. (3.1.9) and (3.1.10) can be expressed in the following theorem for z, μ, ε, $\beta = (\beta_1, \beta_2, \ldots, \beta_d)$, y sufficiently small.

Theorem 3.1.6. *Let U and d be as in Theorem* 3.1.2. *Then there exist small neighborhoods $0 \in Q \subset Z$, $0 \in O \subset \mathbb{R}^d$, $0 \in W \subset \mathbb{R}^m$, $0 \in V \subset \mathbb{R}$ and a C^3-function $H : Q \times O \times W \times V \to \mathbb{R}^d$ denoted by $(y,\beta,\mu,\varepsilon) \to H(y,\beta,\mu,\varepsilon)$ with the following properties:*

(i) *The equation $H(y,\beta,\mu,\varepsilon) = 0$ holds if and only if* (3.1.7) *has a solution near γ and moreover, each such $(y,\beta,\mu,\varepsilon)$ determines only one solution of* (3.1.7)*,*
(ii) $H(0,0,0,0) = 0$*,*
(iii) $\dfrac{\partial H_i}{\partial \mu_j}(0,0,,0,0) = -\sum_{k\in\mathbb{Z}} \left\langle u_i^{\perp}(k), \dfrac{\partial h}{\partial \mu_j}(\gamma_k,0,k) \right\rangle$*,*
(iv) $\dfrac{\partial H_i}{\partial \beta_j}(0,0,0,0) = 0$*,*
(v) $\dfrac{\partial^2 H_i}{\partial \beta_k \partial \beta_j}(0,0,0,0) = -\sum_{l\in\mathbb{Z}} \langle u_i^{\perp}(l), D^2 f(\gamma_l)(u_{d+j}(l), u_{d+k}(l))\rangle$*.*

We introduce the following notations:

$$a_{ij} = -\sum_{l\in\mathbb{Z}} \left\langle u_i^\perp(l), \frac{\partial h}{\partial \mu_j}(\gamma_l, 0, l) \right\rangle,$$
$$b_{ijk} = -\sum_{l\in\mathbb{Z}} \left\langle u_i^\perp(l), D^2 f(\gamma_l)(u_{d+j}(l), u_{d+k}(l)) \right\rangle.$$

Finally, we take the mapping $M_\mu : \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$\big(M_\mu(\beta)\big)_i = \sum_{j=1}^m a_{ij}\mu_j + \frac{1}{2}\sum_{j,k=1}^d b_{ijk}\beta_j\beta_k.$$

Now we can state the main result of this section.

Theorem 3.1.7. *If M_{μ_0} has a simple zero point β_0, i.e. β_0 satisfies $M_{\mu_0}(\beta_0) = 0$ and $D_\beta M_{\mu_0}(\beta_0)$ is a regular matrix, then there is a wedge-shaped region in $\mathbb{R}^m$ for μ of the form*

$$\mathscr{R} = \Big\{ s^2\tilde{\mu} \,:\, s\text{, respectively } \tilde{\mu}\text{, is from a small open neighborhood of } 0 \in \mathbb{R}\text{, respectively of } \mu_0 \in \mathbb{R}^m \Big\}$$

so that for any $\mu \in \mathscr{R}\setminus\{0\}$, Equation (3.1.1) possesses a transversal bounded solution.

Proof. Let us consider the mapping defined by

$$\Phi(y,\tilde{\beta},\tilde{\mu},\tilde{\varepsilon},s) = \begin{cases} \dfrac{1}{s^2} H(y, s\tilde{\beta}, s^2\tilde{\mu}, s^3\tilde{\varepsilon}), & \text{for} \quad s \neq 0, \\ M_{\tilde{\mu}}(\tilde{\beta}), & \text{for} \quad s = 0. \end{cases}$$

According to (ii)–(v) of Theorem 3.1.6, the mapping Φ is C^1-smooth near

$$(y,\tilde{\beta},\tilde{\mu},\tilde{\varepsilon},s) = (0,\beta_0,\mu_0,0,0)$$

with respect to the variable $\tilde{\beta}$. Since

$$M_{\mu_0}(\beta_0) = 0 \quad \text{and} \quad D_\beta M_{\mu_0}(\beta_0) \quad \text{is a regular matrix},$$

we can apply the implicit function theorem to solving locally and uniquely the equation $\Phi = 0$ in the variable $\tilde{\beta}$. This gives for $\varepsilon = 0$, by (i) of Theorem 3.1.6, the existence of $\mathscr{R}$ on which (3.1.1) has a bounded solution.

To prove the transversality of these bounded solutions, we fix $\mu \in \mathscr{R}\setminus\{0\}$ and take

$$y = \tilde{\gamma} - \gamma,$$

where $\tilde{\gamma}$ is the solution of (3.1.7) for which the transversality should be proved. Then we vary $\varepsilon = s^3\tilde{\varepsilon}$ small. Note that $s \neq 0$ is also fixed due to $\mu = s^2\tilde{\mu}$. Since the local uniqueness of solutions of (3.1.7) near $\tilde{\gamma}$ is satisfied for any $\tilde{\varepsilon}$ sufficiently small according to the above application of the implicit function theorem, such equation (3.1.7) (with the fixed $\mu \in \mathscr{R} \setminus \{0\}$, $\varepsilon = s^3\tilde{\varepsilon}$ where $s \neq 0$ is also fixed and the special $y = \tilde{\gamma} - \gamma$) has the only solution $x = \tilde{\gamma}$ near $\tilde{\gamma}$ for any $\tilde{\varepsilon}$ sufficiently small. Now Theorem 2.2.4 gives the invertibility of $DF_{\mu,0,\tilde{\gamma}-\gamma}(\tilde{\gamma})$ and so the only bounded solution on $\mathbb{Z}$ of the equation

$$v_{k+1} = Df(\tilde{\gamma}_k)v_k + D_x h(\tilde{\gamma}_k, \mu, k)v_k$$

is $v_k = 0$, $\forall k \in \mathbb{Z}$. The proof is finished. □

Remark 3.1.8. Note that we can take any bases of bounded solutions of the variational and adjoint variational equations for constructing the Melnikov function M_μ. Similar observations can be applied to detecting of other Melnikov functions in this book.

Remark 3.1.9. Assume that (3.1.1) is autonomous, i.e. h is independent of k, suppose conditions (i)–(iv) and f is a diffeomorphism. Then we have a local diffeomorphism $F_\mu(x) := f(x) + h(x,\mu)$ for μ small. If there is an open bounded subset $\Omega \subset \mathbb{R}^d$ so that $0 \notin M_{\mu_0}(\partial\Omega)$ and $\deg\left(M_{\mu_0}, \Omega, 0\right) \neq 0$ then for any $0 \neq \mu \in \mathscr{R}$ there is a $k_\mu \in \mathbb{N}$ such that for any $k \geq k_\mu$ there is a set $\Lambda_k \subset \mathbb{R}^n$ and a continuous mapping $\varphi_k : \Lambda_k \to \mathscr{E}$ so that $F_\mu^{2k}(\Lambda_k) = \Lambda_k$, φ_k is surjective and injective, and $\varphi_k \circ F_\mu^{2k} = \sigma \circ \varphi_k$. Note that we do not know whether φ_k is a homeomorphism. But we do know that F_μ has infinitely many periodic orbits and quasiperiodic ones and it has positive *topological entropy*. This is a generalization of the Smale-Birkhoff homoclinic theorem 2.5.4 to this case. Particularly, if β_0 is an isolated zero of M_{μ_0} with a nonzero Brouwer index, then we have a chaotic behaviour of F_μ (cf [13]). This remark can be applied to other Melnikov type conditions in this book.

3.1.4 Bifurcation from a Manifold of Homoclinic Solutions

In many cases, (3.1.2) has a manifold of homoclinic solutions. Hence we suppose that

(v) There is an open non-empty subset $\mathscr{O} \subset \mathbb{R}^d$ and C^3-smooth mappings $\gamma_k : \mathscr{O} \to \mathbb{R}^n$, $\omega : \mathscr{O} \to \mathbb{R}^n$, $\forall k \in \mathbb{Z}$ satisfying

$$\begin{aligned} \gamma_{k+1}(\theta) = f(\gamma_k(\theta)), &\quad \forall k \in \mathbb{Z}, \forall \theta \in \mathscr{O}, \\ \omega(\theta) = f(\omega(\theta)), &\quad \forall \theta \in \mathscr{O}, \\ \lim_{k \to \pm\infty} \gamma_k(\theta) = \omega(\theta), &\quad \forall \theta \in \mathscr{O}. \end{aligned}$$

(vi) The eigenvalues of $Df(\omega(\theta)) \forall \theta \in \mathscr{O}$ are non-zero and all lie off the unit circle. Moreover, $Df(\gamma_k(\theta))$ $\forall k \in \mathbb{Z}$, $\forall \theta \in \mathscr{O}$ are nonsingular.

(vii) $\frac{\partial \gamma_k}{\partial \theta_i}$ are uniformly bounded on $\mathscr{O}$ with respect to $k \in \mathbb{Z}$ when $\theta = (\theta_1, \theta_2, \ldots, \theta_d)$.

(viii) From $\gamma_{k+1}(\theta) = f(\gamma_k(\theta))$, we obtain $\frac{\partial \gamma_{k+1}}{\partial \theta_i}(\theta) = Df(\gamma_k(\theta))\frac{\partial \gamma_k}{\partial \theta_i}(\theta)$. We suppose that $\left\{\frac{\partial \gamma_k}{\partial \theta_i}(\theta)\right\}_{i=1,k\in\mathbb{Z}}^{d}$ is a basis of the space of bounded solutions of the difference equation

$$v_{k+1} = Df(\gamma_k(\theta))v_k\,. \tag{3.1.11}$$

We use the approach of Section 3.1.3 by considering θ as a parameter. The difference is only that now $\left\{\frac{\partial \gamma_k}{\partial \theta_i}(\theta)\right\}_{i=1,k\in\mathbb{Z}}^{d}$ provides a natural family of solutions of (3.1.11) corresponding to the projections P_{ss}. Hence we suppose that Theorem 3.1.2 holds parametricaly by $\theta \in \mathscr{O}$, i.e. $U = U(\theta,t)$ is smooth in (θ,t) and columns of $U(\theta,t)$ are numbered so that

$$P_{uu} = \begin{pmatrix} \mathbb{I}_d & 0_d & 0 \\ 0_d & 0_d & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad P_{ss} = \begin{pmatrix} 0_d & 0_d & 0 \\ 0_d & \mathbb{I}_d & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now we take $x = z + \gamma(\theta), \gamma(\theta) = \{\gamma_k(\theta)\}_{k\in\mathbb{Z}}$ in (3.1.7). The corresponding operators of (3.1.8) then depend on θ as well:

$$\begin{aligned} L(z,\theta)_k &= z_{k+1} - Df(\gamma_k(\theta))z_k, \\ G(z,\theta,\mu,\varepsilon,y)_k &= f(z_k + \gamma_k(\theta)) - f(\gamma_k(\theta)) - Df(\gamma_k(\theta))z_k \\ &\quad + h(z_k + \gamma_k(\theta),\mu,k) + \varepsilon|\mu|\mathscr{L}(z-y)\,. \end{aligned}$$

Consequently, (3.1.7) has the form

$$L(z,\theta) = G(z,\theta,\mu,\varepsilon,y)\,,$$

and (3.1.9)–(3.1.10) are replaced by

$$z = \mathscr{K}(\theta)\big(\Pi(\theta)G(z,\theta,\mu,\varepsilon,y)\big), \quad 0 = (\mathbb{I} - \Pi(\theta))G(z,\theta,\mu,\varepsilon,y)\,, \tag{3.1.12}$$

where $\mathscr{K}(\theta)$ and $\Pi(\theta)$ are corresponding mappings to $\mathscr{K}, \Pi$, respectively. We consider in (3.1.12) the variable θ as a bifurcation parameter. We take the mapping $N_\mu : \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$\big(N_\mu(\theta)\big)_i = \sum_{j=1}^{m} a_{ij}(\theta)\mu_j\,,$$

where

$$a_{ij}(\theta) = -\sum_{l\in\mathbb{Z}} \left\langle u_i^\perp(\theta,l), \frac{\partial h}{\partial \mu_j}(\gamma_l(\theta),0,l)\right\rangle.$$

The vectors $u_i^\perp(\theta,l)$ are defined by $\langle u_i^\perp(\theta,l), u_j(\theta,l+1)\rangle = \delta_{ij}$. By repeating the proof of Theorem 3.1.7, we can state the main result of this section.

Theorem 3.1.10. *If N_{μ_0} has a simple zero point θ_0, i.e. θ_0 satisfies $N_{\mu_0}(\theta_0) = 0$ and $D_\theta N_{\mu_0}(\theta_0)$ is a regular matrix, then there is a wedge-shaped region in $\mathbb{R}^m$ for μ of* *the form*

$$\mathscr{R} = \Big\{ s\tilde{\mu} \,:\, s, \text{ respectively } \tilde{\mu}, \text{ is from a small open neighborhood of } 0 \in \mathbb{R}, \text{ respectively of } \mu_0 \in \mathbb{R}^m \Big\}$$

so that for any $\mu \in \mathscr{R} \setminus \{0\}$, *Equation* (3.1.1) *possesses a transversal bounded solution.*

3.1.5 Applications to Impulsive Differential Equations

It is well known that the theory of impulsive differential equations is an important branch of differential equations with many applications [14–20]. For this reason, we consider a 4-dimensional impulsive differential equation given by

$$\begin{aligned} \dot{z} &= g_1(z), \quad \dot{y} = g_2(y), \\ z(i+) &= z(i-) + \mu h_1(z(i-), y(i-), \mu), \\ y(i+) &= y(i-) + \mu h_2(z(i-), y(i-), \mu), \quad i \in \mathbb{Z}, \end{aligned} \tag{3.1.13}$$

where

$$g_{1,2} \in C^3(\mathbb{R}^2, \mathbb{R}^2), \quad h_{1,2} \in C^3(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^2), \quad \mu \in \mathbb{R}$$

and $\dot{z} = g_1(z)$, $\dot{y} = g_2(y)$ are Hamiltonian systems. Let Ψ_1, Ψ_2 be the 1-time Poincarè mappings of $\dot{z} = g_1(z)$, $\dot{y} = g_2(y)$, respectively. Here $z(i\pm) = \lim_{s \to i_\pm} z(s)$. We consider the mapping

$$\begin{aligned} &F(z, y, \mu) = \\ &\Big(\Psi_1(z) + \mu h_1\big(\Psi_1(z), \Psi_2(y), \mu\big), \Psi_2(y) + \mu h_2\big(\Psi_1(z), \Psi_2(y), \mu\big)\Big). \end{aligned} \tag{3.1.14}$$

Clearly the dynamics of (3.1.14) determines the behaviour of (3.1.13). In the notation of (3.1.1), we have

$$\begin{aligned} x = (z, y) \in \mathbb{R}^2 \times \mathbb{R}^2, \quad & f(x) = \big(\Psi_1(z), \Psi_2(y)\big) \\ h(x, \mu, k) = \Big(\mu h_1\big(\Psi_1(z), \Psi_2(y), \mu\big),& \mu h_2\big(\Psi_1(z), \Psi_2(y), \mu\big)\Big). \end{aligned} \tag{3.1.15}$$

We suppose

(a) $g_{1,2}(0) = 0$ and the eigenvalues of $Dg_{1,2}(0)$ lie off the imaginary axis.
(b) There are homoclinic solutions γ_1, γ_2 of $\dot{z} = g_1(z)$, $\dot{y} = g_2(y)$, respectively, to 0.

The conditions (a) and (b) imply that

$$\gamma_k(\theta) = \big(\gamma_1(\theta_1 + k), \gamma_2(\theta_2 + k)\big), \quad k \in \mathbb{Z}$$
$$\omega(\theta) = (0,0), \quad \theta = (\theta_1, \theta_2) \in \mathscr{O} = \mathbb{R}^2$$

satisfy (v)–(viii) of Section 3.1.4 for (3.1.15). Now (3.1.11) has the form

$$v_{k+1} = D\Psi_1\big(\gamma_1(\theta_1 + k)\big)v_k\,, \quad w_{k+1} = D\Psi_2\big(\gamma_2(\theta_2 + k)\big)w_k\,.$$

Hence (3.1.11) is now decomposed into two difference equations. We note that $\Psi_{1,2}$ are area-preserving, i.e. $\det D\Psi_{1,2}(z) = 1$ (cf Sections 2.5.1 and 2.5.3). We can take

$$u_3(\theta,k) = \big(\dot\gamma_1(\theta_1 + k), 0\big)\,, \quad u_4(\theta,k) = \big(0, \dot\gamma_2(\theta_2 + k)\big)\,.$$

Now we need the following result [8, pp. 104–105].

Lemma 3.1.11. *Let* $\{A_k\}_{k\in\mathbb{Z}}$ *be a sequence of invertible* 2×2*-matrices so that* $\det A_k = 1$. *If* $\{x_k\}_{k\in\mathbb{Z}}$ *satisfies* $x_{n+1} = A_k x_k$*, then* $z_k := Jx_{k+1}$ *for* $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *satisfies* $z_{k+1} = (A_{k+1}^*)^{-1} z_k$.

Proof. The result directly follows from the identity $A_k^* \circ J \circ A_k = \det A_k J = J$. □

Using Lemma 3.1.11, we can take

$$u_1^\perp(\theta,k) = \big(\bar{\dot\gamma}_1(\theta_1 + k + 1), 0\big), \quad u_2^\perp(\theta,k) = \big(0, \bar{\dot\gamma}_2(\theta_2 + k + 1)\big)\,,$$

where $\bar z = (z_2, -z_1), \forall z = (z_1, z_2) \in \mathbb{R}^2$, and $u_1(\theta,k)$, $u_2(\theta,k)$ are not required to be known. Consequently, the mapping N_μ of Section 3.1.4 has now the form

$$\begin{aligned}
\big(N_\mu(\theta)\big)_1 &= -\mu \sum_{k\in\mathbb{Z}} h_1\big(\Psi_1(\gamma_1(\theta_1 + k)), \Psi_2(\gamma_2(\theta_2 + k)), 0\big) \wedge \dot\gamma_1(\theta_1 + k + 1) \\
&= \mu \sum_{k\in\mathbb{Z}} \dot\gamma_1(\theta_1 + k) \wedge h_1\big(\gamma_1(\theta_1 + k), \gamma_2(\theta_2 + k), 0\big)\,, \\
\big(N_\mu(\theta)\big)_2 &= -\mu \sum_{k\in\mathbb{Z}} h_2\big(\Psi_1(\gamma_1(\theta_1 + k)), \Psi_2(\gamma_2(\theta_2 + k)), 0\big) \wedge \dot\gamma_2(\theta_2 + k + 1) \\
&= \mu \sum_{k\in\mathbb{Z}} \dot\gamma_2(\theta_2 + k) \wedge h_2\big(\gamma_1(\theta_1 + k), \gamma_2(\theta_2 + k), 0\big)\,,
\end{aligned} \tag{3.1.16}$$

where $\wedge$ is the wedge product defined by $z \wedge y = z_1 y_2 - z_2 y_1$, $z, y \in \mathbb{R}^2$. Theorem 3.1.10 gives the following result.

Theorem 3.1.12. *If there is a simple zero point of* $N_1(\theta)$ *given by* (3.1.16)*, then* (3.1.13) *has a transversal homoclinic solution and so it exhibits chaos for any* $\mu \neq 0$ *sufficiently small.*

Of course, there are h_1, h_2 satisfying the assumptions of Theorem 3.1.12. For simplicity, we assume

$$\begin{aligned} g = g_1 = g_2, \quad h_1(z,y,\mu) &= (1+\mu)y + \alpha \\ h_2(z,y,\mu) &= (1+\mu^2)z + \alpha\,, \end{aligned} \tag{3.1.17}$$

where $\alpha \in \mathbb{R}^2$ is a constant vector. Then we have $\gamma_1 = \gamma_2 = \gamma$ and (3.1.16) possesses the form

$$\begin{aligned}\big(N_\mu(\theta)\big)_1 &= \mu \sum_{k\in\mathbb{Z}} \dot\gamma(\theta_1+k)\wedge\gamma(\theta_2+k) + \mu\Big(\sum_{k\in\mathbb{Z}}\dot\gamma(\theta_1+k)\Big)\wedge\alpha\\ \big(N_\mu(\theta)\big)_2 &= \mu \sum_{k\in\mathbb{Z}} \dot\gamma(\theta_2+k)\wedge\gamma(\theta_1+k) + \mu\Big(\sum_{k\in\mathbb{Z}}\dot\gamma(\theta_2+k)\Big)\wedge\alpha\,.\end{aligned} \tag{3.1.18}$$

We put

$$\Omega(\tau) = \sum_{k\in\mathbb{Z}} \dot\gamma(\tau+k)\wedge\gamma(\tau+k) + \Big(\sum_{k\in\mathbb{Z}}\dot\gamma(\tau+k)\Big)\wedge\alpha\,.$$

We note that Ω is 1-periodic. We clearly for $\theta = (\tau,\tau)$ have

$$\big(N_\mu(\theta)\big)_1 = \big(N_\mu(\theta)\big)_2 = \mu\Omega(\tau),$$
$$\big(DN_\mu(\theta)\big)_1 = \mu\big(\Omega'(\tau),0\big), \quad \big(DN_\mu(\theta)\big)_2 = \mu\big(0,\Omega'(\tau)\big)\,.$$

Simple computations give the following result.

