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PHYSICAL
SCIENCE

Michal Fečkan

Bifurcation and Chaos in Discontinuous and Continuous Systems

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With 30 figures

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To my beloved family

Preface

This book is devoted to the comprehensive bifurcation theory of chaos in nonlinear dynamical systems with applications to mechanics and vibrations. Precise and complete proofs of derived mathematical results are presented with many stimulating and illustrative examples. I study bifurcations of chaotic solutions for perturbed problems from either homoclinic or heteroclinic orbits of unperturbed ones. This method is also known as the Melnikov-type approach. Certainly there are many interesting books in this direction, but all results of this book have not yet been published in any book, since I have collected some results of mine together with my coauthors appeared only in articles and manuscripts. So I hope that this book is a useful contribution to a rapidly developing theory of chaos and it is a good continuation of my recently published book in Springer with similar topics.

The book is intended to be used by scientists interested in the theory of chaos and its applications, like mathematicians, physicists, or engineers. It can also serve as a textbook for a class of nonlinear oscillations and dynamical systems.

Here is a brief outline of each chapter.

Chapter 1 is an introduction to the topic of the book by presenting two well-known chaotic models: damped and driven Duffing and pendulum equations.

To make this book as self-contained as possible, some basic preliminary results are included in Chapter 2.

Chapter 3 studies chaotic bifurcations of discrete dynamical systems including: nonautonomous difference equations; diffeomorphisms; perturbed singular and singularly perturbed impulsive ordinary differential equations (ODEs); and inflated dynamical systems arising in computer assisted proofs and in other numerical methods in dynamical systems, so an extension of Smale horseshoe to inflated dynamical systems is presented.

Chapter 4 deals with proving chaos for parameterized ODEs in arbitrary dimensions. It is shown that if the Melnikov function is identically zero the second order Melnikov function must be derived. I consider a broad variety of ODEs: coupled nonresonant ODEs, resonant systems of ODEs investigated with the help of averaging theory; singularly perturbed ODEs; and inflated ODEs. I also show that the structure of chaotic parameters is related to the Morin singularity of smooth mappings. I end this chapter with infinite dimensional ODEs on lattices by considering a model of two one-dimensional interacting sublattices of harmonically coupled protons and heavy ions.

Chapter 5 shows chaotic vibrations of partial differential equations (PDEs): slowly periodically perturbed and weakly nonlinear beams on elastic bearings; periodically forced and nonresonant buckled elastic beams; and periodically forced compressed beams at resonance.

Chapter 6 is devoted to the study of chaotic oscillations of discontinuous (nonsmooth) differential equations (DDEs). First I consider the case when the homoclinic orbit of the unperturbed DDE transversally crosses discontinuity surfaces. Then I study a chaos for time-perturbed DDEs. I apply our general results to quasiperiodic piecewise linear systems in $\mathbb{R}^3$, and to piecewise smooth forced planar DDEs. Then I extend those result to sliding homoclinic bifurcations, when a part of the homoclinic orbit of the unperturbed DDE lies on a discontinuity surface. A rigorous proof of the existence of chaos for stick-slip systems is presented. I utilize general theoretical results to planar and 3-dimensional sliding homoclinic cases.

In Chapter 7, first I investigate the Melnikov function in general by computing its Fourier coefficients. These computations allow me to find examples when the Melnikov function is ether identically zero or not. I also derive the second order Melnikov function when the (first order) Melnikov function is identically zero. For construction of concrete examples, I solve an inverse problem when the homoclinic orbit is given and a second order ODE is found so that it possesses that homoclinic orbit. The second part of this chapter is devoted to showing chaos near transversal heteroclinic orbits. The third part deals with the blue sky catastrophe for periodic orbits.

In all chapters, derived bifurcation conditions for the existence of chaos are expressed as simple zeroes of corresponding Melnikov functions. Functional analytic approaches are used which are roughly based on a concept of exponential dichotomy together with Lyapunov-Schmidt method. Numerical computations described by figures are given with the help of a computational software program *Mathematica*.

The author is indebted to the coauthors for some results mentioned in this book: Jan Awrejcewicz, Flaviano Battelli, Giovanni Colombo, Matteo Franca, Barnabás M. Garay, Joseph Gruendler, Paweł Olejnik, Weiyao Zeng. Partial support of Grants VEGA-SAV 2/0124/10, VEGA-MS 1/0098/08, an award from Literárny fond and by the Slovak Research and Development Agency under the contract No. APVV-0414-07 are also appreciated.

Michal Fečkan
Bratislava, Slovakia
June 2010

Contents

Chapter 1
Introduction

Many problems in the natural and engineering sciences can be modeled as evolution processes. Mathematically this leads to either discrete or continuous dynamical systems, i.e. to either difference or differential equations. Usually such dynamical systems are nonlinear or even discontinuous and depend on parameters. Consequently the study of qualitative behaviour of their solutions is very difficult. Rather effective method for handling dynamical systems is the bifurcation theory, when the original problem is a perturbation of a solvable problem, and we are interested in qualitative changes of properties of solutions for small parameter variations. Nowadays the bifurcation and perturbation theories are well developed and methods applied by these theories are rather broad including functional-analytical tools and numerical simulations as well [1–13].

Next, one of the fascinating behaviour of nonlinear dynamical systems which may occur is their sensitive dependence on the initial value conditions, which results in a chaotic time behaviour. Chaos is by no means exceptional but a typical property of many dynamical systems in periodically stimulated cardiac cells, in electronic circuits, in chemical reactions, in lasers, in mechanical devices, and in many other models of biology, meteorology, economics and physics. In spite of the fact that it is very difficult to show chaos for general evolution equations, the bifurcation theory based on perturbation methods is a powerful tool for concluding chaos in a rather wide class of parameterized nonlinear dynamical systems. Especially functional-analytical methods are very convenient to show rigorously the existence of chaos in concrete dynamical systems [14–20].

Now we show two well-known simple chaotic mechanical models. First, we consider a periodically forced and damped Duffing equation

$$\dot{x} = y, \quad \dot{y} - x + 2x^3 + \mu_1 y = \mu_2 \cos t \tag{1.0.1}$$

with μ_1, μ_2 being small. Note

$$\ddot{x} + \mu_1 \dot{x} - x + 2x^3 = \mu_2 \cos t$$

describes dynamics of a buckled beam, when only one mode of vibration is considered (cf Section 5.2 and [21]). Particularly, an experimental apparatus in [4, pp. 83–84] is a slender steel beam clamped to a rigid framework which supports two magnets, when x is the beam tip displacement. The apparatus is periodically forced using electromagnetic vibration generator (Figure 1.1).

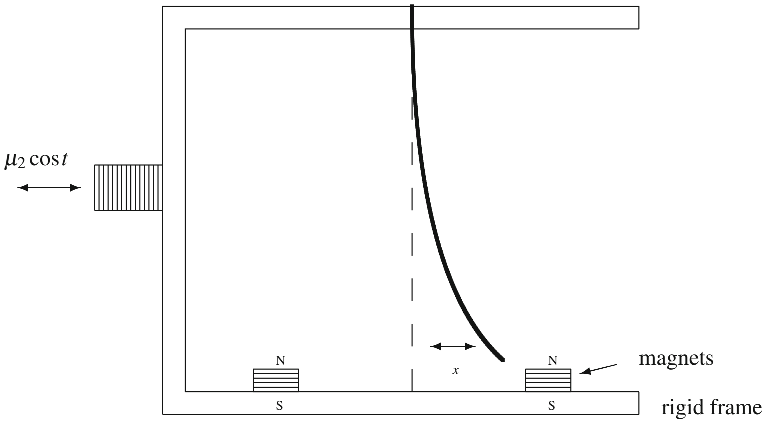

Fig. 1.1 The magneto-elastic beam.

Next, the phase portrait of

$$\dot{x} = y, \quad \dot{y} - x + 2x^3 = 0 \tag{1.0.2}$$

is simply found analytically by analyzing the level sets $\dot{x}^2 - x^2 + x^4 = c \in \mathbb{R}$ [1,4,13]. Here $\mathbb{R}$ denotes the set of real numbers. There are three equilibria: $(0,0)$ is hyperbolic and $(\pm\sqrt{2}/2, 0) \doteq (\pm 0.707107, 0)$ are centers. There is also a symmetric *homoclinic cycle* $\pm\widetilde{\gamma}_d(t)$ with $\widetilde{\gamma}_d(t) = (\gamma_d(t), \dot{\gamma}_d(t))$ and $\gamma_d(t) = \operatorname{sech} t$. The rest are all periodic solutions. These results are consistent with the above experimental model without damping and external forcing as follows: When attractive forces of the magnets overcome the elastic force of the beam, the beam settles with its tip close to one or more of the magnets: these are centers of (1.0.2). There is also an unstable central equilibrium position of the beam at which the magnetic forces are canceled: this is the unstable equilibrium of (1.0.2) (Figure 1.2).

When $\mu_{1,2}$ are small and not identically zero, in spite of the fact that (1.0.1) is a simply looking equation, its dynamics is very difficult. This is demonstrated in Figure 1.3. We see that there are random oscillations of the beam tip between the two magnets. These chaotic vibrations are also observed in the experimental apparatus of Figure 1.1 as shown in [4, p. 84]. Theoretically it is justified by Lemma 7.2.4. Note that almost all trajectories of the damped case $\mu_1 > 0$, $\mu_2 = 0$ tend to one of the stable equilibria $(\pm\sqrt{2}/2, 0)$ (cf case A of Figure 1.3).

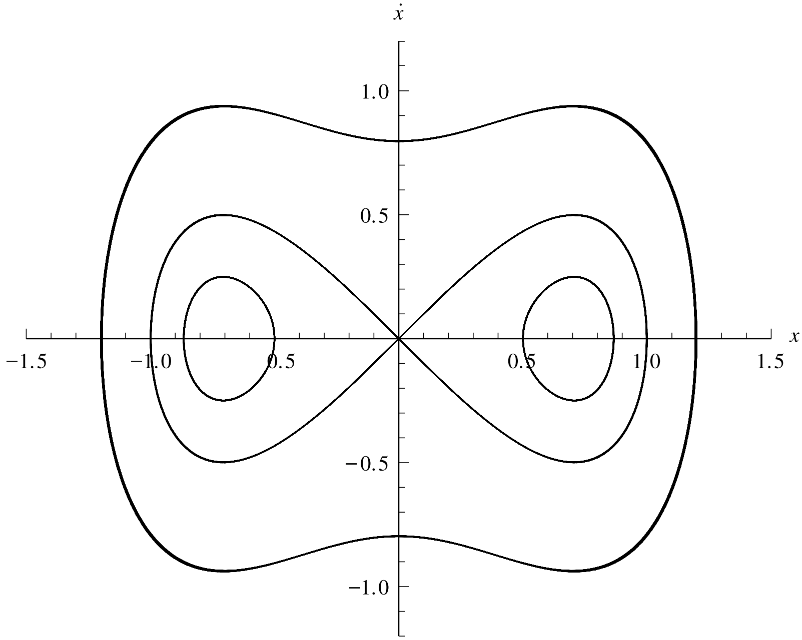

Fig. 1.2 The phase portrait of the Duffing equation (1.0.2).

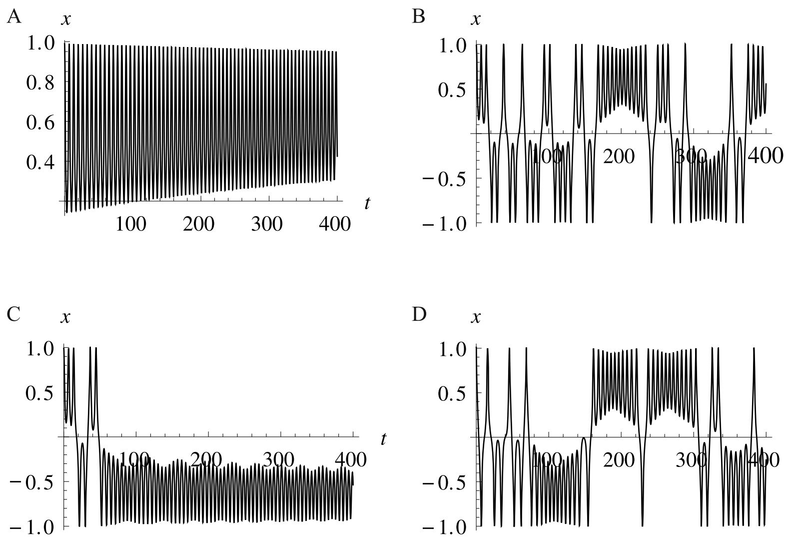

Fig. 1.3 The solution $x(t)$, $0 \leq t \leq 400$ of (1.0.1) for A: $\mu_1 = 0.001$, $\mu_2 = 0$, $x(0) = 0.99$, $\dot{x}(0) = 0$; B: $\mu_1 = 0$, $\mu_2 = 0.01$, $x(0) = 0.99$, $\dot{x}(0) = 0$; C: $\mu_1 = 0.001$, $\mu_2 = 0.01$, $x(0) = 0.99$, $\dot{x}(0) = 0$; D: $\mu_1 = 0.001$, $\mu_2 = 0.01$, $x(0) = 1.01$, $\dot{x}(0) = 0$.

The aim of this book is to show chaos in (1.0.1) analytically. This is presented in Section 4.1 and Subsection 5.2.6: now the *Melnikov function* is given by

$$M(\alpha) = \int_{-\infty}^{\infty} \dot{\gamma}_d(t)\,(\mu_2 \cos(\alpha + t) - \mu_1 \dot{\gamma}_d(t))\,dt = \mu_2 \pi \operatorname{sech}\frac{\pi}{2} \sin\alpha - \mu_1 \frac{2}{3}\,.$$

When μ_1, μ_2 satisfy

$$|\mu_1| < |\mu_2| \frac{3\pi}{2} \operatorname{sech}\frac{\pi}{2} \doteq 1.87806|\mu_2|\,, \tag{1.0.3}$$

clearly there is a *simple zero* α_0 of M, i.e. $M(\alpha_0) = 0$ and $M'(\alpha_0) \neq 0$. Hence by Remark 4.1.6, (1.0.1) is chaotic for μ_1, μ_2 sufficiently small fulfilling (1.0.3). Note (1.0.3) holds for cases B, C, D of Figure 1.3.

The second popular example of chaotic physical model is a damped and forced pendulum consisting of a mass attached to a vertically oscillating pivot point by means of mass-less and inextensible wire described by ODE ([1, p. 278] and [22, p. 216])

$$\ddot{\phi} + \mu_1 \dot{\phi} + \sin\phi = \mu_2 \cos t \sin\phi\,, \tag{1.0.4}$$

where μ_1, μ_2 are parameters (Figure 1.4).

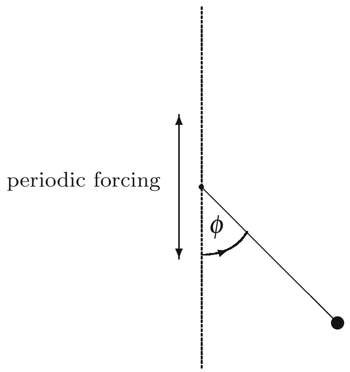

Fig. 1.4 The damped and forced pendulum (1.0.4).

The unperturbed ODE is given by

$$\ddot{\phi} + \sin\phi = 0 \tag{1.0.5}$$

with the phase portrait in Figure 1.5.

Note that $(2k\pi, 0)$ are centers and $((2k+1)\pi, 0)$ are hyperbolic equilibria of (1.0.5) for $k \in \mathbb{Z}$. Here $\mathbb{Z}$ denotes the set of integer numbers. Moreover, $(-\pi, 0)$ and $(\pi, 0)$ are joined by the upper *separatrix* or *heteroclinic orbit* $\widetilde{\gamma}_p(t)$ with $\widetilde{\gamma}_p(t) = (\gamma_p(t), \dot{\gamma}_p(t))$ and $\gamma_p(t) = 2\arctan(\sinh t)$. The lower separatrix is $-\widetilde{\gamma}_p(t)$.

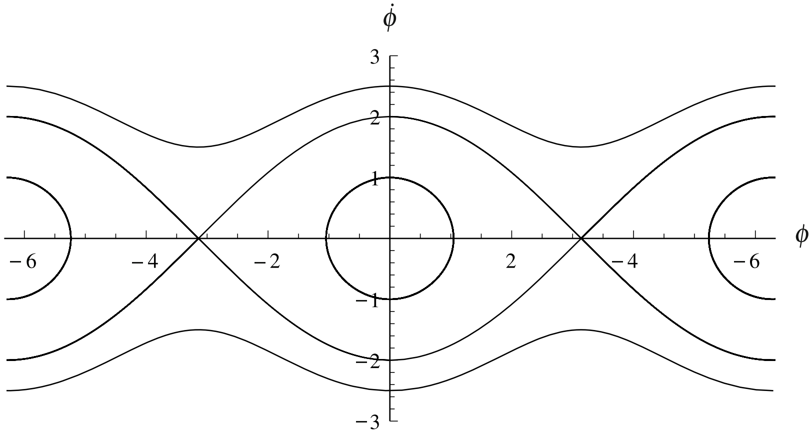

Fig. 1.5 The phase portrait of the pendulum equation (1.0.5).

When $\mu_{1,2}$ are small and not identically zero, (1.0.4) has very difficult dynamics. This is demonstrated in Figure 1.6.

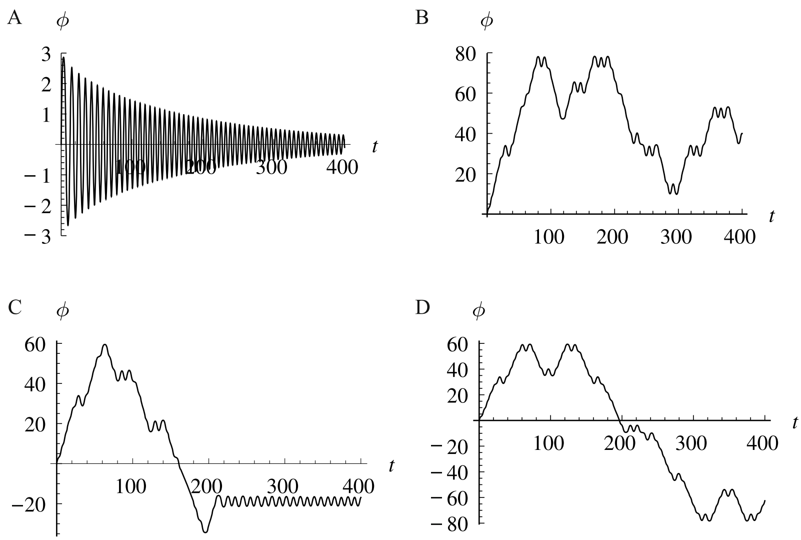

Fig. 1.6 The solution $\phi(t)$, $0 \le t \le 400$ of (1.0.4) for A: $\mu_1 = 0.01$, $\mu_2 = 0$, $\phi(0) = 0$, $\dot\phi(0) = 2$; B: $\mu_1 = 0.001$, $\mu_2 = 0.1$, $\phi(0) = 0$, $\dot\phi(0) = 1.998$; C: $\mu_1 = 0.001$, $\mu_2 = 0.1$, $\phi(0) = 0$, $\dot\phi(0) = 2$; D: $\mu_1 = 0.001$, $\mu_2 = 0.1$, $\phi(0) = 0$, $\dot\phi(0) = 2.002$.

Now the Melnikov function is given by [1, p. 467]

$$M(\alpha) = \int_{-\infty}^{\infty} \dot\gamma_p(t)\,(\mu_2 \cos(\alpha + t) \sin\gamma_p(t) - \mu_1 \dot\gamma_p(t))\,dt = -2\pi\mu_2 \operatorname{csch}\frac{\pi}{2}\sin\alpha - 8\mu_1\,.$$

When μ_1, μ_2 satisfy

$$|\mu_1| < |\mu_2| \frac{\pi}{4} \operatorname{csch} \frac{\pi}{2} \doteq 0.341285 |\mu_2|, \tag{1.0.6}$$

clearly there is a simple zero α_0 of M, i.e. $M(\alpha_0) = 0$ and $M'(\alpha_0) \neq 0$. Hence by Remark 4.1.6, (1.0.4) is chaotic for μ_1, μ_2 sufficiently small fulfilling (1.0.6). Note that (1.0.6) holds for cases B, C, D of Figure 1.6. Note that almost all trajectories of the damped case $\mu_1 > 0$, $\mu_2 = 0$ tend to the one of the stable equilibria $(2k\pi, 0)$, $k \in \mathbb{Z}$ (cf case A of Figure 1.6).

In summary, examples (1.0.1) and (1.0.4) have the following common features: they are simply looking equations with unpredictable dynamics. But deriving their Melnikov functions, it is easy to show their chaotic behaviour. Consequently, the aim of this book is to present many different discrete and continuous dynamical systems defined on spaces with arbitrarily high dimensions including infinite ones when this Melnikov type analysis is shown to be useful, and then we demonstrate abstract results on concrete examples.

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Chapter 2
Preliminary Results

In this chapter, we recall some known mathematical notations, notions and results which will be used later to help readers to understand this book better. For more details, we refer readers to quoted textbooks of nonlinear functional analysis, differential topology, singularities of smooth maps, complex analysis and dynamical systems.

2.1 Linear Functional Analysis

Let X be a *Banach space* with a norm $|\cdot|$. By $\mathbb{N}$ we denote the set of natural numbers. A sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ *converges* to $x_0 \in X$ if $|x_n - x_0| \to 0$ as $n \to \infty$, for short $x_n \to x_0$. We denote by $B_x(r)$ the *closed ball* in X centered at $x \in X$ and with the radius $r > 0$, i.e. $B_x(r) := \{z \in X \mid |z-x| \leq r\}$. Let S be a subset of X, i.e. $S \subset X$. Then S is *convex* if $\lambda s_1 + (1-\lambda)s_2 \in S$ for all $s_1, s_2 \in S$ and $\lambda \in [0,1]$. By $\operatorname{conv} S$ we denote the *convex hull* of S, i.e. the intersection of all convex subsets of X containing S. *Diameter* of S, $\operatorname{diam} S$, is defined as $\operatorname{diam} S := \{\sup |x-y| \mid x, y \in S\}$. S is *open* if any point of S has a closed ball belonging to S. S is *closed* if $X \setminus S$ is open. The *closure* and *interior* of S are denoted by $\bar{S}$ and $\operatorname{int} S$, respectively. Recall that $\bar{S}$ is the smallest closed subset of X containing S, and $\operatorname{int} S$ is the largest open subset of S. Clearly $\operatorname{int} B_x(r) = \{z \in X \mid |z-x| < r\}$ — an *open ball* in X.

Let X and Y be Banach spaces. The set of all *linear bounded/continuous mappings* $A : X \to Y$ is denoted by $L(X,Y)$, while we put $L(X) := L(X,X)$. The norm of A is defined by $\|A\| := \sup_{|x|=1} |Ax|$. More generally, if $Y, X_1, \ldots, X_n$ are Banach spaces, $L(X_1 \times \cdots \times X_n, Y)$ is the Banach space of *bounded/continuous multilinear maps* from $X_1 \times \cdots \times X_n$ into Y.

In using the Lyapunov-Schmidt method, we first need the following *Banach inverse mapping theorem.*

Theorem 2.1.1. *If $A \in L(X,Y)$ is surjective and injective then its inverse $A^{-1} \in L(Y,X)$.*

We also recall the following well-known result.

Lemma 2.1.2. *Let $Z \subset X$ be a linear subspace with either* $\dim Z < \infty$ *or Z to be closed with* $\operatorname{codim} Z < \infty$. *Then there is a bounded projection $P : X \to Z$. Note that* $\operatorname{codim} Z = \dim X/Z$ *and X/Z is the factor space of X with respect to Z.*

Basic Banach spaces are functional ones like $C^m\left([0,1], M^k\right)$ and $L^p\left(\mathscr{I}, M^k\right)$, where $\mathscr{I} \subset \mathbb{R}$ is an interval and $M \in \{\mathbb{R}, \mathbb{C}\}$, with the usual norms:

$\|f\| = \max_{x\in[0,1], i=0,\cdots,m} |D^m f(x)|$ (cf Section 2.2.2) on $C^m\left([0,1], M^k\right)$,
$\|f\|_p = \sqrt[p]{\int_{\mathscr{I}} |f(x)|^p\, dx}$ on $L^p\left(\mathscr{I}, M^k\right)$ for $1 \leq p < \infty$,
$\|f\|_\infty = \text{ess sup}_{x\in\mathscr{I}} |f(x)| = \min\{\lambda \geq 0 \mid |f(x)| \leq \lambda$ for almost all $x \in \mathscr{I}\}$ on $L^\infty\left(\mathscr{I}, M^k\right)$.

Here $\mathbb{C}$ denotes the set of complex numbers. Recall the *Hölder inequality* $\|fg\|_1 \leq \|f\|_p \|g\|_q$ for any $f \in L^p\left(\mathscr{I}, M^k\right)$, $g \in L^q\left(\mathscr{I}, M^k\right)$ and $\frac{1}{p} + \frac{1}{q} = 1$. For $p = q = 2$, we get the *Cauchy-Schwarz-Bunyakovsky inequality*. Discrete analogies of these spaces are as follows: Let $I \in \{\mathbb{N}, \mathbb{Z}\}$. Then we set $\ell^p\left(M^k\right) := \left\{x = \{x_m\}_{m\in I} \subset M^k \mid \sum_{m\in I} |x_m|^p < \infty\right\}$ with the norm $\|x\|_p = \sqrt[p]{\sum_{m\in I} |x_m|^p}$ for $\infty > p \geq 1$, and $\ell^\infty\left(M^k\right) := \left\{x = \{x_m\}_{m\in I} \subset M^k \mid \sup_{m\in I} |x_m| < \infty\right\}$ with the norm $\|x\|_\infty = \sup_{m\in I} |x_m|$. Note that $L^2\left(\mathscr{I}, M^k\right)$ and $\ell^2\left(M^k\right)$ are *Hilbert spaces* with *scalar products* $(f,g) = \int_{\mathscr{I}} f(x)\overline{g(x)}\, dx$ and $(x,y) = \sum_{m\in\mathbb{Z}} x_m \overline{y_m}$, respectively.

Now we state the well-known Arzelà-Ascoli theorem:

Theorem 2.1.3. *Let $\{x_n(t)\}_{n\in\mathbb{N}} \subset C\left([0,1], \mathbb{R}^k\right)$ be a sequence of continuous mappings $x_n : [0,1] \to \mathbb{R}^k$ so that*

(i) *Sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is uniformly bounded, i.e. there is a constant $M > 0$ so that $|x_n(t)| \leq M$ for any $t \in [0,1]$ and $n \in \mathbb{N}$.*
(ii) *Sequence $\{x_n(t)\}_{n\in\mathbb{N}}$ is equicontinuous, i.e. for any $\varepsilon > 0$ there is a $\delta > 0$ so that for any $n \in \mathbb{N}$ and $t, s \in [0,1]$, $|t - s| < \delta$ it holds $|x_n(t) - x_n(s)| \leq \varepsilon$.*

Then there is a subsequence $\{x_{n_i}(t)\}_{i\in\mathbb{N}}$ of $\{x_n(t)\}_{n\in\mathbb{N}}$ therefore $x_{n_i}(t) \rightrightarrows x_0(t)$ uniformly to some $x_0 \in C([0,1], \mathbb{R}^k)$ as $i \to \infty$.

For any $f \in L^2([-\pi,\pi], \mathbb{C})$, we define *Fourier coefficients* of f by the formula:

$$\hat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, \mathrm{e}^{-\imath n x}\, dx$$

and $n \in \mathbb{Z}$. The *Parseval theorem* asserts that

$$2\pi \sum_{m\in\mathbb{Z}} \hat{f}(n)\overline{\hat{g}(n)} = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx$$

and this implies a Hilbert space isomorphism between $L^2([-\pi,\pi], \mathbb{C})$ and $\ell^2(\mathbb{C})$. Note $f = 0$ if and only if $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. More sophisticated Hilbert spaces are *Sobolev spaces* $H^p(\mathbb{C})$, ($H^p(\mathbb{R})$) $p \in \mathbb{N}$ which are all 2π-periodic complex (real) functions $q(t)$ so that $q^{(p)} \in L^2\left([-\pi,\pi], \mathbb{C}\right)$. Next for any $f \in L^1(\mathbb{R}, \mathbb{C})$ we define its *Fourier transform* by the formula:

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-\iota xt}\, dx\,.$$

The *Plancherel theorem* states that the Fourier transform can be extended to $L^2(\mathbb{R},\mathbb{C})$ with $\|\hat{f}\|_2 = \|f\|_2$ and so $f \to \hat{f}$ is a Hilbert space isomorphism from $L^2(\mathbb{R},\mathbb{C})$ to $L^2(\mathbb{R},\mathbb{C})$.

More details and proofs of the above results can be found in [1–3].

2.2 Nonlinear Functional Analysis

2.2.1 Banach Fixed Point Theorem

Let X and Y be Banach spaces. Norms are denoted by $|\cdot|$. Let $U \subset Y$ be open. Consider a mapping $F : B_{x_0}(r) \times U \to X$ for some $x_0 \in X$ and $r > 0$ under the following assumptions

(a) There is an $\alpha \in (0,1)$ so $|F(x_1,y) - F(x_2,y)| \leq \alpha|x_1 - x_2|$ for all $x_1, x_2 \in B_{x_0}(r)$ and $y \in U$.
(b) There is a $0 < \delta < r(1-\alpha)$ so that $|F(x_0,y) - x_0| \leq \delta$ for all $y \in U$.

Set $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Now we can state the *Banach fixed point theorem* or *uniform contraction mapping principle* [1,4,5].

Theorem 2.2.1. *Suppose there exist conditions (a) and (b). Then F has a unique fixed point $\phi(y) \in \operatorname{int} B_{x_0}(r)$ for any $y \in U$, i.e. $\phi(y) = F(\phi(y),y)$ for all $y \in U$. Moreover it holds*

(i) *If there is a constant $\lambda > 0$ so that $|F(x,y_1) - F(x,y_2)| \leq \lambda|y_1 - y_2|$ for all $x \in B_{x_0}(r)$ and $y_1, y_2 \in U$. Then $|\phi(y_1) - \phi(y_2)| \leq \frac{L}{1-\alpha}|y_1 - y_2|$ for all $y_1, y_2 \in U$.*
(ii) *If $F \in C^k\left(B_{x_0}(r) \times U, X\right)$ for a $k \in \mathbb{Z}_+$ then $\phi \in C^k(U,X)$.*

2.2.2 Implicit Function Theorem

Let X and Y be *Banach spaces*. Norms are denoted by $|\cdot|$. Let $\Omega \subset X$ be open. A map $F : \Omega \to Y$ is said to be *(Fréchet) differentiable* at $x_0 \in \Omega$ if there is a $DF(x_0) \in L(X,Y)$ so

$$\lim_{h \to 0} \frac{|F(x_0 + h) - F(x_0) - DF(x_0)h|}{|h|} = 0\,.$$

If F is differentiable at each $x \in \Omega$ and $DF : \Omega \to L(X,Y)$ is continuous then F is said to be continuously differentiable on Ω and we write $F \in C^1(\Omega,Y)$. Higher derivatives D^iF are defined in the usual way by induction. Similarly, the partial derivatives are defined standardly [1, p. 46]. Now we state the *implicit function theorem* [5, p. 26].

Theorem 2.2.2. *Let X, Y, Z be Banach spaces, $U \subset X$, $V \subset Y$ are open subsets and $(x_0, y_0) \in U \times V$. Consider $F \in C^1(U \times V, Z)$ so that $F(x_0, y_0) = 0$ and $D_x F(x_0, y_0) : X \to Z$ has a bounded inverse. Then there is a neighborhood $U_1 \times V_1 \subset U \times V$ of (x_0, y_0) and a function $f \in C^1(V_1, X)$ so that $f(y_0) = x_0$ and $F(x, y) = 0$ for $U_1 \times V_1$ if and only if $x = f(y)$. Moreover, if $F \in C^k(U \times V, Z)$, $k \geq 1$ then $f \in C^k(V_1, X)$.*

We refer the readers to [4, 6] for more applications and generalizations of the implicit function theorem.

2.2.3 Lyapunov-Schmidt Method

Now we recall the well-known *Lyapunov-Schmidt method* for solving locally nonlinear equations when the implicit function theorem fails. So let X, Y, Z be Banach spaces, $U \subset X$, $V \subset Y$ are open subsets and $(x_0, y_0) \in U \times V$. Consider $F \in C^1(U \times V, Z)$ so that $F(x_0, y_0) = 0$. If $D_x F(x_0, y_0) : X \to Z$ has a bounded inverse then the implicit function theorem can be applied to solving

$$F(x, y) = 0 \tag{2.2.1}$$

near (x_0, y_0). So we suppose that $D_x F(x_0, y_0) : X \to Z$ has no a bounded inverse. In general, this situation is difficult. The simplest case is that when $D_x F(x_0, y_0) : X \to Z$ is *Fredholm*, i.e. $\dim \mathscr{N} D_x F(x_0, y_0) < \infty$, $\mathscr{R} D_x F(x_0, y_0)$ is closed in Z and $\operatorname{codim} \mathscr{R} D_x F(x_0, y_0) < \infty$. Here $\mathscr{N} A$ and $\mathscr{R} A$ are the *kernel* and *range* of a linear mapping A. The *index* of $D_x F(x_0, y_0)$ is defined by $\operatorname{index} D_x F(x_0, y_0) := \dim \mathscr{N} D_x F(x_0, y_0) - \operatorname{codim} \mathscr{R} D_x F(x_0, y_0)$. Then by Lemma 2.1.2, there are bounded projections $P : X \to \mathscr{N} D_x F(x_0, y_0)$ and $Q : Z \to \mathscr{R} D_x F(x_0, y_0)$. Hence we split any $x \in X$ as $x = x_0 + u + v$ with $u \in \mathscr{R}(\mathbb{I} - P)$, $v \in \mathscr{R} P$, and decompose (2.2.1) as follows:

$$H(u, v, y) := QF(x_0 + u + v, y) = 0, \tag{2.2.2}$$

$$(\mathbb{I} - Q) F(x_0 + u + v, y) = 0. \tag{2.2.3}$$

Observe that $D_u H(0, 0, y_0) = D_x F(x_0, y_0) | \mathscr{R}(\mathbb{I} - P) \to \mathscr{R} D_x F(x_0, y_0)$. So $D_u H(0, 0, y_0)$ is injective and surjective. So by Banach inverse mapping theorem 2.1.1, $D_u H(0, 0, y_0)$ has a bounded inverse. Since $H(0, 0, y_0) = 0$, the implicit function theorem can be applied to solving (2.2.2) in $u = u(v, y)$ with $u(0, y_0) = 0$. Inserting this solution into (2.2.3) we get the *bifurcation equation*:

$$B(v, y) := (\mathbb{I} - Q) F(x_0 + u(v, y) + v, y) = 0.$$

Since $B(0, y_0) = (\mathbb{I} - Q) F(x_0, y_0) = 0$ and

$$D_v B(0, y_0) = (\mathbb{I} - Q) D_x F(x_0, y_0) \left(D_v u(0, y_0) + \mathbb{I} \right) = 0,$$

the function $B(v,y)$ has a higher singularity at $(0,y_0)$, so the implicit function theorem is not applicable, and the bifurcation theory must be used [5].

2.2.4 Brouwer Degree

Let $\Omega \subset \mathbb{R}^n$ be open bounded subset. A triple (F,Ω,y) is *admissible* if $F \in C(\bar{\Omega},\mathbb{R}^n)$ and $y \in \mathbb{R}^n$ with $y \notin F(\partial\Omega)$, where $\partial\Omega$ is the border of Ω. Now on these admissible triples (F,Ω,y), there is a $\mathbb{Z}$-defined function deg [1, p. 56].

Theorem 2.2.3. *There is a unique mapping* deg *defined on the set of all admissible triples* (F,Ω,y) *determined by the following properties*:

(i) *If* $\deg(F,\Omega,y) \neq 0$ *then there is an* $x \in \Omega$ *consequently* $F(x) = y$.
(ii) $\deg(\mathbb{I},\Omega,y) = 1$ *for any* $y \in \Omega$.
(iii) $\deg(F,\Omega,y) = \deg(F,\Omega_1,y) + \deg(F,\Omega_2,y)$ *whenever* $\Omega_{1,2}$ *are disjoint open subsets of* Ω *so that* $y \notin F\left(\bar{\Omega} \setminus (\Omega_1 \cup \Omega_2)\right)$.
(iv) $\deg(F(\lambda,\cdot),\Omega,y)$ *is constant for* $F \in C\left([0,1] \times \bar{\Omega},X\right)$ *and* $y \notin F([0,1] \times \partial\Omega)$.

The number $\deg(F,\Omega,y)$ is called the *Brouwer degree* of the map F. If x_0 is an isolated zero of F in $\Omega \subset \mathbb{R}^n$ then $I(x_0) := \deg(F,\Omega_0,0)$ is called the *Brouwer index* of F at x_0, where $x_0 \in \Omega_0 \subset \Omega$ is an open subset so x_0 is the only zero point of F on Ω_0 [5, p. 69]. $I(x_0)$ is independent of such Ω_0. Note that if $y \in \mathbb{R}^n$ is a regular value of F, i.e. $\det DF(x) \neq 0$ for any $x \in \Omega$ with $F(x) = y$, and $y \notin F(\partial\Omega)$, then $F^{-1}(y)$ is finite and $\deg(F,\Omega,y) = \sum_{x\in F^{-1}(y)} \operatorname{sgn}\det DF(x)$. Particularly if x_0 is as *simple zero* of $F(x)$, i.e. $F(x_0) = 0$ and $\det DF(x_0) \neq 0$, then $I(x_0) = \operatorname{sgn}\det DF(x_0) = \pm 1$.

2.2.5 Local Invertibility

It is well known that the linear invertibility implies local nonlinear invertibility. More precisely, let us consider a map $F : X \to Y$, $F(0) = 0$, where F is C^1–smooth and X, Y are Banach spaces. If $DF(0)$ is invertible, then any C^1–small perturbation of F has a unique zero point near 0. This follows from the implicit function theorem 2.2.2. Now we shall study a reverse problem [7].

Theorem 2.2.4. *Consider a* C^2*–smooth map* $F : X \to Y$ *satisfying* $F(0) = 0$ *and assume that* $DF(0)$ *is Fredholm with index* 0.

If there exist a neighbourhood $U \subset X$ *of* 0 *and numbers* $K > 0$, $\delta > 0$ *so that for any linear bounded mapping* $B : X \to Y$, $||B|| \leq K$ *the perturbation* $\varepsilon B + F$, $0 \leq \varepsilon \leq \delta$ *has the only zero point* 0 *in* U, *then* $DF(0)$ *is invertible.*

Note that if there is a number K satisfying the assumption of the above theorem, then this assumption holds with any $K > 0$ and the same neighbourhood U. Of course, we must take another $\delta > 0$. If we are interested in the invertibility of $DF(x_0)$

for a general fixed x_0 satisfying $F(x_0) = 0$, then Theorem 2.2.4 is applied with perturbations of the form $\varepsilon\big(B - Bx_0\big) + F$, where B has the properties of Theorem 2.2.4. Indeed, we apply Theorem 2.2.4 to the map $x \to F\big(x + x_0\big)$. The perturbation term $\varepsilon\big(B - Bx_0\big)$ is affinely small.

2.2.6 *Global Invertibility*

Let X, Y be Banach spaces and $f \in C(X,Y)$. Then f is *proper* if the inverse image $f^{-1}(C)$ of any compact subset $C \subset Y$ is compact [4, p. 102].

Theorem 2.2.5. *If X and Y are finitely dimensional, then f is proper if f is coercive, i.e. $|f(x)| \to \infty$ whenever $|x| \to \infty$.*

Now we state the following Banach-Mazur theorem of global invertibility of mappings.

Theorem 2.2.6. (i) *f is a homeomorphism of X onto Y if and only if f is a local homeomorphism and proper.*

(ii) *If $f \in C^1(X,Y)$ then f is a diffeomorphism if and only if f is proper and $Df(x)$ is a linear homeomorphism for each $x \in X$.*

2.3 Multivalued Mappings

Let X, Y be Banach spaces and let $\Omega \subset X$. By 2^Y we denote the family of all subsets of Y. Any mapping $F : \Omega \to 2^Y \setminus \{\emptyset\}$ is called *multivalued or set-valued* mappings. A multivalued mapping $F : \Omega \to 2^Y \setminus \{\emptyset\}$ is *convex (compact)-valued* if $F(x)$ is convex (compact) for any $x \in \Omega$.

By $B(X)$ we denote the family of all nonempty closed bounded subsets of X. Let $A, B \in B(X)$, then their *Hausdorff distance* $d_H(A,B)$ is defined as follows

$$d_H(A,B) := \max\left\{\sup_{a\in A}\left[\inf_{x\in B}|x-a|\right], \sup_{b\in B}\left[\inf_{x\in A}|x-b|\right]\right\}.$$

It is well known that d_H is a metric on $B(X)$ and $B(X)$ is a complete metric space with respect to d_H [8, 9]. A multivalued mapping $F : X \to B(Y)$ is Lipschitz continuous with a constant $\Lambda > 0$, if

$$d_H(F(x_1), F(x_2)) \le \Lambda |x_1 - x_2|$$

for any $x_1, x_2 \in X$. Now we state the *Lojasiewicz-Ornelas parametrization theorem* [10]:

Theorem 2.3.1. *If $G : \mathbb{R}^n \to \mathbb{R}^n$ is a compact convex-valued map which is Lipschitz, then there exists a Lipschitz map $g : \mathbb{R}^n \times \mathscr{B}_{\mathbb{R}^n} \to \mathbb{R}^n$ so that $G(x) = g(x, \mathscr{B}_{\mathbb{R}^n})$ for all $x \in \mathbb{R}^n$, where $\mathscr{B}_{\mathbb{R}^n}$ is a closed unit ball in $\mathbb{R}^n$. Moreover, the Lipschitz constant of $g(= g(x,p))$ with respect to the variable x is proportional to the Lipschitz constant of G, while the Lipschitz constant of g with respect to the second variable p is proportional to the maximal norm of the elements of G.*

2.4 Differential Topology

2.4.1 Differentiable Manifolds

Let M be a subset of $\mathbb{R}^k$. We use the *induced topology* on M, that is, $A \subset M$ is open if there is an open set $\widetilde{A} \subset \mathbb{R}^k$ so that $A = \widetilde{A} \cap M$. We say that $M \subset \mathbb{R}^k$ is a *C^r-manifold ($r \in \mathbb{N}$) of dimension m* if for each $p \in M$ there is a neighborhood $U \subset M$ of p and a homeomorphism $x : U \to U_0$, where U_0 is an open subset in $\mathbb{R}^m$, so that the inverse $x^{-1} \in C^r(U_0, \mathbb{R}^k)$ and $Dx^{-1}(u) : \mathbb{R}^m \to \mathbb{R}^k$ is injective for any $u \in U_0$. Then we say that (x,U) is a *local C^r-chart around p* and U is a *coordinate neighborhood* of p. It is clear that if $x : U \to \mathbb{R}^m$ and $y : V \to \mathbb{R}^m$ are two local C^r-charts in M with $U \cap V \neq \emptyset$ then $y \circ x^{-1} : x(U \cap V) \to y(U \cap V)$ is a C^rdiffeomorphism. This family of local charts is called a *C^r-atlas* for M [11–13].

If there is a C^r-atlas for M so that $\det D(y \circ x^{-1})(z) > 0$ for any $z \in x(U \cap V)$ and any two local C^r-charts $x : U \to \mathbb{R}^m$ and $y : V \to \mathbb{R}^m$ of this atlas with $U \cap V \neq \emptyset$ then M is *oriented*.

Let $\alpha \in C^1((-\varepsilon, \varepsilon), \mathbb{R}^k)$ be a differentiable curve on M, i.e. $\alpha : (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = p$. Then $\alpha'(0)$ is a *tangent vector* to M at p. The set of all tangent vectors to M at p is the *tangent space to M at p* and it is denoted by T_pM. The *tangent bundle* is

$$TM := \left\{ (p,v) \in \mathbb{R}^k \times \mathbb{R}^k \mid p \in M,\ v \in T_pM \right\}$$

with the *natural projection* $\pi : TM \to M$ given as $\pi(p,v) = p$. If M is a C^r-manifold with $r > 1$ then TM is a C^{r-1}-manifold.

Let M and N be two C^r-manifolds. We say that $f : M \to N$ is a *C^r-mapping* if for each $p \in M$ the mapping $y \circ f \circ x^{-1} : x(U) \to y(V)$ is C^r-smooth, where $x : U \to \mathbb{R}^m$ is a local C^r-chart in M around p and $y : V \to \mathbb{R}^s$ is a local C^r-chart in N with $f(U) \subset V$. This definition is independent of the choice of charts. The set of C^r-mappings is denoted by $C^r(M,N)$. Take $f \in C^r(M,N)$. Let $\alpha : (-\varepsilon, \varepsilon) \to M$ be a differentiable curve on M with $\alpha(0) = p$ and $\alpha'(0) = v$. Then $f \circ \alpha : (-\varepsilon, \varepsilon) \to N$ is a differentiable curve on N with $(f \circ \alpha)(0) = f(p)$, so we can define $Df(p)v := D(f \circ \alpha)(0) \in T_{f(p)}N$. This is independent of curve α. The map $Df(p) : T_pM \to T_{f(p)}N$ is linear, and if $r > 1$, $Df : TM \to TN$ defined as $Df(p,v) := (f(p), Df(p)v)$ is C^{r-1}-smooth.

A set $S \subset M \subset \mathbb{R}^k$ is a *C^r-submanifold* of M of dimension s if for each $p \in S$ there are open sets $U \subset M$ containing p, $V \subset \mathbb{R}^s$ containing 0 and $W \subset \mathbb{R}^{m-s}$ containing

0 and a C^r-diffeomorphism $\phi : U \to V \times W$ so that $\phi(S \cap U) = V \times \{0\}$. We put $\operatorname{codim} S = \dim M - \dim S$.

A C^r-mapping $f : M \to N$ is an *immersion (submersion)* if $Df(p)$ is injective (surjective) for all $p \in M$. If $f : M \to N$ is an injective immersion we say that $f(M)$ is an *immersed submanifold*. If, in addition, $f : M \to f(M) \subset N$ is a homeomorphism, where $f(M)$ has the induced topology, then f is an *embedding*. In this case, $f(M)$ is a submanifold of N.

2.4.2 Vector Bundles

A *C^r-vector bundle of dimension n* is a triple (E, p, B) where E, B are C^r-manifolds and $p \in C^r(E, B)$ with the following properties: for each $q \in B$ there is its open neighborhood $U \subset B$ and a C^r-diffeomorphism $\phi : p^{-1}(U) \to U \times \mathbb{R}^n$ so that $p = \pi_1 \circ \phi$ on $p^{-1}(U)$ where $\pi_1 : U \times \mathbb{R}^n \to U$ is defined as $\pi_1(x,y) := x$. Moreover, each $p^{-1}(x)$ is n-dimensional vector spaces and each $\phi_x : p^{-1}(x) \to \mathbb{R}^n$ given by $\phi(y) = (x, \phi_x(y))$ for any $y \in p^{-1}(x)$ is linear isomorphisms. E is called the *total space*, B is the *base space*, p the *projection* of the bundle, the vector space $p^{-1}(x)$ the *fibre* and ϕ a *local trivialization*. So the vector bundle is *locally trivial*. If $U = B$ then the bundle is *trivial*. The family $\mathscr{A} := \{(\phi, U)\}$ of these local trivializations is a *C^r-vector atlas*. The bundle is *oriented* if there is a C^r-vector atlas $\mathscr{A} := \{(\phi, U)\}$ so that for any two local trivializations (ϕ, U) and (ψ, V) with $U \cap V \neq \emptyset$ the linear mapping $\psi_x \circ \phi_x^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is orientation preserving for each $x \in U \cap V$. A C^r-smooth mapping $s : B \to E$ satisfying $p \circ s = \mathbb{I}_B$ is called a *section* of the bundle.

