Chapter 2
Preliminary Results

In this chapter, we recall some known mathematical notations, notions and results which will be used later to help readers to understand this book better. For more details, we refer readers to quoted textbooks of nonlinear functional analysis, differential topology, singularities of smooth maps, complex analysis and dynamical systems.

2.1 Linear Functional Analysis

Let X be a *Banach space* with a norm $|\cdot|$. By $\mathbb{N}$ we denote the set of natural numbers. A sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ *converges* to $x_0 \in X$ if $|x_n - x_0| \to 0$ as $n \to \infty$, for short $x_n \to x_0$. We denote by $B_x(r)$ the *closed ball* in X centered at $x \in X$ and with the radius $r > 0$, i.e. $B_x(r) := \{z \in X \mid |z-x| \leq r\}$. Let S be a subset of X, i.e. $S \subset X$. Then S is *convex* if $\lambda s_1 + (1-\lambda)s_2 \in S$ for all $s_1, s_2 \in S$ and $\lambda \in [0,1]$. By $\operatorname{conv} S$ we denote the *convex hull* of S, i.e. the intersection of all convex subsets of X containing S. *Diameter* of S, $\operatorname{diam} S$, is defined as $\operatorname{diam} S := \{\sup |x-y| \mid x, y \in S\}$. S is *open* if any point of S has a closed ball belonging to S. S is *closed* if $X \setminus S$ is open. The *closure* and *interior* of S are denoted by $\bar{S}$ and $\operatorname{int} S$, respectively. Recall that $\bar{S}$ is the smallest closed subset of X containing S, and $\operatorname{int} S$ is the largest open subset of S. Clearly $\operatorname{int} B_x(r) = \{z \in X \mid |z-x| < r\}$ — an *open ball* in X.

Let X and Y be Banach spaces. The set of all *linear bounded/continuous mappings* $A : X \to Y$ is denoted by $L(X,Y)$, while we put $L(X) := L(X,X)$. The norm of A is defined by $\|A\| := \sup_{|x|=1} |Ax|$. More generally, if $Y, X_1, \ldots, X_n$ are Banach spaces, $L(X_1 \times \cdots \times X_n, Y)$ is the Banach space of *bounded/continuous multilinear maps* from $X_1 \times \cdots \times X_n$ into Y.

In using the Lyapunov-Schmidt method, we first need the following *Banach inverse mapping theorem*.

Theorem 2.1.1. *If $A \in L(X,Y)$ is surjective and injective then its inverse $A^{-1} \in L(Y,X)$.*

M. Fečkan, *Bifurcation and Chaos in Discontinuous and Continuous Systems*

We also recall the following well-known result.

Lemma 2.1.2. *Let $Z \subset X$ be a linear subspace with either* $\dim Z < \infty$ *or Z to be closed with* $\operatorname{codim} Z < \infty$. *Then there is a bounded projection $P : X \to Z$. Note that* $\operatorname{codim} Z = \dim X/Z$ *and X/Z is the factor space of X with respect to Z.*

Basic Banach spaces are functional ones like $C^m\left([0,1],M^k\right)$ and $L^p\left(\mathscr{I},M^k\right)$, where $\mathscr{I} \subset \mathbb{R}$ is an interval and $M \in \{\mathbb{R},\mathbb{C}\}$, with the usual norms:

$\|f\| = \max_{x\in[0,1],i=0,\cdots,m} |D^m f(x)|$ (cf Section 2.2.2) on $C^m\left([0,1],M^k\right)$,
$\|f\|_p = \sqrt[p]{\int_{\mathscr{I}} |f(x)|^p\, dx}$ on $L^p\left(\mathscr{I},M^k\right)$ for $1 \le p < \infty$,
$\|f\|_\infty = \operatorname{ess\,sup}_{x\in\mathscr{I}} |f(x)| = \min\{\lambda \ge 0 \mid |f(x)| \le \lambda$ for almost all $x \in \mathscr{I}\}$ on $L^\infty\left(\mathscr{I},M^k\right)$.

Here $\mathbb{C}$ denotes the set of complex numbers. Recall the *Hölder inequality* $\|fg\|_1 \le \|f\|_p\|g\|_q$ for any $f \in L^p\left(\mathscr{I},M^k\right)$, $g \in L^q\left(\mathscr{I},M^k\right)$ and $\frac{1}{p}+\frac{1}{q}=1$. For $p=q=2$, we get the *Cauchy-Schwarz-Bunyakovsky inequality*. Discrete analogies of these spaces are as follows: Let $I \in \{\mathbb{N},\mathbb{Z}\}$. Then we set $\ell^p\left(M^k\right) := \left\{x = \{x_m\}_{m\in I} \subset M^k \mid \sum_{m\in I} |x_m|^p < \infty\right\}$ with the norm $\|x\|_p = \sqrt[p]{\sum_{m\in I} |x_m|^p}$ for $\infty > p \ge 1$, and $\ell^\infty\left(M^k\right) := \left\{x = \{x_m\}_{m\in I} \subset M^k \mid \sup_{m\in I} |x_m| < \infty\right\}$ with the norm $\|x\|_\infty = \sup_{m\in I} |x_m|$. Note that $L^2\left(\mathscr{I},M^k\right)$ and $\ell^2\left(M^k\right)$ are *Hilbert spaces* with *scalar products* $(f,g) = \int_{\mathscr{I}} f(x)\overline{g(x)}\, dx$ and $(x,y) = \sum_{m\in\mathbb{Z}} x_m\overline{y_m}$, respectively.

Now we state the well-known Arzelà-Ascoli theorem:

Theorem 2.1.3. *Let $\{x_n(t)\}_{n\in\mathbb{N}} \subset C\left([0,1],\mathbb{R}^k\right)$ be a sequence of continuous mappings $x_n : [0,1] \to \mathbb{R}^k$ so that*

(i) *Sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is uniformly bounded, i.e. there is a constant $M > 0$ so that $|x_n(t)| \le M$ for any $t \in [0,1]$ and $n \in \mathbb{N}$.*
(ii) *Sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is equicontinuous, i.e. for any $\varepsilon > 0$ there is a $\delta > 0$ so that for any $n \in \mathbb{N}$ and $t,s \in [0,1]$, $|t-s| < \delta$ it holds $|x_n(t)-x_n(s)| \le \varepsilon$.*

Then there is a subsequence $\{x_{n_i}(t)\}_{i\in\mathbb{N}}$ of $\{x_n(t)\}_{n\in\mathbb{N}}$ therefore $x_{n_i}(t) \rightrightarrows x_0(t)$ uniformly to some $x_0 \in C([0,1],\mathbb{R}^k)$ as $i \to \infty$.

For any $f \in L^2([-\pi,\pi],\mathbb{C})$, we define *Fourier coefficients* of f by the formula:

$$\hat{f}(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, \mathrm{e}^{-\imath nx}\, dx$$

and $n \in \mathbb{Z}$. The *Parseval theorem* asserts that

$$2\pi \sum_{m\in\mathbb{Z}} \hat{f}(n)\overline{\hat{g}(n)} = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx$$

and this implies a Hilbert space isomorphism between $L^2([-\pi,\pi],\mathbb{C})$ and $\ell^2(\mathbb{C})$. Note $f=0$ if and only if $\hat{f}(n)=0$ for all $n \in \mathbb{Z}$. More sophisticated Hilbert spaces are *Sobolev spaces* $H^p(\mathbb{C})$, $(H^p(\mathbb{R}))$ $p \in \mathbb{N}$ which are all 2π-periodic complex (real) functions $q(t)$ so that $q^{(p)} \in L^2\left([-\pi,\pi],\mathbb{C}\right)$. Next for any $f \in L^1(\mathbb{R},\mathbb{C})$ we define its *Fourier transform* by the formula:

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-\imath xt}\, dx\,.$$

The *Plancherel theorem* states that the Fourier transform can be extended to $L^2(\mathbb{R},\mathbb{C})$ with $\|\hat{f}\|_2 = \|f\|_2$ and so $f \to \hat{f}$ is a Hilbert space isomorphism from $L^2(\mathbb{R},\mathbb{C})$ to $L^2(\mathbb{R},\mathbb{C})$.

More details and proofs of the above results can be found in [1–3].