Theorem 3.1.13. *Consider* (3.1.13) *with* (3.1.17). *If τ_0 is a simple root of $\Omega(\tau)$ then $\theta_0 = (\tau_0,\tau_0)$ is a simple zero point of $N_1(\theta)$ given by* (3.1.18).

To be more concrete, we take in (3.1.17)

$$g(x_1,x_2) = \big(x_2, x_1 - 2x_1^3\big), \quad \alpha = (\beta,\beta)\,.$$

Hence (3.1.13) has the form

$$\begin{aligned}\ddot z &= x - 2x^3, \quad \ddot y = y - 2y^3,\\ x(i+) &= x(i-) + \mu\big((1+\mu)y(i-)+\beta\big),\\ \dot x(i+) &= \dot x(i-) + \mu\big((1+\mu)\dot y(i-)+\beta\big),\\ y(i+) &= y(i-) + \mu\big((1+\mu^2)x(i-)+\beta\big),\\ \dot y(i+) &= \dot y(i-) + \mu\big((1+\mu^2)\dot x(i-)+\beta\big), \quad i\in\mathbb{Z}\,.\end{aligned} \tag{3.1.19}$$

(3.1.19) are two Duffing equations coupled by impulsive effects. We now take $\gamma(t) = \big(\operatorname{sech} t, \dot{\operatorname{sech}}\, t\big)$ and Ω has the form

$$\Omega(\tau) = \sum_{k\in\mathbb{Z}} \operatorname{sech}^4(\tau+k) + \beta\sum_{k\in\mathbb{Z}} \frac{3-\mathrm{e}^{2(\tau+k)}}{2}\operatorname{sech}^3(\tau+k)\,.$$

Consequently, we have

$$\Omega(\tau) = \Omega_1(\tau) - \beta\Omega_2(\tau)\,,$$

where

$$\Omega_1(\tau) = \sum_{k\in\mathbb{Z}} \operatorname{sech}^4(\tau+k), \quad \Omega_2(\tau) = \sum_{k\in\mathbb{Z}} \frac{\mathrm{e}^{2(\tau+k)}-3}{2}\operatorname{sech}^3(\tau+k)\,.$$

The functions $\Omega_{1,2}$ are again 1-periodic. Moreover, they are analytic and Ω_1 is positive (cf Section 2.6.5). Clearly, Ω_2/Ω_1 is non-constant. So the image of $\mathbb{R}$ by Ω_2/Ω_1 is an interval $[a_1,a_2]$, $-\infty < a_1 < a_2 < \infty$ and there is only a finite number of $\beta_1,\dots,\beta_{j_0} \in [a_1,a_2]$ so that $\Omega = \Omega_1 - \beta\Omega_2$ does have a simple root for any $\beta \neq 0$ satisfying $1/\beta \in [a_1,a_2] \setminus \{\beta_1,\dots,\beta_{j_0}\}$.

Numerical evaluation of the graph of $\Omega_2(\tau)/\Omega_1(\tau)$ shows that (Figure 3.1)

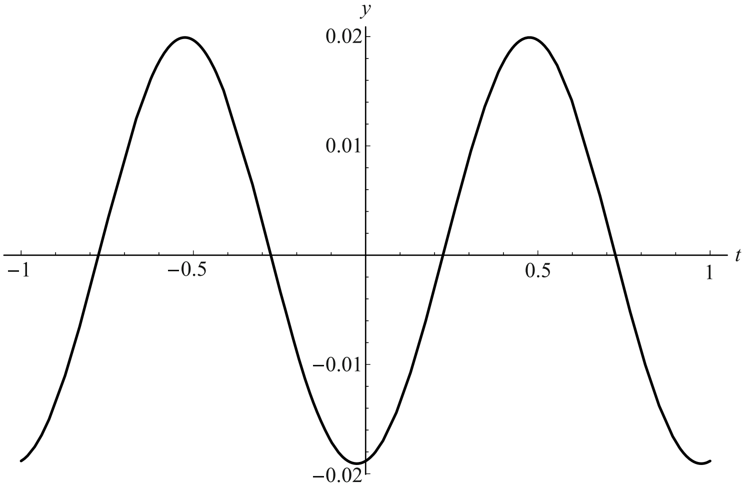

Fig. 3.1 The graph of function $y = \Omega_2(\tau)/\Omega_1(\tau)$.

$$a_1 = \beta_1 \simeq -0.0190729, \quad a_2 = \beta_2 \simeq 0.0199198, \quad j_0 = 2\,.$$

In summary, we arrive at the following result.

Theorem 3.1.14. *If either* $\beta < -52.431$ *or* $\beta > 50.202$ *then impulsive system* (3.1.19) *has a chaotic behaviour for any* $\mu \neq 0$ *sufficiently small.*

We note that a coupled two McMillan mappings (cf Section 3.2.4 and [4,5]) can be similarly studied. In general, after applying our results, the main difficulty is to find an appropriate form of the Melnikov mapping derived in the above way so that one could be able to detect its simple zero point. The Poisson summation formula like in [4] could help to overcome this difficulty.

Remark 3.1.15. Similar to the above, we can study more general impulsive ODEs of the form

$$\begin{aligned} \dot{x} &= f(x,\varepsilon), \\ x(i+0) &= x(i-0) + \varepsilon a\big(x(i-0),\varepsilon\big), \quad i \in \mathbb{Z}, \end{aligned} \tag{3.1.20}$$

where $f,\, a \in C^2(\mathbb{R}^{n+1},\mathbb{R}^n)$, $f(\cdot,0)$ has a hyperbolic fixed point x_0 with a homoclinic orbit $\gamma(\cdot)$. Furthermore, assume that the adjoint variational equation

$$\dot{v} = -\Big(D_x f\big(\gamma(t),0\big)\Big)^* v$$

has only a unique (up to constant multiples) bounded nonzero solution u. Then the Melnikov function of (3.1.20) has the form

$$\mathscr{M}(t) = \sum_{i=-\infty}^{\infty} \big\langle a\big(\gamma(t+i),0\big), u(t+i)\big\rangle + \int_{-\infty}^{\infty} \big\langle D_\varepsilon f\big(\gamma(s),0\big), u(s)\big\rangle\, ds. \qquad (3.1.21)$$

Note that formula (3.1.21) follows also from considerations of Sections 3.3 and 3.4. We see that (3.1.21) consists of the continuous and impulsive parts of (3.1.20) as well.

Finally we note that a different type of chaos is studied in [21] for a special initial value problem of a non-autonomous impulsive differential equation. ODEs with step function coefficients are studied in [22–28], and our theory can be applied to such ODEs.

3.2 Transversal Homoclinic Orbits

3.2.1 Higher Dimensional Difference Equations

This section is a continuation of Section 3.1. So we consider difference equation

$$x_{n+1} = g(x_n) + \varepsilon h(n, x_n, \varepsilon) \qquad (3.2.1)$$

where $x_n \in \mathbb{R}^N$, $\varepsilon \in \mathbb{R}$ is a small parameter. The main purpose of this section is to study the homoclinic bifurcations of difference equations in a degenerate case. We assume the following conditions about the difference equation (3.2.1):

(H1) g, h are C^3-smooth in all continuous variables.
(H2) The unperturbed difference equation

$$x_{n+1} = g(x_n) \qquad (3.2.2)$$

has a hyperbolic fixed poitnt 0, that is, the eigenvalues of $g_x(0)$ are non-zero and they lie off the unit circle.
(H3) The unperturbed difference equation (3.2.2) has a one-parameter family of homoclinic solutions $\gamma(\alpha) = \{\gamma_n(\alpha)\}_{-\infty}^{\infty}$, $\alpha \in \mathbb{R}$ connecting 0. That is, $\{\gamma_n(\alpha)\}_{-\infty}^{\infty}$ is a non-zero sequence of C^3-smooth vector functions satisfying $\gamma_{n+1}(\alpha) = g(\gamma_n(\alpha))$ and $\lim_{n\to\pm\infty} \gamma_n(\alpha) = 0$ uniformly with respect to bounded α. The set $\cup_{n\in\mathbb{Z}} \cup_{\alpha\in\mathbb{R}} \{\gamma_n(\alpha)\}$ is bounded.
(H4) $g_x(\gamma_n(\alpha))$ is invertible, and $\|g_x^{-1}(\gamma_n(\alpha))\|$ is uniformly bounded on $\mathbb{Z}$.

We denote by $W^s(0)$ and $W^u(0)$ the stable and unstable manifolds of the hyperbolic fixed point 0, respectively, and by $T_{\gamma_0(\alpha)}W^s(0)$ and $T_{\gamma_0(\alpha)}W^u(0)$ the tangent spaces to $W^s(0)$ and $W^u(0)$ at $\gamma_0(\alpha)$. We say the homoclinic orbit $\{\gamma_n(\alpha)\}_{-\infty}^{\infty}$ is *degenerate* if the dimension of the linear subspace

$$T_{\gamma_0(\alpha)}W^s(0) \cap T_{\gamma_0(\alpha)}W^u(0)$$

is greater than one. Otherwise, we say the homoclinic orbit $\{\gamma_n(\alpha)\}_{-\infty}^{\infty}$ is *nondegenerate*. We can easily prove that the homoclinic orbit $\{\gamma_n(\alpha)\}_{-\infty}^{\infty}$ is degenerate if and only if the following variational equation along the homoclinic orbit $\{\gamma_n(\alpha)\}_{-\infty}^{\infty}$

$$\xi_{n+1} = g_x(\gamma_n(\alpha))\xi_n \tag{3.2.3}$$

has $d > 1$ linearly independent bounded solutions on $\mathbb{Z}$.

When h is independent of n, i.e. (3.2.1) is a mapping, the existence of a transversal homoclinic solution for (3.2.1) is discussed in [8, 29]. When h depends on n, the existence of a transversal homoclinic solution for (3.2.1) in the degenerate case is discussed in Section 3.1. Now we study (3.2.1) also with $d > 1$ for (3.2.3). Our aim is to find analytic conditions under which the difference equation (3.2.1) has for $\varepsilon \neq 0$ sufficiently small a transversal bounded solution $\{x_n(\varepsilon)\}_{-\infty}^{\infty}$ near the homoclinic solution $\{\gamma_n(\alpha)\}_{-\infty}^{\infty}$. The transversality of $\{x_n(\varepsilon)\}_{-\infty}^{\infty}$ means that the linearization of the difference equation (3.3.1) along $\{x_n(\varepsilon)\}_{-\infty}^{\infty}$ given by

$$\xi_{n+1} = [g_x(x_n(\varepsilon)) + \varepsilon h_x(n, x_n(\varepsilon), \varepsilon)]\,\xi_n$$

admits an exponential dichotomy on $\mathbb{Z}$ (cf Lemma 2.5.2).

The degenerate problem, when $d > 1$ for (3.2.3), can be naturally divided into two cases:

(1) There exists a d-dimensional homoclinic manifold. This is the most natural way to get $d > 1$ for (3.2.3).
(2) The invariant manifolds $W^s(0)$ and $W^u(0)$ meet in only a higher dimensional tangency.

Case (1) is studied in Section 3.1.4 (see also more comments at the end of Section 3.2.2), and Case 2 is treated in this section.

Two-dimensional mappings for nondegenerate cases are considered in [2, 4, 5]. Higher dimensional mappings are studied in [7].

3.2.2 Bifurcation Result

Let

$$X = \Big\{\{x_n\}_{-\infty}^{\infty}\} \mid |x_n \in \mathbb{R}^N \quad \text{and} \quad \sup_{n\in\mathbb{Z}} |x_n| < \infty\Big\}$$

be the Banach space with the norm $|x| = \sup_{n\in\mathbb{Z}} |x_n|$ for $x = \{x_n\}_{-\infty}^{\infty}$. We define a linear operator L as follows:

$$L : X \to X, \quad (L\xi)_n = \xi_{n+1} - g_x(\gamma_n(\alpha))\xi_n$$

where $\xi = \{\xi_n\}_{-\infty}^{\infty}$ and $L\xi = \{(L\xi)_n\}_{-\infty}^{\infty}$. Theorem 3.1.4 has the following equivalent form [29].

Lemma 3.2.1. *Suppose conditions* (*H*1)-(*H*4) *are satisfied. Then*

(i) *The operator L is Fredholm with index zero.*
(ii) $f = \{f_n\}_{-\infty}^{\infty} \in \mathscr{R}L$ *if and only if*

$$\sum_{n=-\infty}^{+\infty} \psi_n^*(\alpha) \cdot f_n = 0 \tag{3.2.4}$$

holds for all bounded solutions $\psi(\alpha) = \{\psi_n(\alpha)\}_{-\infty}^{\infty}$ *of the adjoint variational equation*

$$\xi_{n+1} = (g_x^*(\gamma_{n+1}(\alpha)))^{-1}\xi_n. \tag{3.2.5}$$

(iii) *If* (3.2.4) *holds, then the difference equation*

$$x_{n+1} = g_x(\gamma_n(\alpha))x_n + f_n$$

has a unique bounded solution $x = \{x_n\}_{-\infty}^{\infty}$ *on* $\mathbb{Z}$ *satisfying*

$$\varphi_0^*(\alpha) \cdot x_0 = 0$$

for all bounded solutions $\varphi(\alpha) = \{\varphi_n(\alpha)\}_{-\infty}^{\infty}$ *of the linear difference equation* (3.2.3) *on* $\mathbb{Z}$.

From condition (H3), we have $\gamma_{n+1}(\alpha) = g(\gamma_n(\alpha))$. Differentiating both sides of this difference equation with respect to α, we obtain $\dot{\gamma}_{n+1}(\alpha) = g_x(\gamma_n(\alpha))\dot{\gamma}_n(\alpha)$, where "$\cdot$" $= \frac{d}{d\alpha}$. Hence $\dot{\gamma}(\alpha) = \{\dot{\gamma}_n(\alpha)\}_{-\infty}^{\infty}$ is a nontrivial bounded solution on $\mathbb{Z}$ of the variational equation (3.2.3). That is, $\dot{\gamma}_0(\alpha) \in T_{\gamma_0(\alpha)}W^s(0) \cap T_{\gamma_0(\alpha)}W^u(0)$. We assume that

(H5) $\dim(T_{\gamma_0(\alpha)}W^s(0) \cap T_{\gamma_0(\alpha)}W^u(0)) = d \quad (d \geq 1)$ for a constant d uniformly with respect to α.

Condition (H5) is equivalent to the condition that the variational equation (3.2.3) has d (≥ 1) linearly independent bounded solutions on $\mathbb{Z}$, denoted by

$$\varphi_1(\alpha) = \dot{\gamma}(\alpha) = \{\dot{\gamma}_n(\alpha)\}_{-\infty}^{\infty},$$
$$\varphi_2(\alpha) = \{\varphi_{2,n}(\alpha)\}_{-\infty}^{\infty}, \ \ldots \ , \ \varphi_d(\alpha) = \{\varphi_{d,n}(\alpha)\}_{-\infty}^{\infty}.$$

We let

$$\Phi_n(\alpha) = \Big(\varphi_{1,n}(\alpha), \varphi_{2,n}(\alpha), \ldots, \varphi_{d,n}(\alpha)\Big)$$

be an $N \times d$ matrix and

$$\Phi_n^0(\alpha) = \Big(\varphi_{2,n}(\alpha), \ldots, \varphi_{d,n}(\alpha)\Big)$$

be an $N \times (d-1)$ matrix. From Section 3.1.2 it follows that under conditions (H1)–(H5), the adjoint equation (3.2.5) also has d and only d linearly independent bounded solutions on $\mathbb{Z}$, denoted by

$$\{\psi_{1,n}(\alpha)\}_{-\infty}^{\infty}, \quad \{\psi_{2,n}(\alpha)\}_{-\infty}^{\infty}, \ \ldots \ , \{\psi_{d,n}(\alpha)\}_{-\infty}^{\infty}.$$

We let

$$\Psi_n(\alpha) = \Big(\psi_{1,n}(\alpha), \psi_{2,n}(\alpha), \ldots, \psi_{d,n}(\alpha)\Big)$$

be an $N \times d$ matrix. We suppose that $\Phi_n(\alpha)$ and $\Psi_n(\alpha)$ are C^3-smooth in α for any $n \in \mathbb{Z}$. The main result of this section is the following theorem.

Theorem 3.2.2. *Suppose conditions* (*H*1)–(*H*5) *are satisfied. We define a Melnikov vector mapping by*

$$M(\alpha,\beta) = \sum_{n=-\infty}^{\infty} \Psi_n^*(\alpha) \cdot \Big\{2h(n,\gamma_n(\alpha),0) + g_{xx}(\gamma_n(\alpha))(\Phi_n^0(\alpha)\beta, \Phi_n^0(\alpha)\beta)\Big\}.$$

If there exists $(\alpha_0,\beta_0) \in \mathbb{R} \times \mathbb{R}^{d-1}$ *so that*

$$M(\alpha_0,\beta_0) = 0 \quad \text{and} \quad \det D_{(\alpha,\beta)}M(\alpha_0,\beta_0) \neq 0,$$

then for ε *sufficiently small, there exist two continuously differentiable functions* $\alpha = \alpha(\varepsilon)$, $\beta = \beta(\varepsilon)$, *satisfying* $\alpha(0) = \alpha_0$, $\beta(0) = \beta_0$ *so that for* $\varepsilon \neq 0$ *sufficiently small, the difference equation*

$$x_{n+1} = g(x_n) + \varepsilon^2 h(n,x_n,\varepsilon^2)$$

has a bounded solution $x(\varepsilon) = \{x_n(\varepsilon)\}_{-\infty}^{\infty}$ *so that*

$$|x_n(\varepsilon) - \gamma_n(\alpha(\varepsilon)) - \varepsilon\Phi_n^0(\alpha(\varepsilon))\beta(\varepsilon)| = O(\varepsilon^2) \tag{3.2.6}$$

and the variational equation

$$\xi_{n+1} = \{g_x(x_n(\varepsilon)) + \varepsilon^2 h_x(n,x_n(\varepsilon),\varepsilon^2)\}\xi_n$$

admits an exponential dichotomy on $\mathbb{Z}$.

Proof. First of all, we prove the existence of a bounded solution $\{x_n(\varepsilon)\}_{-\infty}^{\infty}$. We make a change of variables

$$y_n = x_n - \gamma_n(\alpha) - \Phi_n^0(\alpha)\beta$$

for the difference equation (3.2.1), where $\beta \in \mathbb{R}^{d-1}$ is a vector parameter. Then the difference equation (3.2.1) reads

$$\begin{aligned} y_{n+1} = {} & g(y_n + \gamma_n(\alpha) + \Phi_n^0(\alpha)\beta) + \varepsilon h(n, y_n + \gamma_n(\alpha) + \Phi_n^0(\alpha)\beta, \varepsilon) \\ & - g(\gamma_n(\alpha)) - g_x(\gamma_n(\alpha))\Phi_n^0(\alpha)\beta. \end{aligned} \tag{3.2.7}$$

For simplicity, we define

$$\begin{aligned}G(n,y_n,\alpha,\beta,\varepsilon) &= \varepsilon h(n,y_n+\gamma_n(\alpha)+\Phi_n^0(\alpha)\beta,\varepsilon)-g(\gamma_n(\alpha))\\ &\quad +g(y_n+\gamma_n(\alpha)+\Phi_n^0(\alpha)\beta)-g_x(\gamma_n(\alpha))(y_n+\Phi_n^0(\alpha)\beta),\end{aligned}$$

then the difference equation (3.2.7) can be written as

$$y_{n+1}=g_x(\gamma_n(\alpha))y_n+G(n,y_n,\alpha,\beta,\varepsilon)\,. \tag{3.2.8}$$

We put

$$D(\alpha)=\sum_{n=-\infty}^{\infty}\Psi_n^*(\alpha)\cdot\Psi_n(\alpha),$$

so then the $d\times d$ matrix $D(\alpha)$ is invertible [30, p. 129]. Using the Lyapunov-Schmidt method and Lemma 3.2.1, we see that the difference equation (3.2.8) is equivalent to the following two equations

$$\begin{aligned}y_{n+1} &= g_x(\gamma_n(\alpha))y_n+G(n,y_n,\alpha,\beta,\varepsilon)\\ &\quad -\Psi_n(\alpha)D^{-1}(\alpha)\sum_{j=-\infty}^{\infty}\Psi_j^*(\alpha)G(j,y_j,\alpha,\beta,\varepsilon)\,,\end{aligned} \tag{3.2.9}$$

and

$$\sum_{n=-\infty}^{\infty}\Psi_n^*(\alpha)G(n,y_n,\alpha,\beta,\varepsilon)=0\,. \tag{3.2.10}$$

Since

$$\sum_{n=-\infty}^{\infty}\Psi_n^*(\alpha)\Big\{G(n,y_n,\alpha,\beta,\varepsilon)-\Psi_n(\alpha)D^{-1}(\alpha)\sum_{j=-\infty}^{\infty}\Psi_j^*(\alpha)G(j,y_j,\alpha,\beta,\varepsilon)\Big\}=0,$$
$$G(n,0,\alpha,0,0)=0\quad\text{and}\quad G_y(n,0,\alpha,0,0)=0\,,$$

it follows from Lemma 3.2.1 and the implicit function theorem that for ε,β sufficiently small, the difference equation (3.2.9) has a unique small bounded solution $y=y(\alpha,\beta,\varepsilon)=\{y_n(\alpha,\beta,\varepsilon)\}_{-\infty}^{\infty}\in X$ satisfying

$$\Phi_0^*(\alpha)y_0(\alpha,\beta,\varepsilon)=0\,. \tag{3.2.11}$$

Clearly $y(\alpha,0,0)=0$. We substitute

$$y=y(\alpha,\beta,\varepsilon)=\{y_n(\alpha,\beta,\varepsilon)\}_{-\infty}^{\infty}$$

into Eq. (3.2.10) and obtain the following bifurcation equation

$$\overline{B}(\alpha,\beta,\varepsilon)=\sum_{n=-\infty}^{\infty}\Psi_n^*(\alpha)G(n,y_n(\alpha,\beta,\varepsilon),\alpha,\beta,\varepsilon)=0\,. \tag{3.2.12}$$