Typical examples of vector bundles are the tangent bundle (TM, π, M) and the *normal bundle* $(TM^{\perp}, \widetilde{\pi}, M)$ defined as

$$TM^{\perp} := \left\{ (q, v) \in \mathbb{R}^k \times \mathbb{R}^k \mid q \in M, \, v \in T_q M^{\perp} \right\}$$

with the projection $\widetilde{\pi} : TM^{\perp} \to M$ given as $\widetilde{\pi}(q, v) = q$, where $T_x M^{\perp}$ is the orthogonal complement of $T_x M$ in $\mathbb{R}^k$. A section of TM is called a *vector field* on M. When M is oriented, both TM and $TM^{\perp}$ are oriented. Here M is a C^r-manifold with $r > 1$.

2.4.3 Tubular Neighbourhoods

Let M be a submanifold of a smooth manifold N. A *tubular neigbourhood* of M in N is an open subset $\mathscr{O}$ of N together with a submersion $p : \mathscr{O} \to M$ so that [14, pp. 69-71]:

(a) the triple $(\mathscr{O}, p, M)$ is a vector bundle, and
(b) $M \subset \mathscr{O}$ is the zero section of this vector bundle.

Theorem 2.4.1. *Let M be a submanifold of N, then there exists a tubular neighbourhood of M in N.*

If $N = \mathbb{R}^n$ then we can realize a tubular neighbourhood of a submanifold M by using its normal vector bundle $TM^{\perp}$.

2.5 Dynamical Systems

2.5.1 Homogenous Linear Equations

Set $\mathbb{Z}_- := -\mathbb{Z}_+$. Let $J \in \{\mathbb{Z}_+, \mathbb{Z}_-, \mathbb{Z}\}$. Let $A_n \in L(\mathbb{R}^k)$, $n \in J$ be a sequence of invertible matrices. Consider a homogeneous linear difference equation

$$x_{n+1} = A_n x_n \,. \tag{2.5.1}$$

Its *fundamental solution* is defined as $U(n) := A_{n-1} \cdots A_0$ for $n \in \mathbb{N}$, $U(0) = \mathbb{I}$ and $U(n) := A_n^{-1} \cdots A_{-1}^{-1}$ for $-n \in \mathbb{N}$. (2.5.1) has an *exponential dichotomy* on J if there is a projection $P : \mathbb{R}^k \to \mathbb{R}^k$ and constants $L > 0$, $\delta \in (0,1)$ so that

$$\begin{aligned}
\|U(n)PU(m)^{-1}\| &\leq L\delta^{n-m} \text{ for any } m \leq n,\ n,m \in J\,,\\
\|U(n)(\mathbb{I}-P)U(m)^{-1}\| &\leq L\delta^{m-n} \text{ for any } n \leq m,\ n,m \in J\,.
\end{aligned}$$

If $A_n = A$ and its spectrum $\sigma(A)$ has no intersection with the unit circle, i.e. A is *hyperbolic*, then P is the projection onto the generalized eigenspace of eigenvectors inside the unit circle and $\mathscr{N}P$ is the generalized eigenspace of eigenvectors outside the unit circle. Next we have the following *roughness of exponential dichotomies*.

Lemma 2.5.1. *Let $J \in \{\mathbb{Z}_+, \mathbb{Z}_-\}$. Let A be hyperbolic with the dichotomy projection P. Assume that $\{A_n(\xi)\}_{n \in J} \in L(\mathbb{R}^k)$ are invertible matrices and $A_n(\xi) \to A$ in $L(\mathbb{R}^k)$ uniformly with respect to a parameter ξ. Then $x_{n+1} = A_n(\xi)x_n$, with the fundamental solution $U_\xi(n)$, has an exponential dichotomy on J with projection P_ξ and uniform constants $L > 0$, $\delta \in (0,1)$. Moreover, $U_\xi(n)P_\xi U_\xi(n)^{-1} \to P$ as $n \to \pm\infty$ uniformly with respect to ξ.*

Analogical results hold for a homogeneous linear differential equation $\dot{x} = A(t)x$ when $t \in J \in \{(-\infty,0),(0,\infty),\mathbb{R}\}$ and $A(t) \in C(J, L(\mathbb{R}^k))$ is a continuous matrix function. Its *fundamental solution* is a matrix function $U(t)$ satisfying $\dot{U}(t) = A(t)U(t)$ on J. Sometimes we require that $U(0) = \mathbb{I}$ [15]. Now, we recall the *Liouville theorem* that

$$\det U(t) = \det U(t_0)\, \mathrm{e}^{\int_{t_0}^t \operatorname{tr} A(s)\, ds}\,,$$

where $\operatorname{tr} A(t)$ denotes the *trace* which is the sum of diagonal entries of $A(t)$. Finally we mention the *Gronwall inequality* that if

$$\phi(t) \leq \alpha(t) + \int_a^t \psi(s)\phi(s)\, ds$$

for all $t \in [a,b]$ then

$$\phi(t) \leq \alpha(t)\,\mathrm{e}^{\int_a^t \psi(s)\,ds}$$

for all $t \in [a,b]$, where $a < b$, α, ϕ and ψ are nonnegative continuous functions on $[a,b]$, and moreover, α is C^1-smooth satisfying $\alpha'(t) \geq 0$ for any $t \in [a,b]$.

2.5.2 Chaos in Diffeomorphisms

Consider a C^r*-diffeomorphism* f on $\mathbb{R}^m$ with $r \in \mathbb{N}$, i.e. a mapping $f \in C^r(\mathbb{R}^m, \mathbb{R}^m)$ which is invertible and $f^{-1} \in C^r(\mathbb{R}^m, \mathbb{R}^m)$. For any $z \in \mathbb{R}^m$ we define its *k-iteration* as $f^k(z) := f(f^{k-1}(z))$. The set $\{f^n(z)\}_{n=\infty}^{\infty}$ is an *orbit* of f. If $x_0 = f(x_0)$ then x_0 is a *fixed point* of f. It is *hyperbolic* if the *linearization* $Df(x_0)$ of f at x_0 has no eigenvalues on the unit circle. The *global stable (unstable) manifold* $W_{x_0}^{s(u)}$ of a hyperbolic fixed point x_0 is defined by [16]

$$W_{x_0}^{s(u)} := \{z \in \mathbb{R}^m \mid f^n(z) \to x_0 \quad \text{as} \quad n \to \infty(-\infty)\}\,,$$

respectively. Recall that $W_{x_0}^s$ and $W_{x_0}^u$ are immersed C^r-submanifolds in $\mathbb{R}^m$. Furthermore, let y_0 be another hyperbolic fixed point of f. If $x \in W_{x_0}^s \cap W_{y_0}^u \setminus \{x_0, y_0\}$ then it is a *heteroclinic point* of f and then the orbit $\{f^n(x)\}_{n=\infty}^{\infty}$ is called heteroclinic orbit. Clearly $f^n(z) \to x_0$ as $n \to \infty$ and $f^n(z) \to y_0$ as $n \to -\infty$. If $T_xW_{x_0}^s \cap T_xW_{y_0}^u = \{0\}$ then x is a *transversal heteroclinic point* of f. Note the following useful results [15, 17].

Lemma 2.5.2. $x \in W_{x_0}^s \cap W_{y_0}^u \setminus \{x_0, y_0\}$ *is a transversal heteroclinic point if and only if the linear difference equation* $x_{n+1} = Df(f^n(x))x_n$ *has an exponential dichotomy on* $\mathbb{Z}$*, i.e. if and only if the only bounded solution of* $x_{n+1} = Df(f^n(x))x_n$ *on* $\mathbb{Z}$ *is the zero one.*

When $x_0 = y_0$, the word "heteroclinic" is replaced with *homoclinic*. We refer the readers to [15] for more details and proofs of the above subject.

Let $\mathscr{E} = \{0,1\}^{\mathbb{Z}}$ be a compact metric space of the set of doubly infinite sequences of 0 and 1 endowed with the metric [18]

$$d_{\mathscr{E}}(\{e_n\}, \{e'_n\}) := \sum_{n \in \mathbb{Z}} \frac{|e_n - e'_n|}{2^{|n|}}\,.$$

On $\mathscr{E}$ it is defined as the so-called *Bernoulli shift map* $\sigma : \mathscr{E} \to \mathscr{E}$ by $\sigma(\{e_j\}_{j\in\mathbb{Z}}) = \{e_{j+1}\}_{j\in\mathbb{Z}}$ with extremely rich dynamics [19].

Theorem 2.5.3. σ *is a homeomorphism having*

(i) *a countable infinity of periodic orbits of all possible periods,*
(ii) *an uncountable infinity of nonperiodic orbits, and*
(iii) *a dense orbit.*

Now we can state the following result of the existence of *the deterministic chaos* for diffeomorphisms, the *Smale-Birkhoff homoclinic theorem.*

Theorem 2.5.4. *Suppose* $f : \mathbb{R}^m \to \mathbb{R}^m$, $r \in \mathbb{N}$ *are a* C^r*-diffeomorphism having a transversal homoclinic point to a hyperbolic fixed point. Then there is a* $k \in \mathbb{N}$ *so that* f^k *has an invariant set* Λ*, i.e.* $f^k(\Lambda) = \Lambda$*, so* $f^k \circ \varphi = \varphi \circ \sigma$ *for a homeomorphism* $\varphi : \mathscr{E} \to \Lambda$ *(Figure* 2.1).

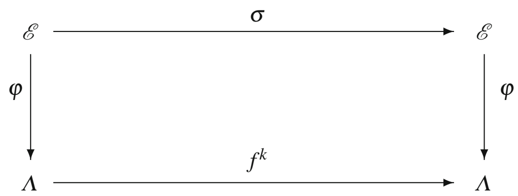

Fig. 2.1 Commutative diagram of deterministic chaos.

The set Λ is the *Smale horseshoe* and we say that f has *horseshoe dynamics* on Λ. By Theorem 2.5.4, f^k on Λ has the same dynamical properties as σ on $\mathscr{E}$, i.e. Theorem 2.5.3 gives chaos for f. Moreover, it is possible to show a *sensitive dependence on initial conditions* of f on Λ in the sense that there is an $\varepsilon_0 > 0$ so that for any $x \in \Lambda$ and any neighborhood U of x, there exists $z \in U \cap \Lambda$ and an integer $q \geq 1$, consequently $|f^q(x) - f^q(z)| > \varepsilon_0$.

2.5.3 Periodic ODEs

It is well known [20] that the Cauchy problem

$$\dot{x} = g(x,t), \quad x(0) = z \in \mathbb{R}^m \tag{2.5.2}$$

for $g \in C^r(\mathbb{R}^m \times \mathbb{R}, \mathbb{R}^m)$, $r \in \mathbb{N}$ has a unique solution $x(t) = \phi(z,t)$ defined in a maximal interval $0 \in I_z \subset \mathbb{R}$. We suppose for simplicity that $I_z = \mathbb{R}$. This is true, for instance, when g is globally Lipschitz continuous in x, i.e. there is a constant $L > 0$ so that $|g(x,t) - g(y,t)| \leq L|x-y|$ for any $x, y \in \mathbb{R}^m$, $t \in \mathbb{R}$. Moreover, we assume that g is T-periodic in t, i.e. $g(x, t+T) = g(x,t)$ for any $x \in \mathbb{R}^m$, $t \in \mathbb{R}$. Then the dynamics of (2.5.2) is determined by the dynamics of the diffeomorphism $f(z) = \phi(z,T)$ which is called the *time or Poincarè map* of (2.5.2). Now we can transform the results of Section 2.5.2 to (2.5.2). So T-periodic solutions (*periodics* for short) of (2.5.2) are fixed points of f. A T-periodic solution of (2.5.2) is hyperbolic if the corresponding fixed point of f is hyperbolic. Periodics of f are *subharmonic solutions* (*subharmonics* for short) of (2.5.2). Similarly we mean a chaos of (2.5.2) as a chaos for f. Finally, let $\gamma_0(t) = \phi(x_0,t)$ be a T-periodic solution of

$$\dot{x} = g(x,t). \tag{2.5.3}$$

Consider its *variational equation* along γ_0 given by $\dot{v} = g_x(\gamma_0(t),t)\,v$ with the fundamental matrix solution $V(t)$. Then $Df(x_0) = V(T)$ [21]. Now we have the following result from the proof of Theorem 2.1 on p. 288 of [22].

Lemma 2.5.5. *Let X be a Banach space. Let $C_b(\mathbb{R},X)$ be the space of all continuous and bounded functions from $\mathbb{R}$ to X endowed with the supremum norm. Consider*

$$\dot{u} = A(t)u \tag{2.5.4}$$

with the fundamental solution $U(t)$, where $A(t) \in C(\mathbb{R},L(X))$ is T-periodic. Then the following statements are equivalent

(i) *The nonhomogeneous equation*

$$\dot{u} = A(t)u + h$$

has a unique solution $u \in C_b(\mathbb{R},X)$ for any $h \in C_b(\mathbb{R},X)$.

(ii) *The zero solution of* (2.5.4) *is hyperbolic, i.e. $\sigma(U(T))$ has no eigenvalues on the unit circle.*

(iii) *Equation* (2.5.4) *has an exponential dichotomy on $\mathbb{R}$.*

Lemma 2.5.5 is useful for verifying the hyperbolicity of γ_0 of (2.5.3).

2.5.4 Vector Fields

When (2.5.2) is *autonomous*, i.e. g is independent of t, (2.5.2) has the form

$$\dot{x} = g(x), \quad x(0) = z \in \mathbb{R}^m. \tag{2.5.5}$$

g is called a *C^r-vector field* on $\mathbb{R}^m$ for $g \in C^r(\mathbb{R}^m,\mathbb{R}^m)$, $r \in \mathbb{N}$. We suppose for simplicity that the unique solution $x(t) = \phi(z,t)$ of (2.5.5) is defined on $\mathbb{R}$. $\phi(z,t)$ is called the *orbit based at z*. Then instead of the time map of (2.5.5), we consider the *flow* $\phi_t : \mathbb{R}^m \to \mathbb{R}^m$ defined as $\phi_t(z) := \phi(z,t)$ with the property $\phi_t(\phi_s(z)) = \phi_{t+s}(z)$.

A point p is an *ω-limit point* of x is there are points $\{\phi_{t_i}(x)\}_{i\in\mathbb{N}}$ on the orbit of x so that $\phi_{t_i}(x) \to p$ and $t_i \to \infty$. A point q is an *α-limit point* if such a sequence exists with $\phi_{t_i}(x) \to q$ and $t_i \to -\infty$. The α- (resp. ω-) limit sets $\alpha(x)$, $\omega(x)$ are the sets of α- and ω-limit points of x.

A point x_0 with $g(x_0) = 0$ is an *equilibrium* of (2.5.5). It is *hyperbolic* if the linearization $Dg(x_0)$ of (2.5.5) at x_0 has no eigenvalues on imaginary axis.

The *global stable (unstable) manifold* $W_{x_0}^{s(u)}$ of a hyperbolic equilibrium x_0 is defined by

$$W_{x_0}^{s(u)} := \{z \in \mathbb{R}^m \mid \phi(z,t) \to x_0 \quad \text{as} \quad t \to \infty(-\infty)\},$$

respectively. These sets are immersed submanifolds of $\mathbb{R}^m$. For any $x \in W_{x_0}^{s(u)}$, we know that

$$T_x W_{x_0}^{s(u)} = \Big\{ v(0) \in \mathbb{R}^m \mid v(t) \text{ is a bounded solution}$$
$$\text{of } \dot{v} = Dg(\phi(x,t))v \text{ on } (0,\infty), ((-\infty,0)), \text{ respectively} \Big\}.$$

Moreover, the set

$$\left(T_x W_{x_0}^s + T_x W_{x_0}^u\right)^\perp$$

is the linear space of initial values $w(0)$ of all bounded solutions $w(t)$ of the *adjoint equation* $\dot{w} = -Dg(\phi(x,t))^* w$ on $\mathbb{R}$ [23].

A local dynamics near a hyperbolic equilibrium x_0 of (2.5.5) is explained by the *Hartman-Grobman theorem for flows* [24].

Theorem 2.5.6. *If $x_0 = 0$ is a hyperbolic equilibrium of (2.5.5) then there is a homeomorphism h defined on a neighborhood U of 0 in $\mathbb{R}^m$ so that*

$$h(\phi(z,t)) = e^{tDg(0)} h(z)$$

for all $z \in U$ and $t \in J_z$ with $\phi(z,t) \in U$, where $0 \in J_z$ is an interval.

For nonhyperbolic equilibria we have the following *center manifold theorem for flows* [24].

Theorem 2.5.7. *Let $x_0 = 0$ be an equilibrium of a C^r-vector field (2.5.5) on $\mathbb{R}^m$. Divide the spectrum of $Dg(0)$ into three parts σ_s, σ_u, σ_c so that $\Re\lambda < 0; > 0; = 0$ if $\lambda \in \sigma_s, \sigma_u, \sigma_c$, respectively. Let the generalized eigenspaces of σ_s, σ_u, σ_c be E^s, E^u, E^c, respectively. Then there are C^r-smooth manifolds: the stable W_0^s, the unstable W_0^u, the center W_0^c tangent at 0 to E^s, E^u, E^c, respectively. These manifolds are invariants for the flow of (2.5.5), i.e. $\phi_t\left(W_0^{s;u;c}\right) \subset W_0^{s;u;c}$ for any $t \in \mathbb{R}$. The stable and unstable ones are unique, but the center one need not be. In addition, when g is embedded into a C^r-smooth family of vector fields g_ε with $g_0 = g$, these invariant manifolds are C^r-smooth also with respect to ε.*

Under the assumptions of Theorem 2.5.7 near $x_0 = 0$ we can write (2.5.5) in the form

$$\begin{aligned} \dot{x}_s &= A_s x_s + g_s(x_s, x_u, x_c, \varepsilon), \\ \dot{x}_u &= A_u x_u + g_u(x_s, x_u, x_c, \varepsilon), \\ \dot{x}_c &= A_c x_s + g_c(x_s, x_u, x_c, \varepsilon), \end{aligned} \tag{2.5.6}$$

where $A_{s;u;c} := Dg(0)/E^{s;u;c}$ and $x_{s;u;c} \in U_{s;u;c}$ for open neighborhoods $U_{s;u;c}$ of 0 in $E^{s;u;c}$, respectively. Here we suppose that (2.5.5) is embedded into a C^r-smooth family. So g_j are C^r-smooth satisfying $g_j(0,0,0,0) = 0$ and $D_{x_j} g_k(0,0,0,0) = 0$ for $j,k = s,u,c$. According to Theorem 2.5.7, the *local center manifold* $W_{loc,\varepsilon}^c$ near $(0,0,0)$ of (2.5.6) is a graph

$$W_{loc,\varepsilon}^c = \{(h_s(x_c,\varepsilon), h_u(x_c,\varepsilon), x_c) \mid x_c \in U_c\}$$

for $h_{s;u} \in C^r\left(U_c \times V, E^{s;u}\right)$ and V is an open neighborhood of $\varepsilon = 0$. Moreover, it holds $h_{s;u}(0,0) = 0$ and $D_{x_c} h_{s;u}(0,0) = 0$. The *reduced equation* is

$$\dot{x}_c = A_c x_s + g_c(h_s(x_c,\varepsilon), h_u(x_c,\varepsilon), x_c, \varepsilon), \tag{2.5.7}$$

which locally determines the dynamics of (2.5.6), i.e. $W^c_{loc,\varepsilon}$ contains all solutions of (2.5.6) staying in $U_s \times U_u \times U_c$ for all $t \in \mathbb{R}$. In particular periodics, homoclinics and heteroclinics of (2.5.6) near $(0,0,0)$ solve (2.5.7).

Finally we say that (2.5.5) has a *first integral* $H : \mathbb{R}^n \to \mathbb{R}$ if $H \circ \phi_t = H$ for any $t \in \mathbb{R}$.

2.5.5 Global Center Manifolds

Let $C^k_b(\mathbb{R}^m, \mathbb{R}^n)$ be the Banach space of C^k functions from $\mathbb{R}^m$ to $\mathbb{R}^n$ which are bounded together with their derivatives, endowed with the usual sup-norm. We consider the following system of ODEs:

$$\begin{aligned} \dot{x} &= A(y)x + F(x,y), \\ \dot{y} &= G(x,y), \end{aligned} \tag{2.5.8}$$

where $x \in \mathbb{R}^n, y \in \mathbb{R}^m$ and assume that the following conditions hold:

(i) $F \in C^r_b(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$, $G \in C^r_b(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^m)$, $A \in C^r_b(\mathbb{R}^m, L(\mathbb{R}^n))$ with $r \geq 1$.
(ii) There exists $\delta > 0$ so that for any $y \in \mathbb{R}^m$ and for any $\lambda(y) \in \sigma(A(y))$, one has $|\Re\lambda(y)| > \delta$. Moreover, the derivatives of order r of $A(y)$, $F(x,y)$, $G(x,y)$ are continuous in x, uniformly with respect to $y \in \mathbb{R}^m$.
(iii) $\sup\limits_{(x,y)\in\mathbb{R}^n\times\mathbb{R}^m} \{|F(0,y)|, |F_x(0,y)|, |G(x,y)|, |G_x(x,y)|, |G_y(x,y)|\} \leq \sigma$.

Now we can state the following result.

Theorem 2.5.8. *There exists a* $\sigma_0 > 0$ *so that, if the above conditions hold with* $\sigma \leq \sigma_0$*, there exists a* C^r*-function* $H(y)$*, defined for* $y \in \mathbb{R}^m$ *so that the manifold*

$$\mathscr{C} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid x = H(y), y \in \mathbb{R}^m\}$$

is invariant for the system (2.5.8) *and has the following property:*

(P) There exists $\rho > 0$ *so that if* $(x(t), y(t))$ *is a solution of* (2.5.8) *satisfying* $||x||_\infty \leq \rho$*, then* $x(t) = H(y(t))$*.*

$\mathscr{C}$ is called the *global center manifold* of (2.5.8). We refer the readers to [25] for more details.

2.5.6 Two-Dimensional Flows

In this section we consider a planar ODE

$$\dot{x} = f(x), \tag{2.5.9}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and $f = (f_1, f_2)$ is smooth. First we have the following useful result of *Poincarè and Bendixson* [20,21].

Theorem 2.5.9. *A nonempty compact ω- or α-limit set of a planar flow, which contains no equilibria, is a closed orbit.*

The next *Bendixson criterion* rules out the occurrence of closed orbits in some cases [20,21].

Theorem 2.5.10. *If in a simply connected region $D \subset \mathbb{R}^2$ the divergence* $\operatorname{div} f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ *of* (2.5.9) *is not identically zero and does not change sign, then* (2.5.9) *has no closed orbits lying entirely in D.*

2.5.7 Averaging Method

In this section, we consider systems of the form [21,24,26]

$$\dot{x} = \varepsilon f(x, t, \varepsilon), \tag{2.5.10}$$

where $f \in C^r\left(\mathbb{R}^{n+2}, \mathbb{R}^n\right)$, $r \geq 2$.

Definition 2.5.11. $f \in C^r\left(\mathbb{R}^{n+2}, \mathbb{R}^n\right)$, $r \geq 2$ is said to be *KBM-vector field*, (KBM stands for Krylov, Bogolyubov and Mitropolsky) if the average

$$f_0(x) := \lim_{t \to \infty} \frac{1}{t} \int_0^t f(x, s, 0)\, ds$$

exists for any $x \in \mathbb{R}^n$. The associated autonomous *averaged system* is defined as

$$\dot{y} = \varepsilon f_0(y). \tag{2.5.11}$$

We have the following results.

Theorem 2.5.12. *Suppose for* (2.5.10) *that f is T-periodic in t. Then f is a KBM-vector field. Moreover, for any $\varepsilon > 0$ sufficiently small, we get*

(i) *If $x(t)$ and $y(t)$ are solutions of* (2.5.10) *and* (2.5.11) *with $|x(0) - y(0)| = O(\varepsilon)$, then $|x(t) - y(t)| = O(\varepsilon)$ on a time scale $t \sim 1/\varepsilon$.*
(ii) *If p_0 is a hyperbolic equilibrium of* (2.5.11) *then* (2.5.10) *possesses a unique hyperbolic periodic orbit $\gamma_\varepsilon(t) = p_0 + O(\varepsilon)$ of the same stability type as p_0.*
(iii) *If $x_s(t) \in W^s(\gamma_\varepsilon)$ is a solution of* (2.5.10) *lying on the stable manifold of γ_ε, $y_s(t) \in W^s(p_0)$ is a solution of* (2.5.11) *lying on the stable manifold of p_0 and $|x(0) - y(0)| = O(\varepsilon)$, then $|x(t) - y(t)| = O(\varepsilon)$ for any $t \geq 0$. Similar results apply to solutions lying in the unstable manifolds in the time interval $t \leq 0$.*

The above theorem can be generalized to more complicated hyperbolic sets [21, 26]. For instance, the following holds:

Theorem 2.5.13. *Suppose f, f_0 are C^1-smooth and $f_0(y_0) = 0$ with $\Re\sigma(Df_0(y_0)) < 0$. If x_0 is in a domain of attraction of y_0, then for any $\varepsilon > 0$ sufficiently small, $|x_\varepsilon(t) - y(t)| = o(1)$ for any $t \geq 0$, where $x_\varepsilon(t)$ and $y(t)$ are solutions of* (2.5.10) *and* (2.5.11) *with $x(0) = y(0) = x_0$, respectively.*

2.5.8 Carathéodory Type ODEs

In this section we recall some results on ODEs only measurable depending on t.

Definition 2.5.14. Let $\mathscr{I}$ be an interval in $\mathbb{R}$. A mapping $f : \mathscr{I} \times \mathbb{R}^n \to \mathbb{R}^n$ is said to have the *Carathéodory property* if the following assumptions hold [27, 28]:

(i) For every $t \in \mathscr{I}$ the mapping $f(t, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is continuous.
(ii) For every $x \in \mathbb{R}^n$ the mapping $f(\cdot, x) : \mathscr{I} \to \mathbb{R}^n$ is measurable with respect to the Borel σ-algebras on $\mathscr{I}$ and $\mathbb{R}^n$.

We note that if f has a Carathéodory property and $x : \mathscr{I} \to \mathbb{R}^n$ is measurable then $f(t, x(t))$ is measurable as well.

Definition 2.5.15. A function $x : \mathscr{I} \to \mathbb{R}^n$ is *absolutely continuous* [2] if for any $\varepsilon > 0$ there is a $\delta > 0$ so that for any $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_k < \beta_k$, $\alpha_i, \beta_i \in \mathscr{I}$ so that $\sum_{i=1}^k (\beta_i - \alpha_i) < \delta$, it holds $\sum_{i=1}^k |x(\beta_i) - x(\alpha_i)| < \varepsilon$.

It is well known that an absolutely continuous function on $\mathscr{I}$ has almost everywhere a derivative. By a solution of an ODE $\dot{x} = f(t, x)$ with a Carathéodory mapping f, we mean an absolutely continuous function $x(t)$ satisfying this ODE almost everywhere.

2.6 Singularities of Smooth Maps

Here we recall some results from the theory of smooth maps [14].

2.6.1 Jet Bundles

Definition 2.6.1. Let M, N be smooth manifolds with dimensions m and n, respectively. Let $f, g \in C^\infty(M, N)$ with $f(p) = g(p) = q$. f has *kth order contact* with g at p if in local coordinates

$$\frac{\partial^{|\alpha|} f_i}{\partial x^\alpha}(p) = \frac{\partial^{|\alpha|} g_i}{\partial x^\alpha}(p)$$

for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $|\alpha| = \alpha_1 + \cdots + \alpha_m \leq k$ and $1 \leq i \leq n$, where f_i, g_i are the coordinate functions of f, g, respectively, and $x = (x_1, \ldots, x_m)$. This is written as $f \sim_k g$ at p.

Let $J^k(M,N)_{p,q}$ denote the set of equivalence classes under "$\sim_k$ at p" in $C^\infty(M,N)$. Let $J^k(M,N) := \bigcup_{(p,q)\in M\times N} J^k(M,N)_{p,q}$ - disjoint union. An element of $J^k(M,N)$ is called a *k-jet* and $J^k(M,N)$ is the *jet bundle*. Note that given $f \in C^\infty(M,N)$ there is a mapping $j^k f : M \to J^k(M,N)$ called the *k-jet of f* defined by $j^k f(p) :=$ the equivalence class of f in $J^k(M,N)_{p,f(p)}$ for every $p \in M$. Note that $J^0(M,N) = M \times N$. For any k-jet $\xi \in J^k(M,N)$, there is its *source* $p \in M$ and the *target* $q \in M$. Let f be the representative of $\xi \in J^1(M,N)$. Then we define the *rank* of ξ as $\operatorname{rank}\xi := \operatorname{rank} Df(p)$ and *corank* as $\operatorname{corank}\xi := \min\{m,n\} - \operatorname{rank}\xi$.

Theorem 2.6.2. *Let* $L^r(\mathbb{R}^m,\mathbb{R}^n) := \{A \in L(\mathbb{R}^m,\mathbb{R}^n) \mid \operatorname{corank} A = r\}$. *Then* $L^r(\mathbb{R}^m,\mathbb{R}^n)$ *is a submanifold of* $L(\mathbb{R}^m,\mathbb{R}^n)$ *with* $\operatorname{codim} L^r(\mathbb{R}^m,\mathbb{R}^n) = (m - \min\{n,m\} + r)(n - \min\{n,m\} + r)$.

Theorem 2.6.3. *Let* $S_r := \left\{\xi \in J^1(M,N) \mid \operatorname{corank}\xi = r\right\}$. *Then* S_r *is a submanifold of* $J^1(M,N)$ *with* $\operatorname{codim} S_r = (m - \min\{n,m\} + r)(n - \min\{n,m\} + r)$.

2.6.2 Whitney C^∞ Topology

Let M,N be smooth manifolds. Let $k \in \mathbb{Z}_0$. Let U be an open subset of $J^k(M,N)$. Then the family of sets

$$\left\{f \in C^\infty(M,N) \mid j^k f(M) \subset U\right\}$$

forms a basis for a *Whitney C^k topology* on $C^\infty(M,N)$. The union of all open subsets of $C^\infty(M,N)$ in some Whitney C^k topology forms a basis of a *Whitney C^∞ topology* on $C^\infty(M,N)$. We note that a subset of topological space is *residual* if it is the countable intersection of open dense subsets. A topological space is a *Baire space* if its every residual set is dense.

Theorem 2.6.4. $C^\infty(M,N)$ *is a Baire space in the Whitney C^∞ topology.*

2.6.3 Transversality

Definition 2.6.5. Let M,N be smooth manifolds and $f : M \to N$ be a smooth map. Let S be a submanifold of N and $x \in M$. Then f *transversally intersects* S at $x \in M$ denoted by $f \pitchfork S$ at x, if either

(i) $f(x) \notin S$, or

(ii) $f(x) \in S$ and $T_{f(x)}N = T_{f(x)}S + Df(x)T_xM$.

If $f \,\overline{\pitchfork}\, S$ for any $x \in M$, then f *transversally intersects* S denoted by $f \,\overline{\pitchfork}\, S$.

Theorem 2.6.6. *If $f \,\overline{\pitchfork}\, S$ then $f^{-1}(S)$ is a smooth submanifold with codimension* $\operatorname{codim} S$.

Now we state the *Thom transversality theorem.*

Theorem 2.6.7. *Let W be a submanifold of $J^k(M,N)$. Then*

$$T_W := \left\{ f \in C^\infty(M,N) \mid j^k f \,\overline{\pitchfork}\, W \right\}$$

is a residual subset of $C^\infty(M,N)$ in the Whitney C^∞ topology. If, in addition, W is closed, then T_W is open.

2.6.4 Malgrange Preparation Theorem

Theorem 2.6.8. *Let F be a smooth real-valued function defined on a neighbourhood of 0 in $\mathbb{R} \times \mathbb{R}^n$ so that $F(t,0) = g(t)t^k$, where $g(0) \neq 0$ and g is smooth on some neighbourhood of 0 in $\mathbb{R}$. Then there is a smooth G with $G(0) \neq 0$ and smooth $\lambda_0, \dots, \lambda_{k-1}$ so that*

$$(GF)(t,x) = t^k + \sum_{i=0}^{k-1} \lambda_i(x)t^i \,.$$

As a consequence of the generalized Malgrange theorem, we have the Whitney theorem [14, p. 108].

Theorem 2.6.9. *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth even function, then there is a smooth function $g : \mathbb{R} \to \mathbb{R}$ satisfying $f(x) = g(x^2)$.*

2.6.5 Complex Analysis

Here we recall some basic results from the theory of complex functions [2]. Let $\Omega \subset \mathbb{C}$ be a *region*, i.e. Ω is open and connected. A complex function $f : \Omega \to \mathbb{C}$ is *holomorphic* if for any $z_0 \in \Omega$ there is a *derivative* $f'(z_0) \in \mathbb{C}$ of f at z_0 defined by

$$\lim_{z \to z_0} \frac{f(z - f(z_0)}{z - z_0} = f'(z_0) \,.$$

The class of all holomorphic functions on Ω is denoted by $H(\Omega)$. Any $f \in H(\Omega)$ is *analytic*, i.e. $f(z) = \sum_{i=0}^{\infty} c_i(z - z_0)^i$ for any $z_0 \in \Omega$ and z near z_0. Next, for any nonzero $f \in H(\Omega)$ the set $Z(f) := \{z \in \Omega \mid f(z) = 0\}$ consists at most of isolated points. Moreover, if $z_0 \in Z(f)$ then $f(z) = (z - z_0)^m g(z)$ for $g \in H(\Omega)$, $g(z_0) \neq 0$, and m is the *order of the zero* which has f at z_0.

A function $f : \Omega \to \mathbb{C}$ has a *pole of order m* in $z_0 \in \Omega$ if

$$f(z) = \sum_{i=-m}^{\infty} c_i (z - z_0)^i$$

with $c_{-m} \neq 0$, for any $z \neq z_0$ near z_0. We denote by $\mathrm{Res}\,(f, z_0) := c_{-1}$ the *complex residue* of $f(z)$ at the pole z_0.

A function $f : \Omega \to \mathbb{C}$ is *meromorphic* if there is a subset $A \subset \Omega$ so that:

1. A consists of isolated points;
2. $f \in H(\Omega \setminus A)$,
3. f has poles in A.

Note that each rational function, i.e. a quotient of two polynomials, is meromorphic on $\mathbb{C}$.

Next z_0 is an *essential singularity* of f if $f(z) = \sum_{i=-\infty}^{\infty} c_i (z - z_0)^i$ for any $z \neq z_0$ near z_0 and with infinitely many nonzero c_m, $m < 0$.

A *path* γ is a piecewise continuously differentiable curve in the plane, i.e. $\gamma \in C([a,b], \mathbb{C})$ and there are finite $a = s_0 < s_1 < \cdots < s_n = b$ so that $\gamma \in C^1([s_i, s_{i+1}], \mathbb{C})$ for each $i = 0, \ldots, n-1$. A path is *closed* if $\gamma(a) = \gamma(b)$. The integral of a holomorphic function f over the path γ is defined as

$$\int_\gamma f(z)\,dz := \sum_{i=0}^{n-1} \int_{s_i}^{s_{i+1}} f(\gamma(t))\gamma'(t)\,dt\,.$$

If a path γ counterclockwise encloses all poles of a meromorphic function $f(z)$, then the *Cauchy residue theorem* states that

$$\int_\gamma f(z)\,dz = 2\pi\imath \sum_{z_0 \in A} \mathrm{Res}\,(f, z_0)\,.$$

Particularly, if a path γ counterclockwise encloses only a pole z_0 of a meromorphic function $f(z)$, then

$$\mathrm{Res}\,(f, z_0) = \frac{1}{2\pi\imath} \int_\gamma f(z)\,dz\,. \tag{2.6.1}$$

Finally we states the *Schwarz reflection principle*.

Theorem 2.6.10. *Suppose L is a segment on the real axis, Ω^+ is a region in $\Pi^+ := \{z \in \mathbb{C} \mid \Im z > 0\}$, and every $z \in L$ is the center of an open disc D_z so that $\Pi^+ \cap D_z$ lies in Ω^+. Let $\Omega^- := \{z \mid \bar{z} \in \Omega^+\}$. Suppose $f \in H(\Omega^+)$ and* $\lim_{n\to\infty} \Im f(z_n) = 0$ *for every sequence $\{z_n\}$ in Ω^+ which converges to a point in L. Then there is a function $F \in H(\Omega^+ \cup L \cup \Omega^-)$, so that $F(z) = f(z)$ in Ω^+ and $F(\bar{z}) = \overline{F(z)}$ for any $z \in \Omega^+ \cup L \cup \Omega^-$.*

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Chapter 3
Chaos in Discrete Dynamical Systems

This chapter is devoted to functional analytical methods for showing chaos in discrete dynamical systems involving difference equations, diffeomorphisms, regular and singular ODEs with impulses, and inflated mappings as well.

3.1 Transversal Bounded Solutions

3.1.1 Difference Equations

In this section, we consider difference equations of the form

$$x_{k+1} = f(x_k) + h(x_k, \mu, k) \tag{3.1.1}$$

with $x_k \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$. We make the following assumptions of (3.1.1):

(i) f, h are C^3-smooth in all non-discrete arguments.
(ii) $f(0) = 0$ and $h(\cdot, 0, \cdot) = 0$.
(iii) The eigenvalues of $Df(0)$ are non-zero and all lie off the unit circle.
(iv) The unperturbed equation $x_{k+1} = f(x_k)$ has a homoclinic solution. That is, there exists a nonzero sequence $\gamma = \{\gamma_k\}_{k\in\mathbb{Z}}$ so that $\lim_{k\to\pm\infty} \gamma_k = 0$ and $\gamma_{k+1} = f(\gamma_k)$. Moreover, $Df(\gamma_k)$, $k \in \mathbb{Z}$ are nonsingular.

Our aim is to find a set of parameters μ for which (3.1.1) has a transversal bounded solution $\{\bar{x}_k\}_{k\in\mathbb{Z}}$ near $\{\gamma_k\}_{k\in\mathbb{Z}}$, i.e. the linearization of (3.1.1) along $\{\bar{x}_k\}_{k\in\mathbb{Z}}$ given by

$$v_{k+1} = \big(Df(\bar{x}_k) + D_x h(\bar{x}_k, \mu, k)\big) v_k, \quad k \in \mathbb{Z}$$

has the only bounded solution $v_k = 0$, $\forall\, k \in \mathbb{Z}$ (cf Lemma 2.5.2). When h is independent of k, i.e. (3.1.1) is a mapping, we know from Section 2.5.2 that the existence of such a bounded solution means the existence of a transversal homoclinic orbit and thus chaos. In general, (3.1.1) can be associated with quasiperiodically perturbed systems [1–3]. To derive these sets, higher dimensional Melnikov mappings are introduced. Simple zero points of those mappings give wedge-shaped regions in $\mathbb{R}^m$ for μ representing the desired sets.

We establish a complete analogy between the Melnikov theories for difference equations and ordinary differential equations (cf Section 4.1). Two-dimensional mappings are considered in [2, 4, 5]. Mappings in arbitrary finite dimensions are considered in [6–8] but the dimension is 1 in [8], which is released in this section, for the intersection of tangent spaces and stable and unstable manifolds along a homoclinic solution to a hyperbolic fixed point of the unperturbed mapping, and while the transversality is not proved in [6]. In this section, no restriction is given on the dimension of the phase space or on the dimension of intersection of stable and unstable manifolds. Other types of homoclinic bifurcations are given in [9].

3.1.2 Variational Equation

The norm and scalar product of $\mathbb{R}^n$ are denoted by $|\cdot|$, $\langle\cdot,\cdot\rangle$, respectively. Let us consider the unperturbed equation

$$x_{k+1} = f(x_k)\,. \tag{3.1.2}$$

For (3.1.2) we adopt the standard notation W^s, W^u for the local stable and local unstable manifolds, respectively, of the origin and $d_s = \dim W^s$, $d_u = \dim W^u$. Since $x = 0$ is a hyperbolic equilibrium, $\{\gamma_k\}_{k\in\mathbb{Z}}$ must approach the origin along W^s as $k \to +\infty$ and along W^u as $k \to -\infty$. By the *variational equation* of (3.1.2) along $\{\gamma_k\}_{k\in\mathbb{Z}}$ we mean the linear difference equation

$$u_{k+1} = Df(\gamma_k)u_k\,. \tag{3.1.3}$$

We note that as $k \to \pm\infty$, $Df(\gamma_k) \to Df(0)$, a hyperbolic matrix. Thus, the following result yields two solutions for (3.1.3), one for $k \in \mathbb{Z}_+$ and one for $k \in \mathbb{Z}_-$.

Lemma 3.1.1. *Let $k \to A(k)$ be a matrix valued function on $\mathbb{Z}_+$ and suppose there exists a constant nonsingular matrix, A_0, and a scalar $a > 0$ so that* $\sup_{k\in\mathbb{Z}_+} |A(k) - A_0|\,\mathrm{e}^{4ak} < \infty$. *Then there exists a fundamental solution, $X(k)$ for k large, to the difference equation $x_{k+1} = A(k)x_k$ so that $\lim\limits_{k\to\infty} X(k)A_0^{-k} = \mathbb{I}$.*

Proof. The proof is very similar to [10, Lemma 3.1.1] and [11, 1. Lemma], but we present it here for the readers' convenience. Let P be a matrix so that $P^{-1}A_0P = J$, where J is the Jordan form with the block-diagonal form $J = \mathrm{diag}\,(J_1, J_2, \ldots, J_r)$. Let k_i be the order of J_i and λ_i is the eigenvalue corresponding to J_i. We arrange the Jordan blocks so that $|\lambda_i| \le |\lambda_{i+1}|$. By putting $y = P^{-1}x$ and $B(k) = P^{-1}A(k)P$, the equation $x_{k+1} = A(k)x_k$ has the form

$$y_{k+1} = B(k)y_k = Jy_k + (B(k) - J)y_k\,. \tag{3.1.4}$$

We fix one block J_i and define $p_i = k_1 + k_2 + \cdots + k_{i-1}$. Similarly we define q_i satisfying $|\lambda_{q_{i-1}}| < |\lambda_i|$ and $|\lambda_{q_i}| = |\lambda_i|$. We split the matrix J^k into $U_1(k), U_2(k)$, where

$$U_1(k) = (J_1^k, J_2^k, \ldots, J_{q_{i-1}}^k, 0, \ldots, 0),$$
$$U_2(k) = (0, 0, \ldots, 0, J_{q_i}^k, \ldots, J_r^k).$$

Since the spectrum $\sigma(U_1(1))$ is contained inside the circle with the radius $|\lambda_{q_{i-1}}|$, we can assume by [12, 3.126 Lemma]

$$|U_1(1)| \le |\lambda_{q_{i-1}}| + b \le |\lambda_i| - b$$

for $b > 0$ sufficiently small. Consequently, we obtain for $k \ge 0$ that $|U_1(k)| \le |U_1(1)|^k \le (|\lambda_i| - b)^k$. Since $\sigma(U_2(-1)) = \big(\sigma(U_2(1))\big)^{-1}$, we similarly have

$$|U_2(k)| \le (|\lambda_i| - b)^k, \quad \forall k \in \mathbb{Z}_-$$

again for $b > 0$ sufficiently small. Let e_k be the k-th column of the $n \times n$ identity matrix. By fixing $k_0 \in \mathbb{N}$ sufficiently large, let us define a mapping T_j for $k = k_0, k_0 + 1, \ldots$ and for $j \in \{1, 2, \ldots, k_i\}$ as follows:

$$T_j(y)_k = J^k e_{p_i+j} + \sum_{j=k_0}^{k-1} U_1(k-1-j)(B(j)-J)y_j - \sum_{j=k}^{\infty} U_2(k-1-j)(B(j)-J)y_j\,. \tag{3.1.5}$$

We consider this mapping on the Banach space:

$$Y = \Big\{ \{y_j\}_{j=k_0}^{\infty} : y_j \in \mathbb{R}^n, \quad \sup_{j \ge k_0} \big|y_j\big| \, (|\lambda_i| + b)^{-j} < \infty \Big\}$$

with the norm $|||y||| = \sup_{k \ge k_0} |y_k| (|\lambda_i| + b)^{-k}$ for $y = \{y_j\}_{j=k_0}^{\infty}$. To show that T_j is well defined, we compute

$$\sup_k |J^k e_{p_i+j}| (|\lambda_i| + b)^{-k} < \infty,$$

since $|J_i^k| < c_1 (|\lambda_i| + d)^k$ for a $0 < d < b$ and $c_1 > 0$. By taking $b > 0$ satisfying

$$\frac{|\lambda_i| + b}{|\lambda_i| - b} < \mathrm{e}^{4a},$$

we have for a constant $c > 0$

$$\sup_k \sum_{j=k_0}^{k-1} \big|U_1(k-1-j)(B(j)-J)y_j\big| \, (|\lambda_i| + b)^{-k}$$
$$\le c(|\lambda_i| - b)^{-1} |||y||| \sup_k \left(\frac{|\lambda_i| - b}{|\lambda_i| + b}\right)^k \sum_{j=k_0}^{k-1} \left(\frac{|\lambda_i| + b}{|\lambda_i| - b} \, \mathrm{e}^{-4a}\right)^j < \infty$$

and

$$\begin{aligned}
&\sup_k \sum_{j=k}^{\infty} \left|U_2(k-1-j)(B(j)-J)y_j\right| (|\lambda_i|+b)^{-k} \le \\
&\le c \sup_k \sum_{j=k}^{\infty} (|\lambda_i|-b)^{k-1-j} |||y|||(|\lambda_i|+b)^j \mathrm{e}^{-4aj}(|\lambda_i|+b)^{-k} \\
&\le c(|\lambda_i|-b)^{-1}|||y||| \sup_k \left(\frac{|\lambda_i|-b}{|\lambda_i|+b}\right)^k \sum_{j=k}^{\infty} \left(\frac{|\lambda_i|+b}{|\lambda_i|-b}\,\mathrm{e}^{-4a}\right)^j < \infty .
\end{aligned}$$

Consequently, we arrive at $|||T_j(y)||| < \infty$, so $T_j : Y \to Y$. Furthermore, we have

$$\forall \varepsilon > 0\ \exists n_0 > k_0 : \left(\frac{|\lambda_i|-b}{|\lambda_i|+b}\right)^k < \varepsilon\ \forall k > n_0 .$$

By using this property, the contraction of T_j follows the same arguments as the well defined T_j. Consequently by Banach fixed point theorem 2.2.1, T_j has a fixed point $y(j)$ satisfying by (3.1.5)

$$|y(j)_k - J^k e_{p_i+j}| \le K_0(|\lambda_i|-b)^k$$

for a constant $K_0 > 0$. By defining the matrix $Y_i(k)$ of the order $n \times k_i$ with $y(j)_k$ in column j, we obtain

$$|Y_i(k) - F_i(k)|(|\lambda_i|-b)^{-k} \le K_0 ,$$

where $F_i(k)$ is the $n \times k_i$-matrix with J_i^k in rows p_i+1 through p_i+k_i and all other rows zero. Let $\overline{G}_i$ be the identity matrix of order $k_i \times k_i$. Then $\lim_{k\to\infty} Y_i(k)J_i^{-k} = G_i$ and G_i is the matrix of order $n \times k_i$ with $\overline{G}_i$ in rows p_i+1 through p_i+k_i and all other rows zero. This construction is done for the block J_i. To get the result, we take the $n \times n$ matrix $Y(k)$ with $Y_i(k)$ in columns p_i+1 through p_i+k_i for $i = 1,2,\dots,r$. So $\lim_{k\to\infty} Y(k)J^{-k} = \mathbb{I}$. Finally, by putting $X(k) = PY(k)P^{-1}$ we arrive at $X(k+1) = A(k)X(k)$ satisfying

$$X(k)A_0^{-k} \to \mathbb{I} \quad \text{as} \quad k \to \infty .$$

The proof is finished. □

Our next result matches at $k = 0$ the two solutions of (3.1.3) provided by the preceding lemma. The proof of the following theorem is a slight extension of [10, Theorem 3.1.2] and [11, Theorem. 2], so we omit the proof.