2.2 Nonlinear Functional Analysis

2.2.1 Banach Fixed Point Theorem

Let X and Y be Banach spaces. Norms are denoted by $|\cdot|$. Let $U \subset Y$ be open. Consider a mapping $F : B_{x_0}(r) \times U \to X$ for some $x_0 \in X$ and $r > 0$ under the following assumptions

(a) There is an $\alpha \in (0,1)$ so $|F(x_1,y) - F(x_2,y)| \leq \alpha |x_1 - x_2|$ for all $x_1, x_2 \in B_{x_0}(r)$ and $y \in U$.
(b) There is a $0 < \delta < r(1-\alpha)$ so that $|F(x_0,y) - x_0| \leq \delta$ for all $y \in U$.

Set $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Now we can state the *Banach fixed point theorem* or *uniform contraction mapping principle* [1,4,5].

Theorem 2.2.1. *Suppose there exist conditions (a) and (b). Then F has a unique fixed point $\phi(y) \in \operatorname{int} B_{x_0}(r)$ for any $y \in U$, i.e. $\phi(y) = F(\phi(y),y)$ for all $y \in U$. Moreover it holds*

(i) *If there is a constant $\lambda > 0$ so that $|F(x,y_1) - F(x,y_2)| \leq \lambda |y_1 - y_2|$ for all $x \in B_{x_0}(r)$ and $y_1, y_2 \in U$. Then $|\phi(y_1) - \phi(y_2)| \leq \frac{L}{1-\alpha}|y_1 - y_2|$ for all $y_1, y_2 \in U$.*
(ii) *If $F \in C^k\left(B_{x_0}(r) \times U, X\right)$ for a $k \in \mathbb{Z}_+$ then $\phi \in C^k(U,X)$.*

2.2.2 Implicit Function Theorem

Let X and Y be *Banach spaces*. Norms are denoted by $|\cdot|$. Let $\Omega \subset X$ be open. A map $F : \Omega \to Y$ is said to be *(Fréchet) differentiable* at $x_0 \in \Omega$ if there is a $DF(x_0) \in L(X,Y)$ so

$$\lim_{h \to 0} \frac{|F(x_0+h) - F(x_0) - DF(x_0)h|}{|h|} = 0\,.$$

If F is differentiable at each $x \in \Omega$ and $DF : \Omega \to L(X,Y)$ is continuous then F is said to be continuously differentiable on Ω and we write $F \in C^1(\Omega,Y)$. Higher derivatives D^iF are defined in the usual way by induction. Similarly, the partial derivatives are defined standardly [1, p. 46]. Now we state the *implicit function theorem* [5, p. 26].

Theorem 2.2.2. *Let X, Y, Z be Banach spaces, $U \subset X$, $V \subset Y$ are open subsets and $(x_0, y_0) \in U \times V$. Consider $F \in C^1(U \times V, Z)$ so that $F(x_0, y_0) = 0$ and $D_x F(x_0, y_0) : X \to Z$ has a bounded inverse. Then there is a neighborhood $U_1 \times V_1 \subset U \times V$ of (x_0, y_0) and a function $f \in C^1(V_1, X)$ so that $f(y_0) = x_0$ and $F(x, y) = 0$ for $U_1 \times V_1$ if and only if $x = f(y)$. Moreover, if $F \in C^k(U \times V, Z)$, $k \geq 1$ then $f \in C^k(V_1, X)$.*

We refer the readers to [4, 6] for more applications and generalizations of the implicit function theorem.

2.2.3 Lyapunov-Schmidt Method

Now we recall the well-known *Lyapunov-Schmidt method* for solving locally nonlinear equations when the implicit function theorem fails. So let X, Y, Z be Banach spaces, $U \subset X$, $V \subset Y$ are open subsets and $(x_0, y_0) \in U \times V$. Consider $F \in C^1(U \times V, Z)$ so that $F(x_0, y_0) = 0$. If $D_x F(x_0, y_0) : X \to Z$ has a bounded inverse then the implicit function theorem can be applied to solving

$$F(x, y) = 0 \tag{2.2.1}$$

near (x_0, y_0). So we suppose that $D_x F(x_0, y_0) : X \to Z$ has no a bounded inverse. In general, this situation is difficult. The simplest case is that when $D_x F(x_0, y_0) : X \to Z$ is *Fredholm*, i.e. $\dim \mathscr{N} D_x F(x_0, y_0) < \infty$, $\mathscr{R} D_x F(x_0, y_0)$ is closed in Z and $\operatorname{codim} \mathscr{R} D_x F(x_0, y_0) < \infty$. Here $\mathscr{N} A$ and $\mathscr{R} A$ are the *kernel* and *range* of a linear mapping A. The *index* of $D_x F(x_0, y_0)$ is defined by $\operatorname{index} D_x F(x_0, y_0) := \dim \mathscr{N} D_x F(x_0, y_0) - \operatorname{codim} \mathscr{R} D_x F(x_0, y_0)$. Then by Lemma 2.1.2, there are bounded projections $P : X \to \mathscr{N} D_x F(x_0, y_0)$ and $Q : Z \to \mathscr{R} D_x F(x_0, y_0)$. Hence we split any $x \in X$ as $x = x_0 + u + v$ with $u \in \mathscr{R}(\mathbb{I} - P)$, $v \in \mathscr{R} P$, and decompose (2.2.1) as follows:

$$H(u, v, y) := QF(x_0 + u + v, y) = 0\,, \tag{2.2.2}$$

$$(\mathbb{I} - Q) F(x_0 + u + v, y) = 0\,. \tag{2.2.3}$$

Observe that $D_u H(0, 0, y_0) = D_x F(x_0, y_0) | \mathscr{R}(\mathbb{I} - P) \to \mathscr{R} D_x F(x_0, y_0)$. So $D_u H(0, 0, y_0)$ is injective and surjective. So by Banach inverse mapping theorem 2.1.1, $D_u H(0, 0, y_0)$ has a bounded inverse. Since $H(0, 0, y_0) = 0$, the implicit function theorem can be applied to solving (2.2.2) in $u = u(v, y)$ with $u(0, y_0) = 0$. Inserting this solution into (2.2.3) we get the *bifurcation equation*:

$$B(v, y) := (\mathbb{I} - Q) F(x_0 + u(v, y) + v, y) = 0\,.$$

Since $B(0, y_0) = (\mathbb{I} - Q) F(x_0, y_0) = 0$ and

$$D_v B(0, y_0) = (\mathbb{I} - Q) D_x F(x_0, y_0) \left(D_v u(0, y_0) + \mathbb{I} \right) = 0\,,$$

the function $B(v,y)$ has a higher singularity at $(0,y_0)$, so the implicit function theorem is not applicable, and the bifurcation theory must be used [5].

2.2.4 Brouwer Degree

Let $\Omega \subset \mathbb{R}^n$ be open bounded subset. A triple (F,Ω,y) is *admissible* if $F \in C(\bar{\Omega},\mathbb{R}^n)$ and $y \in \mathbb{R}^n$ with $y \notin F(\partial\Omega)$, where $\partial\Omega$ is the border of Ω. Now on these admissible triples (F,Ω,y), there is a $\mathbb{Z}$-defined function deg [1, p. 56].

Theorem 2.2.3. *There is a unique mapping* deg *defined on the set of all admissible triples* (F,Ω,y) *determined by the following properties*:

(i) *If* $\deg(F,\Omega,y) \neq 0$ *then there is an* $x \in \Omega$ *consequently* $F(x) = y$.
(ii) $\deg(\mathbb{I},\Omega,y) = 1$ *for any* $y \in \Omega$.
(iii) $\deg(F,\Omega,y) = \deg(F,\Omega_1,y) + \deg(F,\Omega_2,y)$ *whenever* $\Omega_{1,2}$ *are disjoint open subsets of* Ω *so that* $y \notin F\left(\bar{\Omega} \setminus (\Omega_1 \cup \Omega_2)\right)$.
(iv) $\deg(F(\lambda,\cdot),\Omega,y)$ *is constant for* $F \in C\left([0,1] \times \bar{\Omega},X\right)$ *and* $y \notin F([0,1] \times \partial\Omega)$.