To solve Eq. (3.2.12), we consider the equation

$$B(\alpha,\beta,\varepsilon)=\overline{B}(\alpha,\varepsilon\beta,\varepsilon^2)=\sum_{n=-\infty}^{\infty}\Psi_n^*(\alpha)G(n,y_n(\alpha,\varepsilon\beta,\varepsilon^2),\alpha,\varepsilon\beta,\varepsilon^2)=0.$$

If $Y_n(\varepsilon)=y_n(\alpha,\varepsilon\beta,\varepsilon^2)$, then we have

$$\begin{aligned}Y_{n+1}(\varepsilon) &= g_x(\gamma_n(\alpha))Y_n(\varepsilon)+G(n,Y_n(\varepsilon),\alpha,\varepsilon\beta,\varepsilon^2)\\ &\quad -\Psi_n(\alpha)D^{-1}(\alpha)\sum_{j=-\infty}^{\infty}\Psi_j^*(\alpha)G(j,Y_j(\varepsilon),\alpha,\varepsilon\beta,\varepsilon^2).\end{aligned}\tag{3.2.13}$$

Differentiating both sides of the difference equation (3.2.13) with respect to ε and setting $\varepsilon=0$ and noting that $Y_n(0)=0$, we obtain

$$Y_{n+1}^{\varepsilon}(0)=g_x(\gamma_n(\alpha))Y_n^{\varepsilon}(0)$$

where $Y_n^{\varepsilon}(0)=\frac{d}{d\varepsilon}Y_n(\varepsilon)|_{\varepsilon=0}$. Moreover, (3.2.11) implies $\Phi_0^*(\alpha)Y_0^{\varepsilon}(0)=0$. By the uniqueness of the bounded solution of the linear difference equation (3.2.3) satisfying (3.2.11) we have $Y_n^{\varepsilon}(0)=0$. We conclude

$$B(\alpha,\beta,0)=\sum_{n=-\infty}^{\infty}\Psi_n^*(\alpha)G(n,y_n(\alpha,0,0),\alpha,0,0)=\sum_{n=-\infty}^{\infty}\Psi_n^*(\alpha)G(n,0,\alpha,0,0)=0$$

and

$$\begin{aligned}B_{\varepsilon}(\alpha,\beta,\varepsilon) &= \sum_{n=-\infty}^{\infty}\Psi_n^*(\alpha)\Big\{2\varepsilon h\left(n,y_n(\alpha,\varepsilon\beta,\varepsilon^2)+\gamma_n(\alpha)+\varepsilon\Phi_n^0(\alpha)\beta,\varepsilon^2\right)\\ &\quad +\varepsilon^2\frac{d}{d\varepsilon}h\left(n,y_n(\alpha,\varepsilon\beta,\varepsilon^2)+\gamma_n(\alpha)+\varepsilon\Phi_n^0(\alpha)\beta,\varepsilon^2\right)\\ &\quad +g_x\left(y_n(\alpha,\varepsilon\beta,\varepsilon^2)+\gamma_n(\alpha)+\varepsilon\Phi_n^0(\alpha)\beta\right)\cdot\\ &\quad \frac{d}{d\varepsilon}\left[y_n(\alpha,\varepsilon\beta,\varepsilon^2)+\gamma_n(\alpha)+\varepsilon\Phi_n^0(\alpha)\beta\right]\\ &\quad -g_x(\gamma_n(\alpha))\frac{d}{d\varepsilon}\left[y_n(\alpha,\varepsilon\beta,\varepsilon^2)+\gamma_n(\alpha)+\varepsilon\Phi_n^0(\alpha)\beta\right]\Big\}.\end{aligned}\tag{3.2.14}$$

Noting $y_n(\alpha,0,0)=0$ and $Y_n^{\varepsilon}(0)=0$, we have

$$B_{\varepsilon}(\alpha,\beta,0)=0.\tag{3.2.15}$$

From (3.2.14) and $y_n(\alpha,0,0)=0$ and $Y_n^{\varepsilon}(0)=0$, we compute

$$\begin{aligned}B_{\varepsilon\varepsilon}(\alpha,\beta,0) &= \sum_{n=-\infty}^{\infty}\Psi_n^*(\alpha)\Big\{2h(n,\gamma_n(\alpha),0)+g_{xx}(\gamma_n(\alpha))(Y_n^{\varepsilon}(0)+\Phi_n^0(\alpha)\beta,\\ &\quad Y_n^{\varepsilon}(0)+\Phi_n^0(\alpha)\beta)+g_x(\gamma_n(\alpha))Y_n^{\varepsilon\varepsilon}(0)-g_x(\gamma_n(\alpha))Y_n^{\varepsilon\varepsilon}(0)\Big\}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=-\infty}^{\infty} \Psi_n^*(\alpha)\Big\{2h(n,\gamma_n(\alpha),0) + g_{xx}(\gamma_n(\alpha))(\Phi_n^0(\alpha)\beta, \Phi_n^0(\alpha)\beta)\Big\} \\
&= M(\alpha,\beta)
\end{aligned}
$$

where $Y_n^{\varepsilon\varepsilon}(0) = \frac{d^2}{d\varepsilon^2} Y_n(\varepsilon)|_{\varepsilon=0}$. We define the function $H(\alpha,\beta,\varepsilon)$ by

$$
H(\alpha,\beta,\varepsilon) = \begin{cases} \dfrac{B(\alpha,\beta,\varepsilon)}{\varepsilon^2}, & \text{if} \quad \varepsilon \neq 0, \\ \dfrac{1}{2} B_{\varepsilon\varepsilon}(\alpha,\beta,0), & \text{if} \quad \varepsilon = 0. \end{cases}
$$

Since $B(\alpha,\beta,0) = 0$ and (3.2.15) holds, the function $H(\alpha,\beta,\varepsilon)$ is continuously differentiable in α,β,ε. From the conditions of Theorem 3.2.2, we have

$$
H(\alpha_0,\beta_0,0) = \frac{1}{2} B_{\varepsilon\varepsilon}(\alpha_0,\beta_0,0) = \frac{1}{2} M(\alpha_0,\beta_0) = 0
$$

and

$$
\det D_{(\alpha,\beta)} H(\alpha_0,\beta_0,0) = \frac{1}{2^d} \det D_{(\alpha,\beta)} M(\alpha_0,\beta_0) \neq 0.
$$

It follows from the implicit function theorem that for ε sufficiently small, there exist two continuously differentiable functions $\alpha = \alpha(\varepsilon)$ and $\beta = \beta(\varepsilon)$ satisfying $\alpha(0) = \alpha_0$ and $\beta(0) = \beta_0$, respectively, so that $H(\alpha(\varepsilon),\beta(\varepsilon),\varepsilon) = 0$. Hence for $\varepsilon \neq 0$ sufficiently small, we have that $B(\alpha(\varepsilon),\beta(\varepsilon),\varepsilon) = 0$. Thus for $\varepsilon \neq 0$ sufficiently small, the difference equation

$$
x_{n+1} = g(x_n) + \varepsilon^2 h(n,x_n,\varepsilon^2)
$$

has a unique bounded solution $\{x_n(\varepsilon)\}_{-\infty}^{\infty}$ with

$$
x_n(\varepsilon) = y_n(\alpha(\varepsilon),\varepsilon\beta(\varepsilon),\varepsilon^2) + \gamma_n(\alpha(\varepsilon)) + \varepsilon\Phi_n^0(\alpha(\varepsilon))\beta(\varepsilon)
$$

satisfying (3.2.6). This completes the proof of the existence part of the theorem.

Finally, the transversality of the bounded solution $\{x_n(\varepsilon)\}_{-\infty}^{\infty}$ can be proved in the same way as in Theorem 3.1.7, so we omit the proof. □

In the degenerate Case 1 from Section 3.2.1 one would start with a family of homoclinic solutions $\gamma(\alpha) = \{\gamma_n(\alpha)\}_{-\infty}^{\infty}$ with $\alpha \in \mathbb{R}^d$ like in condition (H3). For bounded solutions to the variational equation (3.2.3) in accordance with the above notations one now has

$$
\varphi_i(\alpha) = \left\{ \frac{\partial \gamma_n}{\partial \alpha_i}(\alpha) \right\}_{-\infty}^{\infty}, \quad i = 1,2,\ldots,d.
$$

Using the formula

$$\frac{\partial^2 \gamma_{n+1}}{\partial \alpha_j \partial \alpha_i}(\alpha) = g_x(\gamma_n(\alpha)) \frac{\partial^2 \gamma_n}{\partial \alpha_j \partial \alpha_i}(\alpha) + g_{xx}(\gamma_n(\alpha)) \left(\frac{\partial \gamma_n}{\partial \alpha_j}(\alpha), \frac{\partial \gamma_n}{\partial \alpha_i}(\alpha) \right)$$

it is easy to show by Lemma 3.2.1 that for this case in the Melnikov vector mapping of Theorem 3.2.2 the β terms are identically zero. The Melnikov vector mapping here is

$$M(\alpha) = \sum_{-\infty}^{\infty} \Psi_n^*(\alpha) \cdot h(n, \gamma_n(\alpha), 0), \quad \alpha \in \mathbb{R}^d .$$

We remark that Case 1 is already studied in Section 3.1. We also mention that the vanishing of the β terms in the Melnikov vector mapping of Theorem 3.2.2 is a necessary but not sufficient condition for Case 1. This means that in the general theory, if one computes $d > 1$ for condition (H5) and then finds that all the β terms vanish one cannot apply Theorem 3.2.2 and does not know if Case 1 can be applied or if there is some other higher degeneracy. Then higher-order Melnikov vector mappings could help to study the homoclinic bifurcations of the difference equation (3.2.1).

Finally, we get the above Melnikov vector mapping $M(\alpha)$ also for the case $d = 1$ in condition (H5), but now $\alpha \in \mathbb{R}$. So M is a function.

3.2.3 Applications to McMillan Type Mappings

We consider the following mapping of a McMillan type (cf Section 3.2.4 and [4, 5, 7])

$$\begin{aligned} z_{n+1} &= y_n, \quad y_{n+1} = -z_n + 2K \frac{y_n}{1+y_n^2} + v_n^2 - \varepsilon y_n , \\ u_{n+1} &= v_n, \quad v_{n+1} = -u_n + 2K v_n \frac{1-y_n^2}{(1+y_n^2)^2} + u_n^2 - \varepsilon z_n \end{aligned} \tag{3.2.16}$$

where $K > 1$ is a constant. By Section 3.2.4 we know that

$$\gamma_n(\alpha) = \big(r_n(\alpha), r_{n+1}(\alpha), 0, 0\big),$$
$$r_n(\alpha) = \sinh w \operatorname{sech}(\alpha - nw), \quad w = \cosh^{-1} K, \quad w > 0$$

is a bounded solution of (3.2.16) with $\varepsilon = 0$. Then (3.2.3) has now the form

$$\begin{aligned} a_{n+1} &= b_n, \quad b_{n+1} = -a_n + 2K \frac{1 - r_{n+1}^2(\alpha)}{(1 + r_{n+1}^2(\alpha))^2} b_n , \\ c_{n+1} &= d_n, \quad d_{n+1} = -c_n + 2K \frac{1 - r_{n+1}^2(\alpha)}{(1 + r_{n+1}^2(\alpha))^2} d_n . \end{aligned} \tag{3.2.17}$$

The equilibrium $(0, 0, 0, 0)$ of the unperturbed mapping is hyperbolic with 2-dimensional stable and unstable parts. We can easily verify from (3.2.17) that now $d = 2$ and

$$\Phi_n^0(\alpha) = \left(0, 0, r_n'(\alpha), r_{n+1}'(\alpha)\right).$$

We note that

$$\{\dot{\gamma}_n(\alpha)\}_{-\infty}^{\infty} = \left\{(r_n'(\alpha), r_{n+1}'(\alpha), 0, 0)\right\}_{-\infty}^{\infty}$$

is another solution of (3.2.17) bounded on $\mathbb{Z}$. We also remark that by (3.2.17), the unperturbed mapping of (3.2.16) with $\varepsilon = 0$ is volume preserving on the set $\{\gamma_n(\alpha)\}_{-\infty}^{\infty}$. Then according to Lemma 3.1.11, we find

$$\Psi_n(\alpha) = \begin{pmatrix} r_{n+1}'(\alpha) & 0 \\ -r_n'(\alpha) & 0 \\ 0 & r_{n+1}'(\alpha) \\ 0 & -r_n'(\alpha) \end{pmatrix}.$$

Furthermore, in the notations of the previous section we have

$$g_{xx}(\gamma_n(\alpha))\left(\Phi_n^0(\alpha)\beta, \Phi_n^0(\alpha)\beta\right) = \left(0, 2r_{n+1}'(\alpha)^2\beta^2, 0, 2r_n'(\alpha)^2\beta^2\right),$$
$$h(n, \gamma_n(\alpha), 0) = (0, -r_{n+1}(\alpha), 0, -r_n(\alpha)).$$

Consequently, the Melnikov vector mapping has the form

$$M(\alpha,\beta) = (M_1(\alpha,\beta), M_2(\alpha,\beta))$$

where

$$M_1(\alpha,\beta) = 2\sum_{n=-\infty}^{\infty} r_n'(\alpha) r_{n+1}(\alpha) - 2\beta^2 \sum_{n=-\infty}^{\infty} r_{n+1}'(\alpha)^2 r_n'(\alpha),$$
$$M_2(\alpha,\beta) = 2\sum_{n=-\infty}^{\infty} r_n'(\alpha) r_n(\alpha) - 2\beta^2 \sum_{n=-\infty}^{\infty} r_n'(\alpha)^3.$$

We conclude

$$\begin{aligned} A_1(w) &= \sum_{n=-\infty}^{\infty} r_n'(0) r_{n+1}(0) = \sinh^2 w \sum_{n=1}^{\infty}\left(\operatorname{sech}(n+1)w - \operatorname{sech}(n-1)w\right) \\ &\quad \times \operatorname{sech}^2 nw \sinh nw < 0, \\ A_2(w) &= \sum_{n=-\infty}^{\infty} r_{n+1}'(0)^2 r_n'(0) = \sinh^3 w \sum_{n=1}^{\infty}\left(\operatorname{sech}^4(n+1)w \sinh^2(n+1)w\right. \\ &\quad \left. - \operatorname{sech}^4(n-1)w \sinh^2(n-1)w\right) \times \operatorname{sech}^2 nw \sinh nw, \end{aligned}$$

$$\sum_{n=-\infty}^{\infty}(r_n''(0) r_n(0) + r_n'(0)^2) = \sinh^2 w\left(-1 + 2\sum_{n=1}^{\infty}\operatorname{sech}^4 nw\,(\cosh 2nw - 2)\right),$$

$$\sum_{n=1}^{\infty} r_n'(0)^2 r_n''(0) = \sinh^3 w \sum_{n=1}^{\infty} \operatorname{sech}^7 nw \sinh^2 nw (\cosh^2 nw - 2),$$

$$\sum_{n=-\infty}^{\infty} r_n'(0) r_n(0) = \sinh^2 w \sum_{n=-\infty}^{\infty} \operatorname{sech}^3 nw \sinh nw = 0,$$

$$\sum_{n=-\infty}^{\infty} r_n'(0)^3 = \sinh^3 w \sum_{n=-\infty}^{\infty} \operatorname{sech}^6 nw \sinh^3 nw = 0,$$

$$\frac{\partial}{\partial \alpha} M_2(0,\beta) = 2 \sum_{n=-\infty}^{\infty} (r_n''(0) r_n(0) + r_n'(0)^2) - 12\beta^2 \sum_{n=1}^{\infty} r_n'(0)^2 r_n''(0) = A_3(w,\beta).$$

The above series are very difficult to evaluate and they could be expressed in terms of Jacobi elliptic functions [4]. Instead, we use the following lemmas.

Lemma 3.2.3. *Let $F : [0,\infty) \to \mathbb{R}$ be such that $|F(x)| \le c_1 \mathrm{e}^{-c_2 x}$ for positive constants c_1, c_2. Then*

$$\left| \sum_{n=1}^{\infty} F(n\tilde{w}) \right| \le 2c_1 \mathrm{e}^{-c_2 \tilde{w}}$$

for any $\tilde{w} \ge \ln 2/c_2$.

Lemma 3.2.4. *Let $F, G : [0,\infty) \to \mathbb{R}$ be such that $G(0) = 0$, and*

$$c_1 \mathrm{e}^{-\theta_1 x} \le F(x) \le c_2 \mathrm{e}^{-\theta_1 x}, \quad d_1 \mathrm{e}^{-\theta_2 x} \le G(x) \le d_2 \mathrm{e}^{-\theta_2 x}$$

for any $x \ge 1$ and positive constants $c_i, d_i, \theta_i, i = 1,2$. Then for any $\tilde{w} \ge 1$, we have

$$\begin{aligned} &\frac{c_1 d_1 \mathrm{e}^{-(2\theta_2+\theta_1)\tilde{w}} - c_2 d_2 \mathrm{e}^{-(2\theta_1+\theta_2)\tilde{w}}}{1 - \mathrm{e}^{-(\theta_1+\theta_2)\tilde{w}}} \\ &\le \sum_{n=1}^{\infty} (G((n+1)\tilde{w}) - G((n-1)\tilde{w})) F(n\tilde{w}) \\ &\le \frac{c_2 d_2 \mathrm{e}^{-(2\theta_2+\theta_1)\tilde{w}} - c_1 d_1 \mathrm{e}^{-(2\theta_1+\theta_2)\tilde{w}}}{1 - \mathrm{e}^{-(\theta_1+\theta_2)\tilde{w}}}. \end{aligned}$$

Proofs of the above lemmas are elementary, so we omit them. We apply Lemma 3.2.4 with $G(x) = \operatorname{sech}^4 x \sinh^2 x$, $F(x) = \operatorname{sech}^2 x \sinh x$. Then using

$$\begin{aligned} \mathrm{e}^{-x} \le \operatorname{sech} x \le 2\mathrm{e}^{-x}, \quad x \ge 0, \\ \frac{\mathrm{e}^2 - 1}{2\mathrm{e}^2} \mathrm{e}^x \le \sinh x \le \mathrm{e}^x/2, \quad x \ge 1, \end{aligned}$$

we get $c_1 = \frac{\mathrm{e}^2-1}{2\mathrm{e}^2}$, $c_2 = 2$, $d_1 = \left(\frac{\mathrm{e}^2-1}{2\mathrm{e}^2}\right)^2$, $d_2 = 4$, $\theta_1 = 1$ and $\theta_2 = 2$, and then we obtain

$$A_2(w) \le \sinh^3 w \frac{8\mathrm{e}^{-5w} - \left(\frac{\mathrm{e}^2-1}{2\mathrm{e}^2}\right)^3 \mathrm{e}^{-4w}}{1 - \mathrm{e}^{-3w}} < 0$$

for any $w > \ln\left[\frac{64\,\mathrm{e}^6}{(\mathrm{e}^2-1)^3}\right] \doteq 4.59512$. Similarly, using Lemma 3.2.3, $\cosh^2 1 > 2$ and

$$\left|\operatorname{sech}^4 x(\cosh 2x - 2)\right| \le 32\,\mathrm{e}^{-4x} + 16\,\mathrm{e}^{-2x}, \quad x \ge 0,$$

we derive

$$A_3(w,\beta) \le 2\sinh^2 w\left(-1 + 64\,\mathrm{e}^{-4w} + 32\,\mathrm{e}^{-2w}\right) < 0$$

for any $w > \frac{1}{2}\ln\left[8\left(\sqrt{5}+2\right)\right] \doteq 1.76154$. We already know that $A_1(w) < 0$. Hence $\alpha = 0$, $\beta = \sqrt{A_1(w)/A_2(w)} \ne 0$ is a simple zero of $M(\alpha,\beta) = 0$ for any $w > \ln\left[\frac{64\,\mathrm{e}^6}{(\mathrm{e}^2-1)^3}\right]$, i.e. $K > K_0 := \frac{4096\,\mathrm{e}^{12} + \left(\mathrm{e}^2-1\right)^6}{128\,\mathrm{e}^6\left(\mathrm{e}^2-1\right)^3} \doteq 49.5052$. Now we can apply Theorem 3.2.2 to (3.2.16), and we produce the following result.

Theorem 3.2.5. *For any $K > K_0$, there is an $\varepsilon_0 > 0$ so that* (3.2.16) *exhibits chaos for any* $0 < \varepsilon < \varepsilon_0$.

Of course, either more precise analytical or numerical evaluations of $A_2(w)$ and $A_3(w,\beta)$ could give also partial results for $1 < K \le K_0$. But we do not carry out these computations in this book. We only note that our numerical computations suggest that $K \ge \cosh 0.1 \doteq 1.005$ for obtaining chaos in (3.2.16) for $\varepsilon > 0$ small.