Theorem 3.1.2. *Let* $d_s = \dim W^s$, $d_u = \dim W^u$ *for* (3.1.3) *and let* $\mathbb{I}_s$, $\mathbb{I}_u$ *denote the identity matrices of order* d_s, d_u *respectively. There exists a fundamental solution* $U(k)$, $k \in \mathbb{Z}$ *for* (3.1.3) *along with constants* $M > 1$, $K_0 > 0$ *and four projections* P_{ss}, P_{su}, P_{us}, P_{uu} *so that* $P_{ss}+P_{su}+P_{us}+P_{uu} = \mathbb{I}$ *and the following hold:*

(i) $|U(t)(P_{ss}+P_{us})U(s)^{-1}| \le K_0 M^{(s-t)}$ *for* $0 \le s \le t$,
(ii) $|U(t)(P_{su}+P_{uu})U(s)^{-1}| \le K_0 M^{(t-s)}$ *for* $0 \le t \le s$,
(iii) $|U(t)(P_{ss}+P_{su})U(s)^{-1}| \le K_0 M^{(t-s)}$ *for* $t \le s \le 0$,

(iv) $|U(t)(P_{us}+P_{uu})U(s)^{-1}| \le K_0 M^{(s-t)}$ *for* $s \le t \le 0$.

Also, $\operatorname{rank} P_{ss} = \operatorname{rank} P_{uu} = d$ *for some positive integer* d.

In the language of dichotomies (cf Section 2.5.1) we see that Theorem 3.1.2 provides a two-sided exponential dichotomy. For $k \to -\infty$ an exponential dichotomy is given by the fundamental solution $U(k)$ and the projection $P_{us}+P_{uu}$ while for $k \to +\infty$ such is given by $U(k)$ and $P_{ss}+P_{us}$.

Let $u_j(k)$ denote column j of $U(k)$ and assume that these are numbered so that

$$P_{uu} = \begin{pmatrix} \mathbb{I}_d & 0_d & 0 \\ 0_d & 0_d & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad P_{ss} = \begin{pmatrix} 0_d & 0_d & 0 \\ 0_d & \mathbb{I}_d & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here, $\mathbb{I}_d$ denotes the $d \times d$ identity matrix and 0_d denotes the $d \times d$ zero matrix.

For each $i = 1,\ldots,n$ we define $u_i^{\perp}(k)$ by $\langle u_i^{\perp}(k), u_j(k+1)\rangle = \delta_{ij}$. The vectors $u_i^{\perp}(k)$ can be computed from the formula $U(k)^{\perp *} = U(k+1)^{-1}$ where $U^{\perp}(k)$ denotes the matrix with $u_j^{\perp}(k)$ as column j. By using the identity $U(k+1)U(k)^{\perp *} = \mathbb{I}$ we obtain that $U(k+1)^{\perp} = \big(Df(\gamma_{k+1})^*\big)^{-1} U(k)^{\perp}$. Thus, $U^{\perp}(k)$ is the adjoint of $U(k)$. Note $\{u_i^{\perp}(k)\}_{k\in\mathbb{Z}}$, $i = 1,2,\cdots,d$ is a basis of bounded solutions on $\mathbb{Z}$ to the adjoint variational equation $w_{k+1} = \big(Df(\gamma_{k+1})^*\big)^{-1} w_k$.

We take the Banach space

$$Z = \Big\{\{y_j\}_{j\in\mathbb{Z}} : y_j \in \mathbb{R}^n, \quad \sup_{j\in\mathbb{Z}} |y_j| < \infty\Big\}$$

with the norm $||y|| = \sup_{k\in\mathbb{Z}} |y_k|$ for $y = \{y_j\}_{j\in\mathbb{Z}}$. Summation of the inequalities in Theorem 3.1.2 yields the following result.

Theorem 3.1.3. *Let U be the fundamental solution to* (3.1.3) *along with the projections P_{ss}, P_{su}, P_{us}, P_{uu} as in Theorem* 3.1.2. *Then there exists a constant $K > 0$ so that for any $z \in Z$ the following hold:*

(i) $\sum_{k=0}^{j} |U(j)(P_{ss}+P_{us})U(k)^{-1} z_k| \le K\|z\|$ *for* $j \ge 0$,
(ii) $\sum_{k=j}^{\infty} |U(j)(P_{su}+P_{uu})U(k)^{-1} z_k| \le K\|z\|$ *for* $j \ge 0$,
(iii) $\sum_{k=j}^{0} |U(j)(P_{ss}+P_{su})U(k)^{-1} z_k| \le K\|z\|$ *for* $j \le 0$,
(iv) $\sum_{k=-\infty}^{j} |U(j)(P_{us}+P_{uu})U(k)^{-1} z_k| \le K\|z\|$ *for* $j \le 0$.

Let us define a closed linear subspace of Z given by

$$Z_0 = \Big\{ z \in Z : \sum_{k=-\infty}^{\infty} P_{uu} U(k+1)^{-1} z_k = 0 \Big\}.$$

Note

$$0 = \sum_{k=-\infty}^{\infty} P_{uu} U(k+1)^{-1} z_k = \sum_{k=-\infty}^{\infty} P_{uu} U(k)^{\perp *} z_k \Leftrightarrow \sum_{k=-\infty}^{\infty} \langle u_j^{\perp}(k), z_k\rangle = 0$$

for all $j = 1, 2, \dots, d$. We consider the difference equation:

$$z_{k+1} = Df(\gamma_k)z_k + w_k, \quad \{w_k\}_{k\in\mathbb{Z}} \in Z. \tag{3.1.6}$$

The following result is a Fredholm-like condition for (3.1.6).

Theorem 3.1.4. *Necessary and sufficient condition for the existence of a solution* $\{x_k\}_{k\in\mathbb{Z}} \in Z$ *of* (3.1.6) *is that* $\{w_k\}_{k\in\mathbb{Z}} \in Z_0$.

Proof. "$\Longrightarrow$"

Let $z = \{z_k\}_{k\in\mathbb{Z}}$ be a solution of (3.1.6). Denote $A(k) = Df(\gamma_k)$ and compute

$$P_{uu}U(k+1)^{-1}z_{k+1} = P_{uu}U(k+1)^{-1}A(k)z_k + P_{uu}U(k+1)^{-1}w_k.$$

Since $U(k+1) = A(k)U(k)$, $U(k+1)^{-1} = U(k)^{-1}A(k)^{-1}$, and hence

$$\sum_{k=-\infty}^{\infty} P_{uu}U(k+1)^{-1}z_{k+1} = \sum_{k=-\infty}^{\infty} P_{uu}U(k)^{-1}z_k + \sum_{k=-\infty}^{\infty} P_{uu}U(k+1)^{-1}w_k$$

which implies

$$\sum_{k=-\infty}^{\infty} P_{uu}U(k+1)^{-1}w_k = 0.$$

We note that Theorem 3.1.3 gives the convergence of these series.

"$\Longleftarrow$"

Let $w = \{w_k\}_{k=-\infty}^{\infty} \in Z_0$. We define the mapping $\mathscr{K}$ as follows:

$$\begin{aligned}\mathscr{K}(w)_k = U(k)\Big[&\sum_{j=-\infty}^{-1} P_{us}U(j+1)^{-1}w_j + \sum_{j=0}^{k-1}(P_{ss}+P_{us})U(j+1)^{-1}w_j\\ &-\sum_{j=k}^{\infty}(P_{su}+P_{uu})U(j+1)^{-1}w_j\Big],\end{aligned}$$

for $k \geq 0$,

$$\begin{aligned}\mathscr{K}(w)_k = U(k)\Big[&-\sum_{j=0}^{\infty} P_{su}U(j+1)^{-1}w_j + \sum_{j=-\infty}^{k-1}(P_{us}+P_{uu})U(j+1)^{-1}w_j\\ &-\sum_{j=k}^{-1}(P_{ss}+P_{su})U(j+1)^{-1}w_j\Big],\end{aligned}$$

for $k \leq 0$. Here we define $\sum_{j=0}^{-1} = 0$. Theorem 3.1.3 implies the well defined definition and continuity of $\mathscr{K} : Z_0 \to Z$ and by putting $z_k = \mathscr{K}(w)_k$, $\forall k \in \mathbb{Z}$ in (3.1.6), we easily verify that it is a solution. We note that the general solution of (3.1.6) has the form:

$$z = \sum_{j=1}^{d} \beta_j u_{j+d} + \mathscr{K}(w), \quad \beta_j \in \mathbb{R}.$$

The proof is finished. □

The next result provides an appropriate projection.

Theorem 3.1.5. *Let* U *be as in Theorem* 3.1.2 *and let* Z_0 *be as in Theorem* 3.1.4. *There exists a bounded projection* $\Pi : Z \to Z$ *so that* $\mathscr{R}\Pi = Z_0$.

Proof. We take Π in the form $\mathbb{I} - P$, where P is defined by

$$P(w)_k = \frac{U(k+1)}{a_{k+1}} P_{uu} \sum_{j=-\infty}^{\infty} U(j+1)^{-1} w_j \,,$$

and the sequence $\{a_k\}_{k\in\mathbb{Z}}$ satisfies

$$a_k > 0, \forall k \in \mathbb{Z}, \quad \sum_{k=-\infty}^{\infty} \frac{1}{a_{k+1}} = 1, \quad \sup_{k\in\mathbb{Z}} \frac{U(k+1)}{a_{k+1}} < \infty \,.$$

We verify that this P is a projection, i.e. $P^2 = P$:

$$P(P(w))_k = P\left(\left\{\frac{U(s+1)}{a_{s+1}} P_{uu} \sum_{j=-\infty}^{\infty} U(j+1)^{-1} w_j\right\}_{s\in\mathbb{Z}}\right)$$
$$= \frac{U(k+1)}{a_{k+1}} P_{uu} \sum_{l=-\infty}^{\infty} U(l+1)^{-1} \left(\frac{U(l+1)}{a_{l+1}} P_{uu} \sum_{j=-\infty}^{\infty} U(j+1)^{-1} w_j\right) = P(w)_k \,.$$

Hence P is a projection. Now we verify that $\Pi = \mathbb{I} - P$ is such that $\Pi w \in Z_0$:

$$\sum_{k=-\infty}^{\infty} P_{uu} U(k+1)^{-1} \Pi(w)_k = \sum_{k=-\infty}^{\infty} P_{uu} U(k+1)^{-1} (\mathbb{I} - P)(w)_k$$
$$= \sum_{k=-\infty}^{\infty} P_{uu} U(k+1)^{-1} \left(w_k - \frac{U(k+1)}{a_{k+1}} P_{uu} \sum_{j=-\infty}^{\infty} U(j+1)^{-1} w_j\right)$$
$$= \sum_{k=-\infty}^{\infty} P_{uu} U(k+1)^{-1} w_k$$
$$- \sum_{k=-\infty}^{\infty} P_{uu} U(k+1)^{-1} \left(\frac{U(k+1)}{a_{k+1}} P_{uu} \sum_{j=-\infty}^{\infty} U(j+1)^{-1} w_j\right) = 0 \,.$$

Consequently, Π has the desired properties. □

3.1.3 Perturbation Theory

We study the equation (cf Theorem 2.2.4):

$$\begin{aligned} F_{\mu,\varepsilon,y}(x)_k &= x_{k+1} - f(x_k) - h(x_k,\mu,k) - \varepsilon|\mu|\mathscr{L}(x-y-\gamma) = 0 \\ F_{\mu,\varepsilon,y} &: Z \to Z, \end{aligned} \tag{3.1.7}$$

where $\mathscr{L} : Z \to Z$ is a linear continuous mapping so that $||\mathscr{L}|| \leq 1$, $y \in Z$, and $\varepsilon \in \mathbb{R}$ is small. It is clear that solutions of (3.1.7) with $\varepsilon = 0$ are bounded solutions of (3.1.1). We define mappings $L : Z \to Z$ and $G : Z \times \mathbb{R}^m \times \mathbb{R} \times Z \to Z$ as follows:

$$\begin{aligned} L(z)_k &= z_{k+1} - Df(\gamma_k)z_k, \\ G(z,\mu,\varepsilon,y)_k &= f(z_k+\gamma_k) - f(\gamma_k) - Df(\gamma_k)z_k + h(z_k+\gamma_k,\mu,k) + \varepsilon|\mu|\mathscr{L}(z-y). \end{aligned}$$

By putting $x = z + \gamma$ in (3.1.7), this equation has the form:

$$L(z) = G(z,\mu,\varepsilon,y). \tag{3.1.8}$$

We decompose (3.1.8) in the following way

$$L(z) = \Pi G(z,\mu,\varepsilon,y), \quad 0 = (\mathbb{I} - \Pi)G(z,\mu,\varepsilon,y).$$

By using Theorem 3.1.4, the above pair of equations is equivalent to

$$z = \sum_{j=1}^{d} \beta_j u_{j+d} + \mathscr{K}\big(\Pi G(z,\mu,\varepsilon,y)\big), \quad \beta_j \in \mathbb{R} \tag{3.1.9}$$

and

$$0 = (\mathbb{I} - \Pi)G(z,\mu,\varepsilon,y). \tag{3.1.10}$$

Moreover by using the Lyapunov-Schmidt procedure from Section 2.2.3 like in [11, Theorem 8], the study of Eqs. (3.1.9) and (3.1.10) can be expressed in the following theorem for z, μ, ε, $\beta = (\beta_1,\beta_2,\ldots,\beta_d)$, y sufficiently small.

Theorem 3.1.6. *Let U and d be as in Theorem* 3.1.2. *Then there exist small neighborhoods $0 \in Q \subset Z$, $0 \in O \subset \mathbb{R}^d$, $0 \in W \subset \mathbb{R}^m$, $0 \in V \subset \mathbb{R}$ and a C^3-function $H : Q \times O \times W \times V \to \mathbb{R}^d$ denoted by $(y,\beta,\mu,\varepsilon) \to H(y,\beta,\mu,\varepsilon)$ with the following properties:*

(i) *The equation $H(y,\beta,\mu,\varepsilon) = 0$ holds if and only if* (3.1.7) *has a solution near γ and moreover, each such $(y,\beta,\mu,\varepsilon)$ determines only one solution of* (3.1.7),
(ii) $H(0,0,0,0) = 0$,
(iii) $\dfrac{\partial H_i}{\partial \mu_j}(0,0,,0,0) = -\sum_{k\in\mathbb{Z}} \left\langle u_i^{\perp}(k), \dfrac{\partial h}{\partial \mu_j}(\gamma_k,0,k) \right\rangle$,
(iv) $\dfrac{\partial H_i}{\partial \beta_j}(0,0,0,0) = 0$,
(v) $\dfrac{\partial^2 H_i}{\partial \beta_k \partial \beta_j}(0,0,0,0) = -\sum_{l\in\mathbb{Z}} \langle u_i^{\perp}(l), D^2 f(\gamma_l)(u_{d+j}(l), u_{d+k}(l))\rangle$.

We introduce the following notations:

$$a_{ij} = -\sum_{l\in\mathbb{Z}} \left\langle u_i^{\perp}(l), \frac{\partial h}{\partial \mu_j}(\gamma_l, 0, l) \right\rangle,$$
$$b_{ijk} = -\sum_{l\in\mathbb{Z}} \left\langle u_i^{\perp}(l), D^2 f(\gamma_l)(u_{d+j}(l), u_{d+k}(l)) \right\rangle.$$

Finally, we take the mapping $M_\mu : \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$\big(M_\mu(\beta)\big)_i = \sum_{j=1}^{m} a_{ij}\mu_j + \frac{1}{2}\sum_{j,k=1}^{d} b_{ijk}\beta_j\beta_k.$$

Now we can state the main result of this section.

Theorem 3.1.7. *If M_{μ_0} has a simple zero point β_0, i.e. β_0 satisfies $M_{\mu_0}(\beta_0) = 0$ and $D_\beta M_{\mu_0}(\beta_0)$ is a regular matrix, then there is a wedge-shaped region in $\mathbb{R}^m$ for μ of the form*

$$\mathscr{R} = \Big\{ s^2\tilde{\mu} \,:\, s\text{, respectively } \tilde{\mu}\text{, is from a small open neighborhood of } 0 \in \mathbb{R}\text{, respectively of } \mu_0 \in \mathbb{R}^m \Big\}$$

so that for any $\mu \in \mathscr{R}\setminus\{0\}$, Equation (3.1.1) *possesses a transversal bounded solution.*

Proof. Let us consider the mapping defined by

$$\Phi(y,\tilde{\beta},\tilde{\mu},\tilde{\varepsilon},s) = \begin{cases} \dfrac{1}{s^2}H(y, s\tilde{\beta}, s^2\tilde{\mu}, s^3\tilde{\varepsilon}), & \text{for} \quad s \neq 0, \\ M_{\tilde{\mu}}(\tilde{\beta}), & \text{for} \quad s = 0. \end{cases}$$

According to (ii)–(v) of Theorem 3.1.6, the mapping Φ is C^1-smooth near

$$(y,\tilde{\beta},\tilde{\mu},\tilde{\varepsilon},s) = (0,\beta_0,\mu_0,0,0)$$

with respect to the variable $\tilde{\beta}$. Since

$$M_{\mu_0}(\beta_0) = 0 \quad \text{and} \quad D_\beta M_{\mu_0}(\beta_0) \quad \text{is a regular matrix},$$

we can apply the implicit function theorem to solving locally and uniquely the equation $\Phi = 0$ in the variable $\tilde{\beta}$. This gives for $\varepsilon = 0$, by (i) of Theorem 3.1.6, the existence of $\mathscr{R}$ on which (3.1.1) has a bounded solution.

To prove the transversality of these bounded solutions, we fix $\mu \in \mathscr{R}\setminus\{0\}$ and take

$$y = \tilde{\gamma} - \gamma,$$

where $\tilde{\gamma}$ is the solution of (3.1.7) for which the transversality should be proved. Then we vary $\varepsilon = s^3\tilde{\varepsilon}$ small. Note that $s \neq 0$ is also fixed due to $\mu = s^2\tilde{\mu}$. Since the local uniqueness of solutions of (3.1.7) near $\tilde{\gamma}$ is satisfied for any $\tilde{\varepsilon}$ sufficiently small according to the above application of the implicit function theorem, such equation (3.1.7) (with the fixed $\mu \in \mathscr{R} \setminus \{0\}$, $\varepsilon = s^3\tilde{\varepsilon}$ where $s \neq 0$ is also fixed and the special $y = \tilde{\gamma} - \gamma$) has the only solution $x = \tilde{\gamma}$ near $\tilde{\gamma}$ for any $\tilde{\varepsilon}$ sufficiently small. Now Theorem 2.2.4 gives the invertibility of $DF_{\mu,0,\tilde{\gamma}-\gamma}(\tilde{\gamma})$ and so the only bounded solution on $\mathbb{Z}$ of the equation

$$v_{k+1} = Df(\tilde{\gamma}_k)v_k + D_x h(\tilde{\gamma}_k, \mu, k)v_k$$

is $v_k = 0, \forall k \in \mathbb{Z}$. The proof is finished. □

Remark 3.1.8. Note that we can take any bases of bounded solutions of the variational and adjoint variational equations for constructing the Melnikov function M_μ. Similar observations can be applied to detecting of other Melnikov functions in this book.

Remark 3.1.9. Assume that (3.1.1) is autonomous, i.e. h is independent of k, suppose conditions (i)–(iv) and f is a diffeomorphism. Then we have a local diffeomorphism $F_\mu(x) := f(x) + h(x,\mu)$ for μ small. If there is an open bounded subset $\Omega \subset \mathbb{R}^d$ so that $0 \notin M_{\mu_0}(\partial\Omega)$ and $\deg\left(M_{\mu_0}, \Omega, 0\right) \neq 0$ then for any $0 \neq \mu \in \mathscr{R}$ there is a $k_\mu \in \mathbb{N}$ such that for any $k \geq k_\mu$ there is a set $\Lambda_k \subset \mathbb{R}^n$ and a continuous mapping $\varphi_k : \Lambda_k \to \mathscr{E}$ so that $F_\mu^{2k}(\Lambda_k) = \Lambda_k$, φ_k is surjective and injective, and $\varphi_k \circ F_\mu^{2k} = \sigma \circ \varphi_k$. Note that we do not know whether φ_k is a homeomorphism. But we do know that F_μ has infinitely many periodic orbits and quasiperiodic ones and it has positive *topological entropy*. This is a generalization of the Smale-Birkhoff homoclinic theorem 2.5.4 to this case. Particularly, if β_0 is an isolated zero of M_{μ_0} with a nonzero Brouwer index, then we have a chaotic behaviour of F_μ (cf [13]). This remark can be applied to other Melnikov type conditions in this book.

3.1.4 Bifurcation from a Manifold of Homoclinic Solutions

In many cases, (3.1.2) has a manifold of homoclinic solutions. Hence we suppose that

(v) There is an open non-empty subset $\mathscr{O} \subset \mathbb{R}^d$ and C^3-smooth mappings $\gamma_k : \mathscr{O} \to \mathbb{R}^n$, $\omega : \mathscr{O} \to \mathbb{R}^n$, $\forall k \in \mathbb{Z}$ satisfying

$$\begin{aligned} \gamma_{k+1}(\theta) &= f(\gamma_k(\theta)), && \forall k \in \mathbb{Z}, \forall \theta \in \mathscr{O}, \\ \omega(\theta) &= f(\omega(\theta)), && \forall \theta \in \mathscr{O}, \\ \lim_{k\to\pm\infty} \gamma_k(\theta) &= \omega(\theta), && \forall \theta \in \mathscr{O}. \end{aligned}$$

(vi) The eigenvalues of $Df(\omega(\theta)) \forall \theta \in \mathscr{O}$ are non-zero and all lie off the unit circle. Moreover, $Df(\gamma_k(\theta))\ \forall k \in \mathbb{Z}, \forall \theta \in \mathscr{O}$ are nonsingular.

(vii) $\frac{\partial \gamma_k}{\partial \theta_i}$ are uniformly bounded on $\mathscr{O}$ with respect to $k \in \mathbb{Z}$ when $\theta = (\theta_1, \theta_2, \ldots, \theta_d)$.

(viii) From $\gamma_{k+1}(\theta) = f(\gamma_k(\theta))$, we obtain $\frac{\partial \gamma_{k+1}}{\partial \theta_i}(\theta) = Df(\gamma_k(\theta))\frac{\partial \gamma_k}{\partial \theta_i}(\theta)$. We suppose that $\left\{\frac{\partial \gamma_k}{\partial \theta_i}(\theta)\right\}_{i=1,k\in\mathbb{Z}}^d$ is a basis of the space of bounded solutions of the difference equation

$$v_{k+1} = Df(\gamma_k(\theta))v_k\,. \tag{3.1.11}$$

We use the approach of Section 3.1.3 by considering θ as a parameter. The difference is only that now $\left\{\frac{\partial \gamma_k}{\partial \theta_i}(\theta)\right\}_{i=1,k\in\mathbb{Z}}^d$ provides a natural family of solutions of (3.1.11) corresponding to the projections P_{ss}. Hence we suppose that Theorem 3.1.2 holds parametricaly by $\theta \in \mathscr{O}$, i.e. $U = U(\theta,t)$ is smooth in (θ,t) and columns of $U(\theta,t)$ are numbered so that

$$P_{uu} = \begin{pmatrix} \mathbb{I}_d & 0_d & 0 \\ 0_d & 0_d & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad P_{ss} = \begin{pmatrix} 0_d & 0_d & 0 \\ 0_d & \mathbb{I}_d & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now we take $x = z+\gamma(\theta), \gamma(\theta) = \{\gamma_k(\theta)\}_{k\in\mathbb{Z}}$ in (3.1.7). The corresponding operators of (3.1.8) then depend on θ as well:

$$\begin{aligned} L(z,\theta)_k &= z_{k+1} - Df(\gamma_k(\theta))z_k, \\ G(z,\theta,\mu,\varepsilon,y)_k &= f(z_k+\gamma_k(\theta)) - f(\gamma_k(\theta)) - Df(\gamma_k(\theta))z_k \\ &\quad + h(z_k+\gamma_k(\theta),\mu,k) + \varepsilon|\mu|\mathscr{L}(z-y)\,. \end{aligned}$$

Consequently, (3.1.7) has the form

$$L(z,\theta) = G(z,\theta,\mu,\varepsilon,y)\,,$$

and (3.1.9)–(3.1.10) are replaced by

$$z = \mathscr{K}(\theta)\big(\Pi(\theta)G(z,\theta,\mu,\varepsilon,y)\big)\,, \quad 0 = (\mathbb{I}-\Pi(\theta))G(z,\theta,\mu,\varepsilon,y)\,, \tag{3.1.12}$$

where $\mathscr{K}(\theta)$ and $\Pi(\theta)$ are corresponding mappings to $\mathscr{K}, \Pi$, respectively. We consider in (3.1.12) the variable θ as a bifurcation parameter. We take the mapping $N_\mu : \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$\big(N_\mu(\theta)\big)_i = \sum_{j=1}^m a_{ij}(\theta)\mu_j\,,$$

where

$$a_{ij}(\theta) = -\sum_{l\in\mathbb{Z}} \Big\langle u_i^\perp(\theta,l), \frac{\partial h}{\partial \mu_j}(\gamma_l(\theta),0,l)\Big\rangle\,.$$

The vectors $u_i^\perp(\theta,l)$ are defined by $\langle u_i^\perp(\theta,l), u_j(\theta,l+1)\rangle = \delta_{ij}$. By repeating the proof of Theorem 3.1.7, we can state the main result of this section.

Theorem 3.1.10. *If N_{μ_0} has a simple zero point θ_0, i.e. θ_0 satisfies $N_{\mu_0}(\theta_0) = 0$ and $D_\theta N_{\mu_0}(\theta_0)$ is a regular matrix, then there is a wedge-shaped region in $\mathbb{R}^m$ for μ of* *the form*

$$\mathscr{R} = \Big\{ s\tilde{\mu} \,:\, s, \text{ respectively } \tilde{\mu}, \text{ is from a small open neighborhood of } 0 \in \mathbb{R}, \text{ respectively of } \mu_0 \in \mathbb{R}^m \Big\}$$

so that for any $\mu \in \mathscr{R} \setminus \{0\}$*, Equation* (3.1.1) *possesses a transversal bounded solution.*

3.1.5 Applications to Impulsive Differential Equations

It is well known that the theory of impulsive differential equations is an important branch of differential equations with many applications [14–20]. For this reason, we consider a 4-dimensional impulsive differential equation given by

$$\begin{aligned} \dot{z} &= g_1(z), \quad \dot{y} = g_2(y), \\ z(i+) &= z(i-) + \mu h_1(z(i-), y(i-), \mu), \\ y(i+) &= y(i-) + \mu h_2(z(i-), y(i-), \mu), \quad i \in \mathbb{Z}, \end{aligned} \tag{3.1.13}$$

where

$$g_{1,2} \in C^3(\mathbb{R}^2, \mathbb{R}^2), \quad h_{1,2} \in C^3(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^2), \quad \mu \in \mathbb{R}$$

and $\dot{z} = g_1(z)$, $\dot{y} = g_2(y)$ are Hamiltonian systems. Let Ψ_1, Ψ_2 be the 1-time Poincarè mappings of $\dot{z} = g_1(z)$, $\dot{y} = g_2(y)$, respectively. Here $z(i\pm) = \lim_{s \to i_\pm} z(s)$. We consider the mapping

$$\begin{aligned} &F(z, y, \mu) = \\ &\Big(\Psi_1(z) + \mu h_1\big(\Psi_1(z), \Psi_2(y), \mu\big), \Psi_2(y) + \mu h_2\big(\Psi_1(z), \Psi_2(y), \mu\big) \Big). \end{aligned} \tag{3.1.14}$$

Clearly the dynamics of (3.1.14) determines the behaviour of (3.1.13). In the notation of (3.1.1), we have

$$\begin{aligned} x = (z, y) \in \mathbb{R}^2 \times \mathbb{R}^2, \quad f(x) = \big(\Psi_1(z), \Psi_2(y)\big) \\ h(x, \mu, k) = \Big(\mu h_1\big(\Psi_1(z), \Psi_2(y), \mu\big), \mu h_2\big(\Psi_1(z), \Psi_2(y), \mu\big) \Big). \end{aligned} \tag{3.1.15}$$

We suppose

(a) $g_{1,2}(0) = 0$ and the eigenvalues of $Dg_{1,2}(0)$ lie off the imaginary axis.
(b) There are homoclinic solutions γ_1, γ_2 of $\dot{z} = g_1(z)$, $\dot{y} = g_2(y)$, respectively, to 0.

The conditions (a) and (b) imply that

$$\gamma_k(\theta) = \big(\gamma_1(\theta_1+k), \gamma_2(\theta_2+k)\big), \quad k \in \mathbb{Z}$$
$$\omega(\theta) = (0,0), \quad \theta = (\theta_1, \theta_2) \in \mathscr{O} = \mathbb{R}^2$$

satisfy (v)–(viii) of Section 3.1.4 for (3.1.15). Now (3.1.11) has the form

$$v_{k+1} = D\Psi_1\big(\gamma_1(\theta_1+k)\big)v_k\,, \quad w_{k+1} = D\Psi_2\big(\gamma_2(\theta_2+k)\big)w_k\,.$$

Hence (3.1.11) is now decomposed into two difference equations. We note that $\Psi_{1,2}$ are area-preserving, i.e. $\det D\Psi_{1,2}(z) = 1$ (cf Sections 2.5.1 and 2.5.3). We can take

$$u_3(\theta,k) = \big(\dot\gamma_1(\theta_1+k), 0\big)\,, \quad u_4(\theta,k) = \big(0, \dot\gamma_2(\theta_2+k)\big)\,.$$

Now we need the following result [8, pp. 104–105].

Lemma 3.1.11. *Let $\{A_k\}_{k\in\mathbb{Z}}$ be a sequence of invertible 2×2-matrices so that $\det A_k = 1$. If $\{x_k\}_{k\in\mathbb{Z}}$ satisfies $x_{n+1} = A_k x_k$, then $z_k := Jx_{k+1}$ for $J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ satisfies $z_{k+1} = (A_{k+1}^*)^{-1} z_k$.*

Proof. The result directly follows from the identity $A_k^* \circ J \circ A_k = \det A_k J = J$. □

Using Lemma 3.1.11, we can take

$$u_1^{\perp}(\theta,k) = \big(\bar{\dot\gamma}_1(\theta_1+k+1), 0\big), \quad u_2^{\perp}(\theta,k) = \big(0, \bar{\dot\gamma}_2(\theta_2+k+1)\big)\,,$$

where $\bar z = (z_2, -z_1), \forall z = (z_1, z_2) \in \mathbb{R}^2$, and $u_1(\theta,k)$, $u_2(\theta,k)$ are not required to be known. Consequently, the mapping N_μ of Section 3.1.4 has now the form

$$\begin{aligned}
\big(N_\mu(\theta)\big)_1 &= -\mu \sum_{k\in\mathbb{Z}} h_1\big(\Psi_1(\gamma_1(\theta_1+k)), \Psi_2(\gamma_2(\theta_2+k)), 0\big) \wedge \dot\gamma_1(\theta_1+k+1)\\
&= \mu \sum_{k\in\mathbb{Z}} \dot\gamma_1(\theta_1+k) \wedge h_1\big(\gamma_1(\theta_1+k), \gamma_2(\theta_2+k), 0\big)\,,\\
\big(N_\mu(\theta)\big)_2 &= -\mu \sum_{k\in\mathbb{Z}} h_2\big(\Psi_1(\gamma_1(\theta_1+k)), \Psi_2(\gamma_2(\theta_2+k)), 0\big) \wedge \dot\gamma_2(\theta_2+k+1)\\
&= \mu \sum_{k\in\mathbb{Z}} \dot\gamma_2(\theta_2+k) \wedge h_2\big(\gamma_1(\theta_1+k), \gamma_2(\theta_2+k), 0\big)\,,
\end{aligned} \tag{3.1.16}$$

where $\wedge$ is the wedge product defined by $z \wedge y = z_1 y_2 - z_2 y_1$, $z, y \in \mathbb{R}^2$. Theorem 3.1.10 gives the following result.

Theorem 3.1.12. *If there is a simple zero point of $N_1(\theta)$ given by* (3.1.16)*, then* (3.1.13) *has a transversal homoclinic solution and so it exhibits chaos for any $\mu \neq 0$ sufficiently small.*

Of course, there are h_1, h_2 satisfying the assumptions of Theorem 3.1.12. For simplicity, we assume

$$\begin{aligned} g = g_1 = g_2, \quad h_1(z,y,\mu) &= (1+\mu)y + \alpha \\ h_2(z,y,\mu) &= (1+\mu^2)z + \alpha\,, \end{aligned} \tag{3.1.17}$$

where $\alpha \in \mathbb{R}^2$ is a constant vector. Then we have $\gamma_1 = \gamma_2 = \gamma$ and (3.1.16) possesses the form

$$\begin{aligned}\big(N_\mu(\theta)\big)_1 &= \mu \sum_{k\in\mathbb{Z}} \dot{\gamma}(\theta_1+k)\wedge\gamma(\theta_2+k) + \mu\Big(\sum_{k\in\mathbb{Z}} \dot{\gamma}(\theta_1+k)\Big)\wedge\alpha\\ \big(N_\mu(\theta)\big)_2 &= \mu \sum_{k\in\mathbb{Z}} \dot{\gamma}(\theta_2+k)\wedge\gamma(\theta_1+k) + \mu\Big(\sum_{k\in\mathbb{Z}} \dot{\gamma}(\theta_2+k)\Big)\wedge\alpha\,.\end{aligned} \tag{3.1.18}$$

We put

$$\Omega(\tau) = \sum_{k\in\mathbb{Z}} \dot{\gamma}(\tau+k)\wedge\gamma(\tau+k) + \Big(\sum_{k\in\mathbb{Z}} \dot{\gamma}(\tau+k)\Big)\wedge\alpha\,.$$

We note that Ω is 1-periodic. We clearly for $\theta = (\tau,\tau)$ have

$$\begin{gathered}\big(N_\mu(\theta)\big)_1 = \big(N_\mu(\theta)\big)_2 = \mu\Omega(\tau),\\ \big(DN_\mu(\theta)\big)_1 = \mu\big(\Omega'(\tau),0\big), \quad \big(DN_\mu(\theta)\big)_2 = \mu\big(0,\Omega'(\tau)\big)\,.\end{gathered}$$

Simple computations give the following result.

Theorem 3.1.13. *Consider* (3.1.13) *with* (3.1.17). *If τ_0 is a simple root of $\Omega(\tau)$ then $\theta_0 = (\tau_0,\tau_0)$ is a simple zero point of $N_1(\theta)$ given by* (3.1.18).

To be more concrete, we take in (3.1.17)

$$g(x_1,x_2) = \big(x_2, x_1 - 2x_1^3\big), \quad \alpha = (\beta,\beta)\,.$$

Hence (3.1.13) has the form

$$\begin{aligned}\ddot{z} &= x - 2x^3, \quad \ddot{y} = y - 2y^3,\\ x(i+) &= x(i-) + \mu\big((1+\mu)y(i-)+\beta\big),\\ \dot{x}(i+) &= \dot{x}(i-) + \mu\big((1+\mu)\dot{y}(i-)+\beta\big),\\ y(i+) &= y(i-) + \mu\big((1+\mu^2)x(i-)+\beta\big),\\ \dot{y}(i+) &= \dot{y}(i-) + \mu\big((1+\mu^2)\dot{x}(i-)+\beta\big), \quad i\in\mathbb{Z}\,.\end{aligned} \tag{3.1.19}$$

(3.1.19) are two Duffing equations coupled by impulsive effects. We now take $\gamma(t) = \big(\operatorname{sech} t, \dot{\operatorname{sech}}\, t\big)$ and Ω has the form

$$\Omega(\tau) = \sum_{k\in\mathbb{Z}} \operatorname{sech}^4(\tau+k) + \beta \sum_{k\in\mathbb{Z}} \frac{3-\mathrm{e}^{2(\tau+k)}}{2} \operatorname{sech}^3(\tau+k)\,.$$

Consequently, we have

$$\Omega(\tau) = \Omega_1(\tau) - \beta\Omega_2(\tau)\,,$$

where

$$\Omega_1(\tau) = \sum_{k\in\mathbb{Z}} \operatorname{sech}^4(\tau+k), \quad \Omega_2(\tau) = \sum_{k\in\mathbb{Z}} \frac{\mathrm{e}^{2(\tau+k)}-3}{2} \operatorname{sech}^3(\tau+k)\,.$$

The functions $\Omega_{1,2}$ are again 1-periodic. Moreover, they are analytic and Ω_1 is positive (cf Section 2.6.5). Clearly, Ω_2/Ω_1 is non-constant. So the image of $\mathbb{R}$ by Ω_2/Ω_1 is an interval $[a_1,a_2]$, $-\infty < a_1 < a_2 < \infty$ and there is only a finite number of $\beta_1,\dots,\beta_{j_0} \in [a_1,a_2]$ so that $\Omega = \Omega_1 - \beta\Omega_2$ does have a simple root for any $\beta \neq 0$ satisfying $1/\beta \in [a_1,a_2] \setminus \{\beta_1,\dots,\beta_{j_0}\}$.

Numerical evaluation of the graph of $\Omega_2(\tau)/\Omega_1(\tau)$ shows that (Figure 3.1)

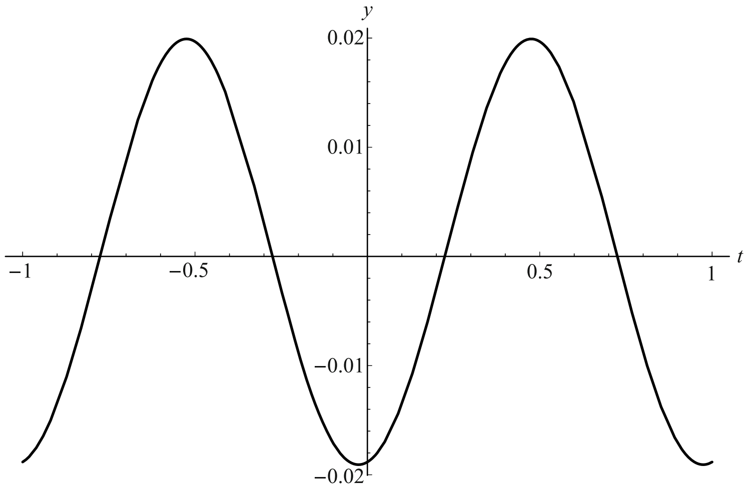

Fig. 3.1 The graph of function $y = \Omega_2(\tau)/\Omega_1(\tau)$.

$$a_1 = \beta_1 \simeq -0.0190729, \quad a_2 = \beta_2 \simeq 0.0199198, \quad j_0 = 2\,.$$

In summary, we arrive at the following result.

Theorem 3.1.14. *If either $\beta < -52.431$ or $\beta > 50.202$ then impulsive system* (3.1.19) *has a chaotic behaviour for any $\mu \neq 0$ sufficiently small.*

We note that a coupled two McMillan mappings (cf Section 3.2.4 and [4,5]) can be similarly studied. In general, after applying our results, the main difficulty is to find an appropriate form of the Melnikov mapping derived in the above way so that one could be able to detect its simple zero point. The Poisson summation formula like in [4] could help to overcome this difficulty.

Remark 3.1.15. Similar to the above, we can study more general impulsive ODEs of the form

$$\begin{aligned} \dot{x} &= f(x,\varepsilon), \\ x(i+0) &= x(i-0) + \varepsilon a\big(x(i-0),\varepsilon\big), \quad i \in \mathbb{Z}, \end{aligned} \tag{3.1.20}$$

where $f, a \in C^2(\mathbb{R}^{n+1},\mathbb{R}^n)$, $f(\cdot,0)$ has a hyperbolic fixed point x_0 with a homoclinic orbit $\gamma(\cdot)$. Furthermore, assume that the adjoint variational equation

$$\dot{v} = -\Big(D_x f\big(\gamma(t),0\big)\Big)^* v$$

has only a unique (up to constant multiples) bounded nonzero solution u. Then the Melnikov function of (3.1.20) has the form

$$\mathscr{M}(t) = \sum_{i=-\infty}^{\infty} \big\langle a\big(\gamma(t+i),0\big), u(t+i)\big\rangle + \int_{-\infty}^{\infty} \big\langle D_\varepsilon f\big(\gamma(s),0\big), u(s)\big\rangle\, ds\,. \qquad (3.1.21)$$

Note that formula (3.1.21) follows also from considerations of Sections 3.3 and 3.4. We see that (3.1.21) consists of the continuous and impulsive parts of (3.1.20) as well.

Finally we note that a different type of chaos is studied in [21] for a special initial value problem of a non-autonomous impulsive differential equation. ODEs with step function coefficients are studied in [22–28], and our theory can be applied to such ODEs.

3.2 Transversal Homoclinic Orbits

3.2.1 Higher Dimensional Difference Equations

This section is a continuation of Section 3.1. So we consider difference equation

$$x_{n+1} = g(x_n) + \varepsilon h(n, x_n, \varepsilon) \qquad (3.2.1)$$

where $x_n \in \mathbb{R}^N$, $\varepsilon \in \mathbb{R}$ is a small parameter. The main purpose of this section is to study the homoclinic bifurcations of difference equations in a degenerate case. We assume the following conditions about the difference equation (3.2.1):

(H1) g, h are C^3-smooth in all continuous variables.

(H2) The unperturbed difference equation

$$x_{n+1} = g(x_n) \qquad (3.2.2)$$

has a hyperbolic fixed poitnt 0, that is, the eigenvalues of $g_x(0)$ are non-zero and they lie off the unit circle.

(H3) The unperturbed difference equation (3.2.2) has a one-parameter family of homoclinic solutions $\gamma(\alpha) = \{\gamma_n(\alpha)\}_{-\infty}^{\infty}$, $\alpha \in \mathbb{R}$ connecting 0. That is, $\{\gamma_n(\alpha)\}_{-\infty}^{\infty}$ is a non-zero sequence of C^3-smooth vector functions satisfying $\gamma_{n+1}(\alpha) = g(\gamma_n(\alpha))$ and $\lim_{n\to\pm\infty} \gamma_n(\alpha) = 0$ uniformly with respect to bounded α. The set $\cup_{n\in\mathbb{Z}} \cup_{\alpha\in\mathbb{R}} \{\gamma_n(\alpha)\}$ is bounded.

(H4) $g_x(\gamma_n(\alpha))$ is invertible, and $\|g_x^{-1}(\gamma_n(\alpha))\|$ is uniformly bounded on $\mathbb{Z}$.

We denote by $W^s(0)$ and $W^u(0)$ the stable and unstable manifolds of the hyperbolic fixed point 0, respectively, and by $T_{\gamma_0(\alpha)}W^s(0)$ and $T_{\gamma_0(\alpha)}W^u(0)$ the tangent spaces to $W^s(0)$ and $W^u(0)$ at $\gamma_0(\alpha)$. We say the homoclinic orbit $\{\gamma_n(\alpha)\}_{-\infty}^{\infty}$ is *degenerate* if the dimension of the linear subspace

$$T_{\gamma_0(\alpha)}W^s(0) \cap T_{\gamma_0(\alpha)}W^u(0)$$

is greater than one. Otherwise, we say the homoclinic orbit $\{\gamma_n(\alpha)\}_{-\infty}^{\infty}$ is *nondegenerate*. We can easily prove that the homoclinic orbit $\{\gamma_n(\alpha)\}_{-\infty}^{\infty}$ is degenerate if and only if the following variational equation along the homoclinic orbit $\{\gamma_n(\alpha)\}_{-\infty}^{\infty}$

$$\xi_{n+1} = g_x(\gamma_n(\alpha))\xi_n \tag{3.2.3}$$

has $d > 1$ linearly independent bounded solutions on $\mathbb{Z}$.

When h is independent of n, i.e. (3.2.1) is a mapping, the existence of a transversal homoclinic solution for (3.2.1) is discussed in [8, 29]. When h depends on n, the existence of a transversal homoclinic solution for (3.2.1) in the degenerate case is discussed in Section 3.1. Now we study (3.2.1) also with $d > 1$ for (3.2.3). Our aim is to find analytic conditions under which the difference equation (3.2.1) has for $\varepsilon \neq 0$ sufficiently small a transversal bounded solution $\{x_n(\varepsilon)\}_{-\infty}^{\infty}$ near the homoclinic solution $\{\gamma_n(\alpha)\}_{-\infty}^{\infty}$. The transversality of $\{x_n(\varepsilon)\}_{-\infty}^{\infty}$ means that the linearization of the difference equation (3.3.1) along $\{x_n(\varepsilon)\}_{-\infty}^{\infty}$ given by

$$\xi_{n+1} = [g_x(x_n(\varepsilon)) + \varepsilon h_x(n, x_n(\varepsilon), \varepsilon)]\,\xi_n$$

admits an exponential dichotomy on $\mathbb{Z}$ (cf Lemma 2.5.2).

The degenerate problem, when $d > 1$ for (3.2.3), can be naturally divided into two cases:

(1) There exists a d-dimensional homoclinic manifold. This is the most natural way to get $d > 1$ for (3.2.3).
(2) The invariant manifolds $W^s(0)$ and $W^u(0)$ meet in only a higher dimensional tangency.

Case (1) is studied in Section 3.1.4 (see also more comments at the end of Section 3.2.2), and Case 2 is treated in this section.

Two-dimensional mappings for nondegenerate cases are considered in [2, 4, 5]. Higher dimensional mappings are studied in [7].

3.2.2 Bifurcation Result

Let

$$X = \Big\{ \{x_n\}_{-\infty}^{\infty} \} \mid \ |x_n \in \mathbb{R}^N \quad \text{and} \quad \sup_{n\in\mathbb{Z}} |x_n| < \infty \Big\}$$

be the Banach space with the norm $|x| = \sup_{n\in\mathbb{Z}} |x_n|$ for $x = \{x_n\}_{-\infty}^{\infty}$. We define a linear operator L as follows:

$$L : X \to X, \quad (L\xi)_n = \xi_{n+1} - g_x(\gamma_n(\alpha))\xi_n$$

where $\xi = \{\xi_n\}_{-\infty}^{\infty}$ and $L\xi = \{(L\xi)_n\}_{-\infty}^{\infty}$. Theorem 3.1.4 has the following equivalent form [29].

Lemma 3.2.1. *Suppose conditions* (*H*1)-(*H*4) *are satisfied. Then*

(i) *The operator L is Fredholm with index zero.*
(ii) $f = \{f_n\}_{-\infty}^{\infty} \in \mathscr{R}L$ *if and only if*

$$\sum_{n=-\infty}^{+\infty} \psi_n^*(\alpha) \cdot f_n = 0 \tag{3.2.4}$$

holds for all bounded solutions $\psi(\alpha) = \{\psi_n(\alpha)\}_{-\infty}^{\infty}$ *of the adjoint variational equation*

$$\xi_{n+1} = (g_x^*(\gamma_{n+1}(\alpha)))^{-1}\xi_n. \tag{3.2.5}$$

(iii) *If* (3.2.4) *holds, then the difference equation*

$$x_{n+1} = g_x(\gamma_n(\alpha))x_n + f_n$$

has a unique bounded solution $x = \{x_n\}_{-\infty}^{\infty}$ *on* $\mathbb{Z}$ *satisfying*

$$\varphi_0^*(\alpha) \cdot x_0 = 0$$

for all bounded solutions $\varphi(\alpha) = \{\varphi_n(\alpha)\}_{-\infty}^{\infty}$ *of the linear difference equation* (3.2.3) *on* $\mathbb{Z}$.