The number $\deg(F,\Omega,y)$ is called the *Brouwer degree* of the map F. If x_0 is an isolated zero of F in $\Omega \subset \mathbb{R}^n$ then $I(x_0) := \deg(F,\Omega_0,0)$ is called the *Brouwer index* of F at x_0, where $x_0 \in \Omega_0 \subset \Omega$ is an open subset so x_0 is the only zero point of F on Ω_0 [5, p. 69]. $I(x_0)$ is independent of such Ω_0. Note that if $y \in \mathbb{R}^n$ is a regular value of F, i.e. $\det DF(x) \neq 0$ for any $x \in \Omega$ with $F(x) = y$, and $y \notin F(\partial\Omega)$, then $F^{-1}(y)$ is finite and $\deg(F,\Omega,y) = \sum_{x \in F^{-1}(y)} \operatorname{sgn}\det DF(x)$. Particularly if x_0 is as *simple zero* of $F(x)$, i.e. $F(x_0) = 0$ and $\det DF(x_0) \neq 0$, then $I(x_0) = \operatorname{sgn}\det DF(x_0) = \pm 1$.

2.2.5 Local Invertibility

It is well known that the linear invertibility implies local nonlinear invertibility. More precisely, let us consider a map $F : X \to Y$, $F(0) = 0$, where F is C^1–smooth and X, Y are Banach spaces. If $DF(0)$ is invertible, then any C^1–small perturbation of F has a unique zero point near 0. This follows from the implicit function theorem 2.2.2. Now we shall study a reverse problem [7].

Theorem 2.2.4. *Consider a* C^2*–smooth map* $F : X \to Y$ *satisfying* $F(0) = 0$ *and assume that* $DF(0)$ *is Fredholm with index* 0.

If there exist a neighbourhood $U \subset X$ *of* 0 *and numbers* $K > 0$, $\delta > 0$ *so that for any linear bounded mapping* $B : X \to Y$, $||B|| \leq K$ *the perturbation* $\varepsilon B + F$, $0 \leq \varepsilon \leq \delta$ *has the only zero point* 0 *in* U, *then* $DF(0)$ *is invertible.*

Note that if there is a number K satisfying the assumption of the above theorem, then this assumption holds with any $K > 0$ and the same neighbourhood U. Of course, we must take another $\delta > 0$. If we are interested in the invertibility of $DF(x_0)$

for a general fixed x_0 satisfying $F(x_0) = 0$, then Theorem 2.2.4 is applied with perturbations of the form $\varepsilon\big(B - Bx_0\big) + F$, where B has the properties of Theorem 2.2.4. Indeed, we apply Theorem 2.2.4 to the map $x \to F\big(x + x_0\big)$. The perturbation term $\varepsilon\big(B - Bx_0\big)$ is affinely small.

2.2.6 Global Invertibility

Let X, Y be Banach spaces and $f \in C(X,Y)$. Then f is *proper* if the inverse image $f^{-1}(C)$ of any compact subset $C \subset Y$ is compact [4, p. 102].

Theorem 2.2.5. *If X and Y are finitely dimensional, then f is proper iff f is coercive, i.e. $|f(x)| \to \infty$ whenever $|x| \to \infty$.*

Now we state the following Banach-Mazur theorem of global invertibility of mappings.

Theorem 2.2.6. (i) *f is a homeomorphism of X onto Y if and only if f is a local homeomorphism and proper.*

(ii) *If $f \in C^1(X,Y)$ then f is a diffeomorphism if and only if f is proper and $Df(x)$ is a linear homeomorphism for each $x \in X$.*

2.3 Multivalued Mappings

Let X, Y be Banach spaces and let $\Omega \subset X$. By 2^Y we denote the family of all subsets of Y. Any mapping $F : \Omega \to 2^Y \setminus \{\emptyset\}$ is called *multivalued or set-valued* mappings. A multivalued mapping $F : \Omega \to 2^Y \setminus \{\emptyset\}$ is *convex (compact)-valued* if $F(x)$ is convex (compact) for any $x \in \Omega$.

By $B(X)$ we denote the family of all nonempty closed bounded subsets of X. Let $A, B \in B(X)$, then their *Hausdorff distance $d_H(A,B)$* is defined as follows

$$d_H(A,B) := \max\left\{\sup_{a\in A}\left[\inf_{x\in B}|x-a|\right], \sup_{b\in B}\left[\inf_{x\in A}|x-b|\right]\right\}.$$

It is well known that d_H is a metric on $B(X)$ and $B(X)$ is a complete metric space with respect to d_H [8,9]. A multivalued mapping $F : X \to B(Y)$ is Lipschitz continuous with a constant $\Lambda > 0$, if

$$d_H(F(x_1), F(x_2)) \leq \Lambda |x_1 - x_2|$$

for any $x_1, x_2 \in X$. Now we state the *Lojasiewicz-Ornelas parametrization theorem* [10]:

Theorem 2.3.1. *If $G : \mathbb{R}^n \to \mathbb{R}^n$ is a compact convex-valued map which is Lipschitz, then there exists a Lipschitz map $g : \mathbb{R}^n \times \mathscr{B}_{\mathbb{R}^n} \to \mathbb{R}^n$ so that $G(x) = g(x, \mathscr{B}_{\mathbb{R}^n})$ for all $x \in \mathbb{R}^n$, where $\mathscr{B}_{\mathbb{R}^n}$ is a closed unit ball in $\mathbb{R}^n$. Moreover, the Lipschitz constant of $g(= g(x,p))$ with respect to the variable x is proportional to the Lipschitz constant of G, while the Lipschitz constant of g with respect to the second variable p is proportional to the maximal norm of the elements of G.*

2.4 Differential Topology

2.4.1 Differentiable Manifolds

Let M be a subset of $\mathbb{R}^k$. We use the *induced topology* on M, that is, $A \subset M$ is open if there is an open set $\widetilde{A} \subset \mathbb{R}^k$ so that $A = \widetilde{A} \cap M$. We say that $M \subset \mathbb{R}^k$ is a *C^r-manifold ($r \in \mathbb{N}$) of dimension m* if for each $p \in M$ there is a neighborhood $U \subset M$ of p and a homeomorphism $x : U \to U_0$, where U_0 is an open subset in $\mathbb{R}^m$, so that the inverse $x^{-1} \in C^r(U_0, \mathbb{R}^k)$ and $Dx^{-1}(u) : \mathbb{R}^m \to \mathbb{R}^k$ is injective for any $u \in U_0$. Then we say that (x,U) is a *local C^r-chart around p* and U is a *coordinate neighborhood* of p. It is clear that if $x : U \to \mathbb{R}^m$ and $y : V \to \mathbb{R}^m$ are two local C^r-charts in M with $U \cap V \neq \emptyset$ then $y \circ x^{-1} : x(U \cap V) \to y(U \cap V)$ is a C^rdiffeomorphism. This family of local charts is called a *C^r-atlas* for M [11–13].

If there is a C^r-atlas for M so that $\det D(y \circ x^{-1})(z) > 0$ for any $z \in x(U \cap V)$ and any two local C^r-charts $x : U \to \mathbb{R}^m$ and $y : V \to \mathbb{R}^m$ of this atlas with $U \cap V \neq \emptyset$ then M is *oriented*.

Let $\alpha \in C^1((-\varepsilon, \varepsilon), \mathbb{R}^k)$ be a differentiable curve on M, i.e. $\alpha : (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = p$. Then $\alpha'(0)$ is a *tangent vector* to M at p. The set of all tangent vectors to M at p is the *tangent space to M at p* and it is denoted by T_pM. The *tangent bundle* is

$$TM := \left\{ (p,v) \in \mathbb{R}^k \times \mathbb{R}^k \mid p \in M, \, v \in T_pM \right\}$$

with the *natural projection* $\pi : TM \to M$ given as $\pi(p,v) = p$. If M is a C^r-manifold with $r > 1$ then TM is a C^{r-1}-manifold.