3.2.4 Planar Integrable Maps with Separatrices

A planar map is called a *standard-like* one if it has a form $F(x,y) = (y, -x + g(y))$ for some smooth g. Note that F is *area-preserving*, i.e. $|\det DF(x,y)| = 1$. A planar map F is *integrable* if there is a function (a first integral) $H : \mathbb{R}^2 \to \mathbb{R}$ so that $H \circ F = H$. An interesting family of standard-like and integrable maps is given by [5]

$$F(x,y) := \left(y, -x + 2y\frac{K+\beta y}{1-2\beta y+y^2}\right), \quad -1 < \beta < 1 < K \tag{3.2.18}$$

with the corresponding first integrals

$$H_{K,\beta}(x,y) = x^2 - 2Kxy + y^2 - 2\beta xy(x+y) + x^2y^2 .$$

Map (3.2.18) with $\beta = 0$ is called *McMillan map*. Next, (3.2.18) has two separatrices $\Gamma^{\pm}_{K,\beta} = \{\gamma^{\pm}_n(\alpha)\}_{n\in\mathbb{Z}}$ contained in the level $H_{K,\beta} = 0$ given by $\gamma^{\pm}_n(\alpha) = (r^{\pm}_n(\alpha), r^{\pm}_{n+1}(\alpha))$ with

$$r^{\pm}_n(\alpha) := \pm\frac{\sinh w \sinh\dfrac{w}{2}}{\sqrt{\beta^2 + \sinh^2\dfrac{w}{2}}\cosh(\alpha - nw) \mp \beta\cosh\dfrac{w}{2}},$$

for $w = \cosh^{-1} K$. Clearly example (3.2.16) can be extended with (3.2.18), but we do not go into details.

3.3 Singular Impulsive ODEs

3.3.1 Singular ODEs with Impulses

The theory of impulsive differential equations is an important branch of differential equations with many applications [16–20]. So in this section, we continue to study such systems by considering the problem

$$\begin{aligned} \varepsilon x' &= f(x) + \varepsilon h(x)\,, \\ x(i+) &= x(i-) + \varepsilon g\big(x(i-)\big)\,, \quad i \in \mathbb{Z}\,, \end{aligned} \tag{3.3.1}$$

when the following assumptions are valid

(H1) $f, g, h \in C^3(\mathbb{R}^m, \mathbb{R}^m)$.
(H2) $0 \in \mathbb{R}^m$ is a hyperbolic equilibrium of $x' = f(x)$.
(H3) The equation $x' = f(x)$ has a homoclinic orbit ϕ to 0.
(H4) The variational equation $v' = Df(\phi)v$ has the unique (up to scalar multiples) bounded solution ϕ' on $\mathbb{R}$.

By Section 4.1.2, we know that (H3) and (H4) imply the uniqueness (up to scalar multiples) of a bounded solution ψ on $\mathbb{R}$ of the adjoint variational equation $\psi' = -\big(Df(\phi)\big)^* \psi$. By a solution of (3.3.1) we mean a function $x(t)$, which is C^1–smooth on $\mathbb{R} \setminus \mathbb{Z}$, satisfies the differential equation in (3.3.1) on this set and the impulsive conditions in (3.3.1) hold as well.

For simplicity, we assume f, h, g to be globally Lipschitz continuous. Let us denote by $\Phi_\varepsilon(t, x_0)$ the unique solution of the differential equation of (3.3.1) with the initial condition $\Phi_\varepsilon(0, x_0) = x_0$ for $\varepsilon > 0$. Then we can define the Poincarè map of (3.3.1) by the formula

$$\pi_\varepsilon(x) = \Phi_\varepsilon\big(1, x + \varepsilon g(x)\big)\,.$$

Of course, the dynamics of (3.3.1) is wholly determined by π_ε.

The purpose of this section is to show the existence of a transversal homoclinic point of π_ε for any $\varepsilon > 0$ sufficiently small (cf Theorem 3.3.10). Then, according to Smale-Birkhoff homoclinic theorem 2.5.4, Equations (3.3.1) will have a chaotic behaviour for $\varepsilon > 0$ sufficiently small. To detect transversal homoclinic orbits of π_ε for $\varepsilon > 0$ small, we derive the Melnikov function of (3.3.1) given by the formula

$$\mathscr{M}(\beta) \equiv \big\langle g\big(\phi(\beta)\big), \psi(\beta)\big\rangle_m + \int_{-\infty}^{\infty} \big\langle h\big(\phi(s)\big), \psi(s)\big\rangle_m ds\,, \tag{3.3.2}$$

where $\langle\cdot,\cdot\rangle_m$ is the usual inner product on $\mathbb{R}^m$. We see from the form of $\mathscr{M}$ that chaos in (3.3.1) can be made only by the impulsive effects, as the integral part of $\mathscr{M}$ containing h is independent of β. Of course, this fact is natural since the ODE (3.3.1) is autonomous. For the readers' convenience, we note that the approach of this section can be simply generalized to study periodic perturbations of (3.3.1), i.e. if $h = h(x,t)$ and $h(\cdot,t+1) = h(\cdot,t)\,\forall t \in \mathbb{R}$. Since the period of h in t is the same as the period of the impulsive conditions, the Poincarè map π_ε can be straightforwardly extended for this case. Then the Melnikov function is

$$\bar{\mathscr{M}}(\beta) = \big\langle g\big(\phi(\beta)\big), \psi(\beta)\big\rangle_m + \int_{-\infty}^{\infty} \big\langle h\big(\phi(s),0\big), \psi(s)\big\rangle_m ds\,, \quad \beta \in \mathbb{R}\,.$$

We are motivated to study such impulsive Duffing–type equations by [31] of the form

$$\begin{aligned} z'' + a^2 p(z) &= aq(z)\,, \\ a\big(z(i+) - z(i-)\big) &= r\big(z(i-)\big)\,, \\ z'(i+) &= z'(i-), \quad i \in \mathbb{Z}\,, \end{aligned} \tag{3.3.3}$$

where $a > 0$ is a large parameter, p, q, $r \in C^3(\mathbb{R},\mathbb{R})$.

3.3.2 Linear Singular ODEs with Impulses

In this section, we derive Fredholm–like alternative results of certain linear impulsive ODEs which are linearizations of (3.3.1). Let $|\cdot|_m$ be the corresponding norm to $\langle\cdot,\cdot\rangle_m$, and set $\mathbb{N}_- = -\mathbb{N}$. Now we introduce several Banach spaces:

$$X^m = \Big\{ x : \mathbb{R}\setminus\mathbb{Z} \to \mathbb{R}^m \,\Big|\, x \text{ is continuous and bounded on } \mathbb{R}\setminus\mathbb{Z}$$
$$\text{and it has } x(i\pm) = \lim_{s\to 0_\pm} x(i+s)\,\forall i \in \mathbb{Z} \Big\}\,,$$

$$X_1^m = \Big\{ x \in X^m \mid x' \in X^m \Big\}\,,$$

$$X_+^m = \Big\{ x : \mathbb{R}_+\setminus\mathbb{N} \to \mathbb{R}^m \,\Big|\, x \text{ is continuous and bounded on } \mathbb{R}_+\setminus\mathbb{N}$$
$$\text{and it has } x(i+), x(i-)\,\forall i \in \mathbb{N} \Big\}\,,$$

$$X_-^m = \Big\{ x : \mathbb{R}_- \setminus \mathbb{N}_- \to \mathbb{R}^m \,\Big|\, x \text{ is continuous and bounded on } \mathbb{R}_- \setminus \mathbb{N}_-$$
$$\text{and it has } x(i+), x(i-) \, \forall\, i \in \mathbb{N}_- \Big\},$$

$$Y_+^m = \Big\{ \{a_n\}_{n\in\mathbb{N}} \,\Big|\, a_n \in \mathbb{R}^m, \sup_n |a_n|_m < \infty \Big\},$$
$$Y_-^m = \Big\{ \{a_n\}_{n\in\mathbb{N}_-} \,\Big|\, a_n \in \mathbb{R}^m, \sup_n |a_n|_m < \infty \Big\},$$
$$Y^m = \Big\{ \{a_n\}_{n\in\mathbb{Z}} \,\Big|\, a_n \in \mathbb{R}^m, \sup_n |a_n|_m < \infty \Big\}.$$

The norms on these spaces are the usual supremum norms. For instance, the norm on X^m is defined by

$$||x||_m = \sup_{s\in\mathbb{R}\setminus\mathbb{Z}} |x(s)|_m .$$

The norm on X_1^m is denoted by $||\cdot||_{m1}$ and on Y^m by $|||\cdot|||_m$. We note that $||x||_{m1} = ||x||_m + ||x'||_m$.

In the first part of this section, we consider the following linear equation suggested by (3.3.1)

$$\begin{aligned} y' &= D_\beta(t)y + q(t), \\ y(i/\varepsilon+) &= y(i/\varepsilon-) + b_i, \quad i \in \mathbb{Z}, \end{aligned} \tag{3.3.4}$$

where $\beta \in \mathbb{R}$, $\varepsilon > 0$ are fixed, $D_\beta(t) = Df\big(\phi(\beta+t)\big)$, $b_i \in \mathbb{R}^m$, $q \in X^m$ and $y(i/\varepsilon\pm) = y(\frac{i}{\varepsilon}\pm)$.

Let $Z_\beta(t)$ be the fundamental solution of $y' = D_\beta(t)y$. Then by Section 2.5.1, this equation has dichotomies on both $\mathbb{R}_+$ and $\mathbb{R}_-$, i.e. there are projections $P_\pm : \mathbb{R}^m \to \mathbb{R}^m$ and constants $K > 0$, $\alpha > 0$ so that

$$\begin{aligned} |Z_\beta(t)P_\pm Z_\beta^{-1}(s)| &\le K\mathrm{e}^{-\alpha(t-s)}, \quad t \ge s, \\ |Z_\beta(t)(\mathbb{I} - P_\pm)Z_\beta^{-1}(s)| &\le K\mathrm{e}^{-\alpha(s-t)}, \quad s \ge t, \end{aligned}$$

where s, t are nonnegative, and nonpositive, for P_+, P_-, respectively . Note that K, α are independent of β, while $P_\pm = P_\pm^\beta = Z_0(\beta)P_\pm^0 Z_0^{-1}(\beta)$.

Theorem 3.3.1. *The problem*

$$\begin{aligned} y' &= D_\beta(t)y + q(t), \\ y(i/\varepsilon+) &= y(i/\varepsilon-) + b_i, \quad i \in \mathbb{N}, \\ P_+ y(0) &= \xi \in \mathscr{R}P_+, \end{aligned} \tag{3.3.5}$$

has a unique solution $y \in X_+^m$ for any $q \in X_+^m$, $\{b_i\}_{i\in\mathbb{N}} \in Y_+^m$. Moreover, for any $0 < \varepsilon < \tilde{c}$ and a fixed constant $\tilde{c} > 0$, it holds

$$||y||_m \leq c\big(|||\{b_i\}_{i\in\mathbb{N}}|||_m + |\xi|_m + ||q||_m\big)\,.$$

Throughout this section c is a generic constant.

Proof. Uniqueness. If $q = 0$, $b_i = 0$, $\xi = 0$ in (3.3.5), then the solution has the form $Z_\beta(t)y_0$, $P_+y_0 = 0$. So $Z_\beta(t)y_0 = Z_\beta(t)(\mathbb{I} - P_+)y_0$. As

$$|y_0|_m = |(\mathbb{I}-P_+)y_0|_m = |(\mathbb{I}-P_+)Z_\beta^{-1}(t)Z_\beta(t)y_0|_m \leq K\mathrm{e}^{-\alpha t}|Z_\beta(t)y_0|_m\,,$$

we have, by the boundedness of $Z_\beta(t)y_0$, $y_0 = 0$. The uniqueness is proved.

Existence. Let us put for $0 \leq n/\varepsilon < t < (n+1)/\varepsilon$ and any $n \in \mathbb{N}\cup\{0\}$

$$\begin{aligned} y(t) &= Z_\beta(t)\xi + \sum_{k=1}^{n} Z_\beta(t)P_+Z_\beta^{-1}(k/\varepsilon)b_k \\ &\quad - \sum_{k=n+1}^{\infty} Z_\beta(t)(\mathbb{I}-P_+)Z_\beta^{-1}(k/\varepsilon)b_k + \int_0^t Z_\beta(t)P_+Z_\beta^{-1}(s)q(s)\,ds \\ &\quad - \int_t^\infty Z_\beta(t)(\mathbb{I}-P_+)Z_\beta^{-1}(s)q(s)\,ds \end{aligned}$$

where we set, for the case $n = 0$, $\sum_{k=1}^n Z_\beta(t)P_+Z_\beta^{-1}(k/\varepsilon)b_k \equiv 0$. Now, we compute for $0 < \varepsilon < \tilde{c}$

$$\begin{aligned} |y(t)|_m &\leq K\mathrm{e}^{-\alpha t}|\xi|_m + \sum_{k=1}^{n} K\mathrm{e}^{-\alpha(t-\frac{k}{\varepsilon})}|b_k|_m \\ &\quad + \sum_{k=n+1}^{\infty} K\mathrm{e}^{-\alpha(\frac{k}{\varepsilon}-t)}|b_k|_m + \int_0^t K\mathrm{e}^{-\alpha(t-s)}||q||_m\,ds + \int_t^\infty K\mathrm{e}^{-\alpha(s-t)}||q||_m\,ds \\ &\leq K|\xi|_m + K\sup_k |b_k|_m \left(\sum_{k=1}^{n} \mathrm{e}^{-\alpha(t-\frac{k}{\varepsilon})} + \sum_{k=n+1}^{\infty} \mathrm{e}^{-\alpha(\frac{k}{\varepsilon}-t)}\right) \\ &\quad + K||q||_m\left(\int_0^t \mathrm{e}^{-\alpha(t-s)}\,ds + \int_t^\infty \mathrm{e}^{-\alpha(s-t)}\,ds\right) \\ &\leq K|\xi|_m + K\sup_k |b_k|_m\left(\frac{\mathrm{e}^{-\alpha(t-\frac{n}{\varepsilon})}}{1-\mathrm{e}^{-\alpha/\varepsilon}} + \frac{\mathrm{e}^{-\alpha(\frac{n+1}{\varepsilon}-t)}}{1-\mathrm{e}^{-\alpha/\varepsilon}}\right) + K||q||_m\frac{2}{\alpha} \\ &\leq K|\xi|_m + \frac{2K}{1-\mathrm{e}^{-\alpha/\varepsilon}}\sup_k |b_k|_m + K||q||_m\frac{2}{\alpha} \\ &\leq K|\xi|_m + \frac{2K}{1-\mathrm{e}^{-\alpha/\tilde{c}}}\sup_k |b_k|_m + K||q||_m\frac{2}{\alpha}\,. \end{aligned}$$

So $y(t)$ satisfies the inequality of this theorem. It is not difficult to see that we can take derivatives with respect to t term by term in the series and with the integral sign so that $y(t)$ satisfies the differential equation in (3.3.4).

To check the impulsive conditions, we compute for $i \in \mathbb{N}$

$$\begin{aligned} y(i/\varepsilon+) - y(i/\varepsilon-) &= \sum_{k=1}^{i} Z_\beta(i/\varepsilon)P_+ Z_\beta^{-1}(k/\varepsilon)b_k - \sum_{k=i+1}^{\infty} Z_\beta(i/\varepsilon)(\mathbb{I}-P_+)Z_\beta^{-1}(k/\varepsilon)b_k \\ &\quad - \sum_{k=1}^{i-1} Z_\beta(i/\varepsilon)P_+ Z_\beta^{-1}(k/\varepsilon)b_k + \sum_{k=i}^{\infty} Z_\beta(i/\varepsilon)(\mathbb{I}-P_+)Z_\beta^{-1}(k/\varepsilon)b_k \\ &= Z_\beta(i/\varepsilon)P_+ Z_\beta^{-1}(i/\varepsilon)b_i + Z_\beta(i/\varepsilon)(\mathbb{I}-P_+)Z_\beta^{-1}(i/\varepsilon)b_i \\ &= Z_\beta(i/\varepsilon)Z_\beta^{-1}(i/\varepsilon)b_i = b_i\,. \end{aligned}$$

Finally

$$P_+y(0) = P_+\xi - P_+\left(\sum_{k=1}^{\infty}(\mathbb{I}-P_+)Z_\beta^{-1}(k/\varepsilon)b_k + \int_0^{\infty}(\mathbb{I}-P_+)Z_\beta^{-1}(s)q(s)\,ds\right) = \xi\,.$$

The proof is finished. □

Theorem 3.3.2. *The problem*

$$\begin{aligned} y' &= D_\beta(t)y + q(t)\,, \\ y(i/\varepsilon+) &= y(i/\varepsilon-) + b_i, \quad i \in \mathbb{N}_-\,, \\ (\mathbb{I}-P_-)y(0) &= \eta \in \mathscr{R}(\mathbb{I}-P_-)\,, \end{aligned} \tag{3.3.6}$$

has a unique solution $y \in X_-^m$ for any $q \in X_-^m$, $\{b_i\}_{i\in\mathbb{N}_-} \in Y_-^m$. Moreover, for any $0 < \varepsilon < \tilde{c}$ and a fixed constant $\tilde{c} > 0$, it holds

$$||y||_m \le c\big(|||\{b_i\}_{i\in\mathbb{N}_-}|||_m + |\eta|_m + ||q||_m\big)\,.$$

Proof. The uniqueness is the same as in the proof of Theorem 3.3.1. For the existence, let us take for $n/\varepsilon < t < (n+1)/\varepsilon \le 0$ and any $n \in \mathbb{N}_-$

$$\begin{aligned} y(t) &= Z_\beta(t)\eta + \sum_{k=-\infty}^{n} Z_\beta(t)P_- Z_\beta^{-1}(k/\varepsilon)b_k \\ &\quad - \sum_{k=n+1}^{-1} Z_\beta(t)(\mathbb{I}-P_-)Z_\beta^{-1}(k/\varepsilon)b_k + \int_{-\infty}^{t} Z_\beta(t)P_- Z_\beta^{-1}(s)q(s)\,ds \\ &\quad - \int_t^0 Z_\beta(t)(\mathbb{I}-P_-)Z_\beta^{-1}(s)q(s)\,ds\,, \end{aligned}$$

where we set again, for the case $n=-1$, $\sum_{k=n+1}^{-1} Z_\beta(t)(\mathbb{I}-P_-)Z_\beta^{-1}(k/\varepsilon)b_k \equiv 0$. The rest of the proof is the same as in Theorem 3.3.1, and so we omit it. The proof is finished. □

Now we can state the main result concerning (3.3.4).

Theorem 3.3.3. *For any* $\{b_i\}_{i\in\mathbb{Z}} \in Y^m$ *and* $q \in X^m$*, Equation* (3.3.4) *has a solution* $y \in X_1^m$ *if and only if*

$$\sum_{i=-\infty}^{\infty} \left\langle b_i, \psi\left(\beta + \frac{i}{\varepsilon}\right)\right\rangle_m + \int_{-\infty}^{\infty} \langle q(s), \psi(\beta+s)\rangle_m \, ds = 0\,. \tag{3.3.7}$$

This solution is unique provided

$$\int_{-\infty}^{\infty} \big\langle y(s), \phi'(\beta+s)\big\rangle_m ds = 0$$

and, for any $0 < \varepsilon < \tilde{c}$ *and a fixed constant* $\tilde{c} > 0$*, it satisfies*

$$||y||_{m1} \le c\big(\sup_i |b_i|_m + ||q||_m\big)\,.$$

Proof. Uniqueness. Assume that $y_1(t), y_2(t)$ are two solutions of (3.3.4) both satisfying the condition

$$\int_{-\infty}^{\infty} \big\langle y(s), \phi'(\beta+s)\big\rangle_m ds = 0.$$

Then $y(t) = y_1(t) - y_2(t)$ satisfies $y'(t) = D_\beta(t)y(t)$ together with $y(i/\varepsilon+) = y(i/\varepsilon-)$, so that $y(t)$ is a C^1-bounded function on $\mathbb{R}$ satisfying the linear homogeneous differential equation $y'(t) = D_\beta(t)y(t)$. Hence $y(0) \in \mathscr{R}P_+ \cap \mathscr{R}(\mathbb{I} - P_-)$ or $y(0) = \lambda\phi'(\beta)$. As a consequence $y(t) = \lambda\phi'(t+\beta)$ and then

$$\lambda \int_{-\infty}^{\infty} |\phi'(\beta+s)|^2 ds = \int_{-\infty}^{\infty} \big\langle y(s), \phi'(\beta+s)\big\rangle_m ds = 0\,.$$

This fact implies $\lambda = 0$ or $y_1(t) = y_2(t)$.