From condition (H3), we have $\gamma_{n+1}(\alpha) = g(\gamma_n(\alpha))$. Differentiating both sides of this difference equation with respect to α, we obtain $\dot{\gamma}_{n+1}(\alpha) = g_x(\gamma_n(\alpha))\dot{\gamma}_n(\alpha)$, where "$\cdot$" $= \frac{d}{d\alpha}$. Hence $\dot{\gamma}(\alpha) = \{\dot{\gamma}_n(\alpha)\}_{-\infty}^{\infty}$ is a nontrivial bounded solution on $\mathbb{Z}$ of the variational equation (3.2.3). That is, $\dot{\gamma}_0(\alpha) \in T_{\gamma_0(\alpha)}W^s(0) \cap T_{\gamma_0(\alpha)}W^u(0)$. We assume that

(H5) $\dim(T_{\gamma_0(\alpha)}W^s(0) \cap T_{\gamma_0(\alpha)}W^u(0)) = d \quad (d \geq 1)$ for a constant d uniformly with respect to α.

Condition (H5) is equivalent to the condition that the variational equation (3.2.3) has d (≥ 1) linearly independent bounded solutions on $\mathbb{Z}$, denoted by

$$\varphi_1(\alpha) = \dot{\gamma}(\alpha) = \{\dot{\gamma}_n(\alpha)\}_{-\infty}^{\infty},$$
$$\varphi_2(\alpha) = \{\varphi_{2,n}(\alpha)\}_{-\infty}^{\infty}, \ \ldots \ , \ \varphi_d(\alpha) = \{\varphi_{d,n}(\alpha)\}_{-\infty}^{\infty}.$$

We let

$$\Phi_n(\alpha) = \Big(\varphi_{1,n}(\alpha), \varphi_{2,n}(\alpha), \ldots, \varphi_{d,n}(\alpha)\Big)$$

be an $N \times d$ matrix and

$$\Phi_n^0(\alpha) = \Big(\varphi_{2,n}(\alpha), \ldots, \varphi_{d,n}(\alpha)\Big)$$

be an $N \times (d-1)$ matrix. From Section 3.1.2 it follows that under conditions (H1)–(H5), the adjoint equation (3.2.5) also has d and only d linearly independent bounded solutions on $\mathbb{Z}$, denoted by

$$\{\psi_{1,n}(\alpha)\}_{-\infty}^{\infty}, \quad \{\psi_{2,n}(\alpha)\}_{-\infty}^{\infty}, \ \dots \ , \{\psi_{d,n}(\alpha)\}_{-\infty}^{\infty}.$$

We let

$$\Psi_n(\alpha) = \Big(\psi_{1,n}(\alpha), \psi_{2,n}(\alpha), \dots, \psi_{d,n}(\alpha)\Big)$$

be an $N \times d$ matrix. We suppose that $\Phi_n(\alpha)$ and $\Psi_n(\alpha)$ are C^3-smooth in α for any $n \in \mathbb{Z}$. The main result of this section is the following theorem.

Theorem 3.2.2. *Suppose conditions* (*H*1)–(*H*5) *are satisfied. We define a Melnikov vector mapping by*

$$M(\alpha,\beta) = \sum_{n=-\infty}^{\infty} \Psi_n^*(\alpha) \cdot \Big\{2h(n,\gamma_n(\alpha),0) + g_{xx}(\gamma_n(\alpha))(\Phi_n^0(\alpha)\beta, \Phi_n^0(\alpha)\beta)\Big\}.$$

If there exists $(\alpha_0,\beta_0) \in \mathbb{R} \times \mathbb{R}^{d-1}$ *so that*

$$M(\alpha_0,\beta_0) = 0 \quad \textit{and} \quad \det D_{(\alpha,\beta)} M(\alpha_0,\beta_0) \neq 0,$$

then for ε *sufficiently small, there exist two continuously differentiable functions* $\alpha = \alpha(\varepsilon)$, $\beta = \beta(\varepsilon)$, *satisfying* $\alpha(0) = \alpha_0$, $\beta(0) = \beta_0$ *so that for* $\varepsilon \neq 0$ *sufficiently small, the difference equation*

$$x_{n+1} = g(x_n) + \varepsilon^2 h(n,x_n,\varepsilon^2)$$

has a bounded solution $x(\varepsilon) = \{x_n(\varepsilon)\}_{-\infty}^{\infty}$ *so that*

$$|x_n(\varepsilon) - \gamma_n(\alpha(\varepsilon)) - \varepsilon\Phi_n^0(\alpha(\varepsilon))\beta(\varepsilon)| = O(\varepsilon^2) \tag{3.2.6}$$

and the variational equation

$$\xi_{n+1} = \{g_x(x_n(\varepsilon)) + \varepsilon^2 h_x(n,x_n(\varepsilon),\varepsilon^2)\}\xi_n$$

admits an exponential dichotomy on $\mathbb{Z}$.

Proof. First of all, we prove the existence of a bounded solution $\{x_n(\varepsilon)\}_{-\infty}^{\infty}$. We make a change of variables

$$y_n = x_n - \gamma_n(\alpha) - \Phi_n^0(\alpha)\beta$$

for the difference equation (3.2.1), where $\beta \in \mathbb{R}^{d-1}$ is a vector parameter. Then the difference equation (3.2.1) reads

$$\begin{aligned} y_{n+1} &= g(y_n + \gamma_n(\alpha) + \Phi_n^0(\alpha)\beta) + \varepsilon h(n, y_n + \gamma_n(\alpha) + \Phi_n^0(\alpha)\beta, \varepsilon) \\ &\quad - g(\gamma_n(\alpha)) - g_x(\gamma_n(\alpha))\Phi_n^0(\alpha)\beta. \end{aligned} \tag{3.2.7}$$

For simplicity, we define

$$\begin{aligned} G(n,y_n,\alpha,\beta,\varepsilon) &= \varepsilon h(n,y_n+\gamma_n(\alpha)+\Phi_n^0(\alpha)\beta,\varepsilon)-g(\gamma_n(\alpha)) \\ &\quad +g(y_n+\gamma_n(\alpha)+\Phi_n^0(\alpha)\beta)-g_x(\gamma_n(\alpha))(y_n+\Phi_n^0(\alpha)\beta), \end{aligned}$$

then the difference equation (3.2.7) can be written as

$$y_{n+1}=g_x(\gamma_n(\alpha))y_n+G(n,y_n,\alpha,\beta,\varepsilon)\,. \tag{3.2.8}$$

We put

$$D(\alpha)=\sum_{n=-\infty}^{\infty}\Psi_n^*(\alpha)\cdot\Psi_n(\alpha),$$

so then the $d\times d$ matrix $D(\alpha)$ is invertible [30, p. 129]. Using the Lyapunov-Schmidt method and Lemma 3.2.1, we see that the difference equation (3.2.8) is equivalent to the following two equations

$$\begin{aligned} y_{n+1} &= g_x(\gamma_n(\alpha))y_n+G(n,y_n,\alpha,\beta,\varepsilon) \\ &\quad -\Psi_n(\alpha)D^{-1}(\alpha)\sum_{j=-\infty}^{\infty}\Psi_j^*(\alpha)G(j,y_j,\alpha,\beta,\varepsilon)\,, \end{aligned} \tag{3.2.9}$$

and

$$\sum_{n=-\infty}^{\infty}\Psi_n^*(\alpha)G(n,y_n,\alpha,\beta,\varepsilon)=0\,. \tag{3.2.10}$$

Since

$$\sum_{n=-\infty}^{\infty}\Psi_n^*(\alpha)\Big\{G(n,y_n,\alpha,\beta,\varepsilon)-\Psi_n(\alpha)D^{-1}(\alpha)\sum_{j=-\infty}^{\infty}\Psi_j^*(\alpha)G(j,y_j,\alpha,\beta,\varepsilon)\Big\}=0\,,$$
$$G(n,0,\alpha,0,0)=0 \quad \text{and} \quad G_y(n,0,\alpha,0,0)=0\,,$$

it follows from Lemma 3.2.1 and the implicit function theorem that for ε,β sufficiently small, the difference equation (3.2.9) has a unique small bounded solution $y=y(\alpha,\beta,\varepsilon)=\{y_n(\alpha,\beta,\varepsilon)\}_{-\infty}^{\infty}\in X$ satisfying

$$\Phi_0^*(\alpha)y_0(\alpha,\beta,\varepsilon)=0\,. \tag{3.2.11}$$

Clearly $y(\alpha,0,0)=0$. We substitute

$$y=y(\alpha,\beta,\varepsilon)=\{y_n(\alpha,\beta,\varepsilon)\}_{-\infty}^{\infty}$$

into Eq. (3.2.10) and obtain the following bifurcation equation

$$\overline{B}(\alpha,\beta,\varepsilon)=\sum_{n=-\infty}^{\infty}\Psi_n^*(\alpha)G(n,y_n(\alpha,\beta,\varepsilon),\alpha,\beta,\varepsilon)=0\,. \tag{3.2.12}$$

To solve Eq. (3.2.12), we consider the equation

$$B(\alpha,\beta,\varepsilon)=\overline{B}(\alpha,\varepsilon\beta,\varepsilon^2)=\sum_{n=-\infty}^{\infty}\Psi_n^*(\alpha)G(n,y_n(\alpha,\varepsilon\beta,\varepsilon^2),\alpha,\varepsilon\beta,\varepsilon^2)=0\,.$$

If $Y_n(\varepsilon)=y_n(\alpha,\varepsilon\beta,\varepsilon^2)$, then we have

$$\begin{aligned}Y_{n+1}(\varepsilon) &= g_x(\gamma_n(\alpha))Y_n(\varepsilon)+G(n,Y_n(\varepsilon),\alpha,\varepsilon\beta,\varepsilon^2)\\ &\quad -\Psi_n(\alpha)D^{-1}(\alpha)\sum_{j=-\infty}^{\infty}\Psi_j^*(\alpha)G(j,Y_j(\varepsilon),\alpha,\varepsilon\beta,\varepsilon^2)\,.\end{aligned} \tag{3.2.13}$$

Differentiating both sides of the difference equation (3.2.13) with respect to ε and setting $\varepsilon=0$ and noting that $Y_n(0)=0$, we obtain

$$Y_{n+1}^{\varepsilon}(0)=g_x(\gamma_n(\alpha))Y_n^{\varepsilon}(0)$$

where $Y_n^{\varepsilon}(0)=\frac{d}{d\varepsilon}Y_n(\varepsilon)|_{\varepsilon=0}$. Moreover, (3.2.11) implies $\Phi_0^*(\alpha)Y_0^{\varepsilon}(0)=0$. By the uniqueness of the bounded solution of the linear difference equation (3.2.3) satisfying (3.2.11) we have $Y_n^{\varepsilon}(0)=0$. We conclude

$$B(\alpha,\beta,0)=\sum_{n=-\infty}^{\infty}\Psi_n^*(\alpha)G(n,y_n(\alpha,0,0),\alpha,0,0)=\sum_{n=-\infty}^{\infty}\Psi_n^*(\alpha)G(n,0,\alpha,0,0)=0$$

and

$$\begin{aligned}B_{\varepsilon}(\alpha,\beta,\varepsilon) &= \sum_{n=-\infty}^{\infty}\Psi_n^*(\alpha)\Big\{2\varepsilon h\left(n,y_n(\alpha,\varepsilon\beta,\varepsilon^2)+\gamma_n(\alpha)+\varepsilon\Phi_n^0(\alpha)\beta,\varepsilon^2\right)\\ &\quad +\varepsilon^2\frac{d}{d\varepsilon}h\left(n,y_n(\alpha,\varepsilon\beta,\varepsilon^2)+\gamma_n(\alpha)+\varepsilon\Phi_n^0(\alpha)\beta,\varepsilon^2\right)\\ &\quad +g_x\left(y_n(\alpha,\varepsilon\beta,\varepsilon^2)+\gamma_n(\alpha)+\varepsilon\Phi_n^0(\alpha)\beta\right)\cdot\\ &\quad \frac{d}{d\varepsilon}\left[y_n(\alpha,\varepsilon\beta,\varepsilon^2)+\gamma_n(\alpha)+\varepsilon\Phi_n^0(\alpha)\beta\right]\\ &\quad -g_x(\gamma_n(\alpha))\frac{d}{d\varepsilon}\left[y_n(\alpha,\varepsilon\beta,\varepsilon^2)+\gamma_n(\alpha)+\varepsilon\Phi_n^0(\alpha)\beta\right]\Big\}\,.\end{aligned} \tag{3.2.14}$$

Noting $y_n(\alpha,0,0)=0$ and $Y_n^{\varepsilon}(0)=0$, we have

$$B_{\varepsilon}(\alpha,\beta,0)=0\,. \tag{3.2.15}$$

From (3.2.14) and $y_n(\alpha,0,0)=0$ and $Y_n^{\varepsilon}(0)=0$, we compute

$$\begin{aligned}B_{\varepsilon\varepsilon}(\alpha,\beta,0) &= \sum_{n=-\infty}^{\infty}\Psi_n^*(\alpha)\Big\{2h(n,\gamma_n(\alpha),0)+g_{xx}(\gamma_n(\alpha))(Y_n^{\varepsilon}(0)+\Phi_n^0(\alpha)\beta,\\ &\quad Y_n^{\varepsilon}(0)+\Phi_n^0(\alpha)\beta)+g_x(\gamma_n(\alpha))Y_n^{\varepsilon\varepsilon}(0)-g_x(\gamma_n(\alpha))Y_n^{\varepsilon\varepsilon}(0)\Big\}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=-\infty}^{\infty} \Psi_n^*(\alpha)\Big\{2h(n,\gamma_n(\alpha),0) + g_{xx}(\gamma_n(\alpha))(\Phi_n^0(\alpha)\beta, \Phi_n^0(\alpha)\beta)\Big\} \\
&= M(\alpha,\beta)
\end{aligned}
$$

where $Y_n^{\varepsilon\varepsilon}(0) = \frac{d^2}{d\varepsilon^2} Y_n(\varepsilon)|_{\varepsilon=0}$. We define the function $H(\alpha,\beta,\varepsilon)$ by

$$
H(\alpha,\beta,\varepsilon) = \begin{cases} \dfrac{B(\alpha,\beta,\varepsilon)}{\varepsilon^2}, & \text{if} \quad \varepsilon \neq 0, \\[2ex] \dfrac{1}{2} B_{\varepsilon\varepsilon}(\alpha,\beta,0), & \text{if} \quad \varepsilon = 0. \end{cases}
$$

Since $B(\alpha,\beta,0) = 0$ and (3.2.15) holds, the function $H(\alpha,\beta,\varepsilon)$ is continuously differentiable in α,β,ε. From the conditions of Theorem 3.2.2, we have

$$
H(\alpha_0,\beta_0,0) = \frac{1}{2} B_{\varepsilon\varepsilon}(\alpha_0,\beta_0,0) = \frac{1}{2} M(\alpha_0,\beta_0) = 0
$$

and

$$
\det D_{(\alpha,\beta)} H(\alpha_0,\beta_0,0) = \frac{1}{2^d} \det D_{(\alpha,\beta)} M(\alpha_0,\beta_0) \neq 0 .
$$

It follows from the implicit function theorem that for ε sufficiently small, there exist two continuously differentiable functions $\alpha = \alpha(\varepsilon)$ and $\beta = \beta(\varepsilon)$ satisfying $\alpha(0) = \alpha_0$ and $\beta(0) = \beta_0$, respectively, so that $H(\alpha(\varepsilon),\beta(\varepsilon),\varepsilon) = 0$. Hence for $\varepsilon \neq 0$ sufficiently small, we have that $B(\alpha(\varepsilon),\beta(\varepsilon),\varepsilon) = 0$. Thus for $\varepsilon \neq 0$ sufficiently small, the difference equation

$$
x_{n+1} = g(x_n) + \varepsilon^2 h(n,x_n,\varepsilon^2)
$$

has a unique bounded solution $\{x_n(\varepsilon)\}_{-\infty}^{\infty}$ with

$$
x_n(\varepsilon) = y_n(\alpha(\varepsilon),\varepsilon\beta(\varepsilon),\varepsilon^2) + \gamma_n(\alpha(\varepsilon)) + \varepsilon \Phi_n^0(\alpha(\varepsilon))\beta(\varepsilon)
$$

satisfying (3.2.6). This completes the proof of the existence part of the theorem.

Finally, the transversality of the bounded solution $\{x_n(\varepsilon)\}_{-\infty}^{\infty}$ can be proved in the same way as in Theorem 3.1.7, so we omit the proof. □

In the degenerate Case 1 from Section 3.2.1 one would start with a family of homoclinic solutions $\gamma(\alpha) = \{\gamma_n(\alpha)\}_{-\infty}^{\infty}$ with $\alpha \in \mathbb{R}^d$ like in condition (H3). For bounded solutions to the variational equation (3.2.3) in accordance with the above notations one now has

$$
\varphi_i(\alpha) = \left\{ \frac{\partial \gamma_n}{\partial \alpha_i}(\alpha) \right\}_{-\infty}^{\infty}, \quad i = 1,2,\ldots,d .
$$

Using the formula

$$\frac{\partial^2 \gamma_{n+1}}{\partial \alpha_j \partial \alpha_i}(\alpha) = g_x(\gamma_n(\alpha)) \frac{\partial^2 \gamma_n}{\partial \alpha_j \partial \alpha_i}(\alpha) + g_{xx}(\gamma_n(\alpha)) \left(\frac{\partial \gamma_n}{\partial \alpha_j}(\alpha), \frac{\partial \gamma_n}{\partial \alpha_i}(\alpha) \right)$$

it is easy to show by Lemma 3.2.1 that for this case in the Melnikov vector mapping of Theorem 3.2.2 the β terms are identically zero. The Melnikov vector mapping here is

$$M(\alpha) = \sum_{-\infty}^{\infty} \Psi_n^*(\alpha) \cdot h(n, \gamma_n(\alpha), 0), \quad \alpha \in \mathbb{R}^d .$$

We remark that Case 1 is already studied in Section 3.1. We also mention that the vanishing of the β terms in the Melnikov vector mapping of Theorem 3.2.2 is a necessary but not sufficient condition for Case 1. This means that in the general theory, if one computes $d > 1$ for condition (H5) and then finds that all the β terms vanish one cannot apply Theorem 3.2.2 and does not know if Case 1 can be applied or if there is some other higher degeneracy. Then higher-order Melnikov vector mappings could help to study the homoclinic bifurcations of the difference equation (3.2.1).

Finally, we get the above Melnikov vector mapping $M(\alpha)$ also for the case $d = 1$ in condition (H5), but now $\alpha \in \mathbb{R}$. So M is a function.

3.2.3 *Applications to McMillan Type Mappings*

We consider the following mapping of a McMillan type (cf Section 3.2.4 and [4, 5, 7])

$$\begin{aligned} z_{n+1} &= y_n, \quad y_{n+1} = -z_n + 2K \frac{y_n}{1+y_n^2} + v_n^2 - \varepsilon y_n , \\ u_{n+1} &= v_n, \quad v_{n+1} = -u_n + 2K v_n \frac{1-y_n^2}{(1+y_n^2)^2} + u_n^2 - \varepsilon z_n \end{aligned} \tag{3.2.16}$$

where $K > 1$ is a constant. By Section 3.2.4 we know that

$$\begin{aligned} \gamma_n(\alpha) &= \big(r_n(\alpha), r_{n+1}(\alpha), 0, 0\big) , \\ r_n(\alpha) &= \sinh w \operatorname{sech}(\alpha - nw), \quad w = \cosh^{-1} K, \quad w > 0 \end{aligned}$$

is a bounded solution of (3.2.16) with $\varepsilon = 0$. Then (3.2.3) has now the form

$$\begin{aligned} a_{n+1} &= b_n, \quad b_{n+1} = -a_n + 2K \frac{1 - r_{n+1}^2(\alpha)}{(1 + r_{n+1}^2(\alpha))^2} b_n , \\ c_{n+1} &= d_n, \quad d_{n+1} = -c_n + 2K \frac{1 - r_{n+1}^2(\alpha)}{(1 + r_{n+1}^2(\alpha))^2} d_n . \end{aligned} \tag{3.2.17}$$

The equilibrium $(0,0,0,0)$ of the unperturbed mapping is hyperbolic with 2-dimensional stable and unstable parts. We can easily verify from (3.2.17) that now $d = 2$ and

$$\Phi_n^0(\alpha) = \left(0, 0, r_n'(\alpha), r_{n+1}'(\alpha)\right).$$

We note that

$$\{\dot{\gamma}_n(\alpha)\}_{-\infty}^{\infty} = \left\{(r_n'(\alpha), r_{n+1}'(\alpha), 0, 0)\right\}_{-\infty}^{\infty}$$

is another solution of (3.2.17) bounded on $\mathbb{Z}$. We also remark that by (3.2.17), the unperturbed mapping of (3.2.16) with $\varepsilon = 0$ is volume preserving on the set $\{\gamma_n(\alpha)\}_{-\infty}^{\infty}$. Then according to Lemma 3.1.11, we find

$$\Psi_n(\alpha) = \begin{pmatrix} r_{n+1}'(\alpha) & 0 \\ -r_n'(\alpha) & 0 \\ 0 & r_{n+1}'(\alpha) \\ 0 & -r_n'(\alpha) \end{pmatrix}.$$

Furthermore, in the notations of the previous section we have

$$g_{xx}(\gamma_n(\alpha))\left(\Phi_n^0(\alpha)\beta, \Phi_n^0(\alpha)\beta\right) = \left(0, 2r_{n+1}'(\alpha)^2\beta^2, 0, 2r_n'(\alpha)^2\beta^2\right),$$
$$h(n, \gamma_n(\alpha), 0) = (0, -r_{n+1}(\alpha), 0, -r_n(\alpha)).$$

Consequently, the Melnikov vector mapping has the form

$$M(\alpha,\beta) = (M_1(\alpha,\beta), M_2(\alpha,\beta))$$

where

$$M_1(\alpha,\beta) = 2\sum_{n=-\infty}^{\infty} r_n'(\alpha) r_{n+1}(\alpha) - 2\beta^2 \sum_{n=-\infty}^{\infty} r_{n+1}'(\alpha)^2 r_n'(\alpha),$$
$$M_2(\alpha,\beta) = 2\sum_{n=-\infty}^{\infty} r_n'(\alpha) r_n(\alpha) - 2\beta^2 \sum_{n=-\infty}^{\infty} r_n'(\alpha)^3.$$

We conclude

$$\begin{aligned} A_1(w) &= \sum_{n=-\infty}^{\infty} r_n'(0) r_{n+1}(0) = \sinh^2 w \sum_{n=1}^{\infty} \left(\operatorname{sech}(n+1)w - \operatorname{sech}(n-1)w\right) \\ &\quad \times \operatorname{sech}^2 nw \sinh nw < 0, \\ A_2(w) &= \sum_{n=-\infty}^{\infty} r_{n+1}'(0)^2 r_n'(0) = \sinh^3 w \sum_{n=1}^{\infty} \left(\operatorname{sech}^4(n+1)w \sinh^2(n+1)w \right. \\ &\quad \left. - \operatorname{sech}^4(n-1)w \sinh^2(n-1)w\right) \times \operatorname{sech}^2 nw \sinh nw, \end{aligned}$$

$$\sum_{n=-\infty}^{\infty} (r_n''(0) r_n(0) + r_n'(0)^2) = \sinh^2 w \left(-1 + 2\sum_{n=1}^{\infty} \operatorname{sech}^4 nw \,(\cosh 2nw - 2)\right),$$

$$\sum_{n=1}^{\infty} r_n'(0)^2 r_n''(0) = \sinh^3 w \sum_{n=1}^{\infty} \operatorname{sech}^7 nw \sinh^2 nw (\cosh^2 nw - 2),$$

$$\sum_{n=-\infty}^{\infty} r_n'(0) r_n(0) = \sinh^2 w \sum_{n=-\infty}^{\infty} \operatorname{sech}^3 nw \sinh nw = 0,$$

$$\sum_{n=-\infty}^{\infty} r_n'(0)^3 = \sinh^3 w \sum_{n=-\infty}^{\infty} \operatorname{sech}^6 nw \sinh^3 nw = 0,$$

$$\frac{\partial}{\partial \alpha} M_2(0,\beta) = 2 \sum_{n=-\infty}^{\infty} (r_n''(0) r_n(0) + r_n'(0)^2) - 12\beta^2 \sum_{n=1}^{\infty} r_n'(0)^2 r_n''(0) = A_3(w,\beta)\,.$$

The above series are very difficult to evaluate and they could be expressed in terms of Jacobi elliptic functions [4]. Instead, we use the following lemmas.

Lemma 3.2.3. *Let $F : [0,\infty) \to \mathbb{R}$ be such that $|F(x)| \le c_1 \mathrm{e}^{-c_2 x}$ for positive constants c_1, c_2. Then*

$$\left| \sum_{n=1}^{\infty} F(n\tilde{w}) \right| \le 2c_1 \mathrm{e}^{-c_2 \tilde{w}}$$

for any $\tilde{w} \ge \ln 2 / c_2$.

Lemma 3.2.4. *Let $F, G : [0,\infty) \to \mathbb{R}$ be such that $G(0) = 0$, and*

$$c_1 \mathrm{e}^{-\theta_1 x} \le F(x) \le c_2 \mathrm{e}^{-\theta_1 x}, \quad d_1 \mathrm{e}^{-\theta_2 x} \le G(x) \le d_2 \mathrm{e}^{-\theta_2 x}$$

for any $x \ge 1$ and positive constants $c_i, d_i, \theta_i, i = 1,2$. Then for any $\tilde{w} \ge 1$, we have

$$\begin{aligned} &\frac{c_1 d_1 \mathrm{e}^{-(2\theta_2+\theta_1)\tilde{w}} - c_2 d_2 \mathrm{e}^{-(2\theta_1+\theta_2)\tilde{w}}}{1 - \mathrm{e}^{-(\theta_1+\theta_2)\tilde{w}}} \\ &\le \sum_{n=1}^{\infty} (G((n+1)\tilde{w}) - G((n-1)\tilde{w})) F(n\tilde{w}) \\ &\le \frac{c_2 d_2 \mathrm{e}^{-(2\theta_2+\theta_1)\tilde{w}} - c_1 d_1 \mathrm{e}^{-(2\theta_1+\theta_2)\tilde{w}}}{1 - \mathrm{e}^{-(\theta_1+\theta_2)\tilde{w}}}\,. \end{aligned}$$

Proofs of the above lemmas are elementary, so we omit them. We apply Lemma 3.2.4 with $G(x) = \operatorname{sech}^4 x \sinh^2 x$, $F(x) = \operatorname{sech}^2 x \sinh x$. Then using

$$\mathrm{e}^{-x} \le \operatorname{sech} x \le 2\mathrm{e}^{-x}, \quad x \ge 0,$$

$$\frac{\mathrm{e}^2 - 1}{2\mathrm{e}^2} \mathrm{e}^x \le \sinh x \le \mathrm{e}^x/2, \quad x \ge 1,$$

we get $c_1 = \frac{\mathrm{e}^2-1}{2\mathrm{e}^2}$, $c_2 = 2$, $d_1 = \left(\frac{\mathrm{e}^2-1}{2\mathrm{e}^2}\right)^2$, $d_2 = 4$, $\theta_1 = 1$ and $\theta_2 = 2$, and then we obtain

$$A_2(w) \le \sinh^3 w \frac{8\mathrm{e}^{-5w} - \left(\frac{\mathrm{e}^2-1}{2\mathrm{e}^2}\right)^3 \mathrm{e}^{-4w}}{1 - \mathrm{e}^{-3w}} < 0$$

for any $w > \ln\left[\frac{64\,\mathrm{e}^6}{(\mathrm{e}^2-1)^3}\right] \doteq 4.59512$. Similarly, using Lemma 3.2.3, $\cosh^2 1 > 2$ and

$$\left|\mathrm{sech}^4 x(\cosh 2x - 2)\right| \le 32\,\mathrm{e}^{-4x} + 16\,\mathrm{e}^{-2x}, \quad x \ge 0,$$

we derive

$$A_3(w,\beta) \le 2\sinh^2 w\left(-1 + 64\,\mathrm{e}^{-4w} + 32\,\mathrm{e}^{-2w}\right) < 0$$

for any $w > \frac{1}{2}\ln\left[8\left(\sqrt{5}+2\right)\right] \doteq 1.76154$. We already know that $A_1(w) < 0$. Hence $\alpha = 0$, $\beta = \sqrt{A_1(w)/A_2(w)} \neq 0$ is a simple zero of $M(\alpha,\beta) = 0$ for any $w > \ln\left[\frac{64\,\mathrm{e}^6}{(\mathrm{e}^2-1)^3}\right]$, i.e. $K > K_0 := \frac{4096\,\mathrm{e}^{12} + \left(\mathrm{e}^2-1\right)^6}{128\,\mathrm{e}^6\left(\mathrm{e}^2-1\right)^3} \doteq 49.5052$. Now we can apply Theorem 3.2.2 to (3.2.16), and we produce the following result.

Theorem 3.2.5. *For any $K > K_0$, there is an $\varepsilon_0 > 0$ so that* (3.2.16) *exhibits chaos for any* $0 < \varepsilon < \varepsilon_0$.

Of course, either more precise analytical or numerical evaluations of $A_2(w)$ and $A_3(w,\beta)$ could give also partial results for $1 < K \le K_0$. But we do not carry out these computations in this book. We only note that our numerical computations suggest that $K \ge \cosh 0.1 \doteq 1.005$ for obtaining chaos in (3.2.16) for $\varepsilon > 0$ small.

3.2.4 Planar Integrable Maps with Separatrices

A planar map is called a *standard-like* one if it has a form $F(x,y) = (y, -x + g(y))$ for some smooth g. Note that F is *area-preserving*, i.e. $|\det DF(x,y)| = 1$. A planar map F is *integrable* if there is a function (a first integral) $H : \mathbb{R}^2 \to \mathbb{R}$ so that $H \circ F = H$. An interesting family of standard-like and integrable maps is given by [5]

$$F(x,y) := \left(y, -x + 2y\frac{K + \beta y}{1 - 2\beta y + y^2}\right), \quad -1 < \beta < 1 < K \tag{3.2.18}$$

with the corresponding first integrals

$$H_{K,\beta}(x,y) = x^2 - 2Kxy + y^2 - 2\beta xy(x+y) + x^2y^2.$$

Map (3.2.18) with $\beta = 0$ is called *McMillan map*. Next, (3.2.18) has two separatrices $\Gamma^{\pm}_{K,\beta} = \{\gamma^{\pm}_n(\alpha)\}_{n\in\mathbb{Z}}$ contained in the level $H_{K,\beta} = 0$ given by $\gamma^{\pm}_n(\alpha) = (r^{\pm}_n(\alpha), r^{\pm}_{n+1}(\alpha))$ with

$$r^{\pm}_n(\alpha) := \pm\frac{\sinh w \sinh\frac{w}{2}}{\sqrt{\beta^2 + \sinh^2\frac{w}{2}\cosh(\alpha - nw) \mp \beta\cosh\frac{w}{2}}},$$

for $w = \cosh^{-1} K$. Clearly example (3.2.16) can be extended with (3.2.18), but we do not go into details.

3.3 Singular Impulsive ODEs

3.3.1 Singular ODEs with Impulses

The theory of impulsive differential equations is an important branch of differential equations with many applications [16–20]. So in this section, we continue to study such systems by considering the problem

$$\begin{aligned} \varepsilon x' &= f(x) + \varepsilon h(x), \\ x(i+) &= x(i-) + \varepsilon g\big(x(i-)\big), \quad i \in \mathbb{Z}, \end{aligned} \tag{3.3.1}$$

when the following assumptions are valid

(H1) $f, g, h \in C^3(\mathbb{R}^m, \mathbb{R}^m)$.
(H2) $0 \in \mathbb{R}^m$ is a hyperbolic equilibrium of $x' = f(x)$.
(H3) The equation $x' = f(x)$ has a homoclinic orbit ϕ to 0.
(H4) The variational equation $v' = Df(\phi)v$ has the unique (up to scalar multiples) bounded solution ϕ' on $\mathbb{R}$.

By Section 4.1.2, we know that (H3) and (H4) imply the uniqueness (up to scalar multiples) of a bounded solution ψ on $\mathbb{R}$ of the adjoint variational equation $\psi' = -\big(Df(\phi)\big)^* \psi$. By a solution of (3.3.1) we mean a function $x(t)$, which is C^1–smooth on $\mathbb{R} \setminus \mathbb{Z}$, satisfies the differential equation in (3.3.1) on this set and the impulsive conditions in (3.3.1) hold as well.

For simplicity, we assume f, h, g to be globally Lipschitz continuous. Let us denote by $\Phi_\varepsilon(t, x_0)$ the unique solution of the differential equation of (3.3.1) with the initial condition $\Phi_\varepsilon(0, x_0) = x_0$ for $\varepsilon > 0$. Then we can define the Poincarè map of (3.3.1) by the formula

$$\pi_\varepsilon(x) = \Phi_\varepsilon\big(1, x + \varepsilon g(x)\big).$$

Of course, the dynamics of (3.3.1) is wholly determined by π_ε.

The purpose of this section is to show the existence of a transversal homoclinic point of π_ε for any $\varepsilon > 0$ sufficiently small (cf Theorem 3.3.10). Then, according to Smale-Birkhoff homoclinic theorem 2.5.4, Equations (3.3.1) will have a chaotic behaviour for $\varepsilon > 0$ sufficiently small. To detect transversal homoclinic orbits of π_ε for $\varepsilon > 0$ small, we derive the Melnikov function of (3.3.1) given by the formula

$$\mathscr{M}(\beta) \equiv \big\langle g\big(\phi(\beta)\big), \psi(\beta)\big\rangle_m + \int_{-\infty}^{\infty} \big\langle h\big(\phi(s)\big), \psi(s)\big\rangle_m \, ds, \tag{3.3.2}$$

where $\langle\cdot,\cdot\rangle_m$ is the usual inner product on $\mathbb{R}^m$. We see from the form of $\mathscr{M}$ that chaos in (3.3.1) can be made only by the impulsive effects, as the integral part of $\mathscr{M}$ containing h is independent of β. Of course, this fact is natural since the ODE (3.3.1) is autonomous. For the readers' convenience, we note that the approach of this section can be simply generalized to study periodic perturbations of (3.3.1), i.e. if $h = h(x,t)$ and $h(\cdot,t+1) = h(\cdot,t)\,\forall t \in \mathbb{R}$. Since the period of h in t is the same as the period of the impulsive conditions, the Poincarè map π_ε can be straightforwardly extended for this case. Then the Melnikov function is

$$\bar{\mathscr{M}}(\beta) = \big\langle g\big(\phi(\beta)\big), \psi(\beta)\big\rangle_m + \int_{-\infty}^{\infty} \big\langle h\big(\phi(s),0\big), \psi(s)\big\rangle_m ds\,, \quad \beta \in \mathbb{R}\,.$$

We are motivated to study such impulsive Duffing–type equations by [31] of the form

$$\begin{aligned} z'' + a^2 p(z) &= aq(z)\,, \\ a\big(z(i+) - z(i-)\big) &= r\big(z(i-)\big)\,, \\ z'(i+) &= z'(i-), \quad i \in \mathbb{Z}\,, \end{aligned} \tag{3.3.3}$$

where $a > 0$ is a large parameter, p, q, $r \in C^3(\mathbb{R},\mathbb{R})$.

3.3.2 Linear Singular ODEs with Impulses

In this section, we derive Fredholm–like alternative results of certain linear impulsive ODEs which are linearizations of (3.3.1). Let $|\cdot|_m$ be the corresponding norm to $\langle\cdot,\cdot\rangle_m$, and set $\mathbb{N}_- = -\mathbb{N}$. Now we introduce several Banach spaces:

$$\begin{aligned} X^m &= \Big\{ x : \mathbb{R}\setminus\mathbb{Z} \to \mathbb{R}^m \,\Big|\, x \text{ is continuous and bounded on } \mathbb{R}\setminus\mathbb{Z} \\ &\qquad \text{and it has } x(i\pm) = \lim_{s\to 0_\pm} x(i+s)\,\forall i \in \mathbb{Z} \Big\}\,, \\ X_1^m &= \Big\{ x \in X^m \mid x' \in X^m \Big\}\,, \\ X_+^m &= \Big\{ x : \mathbb{R}_+\setminus\mathbb{N} \to \mathbb{R}^m \,\Big|\, x \text{ is continuous and bounded on } \mathbb{R}_+\setminus\mathbb{N} \\ &\qquad \text{and it has } x(i+), x(i-)\,\forall i \in \mathbb{N} \Big\}\,, \end{aligned}$$

$$X_-^m = \left\{ x : \mathbb{R}_- \setminus \mathbb{N}_- \to \mathbb{R}^m \,\middle|\, x \text{ is continuous and bounded on } \mathbb{R}_- \setminus \mathbb{N}_- \right.$$
$$\left. \text{and it has } x(i+), x(i-) \,\forall\, i \in \mathbb{N}_- \right\},$$

$$Y_+^m = \left\{ \{a_n\}_{n\in\mathbb{N}} \,\middle|\, a_n \in \mathbb{R}^m, \sup_n |a_n|_m < \infty \right\},$$
$$Y_-^m = \left\{ \{a_n\}_{n\in\mathbb{N}_-} \,\middle|\, a_n \in \mathbb{R}^m, \sup_n |a_n|_m < \infty \right\},$$
$$Y^m = \left\{ \{a_n\}_{n\in\mathbb{Z}} \,\middle|\, a_n \in \mathbb{R}^m, \sup_n |a_n|_m < \infty \right\}.$$

The norms on these spaces are the usual supremum norms. For instance, the norm on X^m is defined by
$$||x||_m = \sup_{s\in\mathbb{R}\setminus\mathbb{Z}} |x(s)|_m .$$

The norm on X_1^m is denoted by $||\cdot||_{m1}$ and on Y^m by $|||\cdot|||_m$. We note that $||x||_{m1} = ||x||_m + ||x'||_m$.

In the first part of this section, we consider the following linear equation suggested by (3.3.1)
$$\begin{aligned} y' &= D_\beta(t)y + q(t), \\ y(i/\varepsilon+) &= y(i/\varepsilon-) + b_i, \quad i \in \mathbb{Z}, \end{aligned} \tag{3.3.4}$$

where $\beta \in \mathbb{R}$, $\varepsilon > 0$ are fixed, $D_\beta(t) = Df\big(\phi(\beta+t)\big)$, $b_i \in \mathbb{R}^m$, $q \in X^m$ and $y(i/\varepsilon\pm) = y(\frac{i}{\varepsilon}\pm)$.

Let $Z_\beta(t)$ be the fundamental solution of $y' = D_\beta(t)y$. Then by Section 2.5.1, this equation has dichotomies on both $\mathbb{R}_+$ and $\mathbb{R}_-$, i.e. there are projections $P_\pm : \mathbb{R}^m \to \mathbb{R}^m$ and constants $K > 0$, $\alpha > 0$ so that

$$\begin{aligned} |Z_\beta(t)P_\pm Z_\beta^{-1}(s)| &\le K\mathrm{e}^{-\alpha(t-s)}, \quad t \ge s, \\ |Z_\beta(t)(\mathbb{I} - P_\pm)Z_\beta^{-1}(s)| &\le K\mathrm{e}^{-\alpha(s-t)}, \quad s \ge t, \end{aligned}$$

where s, t are nonnegative, and nonpositive, for P_+, P_-, respectively . Note that K, α are independent of β, while $P_\pm = P_\pm^\beta = Z_0(\beta)P_\pm^0 Z_0^{-1}(\beta)$.

Theorem 3.3.1. *The problem*

$$\begin{aligned} y' &= D_\beta(t)y + q(t), \\ y(i/\varepsilon+) &= y(i/\varepsilon-) + b_i, \quad i \in \mathbb{N}, \\ P_+y(0) &= \xi \in \mathscr{R}P_+, \end{aligned} \tag{3.3.5}$$

has a unique solution $y \in X^m_+$ *for any* $q \in X^m_+$, $\{b_i\}_{i\in\mathbb{N}} \in Y^m_+$. *Moreover, for any* $0 < \varepsilon < \tilde{c}$ *and a fixed constant* $\tilde{c} > 0$, *it holds*

$$||y||_m \leq c\big(|||\{b_i\}_{i\in\mathbb{N}}|||_m + |\xi|_m + ||q||_m\big)\,.$$

Throughout this section c is a generic constant.

Proof. *Uniqueness.* If $q = 0$, $b_i = 0$, $\xi = 0$ in (3.3.5), then the solution has the form $Z_\beta(t)y_0$, $P_+y_0 = 0$. So $Z_\beta(t)y_0 = Z_\beta(t)(\mathbb{I}-P_+)y_0$. As

$$|y_0|_m = |(\mathbb{I}-P_+)y_0|_m = |(\mathbb{I}-P_+)Z_\beta^{-1}(t)Z_\beta(t)y_0|_m \leq K\mathrm{e}^{-\alpha t}|Z_\beta(t)y_0|_m\,,$$

we have, by the boundedness of $Z_\beta(t)y_0$, $y_0 = 0$. The uniqueness is proved.

Existence. Let us put for $0 \leq n/\varepsilon < t < (n+1)/\varepsilon$ and any $n \in \mathbb{N}\cup\{0\}$

$$\begin{aligned} y(t) &= Z_\beta(t)\xi + \sum_{k=1}^{n} Z_\beta(t)P_+Z_\beta^{-1}(k/\varepsilon)b_k \\ &\quad - \sum_{k=n+1}^{\infty} Z_\beta(t)(\mathbb{I}-P_+)Z_\beta^{-1}(k/\varepsilon)b_k + \int_0^t Z_\beta(t)P_+Z_\beta^{-1}(s)q(s)\,ds \\ &\quad - \int_t^\infty Z_\beta(t)(\mathbb{I}-P_+)Z_\beta^{-1}(s)q(s)\,ds \end{aligned}$$

where we set, for the case $n = 0$, $\sum_{k=1}^{n} Z_\beta(t)P_+Z_\beta^{-1}(k/\varepsilon)b_k \equiv 0$. Now, we compute for $0 < \varepsilon < \tilde{c}$

$$\begin{aligned} |y(t)|_m &\leq K\mathrm{e}^{-\alpha t}|\xi|_m + \sum_{k=1}^{n} K\mathrm{e}^{-\alpha(t-\frac{k}{\varepsilon})}|b_k|_m \\ &\quad + \sum_{k=n+1}^{\infty} K\mathrm{e}^{-\alpha(\frac{k}{\varepsilon}-t)}|b_k|_m + \int_0^t K\mathrm{e}^{-\alpha(t-s)}||q||_m\,ds + \int_t^\infty K\mathrm{e}^{-\alpha(s-t)}||q||_m\,ds \\ &\leq K|\xi|_m + K\sup_k|b_k|_m\left(\sum_{k=1}^{n}\mathrm{e}^{-\alpha(t-\frac{k}{\varepsilon})} + \sum_{k=n+1}^{\infty}\mathrm{e}^{-\alpha(\frac{k}{\varepsilon}-t)}\right) \\ &\quad + K||q||_m\left(\int_0^t \mathrm{e}^{-\alpha(t-s)}\,ds + \int_t^\infty \mathrm{e}^{-\alpha(s-t)}\,ds\right) \\ &\leq K|\xi|_m + K\sup_k|b_k|_m\left(\frac{\mathrm{e}^{-\alpha(t-\frac{n}{\varepsilon})}}{1-\mathrm{e}^{-\alpha/\varepsilon}} + \frac{\mathrm{e}^{-\alpha(\frac{n+1}{\varepsilon}-t)}}{1-\mathrm{e}^{-\alpha/\varepsilon}}\right) + K||q||_m\frac{2}{\alpha} \\ &\leq K|\xi|_m + \frac{2K}{1-\mathrm{e}^{-\alpha/\varepsilon}}\sup_k|b_k|_m + K||q||_m\frac{2}{\alpha} \\ &\leq K|\xi|_m + \frac{2K}{1-\mathrm{e}^{-\alpha/\tilde{c}}}\sup_k|b_k|_m + K||q||_m\frac{2}{\alpha}\,. \end{aligned}$$

So $y(t)$ satisfies the inequality of this theorem. It is not difficult to see that we can take derivatives with respect to t term by term in the series and with the integral sign so that $y(t)$ satisfies the differential equation in (3.3.4).

To check the impulsive conditions, we compute for $i \in \mathbb{N}$

$$\begin{aligned}
y(i/\varepsilon+) - y(i/\varepsilon-) &= \sum_{k=1}^{i} Z_\beta(i/\varepsilon)P_+ Z_\beta^{-1}(k/\varepsilon)b_k - \sum_{k=i+1}^{\infty} Z_\beta(i/\varepsilon)(\mathbb{I}-P_+)Z_\beta^{-1}(k/\varepsilon)b_k \\
&\quad - \sum_{k=1}^{i-1} Z_\beta(i/\varepsilon)P_+ Z_\beta^{-1}(k/\varepsilon)b_k + \sum_{k=i}^{\infty} Z_\beta(i/\varepsilon)(\mathbb{I}-P_+)Z_\beta^{-1}(k/\varepsilon)b_k \\
&= Z_\beta(i/\varepsilon)P_+ Z_\beta^{-1}(i/\varepsilon)b_i + Z_\beta(i/\varepsilon)(\mathbb{I}-P_+)Z_\beta^{-1}(i/\varepsilon)b_i \\
&= Z_\beta(i/\varepsilon)Z_\beta^{-1}(i/\varepsilon)b_i = b_i\,.
\end{aligned}$$

Finally

$$P_+ y(0) = P_+\xi - P_+\left(\sum_{k=1}^{\infty}(\mathbb{I}-P_+)Z_\beta^{-1}(k/\varepsilon)b_k + \int_0^{\infty}(\mathbb{I}-P_+)Z_\beta^{-1}(s)q(s)\,ds\right) = \xi\,.$$

The proof is finished. □

Theorem 3.3.2. *The problem*

$$\begin{aligned}
y' &= D_\beta(t)y + q(t)\,, \\
y(i/\varepsilon+) &= y(i/\varepsilon-) + b_i, \quad i \in \mathbb{N}_-\,, \\
(\mathbb{I}-P_-)y(0) &= \eta \in \mathscr{R}(\mathbb{I}-P_-)\,,
\end{aligned} \tag{3.3.6}$$

has a unique solution $y \in X_-^m$ for any $q \in X_-^m$, $\{b_i\}_{i\in\mathbb{N}_-} \in Y_-^m$. Moreover, for any $0 < \varepsilon < \tilde{c}$ and a fixed constant $\tilde{c} > 0$, it holds

$$||y||_m \le c\big(|||\{b_i\}_{i\in\mathbb{N}_-}|||_m + |\eta|_m + ||q||_m\big)\,.$$

Proof. The uniqueness is the same as in the proof of Theorem 3.3.1. For the existence, let us take for $n/\varepsilon < t < (n+1)/\varepsilon \le 0$ and any $n \in \mathbb{N}_-$

$$\begin{aligned}
y(t) = Z_\beta(t)\eta &+ \sum_{k=-\infty}^{n} Z_\beta(t)P_- Z_\beta^{-1}(k/\varepsilon)b_k \\
&- \sum_{k=n+1}^{-1} Z_\beta(t)(\mathbb{I}-P_-)Z_\beta^{-1}(k/\varepsilon)b_k + \int_{-\infty}^{t} Z_\beta(t)P_- Z_\beta^{-1}(s)q(s)\,ds \\
&- \int_t^0 Z_\beta(t)(\mathbb{I}-P_-)Z_\beta^{-1}(s)q(s)\,ds\,,
\end{aligned}$$

where we set again, for the case $n = -1$, $\sum_{k=n+1}^{-1} Z_\beta(t)(\mathbb{I}-P_-)Z_\beta^{-1}(k/\varepsilon)b_k \equiv 0$. The rest of the proof is the same as in Theorem 3.3.1, and so we omit it. The proof is finished. □

Now we can state the main result concerning (3.3.4).