Let M and N be two C^r-manifolds. We say that $f : M \to N$ is a *C^r-mapping* if for each $p \in M$ the mapping $y \circ f \circ x^{-1} : x(U) \to y(V)$ is C^r-smooth, where $x : U \to \mathbb{R}^m$ is a local C^r-chart in M around p and $y : V \to \mathbb{R}^s$ is a local C^r-chart in N with $f(U) \subset V$. This definition is independent of the choice of charts. The set of C^r-mappings is denoted by $C^r(M,N)$. Take $f \in C^r(M,N)$. Let $\alpha : (-\varepsilon, \varepsilon) \to M$ be a differentiable curve on M with $\alpha(0) = p$ and $\alpha'(0) = v$. Then $f \circ \alpha : (-\varepsilon, \varepsilon) \to N$ is a differentiable curve on N with $(f \circ \alpha)(0) = f(p)$, so we can define $Df(p)v := D(f \circ \alpha)(0) \in T_{f(p)}N$. This is independent of curve α. The map $Df(p) : T_pM \to T_{f(p)}N$ is linear, and if $r > 1$, $Df : TM \to TN$ defined as $Df(p,v) := (f(p), Df(p)v)$ is C^{r-1}-smooth.

A set $S \subset M \subset \mathbb{R}^k$ is a *C^r-submanifold* of M of dimension s if for each $p \in S$ there are open sets $U \subset M$ containing p, $V \subset \mathbb{R}^s$ containing 0 and $W \subset \mathbb{R}^{m-s}$ containing

0 and a C^r-diffeomorphism $\phi : U \to V \times W$ so that $\phi(S \cap U) = V \times \{0\}$. We put $\operatorname{codim} S = \dim M - \dim S$.

A C^r-mapping $f : M \to N$ is an *immersion (submersion)* if $Df(p)$ is injective (surjective) for all $p \in M$. If $f : M \to N$ is an injective immersion we say that $f(M)$ is an *immersed submanifold*. If, in addition, $f : M \to f(M) \subset N$ is a homeomorphism, where $f(M)$ has the induced topology, then f is an *embedding*. In this case, $f(M)$ is a submanifold of N.

2.4.2 Vector Bundles

A *C^r-vector bundle of dimension n* is a triple (E, p, B) where E, B are C^r-manifolds and $p \in C^r(E, B)$ with the following properties: for each $q \in B$ there is its open neighborhood $U \subset B$ and a C^r-diffeomorphism $\phi : p^{-1}(U) \to U \times \mathbb{R}^n$ so that $p = \pi_1 \circ \phi$ on $p^{-1}(U)$ where $\pi_1 : U \times \mathbb{R}^n \to U$ is defined as $\pi_1(x, y) := x$. Moreover, each $p^{-1}(x)$ is n-dimensional vector spaces and each $\phi_x : p^{-1}(x) \to \mathbb{R}^n$ given by $\phi(y) = (x, \phi_x(y))$ for any $y \in p^{-1}(x)$ is linear isomorphisms. E is called the *total space*, B is the *base space*, p the *projection* of the bundle, the vector space $p^{-1}(x)$ the *fibre* and ϕ a *local trivialization*. So the vector bundle is *locally trivial*. If $U = B$ then the bundle is *trivial*. The family $\mathscr{A} := \{(\phi, U)\}$ of these local trivializations is a *C^r-vector atlas*. The bundle is *oriented* if there is a C^r-vector atlas $\mathscr{A} := \{(\phi, U)\}$ so that for any two local trivializations (ϕ, U) and (ψ, V) with $U \cap V \neq \emptyset$ the linear mapping $\psi_x \circ \phi_x^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is orientation preserving for each $x \in U \cap V$. A C^r-smooth mapping $s : B \to E$ satisfying $p \circ s = \mathbb{I}_B$ is called a *section* of the bundle.

Typical examples of vector bundles are the tangent bundle (TM, π, M) and the *normal bundle* $(TM^\perp, \widetilde{\pi}, M)$ defined as

$$TM^\perp := \left\{ (q, v) \in \mathbb{R}^k \times \mathbb{R}^k \mid q \in M,\, v \in T_q M^\perp \right\}$$

with the projection $\widetilde{\pi} : TM^\perp \to M$ given as $\widetilde{\pi}(q, v) = q$, where $T_x M^\perp$ is the orthogonal complement of $T_x M$ in $\mathbb{R}^k$. A section of TM is called a *vector field* on M. When M is oriented, both TM and $TM^\perp$ are oriented. Here M is a C^r-manifold with $r > 1$.

2.4.3 Tubular Neighbourhoods

Let M be a submanifold of a smooth manifold N. A *tubular neigbourhood* of M in N is an open subset $\mathscr{O}$ of N together with a submersion $p : \mathscr{O} \to M$ so that [14, pp. 69-71]:

(a) the triple $(\mathscr{O}, p, M)$ is a vector bundle, and
(b) $M \subset \mathscr{O}$ is the zero section of this vector bundle.

Theorem 2.4.1. *Let M be a submanifold of N, then there exists a tubular neighbourhood of M in N.*

If $N = \mathbb{R}^n$ then we can realize a tubular neighbourhood of a submanifold M by using its normal vector bundle $TM^{\perp}$.

2.5 Dynamical Systems

2.5.1 Homogenous Linear Equations

Set $\mathbb{Z}_- := -\mathbb{Z}_+$. Let $J \in \{\mathbb{Z}_+, \mathbb{Z}_-, \mathbb{Z}\}$. Let $A_n \in L(\mathbb{R}^k)$, $n \in J$ be a sequence of invertible matrices. Consider a homogeneous linear difference equation

$$x_{n+1} = A_n x_n\,. \tag{2.5.1}$$

Its *fundamental solution* is defined as $U(n) := A_{n-1} \cdots A_0$ for $n \in \mathbb{N}$, $U(0) = \mathbb{I}$ and $U(n) := A_n^{-1} \cdots A_{-1}^{-1}$ for $-n \in \mathbb{N}$. (2.5.1) has an *exponential dichotomy* on J if there is a projection $P : \mathbb{R}^k \to \mathbb{R}^k$ and constants $L > 0$, $\delta \in (0,1)$ so that

$$\|U(n)PU(m)^{-1}\| \le L\delta^{n-m} \text{ for any } m \le n,\ n,m \in J\,,$$
$$\|U(n)(\mathbb{I}-P)U(m)^{-1}\| \le L\delta^{m-n} \text{ for any } n \le m,\ n,m \in J\,.$$

If $A_n = A$ and its spectrum $\sigma(A)$ has no intersection with the unit circle, i.e. A is *hyperbolic*, then P is the projection onto the generalized eigenspace of eigenvectors inside the unit circle and $\mathscr{N}P$ is the generalized eigenspace of eigenvectors outside the unit circle. Next we have the following *roughness of exponential dichotomies*.

Lemma 2.5.1. *Let $J \in \{\mathbb{Z}_+, \mathbb{Z}_-\}$. Let A be hyperbolic with the dichotomy projection P. Assume that $\{A_n(\xi)\}_{n\in J} \in L(\mathbb{R}^k)$ are invertible matrices and $A_n(\xi) \to A$ in $L(\mathbb{R}^k)$ uniformly with respect to a parameter ξ. Then $x_{n+1} = A_n(\xi)x_n$, with the fundamental solution $U_\xi(n)$, has an exponential dichotomy on J with projection P_ξ and uniform constants $L > 0$, $\delta \in (0,1)$. Moreover, $U_\xi(n)P_\xi U_\xi(n)^{-1} \to P$ as $n \to \pm\infty$ uniformly with respect to ξ.*

Analogical results hold for a homogeneous linear differential equation $\dot{x} = A(t)x$ when $t \in J \in \{(-\infty,0), (0,\infty), \mathbb{R}\}$ and $A(t) \in C(J, L(\mathbb{R}^k))$ is a continuous matrix function. Its *fundamental solution* is a matrix function $U(t)$ satisfying $\dot{U}(t) = A(t)U(t)$ on J. Sometimes we require that $U(0) = \mathbb{I}$ [15]. Now, we recall the *Liouville theorem* that

$$\det U(t) = \det U(t_0)\, \mathrm{e}^{\int_{t_0}^t \operatorname{tr} A(s)\, ds}\,,$$

where $\operatorname{tr} A(t)$ denotes the *trace* which is the sum of diagonal entries of $A(t)$. Finally we mention the *Gronwall inequality* that if

$$\phi(t) \le \alpha(t) + \int_a^t \psi(s)\phi(s)\, ds$$

for all $t \in [a,b]$ then

$$\phi(t) \leq \alpha(t)\,\mathrm{e}^{\int_a^t \psi(s)\,ds}$$

for all $t \in [a,b]$, where $a < b$, α, ϕ and ψ are nonnegative continuous functions on $[a,b]$, and moreover, α is C^1-smooth satisfying $\alpha'(t) \geq 0$ for any $t \in [a,b]$.