Existence. For any $\xi \in \mathscr{R}P_+$ and $\eta \in \mathscr{R}(\mathbb{I} - P_-)$ let y_+, y_- be the solutions of (3.3.5) and (3.3.6), respectively. We compute

$$\begin{aligned} y_+(0) - y_-(0) = {} & \xi - \sum_{k=1}^{\infty} (\mathbb{I} - P_+)Z_\beta^{-1}(k/\varepsilon)b_k - \int_0^{\infty} (\mathbb{I} - P_+)Z_\beta^{-1}(s)q(s)\,ds \\ & -\eta - \sum_{k=-\infty}^{-1} P_- Z_\beta^{-1}(k/\varepsilon)b_k - \int_{-\infty}^{0} P_- Z_\beta^{-1}(s)q(s)\,ds\,. \end{aligned}$$

As we also require $y_+(0) - y_-(0) = b_0$, we obtain

$$\begin{aligned} \xi - \eta = {} & b_0 + \sum_{k=1}^{\infty} (\mathbb{I} - P_+)Z_\beta^{-1}(k/\varepsilon)b_k + \sum_{k=-\infty}^{-1} P_- Z_\beta^{-1}(k/\varepsilon)b_k \\ & + \int_0^{\infty} (\mathbb{I} - P_+)Z_\beta^{-1}(s)q(s)\,ds + \int_{-\infty}^{0} P_- Z_\beta^{-1}(s)q(s)\,ds\,. \end{aligned} \tag{3.3.8}$$

Equation (3.3.8) is solvable if and only if the right-hand side is in the space

$$\mathscr{R}P_+ + \mathscr{R}(\mathbb{I} - P_-)\,,$$

i.e. if and only if the right-hand side of (3.3.8) is orthogonal to any element of the space

$$\big(\mathscr{R}P_+ + \mathscr{R}(\mathbb{I} - P_-)\big)^\perp = \mathscr{R}P_+^\perp \cap \mathscr{R}(\mathbb{I} - P_-)^\perp = \mathscr{N}P_+^* \cap \mathscr{N}(\mathbb{I} - P_-^*)\,.$$

But it is clear that $\mathscr{N}P_+^* \cap \mathscr{N}(\mathbb{I} - P_-^*)$ is the space of all initial values y_0 for which the solution of the adjoint equation $y' = -D_\beta^*(t)y$ is bounded on $\mathbb{R}$. This assertion follows from the fact that $(Z_\beta^*)^{-1}(t)$ is the fundamental solution of the equation $y' = -D_\beta^*(t)y$ possessing dichotomies on both $\mathbb{R}_+$ and $\mathbb{R}_-$ with the projections $\mathbb{I} - P_+^*$, $\mathbb{I} - P_-^*$, respectively. In our case,

$$\mathscr{N}P_+^* \cap \mathscr{N}(\mathbb{I} - P_-^*) = \operatorname{span}\{\psi(\beta)\}\,.$$

Hence (3.3.8) is solvable if and only if the following holds

$$\begin{aligned}
0 &= \Bigg\langle \psi(\beta), b_0 + \sum_{k=1}^{\infty} (\mathbb{I} - P_+) Z_\beta^{-1}(k/\varepsilon) b_k + \sum_{k=-\infty}^{-1} P_- Z_\beta^{-1}(k/\varepsilon) b_k \\
&\quad + \int_0^\infty (\mathbb{I} - P_+) Z_\beta^{-1}(s) q(s)\, ds + \int_{-\infty}^0 P_- Z_\beta^{-1}(s) q(s)\, ds \Bigg\rangle_m \\
&= \big\langle \psi(\beta), b_0 \big\rangle_m \\
&\quad + \sum_{k=1}^{\infty} \big\langle (Z_\beta^*)^{-1}(k/\varepsilon)(\mathbb{I} - P_+^*)\psi(\beta), b_k \big\rangle_m + \sum_{k=-\infty}^{-1} \big\langle (Z_\beta^*)^{-1}(k/\varepsilon) P_-^* \psi(\beta), b_k \big\rangle_m \\
&\quad + \int_0^\infty \big\langle q(s), (Z_\beta^*)^{-1}(s)(\mathbb{I} - P_+^*)\psi(\beta) \big\rangle_m ds + \int_{-\infty}^0 \big\langle q(s), (Z_\beta^*)^{-1}(s) P_-^* \psi(\beta) \big\rangle_m ds \\
&= \big\langle \psi(\beta), b_0 \big\rangle_m + \sum_{k=1}^{\infty} \left\langle \psi\left(\beta + \frac{k}{\varepsilon}\right), b_k \right\rangle_m + \sum_{k=-\infty}^{-1} \left\langle \psi\left(\beta + \frac{k}{\varepsilon}\right), b_k \right\rangle_m \\
&\quad + \int_0^\infty \big\langle q(s), \psi(\beta + s) \big\rangle_m ds + \int_{-\infty}^0 \big\langle q(s), \psi(\beta + s) \big\rangle_m ds \\
&= \sum_{i=-\infty}^{\infty} \left\langle \psi\left(\beta + \frac{i}{\varepsilon}\right), b_i \right\rangle_m + \int_{-\infty}^{\infty} \big\langle q(s), \psi(\beta + s) \big\rangle_m ds\,.
\end{aligned}$$

We have used the identities

$$\begin{aligned}
(Z_\beta^*)^{-1}(s)(\mathbb{I} - P_+^*)\psi(\beta) &= \psi(\beta + s)\,, \quad \forall s \ge 0\,, \\
(Z_\beta^*)^{-1}(s) P_-^* \psi(\beta) &= \psi(\beta + s)\,, \quad \forall s \le 0\,,
\end{aligned}$$

which follow from the facts that $(Z_\beta^*)^{-1}(t)$ is the fundamental solution of the equation $y' = -D_\beta^*(t)y$ possessing dichotomies on both R_+ and $\mathbb{R}_-$ with the projections

$\mathbb{I}-P_+^*, \mathbb{I}-P_-^*$, respectively, and $\psi(\beta+\cdot)$ is a bounded solution of this equation on $\mathbb{R}$.

So (3.3.8) is solvable if and only if (3.3.7) holds. Moreover, for any $0<\varepsilon<\tilde{c}$, with $\tilde{c}>0$ being a fixed constant, we have

$$|\xi-\eta|_m \le c\big(\sup_n |b_n|_m + ||q||_m\big)\,.$$

Such ξ, η are not unique, since $\mathscr{R}P_+ \cap \mathscr{R}(\mathbb{I}-P_-) = \operatorname{span}\{\phi'(\beta)\}$. However we can obtain uniqueness asking, for example, that η is orthogonal to $\phi'(\beta)$. That is, in Eq. (3.3.8) we take $\xi \in \mathscr{R}P_+$ and $\eta \in \mathscr{S} = \{\eta \in \mathscr{R}(\mathbb{I}-P_-) \,|\, \langle \eta, \phi'(\beta)\rangle_m = 0\}$. Of course, $\mathscr{R}P_+ \oplus \mathscr{S} = \mathscr{R}P_+ + \mathscr{R}(\mathbb{I}-P_-)$, but the direct sum implies the uniqueness. Then we obtain a solution $(\xi_1,\eta_1) \in \mathscr{R}P_+ \oplus \mathscr{S}$ so that

$$|\xi_1|_m + |\eta_1|_m \le c\big(\sup_n |b_n|_m + ||q||_m\big)\,,$$

for any $0<\varepsilon<\tilde{c}$ ($\tilde{c}>0$ being a fixed constant). So (3.3.4) has a solution $y = y_1\big(\{b_n\}_{n=-\infty}^{\infty}, q\big)$ satisfying

$$||y_1||_m \le c\big(\sup_n |b_n|_m + ||q||_m\big)\,,$$

for any $0<\varepsilon<\tilde{c}$, if and only if (3.3.7) holds. As $\phi'(\beta+t)$ is a bounded solution of (3.3.4) with $q=0$, $b_i=0\,\forall i\in\mathbb{Z}$, by putting

$$y(t) = y_1(t) - \phi'(\beta+t)\int_{-\infty}^{\infty} \langle y_1(s), \phi'(\beta+s)\rangle_m ds \Big/ \int_{-\infty}^{\infty} |\phi'(s)|_m^2 ds\,,$$

we obtain another solution of (3.3.4) satisfying

$$\int_{-\infty}^{\infty} \langle y(s), \phi'(\beta+s)\rangle_m ds = 0\,.$$

Of course, we also have

$$||y||_m \le c\big(\sup_n |b_n|_m + ||q||_m\big)\,,$$

for any $0<\varepsilon<\tilde{c}$. As $y'(t) = D_\beta(t)y(t) + q(t)$ we easily obtain the conclusion of this theorem. □

Remark 3.3.4. Let β_0 be a fixed real number. Then the proof of Theorem 3.3.3 can be repeated to obtain a unique solution of (3.3.4) satisfying the condition

$$\int_{-\infty}^{\infty} \langle y(s), \phi'(\beta_0+s)\rangle_m ds = 0\,,$$

provided $|\beta-\beta_0|$ is sufficiently small. This fact will be used in the proof of Theorem 3.3.8.

In the last part of this section, we consider the following linear equation suggested by (3.3.1)

$$\begin{aligned} y' &= Df(0)y + q(t)\,, \\ y(i/\varepsilon+) &= y(i/\varepsilon-) + b_i, \quad i \in \mathbb{Z}\,, \end{aligned} \tag{3.3.9}$$

where $\varepsilon > 0$ is fixed and $b_i \in \mathbb{R}^m$, $q \in X^m$. Let $Z(t)$ be the fundamental solution of $y' = Df(0)y$. Since 0 is hyperbolic for the equation $x' = f(x)$, there is a projection $Q : \mathbb{R}^m \to \mathbb{R}^m$ and constants $M > 0$, $\omega > 0$ so that

$$|Z(t)QZ^{-1}(s)| \le M\mathrm{e}^{-\omega(t-s)}, \quad t \ge s\,,$$
$$|Z(t)(\mathbb{I} - Q)Z^{-1}(s)| \le M\mathrm{e}^{-\omega(s-t)}, \quad s \ge t\,.$$

By repeating the proof of Theorems 3.3.1 and 3.3.2, we obtain the following results.

Theorem 3.3.5. *The problem*

$$\begin{aligned} y' &= Df(0)y + q(t)\,, \\ y(i/\varepsilon+) &= y(i/\varepsilon-) + b_i, \quad i \in \mathbb{N}\,, \\ Qy(0) &= \xi \in \mathscr{R}Q\,, \end{aligned}$$

has a unique solution $y \in X^m_+$ *for any* $q \in X^m_+$, $\{b_i\}_{i\in\mathbb{N}} \in Y^m_+$. *Moreover, for any* $0 < \varepsilon < \tilde{c}$ *and a fixed constant* $\tilde{c} > 0$, *it holds*

$$||y||_m \le c\big(|||\{b_i\}_{i\in\mathbb{N}}|||_m + |\xi|_m + ||q||_m\big)\,.$$

Theorem 3.3.6. *The problem*

$$\begin{aligned} y' &= Df(0)y + q(t)\,, \\ y(i/\varepsilon+) &= y(i/\varepsilon-) + b_i, \quad i \in \mathbb{N}_-\,, \\ (I - Q)y(0) &= \eta \in \mathscr{R}(\mathbb{I} - Q)\,, \end{aligned}$$

has a unique solution $y \in X^m_-$ *for any* $q \in X^m_-$, $\{b_i\}_{i\in\mathbb{N}_-} \in Y^m_-$. *Moreover, for any* $0 < \varepsilon < \tilde{c}$ *and a fixed constant* $\tilde{c} > 0$, *it holds*

$$||y||_m \le c\big(|||\{b_i\}_{i\in\mathbb{N}_-}|||_m + |\eta|_m + ||q||_m\big)\,.$$

Now we can state our main result concerning (3.3.9).

Theorem 3.3.7. *For any* $\{b_i\}_{i\in\mathbb{Z}} \in Y^m$ *and* $q \in X^m$, *Equation* (3.3.9) *has a unique solution* $y \in X^m$ *satisfying*

$$||y||_{m1} \le c\big(\sup_i |b_i|_m + ||q||_m\big)\,,$$

for any $0 < \varepsilon < \tilde{c}$ *and a fixed constant* $\tilde{c} > 0$.

Proof. The proof of Theorem 3.3.3 can be repeated up to Eq. (3.3.8). Now Eq. (3.3.8) is always solvable, since

$$\big(\mathscr{R}Q+\mathscr{R}(\mathbb{I}-Q)\big)^{\perp}=\mathscr{N}Q^{*}\cap\mathscr{N}(\mathbb{I}-Q^{*})=\{0\}\,.$$

Moreover, such a solution is unique, because

$$\mathscr{R}Q\cap\mathscr{R}(\mathbb{I}-Q)=\{0\}\,.$$

So (3.3.9) has the desired solution. The proof is finished. □

3.3.3 Derivation of the Melnikov Function

In this section, we show chaotic behaviour of the Poincarè map π_ε of (3.3.1) for $\varepsilon>0$ small. For this purpose, we derive a Melnikov function for (3.3.1) to show the existence of a transversal homoclinic orbit of π_ε for $\varepsilon>0$ small. By taking the scale of the time $t\leftrightarrow\varepsilon t$, we have

$$\begin{aligned}x'&=f(x)+\varepsilon h(x)\,,\\ x(i/\varepsilon+)&=x(i/\varepsilon-)+\varepsilon g(x(i/\varepsilon-)),\quad i\in\mathbb{Z}\,.\end{aligned}\tag{3.3.10}$$

Equation (3.3.10) can be rewritten in the form $F_\varepsilon=0$, where

$$F_\varepsilon: X_1^m\to X^m\times Y^m=\mathscr{X}^m\,,$$
$$F_\varepsilon(x)=\left(x'-f(x)-\varepsilon h(x),\left\{x(i/\varepsilon+)-x(i/\varepsilon-)-\varepsilon g\big(x(i/\varepsilon-)\big)\right\}_{i\in\mathbb{Z}}\right).$$

We solve $F_\varepsilon=0$ by the Lyapunov–Schmidt method. But this method cannot be applied directly, since F_ε is not defined for $\varepsilon=0$. We overcome this difficulty by Theorems 3.3.3 and 3.3.7. Let β_0 be a fixed real number. Setting

$$x=z+\phi_\beta,\quad \phi_\beta(t)=\phi(\beta+t)\,,$$

we can write (3.3.10) as

$$\begin{aligned}z'&=D_\beta(t)z+\left\{f(z+\phi_\beta)-f(\phi_\beta)-D_\beta(t)z\right\}+\varepsilon h(z+\phi_\beta)\,,\\ z(i/\varepsilon+)&=z(i/\varepsilon-)+\varepsilon g\big(z(i/\varepsilon-)+\phi_\beta(i/\varepsilon)\big),\quad i\in\mathbb{Z}\,,\\ &\int_{-\infty}^{\infty}\big\langle z(s),\phi'(\beta_0+s)\big\rangle_m ds=0\,,\end{aligned}\tag{3.3.11}$$

where $|\beta-\beta_0|$ is sufficiently small. Finally, Equation (3.3.11) is rewritten, by applying the Lyapunov-Schmidt procedure, in the form

$$
\begin{aligned}
z' - D_\beta(t)z &= P(\varepsilon,\beta,z)\Big(\big\{f(z+\phi_\beta)-f(\phi_\beta)-D_\beta(t)z\big\}+\varepsilon h(z+\phi_\beta)\Big),\\
z(i/\varepsilon+)-z(i/\varepsilon-) &= \varepsilon g\big(z(i/\varepsilon-)+\phi_\beta(i/\varepsilon)\big),\quad i\in\mathbb{Z},\\
\int_{-\infty}^{\infty}\big\langle z(s),\phi'(\beta_0+s)\big\rangle_m ds &= 0,
\end{aligned}
\tag{3.3.12}
$$

and

$$
\begin{aligned}
&P(\varepsilon,\beta,z)\Big(\big\{f(z+\phi_\beta)-f(\phi_\beta)-D_\beta(t)z\big\}+\varepsilon h(z+\phi_\beta)\Big)\\
&\quad=\big\{f(z+\phi_\beta)-f(\phi_\beta)-D_\beta(t)z\big\}+\varepsilon h(z+\phi_\beta)
\end{aligned}
\tag{3.3.13}
$$

where

$$
P_d p = -\left[\left(d+\int_{-\infty}^{\infty}\langle p(s),\psi(\beta+s)\rangle_m ds\right)\Big/\int_{-\infty}^{\infty}|\psi(\beta+s)|_m^2 ds\right]\cdot\psi(\beta+\cdot)+p
$$

$$
d=\varepsilon\sum_{i=-\infty}^{\infty}\left\langle g\big(z(i/\varepsilon-)+\phi_\beta(i/\varepsilon)\big),\psi\left(\beta+\frac{i}{\varepsilon}\right)\right\rangle_m,
$$

$$
P(\varepsilon,\beta,z)=P_d,\quad P_d: X^m\to X^m.
$$

Note that

$$
\int_{-\infty}^{\infty}\big\langle P_d p(s),\psi(\beta+s)\big\rangle_m ds=-d.
$$

The term $f(z+\phi_\beta)-f(\phi_\beta)-D_\beta(\cdot)z$ is of order $O(|z|_m^2)$ in (3.3.12) as $|z|_m\to 0$. Moreover, the left-hand side of (3.3.12) defines a linear operator from X_1^m to $\mathscr{X}^m$, which is uniformly invertible for $\varepsilon>0$ small according to Theorem 3.3.3 and Remark 3.3.4. So by applying the uniform contraction principle of Theorem 2.2.1, we can solve (3.3.12) for z, for any $\varepsilon>0$ small and β so that $|\beta-\beta_0|$ is sufficiently small (say $|\beta-\beta_0|<\sigma$). Moreover, for any fixed $\varepsilon\in(0,\tilde{c})$ this solution $z=z(\beta,\varepsilon)$ is C^1–smooth in β and moreover a simple computation shows that $||z(\beta,\varepsilon)||_m, ||z_\beta(\beta,\varepsilon)||_m=O(\varepsilon)$ uniformly in β (here and in the sequel $z_\beta(\beta,\varepsilon)$ will denote $\frac{\partial z(\beta,\varepsilon)}{\partial\beta}$). By putting $z(\beta,\varepsilon)$ into (3.3.13), we obtain the bifurcation equation (see the definition of $P_d p$)

$$
\begin{aligned}
0=\;&\varepsilon\sum_{i=-\infty}^{\infty}\left\langle g\big(z(\beta,\varepsilon)(i/\varepsilon-)+\phi_\beta(i/\varepsilon)\big),\psi\left(\beta+\frac{i}{\varepsilon}\right)\right\rangle_m\\
&+\int_{-\infty}^{\infty}\Big\langle f\big(z(\beta,\varepsilon)(s)+\phi_\beta(s)\big)-f\big(\phi_\beta(s)\big)-D_\beta(s)z(\beta,\varepsilon)(s)\\
&+\varepsilon h\big(z(\varepsilon,\beta)(s)+\phi_\beta(s)\big),\psi(\beta+s)\Big\rangle_m ds.
\end{aligned}
$$

As $||z(\beta,\varepsilon)||_m, ||z_\beta(\beta,\varepsilon)||_m=O(\varepsilon)$, we can divide the above equation by ε to obtain

$$0=\sum_{i=-\infty}^{\infty}\left\langle g\big(z(\beta,\varepsilon)(i/\varepsilon-)+\phi_\beta(i/\varepsilon)\big),\psi\left(\beta+\frac{i}{\varepsilon}\right)\right\rangle_m$$
$$+\int_{-\infty}^{\infty}\big\langle h\big(z(\beta,\varepsilon)(s)+\phi_\beta(s)\big),\psi(\beta+s)\big\rangle_m ds$$
$$+\varepsilon^{-1}\int_{-\infty}^{\infty}\Big\langle f\big(z(\beta,\varepsilon)(s)+\phi_\beta(s)\big)-f\big(\phi_\beta(s)\big)-D_\beta(s)z(\beta,\varepsilon)(s),\psi(\beta+s)\Big\rangle_m ds.$$

Now, the last term in the r.h.s. of the above equation is clearly $O(\varepsilon)$ uniformly in β and it is not difficult to see that it can be differentiated, with respect to β, with the integral sign and that this derivative is also $O(\varepsilon)$, uniformly in β, because of $||z(\beta,\varepsilon)||_m, ||z_\beta(\beta,\varepsilon)||_m = O(\varepsilon)$, uniformly in β. On the other hand, for $i\neq 0$, $\varepsilon>0$ sufficiently small and $|\beta-\beta_0|<\sigma$, we have

$$\left|\psi\left(\beta+\frac{i}{\varepsilon}\right)\right|_m \leq \tilde{K}\mathrm{e}^{-\alpha|\beta+\frac{i}{\varepsilon}|}\leq \tilde{K}\mathrm{e}^{\alpha|\beta|}\mathrm{e}^{-\alpha/\varepsilon}=O(\varepsilon)$$

where $\tilde{K}>0$ is a constant, and a similar inequality holds for $\phi_\beta(i/\varepsilon)$. Using these facts the above equation takes the form

$$\big\langle g\big(\phi(\beta)\big),\psi(\beta)\big\rangle_m+\int_{-\infty}^{\infty}\big\langle h\big(\phi_\beta(s)\big),\psi(\beta+s)\big\rangle_m ds+O(\varepsilon)=0 \tag{3.3.14}$$

where $O(\varepsilon)$ in Equation (3.3.14) has to be considered in the C^1–topology in $\beta\in(\beta_0-\sigma,\beta_0+\sigma)$, i.e. $O(\varepsilon)$ expresses a term which is $O(\varepsilon)$ small, together with the first partial derivative in β, uniformly with respect to $\beta\in(\beta_0-\sigma,\beta_0+\sigma)$. Summing up we see that if β_0 is a simple root of the function (3.3.2) then (3.3.14) has a unique solution near β_0 for $\varepsilon>0$ sufficiently small. This means that (3.3.1) has a bounded solution near ϕ for any $\varepsilon>0$ sufficiently small. So we obtain the following theorem.