Theorem 3.3.3. *For any* $\{b_i\}_{i\in\mathbb{Z}} \in Y^m$ *and* $q \in X^m$*, Equation* (3.3.4) *has a solution* $y \in X_1^m$ *if and only if*

$$\sum_{i=-\infty}^{\infty} \left\langle b_i, \psi\left(\beta + \frac{i}{\varepsilon}\right)\right\rangle_m + \int_{-\infty}^{\infty} \langle q(s), \psi(\beta+s)\rangle_m \, ds = 0\,. \tag{3.3.7}$$

This solution is unique provided

$$\int_{-\infty}^{\infty} \big\langle y(s), \phi'(\beta+s)\big\rangle_m ds = 0$$

and, for any $0 < \varepsilon < \tilde{c}$ *and a fixed constant* $\tilde{c} > 0$*, it satisfies*

$$||y||_{m1} \le c\big(\sup_i |b_i|_m + ||q||_m\big)\,.$$

Proof. Uniqueness. Assume that $y_1(t), y_2(t)$ are two solutions of (3.3.4) both satisfying the condition

$$\int_{-\infty}^{\infty} \big\langle y(s), \phi'(\beta+s)\big\rangle_m ds = 0.$$

Then $y(t) = y_1(t) - y_2(t)$ satisfies $y'(t) = D_\beta(t)y(t)$ together with $y(i/\varepsilon+) = y(i/\varepsilon-)$, so that $y(t)$ is a C^1-bounded function on $\mathbb{R}$ satisfying the linear homogeneous differential equation $y'(t) = D_\beta(t)y(t)$. Hence $y(0) \in \mathscr{R}P_+ \cap \mathscr{R}(\mathbb{I} - P_-)$ or $y(0) = \lambda\phi'(\beta)$. As a consequence $y(t) = \lambda\phi'(t+\beta)$ and then

$$\lambda \int_{-\infty}^{\infty} |\phi'(\beta+s)|^2 ds = \int_{-\infty}^{\infty} \big\langle y(s), \phi'(\beta+s)\big\rangle_m ds = 0\,.$$

This fact implies $\lambda = 0$ or $y_1(t) = y_2(t)$.

Existence. For any $\xi \in \mathscr{R}P_+$ and $\eta \in \mathscr{R}(\mathbb{I} - P_-)$ let y_+, y_- be the solutions of (3.3.5) and (3.3.6), respectively. We compute

$$\begin{aligned} y_+(0) - y_-(0) = {} & \xi - \sum_{k=1}^{\infty} (\mathbb{I} - P_+) Z_\beta^{-1}(k/\varepsilon) b_k - \int_0^{\infty} (\mathbb{I} - P_+) Z_\beta^{-1}(s) q(s)\, ds \\ & -\eta - \sum_{k=-\infty}^{-1} P_- Z_\beta^{-1}(k/\varepsilon) b_k - \int_{-\infty}^{0} P_- Z_\beta^{-1}(s) q(s)\, ds\,. \end{aligned}$$

As we also require $y_+(0) - y_-(0) = b_0$, we obtain

$$\begin{aligned} \xi - \eta = b_0 & + \sum_{k=1}^{\infty} (\mathbb{I} - P_+) Z_\beta^{-1}(k/\varepsilon) b_k + \sum_{k=-\infty}^{-1} P_- Z_\beta^{-1}(k/\varepsilon) b_k \\ & + \int_0^{\infty} (\mathbb{I} - P_+) Z_\beta^{-1}(s) q(s)\, ds + \int_{-\infty}^{0} P_- Z_\beta^{-1}(s) q(s)\, ds\,. \end{aligned} \tag{3.3.8}$$

Equation (3.3.8) is solvable if and only if the right-hand side is in the space

$$\mathscr{R}P_+ + \mathscr{R}(\mathbb{I} - P_-)\,,$$

i.e. if and only if the right-hand side of (3.3.8) is orthogonal to any element of the space

$$\big(\mathscr{R}P_+ + \mathscr{R}(\mathbb{I} - P_-)\big)^\perp = \mathscr{R}P_+^\perp \cap \mathscr{R}(\mathbb{I} - P_-)^\perp = \mathscr{N}P_+^* \cap \mathscr{N}(\mathbb{I} - P_-^*)\,.$$

But it is clear that $\mathscr{N}P_+^* \cap \mathscr{N}(\mathbb{I} - P_-^*)$ is the space of all initial values y_0 for which the solution of the adjoint equation $y' = -D_\beta^*(t)y$ is bounded on $\mathbb{R}$. This assertion follows from the fact that $(Z_\beta^*)^{-1}(t)$ is the fundamental solution of the equation $y' = -D_\beta^*(t)y$ possessing dichotomies on both $\mathbb{R}_+$ and $\mathbb{R}_-$ with the projections $\mathbb{I} - P_+^*$, $\mathbb{I} - P_-^*$, respectively. In our case,

$$\mathscr{N}P_+^* \cap \mathscr{N}(\mathbb{I} - P_-^*) = \operatorname{span}\{\psi(\beta)\}\,.$$

Hence (3.3.8) is solvable if and only if the following holds

$$\begin{aligned}
0 &= \Bigg\langle \psi(\beta), b_0 + \sum_{k=1}^{\infty}(\mathbb{I} - P_+)Z_\beta^{-1}(k/\varepsilon)b_k + \sum_{k=-\infty}^{-1} P_- Z_\beta^{-1}(k/\varepsilon)b_k \\
&\qquad + \int_0^\infty (\mathbb{I} - P_+)Z_\beta^{-1}(s)q(s)\,ds + \int_{-\infty}^0 P_- Z_\beta^{-1}(s)q(s)\,ds \Bigg\rangle_m \\
&= \langle \psi(\beta), b_0\rangle_m \\
&\quad + \sum_{k=1}^{\infty} \langle (Z_\beta^*)^{-1}(k/\varepsilon)(\mathbb{I} - P_+^*)\psi(\beta), b_k\rangle_m + \sum_{k=-\infty}^{-1} \langle (Z_\beta^*)^{-1}(k/\varepsilon)P_-^*\psi(\beta), b_k\rangle_m \\
&\quad + \int_0^\infty \langle q(s), (Z_\beta^*)^{-1}(s)(\mathbb{I} - P_+^*)\psi(\beta)\rangle_m\,ds + \int_{-\infty}^0 \langle q(s), (Z_\beta^*)^{-1}(s)P_-^*\psi(\beta\rangle_m\,ds \\
&= \langle \psi(\beta), b_0\rangle_m + \sum_{k=1}^{\infty}\left\langle \psi\left(\beta + \frac{k}{\varepsilon}\right), b_k\right\rangle_m + \sum_{k=-\infty}^{-1}\left\langle \psi\left(\beta + \frac{k}{\varepsilon}\right), b_k\right\rangle_m \\
&\quad + \int_0^\infty \langle q(s), \psi(\beta + s)\rangle_m\,ds + \int_{-\infty}^0 \langle q(s), \psi(\beta + s)\rangle_m\,ds \\
&= \sum_{i=-\infty}^{\infty}\left\langle \psi\left(\beta + \frac{i}{\varepsilon}\right), b_i\right\rangle_m + \int_{-\infty}^{\infty} \langle q(s), \psi(\beta + s)\rangle_m\,ds\,.
\end{aligned}$$

We have used the identities

$$\begin{aligned}
(Z_\beta^*)^{-1}(s)(\mathbb{I} - P_+^*)\psi(\beta) &= \psi(\beta + s)\,, \quad \forall s \ge 0\,, \\
(Z_\beta^*)^{-1}(s)P_-^*\psi(\beta) &= \psi(\beta + s)\,, \quad \forall s \le 0\,,
\end{aligned}$$

which follow from the facts that $(Z_\beta^*)^{-1}(t)$ is the fundamental solution of the equation $y' = -D_\beta^*(t)y$ possessing dichotomies on both R_+ and $\mathbb{R}_-$ with the projections

$\mathbb{I}-P_+^*, \mathbb{I}-P_-^*$, respectively, and $\psi(\beta+\cdot)$ is a bounded solution of this equation on $\mathbb{R}$.

So (3.3.8) is solvable if and only if (3.3.7) holds. Moreover, for any $0<\varepsilon<\tilde{c}$, with $\tilde{c}>0$ being a fixed constant, we have

$$|\xi-\eta|_m \le c\big(\sup_n |b_n|_m + ||q||_m\big)\,.$$

Such ξ, η are not unique, since $\mathscr{R}P_+ \cap \mathscr{R}(\mathbb{I}-P_-) = \operatorname{span}\{\phi'(\beta)\}$. However we can obtain uniqueness asking, for example, that η is orthogonal to $\phi'(\beta)$. That is, in Eq. (3.3.8) we take $\xi \in \mathscr{R}P_+$ and $\eta \in \mathscr{S} = \{\eta \in \mathscr{R}(\mathbb{I}-P_-) \,|\, \langle \eta, \phi'(\beta)\rangle_m = 0\}$. Of course, $\mathscr{R}P_+ \oplus \mathscr{S} = \mathscr{R}P_+ + \mathscr{R}(\mathbb{I}-P_-)$, but the direct sum implies the uniqueness. Then we obtain a solution $(\xi_1,\eta_1) \in \mathscr{R}P_+ \oplus \mathscr{S}$ so that

$$|\xi_1|_m + |\eta_1|_m \le c\big(\sup_n |b_n|_m + ||q||_m\big)\,,$$

for any $0<\varepsilon<\tilde{c}$ ($\tilde{c}>0$ being a fixed constant). So (3.3.4) has a solution $y = y_1\big(\{b_n\}_{n=-\infty}^{\infty}, q\big)$ satisfying

$$||y_1||_m \le c\big(\sup_n |b_n|_m + ||q||_m\big)\,,$$

for any $0<\varepsilon<\tilde{c}$, if and only if (3.3.7) holds. As $\phi'(\beta+t)$ is a bounded solution of (3.3.4) with $q=0$, $b_i = 0\,\forall i \in \mathbb{Z}$, by putting

$$y(t) = y_1(t) - \phi'(\beta+t)\int_{-\infty}^{\infty}\langle y_1(s), \phi'(\beta+s)\rangle_m ds \Big/ \int_{-\infty}^{\infty}|\phi'(s)|_m^2 ds\,,$$

we obtain another solution of (3.3.4) satisfying

$$\int_{-\infty}^{\infty}\langle y(s), \phi'(\beta+s)\rangle_m ds = 0\,.$$

Of course, we also have

$$||y||_m \le c\big(\sup_n |b_n|_m + ||q||_m\big)\,,$$

for any $0<\varepsilon<\tilde{c}$. As $y'(t) = D_\beta(t)y(t) + q(t)$ we easily obtain the conclusion of this theorem. □

Remark 3.3.4. Let β_0 be a fixed real number. Then the proof of Theorem 3.3.3 can be repeated to obtain a unique solution of (3.3.4) satisfying the condition

$$\int_{-\infty}^{\infty}\langle y(s), \phi'(\beta_0+s)\rangle_m ds = 0\,,$$

provided $|\beta-\beta_0|$ is sufficiently small. This fact will be used in the proof of Theorem 3.3.8.

In the last part of this section, we consider the following linear equation suggested by (3.3.1)

$$\begin{aligned} y' &= Df(0)y + q(t)\,, \\ y(i/\varepsilon+) &= y(i/\varepsilon-) + b_i, \quad i \in \mathbb{Z}\,, \end{aligned} \tag{3.3.9}$$

where $\varepsilon > 0$ is fixed and $b_i \in \mathbb{R}^m$, $q \in X^m$. Let $Z(t)$ be the fundamental solution of $y' = Df(0)y$. Since 0 is hyperbolic for the equation $x' = f(x)$, there is a projection $Q : \mathbb{R}^m \to \mathbb{R}^m$ and constants $M > 0$, $\omega > 0$ so that

$$\begin{aligned} |Z(t)QZ^{-1}(s)| &\le M\mathrm{e}^{-\omega(t-s)}, \quad t \ge s\,, \\ |Z(t)(\mathbb{I}-Q)Z^{-1}(s)| &\le M\mathrm{e}^{-\omega(s-t)}, \quad s \ge t\,. \end{aligned}$$

By repeating the proof of Theorems 3.3.1 and 3.3.2, we obtain the following results.

Theorem 3.3.5. *The problem*

$$\begin{aligned} y' &= Df(0)y + q(t)\,, \\ y(i/\varepsilon+) &= y(i/\varepsilon-) + b_i, \quad i \in \mathbb{N}\,, \\ Qy(0) &= \xi \in \mathscr{R}Q\,, \end{aligned}$$

has a unique solution $y \in X_+^m$ *for any* $q \in X_+^m$, $\{b_i\}_{i\in\mathbb{N}} \in Y_+^m$. *Moreover, for any* $0 < \varepsilon < \tilde{c}$ *and a fixed constant* $\tilde{c} > 0$, *it holds*

$$||y||_m \le c\big(|||\{b_i\}_{i\in\mathbb{N}}|||_m + |\xi|_m + ||q||_m\big)\,.$$

Theorem 3.3.6. *The problem*

$$\begin{aligned} y' &= Df(0)y + q(t)\,, \\ y(i/\varepsilon+) &= y(i/\varepsilon-) + b_i, \quad i \in \mathbb{N}_-\,, \\ (I-Q)y(0) &= \eta \in \mathscr{R}(\mathbb{I}-Q)\,, \end{aligned}$$

has a unique solution $y \in X_-^m$ *for any* $q \in X_-^m$, $\{b_i\}_{i\in\mathbb{N}_-} \in Y_-^m$. *Moreover, for any* $0 < \varepsilon < \tilde{c}$ *and a fixed constant* $\tilde{c} > 0$, *it holds*

$$||y||_m \le c\big(|||\{b_i\}_{i\in\mathbb{N}_-}|||_m + |\eta|_m + ||q||_m\big)\,.$$

Now we can state our main result concerning (3.3.9).

Theorem 3.3.7. *For any* $\{b_i\}_{i\in\mathbb{Z}} \in Y^m$ *and* $q \in X^m$, *Equation* (3.3.9) *has a unique solution* $y \in X^m$ *satisfying*

$$||y||_{m1} \le c\big(\sup_i |b_i|_m + ||q||_m\big)\,,$$

for any $0 < \varepsilon < \tilde{c}$ *and a fixed constant* $\tilde{c} > 0$.

Proof. The proof of Theorem 3.3.3 can be repeated up to Eq. (3.3.8). Now Eq. (3.3.8) is always solvable, since

$$\big(\mathscr{R}Q+\mathscr{R}(\mathbb{I}-Q)\big)^{\perp}=\mathscr{N}Q^{*}\cap\mathscr{N}(\mathbb{I}-Q^{*})=\{0\}\,.$$

Moreover, such a solution is unique, because

$$\mathscr{R}Q\cap\mathscr{R}(\mathbb{I}-Q)=\{0\}\,.$$

So (3.3.9) has the desired solution. The proof is finished. □

3.3.3 Derivation of the Melnikov Function

In this section, we show chaotic behaviour of the Poincarè map π_ε of (3.3.1) for $\varepsilon>0$ small. For this purpose, we derive a Melnikov function for (3.3.1) to show the existence of a transversal homoclinic orbit of π_ε for $\varepsilon>0$ small. By taking the scale of the time $t\leftrightarrow\varepsilon t$, we have

$$\begin{aligned}x'&=f(x)+\varepsilon h(x)\,,\\ x(i/\varepsilon+)&=x(i/\varepsilon-)+\varepsilon g(x(i/\varepsilon-)),\quad i\in\mathbb{Z}\,.\end{aligned}\tag{3.3.10}$$

Equation (3.3.10) can be rewritten in the form $F_\varepsilon=0$, where

$$F_\varepsilon:X_1^m\to X^m\times Y^m=\mathscr{X}^m\,,$$

$$F_\varepsilon(x)=\left(x'-f(x)-\varepsilon h(x),\left\{x(i/\varepsilon+)-x(i/\varepsilon-)-\varepsilon g\big(x(i/\varepsilon-)\big)\right\}_{i\in\mathbb{Z}}\right).$$

We solve $F_\varepsilon=0$ by the Lyapunov–Schmidt method. But this method cannot be applied directly, since F_ε is not defined for $\varepsilon=0$. We overcome this difficulty by Theorems 3.3.3 and 3.3.7. Let β_0 be a fixed real number. Setting

$$x=z+\phi_\beta,\quad \phi_\beta(t)=\phi(\beta+t)\,,$$

we can write (3.3.10) as

$$\begin{aligned}z'&=D_\beta(t)z+\left\{f(z+\phi_\beta)-f(\phi_\beta)-D_\beta(t)z\right\}+\varepsilon h(z+\phi_\beta)\,,\\ z(i/\varepsilon+)&=z(i/\varepsilon-)+\varepsilon g\big(z(i/\varepsilon-)+\phi_\beta(i/\varepsilon)\big),\quad i\in\mathbb{Z}\,,\\ &\int_{-\infty}^{\infty}\big\langle z(s),\phi'(\beta_0+s)\big\rangle_m ds=0\,,\end{aligned}\tag{3.3.11}$$

where $|\beta-\beta_0|$ is sufficiently small. Finally, Equation (3.3.11) is rewritten, by applying the Lyapunov-Schmidt procedure, in the form

$$
\begin{aligned}
z' - D_\beta(t)z &= P(\varepsilon,\beta,z)\Big(\big\{f(z+\phi_\beta)-f(\phi_\beta)-D_\beta(t)z\big\}+\varepsilon h(z+\phi_\beta)\Big),\\
z(i/\varepsilon+)-z(i/\varepsilon-) &= \varepsilon g\big(z(i/\varepsilon-)+\phi_\beta(i/\varepsilon)\big), \quad i\in\mathbb{Z}, \qquad (3.3.12)\\
\int_{-\infty}^{\infty}\big\langle z(s),\phi'(\beta_0+s)\big\rangle_m ds &= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&P(\varepsilon,\beta,z)\Big(\big\{f(z+\phi_\beta)-f(\phi_\beta)-D_\beta(t)z\big\}+\varepsilon h(z+\phi_\beta)\Big)\\
&=\big\{f(z+\phi_\beta)-f(\phi_\beta)-D_\beta(t)z\big\}+\varepsilon h(z+\phi_\beta)
\end{aligned}
\qquad (3.3.13)
$$

where

$$
P_d p = -\left[\left(d+\int_{-\infty}^{\infty}\big\langle p(s),\psi(\beta+s)\big\rangle_m ds\right)\Big/\int_{-\infty}^{\infty}|\psi(\beta+s)|_m^2 ds\right]\cdot\psi(\beta+\cdot)+p
$$

$$
d=\varepsilon\sum_{i=-\infty}^{\infty}\left\langle g\big(z(i/\varepsilon-)+\phi_\beta(i/\varepsilon)\big),\psi\left(\beta+\frac{i}{\varepsilon}\right)\right\rangle_m,
$$

$$
P(\varepsilon,\beta,z)=P_d, \quad P_d: X^m\to X^m.
$$

Note that

$$
\int_{-\infty}^{\infty}\big\langle P_d p(s),\psi(\beta+s)\big\rangle_m ds=-d.
$$

The term $f(z+\phi_\beta)-f(\phi_\beta)-D_\beta(\cdot)z$ is of order $O(|z|_m^2)$ in (3.3.12) as $|z|_m\to 0$. Moreover, the left-hand side of (3.3.12) defines a linear operator from X_1^m to $\mathscr{X}^m$, which is uniformly invertible for $\varepsilon>0$ small according to Theorem 3.3.3 and Remark 3.3.4. So by applying the uniform contraction principle of Theorem 2.2.1, we can solve (3.3.12) for z, for any $\varepsilon>0$ small and β so that $|\beta-\beta_0|$ is sufficiently small (say $|\beta-\beta_0|<\sigma$). Moreover, for any fixed $\varepsilon\in(0,\tilde{c})$ this solution $z=z(\beta,\varepsilon)$ is C^1–smooth in β and moreover a simple computation shows that $||z(\beta,\varepsilon)||_m, ||z_\beta(\beta,\varepsilon)||_m=O(\varepsilon)$ uniformly in β (here and in the sequel $z_\beta(\beta,\varepsilon)$ will denote $\frac{\partial z(\beta,\varepsilon)}{\partial\beta}$). By putting $z(\beta,\varepsilon)$ into (3.3.13), we obtain the bifurcation equation (see the definition of $P_d p$)

$$
\begin{aligned}
0 &= \varepsilon\sum_{i=-\infty}^{\infty}\left\langle g\big(z(\beta,\varepsilon)(i/\varepsilon-)+\phi_\beta(i/\varepsilon)\big),\psi\left(\beta+\frac{i}{\varepsilon}\right)\right\rangle_m\\
&\quad+\int_{-\infty}^{\infty}\Big\langle f\big(z(\beta,\varepsilon)(s)+\phi_\beta(s)\big)-f\big(\phi_\beta(s)\big)-D_\beta(s)z(\beta,\varepsilon)(s)\\
&\quad+\varepsilon h\big(z(\varepsilon,\beta)(s)+\phi_\beta(s)\big),\psi(\beta+s)\Big\rangle_m ds.
\end{aligned}
$$

As $||z(\beta,\varepsilon)||_m, ||z_\beta(\beta,\varepsilon)||_m=O(\varepsilon)$, we can divide the above equation by ε to obtain

$$
\begin{aligned}
0 = &\sum_{i=-\infty}^{\infty} \left\langle g\big(z(\beta,\varepsilon)(i/\varepsilon-)+\phi_\beta(i/\varepsilon)\big), \psi\left(\beta+\frac{i}{\varepsilon}\right)\right\rangle_m \\
&+\int_{-\infty}^{\infty} \big\langle h\big(z(\beta,\varepsilon)(s)+\phi_\beta(s)\big), \psi(\beta+s)\big\rangle_m ds \\
&+\varepsilon^{-1}\int_{-\infty}^{\infty} \Big\langle f\big(z(\beta,\varepsilon)(s)+\phi_\beta(s)\big)-f\big(\phi_\beta(s)\big)-D_\beta(s)z(\beta,\varepsilon)(s), \psi(\beta+s)\Big\rangle_m ds\,.
\end{aligned}
$$

Now, the last term in the r.h.s. of the above equation is clearly $O(\varepsilon)$ uniformly in β and it is not difficult to see that it can be differentiated, with respect to β, with the integral sign and that this derivative is also $O(\varepsilon)$, uniformly in β, because of $||z(\beta,\varepsilon)||_m, ||z_\beta(\beta,\varepsilon)||_m = O(\varepsilon)$, uniformly in β. On the other hand, for $i \neq 0$, $\varepsilon > 0$ sufficiently small and $|\beta-\beta_0| < \sigma$, we have

$$
\left|\psi\left(\beta+\frac{i}{\varepsilon}\right)\right|_m \le \tilde{K}\mathrm{e}^{-\alpha|\beta+\frac{i}{\varepsilon}|} \le \tilde{K}\mathrm{e}^{\alpha|\beta|}\mathrm{e}^{-\alpha/\varepsilon} = O(\varepsilon)
$$

where $\tilde{K} > 0$ is a constant, and a similar inequality holds for $\phi_\beta(i/\varepsilon)$. Using these facts the above equation takes the form

$$
\big\langle g\big(\phi(\beta)\big), \psi(\beta)\big\rangle_m + \int_{-\infty}^{\infty} \big\langle h\big(\phi_\beta(s)\big), \psi(\beta+s)\big\rangle_m ds + O(\varepsilon) = 0 \tag{3.3.14}
$$

where $O(\varepsilon)$ in Equation (3.3.14) has to be considered in the C^1–topology in $\beta \in (\beta_0-\sigma, \beta_0+\sigma)$, i.e. $O(\varepsilon)$ expresses a term which is $O(\varepsilon)$ small, together with the first partial derivative in β, uniformly with respect to $\beta \in (\beta_0-\sigma, \beta_0+\sigma)$. Summing up we see that if β_0 is a simple root of the function (3.3.2) then (3.3.14) has a unique solution near β_0 for $\varepsilon > 0$ sufficiently small. This means that (3.3.1) has a bounded solution near ϕ for any $\varepsilon > 0$ sufficiently small. So we obtain the following theorem.

Theorem 3.3.8. *Assume that the function $\mathscr{M} : \mathbb{R} \to \mathbb{R}$ given by* (3.3.2) *has a simple root at $\beta = \beta_0$. Then* (1.1) *has a unique bounded solution near ϕ_{β_0} for any $\varepsilon > 0$ sufficiently small.*

Let $x(\varepsilon)$ be the solution from Theorem 3.3.8. Then the sequence

$$
\big\{x(\varepsilon)(i/\varepsilon-)\big\}_{i=-\infty}^{\infty}
$$

is a bounded orbit of the Poincarè map π_ε of (3.3.1). In the rest of this section, we show that this orbit is a transversal homoclinic orbit to a hyperbolic fixed point of π_ε. For this purpose (see Lemma 2.5.2), we show that the linearization of (3.3.10) at $x(\varepsilon)$

$$
\begin{aligned}
v' &= Df\big(x(\varepsilon)\big)v + \varepsilon Dh\big(x(\varepsilon)\big)v\,, \\
v(i/\varepsilon+) &= v(i/\varepsilon-) + \varepsilon Dg\big(x(\varepsilon)(i/\varepsilon-)\big)v(i/\varepsilon-), \quad i \in \mathbb{Z}
\end{aligned}
$$

has only the zero bounded solution on $\mathbb{R}$. To show this result, we apply Theorem 2.2.4. So, let $B : X_1^m \to \mathscr{X}^m$ be a bounded linear mapping so that $||B||_{L(X_1^m,\mathscr{X}^m)} \le L$.

Consider the equation

$$F_\varepsilon(x) + \gamma\varepsilon B\big(x - x(\varepsilon_0)\big) = 0 \tag{3.3.15}$$

for a fixed small $\varepsilon_0 > 0$. The perturbation of (3.3.15) is small for γ, $\varepsilon > 0$ small and it is vanishing for $\varepsilon = 0$. Hence we can repeat the proof of Theorem 3.3.8 to obtain a unique solution $\tilde{x}(\varepsilon)$ of (3.3.15) in a neighbourhood of ϕ_{β_0} for $\varepsilon > 0$ and $\gamma > 0$ small. On the other hand,

$$F_{\varepsilon_0}\big(x(\varepsilon_0)\big) + \gamma\varepsilon_0 B\big(x(\varepsilon_0) - x(\varepsilon_0)\big) = 0\,.$$

Hence $x(\varepsilon_0) = \tilde{x}(\varepsilon_0)$. By using Theorem 2.2.4, we obtain that the linear map $DF_{\varepsilon_0}\big(x(\varepsilon_0)\big)$ is invertible, i.e. the above linearized equation of (3.3.10) at $x(\varepsilon_0)$ has only the zero bounded solution on $\mathbb{R}$.

Now we show that π_ε has a hyperbolic fixed point near 0. For this purpose, we solve

$$F_\varepsilon = 0$$

near $x \equiv 0$, i.e. we solve the equation

$$\begin{aligned} z' &= Df(0)z + \big\{f(z) - Df(0)z\big\} + \varepsilon h(z)\,, \\ z(i/\varepsilon+) &= z(i/\varepsilon-) + \varepsilon g\big(z(i/\varepsilon-)\big)\,, \end{aligned} \tag{3.3.16}$$

near $z = 0$. By repeating the above procedure applied to Eqs. (3.3.12)–(3.3.13), when Theorem 3.3.3 is replaced by Theorem 3.3.7, we obtain a unique small solution $\bar{x}(\varepsilon) \in X_1^m$ of (3.3.16). On the other hand, if $\tilde{x}$ is a solution of F_ε then $\tilde{x}(1+\cdot)$ is also a solution. Hence

$$\bar{x}(\varepsilon)(1+\cdot) = \bar{x}(\varepsilon)(\cdot)$$

because of uniqueness. So the point $\bar{x}(1-)$ is a fixed point of π_ε. To show the hyperbolicity of this point, we again apply Lemma 2.5.2 and Theorem 2.2.4 by taking an equation similar to (3.3.15) of the form

$$F_\varepsilon(x) + \gamma\varepsilon B\big(x - \bar{x}(\varepsilon_0)\big) = 0\,,$$

for a fixed small $\varepsilon_0 > 0$. By employing Theorem 3.3.7 as above for (3.3.16), the only small solution of this equation is $\bar{x}(\varepsilon_0)$. So $DF_{\varepsilon_0}\big(\bar{x}(\varepsilon_0)\big)$ is invertible, i.e. $\bar{x}(\varepsilon_0)(1-)$ is a hyperbolic fixed point of π_{ε_0}. Summing up, we obtain

Theorem 3.3.9. *The Poincarè map π_ε of* (1.1) *has a unique hyperbolic fixed point near* 0 *for any $\varepsilon > 0$ sufficiently small.*

Summarizing our results we see that the set $\big\{x(\varepsilon)(i/\varepsilon-)\big\}_{i=-\infty}^{\infty}$ is a transversal homoclinic orbit of π_ε to the hyperbolic fixed point $\bar{x}(\varepsilon)(1-)$ for any $\varepsilon > 0$ sufficiently small. This gives the main result of this section.

Theorem 3.3.10. *If there is a simple root of $\mathscr{M}(\beta) = 0$, then π_ε - the Poincarè map of* (3.3.1) *- possesses a transversal homoclinic point for any $\varepsilon > 0$ sufficiently small.*

3.3.4 Examples of Singular Impulsive ODEs

Consider

$$\begin{aligned} \varepsilon x' &= f(x) + \varepsilon h(x)\,, \\ x(i+) &= x(i-) + \varepsilon\tau a, \quad i \in \mathbb{Z}\,, \end{aligned} \tag{3.3.17}$$

where $a \in \mathbb{R}^m$ is fixed, $\tau \in \mathbb{R}$ is a parameter and f, h satisfy the assumptions (H1)–(H4).

Theorem 3.3.11. *If* $\int_{-\infty}^{\infty} \langle h(\phi(s)), \psi(s) \rangle_m ds \neq 0$ *and there is* $\beta_0 \in \mathbb{R}$ *satisfying*

$$\langle a, \psi(\beta_0) \rangle_m \neq 0, \quad \langle a, \psi'(\beta_0) \rangle_m \neq 0\,.$$

Then, for any $\varepsilon > 0$ *sufficiently small, the Poincarè map of* (3.3.17) *has a transversal homoclinic orbit for* $\tau_0 = -\int_{-\infty}^{\infty} \langle h(\phi(s)), \psi(s) \rangle_m ds / \langle a, \psi(\beta_0) \rangle_m$.

Proof. In this case, the Melnikov function (3.3.2) for (3.3.17) with $\tau = \tau_0$ has the form

$$\mathscr{M}(\beta) = \tau_0 \langle a, \psi(\beta) \rangle_m + \int_{-\infty}^{\infty} \langle h(\phi(s)), \psi(s) \rangle_m ds\,.$$

It is clear that $\mathscr{M}(\beta_0) = 0$, $\mathscr{M}'(\beta_0) \neq 0$. So Theorem 3.3.10 implies the assertion. The proof is finished. □

We note that under the assumptions of Theorem 3.3.11, the Poincarè map of (3.3.17) has a transversal homoclinic orbit for any τ near τ_0 and any $\varepsilon > 0$ sufficiently small.

Theorem 3.3.12. *If* $\int_{-\infty}^{\infty} \langle h(\phi(s)), \psi(s) \rangle_m ds = 0$ *and there is* $\beta_0 \in \mathbb{R}$ *satisfying*

$$\langle a, \psi(\beta_0) \rangle_m = 0, \quad \langle a, \psi'(\beta_0) \rangle_m \neq 0\,.$$

Then, for any $\varepsilon > 0$ *sufficiently small, the Poincarè map of* (3.3.17) *has a transversal homoclinic orbit for any* $\tau \neq 0$ *fixed.*

Proof. In this case,

$$\mathscr{M}(\beta) = \tau \langle a, \psi(\beta) \rangle_m\,.$$

So $\mathscr{M}(\beta_0) = 0$, $\mathscr{M}'(\beta_0) \neq 0$. The proof is finished by Theorem 3.3.10. □

Finally, let us consider an impulsive Duffing–type equation of the form (3.3.3).

Theorem 3.3.13. *Assume that* $p(0) = 0$, $p'(0) < 0$ *and the second–order ODE*

$$z'' + p(z) = 0$$

has a nonconstant solution $\gamma(t)$ *so that* $\gamma(t) \to 0$ *as* $t \to \pm\infty$. *If there is* $\beta_0 \in \mathbb{R}$ *so that* $\gamma''(\beta_0) = 0$, $\gamma'''(\beta_0) \neq 0$ *and* $r\big(\gamma(\beta_0)\big) \neq 0$, *then* (3.3.3) *has chaotic behaviour for any* $a > 0$ *sufficiently large.*

Proof. The equation can be rewritten in the form

$$\begin{aligned} \varepsilon x' &= f(x) + \varepsilon h(x), \\ x(i+) &= x(i-) + \varepsilon g\big(x(i-)\big), \end{aligned} \tag{3.3.18}$$

where

$$\varepsilon = 1/a, \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad f(x_1, x_2) = \big(x_2, -p(x_1)\big),$$
$$h(x_1, x_2) = \big(0, q(x_1)\big), \quad g(x_1, x_2) = \big(r(x_1), 0\big).$$

We note [31] that in this case

$$\phi(\beta) = \big(\gamma(\beta), \gamma'(\beta)\big), \quad \psi(\beta) = \big(-\gamma''(\beta), \gamma'(\beta)\big).$$

So the Melnikov function of Theorem 3.3.10 has the form:

$$\mathscr{M}(\beta) = -r\big(\gamma(\beta)\big)\gamma''(\beta) + \int_{-\infty}^{\infty} q\big(\gamma(s)\big)\gamma'(s)\, ds = r\big(\gamma(\beta)\big)p(\gamma(\beta)).$$

By $\mathscr{M}(\beta_0) = 0$ and $\mathscr{M}'(\beta_0) \neq 0$, the conclusion follows from Theorem 3.3.10. □

Remark 3.3.14. Consider

$$\begin{aligned} z'' + a^2 p(z) &= q(z), \\ a\big(z(i+) - z(i-)\big) &= r\big(z(i-)\big), \\ z'(i+) = z'(i-), &\quad i \in \mathbb{Z} \end{aligned} \tag{3.3.19}$$

instead of (3.3.3). Then the statement of Theorem 3.3.13 holds, since (3.3.18) is replaced by

$$\begin{aligned} \varepsilon x' &= f(x) + \varepsilon^2 h(x), \\ x(i+) &= x(i-) + \varepsilon g\big(x(i-)\big). \end{aligned}$$

It easily follows, from the proof of Theorem 3.3.13, that $\mathscr{M}(\beta) = r\big(\gamma(\beta)\big)p\big(\gamma(\beta)\big)$ in this case too, hence Theorem 3.3.13 still holds.

Remark 3.3.15. Consider

$$\begin{aligned} z'' + a^2 p(z) &= q(z), \\ a^2\big(z(i+) - z(i-)\big) &= r\big(z(i-)\big), \\ z'(i+) = z'(i-), &\quad i \in \mathbb{Z} \end{aligned} \tag{3.3.20}$$

instead of (3.3.3). Then the statement of Theorem 3.3.13 holds, since (3.3.18) is replaced by

$$\begin{aligned} \varepsilon x' &= f(x) + \varepsilon^2 h(x), \\ x(i+) &= x(i-) + \varepsilon^2 g\big(x(i-)\big). \end{aligned} \tag{3.3.21}$$

Of course, the Melnikov function for (3.3.21) is vanishing, since we derived in Theorem 3.3.10 the first–order Melnikov function. However the factor ε^2 in both the perturbation and the jumping term allow us to repeat the arguments of Section 3.3.3

showing, then that the solution of system (3.3.12) is $O(\varepsilon^2)$-bounded, uniformly in β and the same holds for its derivative with respect to β. Thus, we can divide the bifurcation function by ε^2 and take the limit as $\varepsilon \to 0$ (uniformly in β), getting the same bifurcation function as in (3.3.2). Hence [31, p. 284] we see that a simple root of the above Melnikov function of (3.3.18) ensures the validity of Theorem 3.3.13 also for (3.3.20).

3.4 Singularly Perturbed Impulsive ODEs

3.4.1 Singularly Perturbed ODEs with Impulses

In this section we proceed with the study of chaotic behaviour of dynamical systems with impulses. More precisely, we study the chaotic behavior of the equation

$$\begin{aligned} \varepsilon y' &= f(x,y,\varepsilon)\,, \\ x' &= g(x,y,\varepsilon)\,, \end{aligned} \tag{3.4.1}$$

with the impulsive effects

$$\begin{aligned} x(i+0) &= x(i-0) + \varepsilon a\big(x(i-0), y(i-0), \varepsilon\big)\,, \\ y(i+0) &= y(i-0) + \varepsilon b\big(x(i-0), y(i-0), \varepsilon\big), \quad i \in \mathbb{Z}\,, \end{aligned} \tag{3.4.2}$$

where as usual $\lim\limits_{t\to i_\pm} x(t) = x(i \pm 0)$. Here $y \in \mathbb{R}^p$, $x \in \mathbb{R}^m$ and $\varepsilon > 0$ is a small parameter. We assume that

(H1) f, g, a, b are C^3–smooth;
(H2) $f(\cdot,0,0) = 0$, $D_y f(\cdot,0,0) = \big(A(\cdot), B(\cdot)\big)$, where $A(\cdot) \in L(\mathbb{R}^{k_1})$, $B(\cdot) \in L(\mathbb{R}^{k_2})$, $k_1 + k_2 = p$;
(H3) $\big\{\Re\tau \mid \tau \in \sigma\big(A(\cdot)\big)\big\} \subset (-\infty, -\gamma)$ and $\big\{\Re\tau \mid \tau \in \sigma\big(B(\cdot)\big)\big\} \subset (\gamma, \infty)$ for some constant $\gamma > 0$;
(H4) The *reduced equation* $x' = g(x,0,0)$ has a hyperbolic equilibrium $\bar{x}_0$ with a homoclinic orbit $x(t)$;
(H5) The variational equation $v' = D_x g\big(x(t),0,0\big)v$ has the only unique (up to constant multiples) bounded solution $x'(\cdot)$.

By a solution of (3.4.1)–(3.4.2) we mean some (x,y) which is C^1–smooth in $\mathbb{R}\setminus\mathbb{Z}$ satisfying (3.4.1) on this set and moreover, (3.4.2) holds for any $i \in \mathbb{Z}$. For simplicity, we assume that f, g, a, b are globally Lipschitz continuous. Then (3.4.1)–(3.4.2) with any initial condition $x(t_0) = x_0$, $y(t_0) = y_0$, $t_0 \notin \mathbb{Z}$ has a unique global solution. Furthermore, we can define a Poincarè map H_ε of (3.4.1)–(3.4.2) in the following way. Let $\phi_\varepsilon\big(t,(x_0,y_0)\big)$ be the unique solution of (3.4.1) with the initial point (x_0,y_0). Then we put

$$H_\varepsilon(x_0,y_0) = \phi_\varepsilon\Big(1,\big(x_0+\varepsilon a(x_0,y_0,\varepsilon),y_0+\varepsilon b(x_0,y_0,\varepsilon)\big)\Big).$$

Of course, the dynamics of (3.4.1)–(3.4.2) is wholly determined by H_ε. The aim of this section is to find assumptions for f, g, a, b which give the existence of transversal homoclinic point of H_ε for any $\varepsilon > 0$ small. For this purpose, we derive a Melnikov function for (3.4.1)–(3.4.2). Then such Eqs. (3.4.1)–(3.4.2) will have a chaotic behaviour for $\varepsilon > 0$ small. The chaotic behaviour of small periodic perturbations of (3.4.1) is studied in Section 4.4.

3.4.2 Melnikov Function

We know by Section 4.1.2 that (H4) and (H5) imply the uniqueness (up to constant multiples) of a bounded nonzero solution u of the adjoint variational equation

$$u' = -\Big(D_x g\big(x(t),0,0\big)\Big)^* u.$$

Since the derivation of a Melnikov function for (3.4.1)–(3.4.2) is very similar to results of Section 3.3, we omit further details and refer to [32]. Hence the Melnikov function is now:

$$\begin{aligned}\mathscr{M}(t) = &\sum_{i=-\infty}^{\infty} \langle a\big(x(t+i),0,0\big),u(t+i)\rangle_m \\ &+\int_{-\infty}^{\infty}\Big\langle -D_y g\big(x(s),0,0\big)D_y f\big(x(s),0,0\big)^{-1}D_\varepsilon f\big(x(s),0,0\big)+ \\ &+D_\varepsilon g\big(x(s),0,0\big),u(s)\Big\rangle_m ds\end{aligned} \tag{3.4.3}$$

where $\langle\cdot,\cdot\rangle_m$ is the usual inner product on $\mathbb{R}^m$. Now we are ready to state the main result of this section.

Theorem 3.4.1. *Assume that there is t_0 so that*

$$\mathscr{M}(t_0) = 0, \quad \mathscr{M}'(t_0) \neq 0.$$

Then (3.4.1)–(3.4.2) *have transversal homoclinic orbit for any* $\varepsilon > 0$ *small.*

Remark 3.4.2. We have considered only the case of the uniform distribution of impulsive effects. We may study (3.4.1) similarly as above with impulsive effects of the form (3.4.2) at t_i, $i \in \mathbb{Z}$ for a fixed sequence $\{t_i\}_{i=-\infty}^{\infty}$, $t_i < t_{i+1}$ so that

$$\begin{gathered} t_i \to \pm\infty \quad \text{as} \quad i \to \pm\infty \\ \sup_i (t_{i+1}-t_i) < \infty, \quad \inf_i (t_{i+1}-t_i) > 0. \end{gathered}$$

Then, of course, (3.4.1)–(3.4.2) do not define any Poincarè map for general $\{t_i\}_{i=-\infty}^{\infty}$. A line of the paper [33] may be followed for the above general impulsive effects.

Remark 3.4.3. The second term of the Melnikov function $\mathscr{M}$ (see (3.4.3)), which does not depend on t, is only a contribution of (3.4.1) (see Section 4.4). While the first term of $\mathscr{M}$ is determined by both (3.4.1) and (3.4.2).

3.4.3 Second Order Singularly Perturbed ODEs with Impulses

In this section, we consider

$$\begin{aligned} \varepsilon x'' &= x' - f(x)\,,\\ x(i+0) &= x(i-0) + \varepsilon a\big(x(i-0), x'(i-0)\big)\,,\\ x'(i+0) &= x'(i-0) + \varepsilon b\big(x(i-0), x'(i-0)\big) \end{aligned} \tag{3.4.4}$$

where $f:\mathbb{R}^m \to \mathbb{R}^m$ and f, a, b are C^2–smooth. Moreover, assume that the equation $x' = f(x)$ has a hyperbolic equilibrium $\bar{x}_0$ with a homoclinic orbit $x(\cdot)$. Furthermore, suppose the adjoint variational equation $v' = -(Df(x(t)))^* v$ has a unique (up to constant multiples) bounded nonzero solution u. Taking $x' = y + f(x)$ we obtain from (3.4.4)

$$\begin{aligned} \varepsilon y' &= y - \varepsilon Df(x)\big(y + f(x)\big)\,,\\ x' &= y + f(x)\,,\\ x(i+0) &= x(i-0) + \varepsilon a\big(x(i-0), y(i-0) + f(x(i-0))\big)\,,\\ y(i+0) &= y(i-0) + \varepsilon b\big(x(i-0), y(i-0) + f(x(i-0))\big)\\ &\quad + f\big(x(i-0)\big) - f\Big(x(i-0) + \varepsilon a\big(x(i-0), y(i-0) + f(x(i-0))\big)\Big)\,. \end{aligned} \tag{3.4.5}$$

We see (3.4.5) is of the form (3.4.1)–(3.4.2), and the Melnikov function $\mathscr{M}$, for this case, has the form (see (3.4.3))

$$\begin{aligned} \bar{\mathscr{M}}(t) &= \sum_{i=-\infty}^{\infty} \langle a\big(x(t+i), f(x(t+i))\big), u(t+i)\rangle_m + \int_{-\infty}^{\infty} \langle Df\big(x(s)\big) f\big(x(s)\big), u(s)\rangle_m\, ds\\ &= \sum_{i=-\infty}^{\infty} \langle a\big(x(t+i), f(x(t+i))\big), u(t+i)\rangle_m + \int_{-\infty}^{\infty} \langle Df\big(x(s)\big) x'(s), u(s)\rangle_m\, ds\\ &= \sum_{i=-\infty}^{\infty} \langle a\big(x(t+i), f(x(t+i))\big), u(t+i)\rangle_m + \int_{-\infty}^{\infty} \langle x''(s), u(s)\rangle_m\, ds\\ &= \sum_{i=-\infty}^{\infty} \langle a\big(x(t+i), f(x(t+i))\big), u(t+i)\rangle_m - \int_{-\infty}^{\infty} \langle x'(s), u'(s)\rangle_m\, ds\,. \end{aligned}$$

Hence

$$\bar{\mathscr{M}}(t) = \sum_{i=-\infty}^{\infty} \langle a\big(x(t+i), f(x(t+i))\big), u(t+i)\rangle_m - \int_{-\infty}^{\infty} \langle x'(s), u'(s)\rangle_m \, ds. \quad (3.4.6)$$

By applying Theorem 3.4.1 we obtain.

Theorem 3.4.4. *Assume that there is t_0 so that*

$$\bar{\mathscr{M}}(t_0) = 0, \quad \bar{\mathscr{M}}'(t_0) \neq 0.$$

Then (3.4.4) *has a chaotic behaviour for any $\varepsilon > 0$ small.*

3.5 Inflated Deterministic Chaos

3.5.1 Inflated Dynamical Systems

The following problem arises in computer-assisted proofs and other numerical methods in dynamical systems [34–37]. Let $\mathscr{B}_{\mathbb{R}^n}$ be a unit closed ball of $\mathbb{R}^n$. For a homeomorphism $f : \mathbb{R}^n \mapsto \mathbb{R}^n$, we consider an orbit $\{x_j\}_{j\in\mathbb{Z}}$ of an *ε-inflated mapping $x \longrightarrow f(x) + \varepsilon\mathscr{B}_{\mathbb{R}^n}$* for $\varepsilon > 0$. Then we deal with a difference inclusion

$$x_{j+1} \in f(x_j) + \varepsilon\mathscr{B}_{\mathbb{R}^n}, \quad j \in \mathbb{Z}. \quad (3.5.1)$$

The concept of ε–inflated dynamics was introduced in [36] and was used in a fairly large number of papers since then. For details, see the monograph [38] and the references therein. Consequently, the theory of generalized nonautonomous attractors in the ε–inflated dynamics can be considered to be complete by now.