2.5.2 Chaos in Diffeomorphisms

Consider a *C^r-diffeomorphism* f on $\mathbb{R}^m$ with $r \in \mathbb{N}$, i.e. a mapping $f \in C^r(\mathbb{R}^m,\mathbb{R}^m)$ which is invertible and $f^{-1} \in C^r(\mathbb{R}^m,\mathbb{R}^m)$. For any $z \in \mathbb{R}^m$ we define its *k-iteration* as $f^k(z) := f(f^{k-1}(z))$. The set $\{f^n(z)\}_{n=\infty}^{\infty}$ is an *orbit* of f. If $x_0 = f(x_0)$ then x_0 is a *fixed point* of f. It is *hyperbolic* if the *linearization* $Df(x_0)$ of f at x_0 has no eigenvalues on the unit circle. The *global stable (unstable) manifold* $W_{x_0}^{s(u)}$ of a hyperbolic fixed point x_0 is defined by [16]

$$W_{x_0}^{s(u)} := \{z \in \mathbb{R}^m \mid f^n(z) \to x_0 \quad \text{as} \quad n \to \infty(-\infty)\}\ ,$$

respectively. Recall that $W_{x_0}^s$ and $W_{x_0}^u$ are immersed C^r-submanifolds in $\mathbb{R}^m$. Furthermore, let y_0 be another hyperbolic fixed point of f. If $x \in W_{x_0}^s \cap W_{y_0}^u \setminus \{x_0,y_0\}$ then it is a *heteroclinic point* of f and then the orbit $\{f^n(x)\}_{n=\infty}^{\infty}$ is called heteroclinic orbit. Clearly $f^n(z) \to x_0$ as $n \to \infty$ and $f^n(z) \to y_0$ as $n \to -\infty$. If $T_xW_{x_0}^s \cap T_xW_{y_0}^u = \{0\}$ then x is a *transversal heteroclinic point* of f. Note the following useful results [15, 17].

Lemma 2.5.2. $x \in W_{x_0}^s \cap W_{y_0}^u \setminus \{x_0,y_0\}$ *is a transversal heteroclinic point if and only if the linear difference equation* $x_{n+1} = Df(f^n(x))x_n$ *has an exponential dichotomy on* $\mathbb{Z}$*, i.e. if and only if the only bounded solution of* $x_{n+1} = Df(f^n(x))x_n$ *on* $\mathbb{Z}$ *is the zero one.*

When $x_0 = y_0$, the word "heteroclinic" is replaced with *homoclinic*. We refer the readers to [15] for more details and proofs of the above subject.

Let $\mathscr{E} = \{0,1\}^{\mathbb{Z}}$ be a compact metric space of the set of doubly infinite sequences of 0 and 1 endowed with the metric [18]

$$d_{\mathscr{E}}(\{e_n\},\{e'_n\}) := \sum_{n\in\mathbb{Z}} \frac{|e_n - e'_n|}{2^{|n|}}\,.$$

On $\mathscr{E}$ it is defined as the so-called *Bernoulli shift map* $\sigma : \mathscr{E} \to \mathscr{E}$ by $\sigma(\{e_j\}_{j\in\mathbb{Z}}) = \{e_{j+1}\}_{j\in\mathbb{Z}}$ with extremely rich dynamics [19].

Theorem 2.5.3. σ *is a homeomorphism having*

(i) *a countable infinity of periodic orbits of all possible periods,*
(ii) *an uncountable infinity of nonperiodic orbits, and*
(iii) *a dense orbit.*

Now we can state the following result of the existence of *the deterministic chaos* for diffeomorphisms, the *Smale-Birkhoff homoclinic theorem*.

Theorem 2.5.4. *Suppose* $f : \mathbb{R}^m \to \mathbb{R}^m$, $r \in \mathbb{N}$ *are a* C^r*-diffeomorphism having a transversal homoclinic point to a hyperbolic fixed point. Then there is a* $k \in \mathbb{N}$ *so that* f^k *has an invariant set* Λ*, i.e.* $f^k(\Lambda) = \Lambda$*, so* $f^k \circ \varphi = \varphi \circ \sigma$ *for a homeomorphism* $\varphi : \mathscr{E} \to \Lambda$ (*Figure* 2.1).

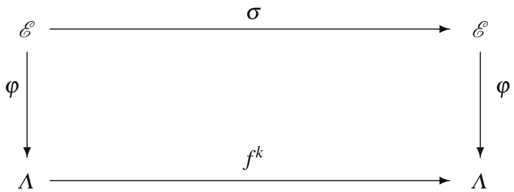

Fig. 2.1 Commutative diagram of deterministic chaos.

The set Λ is the *Smale horseshoe* and we say that f has *horseshoe dynamics* on Λ. By Theorem 2.5.4, f^k on Λ has the same dynamical properties as σ on $\mathscr{E}$, i.e. Theorem 2.5.3 gives chaos for f. Moreover, it is possible to show a *sensitive dependence on initial conditions* of f on Λ in the sense that there is an $\varepsilon_0 > 0$ so that for any $x \in \Lambda$ and any neighborhood U of x, there exists $z \in U \cap \Lambda$ and an integer $q \geq 1$, consequently $|f^q(x) - f^q(z)| > \varepsilon_0$.

2.5.3 Periodic ODEs

It is well known [20] that the Cauchy problem

$$\dot{x} = g(x,t), \quad x(0) = z \in \mathbb{R}^m \tag{2.5.2}$$

for $g \in C^r(\mathbb{R}^m \times \mathbb{R}, \mathbb{R}^m)$, $r \in \mathbb{N}$ has a unique solution $x(t) = \phi(z,t)$ defined in a maximal interval $0 \in I_z \subset \mathbb{R}$. We suppose for simplicity that $I_z = \mathbb{R}$. This is true, for instance, when g is globally Lipschitz continuous in x, i.e. there is a constant $L > 0$ so that $|g(x,t) - g(y,t)| \leq L|x - y|$ for any $x, y \in \mathbb{R}^m$, $t \in \mathbb{R}$. Moreover, we assume that g is T-periodic in t, i.e. $g(x, t+T) = g(x,t)$ for any $x \in \mathbb{R}^m$, $t \in \mathbb{R}$. Then the dynamics of (2.5.2) is determined by the dynamics of the diffeomorphism $f(z) = \phi(z,T)$ which is called the *time or Poincarè map* of (2.5.2). Now we can transform the results of Section 2.5.2 to (2.5.2). So T-periodic solutions (*periodics* for short) of (2.5.2) are fixed points of f. A T-periodic solution of (2.5.2) is hyperbolic if the corresponding fixed point of f is hyperbolic. Periodics of f are *subharmonic solutions* (*subharmonics* for short) of (2.5.2). Similarly we mean a chaos of (2.5.2) as a chaos for f. Finally, let $\gamma_0(t) = \phi(x_0, t)$ be a T-periodic solution of

$$\dot{x} = g(x,t)\,. \tag{2.5.3}$$

Consider its *variational equation* along γ_0 given by $\dot{v} = g_x(\gamma_0(t),t)\,v$ with the fundamental matrix solution $V(t)$. Then $Df(x_0) = V(T)$ [21]. Now we have the following result from the proof of Theorem 2.1 on p. 288 of [22].

Lemma 2.5.5. *Let X be a Banach space. Let $C_b(\mathbb{R},X)$ be the space of all continuous and bounded functions from $\mathbb{R}$ to X endowed with the supremum norm. Consider*

$$\dot{u} = A(t)u \tag{2.5.4}$$

with the fundamental solution $U(t)$, where $A(t) \in C(\mathbb{R},L(X))$ is T-periodic. Then the following statements are equivalent

(i) *The nonhomogeneous equation*

$$\dot{u} = A(t)u + h$$

has a unique solution $u \in C_b(\mathbb{R},X)$ for any $h \in C_b(\mathbb{R},X)$.