Theorem 3.3.8. *Assume that the function $\mathscr{M}:\mathbb{R}\to\mathbb{R}$ given by* (3.3.2) *has a simple root at $\beta=\beta_0$. Then* (1.1) *has a unique bounded solution near ϕ_{β_0} for any $\varepsilon>0$ sufficiently small.*

Let $x(\varepsilon)$ be the solution from Theorem 3.3.8. Then the sequence

$$\big\{x(\varepsilon)(i/\varepsilon-)\big\}_{i=-\infty}^{\infty}$$

is a bounded orbit of the Poincarè map π_ε of (3.3.1). In the rest of this section, we show that this orbit is a transversal homoclinic orbit to a hyperbolic fixed point of π_ε. For this purpose (see Lemma 2.5.2), we show that the linearization of (3.3.10) at $x(\varepsilon)$

$$v'=Df\big(x(\varepsilon)\big)v+\varepsilon Dh\big(x(\varepsilon)\big)v,$$
$$v(i/\varepsilon+)=v(i/\varepsilon-)+\varepsilon Dg\big(x(\varepsilon)(i/\varepsilon-)\big)v(i/\varepsilon-),\quad i\in\mathbb{Z}$$

has only the zero bounded solution on $\mathbb{R}$. To show this result, we apply Theorem 2.2.4. So, let $B:X_1^m\to\mathscr{X}^m$ be a bounded linear mapping so that $||B||_{L(X_1^m,\mathscr{X}^m)}\leq L$.

Consider the equation
$$F_\varepsilon(x) + \gamma\varepsilon B\big(x - x(\varepsilon_0)\big) = 0 \tag{3.3.15}$$
for a fixed small $\varepsilon_0 > 0$. The perturbation of (3.3.15) is small for γ, $\varepsilon > 0$ small and it is vanishing for $\varepsilon = 0$. Hence we can repeat the proof of Theorem 3.3.8 to obtain a unique solution $\tilde{x}(\varepsilon)$ of (3.3.15) in a neighbourhood of ϕ_{β_0} for $\varepsilon > 0$ and $\gamma > 0$ small. On the other hand,

$$F_{\varepsilon_0}\big(x(\varepsilon_0)\big) + \gamma\varepsilon_0 B\big(x(\varepsilon_0) - x(\varepsilon_0)\big) = 0\,.$$

Hence $x(\varepsilon_0) = \tilde{x}(\varepsilon_0)$. By using Theorem 2.2.4, we obtain that the linear map $DF_{\varepsilon_0}\big(x(\varepsilon_0)\big)$ is invertible, i.e. the above linearized equation of (3.3.10) at $x(\varepsilon_0)$ has only the zero bounded solution on $\mathbb{R}$.

Now we show that π_ε has a hyperbolic fixed point near 0. For this purpose, we solve
$$F_\varepsilon = 0$$
near $x \equiv 0$, i.e. we solve the equation

$$\begin{aligned} z' &= Df(0)z + \big\{f(z) - Df(0)z\big\} + \varepsilon h(z)\,,\\ z(i/\varepsilon+) &= z(i/\varepsilon-) + \varepsilon g\big(z(i/\varepsilon-)\big)\,, \end{aligned} \tag{3.3.16}$$

near $z = 0$. By repeating the above procedure applied to Eqs. (3.3.12)–(3.3.13), when Theorem 3.3.3 is replaced by Theorem 3.3.7, we obtain a unique small solution $\bar{x}(\varepsilon) \in X_1^m$ of (3.3.16). On the other hand, if $\tilde{x}$ is a solution of F_ε then $\tilde{x}(1+\cdot)$ is also a solution. Hence
$$\bar{x}(\varepsilon)(1+\cdot) = \bar{x}(\varepsilon)(\cdot)$$
because of uniqueness. So the point $\bar{x}(1-)$ is a fixed point of π_ε. To show the hyperbolicity of this point, we again apply Lemma 2.5.2 and Theorem 2.2.4 by taking an equation similar to (3.3.15) of the form

$$F_\varepsilon(x) + \gamma\varepsilon B\big(x - \bar{x}(\varepsilon_0)\big) = 0\,,$$

for a fixed small $\varepsilon_0 > 0$. By employing Theorem 3.3.7 as above for (3.3.16), the only small solution of this equation is $\bar{x}(\varepsilon_0)$. So $DF_{\varepsilon_0}\big(\bar{x}(\varepsilon_0)\big)$ is invertible, i.e. $\bar{x}(\varepsilon_0)(1-)$ is a hyperbolic fixed point of π_{ε_0}. Summing up, we obtain

Theorem 3.3.9. *The Poincarè map π_ε of* (1.1) *has a unique hyperbolic fixed point near* 0 *for any $\varepsilon > 0$ sufficiently small.*

Summarizing our results we see that the set $\big\{x(\varepsilon)(i/\varepsilon-)\big\}_{i=-\infty}^{\infty}$ is a transversal homoclinic orbit of π_ε to the hyperbolic fixed point $\bar{x}(\varepsilon)(1-)$ for any $\varepsilon > 0$ sufficiently small. This gives the main result of this section.

Theorem 3.3.10. *If there is a simple root of $\mathscr{M}(\beta) = 0$, then π_ε - the Poincarè map of* (3.3.1) *- possesses a transversal homoclinic point for any $\varepsilon > 0$ sufficiently small.*

3.3.4 Examples of Singular Impulsive ODEs

Consider

$$\begin{aligned} \varepsilon x' &= f(x) + \varepsilon h(x)\,, \\ x(i+) &= x(i-) + \varepsilon \tau a, \quad i \in \mathbb{Z}\,, \end{aligned} \tag{3.3.17}$$

where $a \in \mathbb{R}^m$ is fixed, $\tau \in \mathbb{R}$ is a parameter and f, h satisfy the assumptions (H1)–(H4).

Theorem 3.3.11. *If $\int_{-\infty}^{\infty} \big\langle h(\phi(s)), \psi(s)\big\rangle_m ds \neq 0$ and there is $\beta_0 \in \mathbb{R}$ satisfying*

$$\big\langle a, \psi(\beta_0)\big\rangle_m \neq 0, \quad \big\langle a, \psi'(\beta_0)\big\rangle_m \neq 0\,.$$

Then, for any $\varepsilon > 0$ sufficiently small, the Poincarè map of (3.3.17) *has a transversal homoclinic orbit for $\tau_0 = -\int_{-\infty}^{\infty} \big\langle h(\phi(s)), \psi(s)\big\rangle_m ds / \big\langle a, \psi(\beta_0)\big\rangle_m$.*

Proof. In this case, the Melnikov function (3.3.2) for (3.3.17) with $\tau = \tau_0$ has the form

$$\mathscr{M}(\beta) = \tau_0 \big\langle a, \psi(\beta)\big\rangle_m + \int_{-\infty}^{\infty} \big\langle h(\phi(s)), \psi(s)\big\rangle_m ds\,.$$

It is clear that $\mathscr{M}(\beta_0) = 0$, $\mathscr{M}'(\beta_0) \neq 0$. So Theorem 3.3.10 implies the assertion. The proof is finished. □

We note that under the assumptions of Theorem 3.3.11, the Poincarè map of (3.3.17) has a transversal homoclinic orbit for any τ near τ_0 and any $\varepsilon > 0$ sufficiently small.

Theorem 3.3.12. *If $\int_{-\infty}^{\infty} \big\langle h(\phi(s)), \psi(s)\big\rangle_m ds = 0$ and there is $\beta_0 \in \mathbb{R}$ satisfying*

$$\big\langle a, \psi(\beta_0)\big\rangle_m = 0, \quad \big\langle a, \psi'(\beta_0)\big\rangle_m \neq 0\,.$$

Then, for any $\varepsilon > 0$ sufficiently small, the Poincarè map of (3.3.17) *has a transversal homoclinic orbit for any $\tau \neq 0$ fixed.*

Proof. In this case,

$$\mathscr{M}(\beta) = \tau \big\langle a, \psi(\beta)\big\rangle_m\,.$$

So $\mathscr{M}(\beta_0) = 0$, $\mathscr{M}'(\beta_0) \neq 0$. The proof is finished by Theorem 3.3.10. □

Finally, let us consider an impulsive Duffing–type equation of the form (3.3.3).

Theorem 3.3.13. *Assume that $p(0) = 0$, $p'(0) < 0$ and the second–order ODE*

$$z'' + p(z) = 0$$

has a nonconstant solution $\gamma(t)$ so that $\gamma(t) \to 0$ as $t \to \pm\infty$. If there is $\beta_0 \in \mathbb{R}$ so that $\gamma''(\beta_0) = 0$, $\gamma'''(\beta_0) \neq 0$ and $r\big(\gamma(\beta_0)\big) \neq 0$, then (3.3.3) *has chaotic behaviour for any $a > 0$ sufficiently large.*

Proof. The equation can be rewritten in the form

$$\begin{aligned} \varepsilon x' &= f(x) + \varepsilon h(x)\,, \\ x(i+) &= x(i-) + \varepsilon g\big(x(i-)\big)\,, \end{aligned} \tag{3.3.18}$$

where

$$\varepsilon = 1/a, \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad f(x_1,x_2) = \big(x_2, -p(x_1)\big)\,,$$
$$h(x_1,x_2) = \big(0, q(x_1)\big), \quad g(x_1,x_2) = \big(r(x_1), 0\big)\,.$$

We note [31] that in this case

$$\phi(\beta) = \big(\gamma(\beta), \gamma'(\beta)\big)\,, \quad \psi(\beta) = \big(-\gamma''(\beta), \gamma'(\beta)\big)\,.$$

So the Melnikov function of Theorem 3.3.10 has the form:

$$\mathscr{M}(\beta) = -r\big(\gamma(\beta)\big)\gamma''(\beta) + \int_{-\infty}^{\infty} q\big(\gamma(s)\big)\gamma'(s)\,ds = r\big(\gamma(\beta)\big)p(\gamma(\beta))\,.$$

By $\mathscr{M}(\beta_0) = 0$ and $\mathscr{M}'(\beta_0) \neq 0$, the conclusion follows from Theorem 3.3.10. □

Remark 3.3.14. Consider

$$\begin{aligned} z'' + a^2 p(z) &= q(z)\,, \\ a\big(z(i+) - z(i-)\big) &= r\big(z(i-)\big)\,, \\ z'(i+) &= z'(i-), \quad i \in \mathbb{Z} \end{aligned} \tag{3.3.19}$$

instead of (3.3.3). Then the statement of Theorem 3.3.13 holds, since (3.3.18) is replaced by

$$\begin{aligned} \varepsilon x' &= f(x) + \varepsilon^2 h(x)\,, \\ x(i+) &= x(i-) + \varepsilon g\big(x(i-)\big)\,. \end{aligned}$$

It easily follows, from the proof of Theorem 3.3.13, that $\mathscr{M}(\beta) = r\big(\gamma(\beta)\big)p\big(\gamma(\beta)\big)$ in this case too, hence Theorem 3.3.13 still holds.

Remark 3.3.15. Consider

$$\begin{aligned} z'' + a^2 p(z) &= q(z)\,, \\ a^2\big(z(i+) - z(i-)\big) &= r\big(z(i-)\big)\,, \\ z'(i+) &= z'(i-), \quad i \in \mathbb{Z} \end{aligned} \tag{3.3.20}$$

instead of (3.3.3). Then the statement of Theorem 3.3.13 holds, since (3.3.18) is replaced by

$$\begin{aligned} \varepsilon x' &= f(x) + \varepsilon^2 h(x)\,, \\ x(i+) &= x(i-) + \varepsilon^2 g\big(x(i-)\big)\,. \end{aligned} \tag{3.3.21}$$

Of course, the Melnikov function for (3.3.21) is vanishing, since we derived in Theorem 3.3.10 the first–order Melnikov function. However the factor ε^2 in both the perturbation and the jumping term allow us to repeat the arguments of Section 3.3.3

showing, then that the solution of system (3.3.12) is $O(\varepsilon^2)$-bounded, uniformly in β and the same holds for its derivative with respect to β. Thus, we can divide the bifurcation function by ε^2 and take the limit as $\varepsilon \to 0$ (uniformly in β), getting the same bifurcation function as in (3.3.2). Hence [31, p. 284] we see that a simple root of the above Melnikov function of (3.3.18) ensures the validity of Theorem 3.3.13 also for (3.3.20).

3.4 Singularly Perturbed Impulsive ODEs

3.4.1 Singularly Perturbed ODEs with Impulses

In this section we proceed with the study of chaotic behaviour of dynamical systems with impulses. More precisely, we study the chaotic behavior of the equation

$$\begin{aligned} \varepsilon y' &= f(x,y,\varepsilon)\,,\\ x' &= g(x,y,\varepsilon)\,, \end{aligned} \tag{3.4.1}$$

with the impulsive effects

$$\begin{aligned} x(i+0) &= x(i-0)+\varepsilon a\big(x(i-0),y(i-0),\varepsilon\big)\,,\\ y(i+0) &= y(i-0)+\varepsilon b\big(x(i-0),y(i-0),\varepsilon\big), \quad i\in\mathbb{Z}\,, \end{aligned} \tag{3.4.2}$$

where as usual $\lim\limits_{t\to i_\pm} x(t) = x(i\pm 0)$. Here $y\in\mathbb{R}^p$, $x\in\mathbb{R}^m$ and $\varepsilon>0$ is a small parameter. We assume that

(H1) f,g,a,b are C^3–smooth;
(H2) $f(\cdot,0,0)=0$, $D_yf(\cdot,0,0)=\big(A(\cdot),B(\cdot)\big)$, where $A(\cdot)\in L(\mathbb{R}^{k_1})$, $B(\cdot)\in L(\mathbb{R}^{k_2})$, $k_1+k_2=p$;
(H3) $\big\{\Re\tau \mid \tau\in\sigma\big(A(\cdot)\big)\big\}\subset(-\infty,-\gamma)$ and $\big\{\Re\tau \mid \tau\in\sigma\big(B(\cdot)\big)\big\}\subset(\gamma,\infty)$ for some constant $\gamma>0$;
(H4) The *reduced equation* $x'=g(x,0,0)$ has a hyperbolic equilibrium $\bar{x}_0$ with a homoclinic orbit $x(t)$;
(H5) The variational equation $v'=D_xg\big(x(t),0,0\big)v$ has the only unique (up to constant multiples) bounded solution $x'(\cdot)$.

By a solution of (3.4.1)–(3.4.2) we mean some (x,y) which is C^1–smooth in $\mathbb{R}\setminus\mathbb{Z}$ satisfying (3.4.1) on this set and moreover, (3.4.2) holds for any $i\in\mathbb{Z}$. For simplicity, we assume that f,g,a,b are globally Lipschitz continuous. Then (3.4.1)–(3.4.2) with any initial condition $x(t_0)=x_0$, $y(t_0)=y_0$, $t_0\notin\mathbb{Z}$ has a unique global solution. Furthermore, we can define a Poincarè map H_ε of (3.4.1)–(3.4.2) in the following way. Let $\phi_\varepsilon\big(t,(x_0,y_0)\big)$ be the unique solution of (3.4.1) with the initial point (x_0,y_0). Then we put

$$H_\varepsilon(x_0,y_0) = \phi_\varepsilon\Big(1, \big(x_0+\varepsilon a(x_0,y_0,\varepsilon), y_0+\varepsilon b(x_0,y_0,\varepsilon)\big)\Big)\,.$$

Of course, the dynamics of (3.4.1)–(3.4.2) is wholly determined by H_ε. The aim of this section is to find assumptions for f, g, a, b which give the existence of transversal homoclinic point of H_ε for any $\varepsilon > 0$ small. For this purpose, we derive a Melnikov function for (3.4.1)–(3.4.2). Then such Eqs. (3.4.1)–(3.4.2) will have a chaotic behaviour for $\varepsilon > 0$ small. The chaotic behaviour of small periodic perturbations of (3.4.1) is studied in Section 4.4.

3.4.2 Melnikov Function

We know by Section 4.1.2 that (H4) and (H5) imply the uniqueness (up to constant multiples) of a bounded nonzero solution u of the adjoint variational equation

$$u' = -\Big(D_x g\big(x(t),0,0\big)\Big)^* u\,.$$

Since the derivation of a Melnikov function for (3.4.1)–(3.4.2) is very similar to results of Section 3.3, we omit further details and refer to [32]. Hence the Melnikov function is now:

$$\begin{aligned}\mathscr{M}(t) = &\sum_{i=-\infty}^{\infty} \langle a\big(x(t+i),0,0\big), u(t+i)\rangle_m \\ &+\int_{-\infty}^{\infty}\Big\langle -D_y g\big(x(s),0,0\big)D_y f\big(x(s),0,0\big)^{-1}D_\varepsilon f\big(x(s),0,0\big)+ \\ &+D_\varepsilon g\big(x(s),0,0\big), u(s)\Big\rangle_m ds\end{aligned} \tag{3.4.3}$$

where $\langle\cdot,\cdot\rangle_m$ is the usual inner product on $\mathbb{R}^m$. Now we are ready to state the main result of this section.

Theorem 3.4.1. *Assume that there is t_0 so that*

$$\mathscr{M}(t_0) = 0, \quad \mathscr{M}'(t_0) \neq 0\,.$$

Then (3.4.1)–(3.4.2) *have transversal homoclinic orbit for any $\varepsilon > 0$ small.*

Remark 3.4.2. We have considered only the case of the uniform distribution of impulsive effects. We may study (3.4.1) similarly as above with impulsive effects of the form (3.4.2) at t_i, $i \in \mathbb{Z}$ for a fixed sequence $\{t_i\}_{i=-\infty}^{\infty}$, $t_i < t_{i+1}$ so that

$$\begin{gathered} t_i \to \pm\infty \quad \text{as} \quad i \to \pm\infty \\ \sup_i (t_{i+1}-t_i) < \infty, \quad \inf_i (t_{i+1}-t_i) > 0\,. \end{gathered}$$

Then, of course, (3.4.1)–(3.4.2) do not define any Poincarè map for general $\{t_i\}_{i=-\infty}^{\infty}$. A line of the paper [33] may be followed for the above general impulsive effects.

Remark 3.4.3. The second term of the Melnikov function $\mathscr{M}$ (see (3.4.3)), which does not depend on t, is only a contribution of (3.4.1) (see Section 4.4). While the first term of $\mathscr{M}$ is determined by both (3.4.1) and (3.4.2).

3.4.3 Second Order Singularly Perturbed ODEs with Impulses

In this section, we consider

$$\begin{aligned} \varepsilon x'' &= x' - f(x)\,, \\ x(i+0) &= x(i-0) + \varepsilon a\big(x(i-0), x'(i-0)\big)\,, \\ x'(i+0) &= x'(i-0) + \varepsilon b\big(x(i-0), x'(i-0)\big) \end{aligned} \tag{3.4.4}$$

where $f : \mathbb{R}^m \to \mathbb{R}^m$ and f, a, b are C^2–smooth. Moreover, assume that the equation $x' = f(x)$ has a hyperbolic equilibrium $\bar{x}_0$ with a homoclinic orbit $x(\cdot)$. Furthermore, suppose the adjoint variational equation $v' = -(Df(x(t)))^* v$ has a unique (up to constant multiples) bounded nonzero solution u. Taking $x' = y + f(x)$ we obtain from (3.4.4)

$$\begin{aligned} \varepsilon y' &= y - \varepsilon Df(x)\big(y + f(x)\big)\,, \\ x' &= y + f(x)\,, \\ x(i+0) &= x(i-0) + \varepsilon a\big(x(i-0), y(i-0) + f(x(i-0))\big)\,, \\ y(i+0) &= y(i-0) + \varepsilon b\big(x(i-0), y(i-0) + f(x(i-0))\big) \\ &\quad + f\big(x(i-0)\big) - f\Big(x(i-0) + \varepsilon a\big(x(i-0), y(i-0) + f(x(i-0))\big)\Big)\,. \end{aligned} \tag{3.4.5}$$

We see (3.4.5) is of the form (3.4.1)–(3.4.2), and the Melnikov function $\mathscr{M}$, for this case, has the form (see (3.4.3))

$$\begin{aligned} \bar{\mathscr{M}}(t) &= \sum_{i=-\infty}^{\infty} \langle a\big(x(t+i), f(x(t+i))\big), u(t+i)\rangle_m + \int_{-\infty}^{\infty} \langle Df\big(x(s)\big) f\big(x(s)\big), u(s)\rangle_m\, ds \\ &= \sum_{i=-\infty}^{\infty} \langle a\big(x(t+i), f(x(t+i))\big), u(t+i)\rangle_m + \int_{-\infty}^{\infty} \langle Df\big(x(s)\big) x'(s), u(s)\rangle_m\, ds \\ &= \sum_{i=-\infty}^{\infty} \langle a\big(x(t+i), f(x(t+i))\big), u(t+i)\rangle_m + \int_{-\infty}^{\infty} \langle x''(s), u(s)\rangle_m\, ds \\ &= \sum_{i=-\infty}^{\infty} \langle a\big(x(t+i), f(x(t+i))\big), u(t+i)\rangle_m - \int_{-\infty}^{\infty} \langle x'(s), u'(s)\rangle_m\, ds\,. \end{aligned}$$

Hence

$$\bar{\mathscr{M}}(t) = \sum_{i=-\infty}^{\infty} \langle a\big(x(t+i), f(x(t+i))\big), u(t+i)\rangle_m - \int_{-\infty}^{\infty} \langle x'(s), u'(s)\rangle_m \, ds\,. \quad (3.4.6)$$

By applying Theorem 3.4.1 we obtain.