We are not interested in the existence of one solution of (3.5.1), but in the set of all trajectories of (3.5.1). So, for instance, to fix the initial point x_0, we consider a single-valued difference equation

$$x_{j+1} = f(x_j) + \varepsilon p_j, \quad p_j \in \mathscr{B}_{\mathbb{R}^n}, \quad j \in \mathbb{Z}, \quad (3.5.2)$$

where $\mathbf{p} = \{p_j\}_{j\in\mathbb{Z}} \in \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}$ is considered as a parameter. This orbit of (3.5.2) is denoted by $\mathbf{x}(\mathbf{p}) = \{x_j(\mathbf{p})\}_{j\in\mathbb{Z}}$. Then we define an ε-inflated orbit of (3.5.1) given by

$$\mathbf{x}^\varepsilon(x_0) = \{x_j^\varepsilon\}_{j\in\mathbb{Z}}, \quad x_j^\varepsilon = \left\{x_j(\mathbf{p}) \mid \mathbf{p} \in \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}\right\}.$$

Here

$$\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n) = \left\{ p = \{p_j\}_{j\in\mathbb{Z}} \mid p_j \in \mathbb{R}^n, \forall j \in \mathbb{Z} \text{ and } \|p\| := \sup_{j\in\mathbb{Z}} |p_j| < \infty \right\}$$

is the usual Banach space and $\mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}$ is its closed unit ball. Certainly it holds

$$x_{j+1}^{\varepsilon} = f(x_j^{\varepsilon}) + \varepsilon \mathscr{B}_{\mathbb{R}^n}, \quad j \in \mathbb{Z}.$$

Hence x_j^{ε} are contractible into themselves to $x_j^0 = f^j(x_0)$. The iteration $f^j(x_0)$, $j \neq 0$ is in the interior of x_j^{ε}. Note that $x_0^{\varepsilon} = x_0$. Moreover, x_j^{ε} are compact.

This approach of considering parameterized difference equation (3.5.2) instead of difference inclusion (3.5.1) is used in [39] for investigation of ε-inflated dynamics near either to a hyperbolic fixed point of a diffeomorphism or to a hyperbolic equilibrium of a differential equation. More precisely, we construct analogues of the stable and unstable manifolds, which are typical of a single-valued hyperbolic dynamics; moreover, we construct the maximal weakly invariant bounded set and prove that all such sets are graphs of Lipschitz maps. Then a parameterized generalization of Hartman-Grobman lemma is shown. Inflated ODEs are studied in Section 4.6.

3.5.2 Inflated Chaos

We consider a C^1-diffeomorphism $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ possessing a hyperbolic fixed point x_0. Then we take its g-inflated perturbation

$$x \longrightarrow f(x) + g(x, \mathscr{B}_{\mathbb{R}^n}) \tag{3.5.3}$$

where $g : \mathbb{R}^n \times \mathscr{B}_{\mathbb{R}^n} \to \mathbb{R}^n$ is Lipschitz in the both variables, i.e. the following holds: There are positive constants λ, Λ and L so that

$$|g(x,p) - g(\tilde{x},\tilde{p})| \leq \lambda|x - \tilde{x}| + \Lambda|p - \tilde{p}| \quad \text{and} \quad |g(x,0)| \leq L \tag{3.5.4}$$

whenever $x, \tilde{x} \in \mathbb{R}^n$ and $p, \tilde{p} \in \mathscr{B}_{\mathbb{R}^n}$. We suppose, in addition, that diffeomorphism f possesses a transversal homoclinic orbit $\{x_k^0\}_{k\in\mathbb{Z}}$ to hyperbolic fixed point x_0. Then f is chaotic by the Smale-Birkhoff homoclinic theorem 2.5.4. Our aim is to extend this theorem to (3.5.3).

Our multivalued perturbation takes the special form $G(x) = g(x, \mathscr{B}_{\mathbb{R}^n})$. So (3.5.3) has the form $x \longrightarrow f(x) + G(x)$. In view of the Lojasiewicz-Ornelas parametrization theorem 2.3.1, this is not a loss of generality if the values of G are convex and compact. However, in the general case a parameterization of G does not exist. We mention that some nonconvex versions exist as well [40], but in general, a parameterization cannot be available, since continuous selections may not exists (see [41], Section 1.6). Hence, we consider

$$x_{k+1} \in f(x_k) + g(x_k, \mathscr{B}_{\mathbb{R}^n}), \quad k \in \mathbb{Z}. \tag{3.5.5}$$

Like in [39], we take $\mathbf{p} = \{p_k\}_{k\in\mathbb{Z}} \in \ell_{\mathbb{Z}}^{\infty}(\mathbb{R}^n)$, $\|\mathbf{p}\| \leq 1$ and consider the system

$$x_{k+1} = f(x_k) + g(x_k, p_k), \quad k \in \mathbb{Z}. \tag{3.5.6}$$

First, we know by Lemma 2.5.2 that the transversality of a homoclinic orbit $\{x_k^0\}_{k\in\mathbb{Z}}$ is equivalent to the existence of an exponential dichotomy of $w_{k+1} = Df(x_k^0)w_k$ on $\mathbb{Z}$, i.e. setting the fundamental solution

$$W(k) := \begin{cases} Df(x_{k-1}^0)\cdots Df(x_0^0), & \text{if } k>0, \\ \mathbb{I}, & \text{if } k=0, \\ Df(x_k^0)^{-1}\cdots Df(x_{-1}^0)^{-1}, & \text{if } k<0, \end{cases}$$

there are a projection $P:\mathbb{R}^n \mapsto \mathbb{R}^n$ and positive constants $K>0$, $\delta\in(0,1)$ so that

$$\begin{aligned} \left|W(k)PW(r)^{-1}\right| &\le K\delta^{k-r}, & \text{for } k\ge r, \\ \left|W(k)(\mathbb{I}-P)W(r)^{-1}\right| &\le K\delta^{k-r}, & \text{for } k\le r. \end{aligned}$$

Now we fix $\omega\in\mathbb{N}$ large and for any $\xi\in\mathscr{E}$, $\xi=\{e_j\}_{j\in\mathbb{Z}}$ we define a pseudo-orbit $\mathbf{x}^\xi=\{x_k^\xi\}_{k\in\mathbb{Z}}$ as follows for $k\in\{2j\omega,\dots,2(j+1)\omega-1\}$, $j\in\mathbb{Z}$:

$$x_k^\xi := \begin{cases} x_{k-(2j+1)\omega}^0, & \text{for } e_j=1, \\ x_0, & \text{for } e_j=0. \end{cases}$$

Let $|x_{k_0}^0 - x_0| = \max_{k\in\mathbb{Z}} |x_k^0 - x_0|$. Following [10, pp. 148–151] and [13], we have the following result.

Lemma 3.5.1. *There exist $\omega_0\in\mathbb{N}$, $\omega_0>|k_0|$ and a constant $c>0$ so that for any $\xi\in\mathscr{E}$, $\mathbf{h}=\{h_k\}_{k\in\mathbb{Z}}\in\ell_{\mathbb{Z}}^\infty(\mathbb{R}^n)$, there is a unique solution $\mathbf{w}=\{w_k\}_{k\in\mathbb{Z}}\in\ell_{\mathbb{Z}}^\infty(\mathbb{R}^n)$ of the linear system*

$$w_{k+1} = Df(x_k^\xi)w_k + h_k, \quad k\in\mathbb{Z}.$$

Moreover, $\mathbf{w}$ *is linear in* $\mathbf{h}$ *and it holds* $\|\mathbf{w}\|\le c\|\mathbf{h}\|$.

We denote that $K(\xi)h=\mathbf{w}$ is the unique solution from Lemma 3.5.1. Certainly $K(\xi)\in L\left(\ell_{\mathbb{Z}}^\infty(\mathbb{R}^n)\right)$ with $\|K(\xi)\|\le c$, and $K(\xi)^{-1}\mathbf{w}=\left\{w_{k+1}-Df(x_k^\xi)w_k\right\}_{k\in\mathbb{Z}}$, so $K(\xi)^{-1}\in L\left(\ell_{\mathbb{Z}}^\infty(\mathbb{R}^n)\right)$.

Now we look for a solution of (3.5.6) near $\mathbf{x}^\xi$. For this reason, we make a change of variables $x_k = w_k + x_k^\xi$, $k\in\mathbb{Z}$ to get the equation

$$w_{k+1} = Df\left(x_k^\xi\right)w_k + f\left(w_k+x_k^\xi\right) - x_{k+1}^\xi - Df\left(x_k^\xi\right)w_k + g\left(w_k+x_k^\xi, p_k\right) \quad (3.5.7)$$

for $k\in\mathbb{Z}$. To solve (3.5.7), we introduce a mapping

$$G:\mathscr{E}\times\mathscr{B}_{\ell_{\mathbb{Z}}^\infty(\mathbb{R}^n)}\times\ell_{\mathbb{Z}}^\infty(\mathbb{R}^n)\mapsto\ell_{\mathbb{Z}}^\infty(\mathbb{R}^n)$$

as follows:

$$G(\xi,\mathbf{p},\mathbf{w}) := \left\{f\left(w_k+x_k^\xi\right) - x_{k+1}^\xi - Df\left(x_k^\xi\right)w_k + g\left(w_k+x_k^\xi,p_k\right)\right\}_{k\in\mathbb{Z}}.$$

Now for any $\xi \in \mathscr{E}$, $\mathbf{w}^1, \mathbf{w}^2 \in \ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)$, $\|\mathbf{w}^{1,2}\| \le \rho$ and $\mathbf{p}^1, \mathbf{p}^2 \in \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}$, we derive

$$\|G(\xi,\mathbf{p}^1,\mathbf{w}^1) - G(\xi,\mathbf{p}^2,\mathbf{w}^2)\| \le (\triangle(\rho)+\lambda)\,\|\mathbf{w}^1-\mathbf{w}^2\| + \Lambda\|\mathbf{p}^1-\mathbf{p}^2\| \tag{3.5.8}$$

for

$$\triangle(\rho) := \sup\left\{|Df(w+x)-Df(x)| : |x-x_0| \le 2|x^0_{k_0}-x_0|, |w| \le \rho\right\}.$$

Note that $\triangle(0)=0$. Since $\{x^0_k\}_{k\in\mathbb{Z}}$ is a homoclinic orbit of f to x_0, by [42, p. 148], we also get

$$\|G(\xi,0,0)\| \le L + \sup_{k\in\mathbb{Z},\xi\in\mathscr{E}} |x^\xi_{k+1} - f(x^\xi_k)| \le L + \widetilde{c}\left(\frac{\delta+1}{2}\right)^\omega \tag{3.5.9}$$

for a constant $\widetilde{c} > 0$ and any $\xi \in \mathscr{E}$. Now we are ready to rewrite (3.5.7) as the following fixed point problem

$$\mathbf{w} = F(\xi,\mathbf{p},\mathbf{w}) := K(\xi)G(\xi,\mathbf{p},\mathbf{w}).$$

By Lemma 3.5.1, (3.5.8) and (3.5.9), we obtain

$$\begin{aligned}\|F(\xi,\mathbf{p}^1,\mathbf{w}^1) - F(\xi,\mathbf{p}^2,\mathbf{w}^2)\| &\le c\,(\triangle(\rho)+\lambda)\,\|\mathbf{w}^1-\mathbf{w}^2\| + \Lambda c\|\mathbf{p}^1-\mathbf{p}^2\|,\\ \|F(\xi,\mathbf{p}^1,\mathbf{w}^1)\| &\le c\,(\triangle(\rho)+\lambda)\,\|\mathbf{w}^1\| + \Lambda c\|\mathbf{p}^1\| + Lc + c\widetilde{c}\left(\frac{\delta+1}{2}\right)^\omega\end{aligned} \tag{3.5.10}$$

for any $\xi \in \mathscr{E}$, $\mathbf{w}^1, \mathbf{w}^2 \in \ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)$, $\|\mathbf{w}^{1,2}\| \le \rho$ and $\mathbf{p}^1, \mathbf{p}^2 \in \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}$. Assuming that

$$c\lambda < 1, \tag{3.5.11}$$

we set

$$\widetilde{\kappa}_0 := \min\left\{1, c\lambda + c\triangle\left(\frac{|x^0_{k_0}-x_0|}{4}\right)\right\},$$

$$M_0(c,\lambda) := \max_{c\lambda\le\kappa\le\widetilde{\kappa}_0}\left\{\frac{1-\kappa}{c}\min\left\{\triangle^{-1}\left(\frac{\kappa-c\lambda}{c}\right)\right\}\right\}$$

and the above maximum is achieved at $\kappa_0 \in (c\lambda, 1)$. Here $\triangle^{-1} : \mathbb{R}_+ \to 2^{\mathbb{R}_+} \setminus \{\emptyset\}$ is considered as an upper semicontinuous mapping which is increasing with increasing compact interval set values. Put

$$\rho_0 := \min\left\{\triangle^{-1}\left(\frac{\kappa_0-c\lambda}{c}\right)\right\}.$$

Note that

$$0 < \rho_0 = \min\left\{\triangle^{-1}\left(\frac{\kappa_0 - c\lambda}{c}\right)\right\} \le \min\left\{\triangle^{-1}\left(\frac{\widetilde{\kappa}_0 - c\lambda}{c}\right)\right\} \le \frac{|x^0_{k_0} - x_0|}{4},$$
$$\kappa_0 = c\left(\triangle(\rho_0) + \lambda\right).$$

If

$$\Lambda + L < M_0(c, \lambda), \tag{3.5.12}$$

then $\Lambda + L < M_0(c,\lambda) = \frac{1-\kappa_0}{c}\rho_0$ and so

$$c\Lambda + cL + c\left(\triangle(\rho_0) + \lambda\right)\rho_0 = c\Lambda + cL + \kappa_0\rho_0 < \rho_0.$$

Consequently, we find $\mathbb{N} \ni \omega_1 > \omega_0$ so that

$$c\widetilde{c}\left(\frac{\delta+1}{2}\right)^{\omega_1} + c\Lambda + cL + \kappa_0\rho_0 \le \rho_0. \tag{3.5.13}$$

Then for any fixed $\mathbb{N} \ni \omega \ge \omega_1$, mapping:

$$F : \mathscr{E} \times \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)} \times \mathscr{B}^{\rho_0}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)} \mapsto \mathscr{B}^{\rho_0}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}$$

is a contraction with a constant κ_0, where $\mathscr{B}^{\rho_0}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}$ is the ball of $\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)$ centered at 0 with the radius ρ_0. By the Banach fixed point theorem 2.2.1 we get the following result.

Theorem 3.5.2. *Assume* (3.5.11) *and* (3.5.12). *Then there are* $\omega_1 > \omega_0$, $\frac{|x^0_{k_0} - x_0|}{4} \ge \rho_0 > 0$ *so that for any* $\mathbb{N} \ni \omega \ge \omega_1$ *but fixed and for any* $\xi \in \mathscr{E}$, $\mathbf{p} \in \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}$, *there is a unique solution* $\mathbf{x}(\mathbf{p},\xi) = \{x_k(\mathbf{p},\xi)\}_{k\in\mathbb{Z}} \in \ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)$ *of* (3.5.6) *so that*

$$\|\mathbf{x}(\mathbf{p},\xi) - \mathbf{x}^\xi\| \le \rho_0. \tag{3.5.14}$$

By (3.5.10), mapping:

$$\mathbf{x} : \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)} \times \mathscr{E} \mapsto \ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)$$

is Lipschitzian in $\mathbf{p}$:

$$\|\mathbf{x}(\mathbf{p}^1,\xi) - \mathbf{x}(\mathbf{p}^2,\xi)\| \le \frac{c\Lambda}{1-\kappa_0}\|\mathbf{p}^1 - \mathbf{p}^2\| \tag{3.5.15}$$

for any $\xi \in \mathscr{E}$ and $\mathbf{p}^1, \mathbf{p}^2 \in \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}$. Let

$$\ell_{\mathbb{Z}}(\mathbb{R}^n) := \{\{x_k\}_{k\in\mathbb{Z}} \mid x_k \in \mathbb{R}^n\}$$

be a metric space with a norm

$$d(\{e_{k\in\mathbb{Z}}\}, \{e'_{k\in\mathbb{Z}}\}) := \sum_{k\in\mathbb{Z}} \frac{|e_k - e'_k|}{2^{|k|+1}(1 + |e_k - e'_k|)}.$$

Clearly $\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n) \subset \ell_{\mathbb{Z}}(\mathbb{R}^n)$. Now we prove several useful results.

Theorem 3.5.3. *Mapping* $\mathbf{x} : \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)} \times \mathscr{E} \mapsto \ell_{\mathbb{Z}}(\mathbb{R}^n)$ *is continuous.*

Proof. Let $\mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)} \ni \mathbf{p}^i = \{p^i_j\}_{j\in\mathbb{Z}} \to \mathbf{p}^0 = \{p^0_j\}_{j\in\mathbb{Z}} \in \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}$, $\mathscr{E} \ni \xi_i = \{e^i_j\}_{j\in\mathbb{Z}} \to \xi_0 = \{e^0_j\}_{j\in\mathbb{Z}} \in \mathscr{E}$ as $i \to \infty$. Then using (3.5.14) and the Cantor diagonal procedure, we can suppose, by passing to subsequences, that

$$x_j(\mathbf{p}^i, \xi_i) \to \widetilde{x}^0_j\,, \quad \forall j \in \mathbb{Z}\,,$$

as $i \to \infty$. We note that $e^i_j \to e^0_j$ as $i \to \infty$ $\forall j \in \mathbb{Z}$ and $\mathbf{x}(\mathbf{p}^i, \xi_i)$, $i \in \mathbb{Z}$ solving (3.5.6) along with (3.5.14) holds as well. By passing to the limit $i \to \infty$, we obtain

$$\widetilde{x}^0_{k+1} = f(\widetilde{x}^0_k) + g(\widetilde{x}^0_k, p^0_k), \quad k \in \mathbb{Z}$$

and $\widetilde{\mathbf{x}} = \{\widetilde{x}^0_j\}_{j\in\mathbb{Z}}$ satisfies (3.5.14) with $\xi = \xi_0$. The uniqueness property of Theorem 3.5.2 implies $\widetilde{\mathbf{x}} = \mathbf{x}(\mathbf{p}^0, \xi_0)$. The continuity of $\mathbf{x}$ is proved. □

Theorem 3.5.4. *It holds*

$$x_k(\widetilde{\mathbf{p}}, \sigma(\xi)) = x_{k+2\omega}(\mathbf{p}, \xi)\,, \quad \forall k \in \mathbb{Z}\,, \tag{3.5.16}$$

for $\widetilde{\mathbf{p}} := \{p_{k+2\omega}\}_{k\in\mathbb{Z}}$.

Proof. Taking $z_k := x_{k+2\omega}(\mathbf{p}, \xi)$ for any $k \in \mathbb{Z}$, by $x^{\sigma(\xi)}_k = x^\xi_{k+2\omega}$ $\forall k \in \mathbb{Z}$, (3.5.6) and (3.5.14) we derive

$$\begin{gathered} z_{k+1} = f(z_k) + g(z_k, p_{k+2\omega})\,, \\ \left|z_k - x^{\sigma(\xi)}_k\right| = \left|x_{k+2\omega}(\mathbf{p}, \xi) - x^\xi_{k+2\omega}\right| \le \rho_0\,, \end{gathered}$$

for any $k \in \mathbb{Z}$. The uniqueness property of Theorem 3.5.2 implies $z_k = x_k(\widetilde{\mathbf{p}}, \sigma(\xi))$ for any $k \in \mathbb{Z}$, so (3.5.16) is shown. □

Then (3.5.16) implies

$$x_{2k\omega}(\mathbf{p}, \xi) = x_0\left(\widetilde{\sigma}^k(\mathbf{p}), \sigma^k(\xi)\right)\,, \quad \forall k \in \mathbb{Z}\,, \tag{3.5.17}$$

for a shift homeomorphism

$$\widetilde{\sigma} : \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)} \mapsto \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}$$

given by $\widetilde{\sigma}(\mathbf{p}) := \widetilde{\mathbf{p}}$. Note that

$$x_{2(k+1)\omega}(\mathbf{p}, \xi) = F^{2(k+1)\omega}_{2k\omega,\mathbf{p}}\left(x_{2k\omega}(\mathbf{p}, \xi)\right)\,, \quad \forall k \in \mathbb{Z}\,, \tag{3.5.18}$$

for continuous mappings

$$F_{2k\omega,\mathbf{p}}^{2(k+1)\omega}(x) := \left(f+g(\cdot,p_{2(k+1)\omega-1})\right)\cdots(f+g(\cdot,p_{2k\omega+1}))\,(f+g(\cdot,p_{2k\omega}))\,(x)\,.$$

Then (3.5.17) and (3.5.18) imply

$$x_0\left(\widetilde{\sigma}^{k+1}(\mathbf{p}),\sigma^{k+1}(\xi)\right) = F_{2k\omega,\mathbf{p}}^{2(k+1)\omega}\left(x_0\left(\widetilde{\sigma}^{k}(\mathbf{p}),\sigma^{k}(\xi)\right)\right), \quad \forall k\in\mathbb{Z}, \tag{3.5.19}$$

and since $\sigma^k : \mathscr{E} \mapsto \mathscr{E}$ is a homeomorphism, (3.5.19) gives

$$x_0\left(\widetilde{\sigma}^{k+1}(\mathbf{p}),\sigma(\xi)\right) = F_{2k\omega,\mathbf{p}}^{2(k+1)\omega}\left(x_0\left(\widetilde{\sigma}^{k}(\mathbf{p}),\xi\right)\right), \quad \forall k\in\mathbb{Z}. \tag{3.5.20}$$

Next, introducing the following mappings

$$\begin{gathered}
\Sigma : \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)} \times \mathscr{E} \times \mathbb{Z} \mapsto \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)} \times \mathscr{E} \times \mathbb{Z}\,, \\
\Sigma(\mathbf{p},\xi,k) := (\mathbf{p},\sigma(\xi),k+1)\,, \\[1ex]
\Phi : \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)} \times \mathscr{E} \times \mathbb{Z} \mapsto \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)} \times \mathbb{R}^n \times \mathbb{Z}\,, \\
\Phi(\mathbf{p},\xi,k) := \left(\mathbf{p},x_0\left(\widetilde{\sigma}^{k}(\mathbf{p}),\xi\right),k\right), \\
F^{2\omega} : \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)} \times \mathbb{R}^n \times \mathbb{Z} \mapsto \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)} \times \mathbb{R}^n \times \mathbb{Z}\,, \\
F^{2\omega}(\mathbf{p},x,k) := \left(\mathbf{p},F_{2k\omega,\mathbf{p}}^{2(k+1)\omega}(x),k+1\right),
\end{gathered}$$

and the set

$$\Lambda := \Phi\left(\mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)} \times \mathscr{E} \times \mathbb{Z}\right),$$

we obtain the main result of this section.

Theorem 3.5.5. *The diagram of Figure 3.2 is commutative. Moreover, mappings Σ and Φ are homeomorphisms.*

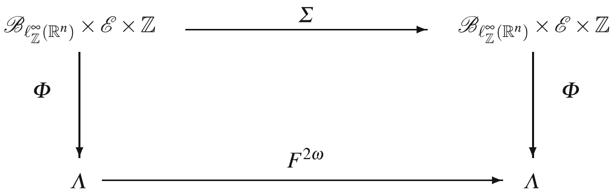

Fig. 3.2 Commutative diagram of inflated deterministic chaos.

Proof. The commutativity of diagram in Figure 3.2 follows directly from (3.5.20). Since $\sigma : \mathscr{E} \mapsto \mathscr{E}$ is a homeomorphism, Σ is also a homeomorphism. Now we show the injectivity of the mapping $x_0(\mathbf{p},\cdot) : \mathscr{E} \mapsto \mathbb{R}^n$. If there exist $\mathscr{E} \ni \xi^1 = \{e_j^1\}_{j\in\mathbb{Z}} \neq \xi^2 = \{e_j^2\}_{j\in\mathbb{Z}} \in \mathscr{E}$ and $x_0(\mathbf{p},\xi^1) = x_0(\mathbf{p},\xi^2)$, then $x_k(\mathbf{p},\xi^1) = x_k(\mathbf{p},\xi^2)$ for any $k\in\mathbb{Z}$ and $j_0\in\mathbb{Z}$ exists so that $e_{j_0}^1 \neq e_{j_0}^2$. Then (3.5.14) gives

$$|x_{k_0}^0 - x_0| = \left|x_{(2j_0+1)\omega+k_0}^{\xi^1} - x_{(2j_0+1)\omega+k_0}^{\xi^2}\right| \le \left|x_{(2j_0+1)\omega+k_0}(\mathbf{p},\xi^1) - x_{(2j_0+1)\omega+k_0}^{\xi^1}\right| + \left|x_{(2j_0+1)\omega+k_0}(\mathbf{p},\xi^2) - x_{(2j_0+1)\omega+k_0}^{\xi^2}\right| \le 2\rho_0 < |x_{k_0}^0 - x_0|\,,$$

which is a contradiction. Consequently $x_0(\mathbf{p},\cdot)$ is injective. Now suppose $\Phi(\mathbf{p}^1,\xi^1,k_1) = \Phi(\mathbf{p}^2,\xi^2,k_2)$. Then $\mathbf{p}^1 = \mathbf{p}^2 = \mathbf{p}$, $k_1 = k_2 = k$ and

$$x_0\left(\widetilde{\sigma}^k(\mathbf{p}),\xi^1\right) = x_0\left(\widetilde{\sigma}^k(\mathbf{p}),\xi^2\right)$$

and thus $\xi^1 = \xi^2$. Hence Φ is also injective. Finally assume that $\Phi(\mathbf{p}^i,\xi^i,k_i) \to \Phi(\mathbf{p}^0,\xi^0,k_0)$ as $i \to \infty$. Then $k^i = k^0$ for large i, $\mathbf{p}^i \to \mathbf{p}^0$ and

$$x_0\left(\widetilde{\sigma}^{k_0}(\mathbf{p}^i),\xi^i\right) \to x_0\left(\widetilde{\sigma}^{k_0}(\mathbf{p}^0),\xi^0\right).$$

Since $\mathscr{E}$ is compact, we can suppose $\xi^i \to \widetilde{\xi}^0$ and then

$$x_0\left(\widetilde{\sigma}^{k_0}(\mathbf{p}^0),\xi^0\right) = x_0\left(\widetilde{\sigma}^{k_0}(\mathbf{p}^0),\widetilde{\xi}^0\right)$$

and so $\widetilde{\xi}^0 = \xi^0$, i.e. Φ^{-1} is continuous. In summary, Φ is a homeomorphism. The proof is finished. □

Figure 3.2 has the following more transparent form in Figure 3.3 where

$$\widetilde{\Sigma} : \mathscr{B}_{\ell_{\mathbb{Z}}^\infty(\mathbb{R}^n)} \times \mathscr{E} \mapsto \mathscr{B}_{\ell_{\mathbb{Z}}^\infty(\mathbb{R}^n)} \times \mathscr{E}\,, \quad \widetilde{\Sigma}(\mathbf{p},\xi) := (\mathbf{p},\sigma(\xi))\,,$$

$$\Phi_k : \mathscr{B}_{\ell_{\mathbb{Z}}^\infty(\mathbb{R}^n)} \times \mathscr{E} \mapsto \mathscr{B}_{\ell_{\mathbb{Z}}^\infty(\mathbb{R}^n)} \times \mathbb{R}^n\,, \quad \Phi_k(\mathbf{p},\xi) := \left(\mathbf{p}, x_0\left(\widetilde{\sigma}^k(\mathbf{p}),\xi\right)\right),$$

$$\Lambda_k := \Phi_k\left(\mathscr{B}_{\ell_{\mathbb{Z}}^\infty(\mathbb{R}^n)} \times \mathscr{E}\right),$$

$$F_k^{2\omega} : \mathscr{B}_{\ell_{\mathbb{Z}}^\infty(\mathbb{R}^n)} \times \mathbb{R}^n \mapsto \mathscr{B}_{\ell_{\mathbb{Z}}^\infty(\mathbb{R}^n)} \times \mathbb{R}^n\,, \quad F_k^{2\omega}(\mathbf{p},x) := \left(\mathbf{p}, F_{2k\omega,\mathbf{p}}^{2(k+1)\omega}(x)\right).$$

By putting

$$\Phi_k^{\mathbf{p}} : \mathscr{E} \mapsto \mathbb{R}^n\,, \quad \Phi_k^{\mathbf{p}}(\xi) := x_0\left(\widetilde{\sigma}^k(\mathbf{p}),\xi\right), \quad \Lambda_k^{\mathbf{p}} := \Phi_k^{\mathbf{p}}(\mathscr{E})\,,$$

Figure 3.3 has also more transparent forms described in Figure 3.4. All mappings in Figures 3.3 and 3.4 are again homeomorphisms, and sets $\Lambda_k^{\mathbf{p}}$ are compact. So Figure 3.4 is a two-parameterized analogy of Figure 2.1 of Section 2.5.2 by parameters $\mathbf{p} \in \mathscr{B}_{\ell_{\mathbb{Z}}^\infty(\mathbb{R}^n)}$ and $k \in \mathbb{Z}$.

Set

$$\varphi_0(\xi) = \Phi_0^0(\xi) = x_0(0,\xi), \quad \Lambda_0 = \Lambda_0^0 = x_0(0,\mathscr{E}), \quad m = 2\omega\,. \tag{3.5.21}$$

By (3.5.15), all sets $\Lambda_k^{\mathbf{p}}$ are in a $\frac{c\Lambda}{1-\kappa_0}$-neighborhood of Λ_0. If

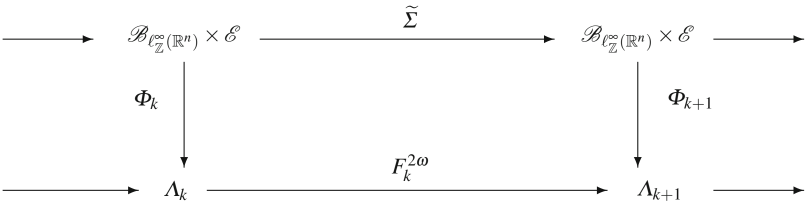

Fig. 3.3 A sequence of commutative diagrams from Figure 3.2.

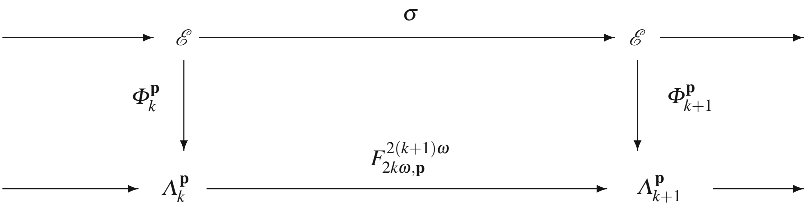

Fig. 3.4 A parameterized sequence of commutative diagrams from Figure 3.3.

$$g(x,0) = 0 \quad \forall x \in \mathbb{R}^n \tag{3.5.22}$$

then $L = 0$ in (3.5.4), $\varphi = \varphi_0$, $\Lambda = \Lambda_0$ in (3.5.21) and Figure 2.1 of Section 2.5.2 is derived from Figure 3.4 by setting $\mathbf{p} = 0$. Moreover, inequality (3.5.13) gives $\widetilde{\rho}_0 := c\widetilde{c}\left(\frac{\delta+1}{2}\right)^{\omega_0} + \kappa_0\rho < \rho_0$. Clearly $\triangle(\widetilde{\rho}_0) \leq \triangle(\rho_0)$ and so $\widetilde{\kappa}_0 := c\triangle(\widetilde{\rho}_0) \leq \kappa_0$. Repeating the proof of Theorem 3.5.2 we get $\|\mathbf{x}(0,\xi) - \mathbf{x}^\xi\| \leq \widetilde{\rho}_0$ for any $\xi \in \mathscr{E}$.

Note, the above diagrams are generalizations of similar results of [33,43,44] for non-autonomous sequences of diffeomorphisms, ordinary differential equations and inclusions. Now we put

$$\widetilde{\Lambda} := \bigcup_{\mathbf{p}\in\mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)},k\in\mathbb{Z}} \Lambda_k^{\mathbf{p}}.$$

Note that $\widetilde{\Lambda} = x_0\left(\mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}, \mathscr{E}\right)$. We can consider $\widetilde{\Lambda}$ as an *inflated Smale horseshoe* of f.

Theorem 3.5.6. *Assume* (3.5.11)*,* (3.5.12) *and* (3.5.22)*. If* $\omega \in \mathbb{N}$ *is sufficiently large, then the following properties hold:*

(i) $\Lambda \subset \widetilde{\Lambda}$ *and if in addition*

$$g_x := g(x,\cdot) : \mathscr{B}_{\mathbb{R}^n} \to \mathbb{R}^n \quad \textit{is injective} \quad \forall x \in \mathbb{R}^n, \tag{3.5.23}$$

then Λ *is in the interior of* $\widetilde{\Lambda}$*.*

(ii) $\widetilde{\Lambda}$ *is contractible into* Λ *in itself.*

(iii) $\widetilde{\Lambda}$ *is in a* $\frac{c\Lambda}{1-\kappa_0}$*-neighborhood of* Λ*.*

(iv) $\widetilde{\Lambda}$ *is back and forward weakly invariant with respect to an m-iteration of* (3.5.3)*, i.e.* $\exists m \in \mathbb{N}$ *so that* $\forall \bar{x}_0 \in \widetilde{\Lambda}$*,* $\exists\{\bar{x}_k\}_{k\in\mathbb{Z}}$ *satisfying* $\bar{x}_{k+1} \in f(\bar{x}_k) + g(\bar{x}_k, \mathscr{B}_{\mathbb{R}^n})$ *and* $\bar{x}_{km} \in \widetilde{\Lambda}$*,* $\forall k \in \mathbb{Z}$*.*

(v) *Dynamics of* (3.5.3) *back and forward sensitively depends on* $\widetilde{\Lambda}$*, i.e. there is a constant* $\eta > 0$ *so that for any* $\bar{x}_0 \in \widetilde{\Lambda}$ *and any open neighborhood* $\bar{x}_0 \in U \subset \mathbb{R}^n$*, there is* $\widetilde{x}_0 \in U \cap \widetilde{\Lambda}$ *and* $\{\bar{x}_k\}_{k\in\mathbb{Z}}$*,* $\{\widetilde{x}_k\}_{k\in\mathbb{Z}}$ *satisfying* $\bar{x}_{k+1} \in f(\bar{x}_k) + g(\bar{x}_k, \mathscr{B}_{\mathbb{R}^n})$ *and* $\widetilde{x}_{k+1} \in f(\widetilde{x}_k) + g(\widetilde{x}_k, \mathscr{B}_{\mathbb{R}^n})$*,* $\forall k \in \mathbb{Z}$*, and there exist* $j_0, j_1 \in \mathbb{Z}$*,* $j_0 < 0 < j_1$ *so that* $\left|\bar{x}_{j_0} - \widetilde{x}_{j_0}\right| \geq \eta$ *and* $\left|\bar{x}_{j_1} - \widetilde{x}_{j_1}\right| \geq \eta$*.*

(vi) (3.5.3) *has a chaotic/oscillatory behavior on* $\widetilde{\Lambda}$*.*

where we consider Theorem 2.5.4 in the sense of (3.5.21).

Proof. Since $\Lambda_0 = \Lambda$, we get $\Lambda \subset \widetilde{\Lambda}$. Next we fix $\xi \in \mathscr{E}$ and consider a mapping $\Theta_\xi : \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)} \mapsto \ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)$ given by $\Theta_\xi(\mathbf{p}) = \mathbf{x}(\mathbf{p}, \xi)$. We study Θ_ξ for $\mathbf{p}$ near 0. From (3.5.23), there are open neighborhoods $0 \in V \subset \mathbb{R}^n$ and $\overline{\widetilde{\Lambda}} \subset W$ so that

$$V \subset g_x(\mathscr{B}_{\mathbb{R}^n}), \quad \forall x \in W.$$

So we have $\psi_x := g_x^{-1} : V \to \mathscr{B}_{\mathbb{R}^n}$, $\forall x \in W$. Clearly $\psi(x,z) := \psi_x(z)$, $\psi : W \times V \to \mathbb{R}^n$ is continuous. We continuously extend ψ on $\mathbb{R}^n \times \mathbb{R}^n$. Then we define $R : \ell^\infty_{\mathbb{Z}}(\mathbb{R}^n) \to \ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)$ as follows

$$R(\mathbf{x}) := \{\psi(x_k, x_{k+1} - f(x_k)\}_{k\in\mathbb{Z}}.$$

R is continuous. If $\|\mathbf{p}\|$ is small then $x_{k+1} - f(x_k) = g(x_k, p_k) \in V$ for $\mathbf{x}(\mathbf{p}, \xi) = \{x_k\}_{k\in\mathbb{Z}}$, so $p_k = g_{x_k}^{-1}(x_{k+1} - f(x_k)) = \psi(x_k, x_{k+1} - f(x_k))$, i.e. $R(\Theta_\xi(\mathbf{p})) = \mathbf{p}$ for any $\mathbf{p}$ small. Note that $\Theta_\xi(0) = \mathbf{x}(0, \xi) = \left\{f^k(\varphi(\xi))\right\}_{k\in\mathbb{Z}}$ and $\|\mathbf{x}(0,\xi) - \mathbf{x}^\xi\| \leq \widetilde{\rho}_0 < \rho_0$ for any $\xi \in \mathscr{E}$. On the other hand, if $\mathbf{x} = \{x_k\}_{k\in\mathbb{Z}}$ is close to $\Theta_\xi(0)$ then $x_{k+1} - f(x_k) \in V$ $\forall k \in \mathbb{Z}$ along with $\|\mathbf{x} - \mathbf{x}^\xi\| \leq \rho_0$, so we can put $p_k := \psi(x_k, x_{k+1} - f(x_k)) \in \mathscr{B}_{\mathbb{R}^n}$. Then $x_{k+1} = f(x_k) + g(x_k, p_k)$. From the uniqueness we derive $\mathbf{x} = \Theta_\xi(\mathbf{p}) = \Theta_\xi(R(\mathbf{x}))$. In summary, Θ_ξ is a local homeomorphism at $\mathbf{p} = 0$. Now, a projection $P_0 : \ell^\infty_{\mathbb{Z}}(\mathbb{R}^n) \mapsto \mathbb{R}^n$ given by $P_0(\{\bar{x}_k\}_{k\in\mathbb{Z}}) := \bar{x}_0$ is an open linear mapping. Consequently, a mapping $P_0 \circ \Theta_\xi(\mathbf{p}) = x_0(\mathbf{p}, \xi)$ maps a small open neighborhood of $\mathbf{p} = 0$ onto a small open neighborhood of $\varphi(\xi) = P_0 \circ \Theta_\xi(0) \in \Lambda$. This implies property (i). By taking

$$\widetilde{\Lambda}_\lambda := \left\{x_0(\lambda\mathbf{p}, \xi) : \mathbf{p} \in \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}, \xi \in \mathscr{E}\right\}$$

for $\lambda \in [0,1]$, we get property (ii), since clearly $\widetilde{\Lambda}_\lambda \subset \widetilde{\Lambda}$ and $\widetilde{\Lambda}_0 = \Lambda$. Property (iii) follows from (3.5.15). The definition of $\widetilde{\Lambda}$ implies property (iv). Now we show property (v). Take $\eta := \left|x_{k_0} - x_0\right| - 2\rho_0 > 0$. Then for any $\bar{x}_0 \in \widetilde{\Lambda}$ we have $\bar{x}_0 = x_0(\mathbf{p}, \bar{\xi})$ for some $\mathbf{p} \in \mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}$ and $\bar{\xi} \in \mathscr{E}$. Let $\bar{x}_0 \in U \subset \mathbb{R}^n$ be an open neighborhood. From the continuity of mapping $\xi \to x_0(\mathbf{p}, \xi)$ (see Theorem 3.5.3), there is $\widetilde{\xi} \in \mathscr{E}$ close to $\bar{\xi}$ so that $\widetilde{x}_0 = x_0(\mathbf{p}, \widetilde{\xi}) \in U \cap \widetilde{\Lambda}$ and there exist $i_0, i_1 \in \mathbb{Z}$, $i_0 < -\frac{k_0+\omega}{2\omega} < i_1$ so that $\bar{e}_{i_0} \neq \widetilde{e}_{i_0}$, $\bar{e}_{i_1} \neq \widetilde{e}_{i_1}$ for $\bar{\xi} = \{\bar{e}_i\}_{i\in\mathbb{Z}}$ and $\widetilde{\xi} = \{\widetilde{e}_i\}_{i\in\mathbb{Z}}$. Then for $j_0 = (2i_0 + 1)\omega + k_0 < 0$, (3.5.14) gives

$$\left|x_{j_0}(\mathbf{p},\bar{\xi})-x_{j_0}(\mathbf{p},\widetilde{\xi})\right| \geq \left|x_{j_0}^{\bar{\xi}}-x_{j_0}^{\widetilde{\xi}}\right| - \left|x_{j_0}(\mathbf{p},\bar{\xi})-x_{j_0}^{\bar{\xi}}\right| - \left|x_{j_0}(\mathbf{p},\widetilde{\xi})-x_{j_0}^{\widetilde{\xi}}\right|$$
$$\geq |x_{k_0}^0-x_0|-2\rho_0=\eta>0\,.$$

The same estimates hold for $j_1=(2i_1+1)\omega+k_0>0$. Property (v) is shown. Diagram in Figure 3.4 gives property (vi). The proof is completed. □

With property (v), we can construct many continuum orbits of (3.5.3) starting from U and oscillating back and forward on $\mathbb{Z}$ between x_0 and $x_{k_0}^0$ in any order. Of course, results of this section can be directly extended to more ε-inflated systems of the form $x_{k+1}=f(x_k+\varepsilon q_k)+g(x_k,p_k)$, $k\in\mathbb{Z}$ for any $\{p_k\}_{k\in\mathbb{Z}}$, $\{q_k\}_{k\in\mathbb{Z}}\in\mathscr{B}_{\ell^\infty_{\mathbb{Z}}(\mathbb{R}^n)}$ and $\varepsilon>0$ small fixed.

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Chapter 4
Chaos in Ordinary Differential Equations

Functional analytical methods are presented in this chapter to predict chaos for ODEs depending on parameters. Several types of ODEs are considered. We also study multivalued perturbations of ODEs, and coupled infinite-dimensional ODEs on the lattice $\mathbb{Z}$ as well. Moreover, the structure of bifurcation parameters for homoclinic orbits is investigated.

4.1 Higher Dimensional ODEs

4.1.1 Parameterized Higher Dimensional ODEs

In this section, we consider ODEs of the form

$$\dot{x} = f(x) + h(x, \mu, t) \tag{4.1.1}$$

with $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$. We make the following assumptions of (4.1.1):

(i) f and h are C^3 in all arguments.
(ii) $f(0) = 0$ and $h(\cdot, 0, \cdot) = 0$.
(iii) The eigenvalues of $Df(0)$ lie off the imaginary axis.
(iv) The unperturbed equation has a homoclinic solution, i.e. there is a nonzero differentiable function $\gamma(t)$ so that $\lim_{t\to\pm\infty} \gamma(t) = 0$ and $\dot{\gamma}(t) = f(\gamma(t))$.
(v) $h(x, \mu, t+1) = h(x, \mu, t)$ for $t \in \mathbb{R}$.

Let Ψ_μ be the period map of (4.1.1), i.e. $\Psi_\mu(x) = \phi_\mu(x, 1)$ where $\phi_\mu(x,t)$ is the solution of (4.1.1) with the initial condition $\phi_\mu(x, 0) = x$. The purpose of this section is to find a set of parameters μ for which the period map Ψ_μ of (4.1.1) has a transversal homoclinic orbit. For this reason, higher dimensional Melnikov mappings are introduced. Simple zero points of those mappings give wedge-shaped regions in $\mathbb{R}^m$ for μ where Ψ_μ possesses transversal homoclinic orbits. This result is a continuous version of Section 3.1, where difference equations are studied. Melnikov theory for

ODEs is also given in a lot of work [1–7]. This method is usually applied when the unperturbed equation

$$\dot{x} = f(x) \tag{4.1.2}$$

is integrable [8].

4.1.2 Variational Equations

For (4.1.2) we adopt the standard notations W^s, W^u for the stable and unstable manifolds, respectively, of the origin and $d_s = \dim W^s$, $d_u = \dim W^u$. Since $x = 0$ is a hyperbolic equilibrium, γ lies on $W^s \cap W^u$. By the *variational equation* along γ we mean the linear differential equation

$$\dot{u} = Df(\gamma(t))u. \tag{4.1.3}$$

Now, we can repeat the arguments of Section 3.1.2 to (4.1.3), but since it is straightforward, we do not go into details, and we refer the readers to [3, Theorem 2] and [9, Theorem 3.1.2]. Consequently, the following results hold.

Theorem 4.1.1. *There exists a fundamental solution U for* (4.1.3) *along with constants $M > 0$, $K_0 > 0$ and four projections P_{ss}, P_{su}, P_{us}, P_{uu} so that $P_{ss} + P_{su} + P_{us} + P_{uu} = \mathbb{I}$ and the following hold:*

(i) $|U(t)(P_{ss} + P_{us})U(s)^{-1}| \leq K_0 e^{2M(s-t)}$, *for* $0 \leq s \leq t$,
(ii) $|U(t)(P_{su} + P_{uu})U(s)^{-1}| \leq K_0 e^{2M(t-s)}$, *for* $0 \leq t \leq s$,
(iii) $|U(t)(P_{ss} + P_{su})U(s)^{-1}| \leq K_0 e^{2M(t-s)}$, *for* $t \leq s \leq 0$,
(iv) $|U(t)(P_{us} + P_{uu})U(s)^{-1}| \leq K_0 e^{2M(s-t)}$, *for* $s \leq t \leq 0$.

Also $\operatorname{rank} P_{ss} = \operatorname{rank} P_{uu} = d$.

In the language of exponential dichotomies we see that Theorem 4.1.1 provides a two-sided exponential dichotomy. For $t \to -\infty$ an exponential dichotomy is given by the fundamental solution U and the projection $P_{us} + P_{uu}$ while for $t \to +\infty$ such an exponential dichotomy is given by U and $P_{ss} + P_{us}$.

Let u_j denote column j of U and assume that these are numbered so that

$$P_{uu} = \begin{pmatrix} \mathbb{I}_d & 0_d & 0 \\ 0_d & 0_d & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad P_{ss} = \begin{pmatrix} 0_d & 0_d & 0 \\ 0_d & \mathbb{I}_d & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here, $\mathbb{I}_d$ denotes the $d \times d$ identity matrix and 0_d denotes the $d \times d$ zero matrix.

For each $i = 1, \ldots, n$ we define $u_i^\perp(t)$ by $\langle u_i^\perp(t), u_j(t) \rangle = \delta_{ij}$, where $\langle \cdot, \cdot \rangle$ is the scalar product on $\mathbb{R}^n$. The vectors $u_i^\perp$ can be computed from the formula $U^{\perp *} = U^{-1}$ where $U^\perp$ denotes the matrix with $u_j^\perp$ as column j. Differentiating $UU^{\perp *} = \mathbb{I}$ we obtain $\dot{U}U^{\perp *} + U\dot{U}^{\perp *} = 0$ so that $\dot{U}^\perp = -(U^{-1}\dot{U}U^{\perp *})^* = -Df(\gamma)^* U^\perp$. Thus, $U^\perp$

is the adjoint of U. Note that $\{u_i^{\perp}(t) \mid i=1,2,\ldots,d\}$ is a basis of bounded solutions on $\mathbb{R}$ of the adjoint variational equation $\dot{w} = -Df(\gamma)^* w$. The function $\dot{\gamma}$ is always a solution to the variational equation (4.1.3) and we may assume that $u_{2d} = \dot{\gamma}$, since $\dot{\gamma}$ is a linear combination of columns u_{d+1} through u_{2d} of U and a linear transformation of these columns preserves the projections.