(ii) *The zero solution of* (2.5.4) *is hyperbolic, i.e. $\sigma(U(T))$ has no eigenvalues on the unit circle.*

(iii) *Equation* (2.5.4) *has an exponential dichotomy on $\mathbb{R}$.*

Lemma 2.5.5 is useful for verifying the hyperbolicity of γ_0 of (2.5.3).

2.5.4 Vector Fields

When (2.5.2) is *autonomous*, i.e. g is independent of t, (2.5.2) has the form

$$\dot{x} = g(x), \quad x(0) = z \in \mathbb{R}^m. \tag{2.5.5}$$

g is called a *C^r-vector field* on $\mathbb{R}^m$ for $g \in C^r(\mathbb{R}^m,\mathbb{R}^m)$, $r \in \mathbb{N}$. We suppose for simplicity that the unique solution $x(t) = \phi(z,t)$ of (2.5.5) is defined on $\mathbb{R}$. $\phi(z,t)$ is called the *orbit based at z*. Then instead of the time map of (2.5.5), we consider the *flow* $\phi_t : \mathbb{R}^m \to \mathbb{R}^m$ defined as $\phi_t(z) := \phi(z,t)$ with the property $\phi_t(\phi_s(z)) = \phi_{t+s}(z)$.

A point p is an *ω-limit point* of x is there are points $\{\phi_{t_i}(x)\}_{i\in\mathbb{N}}$ on the orbit of x so that $\phi_{t_i}(x) \to p$ and $t_i \to \infty$. A point q is an *α-limit point* if such a sequence exists with $\phi_{t_i}(x) \to q$ and $t_i \to -\infty$. The α- (resp. ω-) limit sets $\alpha(x)$, $\omega(x)$ are the sets of α- and ω-limit points of x.

A point x_0 with $g(x_0) = 0$ is an *equilibrium* of (2.5.5). It is *hyperbolic* if the linearization $Dg(x_0)$ of (2.5.5) at x_0 has no eigenvalues on imaginary axis.

The *global stable (unstable) manifold* $W^{s(u)}_{x_0}$ of a hyperbolic equilibrium x_0 is defined by

$$W^{s(u)}_{x_0} := \{z \in \mathbb{R}^m \mid \phi(z,t) \to x_0 \quad \text{as} \quad t \to \infty(-\infty)\},$$

respectively. These sets are immersed submanifolds of $\mathbb{R}^m$. For any $x \in W^{s(u)}_{x_0}$, we know that

$$T_x W_{x_0}^{s(u)} = \Big\{ v(0) \in \mathbb{R}^m \mid v(t) \text{ is a bounded solution}$$
$$\text{of } \dot{v} = Dg(\phi(x,t))v \text{ on } (0,\infty), ((-\infty,0)), \text{respectively} \Big\}.$$

Moreover, the set
$$\left(T_x W_{x_0}^s + T_x W_{x_0}^u\right)^{\perp}$$
is the linear space of initial values $w(0)$ of all bounded solutions $w(t)$ of the *adjoint equation* $\dot{w} = -Dg(\phi(x,t))^* w$ on $\mathbb{R}$ [23].

A local dynamics near a hyperbolic equilibrium x_0 of (2.5.5) is explained by the *Hartman-Grobman theorem for flows* [24].

Theorem 2.5.6. *If* $x_0 = 0$ *is a hyperbolic equilibrium of* (2.5.5) *then there is a homeomorphism* h *defined on a neighborhood* U *of* 0 *in* $\mathbb{R}^m$ *so that*

$$h(\phi(z,t)) = e^{tDg(0)} h(z)$$

for all $z \in U$ *and* $t \in J_z$ *with* $\phi(z,t) \in U$*, where* $0 \in J_z$ *is an interval.*

For nonhyperbolic equilibria we have the following *center manifold theorem for flows* [24].

Theorem 2.5.7. *Let* $x_0 = 0$ *be an equilibrium of a* C^r*-vector field* (2.5.5) *on* $\mathbb{R}^m$*. Divide the spectrum of* $Dg(0)$ *into three parts* σ_s, σ_u, σ_c *so that* $\Re\lambda < 0; > 0; = 0$ *if* $\lambda \in \sigma_s, \sigma_u, \sigma_c$*, respectively. Let the generalized eigenspaces of* σ_s, σ_u, σ_c *be* E^s, E^u, E^c*, respectively. Then there are* C^r*-smooth manifolds: the stable* W_0^s*, the unstable* W_0^u*, the center* W_0^c *tangent at* 0 *to* E^s, E^u, E^c*, respectively. These manifolds are invariants for the flow of* (2.5.5)*, i.e.* $\phi_t\left(W_0^{s;u;c}\right) \subset W_0^{s;u;c}$ *for any* $t \in \mathbb{R}$*. The stable and unstable ones are unique, but the center one need not be. In addition, when* g *is embedded into a* C^r*-smooth family of vector fields* g_ε *with* $g_0 = g$*, these invariant manifolds are* C^r*-smooth also with respect to* ε*.*

Under the assumptions of Theorem 2.5.7 near $x_0 = 0$ we can write (2.5.5) in the form

$$\begin{aligned} \dot{x}_s &= A_s x_s + g_s(x_s, x_u, x_c, \varepsilon), \\ \dot{x}_u &= A_u x_u + g_u(x_s, x_u, x_c, \varepsilon), \\ \dot{x}_c &= A_c x_s + g_c(x_s, x_u, x_c, \varepsilon), \end{aligned} \tag{2.5.6}$$

where $A_{s;u;c} := Dg(0)/E^{s;u;c}$ and $x_{s;u;c} \in U_{s;u;c}$ for open neighborhoods $U_{s;u;c}$ of 0 in $E^{s;u;c}$, respectively. Here we suppose that (2.5.5) is embedded into a C^r-smooth family. So g_j are C^r-smooth satisfying $g_j(0,0,0,0) = 0$ and $D_{x_j} g_k(0,0,0,0) = 0$ for $j,k = s,u,c$. According to Theorem 2.5.7, the *local center manifold* $W_{loc,\varepsilon}^c$ near $(0,0,0)$ of (2.5.6) is a graph

$$W_{loc,\varepsilon}^c = \{(h_s(x_c,\varepsilon), h_u(x_c,\varepsilon), x_c) \mid x_c \in U_c\}$$

for $h_{s;u} \in C^r\left(U_c \times V, E^{s;u}\right)$ and V is an open neighborhood of $\varepsilon = 0$. Moreover, it holds $h_{s;u}(0,0) = 0$ and $D_{x_c} h_{s;u}(0,0) = 0$. The *reduced equation* is

$$\dot{x}_c = A_c x_s + g_c(h_s(x_c,\varepsilon), h_u(x_c,\varepsilon), x_c, \varepsilon), \tag{2.5.7}$$

which locally determines the dynamics of (2.5.6), i.e. $W^c_{loc,\varepsilon}$ contains all solutions of (2.5.6) staying in $U_s \times U_u \times U_c$ for all $t \in \mathbb{R}$. In particular periodics, homoclinics and heteroclinics of (2.5.6) near $(0,0,0)$ solve (2.5.7).

Finally we say that (2.5.5) has a *first integral* $H : \mathbb{R}^n \to \mathbb{R}$ if $H \circ \phi_t = H$ for any $t \in \mathbb{R}$.

2.5.5 Global Center Manifolds

Let $C^k_b(\mathbb{R}^m, \mathbb{R}^n)$ be the Banach space of C^k functions from $\mathbb{R}^m$ to $\mathbb{R}^n$ which are bounded together with their derivatives, endowed with the usual sup-norm. We consider the following system of ODEs:

$$\begin{aligned} \dot{x} &= A(y)x + F(x,y), \\ \dot{y} &= G(x,y), \end{aligned} \tag{2.5.8}$$

where $x \in \mathbb{R}^n, y \in \mathbb{R}^m$ and assume that the following conditions hold:

(i) $F \in C^r_b(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$, $G \in C^r_b(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^m)$, $A \in C^r_b(\mathbb{R}^m, L(\mathbb{R}^n))$ with $r \geq 1$.
(ii) There exists $\delta > 0$ so that for any $y \in \mathbb{R}^m$ and for any $\lambda(y) \in \sigma(A(y))$, one has $|\Re\lambda(y)| > \delta$. Moreover, the derivatives of order r of $A(y)$, $F(x,y)$, $G(x,y)$ are continuous in x, uniformly with respect to $y \in \mathbb{R}^m$.
(iii) $\sup_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m} \left\{|F(0,y)|, |F_x(0,y)|, |G(x,y)|, |G_x(x,y)|, |G_y(x,y)|\right\} \leq \sigma$.