Theorem 3.4.4. *Assume that there is t_0 so that*

$$\bar{\mathscr{M}}(t_0) = 0, \quad \bar{\mathscr{M}}'(t_0) \neq 0\,.$$

Then (3.4.4) *has a chaotic behaviour for any $\varepsilon > 0$ small.*

3.5 Inflated Deterministic Chaos

3.5.1 Inflated Dynamical Systems

The following problem arises in computer-assisted proofs and other numerical methods in dynamical systems [34–37]. Let $\mathscr{B}_{\mathbb{R}^n}$ be a unit closed ball of $\mathbb{R}^n$. For a homeomorphism $f : \mathbb{R}^n \mapsto \mathbb{R}^n$, we consider an orbit $\{x_j\}_{j\in\mathbb{Z}}$ of an *ε-inflated mapping $x \longrightarrow f(x) + \varepsilon\mathscr{B}_{\mathbb{R}^n}$* for $\varepsilon > 0$. Then we deal with a difference inclusion

$$x_{j+1} \in f(x_j) + \varepsilon\mathscr{B}_{\mathbb{R}^n}\,, \quad j \in \mathbb{Z}\,. \quad (3.5.1)$$

The concept of ε–inflated dynamics was introduced in [36] and was used in a fairly large number of papers since then. For details, see the monograph [38] and the references therein. Consequently, the theory of generalized nonautonomous attractors in the ε–inflated dynamics can be considered to be complete by now.

We are not interested in the existence of one solution of (3.5.1), but in the set of all trajectories of (3.5.1). So, for instance, to fix the initial point x_0, we consider a single-valued difference equation

$$x_{j+1} = f(x_j) + \varepsilon p_j, \quad p_j \in \mathscr{B}_{\mathbb{R}^n}\,, \quad j \in \mathbb{Z}\,, \quad (3.5.2)$$

where $\mathbf{p} = \{p_j\}_{j\in\mathbb{Z}} \in \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}$ is considered as a parameter. This orbit of (3.5.2) is denoted by $\mathbf{x}(\mathbf{p}) = \{x_j(\mathbf{p})\}_{j\in\mathbb{Z}}$. Then we define an ε-inflated orbit of (3.5.1) given by

$$\mathbf{x}^\varepsilon(x_0) = \{x_j^\varepsilon\}_{j\in\mathbb{Z}}, \quad x_j^\varepsilon = \left\{x_j(\mathbf{p}) \mid \mathbf{p} \in \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}\right\}\,.$$

Here

$$\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n) = \left\{p = \{p_j\}_{j\in\mathbb{Z}} \mid p_j \in \mathbb{R}^n, \forall j \in \mathbb{Z} \text{ and } \|p\| := \sup_{j\in\mathbb{Z}} |p_j| < \infty\right\}$$

is the usual Banach space and $\mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}$ is its closed unit ball. Certainly it holds

$$x_{j+1}^{\varepsilon} = f(x_j^{\varepsilon}) + \varepsilon \mathscr{B}_{\mathbb{R}^n}, \quad j \in \mathbb{Z}.$$

Hence x_j^{ε} are contractible into themselves to $x_j^0 = f^j(x_0)$. The iteration $f^j(x_0)$, $j \neq 0$ is in the interior of x_j^{ε}. Note that $x_0^{\varepsilon} = x_0$. Moreover, x_j^{ε} are compact.

This approach of considering parameterized difference equation (3.5.2) instead of difference inclusion (3.5.1) is used in [39] for investigation of ε-inflated dynamics near either to a hyperbolic fixed point of a diffeomorphism or to a hyperbolic equilibrium of a differential equation. More precisely, we construct analogues of the stable and unstable manifolds, which are typical of a single-valued hyperbolic dynamics; moreover, we construct the maximal weakly invariant bounded set and prove that all such sets are graphs of Lipschitz maps. Then a parameterized generalization of Hartman-Grobman lemma is shown. Inflated ODEs are studied in Section 4.6.

3.5.2 Inflated Chaos

We consider a C^1-diffeomorphism $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ possessing a hyperbolic fixed point x_0. Then we take its g-inflated perturbation

$$x \longrightarrow f(x) + g(x, \mathscr{B}_{\mathbb{R}^n}) \tag{3.5.3}$$

where $g : \mathbb{R}^n \times \mathscr{B}_{\mathbb{R}^n} \to \mathbb{R}^n$ is Lipschitz in the both variables, i.e. the following holds: There are positive constants λ, Λ and L so that

$$|g(x,p) - g(\tilde{x},\tilde{p})| \leq \lambda |x - \tilde{x}| + \Lambda |p - \tilde{p}| \quad \text{and} \quad |g(x,0)| \leq L \tag{3.5.4}$$

whenever $x, \tilde{x} \in \mathbb{R}^n$ and $p, \tilde{p} \in \mathscr{B}_{\mathbb{R}^n}$. We suppose, in addition, that diffeomorphism f possesses a transversal homoclinic orbit $\{x_k^0\}_{k\in\mathbb{Z}}$ to hyperbolic fixed point x_0. Then f is chaotic by the Smale-Birkhoff homoclinic theorem 2.5.4. Our aim is to extend this theorem to (3.5.3).

Our multivalued perturbation takes the special form $G(x) = g(x, \mathscr{B}_{\mathbb{R}^n})$. So (3.5.3) has the form $x \longrightarrow f(x) + G(x)$. In view of the Lojasiewicz-Ornelas parametrization theorem 2.3.1, this is not a loss of generality if the values of G are convex and compact. However, in the general case a parameterization of G does not exist. We mention that some nonconvex versions exist as well [40], but in general, a parameterization cannot be available, since continuous selections may not exists (see [41], Section 1.6). Hence, we consider

$$x_{k+1} \in f(x_k) + g(x_k, \mathscr{B}_{\mathbb{R}^n}), \quad k \in \mathbb{Z}. \tag{3.5.5}$$

Like in [39], we take $\mathbf{p} = \{p_k\}_{k\in\mathbb{Z}} \in \ell_{\mathbb{Z}}^{\infty}(\mathbb{R}^n)$, $\|\mathbf{p}\| \leq 1$ and consider the system

$$x_{k+1} = f(x_k) + g(x_k, p_k), \quad k \in \mathbb{Z}. \tag{3.5.6}$$

First, we know by Lemma 2.5.2 that the transversality of a homoclinic orbit $\{x_k^0\}_{k\in\mathbb{Z}}$ is equivalent to the existence of an exponential dichotomy of $w_{k+1} = Df(x_k^0)w_k$ on $\mathbb{Z}$, i.e. setting the fundamental solution

$$W(k) := \begin{cases} Df(x_{k-1}^0)\cdots Df(x_0^0), & \text{if } k>0, \\ \mathbb{I}, & \text{if } k=0, \\ Df(x_k^0)^{-1}\cdots Df(x_{-1}^0)^{-1}, & \text{if } k<0, \end{cases}$$

there are a projection $P:\mathbb{R}^n \mapsto \mathbb{R}^n$ and positive constants $K>0$, $\delta\in(0,1)$ so that

$$\begin{aligned} \left|W(k)PW(r)^{-1}\right| &\le K\delta^{k-r}, &\quad \text{for } k\ge r, \\ \left|W(k)(\mathbb{I}-P)W(r)^{-1}\right| &\le K\delta^{k-r}, &\quad \text{for } k\le r. \end{aligned}$$

Now we fix $\omega\in\mathbb{N}$ large and for any $\xi\in\mathscr{E}$, $\xi=\{e_j\}_{j\in\mathbb{Z}}$ we define a pseudo-orbit $\mathbf{x}^\xi=\{x_k^\xi\}_{k\in\mathbb{Z}}$ as follows for $k\in\{2j\omega,\dots,2(j+1)\omega-1\}$, $j\in\mathbb{Z}$:

$$x_k^\xi := \begin{cases} x_{k-(2j+1)\omega}^0, & \text{for } e_j=1, \\ x_0, & \text{for } e_j=0. \end{cases}$$

Let $|x_{k_0}^0-x_0| = \max_{k\in\mathbb{Z}}|x_k^0-x_0|$. Following [10, pp. 148–151] and [13], we have the following result.

Lemma 3.5.1. *There exist $\omega_0\in\mathbb{N}$, $\omega_0>|k_0|$ and a constant $c>0$ so that for any $\xi\in\mathscr{E}$, $\mathbf{h}=\{h_k\}_{k\in\mathbb{Z}}\in\ell_{\mathbb{Z}}^\infty(\mathbb{R}^n)$, there is a unique solution $\mathbf{w}=\{w_k\}_{k\in\mathbb{Z}}\in\ell_{\mathbb{Z}}^\infty(\mathbb{R}^n)$ of the linear system*

$$w_{k+1} = Df(x_k^\xi)w_k + h_k, \quad k\in\mathbb{Z}.$$

Moreover, $\mathbf{w}$ *is linear in* $\mathbf{h}$ *and it holds* $\|\mathbf{w}\|\le c\|\mathbf{h}\|$.

We denote that $K(\xi)h=\mathbf{w}$ is the unique solution from Lemma 3.5.1. Certainly $K(\xi)\in L\left(\ell_{\mathbb{Z}}^\infty(\mathbb{R}^n)\right)$ with $\|K(\xi)\|\le c$, and $K(\xi)^{-1}\mathbf{w} = \left\{w_{k+1}-Df(x_k^\xi)w_k\right\}_{k\in\mathbb{Z}}$, so $K(\xi)^{-1}\in L\left(\ell_{\mathbb{Z}}^\infty(\mathbb{R}^n)\right)$.

Now we look for a solution of (3.5.6) near $\mathbf{x}^\xi$. For this reason, we make a change of variables $x_k = w_k + x_k^\xi$, $k\in\mathbb{Z}$ to get the equation

$$w_{k+1} = Df\left(x_k^\xi\right)w_k + f\left(w_k+x_k^\xi\right) - x_{k+1}^\xi - Df\left(x_k^\xi\right)w_k + g\left(w_k+x_k^\xi,p_k\right) \quad (3.5.7)$$

for $k\in\mathbb{Z}$. To solve (3.5.7), we introduce a mapping

$$G:\mathscr{E}\times\mathscr{B}_{\ell_{\mathbb{Z}}^\infty(\mathbb{R}^n)}\times\ell_{\mathbb{Z}}^\infty(\mathbb{R}^n)\mapsto\ell_{\mathbb{Z}}^\infty(\mathbb{R}^n)$$

as follows:

$$G(\xi,\mathbf{p},\mathbf{w}) := \left\{f\left(w_k+x_k^\xi\right) - x_{k+1}^\xi - Df\left(x_k^\xi\right)w_k + g\left(w_k+x_k^\xi,p_k\right)\right\}_{k\in\mathbb{Z}}.$$

Now for any $\xi \in \mathscr{E}$, $\mathbf{w}^1, \mathbf{w}^2 \in \ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)$, $\|\mathbf{w}^{1,2}\| \le \rho$ and $\mathbf{p}^1, \mathbf{p}^2 \in \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}$, we derive

$$\|G(\xi,\mathbf{p}^1,\mathbf{w}^1) - G(\xi,\mathbf{p}^2,\mathbf{w}^2)\| \le (\triangle(\rho)+\lambda)\,\|\mathbf{w}^1-\mathbf{w}^2\| + \Lambda\|\mathbf{p}^1-\mathbf{p}^2\| \tag{3.5.8}$$

for

$$\triangle(\rho) := \sup\left\{|Df(w+x)-Df(x)| : |x-x_0| \le 2|x^0_{k_0}-x_0|, |w|\le\rho\right\}.$$

Note that $\triangle(0)=0$. Since $\{x^0_k\}_{k\in\mathbb{Z}}$ is a homoclinic orbit of f to x_0, by [42, p. 148], we also get

$$\|G(\xi,0,0)\| \le L + \sup_{k\in\mathbb{Z},\xi\in\mathscr{E}} |x^\xi_{k+1} - f(x^\xi_k)| \le L + \widetilde{c}\left(\frac{\delta+1}{2}\right)^\omega \tag{3.5.9}$$

for a constant $\widetilde{c}>0$ and any $\xi\in\mathscr{E}$. Now we are ready to rewrite (3.5.7) as the following fixed point problem

$$\mathbf{w} = F(\xi,\mathbf{p},\mathbf{w}) := K(\xi)G(\xi,\mathbf{p},\mathbf{w}).$$

By Lemma 3.5.1, (3.5.8) and (3.5.9), we obtain

$$\begin{aligned}\|F(\xi,\mathbf{p}^1,\mathbf{w}^1)-F(\xi,\mathbf{p}^2,\mathbf{w}^2)\| &\le c\,(\triangle(\rho)+\lambda)\,\|\mathbf{w}^1-\mathbf{w}^2\| + \Lambda c\|\mathbf{p}^1-\mathbf{p}^2\|,\\ \|F(\xi,\mathbf{p}^1,\mathbf{w}^1)\| &\le c\,(\triangle(\rho)+\lambda)\,\|\mathbf{w}^1\| + \Lambda c\|\mathbf{p}^1\| + Lc + c\widetilde{c}\left(\frac{\delta+1}{2}\right)^\omega\end{aligned} \tag{3.5.10}$$

for any $\xi \in \mathscr{E}$, $\mathbf{w}^1, \mathbf{w}^2 \in \ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)$, $\|\mathbf{w}^{1,2}\| \le \rho$ and $\mathbf{p}^1, \mathbf{p}^2 \in \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}$. Assuming that

$$c\lambda < 1, \tag{3.5.11}$$

we set

$$\widetilde{\kappa}_0 := \min\left\{1, c\lambda + c\triangle\left(\frac{|x^0_{k_0}-x_0|}{4}\right)\right\},$$

$$M_0(c,\lambda) := \max_{c\lambda\le\kappa\le\widetilde{\kappa}_0}\left\{\frac{1-\kappa}{c}\min\left\{\triangle^{-1}\left(\frac{\kappa-c\lambda}{c}\right)\right\}\right\}$$

and the above maximum is achieved at $\kappa_0 \in (c\lambda,1)$. Here $\triangle^{-1}:\mathbb{R}_+ \to 2^{\mathbb{R}_+}\setminus\{\emptyset\}$ is considered as an upper semicontinuous mapping which is increasing with increasing compact interval set values. Put

$$\rho_0 := \min\left\{\triangle^{-1}\left(\frac{\kappa_0-c\lambda}{c}\right)\right\}.$$

Note that

$$0<\rho_0=\min\left\{\triangle^{-1}\left(\frac{\kappa_0-c\lambda}{c}\right)\right\}\le\min\left\{\triangle^{-1}\left(\frac{\widetilde{\kappa}_0-c\lambda}{c}\right)\right\}\le\frac{|x_{k_0}^0-x_0|}{4},$$
$$\kappa_0=c\,(\triangle(\rho_0)+\lambda)\,.$$

If

$$\Lambda+L<M_0(c,\lambda)\,, \tag{3.5.12}$$

then $\Lambda+L<M_0(c,\lambda)=\frac{1-\kappa_0}{c}\rho_0$ and so

$$c\Lambda+cL+c\,(\triangle(\rho_0)+\lambda)\,\rho_0=c\Lambda+cL+\kappa_0\rho_0<\rho_0\,.$$

Consequently, we find $\mathbb{N}\ni\omega_1>\omega_0$ so that

$$c\widetilde{c}\left(\frac{\delta+1}{2}\right)^{\omega_1}+c\Lambda+cL+\kappa_0\rho_0\le\rho_0\,. \tag{3.5.13}$$

Then for any fixed $\mathbb{N}\ni\omega\ge\omega_1$, mapping:

$$F:\mathscr{E}\times\mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}\times\mathscr{B}^{\rho_0}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}\mapsto\mathscr{B}^{\rho_0}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}$$

is a contraction with a constant κ_0, where $\mathscr{B}^{\rho_0}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}$ is the ball of $\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)$ centered at 0 with the radius ρ_0. By the Banach fixed point theorem 2.2.1 we get the following result.

Theorem 3.5.2. *Assume* (3.5.11) *and* (3.5.12). *Then there are* $\omega_1>\omega_0$, $\frac{|x_{k_0}^0-x_0|}{4}\ge\rho_0>0$ *so that for any* $\mathbb{N}\ni\omega\ge\omega_1$ *but fixed and for any* $\xi\in\mathscr{E}$, $\mathbf{p}\in\mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}$, *there is a unique solution* $\mathbf{x}(\mathbf{p},\xi)=\{x_k(\mathbf{p},\xi)\}_{k\in\mathbb{Z}}\in\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)$ *of* (3.5.6) *so that*

$$\|\mathbf{x}(\mathbf{p},\xi)-\mathbf{x}^\xi\|\le\rho_0\,. \tag{3.5.14}$$

By (3.5.10), mapping:

$$\mathbf{x}:\mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}\times\mathscr{E}\mapsto\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)$$

is Lipschitzian in $\mathbf{p}$:

$$\|\mathbf{x}(\mathbf{p}^1,\xi)-\mathbf{x}(\mathbf{p}^2,\xi)\|\le\frac{c\Lambda}{1-\kappa_0}\|\mathbf{p}^1-\mathbf{p}^2\| \tag{3.5.15}$$

for any $\xi\in\mathscr{E}$ and $\mathbf{p}^1,\mathbf{p}^2\in\mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}$. Let

$$\ell_{\mathbb{Z}}(\mathbb{R}^n):=\{\{x_k\}_{k\in\mathbb{Z}}\mid x_k\in\mathbb{R}^n\}$$

be a metric space with a norm

$$d(\{e_{k\in\mathbb{Z}}\},\{e'_{k\in\mathbb{Z}}\}):=\sum_{k\in\mathbb{Z}}\frac{|e_k-e'_k|}{2^{|k|+1}(1+|e_k-e'_k|)}\,.$$

Clearly $\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n) \subset \ell_{\mathbb{Z}}(\mathbb{R}^n)$. Now we prove several useful results.

Theorem 3.5.3. *Mapping* $\mathbf{x} : \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)} \times \mathscr{E} \mapsto \ell_{\mathbb{Z}}(\mathbb{R}^n)$ *is continuous.*

Proof. Let $\mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)} \ni \mathbf{p}^i = \{p^i_j\}_{j\in\mathbb{Z}} \to \mathbf{p}^0 = \{p^0_j\}_{j\in\mathbb{Z}} \in \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}$, $\mathscr{E} \ni \xi_i = \{e^i_j\}_{j\in\mathbb{Z}} \to \xi_0 = \{e^0_j\}_{j\in\mathbb{Z}} \in \mathscr{E}$ as $i \to \infty$. Then using (3.5.14) and the Cantor diagonal procedure, we can suppose, by passing to subsequences, that

$$x_j(\mathbf{p}^i, \xi_i) \to \widetilde{x}^0_j, \quad \forall j \in \mathbb{Z},$$

as $i \to \infty$. We note that $e^i_j \to e^0_j$ as $i \to \infty$ $\forall j \in \mathbb{Z}$ and $\mathbf{x}(\mathbf{p}^i, \xi_i)$, $i \in \mathbb{Z}$ solving (3.5.6) along with (3.5.14) holds as well. By passing to the limit $i \to \infty$, we obtain

$$\widetilde{x}^0_{k+1} = f(\widetilde{x}^0_k) + g(\widetilde{x}^0_k, p^0_k), \quad k \in \mathbb{Z}$$

and $\widetilde{\mathbf{x}} = \{\widetilde{x}^0_j\}_{j\in\mathbb{Z}}$ satisfies (3.5.14) with $\xi = \xi_0$. The uniqueness property of Theorem 3.5.2 implies $\widetilde{\mathbf{x}} = \mathbf{x}(\mathbf{p}^0, \xi_0)$. The continuity of $\mathbf{x}$ is proved. □

Theorem 3.5.4. *It holds*

$$x_k(\widetilde{\mathbf{p}}, \sigma(\xi)) = x_{k+2\omega}(\mathbf{p}, \xi), \quad \forall k \in \mathbb{Z}, \tag{3.5.16}$$

for $\widetilde{\mathbf{p}} := \{p_{k+2\omega}\}_{k\in\mathbb{Z}}$.