Now we define the following Banach spaces

$$Z = \left\{ z \in C\big((-\infty,\infty),\mathbb{R}^n\big) \mid \sup_{t\in\mathbb{R}} |z(t)| < \infty \right\},$$

$$Y = \left\{ z \in C^1\left((-\infty,\infty),\mathbb{R}^n\right) \mid z, \dot{z} \in Z \right\},$$

with the usual supremum norms.

Theorem 4.1.2. *The linear equation*

$$\dot{u} = Df(\gamma(t))u + z, \quad z \in Z.$$

has a solution $u = K(z)(t) \in Y$ if and only if

$$z \in \widetilde{Z} := \left\{ z \in Z \mid \int_{-\infty}^{\infty} P_{uu} U(s)^{-1} z(s)\, ds = 0 \right\}.$$

Moreover, if $z \in \widetilde{Z}$ then we can take

$$K(z)(t) = \begin{cases} U(t)\Big[\int_{-\infty}^{0} P_{su} U(s)^{-1} z(s)\, ds + \int_0^t (P_{ss}+P_{su}) U(s)^{-1} z(s)\, ds \\ \qquad - \int_t^{\infty} (P_{us}+P_{uu}) U(s)^{-1} z(s)\, ds\Big], \quad \text{for } t \geq 0, \\ U(t)\Big[-\int_0^{\infty} P_{us} U(s)^{-1} z(s)\, ds + \int_0^t (P_{ss}+P_{us}) U(s)^{-1} z(s)\, ds \\ \qquad + \int_{-\infty}^{t} (P_{su}+P_{uu}) U(s)^{-1} z(s)\, ds\Big], \quad \text{for } t \leq 0. \end{cases}$$

Note that $z \in \widetilde{Z} \Leftrightarrow \int_{-\infty}^{\infty} \langle u_i^{\perp}(t), z(s)\rangle\, ds = 0$ for all $i = 1,2,\ldots,d$.

Theorem 4.1.3. *Define a projection $\Pi : Z \to Z$ by*

$$\Pi(z)(t) := \varphi(t) \int_{-\infty}^{\infty} U(t) P_{uu} U(s)^{-1} z(s)\, ds,$$

for a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$ satisfying $\sup_t \left|\varphi(t) u_j(t)\right| < \infty$ for all j and $\int_{-\infty}^{\infty} \varphi(s)\, ds = 1$. Then $\mathscr{R}(\mathbb{I} - \Pi) = \widetilde{Z}$.

4.1.3 Melnikov Mappings

Without loss of generality, we can suppose that f and h as well as all their partial derivatives up to the order 3 are uniformly bounded on the whole spaces of definition. We study the equation (cf Theorem 2.2.4)

$$\begin{gathered} F_{\mu,\varepsilon,y}(x) = \dot{x} - f(x) - h(x,\mu,t) - \varepsilon|\mu|L(x-y) = 0, \\ F_{\mu,\varepsilon,y} : Y \to Z, \end{gathered} \tag{4.1.4}$$

where $L : Y \to Z$ is a linear continuous mapping so that $||L|| \leq 1$, $y \in Y$ and $\varepsilon \in \mathbb{R}$ is small. It is clear that solutions of (4.1.4) near γ with $\varepsilon = 0$ are homoclinic ones of (4.1.1). We make in (4.1.4) the change of variable

$$x(t) = \gamma(t-\alpha) + w(t), \quad \langle w(0), \dot{\gamma}^{\perp}(-\alpha) \rangle = 0, \tag{4.1.5}$$

where $\alpha \in \mathscr{I} \subset \mathbb{R}$ and $\mathscr{I}$ is a given bounded open interval. We note that (4.1.5) defines a tubular neighbourhood of the manifold $\{\gamma(t-\alpha)\}_{\alpha \in \mathscr{I}}$ in Y when w is sufficiently small (cf Section 2.4.3). Hence (4.1.4) has the form

$$\begin{aligned} G_{\alpha,\mu,\varepsilon,y}(w) = {} & \dot{w} - f(\gamma(t-\alpha) + w)) + f(\gamma(t-\alpha)) \\ & - h(\gamma(t-\alpha) + w, \mu, t) - \varepsilon|\mu| L\big(w + \gamma(t-\alpha) - y\big) = 0, \\ & G_{\alpha,\mu,\varepsilon,y} : Y \to Z. \end{aligned}$$

We have

$$D_w G_{\alpha,0,0,y}(0)u = \dot{u} - Df(\gamma(t-\alpha))u.$$

By putting

$$U_\alpha(t) = U(t-\alpha), \quad U_\alpha^{\perp}(t) = U^{\perp}(t-\alpha),$$

Theorem 4.1.1 is valid when U is replaced by U_α and (4.1.3) by

$$\dot{u} = Df(\gamma(t-\alpha))u,$$

respectively, but $K_0 > 0$ should be enlarged. Moreover, we put

$$\gamma_\alpha(t) = \gamma(t-\alpha), \quad u_{j,\alpha} = u_j(t-\alpha), \quad u_{j,\alpha}^{\perp} = u_j^{\perp}(t-\alpha).$$

Consequently, by taking

$$Q = \Big\{ y \in Y \mid \sup_{t \in \mathbb{R}} \big(|y(t)| + |\dot{y}(t)|\big) < \sup_{t \in \mathbb{R}} \big(|\gamma(t)| + |\dot{\gamma}(t)|\big) + 1 \Big\}$$

and by using the same approach as in [3], [5, p. 709] and Section 3.1.3 along with Theorems 4.1.2 and 4.1.3, there are open small neighborhoods $0 \in O \subset \mathbb{R}^{d-1}$, $0 \in V \subset \mathbb{R}$, $0 \in W \subset \mathbb{R}^m$ and a mapping

$$G \in C^3\left(Y \times O \times \mathscr{I} \times W \times V \times Q, Z\right),$$

so that any solution of (4.1.4) near γ_α for $\mu \in W$, $\varepsilon \in V$, $y \in Q$ is determined by the equation $G(z,\beta,\alpha,\mu,\varepsilon,y) = 0$ and this solution has the form

$$x = \gamma_\alpha + z, \quad P_{ss}U_\alpha^{-1}(0)\big(z(0) - \sum_{j=1}^{d-1} \beta_j u_{j+d,\alpha}(0)\big) = 0, \tag{4.1.6}$$

where $\beta = (\beta_1, \ldots, \beta_{d-1})$. We remark that $\{u_{j,\alpha}(0)\}_{j=1}^n$ are linearly independent, $u_{2d,\alpha}(0) = \dot{\gamma}_\alpha(0) = \dot{\gamma}(-\alpha)$, as well as

$$\left\{v \in \mathbb{R}^n \mid \langle v, \dot{\gamma}^\perp(-\alpha)\rangle = 0\right\} = \operatorname{span}\left\{\{u_{j,\alpha}(0)\}_{j=1}^n \setminus \{u_{2d,\alpha}(0)\}\right\},$$

and

$$0 = P_{ss}U_\alpha^{-1}(0)w = P_{ss}U_\alpha^{\perp *}(0)w \iff \langle u_{j+d,\alpha}^\perp(0), w\rangle = 0, \quad \forall j, 1 \le j \le d\,.$$

Hence (4.1.5) and (4.1.6) provide a suitable decomposition of any x in (4.1.4) near the manifold $\{\gamma(t-\alpha)\}_{\alpha\in\mathscr{I}}$. Now by using the Lyapunov-Schmidt procedure (see again [3, Theorem 8], [5, p. 709] and Section 3.1.3), the study of the equation $G(z,\beta,\alpha,\mu,\varepsilon,y) = 0$ can be expressed in the following theorem for z, μ, ε, β small, $y \in Q$ and $\alpha \in \mathscr{I}$.

Theorem 4.1.4. *U and d are the same as in Theorem 4.1.1. Then there exist small neighborhoods $0 \in O_1 \subset \mathbb{R}^{d-1}$, $0 \in W_1 \subset \mathbb{R}^m$, $0 \in V_1 \subset \mathbb{R}$ and a C^3 function $H : Q \times O_1 \times \mathscr{I} \times W_1 \times V_1 \to \mathbb{R}^d$ denoted $(y,\beta,\alpha,\mu,\varepsilon) \to H(y,\beta,\alpha,\mu,\varepsilon)$ with the following properties:*

(i) *The equation $H(y,\beta,\alpha,\mu,\varepsilon) = 0$ holds if and only if* (4.1.4) *has a solution near γ_α and each such $(y,\beta,\alpha,\mu,\varepsilon)$ determines only one solution of* (4.1.4)*,*
(ii) *$H(y,0,\alpha,0,0) = 0$,*
(iii) *$\frac{\partial H_i}{\partial \mu_j}(y,0,\alpha,0,0) = -\int_{-\infty}^{\infty}\left\langle u_i^\perp(t), \frac{\partial h}{\partial \mu_j}(\gamma(t),0,t+\alpha)\right\rangle dt$,*
(iv) *$\frac{\partial H_i}{\partial \beta_j}(y,0,\alpha,0,0) = 0$,*
(v) *$\frac{\partial^2 H_i}{\partial \beta_k \partial \beta_j}(y,0,\alpha,0,0) = -\int_{-\infty}^{\infty}\langle u_i^\perp(t), D^2 f(\gamma(t))u_{d+j}(t)u_{d+k}(t)\rangle\, dt.$*

We introduce the following notations:

$$a_{ij}(\alpha) = -\int_{-\infty}^{\infty}\left\langle u_i^\perp(t), \frac{\partial h}{\partial \mu_j}(\gamma(t),0,t+\alpha)\right\rangle dt,$$
$$b_{ijk} = -\int_{-\infty}^{\infty}\left\langle u_i^\perp, D^2 f(\gamma)u_{d+j}u_{d+k}\right\rangle dt\,.$$

Finally, we take the mapping $M_\mu : \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$\big(M_\mu(\alpha,\beta)\big)_i = \sum_{j=1}^{m} a_{ij}(\alpha)\mu_j + \frac{1}{2}\sum_{j,k=1}^{d-1} b_{ijk}\beta_j\beta_k\,.$$

Note that we can take any bases of bounded solutions of the adjoint and adjoint variational equations (with $u_{2d} = \dot\gamma$) for constructing the Melnikov function M_μ. Now we can state the main result of this section.

Theorem 4.1.5. *Let $d > 1$. If M_{μ_0} has a simple root (α_0,β_0), i.e. (α_0,β_0) satisfies $M_{\mu_0}(\alpha_0,\beta_0) = 0$ and $D_{(\alpha,\beta)}M_{\mu_0}(\alpha_0,\beta_0)$ is a regular matrix, then there is a wedge-shaped region in $\mathbb{R}^m$ for μ of the form*

$$\mathscr{R} = \Big\{ s^2\tilde\mu \,\Big|\, s \text{ is from a small open neighborhood of } 0 \in \mathbb{R} \text{ and } \tilde\mu \text{ is from a small open neighborhood of } \mu_0 \in \mathbb{R}^m \text{ satisfying } |\tilde\mu| = |\mu_0| \Big\},$$

so that for any $\mu \in \mathscr{R}\setminus\{0\}$, period map Ψ_μ of (4.1.1) *possesses a transversal homoclinic orbit.*

Proof. Let us take $\mathscr{I} = (\alpha_0 - 1, \alpha_0 + 1)$ and let us consider the mapping defined by

$$\Phi(y,\tilde\beta,\alpha,\tilde\mu,\tilde\varepsilon,s) = \begin{cases} \dfrac{1}{s^2}H(y,s\tilde\beta,\alpha,s^2\tilde\mu,s^3\tilde\varepsilon), & \text{for} \quad s \neq 0,\\ M_{\tilde\mu}(\alpha,\tilde\beta), & \text{for} \quad s = 0\,. \end{cases}$$

According to (ii)–(v) of Theorem 4.1.4, the mapping Φ is C^1-smooth near

$$(y,\tilde\beta,\alpha,\tilde\mu,\tilde\varepsilon,s) = (y,\beta_0,\alpha_0,\mu_0,0,0), \quad y \in Q$$

with respect to the variables $\tilde\beta,\alpha$. Since

$$M_{\mu_0}(\alpha_0,\beta_0) = 0 \quad \text{and} \quad D_{(\alpha,\beta)}M_{\mu_0}(\alpha_0,\beta_0) \quad \text{is a regular matrix}\,,$$

we can apply the implicit function theorem to solving locally and uniquely the equation $\Phi = 0$ in the variables $\tilde\beta,\alpha$, where $\tilde\mu$ is near $\tilde\mu_0$ satisfying $|\tilde\mu| = |\mu_0|$. This gives for $\varepsilon = 0$, by (i) of Theorem 4.1.4, the existence of $\mathscr{R}$ on which Ψ_μ has a homoclinic orbit. Moreover, we can suppose that the corresponding solutions of (4.1.4) lie in Q.

To prove the transversality of these homoclinic orbits, we fix $\mu \in \mathscr{R}\setminus\{0\}$ and take $y = \tilde\gamma$, where $\tilde\gamma$ is the solution of (4.1.4) for which the transversality of the corresponding homoclinic orbit of Ψ_μ should be proved. Then we vary $\varepsilon = s^3\tilde\varepsilon$ small. Note that $s \neq 0$ is also fixed due to $\mu = s^2\tilde\mu$ and $|\tilde\mu| = |\mu_0|$ as well. Since the local uniqueness of solutions of (4.1.4) near $\tilde\gamma$ is satisfied for any $\tilde\varepsilon$ sufficiently small according to the above application of the implicit function theorem, such equation (4.1.4) (with the fixed $\mu \in \mathscr{R}\setminus\{0\}$, $\varepsilon = s^3\tilde\varepsilon$ where $s \neq 0$ is also fixed and the special $y = \tilde\gamma$) has the only solution $x = \tilde\gamma$ near $\tilde\gamma$ for any $\tilde\varepsilon$ sufficiently small. Hence Theorem 2.2.4 gives the invertibility of $DF_{\mu,0,\tilde\gamma}(\tilde\gamma)$, so the only bounded solution on $\mathbb{R}$ of the equation $\dot v = Df(\tilde\gamma)v + D_xh(\tilde\gamma,\mu,t)v$ is $v = 0$. Then Lemma 2.5.2 implies the transversality of these homoclinic orbits of Ψ_μ for $\mu \in \mathscr{R}\setminus\{0\}$. □

Remark 4.1.6. (a) If M_{μ_0} has a simple zero point (α_0, β_0), then $M_{r^2\mu_0}$ has also a simple zero point at $(\alpha_0, r\beta_0)$ for any $r \in \mathbb{R} \setminus \{0\}$.

(b) If $d = 1$ then we take the function $M_\mu(\alpha) = \sum_{j=1}^m a_{1j}(\alpha)\mu_j$, which is the usual Melnikov function. So for any simple zero α_0 of $M_{\mu_0}(\alpha) = 0$, when μ_0 is fixed, there is a two-sided wedge-shaped region in $\mathbb{R}^m$ for μ of the form

$$\mathscr{R} = \Big\{ s\tilde{\mu} \Big| \ s \text{ is from a small open neighborhood of } 0 \in \mathbb{R} \text{ and } \tilde{\mu} \text{ is from}$$
$$\text{a small open neighborhood of } \mu_0 \in \mathbb{R}^m \text{ satisfying } |\tilde{\mu}| = |\mu_0| \Big\}$$

so that for any $\mu \in \mathscr{R} \setminus \{0\}$, the period map Ψ_μ of Eq. (4.1.1) possesses a transversal homoclinic orbit.

Remark 4.1.7. A standard perturbation theory [10–13], which can be verified by repeating the above arguments, implies the existence of a unique 1-periodic solution of (4.1.1) for any μ small, which is, in addition, hyperbolic. Then the transversal homoclinic solution of Theorem 4.1.5 is exponentially asymptotic to this periodic orbit.

Remark 4.1.8. Note that we can take any bases of bounded solutions of the adjoint variational and variational equations (with $u_{2d} = \dot{\gamma}$) for constructing the Melnikov function M_μ. Similar observations can be applied to detecting the other continuous Melnikov functions in this book.

Remark 4.1.9. The above results can be generalized to ODEs possessing heteroclinic orbits to *semi-hyperbolic equilibria* [14].

4.1.4 The Second Order Melnikov Function

When Melnikov function M_μ is identically zero then we need to compute the *second order Melnikov function*. Since in general computations are awkward, we consider the simplest case given by a C^3-equation

$$\ddot{x} = f(x) + \varepsilon q(t) \tag{4.1.7}$$

with 2π-periodic $q(t)$, and $\ddot{x} = f(x)$ has a homoclinic solution $p(t)$ to 0 with $f'(0) > 0$. We can suppose $\dot{p}(0) = 0$. We are looking for bounded solutions of (4.1.7) near $p(t)$. We briefly repeat the above arguments, so we shift $t \leftrightarrow t + \alpha$ and take $x = p + v$ in (4.1.7) with $v \in Y_0 := \{v \in Y \mid \dot{v}(0) = 0\}$ to obtain

$$\ddot{v} - f'(p)v = f(p+v) - f'(p)v - f(p) + \varepsilon q(t+\alpha)\,.$$

By introducing the projection $\Pi : X \to X$ as $\Pi z := \int_{-\infty}^{\infty} z(t)\dot{p}(t)\,dt / \int_{-\infty}^{\infty} \dot{p}^2(t)\,dt \cdot p$, the Lyapunov-Schmidt method splits (4.1.7) into two equations

$$\ddot{v} - f'(p)v = (\mathbb{I} - \Pi)\left\{f(p+v) - f'(p)v - f(p) + \varepsilon q(t+\alpha)\right\} \tag{4.1.8}$$

and

$$\int_{-\infty}^{\infty} \left\{f(p(t)+v(t)) - f'(p(t))v(t) - f(p(t)) + \varepsilon q(t+\alpha)\right\} \dot{p}(t)\,dt = 0\,. \tag{4.1.9}$$

By the implicit function theorem, we can uniquely solve (4.1.8) to get $v = v(\varepsilon,\alpha) \in Y_0$ with $v(0,\alpha) = 0$, so we put $v(\varepsilon,\alpha) = \varepsilon w(\varepsilon,\alpha)$, and inserting this into (4.1.9), we get the scalar bifurcation equation

$$B(\varepsilon,\alpha) := \int_{-\infty}^{\infty} \Big\{f(p(t)+\varepsilon w(\varepsilon,\alpha)(t)) - f'(p(t))\varepsilon w(\varepsilon,\alpha)(t)$$
$$-f(p(t)) + \varepsilon q(t+\alpha)\Big\}\dot{p}(t)\,dt = 0\,.$$

Clearly $B(0,\alpha) = 0$ and $B_\varepsilon(0,\alpha) = \int_{-\infty}^{\infty} q(t+\alpha)\dot{p}(t)\,dt = M(\alpha)$, where $M(\alpha)$ is the Melnikov function for (4.1.7). We have until now repeated arguments of Section 4.1.3 to (4.1.7). When $M(\alpha) = 0$, then we proceed further to derive

$$B_{\varepsilon\varepsilon}(0,\alpha) = \int_{-\infty}^{+\infty} \dot{p}(t) f''(p(t)) w(0,\alpha)^2(t)\,dt\,.$$

Note that by (4.1.8), $w(0,\alpha)$ solves $\ddot{w}(0,\alpha)(t) = f'(p(t))w(0,\alpha)(t) + q(t+\alpha)$. Summarizing the second order Melnikov function is given by

$$M_2(\alpha) := \int_{-\infty}^{+\infty} \dot{p}(t) f''(p(t)) v_\alpha^2(t) dt\,, \tag{4.1.10}$$

where $v_\alpha(t)$ is any fixed bounded solution of the equation

$$\ddot{x} = f'(p(t))x + q(t+\alpha).$$

This solution exists thanks to the fact that $M(\alpha) = 0$ (cf Theorem 4.1.2). Note that any two of these bounded solutions differ for a multiple of $\dot{p}(t)$, and hence $v_{\alpha+2\pi}(t) = v_\alpha(t) + \lambda \dot{p}(t)$, for some $\lambda \in \mathbb{R}$. On the other hand, $M_2(\alpha)$ does not depend on the particular solution $v_\alpha(t)$ we choose. This easily follows from that $\ddot{p}(t)$ is a bounded solution of the non homogeneous system

$$\ddot{x} = f'(p(t))x + f''(p(t))\dot{p}(t)^2$$

and $\dot{v}_\alpha(t)$ is a bounded solution of

$$\ddot{x} = f'(p(t))x + f''(p(t))\dot{p}(t)v_\alpha + \dot{q}(t+\alpha).$$

Hence:

$$\int_{-\infty}^{+\infty} \dot{p}(t) f''(p(t)) \dot{p}(t)^2 dt = 0$$

and

$$\int_{-\infty}^{+\infty} \dot{p}(t) f''(p(t)) \dot{p}(t) v_\alpha(t) dt = -\int_{-\infty}^{+\infty} \dot{p}(t) \dot{q}(t+\alpha) = M'(\alpha) = 0.$$

Note that $M_2(\alpha)$ is 2π-periodic since the bifurcation function itself is 2π-periodic.

4.1.5 Application to Periodically Perturbed ODEs

We illustrate our theory on the following example. Consider the equation

$$\begin{aligned}
\ddot{x} &= x - 2xz^2 + \dot{x}^2 + \mu_1 \cos \omega t - \mu_2 z, \\
\ddot{y} &= y - 2yz^2 + \dot{x}\dot{y}, \\
\ddot{z} &= z - 2z^3 + y\dot{y} + \mu_1 \cos \omega t + (\mu_2 - \mu_1)\dot{z}.
\end{aligned} \tag{4.1.11}$$

This equation is studied in Example 1 of [3]. In the space $(x, \dot{x}, y, \dot{y}, z, \dot{z})$, the eigenvalues of $Df(0)$ are $\lambda_1 = \lambda_2 = \lambda_3 = -1$, $\lambda_4 = \lambda_5 = \lambda_6 = 1$. A homoclinic solution when $\mu = 0$ is given by $x = 0, y = 0, z = r$, i.e. $\gamma = (0,0,0,0,r,\dot{r})$ where $r(t) = \operatorname{sech} t$. Note that $\ddot{r} = r - r^3$ and $\ddot{z} = z - z^3$ is the familiar Duffing equation (cf Chapter 1). The linearization of (4.1.11) at γ has the form

$$\ddot{x} = \left(1 - 2\gamma^2\right) x, \quad \ddot{y} = \left(1 - 2\gamma^2\right) y, \quad \ddot{z} = \left(1 - 6\gamma^2\right) z.$$

Clearly $d = 3$ and by Remark 4.1.8, it is readily to find

$$u_4 = (r, \dot{r}, 0, 0, 0, 0), \quad u_5 = (0, 0, r, \dot{r}, 0, 0), \quad u_6 = (0, 0, 0, 0, \dot{r}, \ddot{r})$$

$$u_1^\perp = (-\dot{r}, r, 0, 0, 0, 0), \quad u_2^\perp = (0, 0, -\dot{r}, r, 0, 0), \quad u_3^\perp = (0, 0, 0, 0, -\ddot{r}, \dot{r}).$$

Using these results, we easily get

$$M_\mu(\alpha, \beta_1, \beta_2) = \begin{cases} a_{11}(\alpha)\mu_1 + 2\mu_2 - \frac{\pi}{8}\beta_1^2, \\ -\frac{\pi}{8}\beta_1\beta_2, \\ a_{31}(\alpha)\mu_1 - \frac{2}{3}\mu_2 - \frac{\pi}{8}\beta_2^2, \end{cases}$$

where

$$a_{11}(\alpha) = -\pi \cos \omega\alpha \operatorname{sech} \frac{\pi\omega}{2}, \quad a_{31}(\alpha) = \frac{2}{3} - \pi\omega \sin \omega\alpha \operatorname{sech} \frac{\pi\omega}{2}.$$

There are the following solutions of $M_\mu(\alpha, \beta) = 0$ (see Remark 4.1.6 (a))

$$\beta(\alpha)=\left(\sqrt{\frac{8}{\pi}(a_{11}+3a_{31})},0\right),\quad \mu(\alpha)=\left(1,\frac{3}{2}a_{31}\right) \tag{4.1.12}$$

$$\beta(\alpha)=\left(0,\sqrt{\frac{8}{3\pi}(a_{11}+3a_{31})}\right),\quad \mu(\alpha)=\left(1,-\frac{1}{2}a_{11}\right). \tag{4.1.13}$$

The linearization $D_{(\alpha,\beta)}M_\mu(\alpha,\beta)$ at the points (4.1.12) reads

$$\begin{pmatrix} a'_{11} & -\frac{\pi}{4}\sqrt{\frac{8}{\pi}(a_{11}+3a_{31})} & 0 \\ 0 & 0 & -\frac{\pi}{8}\sqrt{\frac{8}{\pi}(a_{11}+3a_{31})} \\ a'_{31} & 0 & 0 \end{pmatrix},$$

and at the points (4.1.13) it has the form

$$\begin{pmatrix} a'_{11} & 0 & 0 \\ 0 & -\frac{\pi}{8}\sqrt{\frac{8}{3\pi}(a_{11}+3a_{31})} & 0 \\ a'_{31} & 0 & -\frac{\pi}{4}\sqrt{\frac{8}{3\pi}(a_{11}+3a_{31})} \end{pmatrix}.$$

Next, we have $a_{11}(\alpha)+3a_{31}(\alpha)\geq 2-\pi(3\omega+1)\operatorname{sech}\frac{\pi\omega}{2}>0$ for $\omega>\omega_0$, where $\omega_0\doteq 1.95332$ is the only positive root of $\pi(3\omega_0+1)\operatorname{sech}\frac{\pi\omega_0}{2}=2$. So for $\omega>\omega_0$ the points (4.1.12), involving (4.1.13), are simple zero points of $M_\mu(\alpha,\beta)$ when $\alpha\neq\frac{\pi(2k+1)}{2\omega}$, $\alpha\neq\frac{\pi k}{\omega}$, $k\in\mathbb{Z}$. Hence for $\omega>\omega_0$, there are two small open wedge-shaped regions in the μ_1–μ_2 plane with the limit slopes given by

$$1\pm\frac{3}{2}\pi\omega\operatorname{sech}\frac{\pi\omega}{2}\quad\text{and}\quad\pm\frac{\pi}{2}\operatorname{sech}\frac{\pi\omega}{2}$$

containing parameters for which the period map of (4.1.11) possesses a transversal homoclinic orbit near γ. Since $1\pm\frac{3}{2}\pi\omega\operatorname{sech}\frac{\pi\omega}{2}\sim 1\pm 3\pi\omega\,\mathrm{e}^{-\pi\omega/2}$ and $\pm\frac{\pi}{2}\operatorname{sech}\frac{\pi\omega}{2}\sim\pm\pi\,\mathrm{e}^{-\pi\omega/2}$ for large values of ω, i.e. for rapid forcing, these wedge-shaped regions become very narrow as $\omega\to\infty$. For instance, if $\omega=10$ then $\frac{3}{2}\pi\omega\operatorname{sech}\frac{\pi\omega}{2}\doteq 0.0000142033$ while $\frac{\pi}{2}\operatorname{sech}\frac{\pi\omega}{2}\doteq 4.73443\times 10^{-7}$. Finally note that $1+\frac{3}{2}\pi\,\omega_0\ \operatorname{sech}\frac{\pi\omega_0}{2}\doteq 1.85423$, $1-\frac{3}{2}\pi\omega_0\operatorname{sech}\frac{\pi\omega_0}{2}=\frac{\pi}{2}\operatorname{sech}\frac{\pi\omega_0}{2}\doteq 0.145773$ and functions $\frac{3}{2}\pi\omega\operatorname{sech}\frac{\pi\omega}{2}$, $\frac{\pi}{2}\operatorname{sech}\frac{\pi\omega}{2}$ are rapidly decreasing on $[\omega_0,\infty)$.

4.2 ODEs with Nonresonant Center Manifolds

4.2.1 Parameterized Coupled Oscillators

To illustrate the ideas of this section consider the equations

$$\ddot{x} = x - 2x(x^2+y^2) - 2\mu_2\dot{x} + \mu_1 \cos \omega t, \tag{4.2.1a}$$

$$\ddot{y} = (1-k)y - 2y(x^2+y^2) - 2\mu_2\dot{y} + \mu_1 \cos p\omega t, \tag{4.2.1b}$$

where $p \in \mathbb{N}$ and $\omega > 0$. This system consists of a radially symmetric Duffing oscillator with an additional spring of stiffness k in the y equation along with damping and external forces added as perturbation terms. Let us assume $k > 1$ in (4.2.1b). Then, for the unperturbed equation, i.e. when $\mu_1 = \mu_2 = 0$, the linear part of (4.2.1a) has a hyperbolic equilibrium and the linear part of (4.2.1b) has a center. Furthermore, for small μ_2, the eigenvalues of $\ddot{y} = (1-k)y - 2\mu_2\dot{y}$ are complex functions, $\lambda(\mu_2)$, with $\Re(\lambda(\mu_2)) = -\mu_2$ so that we have $\Re(\lambda(0)) = 0$ and $\Re(\lambda'(0)) = -1$. Thus, for small $\mu_2 \neq 0$, the equilibrium of (4.2.1b) is weakly hyperbolic.

If we set $y = 0$ in (4.2.1a) we get the standard forced, and damped Duffing equation

$$\ddot{x} = x - 2x^3 - 2\mu_2\dot{x} + \mu_1 \cos \omega t\,. \tag{4.2.2}$$

Using Melnikov theory of Section 4.1 one can show (see Example 4.2.6 below) that for small $\mu_1 \neq 0$ and for $\mu_2 \neq 0$, within a range

$$|\mu_2| < \frac{3\pi\omega}{4}|\mu_1|\operatorname{sech}\frac{\pi\omega}{2}, \tag{4.2.3}$$

Equation (4.2.2) has a transverse homoclinic orbit and hence exhibits chaos. The purpose of this section is to show that if $\mu_1 \neq 0$, $\mu_2 \neq 0$ are chosen to produce chaos in (4.2.1a) when $y = 0$ and if $p\omega \neq \sqrt{k-1}$ then, as a consequence of the weak hyperbolicity in the y equation, there exists chaos in the full Eq. (4.2.1) which, in some sense, shadows the chaos obtained in (4.2.1a) with $y = 0$. Condition $p\omega \neq \sqrt{k-1}$ means non-resonance in (4.2.1b). Resonant systems of ODEs are studied in Section 4.3.

As an abstract version of (4.2.1) we consider differential equations of the form

$$\dot{x} = f(x,y,\mu,t) = f_0(x,y) + \mu_1 f_1(x,y,\mu,t) + \mu_2 f_2(x,y,\mu,t), \tag{4.2.4a}$$

$$\dot{y} = g(x,y,\mu,t) = g_0(x,y) + \mu_1 g_1(x,y,\mu,t) + \mu_2 g_2(x,y,\mu), \tag{4.2.4b}$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\mu = (\mu_1,\mu_2) \in \mathbb{R}^2$. We make the following assumptions of (4.2.4):

(i) Each f_i, g_i is C^4-smooth in all arguments.
(ii) f_1, f_2 and g_1 are periodic in t with period T.
(iii) $D_2 f_0(x,0) = 0$.

(iv) The eigenvalues of $D_1 f_0(0,0)$ lie off the imaginary axis.
(v) The equation $\dot{x} = f_0(x,0)$ has a homoclinic solution γ.
(vi) $g_0(x,0) = g_2(x,0,\mu) = 0$, $D_{21}g_0(0,0) = 0$ and $D_{22}g_0(0,0) = 0$.
(vii) The eigenvalues of $D_2 g_0(0,0)$ lie on the imaginary axis.
(viii) If a function $\lambda(\mu_2)$ is an eigenvalue of the matrix $D_2 g_0(0,0) + \mu_2 D_2 g_2(0,0,0)$ then $\Re(\lambda'(0)) < 0$.
(ix) $D_2 g_1(0,0,0,t) = 0$.

Hypothesis (viii) is based on the examples for which the μ_2 perturbation represents damping which cases all the eigenvalues of (4.2.4b) to move to the left of the imaginary axis. In fact, it is sufficient to assume that $\Re(\lambda'(0)) \neq 0$. In other words, (4.2.4b) is weakly hyperbolic. This more general assumption requires a little more work since it is necessary to include a nontrivial projection in Lemma 4.2.4 below.

4.2.2 Chaotic Dynamics on the Hyperbolic Subspace

In this section we consider the equation

$$\dot{x} = f(x,0,\mu,t) = f_0(x,0) + \mu_1 f_1(x,0,\mu,t) + \mu_2 f_2(x,0,\mu,t) \tag{4.2.5}$$

obtained by setting $y = 0$ in (4.2.4a). Equation (4.2.5) will be called the *reduced equation* obtained from (4.2.4). We apply to this equation Melnikov theory from Section 4.1 which we summarize here for the readers' convenience. By hypothesis, the equation $\dot{x} = f_0(x,0)$ has a hyperbolic equilibrium and a homoclinic solution γ. Then (4.2.5) has a unique small hyperbolic T-periodic solution $p_\mu(t)$ for $|\mu|$ small (cf [11], Remark 4.1.7). Let $\{u_1,\ldots,u_d\}$ denote a basis for the vector space of bounded solutions to the variational equation $\dot{u} = D_1 f_0(\gamma,0)u$ with $u_d = \dot{\gamma}$ and let $\{v_1,\ldots,v_d\}$ denote a basis for the vector space of bounded solutions to the adjoint variational equation $\dot{v} = -D_1 f_0(\gamma,0)^t v$. Now define the functions $a_{ij} : \mathbb{R} \to \mathbb{R}$, constants b_{ijk} and function $M : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^{d-1} \to \mathbb{R}^d$ by

$$\begin{aligned}
a_{ij}(\alpha) &= \int_{-\infty}^{\infty} \langle v_i(t), f_j(\gamma(t),0,0,t+\alpha)\rangle\, dt, \quad \begin{cases} 1 \le i \le d, \\ 1 \le j \le 2; \end{cases} \\
b_{ijk} &= \int_{-\infty}^{\infty} \langle v_i, D_{11} f_0(\gamma,0) u_j u_k\rangle\, dt, \quad \begin{cases} 1 \le i \le d, \\ 1 \le j,k \le d-1; \end{cases} \\
M_i(\mu,\alpha,\beta) &= \sum_{j=1}^{2} a_{ij}(\alpha)\mu_j + \frac{1}{2}\sum_{j,k=1}^{d-1} b_{ijk}\beta_j\beta_k, \quad 1 \le i \le d.
\end{aligned} \tag{4.2.6}$$

The function M is our bifurcation function and is used in Theorem 4.2.1 below. The integer d has a geometric interpretation. Let $P = \gamma(0)$ and let W^s, W^u denote the stable, unstable manifolds respectively of the origin for the unperturbed equation from (4.2.5). Then the entire orbit of γ lies in $W^s \cap W^u$ so that $P \in W^s \cap W^u$ and $\dot{\gamma}(0) \in T_P W^s \cap T_P W^u$. The vectors $\{u_1(0), \dots, u_d(0)\}$ are a basis for $T_P W^s \cap T_P W^u$ and $d = \dim(T_P W^s \cap T_P W^u)$.

Suppose that $W^s \cap W^u$ has a connected component which is a manifold of dimension d and contains the orbit of γ. Then in (4.2.6), all $b_{ijk} = 0$, the hypotheses of Theorem 4.2.1 below cannot be satisfied and an alternate bifurcation function is required. Let W^h denote a homoclinic d-manifold containing γ, let U_0 be an open neighborhood of the origin in $\mathbb{R}^{d-1}$, let $\eta : U_0 \to W^h$ be a differentiable function-denoted $\beta \to \eta(\beta)$ with $\eta(0) = P$, let $t \to \gamma_\beta(t)$ be the solution to the unperturbed equation (4.2.5) satisfying $\gamma_\beta(0) = \eta(\beta)$, and assume that η is constructed so that $(\beta, t) \to \gamma_\beta(t)$ establishes local coordinates on W^h. In other words, the original orbit γ is embedded in a $(d-1)$-parameter family of homoclinic orbits. We suppose that $\left\{\dot{\gamma}_\beta(t), \frac{\partial \gamma_\beta}{\partial \beta_i}(t), i = 1, \dots, d-1\right\}$, $\beta = (\beta_1, \dots, \beta_{d-1})$, is a basis of bounded solutions of the variational equation $\dot{v} = D_1 f_0(\gamma_\beta, 0)v$. For each fixed β we let $\{v_{\beta 1}, \dots, v_{\beta d}\}$ denote a basis for the vector space of bounded solutions to the adjoint variational equation $\dot{v} = -D_1 f_0(\gamma_\beta, 0)^t v$. Without loss of generality we can assume that each $v_{\beta i}$ depends differentially on β. Now define functions $a_{ij} : \mathbb{R} \times U_0 \to \mathbb{R}$ and $M : \mathbb{R}^2 \times \mathbb{R} \times U_0 \to \mathbb{R}^d$ by

$$\begin{aligned} a_{ij}(\alpha, \beta) &= \int_{-\infty}^{\infty} \langle v_{\beta i}(t), f_j(\gamma_\beta(t), 0, 0, t+\alpha) \rangle \, dt, \quad \begin{cases} 1 \le i \le d, \\ 1 \le j \le 2, \end{cases} \\ M_i(\mu, \alpha, \beta) &= \sum_{j=1}^{2} a_{ij}(\alpha, \beta)\mu_j, \quad 1 \le i \le d. \end{aligned} \tag{4.2.7}$$

This is our bifurcation function for the homoclinic manifold case. By combining results from Section 4.1 we now get the following result.

Theorem 4.2.1. *M is the same as in* (4.2.6) *or* (4.2.7) *and suppose $(\mu_0, \alpha_0, \beta_0)$ are such that $M(\mu_0, \alpha_0, \beta_0) = 0$ and $D_{(\alpha,\beta)} M(\mu_0, \alpha_0, \beta_0)$ is nonsingular. Then there exists $\xi_0 > 0$ so that if $0 < \xi < \xi_0$ the equation $\dot{x} = f(x, 0, \xi\mu_0, t)$ has a homoclinic solution γ_ξ to $p_{\xi\mu_0}$. Furthermore, $\gamma_\xi(t) \to p_{\xi\mu_0}$ at an exponential rate as $t \to \pm\infty$, γ_ξ depends continuously on ξ, $\lim_{\xi \to 0} \gamma_\xi(t) = \gamma(t - \alpha_0)$ (or $= \gamma_{\beta_0}(t - \alpha_0)$), uniformly in t and the variational equation along γ_ξ has an exponential dichotomy for the whole line when $\xi \neq 0$.*

Following Sections 2.5.2 and 2.5.3, Theorem 4.2.1 establishes chaos for the differential equation $\dot{x} = f(x, 0, \xi\mu_0, t)$.

We remark that the constant K_ξ of the exponential dichotomy for the variational equation $\dot{u} = D_1 f(\gamma_\xi, 0, \xi\mu_0, t)u$ along $\gamma_\xi(t)$ tends to infinity as $\xi \to 0$. Indeed, let a_ξ, P_ξ, U_ξ be the corresponding constant, projection and fundamental solution from the definition of exponential dichotomy from Section 2.5.1, respectively. The roughness result for exponential dichotomies (cf Lemma 2.5.1) implies that we can take $a_\xi = a_0 > 0$ for some constant a_0. If $\sup_{\xi > 0} K_\xi < \infty$, then there is a sequence $\{\xi_i\}_{i=1}^{\infty}$ so that $\xi_i \to 0$, $K_{\xi_i} \to K_0$, $P_{\xi_i} \to P_0$ and $U_{\xi_i}(t) \to U_0(t)$ pointwise. Clearly, P_0 is a projection and $U_0(t)$ is the fundamental solution of $\dot{u} = D_1 f_0(\gamma,0)u$ creating an exponential dichotomy for this equation on the whole line $\mathbb{R}$ with constants (K_0, a_0). This contradicts the existence of a bounded solution $\dot{\gamma}$ for this equation. Consequently, $K_\xi \to \infty$ as $\xi \to 0$.

4.2.3 Chaos in the Full Equation

We construct the bifurcation function M from (4.2.6) or (4.2.7), as in the preceding section, from the reduced equation (4.2.5). If M satisfies the hypotheses for Theorem 4.2.1 we have a transverse homoclinic solution and hence chaos for (4.2.5) when $\mu = \xi\mu_0$, $0 < \xi < \xi_0$. We now establish a condition for chaos to exist in the full equation (4.2.4). Since the exponential constant K_ξ of $\dot{u} = D_1 f(\gamma_\xi, 0, \xi\mu_0, t)u$ tends to infinity as $\xi \to 0$, as we showed in previous section, we have to deal with the full system (4.2.4). For this we consider the modification of (4.2.4) in the form

$$\begin{gathered} \dot{x} = f(x, \lambda y, \mu, t)\,, \\ \dot{y} = g_0(x,y) + \lambda\mu_1 g_1(x,y,\mu,t) + \mu_2 g_2(x,y,\mu), \\ 0 \le \lambda \le 1\,. \end{gathered} \tag{4.2.8}$$

The changes $x = \gamma + \sum_{i=1}^{d-1} \xi\beta_i u_i + \xi^2 u$, $y = \xi^2 v$, $\mu = \xi^2\mu_0$ with $\mu_0 \neq 0$ into (4.2.8) yield

$$\begin{aligned} \dot{u} = {} & D_1 f_0(\gamma,0)u + \frac{1}{2}\sum_{i,j=1}^{d-1} D_{11} f_0(\gamma,0)\beta_i\beta_j u_i u_j \\ & + \mu_{0,1} f_1(\gamma,0,0,t+\alpha) + \mu_{0,2} f_2(\gamma,0,0,t+\alpha) + O(\xi), \end{aligned} \tag{4.2.9a}$$

$$\begin{aligned} \dot{v} = {} & \left[D_2 g_0(\gamma,0) + \xi^2\mu_{0,2} D_2 g_2(\gamma,0,0)\right] v \\ & + \left[D_2 g_0\left(\gamma + \xi\sum_{i=1}^{d-1}\beta_i u_i + \xi^2 u, 0\right) - D_2 g_0(\gamma,0)\right. \\ & \left. + D_{22} g_0\left(\gamma + \xi\sum_{i=1}^{d-1}\beta_i u_i + \xi^2 u, 0\right)\xi^2 v + O(\xi^4 v^2)\right] v + \lambda\mu_{0,1} g_1(0,0,0,t+\alpha) \\ & + \lambda\mu_{0,1}\left\{g_1\left(\gamma + \xi\sum_{i=1}^{d-1}\beta_i u_i + \xi^2 u, \xi^2 v, \xi^2\mu_0, t+\alpha\right) - g_1(0,0,0,t+\alpha)\right\} \\ & + \xi^2\mu_{0,2}\left\{D_2 g_2\left(\gamma + \xi\sum_{i=1}^{d-1}\beta_i u_i + \xi^2 u, 0, \xi^2\mu_0\right) - D_2 g_2(\gamma,0,0) + O(\xi^2 v)\right\} v\,. \end{aligned} \tag{4.2.9b}$$

We consider the Banach spaces

$$X_n = \left\{ x \in C(\mathbb{R}, \mathbb{R}^n) \Big| \sup_{t \in \mathbb{R}} |x| < \infty \right\},$$

$$Y_n = \left\{ y \in X_n \Big| \int_{-\infty}^{\infty} \langle y(t), v(t) \rangle \, dt \text{ for every solution } v \in X_n \text{ of } \dot{v} = -Df_0(\gamma, 0)^t v \right\}$$

with the supremum norm $||x|| = \sup_{t \in \mathbb{R}} |x(t)|$. To solve (4.2.9a), we recall Theorems 4.1.2 and 4.1.3.

Lemma 4.2.2. *Given $h \in Y_n$, the equation $\dot{u} = D_1 f_0(\gamma(t), 0)u + h$ has a unique solution $u \in X_n$ satisfying $\langle u(0), u_i(0) \rangle = 0$ for every $i = 1, 2, \ldots, d$.*

Lemma 4.2.3. *There exists a projection $\Pi : X_n \to X_n$ so that $\mathscr{R}(\mathbb{I} - \Pi) = Y_n$.*

We also need the following lemma.

Lemma 4.2.4. *There exist constants $b > 0$, $B > 0$ and $\widetilde{\xi}_0 > 0$ so that given $\mu_{0,2} > 0$, for any $0 < \xi \leq \widetilde{\xi}_0$ the variational equation*

$$\dot{v} = \left[D_2 g_0(\gamma(t), 0) + \xi^2 \mu_{0,2} D_2 g_2(\gamma(t), 0, 0) \right] v$$

has an exponential dichotomy on $\mathbb{R}$ with constants $(B, b\xi^2 \mu_{0,2})$.