Now we can state the following result.

Theorem 2.5.8. *There exists a* $\sigma_0 > 0$ *so that, if the above conditions hold with* $\sigma \leq \sigma_0$*, there exists a* C^r*-function* $H(y)$*, defined for* $y \in \mathbb{R}^m$ *so that the manifold*

$$\mathscr{C} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid x = H(y), y \in \mathbb{R}^m\}$$

is invariant for the system (2.5.8) *and has the following property:*

(P) There exists $\rho > 0$ *so that if* $(x(t), y(t))$ *is a solution of* (2.5.8) *satisfying* $||x||_\infty \leq \rho$*, then* $x(t) = H(y(t))$.

$\mathscr{C}$ is called the *global center manifold* of (2.5.8). We refer the readers to [25] for more details.

2.5.6 Two-Dimensional Flows

In this section we consider a planar ODE

$$\dot{x} = f(x), \tag{2.5.9}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and $f = (f_1, f_2)$ is smooth. First we have the following useful result of *Poincarè and Bendixson* [20, 21].

Theorem 2.5.9. *A nonempty compact ω- or α-limit set of a planar flow, which contains no equilibria, is a closed orbit.*

The next *Bendixson criterion* rules out the occurrence of closed orbits in some cases [20, 21].

Theorem 2.5.10. *If in a simply connected region $D \subset \mathbb{R}^2$ the divergence* $\operatorname{div} f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ *of* (2.5.9) *is not identically zero and does not change sign, then* (2.5.9) *has no closed orbits lying entirely in D.*

2.5.7 Averaging Method

In this section, we consider systems of the form [21, 24, 26]

$$\dot{x} = \varepsilon f(x, t, \varepsilon), \tag{2.5.10}$$

where $f \in C^r\left(\mathbb{R}^{n+2}, \mathbb{R}^n\right)$, $r \geq 2$.

Definition 2.5.11. $f \in C^r\left(\mathbb{R}^{n+2}, \mathbb{R}^n\right)$, $r \geq 2$ is said to be *KBM-vector field*, (KBM stands for Krylov, Bogolyubov and Mitropolsky) if the average

$$f_0(x) := \lim_{t \to \infty} \frac{1}{t} \int_0^t f(x, s, 0)\, ds$$

exists for any $x \in \mathbb{R}^n$. The associated autonomous *averaged system* is defined as

$$\dot{y} = \varepsilon f_0(y). \tag{2.5.11}$$

We have the following results.

Theorem 2.5.12. *Suppose for* (2.5.10) *that f is T-periodic in t. Then f is a KBM-vector field. Moreover, for any $\varepsilon > 0$ sufficiently small, we get*

(i) *If $x(t)$ and $y(t)$ are solutions of* (2.5.10) *and* (2.5.11) *with $|x(0) - y(0)| = O(\varepsilon)$, then $|x(t) - y(t)| = O(\varepsilon)$ on a time scale $t \sim 1/\varepsilon$.*
(ii) *If p_0 is a hyperbolic equilibrium of* (2.5.11) *then* (2.5.10) *possesses a unique hyperbolic periodic orbit $\gamma_\varepsilon(t) = p_0 + O(\varepsilon)$ of the same stability type as p_0.*
(iii) *If $x_s(t) \in W^s(\gamma_\varepsilon)$ is a solution of* (2.5.10) *lying on the stable manifold of γ_ε, $y_s(t) \in W^s(p_0)$ is a solution of* (2.5.11) *lying on the stable manifold of p_0 and $|x(0) - y(0)| = O(\varepsilon)$, then $|x(t) - y(t)| = O(\varepsilon)$ for any $t \geq 0$. Similar results apply to solutions lying in the unstable manifolds in the time interval $t \leq 0$.*

The above theorem can be generalized to more complicated hyperbolic sets [21, 26]. For instance, the following holds:

Theorem 2.5.13. *Suppose f, f_0 are C^1-smooth and $f_0(y_0) = 0$ with $\Re\sigma(Df_0(y_0)) < 0$. If x_0 is in a domain of attraction of y_0, then for any $\varepsilon > 0$ sufficiently small, $|x_\varepsilon(t) - y(t)| = o(1)$ for any $t \geq 0$, where $x_\varepsilon(t)$ and $y(t)$ are solutions of* (2.5.10) *and* (2.5.11) *with $x(0) = y(0) = x_0$, respectively.*

2.5.8 Carathéodory Type ODEs

In this section we recall some results on ODEs only measurable depending on t.

Definition 2.5.14. Let $\mathscr{I}$ be an interval in $\mathbb{R}$. A mapping $f : \mathscr{I} \times \mathbb{R}^n \to \mathbb{R}^n$ is said to have the *Carathéodory property* if the following assumptions hold [27, 28]:

(i) For every $t \in \mathscr{I}$ the mapping $f(t,\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is continuous.
(ii) For every $x \in \mathbb{R}^n$ the mapping $f(\cdot,x) : \mathscr{I} \to \mathbb{R}^n$ is measurable with respect to the Borel σ-algebras on $\mathscr{I}$ and $\mathbb{R}^n$.

We note that if f has a Carathéodory property and $x : \mathscr{I} \to \mathbb{R}^n$ is measurable then $f(t,x(t))$ is measurable as well.

Definition 2.5.15. A function $x : \mathscr{I} \to \mathbb{R}^n$ is *absolutely continuous* [2] if for any $\varepsilon > 0$ there is a $\delta > 0$ so that for any $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_k < \beta_k$, $\alpha_i, \beta_i \in \mathscr{I}$ so that $\sum_{i=1}^k (\beta_i - \alpha_i) < \delta$, it holds $\sum_{i=1}^k |x(\beta_i) - x(\alpha_i)| < \varepsilon$.

It is well known that an absolutely continuous function on $\mathscr{I}$ has almost everywhere a derivative. By a solution of an ODE $\dot{x} = f(t,x)$ with a Carathéodory mapping f, we mean an absolutely continuous function $x(t)$ satisfying this ODE almost everywhere.

2.6 Singularities of Smooth Maps

Here we recall some results from the theory of smooth maps [14].

2.6.1 Jet Bundles

Definition 2.6.1. Let M,N be smooth manifolds with dimensions m and n, respectively. Let $f,g \in C^\infty(M,N)$ with $f(p) = g(p) = q$. f has *kth order contact* with g at p if in local coordinates

$$\frac{\partial^{|\alpha|} f_i}{\partial x^\alpha}(p) = \frac{\partial^{|\alpha|} g_i}{\partial x^\alpha}(p)$$

for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $|\alpha| = \alpha_1 + \cdots + \alpha_m \leq k$ and $1 \leq i \leq n$, where f_i, g_i are the coordinate functions of f, g, respectively, and $x = (x_1, \ldots, x_m)$. This is written as $f \sim_k g$ at p.

Let $J^k(M,N)_{p,q}$ denote the set of equivalence classes under "$\sim_k$ at p" in $C^\infty(M,N)$. Let $J^k(M,N) := \bigcup_{(p,q)\in M\times N} J^k(M,N)_{p,q}$ - disjoint union. An element of $J^k(M,N)$ is called a *k-jet* and $J^k(M,N)$ is the *jet bundle*. Note that given $f \in C^\infty(M,N)$ there is a mapping $j^k f : M \to J^k(M,N)$ called the *k-jet of f* defined by $j^k f(p) :=$ the equivalence class of f in $J^k(M,N)_{p,f(p)}$ for every $p \in M$. Note that $J^0(M,N) = M \times N$. For any k-jet $\xi \in J^k(M,N)$, there is its *source* $p \in M$ and the *target* $q \in M$. Let f be the representative of $\xi \in J^1(M,N)$. Then we define the *rank* of ξ as $\operatorname{rank}\xi := \operatorname{rank} Df(p)$ and *corank* as $\operatorname{corank}\xi := \min\{m,n\} - \operatorname{rank}\xi$.