Proof. Taking $z_k := x_{k+2\omega}(\mathbf{p}, \xi)$ for any $k \in \mathbb{Z}$, by $x^{\sigma(\xi)}_k = x^{\xi}_{k+2\omega}$ $\forall k \in \mathbb{Z}$, (3.5.6) and (3.5.14) we derive

$$z_{k+1} = f(z_k) + g(z_k, p_{k+2\omega}),$$
$$\left| z_k - x^{\sigma(\xi)}_k \right| = \left| x_{k+2\omega}(\mathbf{p}, \xi) - x^{\xi}_{k+2\omega} \right| \le \rho_0,$$

for any $k \in \mathbb{Z}$. The uniqueness property of Theorem 3.5.2 implies $z_k = x_k(\widetilde{\mathbf{p}}, \sigma(\xi))$ for any $k \in \mathbb{Z}$, so (3.5.16) is shown. □

Then (3.5.16) implies

$$x_{2k\omega}(\mathbf{p}, \xi) = x_0\left(\widetilde{\sigma}^k(\mathbf{p}), \sigma^k(\xi)\right), \quad \forall k \in \mathbb{Z}, \tag{3.5.17}$$

for a shift homeomorphism

$$\widetilde{\sigma} : \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)} \mapsto \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}$$

given by $\widetilde{\sigma}(\mathbf{p}) := \widetilde{\mathbf{p}}$. Note that

$$x_{2(k+1)\omega}(\mathbf{p}, \xi) = F^{2(k+1)\omega}_{2k\omega,\mathbf{p}}\left(x_{2k\omega}(\mathbf{p}, \xi)\right), \quad \forall k \in \mathbb{Z}, \tag{3.5.18}$$

for continuous mappings

$$F_{2k\omega,\mathbf{p}}^{2(k+1)\omega}(x) := \left(f+g(\cdot,p_{2(k+1)\omega-1})\right)\cdots\left(f+g(\cdot,p_{2k\omega+1})\right)\left(f+g(\cdot,p_{2k\omega})\right)(x).$$

Then (3.5.17) and (3.5.18) imply

$$x_0\left(\widetilde{\sigma}^{k+1}(\mathbf{p}),\sigma^{k+1}(\xi)\right) = F_{2k\omega,\mathbf{p}}^{2(k+1)\omega}\left(x_0\left(\widetilde{\sigma}^{k}(\mathbf{p}),\sigma^{k}(\xi)\right)\right), \quad \forall k\in\mathbb{Z}, \tag{3.5.19}$$

and since $\sigma^k : \mathscr{E} \mapsto \mathscr{E}$ is a homeomorphism, (3.5.19) gives

$$x_0\left(\widetilde{\sigma}^{k+1}(\mathbf{p}),\sigma(\xi)\right) = F_{2k\omega,\mathbf{p}}^{2(k+1)\omega}\left(x_0\left(\widetilde{\sigma}^{k}(\mathbf{p}),\xi\right)\right), \quad \forall k\in\mathbb{Z}. \tag{3.5.20}$$

Next, introducing the following mappings

$$\begin{gathered}\Sigma : \mathscr{B}_{\ell^\infty_\mathbb{Z}(\mathbb{R}^n)}\times\mathscr{E}\times\mathbb{Z}\mapsto\mathscr{B}_{\ell^\infty_\mathbb{Z}(\mathbb{R}^n)}\times\mathscr{E}\times\mathbb{Z},\\ \Sigma(\mathbf{p},\xi,k) := (\mathbf{p},\sigma(\xi),k+1),\end{gathered}$$

$$\begin{gathered}\Phi : \mathscr{B}_{\ell^\infty_\mathbb{Z}(\mathbb{R}^n)}\times\mathscr{E}\times\mathbb{Z}\mapsto\mathscr{B}_{\ell^\infty_\mathbb{Z}(\mathbb{R}^n)}\times\mathbb{R}^n\times\mathbb{Z},\\ \Phi(\mathbf{p},\xi,k) := \left(\mathbf{p},x_0\left(\widetilde{\sigma}^k(\mathbf{p}),\xi\right),k\right),\\ F^{2\omega} : \mathscr{B}_{\ell^\infty_\mathbb{Z}(\mathbb{R}^n)}\times\mathbb{R}^n\times\mathbb{Z}\mapsto\mathscr{B}_{\ell^\infty_\mathbb{Z}(\mathbb{R}^n)}\times\mathbb{R}^n\times\mathbb{Z},\\ F^{2\omega}(\mathbf{p},x,k) := \left(\mathbf{p},F_{2k\omega,\mathbf{p}}^{2(k+1)\omega}(x),k+1\right),\end{gathered}$$

and the set

$$\Lambda := \Phi\left(\mathscr{B}_{\ell^\infty_\mathbb{Z}(\mathbb{R}^n)}\times\mathscr{E}\times\mathbb{Z}\right),$$

we obtain the main result of this section.

Theorem 3.5.5. *The diagram of Figure 3.2 is commutative. Moreover, mappings Σ and Φ are homeomorphisms.*

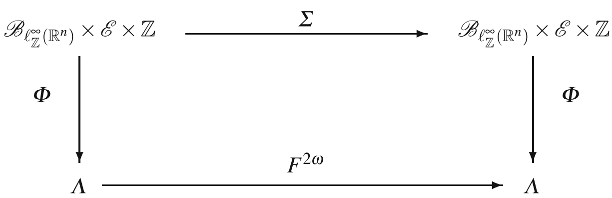

Fig. 3.2 Commutative diagram of inflated deterministic chaos.

Proof. The commutativity of diagram in Figure 3.2 follows directly from (3.5.20). Since $\sigma : \mathscr{E}\mapsto\mathscr{E}$ is a homeomorphism, Σ is also a homeomorphism. Now we show the injectivity of the mapping $x_0(\mathbf{p},\cdot) : \mathscr{E}\mapsto\mathbb{R}^n$. If there exist $\mathscr{E}\ni\xi^1 = \{e_j^1\}_{j\in\mathbb{Z}} \neq \xi^2 = \{e_j^2\}_{j\in\mathbb{Z}}\in\mathscr{E}$ and $x_0(\mathbf{p},\xi^1) = x_0(\mathbf{p},\xi^2)$, then $x_k(\mathbf{p},\xi^1) = x_k(\mathbf{p},\xi^2)$ for any $k\in\mathbb{Z}$ and $j_0\in\mathbb{Z}$ exists so that $e_{j_0}^1 \neq e_{j_0}^2$. Then (3.5.14) gives

$$|x^0_{k_0} - x_0| = \left|x^{\xi^1}_{(2j_0+1)\omega+k_0} - x^{\xi^2}_{(2j_0+1)\omega+k_0}\right| \leq \left|x_{(2j_0+1)\omega+k_0}(\mathbf{p},\xi^1) - x^{\xi^1}_{(2j_0+1)\omega+k_0}\right|$$
$$+ \left|x_{(2j_0+1)\omega+k_0}(\mathbf{p},\xi^2) - x^{\xi^2}_{(2j_0+1)\omega+k_0}\right| \leq 2\rho_0 < |x^0_{k_0} - x_0|\,,$$

which is a contradiction. Consequently $x_0(\mathbf{p},\cdot)$ is injective. Now suppose $\Phi(\mathbf{p}^1,\xi^1,k_1) = \Phi(\mathbf{p}^2,\xi^2,k_2)$. Then $\mathbf{p}^1 = \mathbf{p}^2 = \mathbf{p}$, $k_1 = k_2 = k$ and

$$x_0\left(\widetilde{\sigma}^k(\mathbf{p}),\xi^1\right) = x_0\left(\widetilde{\sigma}^k(\mathbf{p}),\xi^2\right)$$

and thus $\xi^1 = \xi^2$. Hence Φ is also injective. Finally assume that $\Phi(\mathbf{p}^i,\xi^i,k_i) \to \Phi(\mathbf{p}^0,\xi^0,k_0)$ as $i \to \infty$. Then $k^i = k^0$ for large i, $\mathbf{p}^i \to \mathbf{p}^0$ and

$$x_0\left(\widetilde{\sigma}^{k_0}(\mathbf{p}^i),\xi^i\right) \to x_0\left(\widetilde{\sigma}^{k_0}(\mathbf{p}^0),\xi^0\right).$$

Since $\mathscr{E}$ is compact, we can suppose $\xi^i \to \widetilde{\xi}^0$ and then

$$x_0\left(\widetilde{\sigma}^{k_0}(\mathbf{p}^0),\xi^0\right) = x_0\left(\widetilde{\sigma}^{k_0}(\mathbf{p}^0),\widetilde{\xi}^0\right)$$

and so $\widetilde{\xi}^0 = \xi^0$, i.e. Φ^{-1} is continuous. In summary, Φ is a homeomorphism. The proof is finished. □

Figure 3.2 has the following more transparent form in Figure 3.3 where

$$\widetilde{\Sigma} : \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)} \times \mathscr{E} \mapsto \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)} \times \mathscr{E}\,, \quad \widetilde{\Sigma}(\mathbf{p},\xi) := (\mathbf{p},\sigma(\xi))\,,$$

$$\Phi_k : \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)} \times \mathscr{E} \mapsto \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)} \times \mathbb{R}^n\,, \quad \Phi_k(\mathbf{p},\xi) := \left(\mathbf{p}, x_0\left(\widetilde{\sigma}^k(\mathbf{p}),\xi\right)\right),$$
$$\Lambda_k := \Phi_k\left(\mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)} \times \mathscr{E}\right),$$
$$F^{2\omega}_k : \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)} \times \mathbb{R}^n \mapsto \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)} \times \mathbb{R}^n\,, \quad F^{2\omega}_k(\mathbf{p},x) := \left(\mathbf{p}, F^{2(k+1)\omega}_{2k\omega,\mathbf{p}}(x)\right).$$

By putting

$$\Phi^{\mathbf{p}}_k : \mathscr{E} \mapsto \mathbb{R}^n\,, \quad \Phi^{\mathbf{p}}_k(\xi) := x_0\left(\widetilde{\sigma}^k(\mathbf{p}),\xi\right), \quad \Lambda^{\mathbf{p}}_k := \Phi^{\mathbf{p}}_k(\mathscr{E})\,,$$

Figure 3.3 has also more transparent forms described in Figure 3.4. All mappings in Figures 3.3 and 3.4 are again homeomorphisms, and sets $\Lambda^{\mathbf{p}}_k$ are compact. So Figure 3.4 is a two-parameterized analogy of Figure 2.1 of Section 2.5.2 by parameters $\mathbf{p} \in \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}$ and $k \in \mathbb{Z}$.

Set

$$\varphi_0(\xi) = \Phi^0_0(\xi) = x_0(0,\xi), \quad \Lambda_0 = \Lambda^0_0 = x_0(0,\mathscr{E}), \quad m = 2\omega\,. \tag{3.5.21}$$

By (3.5.15), all sets $\Lambda^{\mathbf{p}}_k$ are in a $\frac{c\Lambda}{1-\kappa_0}$-neighborhood of Λ_0. If

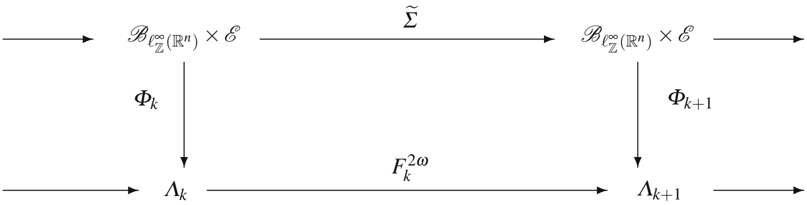

Fig. 3.3 A sequence of commutative diagrams from Figure 3.2.

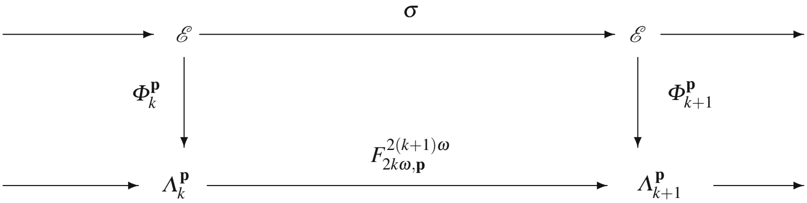

Fig. 3.4 A parameterized sequence of commutative diagrams from Figure 3.3.

$$g(x,0) = 0 \quad \forall x \in \mathbb{R}^n \tag{3.5.22}$$

then $L = 0$ in (3.5.4), $\varphi = \varphi_0$, $\Lambda = \Lambda_0$ in (3.5.21) and Figure 2.1 of Section 2.5.2 is derived from Figure 3.4 by setting $\mathbf{p} = 0$. Moreover, inequality (3.5.13) gives $\widetilde{\rho}_0 := c\widetilde{c}\left(\frac{\delta+1}{2}\right)^{\omega_0} + \kappa_0\rho < \rho_0$. Clearly $\triangle(\widetilde{\rho}_0) \leq \triangle(\rho_0)$ and so $\widetilde{\kappa}_0 := c\triangle(\widetilde{\rho}_0) \leq \kappa_0$. Repeating the proof of Theorem 3.5.2 we get $\|\mathbf{x}(0,\xi) - \mathbf{x}^\xi\| \leq \widetilde{\rho}_0$ for any $\xi \in \mathscr{E}$.

Note, the above diagrams are generalizations of similar results of [33, 43, 44] for non-autonomous sequences of diffeomorphisms, ordinary differential equations and inclusions. Now we put

$$\widetilde{\Lambda} := \bigcup_{\mathbf{p} \in \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}, k \in \mathbb{Z}} \Lambda_k^{\mathbf{p}}.$$

Note that $\widetilde{\Lambda} = x_0\left(\mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}, \mathscr{E}\right)$. We can consider $\widetilde{\Lambda}$ as an *inflated Smale horseshoe* of f.

Theorem 3.5.6. *Assume* (3.5.11), (3.5.12) *and* (3.5.22). *If* $\omega \in \mathbb{N}$ *is sufficiently large, then the following properties hold:*

(i) $\Lambda \subset \widetilde{\Lambda}$ *and if in addition*

$$g_x := g(x,\cdot) : \mathscr{B}_{\mathbb{R}^n} \to \mathbb{R}^n \quad \textit{is injective} \quad \forall x \in \mathbb{R}^n, \tag{3.5.23}$$

then Λ *is in the interior of* $\widetilde{\Lambda}$.

(ii) $\widetilde{\Lambda}$ *is contractible into* Λ *in itself.*

(iii) $\widetilde{\Lambda}$ *is in a* $\frac{c\Lambda}{1-\kappa_0}$*-neighborhood of* Λ.

(iv) $\widetilde{\Lambda}$ *is back and forward weakly invariant with respect to an m-iteration of* (3.5.3), *i.e.* $\exists m \in \mathbb{N}$ *so that* $\forall \bar{x}_0 \in \widetilde{\Lambda}$, $\exists \{\bar{x}_k\}_{k\in\mathbb{Z}}$ *satisfying* $\bar{x}_{k+1} \in f(\bar{x}_k) + g(\bar{x}_k, \mathscr{B}_{\mathbb{R}^n})$ *and* $\bar{x}_{km} \in \widetilde{\Lambda}$, $\forall k \in \mathbb{Z}$.

(v) *Dynamics of* (3.5.3) *back and forward sensitively depends on* $\widetilde{\Lambda}$, *i.e. there is a constant* $\eta > 0$ *so that for any* $\bar{x}_0 \in \widetilde{\Lambda}$ *and any open neighborhood* $\bar{x}_0 \in U \subset \mathbb{R}^n$, *there is* $\widetilde{x}_0 \in U \cap \widetilde{\Lambda}$ *and* $\{\bar{x}_k\}_{k\in\mathbb{Z}}$, $\{\widetilde{x}_k\}_{k\in\mathbb{Z}}$ *satisfying* $\bar{x}_{k+1} \in f(\bar{x}_k) + g(\bar{x}_k, \mathscr{B}_{\mathbb{R}^n})$ *and* $\widetilde{x}_{k+1} \in f(\widetilde{x}_k) + g(\widetilde{x}_k, \mathscr{B}_{\mathbb{R}^n})$, $\forall k \in \mathbb{Z}$, *and there exist* $j_0, j_1 \in \mathbb{Z}$, $j_0 < 0 < j_1$ *so that* $\left|\bar{x}_{j_0} - \widetilde{x}_{j_0}\right| \geq \eta$ *and* $\left|\bar{x}_{j_1} - \widetilde{x}_{j_1}\right| \geq \eta$.

(vi) (3.5.3) *has a chaotic/oscillatory behavior on* $\widetilde{\Lambda}$.

where we consider Theorem 2.5.4 in the sense of (3.5.21).

Proof. Since $\Lambda_0 = \Lambda$, we get $\Lambda \subset \widetilde{\Lambda}$. Next we fix $\xi \in \mathscr{E}$ and consider a mapping $\Theta_\xi : \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)} \mapsto \ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)$ given by $\Theta_\xi(\mathbf{p}) = \mathbf{x}(\mathbf{p}, \xi)$. We study Θ_ξ for $\mathbf{p}$ near 0. From (3.5.23), there are open neighborhoods $0 \in V \subset \mathbb{R}^n$ and $\overline{\widetilde{\Lambda}} \subset W$ so that

$$V \subset g_x(\mathscr{B}_{\mathbb{R}^n}), \quad \forall x \in W.$$

So we have $\psi_x := g_x^{-1} : V \to \mathscr{B}_{\mathbb{R}^n}$, $\forall x \in W$. Clearly $\psi(x,z) := \psi_x(z)$, $\psi : W \times V \to \mathbb{R}^n$ is continuous. We continuously extend ψ on $\mathbb{R}^n \times \mathbb{R}^n$. Then we define $R : \ell^\infty_{\mathbb{Z}}(\mathbb{R}^n) \to \ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)$ as follows

$$R(\mathbf{x}) := \{\psi(x_k, x_{k+1} - f(x_k)\}_{k\in\mathbb{Z}}.$$

R is continuous. If $\|\mathbf{p}\|$ is small then $x_{k+1} - f(x_k) = g(x_k, p_k) \in V$ for $\mathbf{x}(\mathbf{p}, \xi) = \{x_k\}_{k\in\mathbb{Z}}$, so $p_k = g_{x_k}^{-1}(x_{k+1} - f(x_k)) = \psi(x_k, x_{k+1} - f(x_k))$, i.e. $R(\Theta_\xi(\mathbf{p})) = \mathbf{p}$ for any $\mathbf{p}$ small. Note that $\Theta_\xi(0) = \mathbf{x}(0,\xi) = \left\{f^k(\varphi(\xi))\right\}_{k\in\mathbb{Z}}$ and $\|\mathbf{x}(0,\xi) - \mathbf{x}^\xi\| \leq \widetilde{\rho}_0 < \rho_0$ for any $\xi \in \mathscr{E}$. On the other hand, if $\mathbf{x} = \{x_k\}_{k\in\mathbb{Z}}$ is close to $\Theta_\xi(0)$ then $x_{k+1} - f(x_k) \in V$ $\forall k \in \mathbb{Z}$ along with $\|\mathbf{x} - \mathbf{x}^\xi\| \leq \rho_0$, so we can put $p_k := \psi(x_k, x_{k+1} - f(x_k)) \in \mathscr{B}_{\mathbb{R}^n}$. Then $x_{k+1} = f(x_k) + g(x_k, p_k)$. From the uniqueness we derive $\mathbf{x} = \Theta_\xi(\mathbf{p}) = \Theta_\xi(R(\mathbf{x}))$. In summary, Θ_ξ is a local homeomorphism at $\mathbf{p} = 0$. Now, a projection $P_0 : \ell^\infty_{\mathbb{Z}}(\mathbb{R}^n) \mapsto \mathbb{R}^n$ given by $P_0(\{\bar{x}_k\}_{k\in\mathbb{Z}}) := \bar{x}_0$ is an open linear mapping. Consequently, a mapping $P_0 \circ \Theta_\xi(\mathbf{p}) = x_0(\mathbf{p}, \xi)$ maps a small open neighborhood of $\mathbf{p} = 0$ onto a small open neighborhood of $\varphi(\xi) = P_0 \circ \Theta_\xi(0) \in \Lambda$. This implies property (i). By taking

$$\widetilde{\Lambda}_\lambda := \left\{x_0(\lambda\mathbf{p}, \xi) : \mathbf{p} \in \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}, \xi \in \mathscr{E}\right\}$$

for $\lambda \in [0,1]$, we get property (ii), since clearly $\widetilde{\Lambda}_\lambda \subset \widetilde{\Lambda}$ and $\widetilde{\Lambda}_0 = \Lambda$. Property (iii) follows from (3.5.15). The definition of $\widetilde{\Lambda}$ implies property (iv). Now we show property (v). Take $\eta := \left|x_{k_0} - x_0\right| - 2\rho_0 > 0$. Then for any $\bar{x}_0 \in \widetilde{\Lambda}$ we have $\bar{x}_0 = x_0(\mathbf{p}, \bar{\xi})$ for some $\mathbf{p} \in \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}$ and $\bar{\xi} \in \mathscr{E}$. Let $\bar{x}_0 \in U \subset \mathbb{R}^n$ be an open neighborhood. From the continuity of mapping $\xi \to x_0(\mathbf{p}, \xi)$ (see Theorem 3.5.3), there is $\widetilde{\xi} \in \mathscr{E}$ close to $\bar{\xi}$ so that $\widetilde{x}_0 = x_0(\mathbf{p}, \widetilde{\xi}) \in U \cap \widetilde{\Lambda}$ and there exist $i_0, i_1 \in \mathbb{Z}$, $i_0 < -\frac{k_0+\omega}{2\omega} < i_1$ so that $\bar{e}_{i_0} \neq \widetilde{e}_{i_0}$, $\bar{e}_{i_1} \neq \widetilde{e}_{i_1}$ for $\bar{\xi} = \{\bar{e}_i\}_{i\in\mathbb{Z}}$ and $\widetilde{\xi} = \{\widetilde{e}_i\}_{i\in\mathbb{Z}}$. Then for $j_0 = (2i_0 + 1)\omega + k_0 < 0$, (3.5.14) gives

$$\left|x_{j_0}(\mathbf{p},\bar{\xi}) - x_{j_0}(\mathbf{p},\widetilde{\xi})\right| \geq \left|x_{j_0}^{\bar{\xi}} - x_{j_0}^{\widetilde{\xi}}\right| - \left|x_{j_0}(\mathbf{p},\bar{\xi}) - x_{j_0}^{\bar{\xi}}\right| - \left|x_{j_0}(\mathbf{p},\widetilde{\xi}) - x_{j_0}^{\widetilde{\xi}}\right|$$
$$\geq |x_{k_0}^0 - x_0| - 2\rho_0 = \eta > 0\,.$$

The same estimates hold for $j_1 = (2i_1+1)\omega + k_0 > 0$. Property (v) is shown. Diagram in Figure 3.4 gives property (vi). The proof is completed. □

With property (v), we can construct many continuum orbits of (3.5.3) starting from U and oscillating back and forward on $\mathbb{Z}$ between x_0 and $x_{k_0}^0$ in any order. Of course, results of this section can be directly extended to more ε-inflated systems of the form $x_{k+1} = f(x_k + \varepsilon q_k) + g(x_k, p_k)$, $k \in \mathbb{Z}$ for any $\{p_k\}_{k\in\mathbb{Z}}$, $\{q_k\}_{k\in\mathbb{Z}} \in \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}$ and $\varepsilon > 0$ small fixed.

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