Theorem 2.6.2. *Let* $L^r(\mathbb{R}^m,\mathbb{R}^n) := \{A \in L(\mathbb{R}^m,\mathbb{R}^n) \mid \operatorname{corank} A = r\}$. *Then* $L^r(\mathbb{R}^m,\mathbb{R}^n)$ *is a submanifold of* $L(\mathbb{R}^m,\mathbb{R}^n)$ *with* $\operatorname{codim} L^r(\mathbb{R}^m,\mathbb{R}^n) = (m - \min\{n,m\} + r)(n - \min\{n,m\} + r)$.

Theorem 2.6.3. *Let* $S_r := \left\{\xi \in J^1(M,N) \mid \operatorname{corank}\xi = r\right\}$. *Then* S_r *is a submanifold of* $J^1(M,N)$ *with* $\operatorname{codim} S_r = (m - \min\{n,m\} + r)(n - \min\{n,m\} + r)$.

2.6.2 Whitney C^∞ Topology

Let M,N be smooth manifolds. Let $k \in \mathbb{Z}_0$. Let U be an open subset of $J^k(M,N)$. Then the family of sets

$$\left\{f \in C^\infty(M,N) \mid j^k f(M) \subset U\right\}$$

forms a basis for a *Whitney C^k topology* on $C^\infty(M,N)$. The union of all open subsets of $C^\infty(M,N)$ in some Whitney C^k topology forms a basis of a *Whitney C^∞ topology* on $C^\infty(M,N)$. We note that a subset of topological space is *residual* if it is the countable intersection of open dense subsets. A topological space is a *Baire space* if its every residual set is dense.

Theorem 2.6.4. $C^\infty(M,N)$ *is a Baire space in the Whitney C^∞ topology.*

2.6.3 Transversality

Definition 2.6.5. Let M,N be smooth manifolds and $f : M \to N$ be a smooth map. Let S be a submanifold of N and $x \in M$. Then f *transversally intersects* S at $x \in M$ denoted by $f \pitchfork S$ at x, if either

(i) $f(x) \notin S$, or

(ii) $f(x) \in S$ and $T_{f(x)}N = T_{f(x)}S + Df(x)T_xM$.

If $f \overline{\pitchfork} S$ for any $x \in M$, then f *transversally intersects* S denoted by $f \overline{\pitchfork} S$.

Theorem 2.6.6. *If $f \overline{\pitchfork} S$ then $f^{-1}(S)$ is a smooth submanifold with codimension* $\operatorname{codim} S$.

Now we state the *Thom transversality theorem*.

Theorem 2.6.7. *Let W be a submanifold of $J^k(M,N)$. Then*

$$T_W := \left\{ f \in C^\infty(M,N) \mid j^k f \overline{\pitchfork} W \right\}$$

is a residual subset of $C^\infty(M,N)$ in the Whitney C^∞ topology. If, in addition, W is closed, then T_W is open.

2.6.4 Malgrange Preparation Theorem

Theorem 2.6.8. *Let F be a smooth real-valued function defined on a neighbourhood of* 0 *in* $\mathbb{R} \times \mathbb{R}^n$ *so that $F(t,0) = g(t)t^k$, where $g(0) \neq 0$ and g is smooth on some neighbourhood of* 0 *in* $\mathbb{R}$. *Then there is a smooth G with $G(0) \neq 0$ and smooth $\lambda_0, \dots, \lambda_{k-1}$ so that*

$$(GF)(t,x) = t^k + \sum_{i=0}^{k-1} \lambda_i(x)t^i \,.$$

As a consequence of the generalized Malgrange theorem, we have the Whitney theorem [14, p. 108].

Theorem 2.6.9. *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth even function, then there is a smooth function $g : \mathbb{R} \to \mathbb{R}$ satisfying $f(x) = g(x^2)$.*

2.6.5 Complex Analysis

Here we recall some basic results from the theory of complex functions [2]. Let $\Omega \subset \mathbb{C}$ be a *region*, i.e. Ω is open and connected. A complex function $f : \Omega \to \mathbb{C}$ is *holomorphic* if for any $z_0 \in \Omega$ there is a *derivative* $f'(z_0) \in \mathbb{C}$ of f at z_0 defined by

$$\lim_{z \to z_0} \frac{f(z - f(z_0)}{z - z_0} = f'(z_0) \,.$$

The class of all holomorphic functions on Ω is denoted by $H(\Omega)$. Any $f \in H(\Omega)$ is *analytic*, i.e. $f(z) = \sum_{i=0}^{\infty} c_i(z - z_0)^i$ for any $z_0 \in \Omega$ and z near z_0. Next, for any nonzero $f \in H(\Omega)$ the set $Z(f) := \{z \in \Omega \mid f(z) = 0\}$ consists at most of isolated points. Moreover, if $z_0 \in Z(f)$ then $f(z) = (z - z_0)^m g(z)$ for $g \in H(\Omega)$, $g(z_0) \neq 0$, and m is the *order of the zero* which has f at z_0.

A function $f : \Omega \to \mathbb{C}$ has a *pole of order m* in $z_0 \in \Omega$ if

$$f(z) = \sum_{i=-m}^{\infty} c_i (z - z_0)^i$$

with $c_{-m} \neq 0$, for any $z \neq z_0$ near z_0. We denote by $\operatorname{Res}(f, z_0) := c_{-1}$ the *complex residue* of $f(z)$ at the pole z_0.

A function $f : \Omega \to \mathbb{C}$ is *meromorphic* if there is a subset $A \subset \Omega$ so that:

1. A consists of isolated points;
2. $f \in H(\Omega \setminus A)$,
3. f has poles in A.

Note that each rational function, i.e. a quotient of two polynomials, is meromorphic on $\mathbb{C}$.

Next z_0 is an *essential singularity* of f if $f(z) = \sum_{i=-\infty}^{\infty} c_i (z - z_0)^i$ for any $z \neq z_0$ near z_0 and with infinitely many nonzero c_m, $m < 0$.

A *path* γ is a piecewise continuously differentiable curve in the plane, i.e. $\gamma \in C([a, b], \mathbb{C})$ and there are finite $a = s_0 < s_1 < \cdots < s_n = b$ so that $\gamma \in C^1([s_i, s_{i+1}], \mathbb{C})$ for each $i = 0, \ldots, n-1$. A path is *closed* if $\gamma(a) = \gamma(b)$. The integral of a holomorphic function f over the path γ is defined as

$$\int_\gamma f(z)\, dz := \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} f(\gamma(t)) \gamma'(t)\, dt\,.$$

If a path γ counterclockwise encloses all poles of a meromorphic function $f(z)$, then the *Cauchy residue theorem* states that

$$\int_\gamma f(z)\, dz = 2\pi\imath \sum_{z_0 \in A} \operatorname{Res}(f, z_0)\,.$$

Particularly, if a path γ counterclockwise encloses only a pole z_0 of a meromorphic function $f(z)$, then

$$\operatorname{Res}(f, z_0) = \frac{1}{2\pi\imath} \int_\gamma f(z)\, dz. \tag{2.6.1}$$

Finally we states the *Schwarz reflection principle*.

Theorem 2.6.10. *Suppose L is a segment on the real axis, Ω^+ is a region in $\Pi^+ := \{z \in \mathbb{C} \mid \Im z > 0\}$, and every $z \in L$ is the center of an open disc D_z so that $\Pi^+ \cap D_z$ lies in Ω^+. Let $\Omega^- := \{z \mid \bar{z} \in \Omega^+\}$. Suppose $f \in H(\Omega^+)$ and $\lim_{n\to\infty} \Im f(z_n) = 0$ for every sequence $\{z_n\}$ in Ω^+ which converges to a point in L. Then there is a function $F \in H(\Omega^+ \cup L \cup \Omega^-)$, so that $F(z) = f(z)$ in Ω^+ and $F(\bar{z}) = \overline{F(z)}$ for any $z \in \Omega^+ \cup L \cup \Omega^-$.*

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