Proof. Write the given equation in the form $\dot{v} = Rv + S(t)v$ where

$$R = D_2 g_0(0,0) + \xi^2 \mu_{0,2} D_2 g_2(0,0,0),$$
$$S(t) = D_2 g_0(\gamma(t), 0) - D_2 g_0(0,0) + \xi^2 \mu_{0,2} \left[D_2 g_2(\gamma(t), 0, 0) - D_2 g_2(0,0,0) \right].$$

Let V_ξ be the fundamental solution for $\dot{v} = Rv + S(t)v$ with $V_\xi(0) = \mathbb{I}$. Then for $s \leq t$ we have

$$V_\xi(t) = \mathrm{e}^{(t-s)R} V_\xi(s) + \int_s^t \mathrm{e}^{(t-\tau)R} S(\tau) V_\xi(\tau) \, d\tau.$$

Using (vii) and (viii) for (4.2.4) we can, for $\widetilde{\xi}_0$ sufficiently small, find $K_1, b > 0$ so that $|\, \mathrm{e}^{(t-s)R} \,| \leq K_1 \, \mathrm{e}^{b\xi^2 \mu_{0,2}(s-t)}$ when $0 < \xi \leq \widetilde{\xi}_0$ and $s \leq t$. Now define

$$x(t) = |V_\xi(t) V_\xi(s)^{-1}| \, \mathrm{e}^{b\xi^2 \mu_{0,2}(t-s)}.$$

Then from the preceding equation for V_ξ we get

$$x(t) \leq K_1 + \int_s^t K_1 |S(\tau)| x(\tau) \, d\tau.$$

Hence, from the Gronwall inequality (cf Section 2.5.1),

$$x(t) \leq K_1 \, \mathrm{e}^{K_1 \int_s^t |S(\tau)| \, d\tau} \leq B$$

for a constant $B > 0$. □

We define the linear map $\mathscr{K} : Y_n \to X_n$ by $\mathscr{K}h = u$ where h, u are as in Lemma 4.2.2. Using the projection Π and the exponential dichotomy V_ξ from Lemma 4.2.4, where we suppose $\mu_{0,2} > 0$ (the case $\mu_{0,2} < 0$ can be handled analogously), we can rewrite (4.2.9) as the fixed point problem

$$u = \mathscr{K}(\mathbb{I} - \Pi)\Bigg(\frac{1}{2}\sum_{i,j=1}^{d-1} D_{11}f_0(\gamma,0)\beta_i\beta_j u_i u_j + \mu_{0,1}f_1(\gamma,0,0,,t+\alpha) + \mu_{0,2}f_2(\gamma,0,0,t+\alpha) + O(\xi)\Bigg), \tag{4.2.10a}$$

$$\begin{aligned} v(t) = \int_{-\infty}^{t} V_\xi(t)V_\xi(s)^{-1}\Bigg\{\Bigg[D_2g_0\left(\gamma(s) + \xi\sum_{i=1}^{d-1}\beta_i u_i(s) + \xi^2 u(s), 0\right) \\ + D_{22}g_0\left(\gamma(s) + \xi\sum_{i=1}^{d-1}\beta_i u_i(s) + \xi^2 u(s), 0\right)\xi^2 v(s) \\ - D_2g_0(\gamma(s),0) + O(\xi^4 v(s)^2)\Bigg]v(s) + \lambda\mu_{0,1}g_1(0,0,0,s+\alpha) \\ + \lambda\mu_{0,1}\Bigg\{g_1\left(\gamma(s) + \xi\sum_{i=1}^{d-1}\beta_i u_i(s) + \xi^2 u(s), \xi^2 v(s), \xi^2\mu_0, s+\alpha\right) \\ - g_1(0,0,0,s+\alpha)\Bigg\} \\ + \xi^2\mu_{0,2}\Bigg\{D_2g_2\left(\gamma(s) + \xi\sum_{i=1}^{d-1}\beta_i u_i(s) + \xi^2 u(s), 0, \xi^2\mu_0\right) \\ - D_2g_2(\gamma(s),0,0) + O(\xi^2 v)\Bigg\}v(s)\Bigg\}ds \end{aligned} \tag{4.2.10b}$$

along with the system of bifurcation equations

$$\int_{-\infty}^{\infty}\Big\langle v_i(t), \frac{1}{2}\sum_{i,j=1}^{d-1} D_{11}f_0(\gamma(t),0)\beta_i\beta_j u_i(t)u_j(t) + \mu_{0,1}f_1(\gamma(t),0,0,t+\alpha) + \mu_{0,2}f_2(\gamma(t),0,0,t+\alpha) + O(\xi)\Big\rangle dt = 0, \quad i = 1,2,\ldots,d \tag{4.2.11}$$

where $\{v_1,\ldots,v_d\}$ is a basis for the space of bounded solutions to the adjoint equation. Using (ix) we have

$$\begin{aligned} D_2g_0\left(\gamma + \xi\sum_{i=1}^{d-1}\beta_i u_i + \xi^2 u, 0\right) - D_2g_0(\gamma,0) + D_{22}g_0\left(\gamma + \xi\sum_{i=1}^{d-1}\beta_i u_i + \xi^2 u, 0\right)\xi^2 v \\ = O\left(\xi^2|\gamma||v|\right) + O\left(\xi^4|u||v|\right) + O\left(\xi^2|\gamma||u|\right) + O\left(\xi\sum_{i=1}^{d-1}\beta_i|u_i|\right), \end{aligned}$$

$$
\begin{aligned}
& g_1\left(\gamma+\xi\sum_{i=1}^{d-1}\beta_i u_i+\xi^2 u,\xi^2 v,\xi^2\mu_0,t+\alpha\right)-g_1(0,0,0,t+\alpha)\\
&\quad = O\left(\xi^2|\gamma||v|\right)+O\left(\xi^4|u||v|\right)+O\left(\xi^2\right)+O\left(\xi^4|v|^2\right)\\
&\qquad + O\left(\xi^2|u|\right)+O(|\gamma|)+O\left(\xi\sum_{i=1}^{d-1}\beta_i|u_i|\right),\\
& D_2 g_2\left(\gamma+\xi\sum_{i=1}^{d-1}\beta_i u_i+\xi^2 u,0,\xi^2\mu_0\right)-D_2 g_2(\gamma,0,0)\\
&\quad = O\left(\xi^2\right)+O\left(\xi^2|u|\right)+O\left(\xi\sum_{i=1}^{d-1}\beta_i|u_i|\right).
\end{aligned}
$$

We note that $|\gamma(t)|\le c\mathrm{e}^{-a|t|}$ and $|u_i(t)|\le c\mathrm{e}^{-a|t|}$, $i=1,2,\ldots,d$ for constants $c>0$, $a>0$. Moreover, it holds that

$$
\begin{aligned}
\int_{-\infty}^{t}\mathrm{e}^{-b\xi^2\mu_{0,2}(t-s)}\,ds &= \frac{1}{b\xi^2\mu_{0,2}},\\
\int_{-\infty}^{t}\mathrm{e}^{-b\xi^2\mu_{0,2}(t-s)-a|s|}\,ds &\le \int_{-\infty}^{\infty}\mathrm{e}^{-a|s|}\,ds = 2/a\,.
\end{aligned}
$$

Consequently, if we assume that

$$
\begin{aligned}
&\sup_{0\le\alpha\le T}\sup_{\xi>0}\int_{-\infty}^{t}\left|V_\xi(t)V_\xi(s)^{-1}g_1(0,0,0,s+\alpha)\,ds\right|<\infty,\\
&\sup_{0\le\alpha\le T}\sup_{\xi>0}\int_{-\infty}^{t}\left|V_\xi(t)V_\xi(s)^{-1}D_4 g_1(0,0,0,s+\alpha)ds\right|<\infty
\end{aligned}
\tag{4.2.12}
$$

then we can apply the Banach fixed point theorem 2.2.1 on a ball centered at 0 in the space $X_n\times X_m$ to solving (4.2.10) for $\xi>0$ sufficiently small. Substituting this solution into (4.2.11) yields a system of bifurcation equations of the form

$$
M(\mu,\alpha,\beta)+O(\xi)=0, \tag{4.2.13}
$$

where M is as in (4.2.6) or (4.2.7). The case for (4.2.7) can be handled like above.

The assumptions of Theorem 4.2.1 imply the solvability of (4.2.13). This gives a transverse homoclinic orbit $\Gamma(\lambda,\xi^2\mu_0)(t)=\big(\Gamma_1(\lambda,\xi^2\mu_0)(t),\Gamma_2(\lambda,\xi^2\mu_0)(t)\big)$ of (4.2.8) near γ so that $\Gamma_1(\lambda,\xi^2\mu_0)(t)=\gamma(t)+O(\xi)$. The transversality follows exactly as in Section 4.1.3, so we omit its proof. Moreover, we have $\Gamma(0,\xi^2\mu_0)=(\gamma_\xi,0)$ for γ_ξ from Theorem 4.2.1, and $\Gamma(1,\xi^2\mu_0)$ is a homoclinic solution for (4.2.4). The dichotomy constants of the linearized system of (4.2.8) along $\Gamma(\lambda,\xi^2\mu_0)(t)$ are uniform for $0\le\lambda\le1$ and fixed ξ. This follows from the roughness result of exponential dichotomies from Lemma 2.5.1. Now we can follow directly a construction of a Smale horseshoe of Section 3.5.2 [7] along $\Gamma(\lambda,\xi^2\mu_0)(t)$ for fixed small

ξ. Thus we have a continuous family Σ_λ of Smale horseshoes for (4.2.8). This gives us the lifting of the Smale horseshoe of the reduced system to the full one.

The conditions (4.2.12) are, in fact, ones of nonresonance. To see this consider the equations

$$\dot{v} = [D_2 g_0(\gamma,0) + \xi^2 \mu_{0,2} D_2 g_2(\gamma,0,0)]v + h,$$
$$\dot{w} = [D_2 g_0(0,0) + \xi^2 \mu_{0,2} D_2 g_2(0,0,0)]w + h,$$

where $v,w,h \in X_m$. Then we get

$$\frac{d}{dt}(v-w) = \left[D_2 g_0(0,0) + \xi^2 \mu_{0,2} D_2 g_2(0,0,0)\right](v-w)$$
$$+\Big[D_2 g_0(\gamma,0) - D_2 g_0(0,0) + \xi^2 \mu_{0,2}(D_2 g_2(\gamma,0,0) - D_2 g_2(\gamma,0,0))\Big]v\,.$$

This gives

$$|v(t)-w(t)| \le ||v||K_1 \int_{-\infty}^{t} \mathrm{e}^{-b\xi^2\mu_{0,2}(t-s)-a|s|}\, ds \le 2||v||K_1/a$$

for constants $K_1 > 0$, $a > 0$. Hence there is a constant $K_2 > 0$ so that

$$||w-v|| \le K_2||v||, \quad ||w-v|| \le K_2||w||\,.$$

These inequalities imply that assumption (4.2.12) is equivalent to the condition that when $\xi > 0$ the only bounded solution, $v_{\alpha,\xi}$, of

$$\dot{v} = \left[D_2 g_0(0,0) + \xi^2 \mu_{0,2} D_2 g_2(0,0,0)\right]v + g_1(0,0,0,t+\alpha) \tag{4.2.14}$$

satisfies $\sup_{0\le\alpha\le T}\sup_{\xi>0}||v_{\alpha,\xi}|| < \infty$. Then also $\sup_{0\le\alpha\le T}\sup_{\xi>0}||\dot{v}_{\alpha,\xi}|| < \infty$. Hence by the Arzelà-Ascoli theorem 2.1.3, there is a sequence $\{\xi_i\}_{i=1}^{\infty}$, $\xi_i > 0$, $\xi_i \to 0$ so that $v_{\alpha,\xi_i} \to v_0$ and $\dot{v}_{\alpha,\xi_i} \to \dot{v}_0$ uniformly in compact intervals. Consequently, we get

$$\dot{v}_0 = D_2 g_0(0,0)v_0 + g_1(0,0,0,t+\alpha)\,. \tag{4.2.15}$$

We note that $v_{\alpha,\xi}$, v_0 are T-periodic. We know [11] that (4.2.15) has a T-periodic solution if and only if

$$\int_0^T \langle w_i(t), g_1(0,0,0,t)\rangle\, dt = 0, \quad i = 1,2,\dots,d_1, \tag{4.2.16}$$

where $\{w_1,\dots,w_{d_1}\}$ is a basis of T-periodic solutions of the adjoint variational equation $\dot{w} = -D_2 g_0(0,0)^t w$. Hence assumption (4.2.12) implies the validity of (4.2.16).

Conversely, let (4.2.16) hold. Then (4.2.15) has a T-periodic solution and we put $v = v_0 + w$ into (4.2.14) to get

$$\dot{w} = [D_2 g_0(0,0) + \xi^2 \mu_{0,2} D_2 g_2(0,0,0)]w + \xi^2 \mu_{0,2} D_2 g_2(0,0,0)v_0\,. \tag{4.2.17}$$

The above arguments and Lemma 4.2.4 give that the unique solution $w_{\alpha,\xi} \in X_m$ of (4.2.17) satisfies $\sup_{0\leq\alpha\leq T}\sup_{\xi>0}||w_{\alpha,\xi}|| < \infty$. In summary, we see that assumption (4.2.12) is equivalent to condition (4.2.16).

Now we can state our results in the form of the next theorem.

Theorem 4.2.5. *Let* (i)-(ix) *hold. Let* M *be the same as in* (4.2.6) *or* (4.2.7) *and suppose* $(\mu_0, \alpha_0, \beta_0)$ *are such that*

$$M(\mu_0, \alpha_0, \beta_0) = 0 \text{ and } D_{(\alpha,\beta)}M(\mu_0, \alpha_0, \beta_0) \text{ is nonsingular.}$$

If condition (4.2.16) *holds then there exist* $\bar{\xi}_0 > 0$, $K > 0$ *so that if* $0 < \xi \leq \bar{\xi}_0$ *and if the parameters in* (4.2.4) *are given by* $\mu = \xi\mu_0$, *then there exists a continuous map* $\phi : \mathscr{E} \times [0,1] \to \mathbb{R}^{n+m}$ *(cf Section 2.5.2) and* $m_0 \in \mathbb{N}$ *so that:*

(i) $\phi_\lambda = \phi(\cdot,\lambda) : \mathscr{E} \to \mathbb{R}^{n+m}$ *is a homeomorphism of* $\mathscr{E}$ *onto a compact subset of* $\mathbb{R}^{n+m}$ *on which the* m_0*th iterate* $F_\lambda^{m_0}$ *of the period map* F_λ *of* (4.2.8) *is invariant and satisfies* $F_\lambda^{2m_0} \circ \phi_\lambda = \phi_\lambda \circ \sigma$ *where* σ *is the Bernoulli shift on* $\mathscr{E}$.
(ii) $\phi_0 = \phi(\cdot,0) : \mathscr{E} \to \mathbb{R}^n \times \{0\}$ *and* $F_0 = (G_0, 0)$ *for the period map* G_0 *of the reduced equation* (4.2.5).
(iii) F_1 *is the period map of the full system* (4.2.4).
(iv) $|\phi(x,\lambda) - \phi(x,0)| \leq K\sqrt{\xi}$ *for any* $(x,\lambda) \in \mathscr{E} \times [0,1]$.

Theorem 4.2.5 roughly states that the Smale horseshoe of the reduced equation (4.2.5) can be shadowed and continued to the full system (4.2.4).

4.2.4 Applications to Nonlinear ODEs

We now illustrate the above theory with two examples. For convenience in our calculations let us denote $r(t) = \operatorname{sech} t$. Note that $\ddot{r} = r - 2r^3$ and $\dddot{r} = (1 - 6r^2)\dot{r}$.

Example 4.2.6. As our first example consider the equations (4.2.1) from the introduction. The reduced equation is

$$\ddot{x} = x - 2x^3 - 2\mu_2\dot{x} + \mu_1 \cos\omega t$$

which we consider as a first order system in the phase space $(x, \dot{x})$. Since this system is in $\mathbb{R}^2$ we necessarily have $d = 1$. A bounded solution to the adjoint equation is $v = (-\ddot{r}, \dot{r})$ and from this we compute

$$a_{11}(\alpha) = \int_{-\infty}^{\infty} \dot{r} \cos\omega(t+\alpha)\,dt = \pi\omega \operatorname{sech}\frac{\pi\omega}{2} \sin\omega\alpha,$$
$$a_{12} = \int_{-\infty}^{\infty} -2\dot{r}^2\,dt = -\frac{4}{3}.$$

The bifurcation equation obtained from (4.2.6) is

$$M(\alpha,\mu) = \left(\pi\omega \operatorname{sech}\frac{\pi\omega}{2}\sin\omega\alpha\right)\mu_1 - \frac{4}{3}\mu_2 = 0.$$

We can satisfy this equation by choosing $\alpha_0 \in \left[-\frac{\pi}{2\omega}, \frac{\pi}{2\omega}\right]$ and then taking $\mu_{0,1} \neq 0$ and

$$\frac{\mu_{0,2}}{\mu_{0,1}} = \frac{3\pi\omega}{4}\operatorname{sech}\frac{\pi\omega}{2}\sin\omega\alpha_0.$$

Since in (4.2.6), $d = 1$, the transversality condition is

$$D_\alpha M(\alpha_0,\mu_0) = \pi\omega^2\mu_{0,1}\operatorname{sech}\frac{\pi\omega}{2}\cos\omega\alpha_0 \neq 0$$

which is satisfied for $\alpha_0 \in \left(-\frac{\pi}{2\omega}, \frac{\pi}{2\omega}\right)$. Let $m_0 := (3\pi\omega/4)\operatorname{sech}\pi\omega/2$. By varying α_0 we see that the reduced equation exhibits chaos for all sufficiently small $|\mu_0|$ satisfying $-m_0 < \mu_{0,2}/\mu_{0,1} < m_0$. Theorem 4.2.5 gives another result.

Theorem 4.2.7. *If* $p\omega \neq \sqrt{k-1}$ *then the full equation* (4.2.1) *exhibits chaos for all sufficiently small* $\mu_1 \neq 0, \mu_2$ *satisfying* (4.2.3).

Example 4.2.8. As a generalization of the preceding example consider the equations

$$\begin{aligned}\ddot{x} &= x - 2x(x^2+y^2+z^2) - \mu_2(\dot{x}+\dot{y}) + \mu_1\cos\omega t,\\ \ddot{y} &= y - 2y(x^2+y^2+z^2) - \mu_2(\dot{x}+\dot{y}),\\ \ddot{z} &= (1-k)z - 2z(x^2+y^2+z^2) - \mu_2\dot{z} + \mu_1\cos p\omega t\end{aligned} \tag{4.2.18}$$

where, as before, we assume that $k > 1$ and $p \in \mathbb{N}$. We consider these equations as a first order system in the phase space $(x,\dot{x},y,\dot{y},z,\dot{z})$. The reduced equations of (4.2.18) are

$$\begin{aligned}\ddot{x} &= x - 2x(x^2+y^2) - \mu_2(\dot{x}+\dot{y}) + \mu_1\cos\omega t,\\ \ddot{y} &= y - 2y(x^2+y^2) - \mu_2(\dot{x}+\dot{y}).\end{aligned} \tag{4.2.19}$$

The unperturbed motion of (4.2.19) has a homoclinic 2-manifold with a family of homoclinic orbits given by $x = r(t)\cos\beta$, $y = r(t)\sin\beta$ (cf [9, p. 133]). Writing out the adjoint equation in $\mathbb{R}^4$ we obtain as a basis for the space of bounded solutions

$$v_{\beta 1} = (-\ddot{r}\cos\beta, \dot{r}\cos\beta, -\ddot{r}\sin\beta, \dot{r}\sin\beta),$$

$$v_{\beta 2} = (-\dot{r}\sin\beta, r\sin\beta, \dot{r}\cos\beta, -r\cos\beta).$$

Next we compute

$$\begin{aligned}a_{11}(\alpha,\beta) &= \int_{-\infty}^{\infty}\dot{r}\cos\beta\cos\omega(t+\alpha)\,dt = \pi\omega\operatorname{sech}\frac{\pi\omega}{2}\sin\omega\alpha\cos\beta,\\ a_{12}(\alpha,\beta) &= \int_{-\infty}^{\infty} -\dot{r}\cos\beta(\dot{r}\cos\beta + \dot{r}\sin\beta) - \dot{r}\sin\beta(\dot{r}\cos\beta + \dot{r}\sin\beta)\,dt\\ &= -\frac{2}{3}(\cos\beta+\sin\beta)^2,\end{aligned}$$

$$a_{21}(\alpha,\beta) = \int_{-\infty}^{\infty} r\sin\beta\cos\omega(t+\alpha)\,dt = \pi\,\mathrm{sech}\,\frac{\pi\omega}{2}\cos\omega\alpha\sin\beta,$$

$$a_{22}(\alpha,\beta) = \int_{-\infty}^{\infty} -r\sin\beta(\dot r\cos\beta+\dot r\sin\beta)+r\cos\beta(\dot r\cos\beta+\dot r\sin\beta)\,dt = 0.$$

In (4.2.7), $d=2$, β is a scalar and the bifurcation equation $M(\alpha,\beta,\mu)=0$ takes the form

$$a_{11}(\alpha,\beta)\mu_1+a_{12}(\alpha,\beta)\mu_2=0,\quad a_{21}(\alpha,\beta)\mu_1=0.$$

A sufficient condition for a nontrivial solution is $a_{21}=0$ which is satisfied by $\omega\alpha_0^{\pm}=\pm\pi/2$. We then have

$$\frac{\mu_2}{\mu_1} = -\frac{a_{11}(\alpha_0^{\pm},\beta_0)}{a_{12}(\alpha_0^{\pm},\beta_0)} = \pm\frac{3\pi\omega\,\mathrm{sech}\,\frac{\pi\omega}{2}\cos\beta_0}{2(\cos\beta_0+\sin\beta_0)^2}.$$

We see from Figure 4.1 that the range is $\mathbb{R}$ of the function $H(\beta) := \frac{\cos\beta}{(\cos\beta+\sin\beta)^2}$ as $\beta\in[0,2\pi]\setminus\left\{\frac{3\pi}{4},\frac{7\pi}{4}\right\}$.

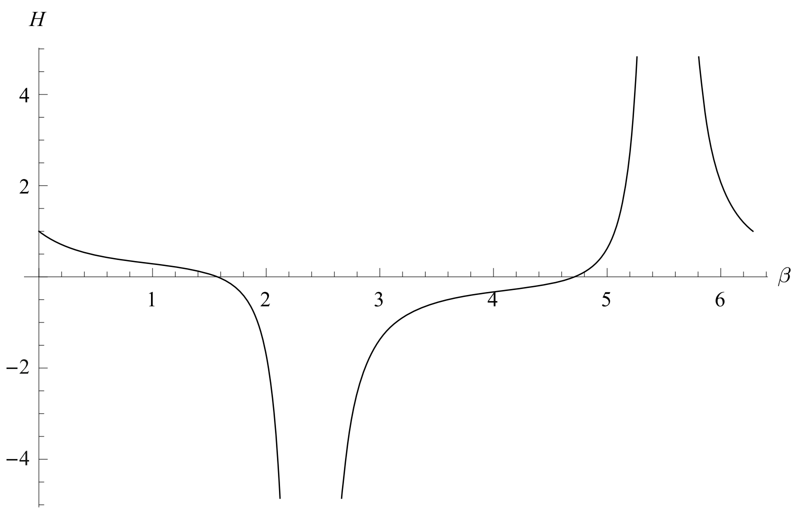

Fig. 4.1 The graph of the function $H(\beta)$ over $[0,2\pi]$.

It remains checking the transversality condition which takes the forms

$$\det D_{(\alpha,\beta)}M(\alpha_0^{+},\beta_0,\mu) = -\frac{\mu_1^2\pi^2\omega^2(\sin\beta_0+2\cos^3\beta_0)\sin\beta_0\,\mathrm{sech}^2\,\frac{\pi\omega}{2}}{(\cos\beta_0+\sin\beta_0)^2}\neq 0, \tag{4.2.20}$$

and

$$\det D_{(\alpha,\beta)} M(\alpha_0^-,\beta_0,\mu) = \frac{\mu_1^2\pi^2\omega^2 \operatorname{sech}^2 \frac{\pi\omega}{2}\,(2\cos 2\beta_0 - 2 - 2\sin 2\beta_0 + 3\sin 4\beta_0)}{4(\cos\beta_0+\sin\beta_0)^2} \neq 0\,, \tag{4.2.21}$$

and (4.2.20) is satisfied for $\beta_0 \in [0,2\pi]\setminus\left\{0,\frac{3\pi}{4},\pi,\frac{7\pi}{4},2\pi\right\}$, while (4.2.21) holds for

$$\beta_0 \in [0,2\pi]\setminus\left\{0,\frac{1}{2}\arccos\left(\frac{\sqrt{17}-1}{6}\right),-\frac{1}{2}\arccos\left(\frac{-\sqrt{17}-1}{6}\right)+\pi,\right.$$
$$\left.\frac{1}{2}\arccos\left(\frac{\sqrt{17}-1}{6}\right)+\pi,-\frac{1}{2}\arccos\left(\frac{-\sqrt{17}-1}{6}\right)+2\pi,\frac{3\pi}{4},\pi,\frac{7\pi}{4},2\pi\right\}.$$

Thus, the reduced equation exhibits chaos for all sufficiently small μ in the μ_1-μ_2 plane except along three lines of slopes $m = \pm m_0$, ∞, where $m_0 = \frac{3\pi\omega}{2}\operatorname{sech}\frac{\pi\omega}{2}$. From Theorem 4.2.5, if $p\omega \neq \sqrt{k-1}$ then the full equation exhibits chaos for all sufficiently small μ lying except along three lines of slopes $m = \pm m_0$, ∞. We obtain these transversal homoclinic orbits from (α_0^+,β_0). Moreover, we see from Figure 4.1 that the equation $H(\beta_0) = y$ has two solutions in $[0,2\pi)$ for any $y\in\mathbb{R}$. So we get two different transversal homoclinic orbits. Furthermore excluding also the next four lines of the slopes $\pm m_\pm$ with $m_\pm = \frac{3\pi\omega\sqrt{69\pm 3\sqrt{17}}}{32}\operatorname{sech}\frac{\pi\omega}{2}$ we can involve also the point (α_0^-,β_0), and consequently we get four different transversal homoclinic orbits. Note that $H(\beta+\pi) = -H(\beta)$, $H(0) = 1$ and $H\left(\mp\frac{1}{2}\arccos\left(\frac{\pm\sqrt{17}-1}{6}\right)\right) = \frac{\sqrt{69\pm 3\sqrt{17}}}{16}$.

4.3 ODEs with Resonant Center Manifolds

4.3.1 ODEs with Saddle-Center Parts

We consider differential equations of the form

$$\dot{x} = f(x,y,\mu,t) = f_0(x,y) + \mu_1 f_1(x,y,\mu,t) + \mu_2 f_2(x,y,\mu,t), \tag{4.3.1a}$$

$$\dot{y} = g(x,y,\mu,t) = g_0(x,y) + \mu_1 g_1(x,y,\mu,t) + \mu_2 g_2(x,y,\mu) \tag{4.3.1b}$$

with $x\in\mathbb{R}^n$, $y\in\mathbb{R}^m$, $\mu = (\mu_1,\mu_2)\in\mathbb{R}^2$. We make the following assumptions of (4.3.1):

(i) Each f_i, g_i are C^4-smooth in all arguments.
(ii) f_1, f_2 and g_1 are periodic in t with period T.
(iii) $D_2 f_0(x,0) = 0$.

(iv) The eigenvalues of $D_1 f_0(0,0)$ lie off the imaginary axis.
(v) The equation $\dot{x} = f_0(x,0)$ has a homoclinic solution γ.
(vi) $g_0(x,0) = g_2(x,0,\mu) = 0$, $D_{21}g_0(0,0) = 0$ and $D_{22}g_0(0,0) = 0$.
(vii) The eigenvalues of $D_2 g_0(0,0)$ lie on the imaginary axis.
(viii) If $\lambda(\mu_2)$ is an eigenvalue function of $D_2 g_0(0,0) + \mu_2 D_2 g_2(0,0,0)$ then $\Re(\lambda'(0)) < 0$.

In the hypothesis (viii), it is sufficient to assume that $\Re(\lambda'(0)) \neq 0$. In other words, (4.3.1b) is weakly hyperbolic with respect to μ_2. This more general assumption requires a little more work since it is necessary to include a nontrivial projection in Lemma 4.3.4 below. Consider the *reduced equation*

$$\dot{x} = f_0(x,0) + \mu_1 f_1(x,0,\mu,t) + \mu_2 f_2(x,0,\mu,t) \tag{4.3.2}$$

obtained by setting $y = 0$ in (4.3.1a). By hypothesis, the equation $\dot{x} = f_0(x,0)$ has a hyperbolic equilibrium and a homoclinic solution γ. Melnikov theory is used in Section 4.1 to obtain a transverse homoclinic solution in the reduced equation. The problem which naturally arises is showing that a transverse homoclinic solution for the reduced equation is shadowed by a transverse homoclinic solution for the full equation (4.3.1). This is done in Section 4.2 when the *center equation*

$$\dot{y} = g_0(0,y) + \mu_1 g_1(0,y,\mu,t) + \mu_2 g_2(0,y,\mu) \tag{4.3.3}$$

is not resonant at $y = 0$. The purpose of this section is to treat the resonant case and to detect a transverse homoclinic solution for the full system from a Melnikov function derived from the reduced and center equations. But the situation in this section is much more delicate than in Section 4.2.

Finally we note that a related problem is studied also in [15], where a three-dimensional ODE is considered with slowly varying one-dimensional variable. The approach in [15] is more geometrical than ours in this section.

4.3.2 Example of Coupled Oscillators at Resonance

We start with the equations

$$\begin{aligned} \ddot{x} &= x - 2x(x^2 + \xi y^2) - 2\mu_2 \delta \dot{x} + \mu_4 \cos(t+\alpha) + \mu_5 \sin(t+\alpha), \\ \ddot{y} &= -y - 2y(x^2 + y^2) - 2\mu_2 \dot{y} + \mu_1 \cos(t+\alpha) + \mu_3 \sin(t+\alpha)\,. \end{aligned} \tag{4.3.4}$$

Here δ, ξ are positive constants and μ_i, $i = 1,\dots,5$ are small parameters. We put $\gamma(t) = \operatorname{sech} t$, $x = \gamma + \varepsilon^2 u$, $y = \varepsilon v$, $\mu_1 = \varepsilon^3 a_1$, $\mu_2 = \varepsilon^2$, $\mu_3 = \varepsilon^3 a_2$, $\mu_4 = \varepsilon^2 a_3$ and $\mu_5 = \varepsilon^2 a_4$, with $a_1, a_2, a_3, a_4 \in \mathbb{R}$ into (4.3.4) to get

$$\begin{aligned}\ddot{u} &= (1-6\gamma^2)u - 2\delta\dot{\gamma} - 2\xi\gamma v^2 + a_3\cos(t+\alpha) + a_4\sin(t+\alpha) + O(\varepsilon^2),\\ \ddot{v} &= -(1+2\gamma^2)v - 2\varepsilon^2\dot{v} - 4\varepsilon^2\gamma uv \\ &\quad -2\varepsilon^4u^2v - 2\varepsilon^2v^3 + \varepsilon^2a_1\cos(t+\alpha) + \varepsilon^2a_2\sin(t+\alpha)\,.\end{aligned} \tag{4.3.5}$$

First, we look for a 2π-periodic solution of the equation

$$\ddot{v}_{\varepsilon,\alpha,a} = -v_{\varepsilon,\alpha,a} - 2\varepsilon^2\dot{v}_{\varepsilon,\alpha,a} - 2\varepsilon^2v^3_{\varepsilon,\alpha,a} + \varepsilon^2a_1\cos(t+\alpha) + \varepsilon^2a_2\sin(t+\alpha)\,. \tag{4.3.6}$$

Clearly $v_{\varepsilon,\alpha,a}(t) = w_{\varepsilon,a}(t+\alpha)$ where $w_{\varepsilon,a}$ is a 2π-periodic solution of

$$\ddot{w}_{\varepsilon,a} = -w_{\varepsilon,a} - 2\varepsilon^2\dot{w}_{\varepsilon,a} - 2\varepsilon^2w^3_{\varepsilon,a} + \varepsilon^2a_1\cos t + \varepsilon^2a_2\sin t\,. \tag{4.3.7}$$

Consider the operator $L: C^2_{2\pi}(\mathbb{R}) \to C_{2\pi}(\mathbb{R})$ defined as $Lw = \ddot{w} + w$. Here $C^r_{2\pi}(\mathbb{R})$, $r \in \mathbb{Z}_+$, is the Banach space of C^r-smooth and 2π-periodic functions endowed with the maximum norm. We have

$$\begin{aligned}\mathscr{N}L &= \operatorname{span}\{\cos t, \sin t\},\\ \mathscr{R}L &= \left\{h \in C_{2\pi}(\mathbb{R}) \mid \int_0^{2\pi} h(t)\cos t\,dt = 0,\ \int_0^{2\pi} h(t)\sin t\,dt = 0\right\}.\end{aligned}$$

Let $Q: C_{2\pi}(\mathbb{R}) \to \mathscr{R}L$ be the continuous projection

$$(Qw)(t) = w - \frac{1}{\pi}\cos t\int_0^{2\pi} w(t)\cos t\,dt - \frac{1}{\pi}\sin t\int_0^{2\pi} w(t)\sin t\,dt.$$

Equation (4.3.7) can now be split into a new differential equation

$$\ddot{w} + w = Q(-2\varepsilon^2\dot{w} - 2\varepsilon^2w^3 + \varepsilon^2a_1\cos t + \varepsilon^2a_2\sin t) = Q(-2\varepsilon^2\dot{w} - 2\varepsilon^2w^3)$$

and a bifurcation equation

$$\begin{aligned}(\mathbb{I}-Q)&(-2\varepsilon^2\dot{w} - 2\varepsilon^2w^3 + \varepsilon^2a_1\cos t + \varepsilon^2a_2\sin t)\\ &= \varepsilon^2\left[a_1 - \frac{1}{\pi}\int_0^{2\pi}(2\dot{w} + 2w^3)\cos t\,dt\right]\cos t\\ &\quad + \varepsilon^2\left[a_2 - \frac{1}{\pi}\int_0^{2\pi}(2\dot{w} + 2w^3)\sin t\,dt\right]\sin t = 0.\end{aligned}$$

The differential equation has a solution $w \in C^2_{2\pi}(\mathbb{R})$ of the form

$$w(t) = \varphi(\varepsilon, c_1, c_2)(t) + c_1\cos t + c_2\sin t$$

where c_1, c_2 are arbitrary and $\varphi = O(\varepsilon^2)$. Substituting this into the bifurcation equation gives

$$a_2 - \frac{1}{\pi}\int_0^{2\pi} \left[2(-c_1 \sin t + c_2 \cos t) + 2(c_1 \cos t + c_2 \sin t)^3\right] \sin t\, dt + O(\varepsilon^2) = 0,$$
$$a_1 - \frac{1}{\pi}\int_0^{2\pi} \left[2(-c_1 \sin t + c_2 \cos t) + 2(c_1 \cos t + c_2 \sin t)^3\right] \cos t\, dt + O(\varepsilon^2) = 0$$

or

$$\begin{aligned} 4c_1 - 3c_2^3 - 3c_1^2 c_2 &= -2a_2 + O(\varepsilon^2), \\ 4c_2 + 3c_1^3 + 3c_2^2 c_1 &= 2a_1 + O(\varepsilon^2). \end{aligned} \tag{4.3.8}$$

The determinant of the Jacobian of the left hand side of (4.3.8) is

$$16 + 27(c_1^2 + c_2^2)^2 \neq 0.$$

Now we have

$$|4c_1 - 3c_1^2 c_2 - 3c_2^3| + |4c_2 + 3c_1^3 + 3c_1 c_2^2| \geq (3(c_1^2 + c_2^2) - 4)(|c_1| + |c_2|)\,.$$

Hence the map

$$(c_1, c_2) \to (4c_1 - 3c_2^3 - 3c_1^2 c_2, 4c_2 + 3c_1^3 + 3c_2^2 c_1)$$

from $\mathbb{R}^2$ to $\mathbb{R}^2$ is proper and locally invertible and thus a diffeomorphism by the Banach-Mazur Theorem 2.2.6. Hence we can use the implicit function theorem to get solutions $c_1(a,\varepsilon)$ and $c_2(a,\varepsilon)$ to (4.3.8) for ε small and $a = (a_1, a_2) \in \mathbb{R}^2$ from bounded subsets. In summary, we have the following result:

Lemma 4.3.1. *For any $n \in \mathbb{N}$, there exist $\varepsilon_0 = \varepsilon_0(n) > 0$ and a differentiable function $c : (-n,n)^2 \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^2$ denoted $(a,\varepsilon) \to c(a,\varepsilon)$ so that* (4.3.6) *has a 2π-periodic solution of the form*:

$$v_{\varepsilon,\alpha,a}(t) = c_1(a,\varepsilon)\cos(t+\alpha) + c_2(a,\varepsilon)\sin(t+\alpha) + O(\varepsilon^2)\,. \tag{4.3.9}$$

We note that the function $c(a,\varepsilon)$ may also depend on n, but when $m > n$ these two functions $c(a,\varepsilon)$ from Lemma 4.3.1 coincide on the set $(-n,n)^2 \times (-\bar{\varepsilon}_0, \bar{\varepsilon}_0)$ with $\bar{\varepsilon}_0 = \min\{\varepsilon_0(n), \varepsilon_0(m)\}$.

We now substitute $v = w + v_{\varepsilon,\alpha,a}$ into (4.3.5) to get

$$\begin{aligned} \ddot{u} = {} & (1 - 6\gamma^2)u - 2\delta\dot{\gamma} - 2\xi\gamma\big(w + v_{\varepsilon,\alpha,a}(t)\big)^2 \\ & + a_3\cos(t+\alpha) + a_4\sin(t+\alpha) + O(\varepsilon), \end{aligned} \tag{4.3.10a}$$
$$\begin{aligned} \ddot{w} = {} & -(1 + 6\varepsilon^2 v_{\varepsilon,\alpha,a}^2)w - 2\varepsilon^2\dot{w} - 2\gamma^2 w \\ & - 2\gamma^2 v_{\varepsilon,\alpha,a} - 4\varepsilon^2\gamma u(w + v_{\varepsilon,\alpha,a}) - 2\varepsilon^4 u^2(w + v_{\varepsilon,\alpha,a}) \\ & - 6\varepsilon^2 w^2 v_{\varepsilon,\alpha,a} - 2\varepsilon^2 w^3. \end{aligned} \tag{4.3.10b}$$

To study (4.3.10) we must establish the existence of properties for an exponential dichotomy for the linear part of (4.3.10b) in three steps.

We first study the equation

$$\ddot{w} = -[1+\varepsilon^2\phi_\varepsilon(t)^2]w - 2\varepsilon^2\dot{w}, \tag{4.3.11}$$

where $\phi_\varepsilon(t) = \sqrt{6}v_{\varepsilon,\alpha,a}$.

Step 1. We put $w = \mathrm{e}^{-\varepsilon^2 t} z_1$ to get

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= -[1+\varepsilon^2\phi_\varepsilon(t)^2 - \varepsilon^4]z_1. \end{aligned} \tag{4.3.12}$$

By Floquet theory [12,13] (4.3.12) has a solution, Z_ε, of the form $Z_\varepsilon = U_\varepsilon(t)\,\mathrm{e}^{tB_\varepsilon}$ where $U_\varepsilon(0) = \mathbb{I}$, $U_\varepsilon(t+2\pi) = U_\varepsilon(t)$ and

$$U_0(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

so that $\|U_0(t)\| = 1$. Stability is determined by the matrix B_ε and $Z_\varepsilon(2\pi) = \mathrm{e}^{2\pi B_\varepsilon}$, so we are interested in $Z_\varepsilon(2\pi)$. We have $Z_\varepsilon(t+2\pi) = Z_\varepsilon(t)Z_\varepsilon(2\pi)$ and from Liouville's formula (cf Section 2.5.1 and [12]) $\det Z_\varepsilon(2\pi) = 1$. Hence the eigenvalues of $Z_\varepsilon(2\pi)$ are a complex conjugate pair with norm 1 if and only if $|\operatorname{tr} Z_\varepsilon(2\pi)| < 2$. To compute an estimate for Z_ε we expand

$$\begin{aligned} z_1 &= u_0 + \varepsilon^2 u_1 + O(\varepsilon^4), \\ z_2 &= v_0 + \varepsilon^2 v_1 + O(\varepsilon^4), \\ \phi_\varepsilon &= \phi_0 + O(\varepsilon^2). \end{aligned}$$

Substituting these expansions into (4.3.12) we get

$$\dot{u}_0 = v_0, \quad \dot{v}_0 = -u_0, \qquad \dot{u}_1 = v_1, \quad \dot{v}_1 = -u_1 - \phi_0^2 u_0$$

and $Z_\varepsilon(0) = \mathbb{I}$ requires $u_1(0) = v_1(0) = 0$. By choosing either $u_0 = \cos t$, $v_0 = -\sin t$ or $u_0 = \sin t$, $v_0 = \cos t$, we find u_1, v_1 and then a computation shows that

$$Z_\varepsilon(2\pi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} \frac{1}{2}\int_0^{2\pi} \phi_0^2(s)\sin 2s\,ds & \int_0^{2\pi} \phi_0^2(s)\sin^2 s\,ds \\ -\int_0^{2\pi} \phi_0^2(s)\cos^2 s\,ds & -\frac{1}{2}\int_0^{2\pi} \phi_0^2(s)\sin 2s\,ds \end{pmatrix} + O(\varepsilon^4).$$

We have $\phi_0(t) = \sqrt{6}v_{0,\alpha,a}(t) = \sqrt{6}(c_1(a,0)\cos(t+\alpha) + c_2(a,0)\sin(t+\alpha))$. Thus, as long as $a \neq 0$ it follows from (4.3.8) that $c_1(a,0)^2 + c_2(a,0)^2 \neq 0$ and we can write

$$v_{0,a,\alpha} = c_5(a)\sin(t+\alpha+c_4(a))$$

where $c_5(a) = \sqrt{c_1(a,0)^2 + c_2(a,0)^2}$ and $c_4(a)$ is defined by the equality. Then $\phi_0(t) = c_3(a)\sin(t+\alpha+c_4(a))$ where $c_3(a) = \sqrt{6}c_5(a)$ and

$$
\begin{aligned}
\int_0^{2\pi} \phi_0^2(s)\sin 2s\,ds &= c_3(a)^2 \int_0^{2\pi} \sin 2s \sin^2(s+\alpha+c_4(a))\,ds \\
&= \frac{\pi}{2} c_3(a)^2 \sin 2(\alpha+c_4(a)), \\
\int_0^{2\pi} \phi_0^2(s)\sin^2 s\,ds &= c_3(a)^2 \int_0^{2\pi} \sin^2 s \sin^2(s+\alpha+c_4(a))\,ds \\
&= c_3(a)^2 \Big(\frac{\pi}{2} + \frac{\pi}{4}\cos 2(\alpha+c_4(a))\Big), \\
\int_0^{2\pi} \phi_0^2(s)\cos^2 s\,ds &= c_3(a)^2 \int_0^{2\pi} \cos^2 s \sin^2(s+\alpha+c_4(a))\,ds \\
&= c_3(a)^2 \Big(\frac{\pi}{2} - \frac{\pi}{4}\cos 2(\alpha+c_4(a))\Big).
\end{aligned}
$$

Hence

$$
\begin{aligned}
Z_\varepsilon(2\pi) &= \mathbb{I} + \varepsilon^2 c_3(a)^2 \frac{\pi}{4} \begin{pmatrix} \sin 2(\alpha+c_4(a)) & 2+\cos 2(\alpha+c_4(a)) \\ -2+\cos 2(\alpha+c_4(a)) & -\sin 2(\alpha+c_4(a)) \end{pmatrix} + O(\varepsilon^4) \\
&= \mathbb{I} + \varepsilon^2 A_\varepsilon
\end{aligned}
$$

where the second equality defines the 2×2 matrix A_ε whose entries we denote are a_{ij}. If λ_A denotes an eigenvalue of A_ε then we can take

$$
2\lambda_A = \mathrm{tr} A_\varepsilon + \sqrt{(\mathrm{tr} A_\varepsilon)^2 - 4\det A_\varepsilon}.
$$

A direct computation shows $\det A_\varepsilon = \frac{3\pi^2}{16} c_3(a)^4 + O(\varepsilon^2)$. Also, $\det Z_\varepsilon(2\pi) = 1$ previously so that another calculation yields $\det Z_\varepsilon(2\pi) = 1 + \varepsilon^2 \,\mathrm{tr} A_\varepsilon + \varepsilon^4 \det A_\varepsilon = 1$ and we get $\mathrm{tr} A_\varepsilon = -\varepsilon^2 \det A_\varepsilon = -\varepsilon^2 \frac{3\pi^2}{16} c_3(a)^4 + O(\varepsilon^4)$. If we denote $\lambda_A = \varepsilon^2 \lambda_A^R + i\lambda_A^I$ then

$$
\begin{aligned}
\lambda_A^R &= \frac{1}{2\varepsilon^2}\,\mathrm{tr} A_\varepsilon = -\frac{3\pi^2}{32} c_3(a)^4 + O(\varepsilon^2), \\
\lambda_A^I &= \sqrt{\det A_\varepsilon - \Big(\frac{1}{2}\,\mathrm{tr} A_\varepsilon\Big)^2} = \frac{\sqrt{3}\pi}{4} c_3(a)^2 + O(\varepsilon).
\end{aligned}
$$

Also, an eigenvalue, λ_Z, of $Z_\varepsilon(2\pi)$ is given by $\lambda_Z = 1 + \varepsilon^2 \lambda_A$. The corresponding transformation matrix P_ε is

$$
P_\varepsilon = \begin{pmatrix} a_{12} & 0 \\ -a_{11} + \varepsilon^2 \lambda_A^R & \lambda_A^I \end{pmatrix} \quad \text{with} \quad P_\varepsilon^{-1} = \begin{pmatrix} 1/a_{12} & 0 \\ \dfrac{a_{11} - \varepsilon^2 \lambda_A^R}{a_{12}\lambda_A^I} & \dfrac{1}{\lambda_A^I} \end{pmatrix}.
$$

We have $\lambda_A^I > 0$ for small ε, $\frac{\pi}{4} c_3(a)^2 \le a_{12} \le \frac{3\pi}{4} c_3(a)^2$ and

$$P_\varepsilon^{-1} Z_\varepsilon(2\pi) P_\varepsilon = \begin{pmatrix} \Re\lambda_Z & \Im\lambda_Z \\ -\Im\lambda_Z & \Re\lambda_Z \end{pmatrix}.$$

Since $|\lambda_Z| = 1$ we can write

$$\begin{pmatrix} \Re\lambda_Z & \Im\lambda_Z \\ -\Im\lambda_Z & \Re\lambda_Z \end{pmatrix} = \mathrm{e}^{\Phi_\varepsilon} \quad \text{where} \quad \Phi_\varepsilon = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \quad \text{with} \quad \theta = \operatorname{Arg}\lambda_Z.$$

Now, we observe that the operator norm of a 2×2 square matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(that is the square root of the greatest eigenvalue of the symmetric matrix A^*A) is given by

$$\|A\|^2 = \frac{1}{2}\left[(a^2+b^2+c^2+d^2) + \sqrt{[(a-d)^2+(b+c)^2][(a+d)^2+(b-c)^2]}\right]$$

and hence $\|A^{-1}\| = \frac{1}{|\det A|}\|A\|$ since

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Using these formulas we get

$$\begin{aligned}
\|P_0\|^2 &= \frac{3\pi^2}{16} c_3(a)^4 \left[2 + \cos 2(\alpha + c_4(a))\right], \\
\det P_0 &= \frac{\sqrt{3}\pi^2}{16} c_3(a)^4 \left[2 + \cos 2(\alpha + c_4(a))\right], \\
\|P_0\|\|P_0^{-1}\| &= \sqrt{3}.
\end{aligned}$$

We see that $\|P_\varepsilon\|$ and $\|P_\varepsilon^{-1}\|$ are both uniformly bounded for ε small and a bounded. Finally, we have

$$\begin{aligned}
Z_\varepsilon(t) Z_\varepsilon(s)^{-1} &= U_\varepsilon(t)\, \mathrm{e}^{(t-s)B_\varepsilon}\, U_\varepsilon(s)^{-1} = U_\varepsilon(t) \exp\left(\frac{t-s}{2\pi} P_\varepsilon \Phi_\varepsilon P_\varepsilon^{-1}\right) U_\varepsilon(s)^{-1} \\
&= U_\varepsilon(t) P_\varepsilon\, \mathrm{e}^{\frac{t-s}{2\pi}\Phi_\varepsilon}\, P_\varepsilon^{-1} U_\varepsilon(s)^{-1} \\
&= U_\varepsilon(t) P_\varepsilon \begin{pmatrix} \cos\left(\dfrac{(t-s)\theta}{2\pi}\right) & \sin\left(\dfrac{(t-s)\theta}{2\pi}\right) \\ -\sin\left(\dfrac{(t-s)\theta}{2\pi}\right) & \cos\left(\dfrac{(t-s)\theta}{2\pi}\right) \end{pmatrix} P_\varepsilon^{-1} U_\varepsilon(s)^{-1}.
\end{aligned}$$

Taking norms we get $\|Z_\varepsilon(t)Z_\varepsilon(s)^{-1}\| \le \sqrt{3}+\delta$ where $\delta \to 0$ as $\varepsilon \to 0$. This completes our study of (4.3.12).

Step 2. Next we write (4.3.11) as the system

$$\begin{aligned} \dot{w}_1 &= w_2, \\ \dot{w}_2 &= -w_1\left(1+\varepsilon^2\phi_\varepsilon(t)^2\right) - 2\varepsilon^2 w_2. \end{aligned} \tag{4.3.13}$$

Then the fundamental solution $\bar{W}_\varepsilon$ of (4.3.13) is given by

$$\bar{W}_\varepsilon(t) = \mathrm{e}^{-\varepsilon^2 t}\begin{pmatrix} 1 & 0 \\ -\varepsilon^2 & 1 \end{pmatrix} Z_\varepsilon(t).$$

This implies

$$\bar{W}_\varepsilon(t)\bar{W}_\varepsilon(s)^{-1} = \mathrm{e}^{-\varepsilon^2(t-s)}\begin{pmatrix} 1 & 0 \\ -\varepsilon^2 & 1 \end{pmatrix} Z_\varepsilon(t)Z_\varepsilon(s)^{-1}\begin{pmatrix} 1 & 0 \\ \varepsilon^2 & 1 \end{pmatrix}$$

and hence $\|\bar{W}_\varepsilon(t)\bar{W}_\varepsilon(s)^{-1}\| \le (\sqrt{3}+\delta)\,\mathrm{e}^{-\varepsilon^2(t-s)}$.

Step 3. Finally, we consider

$$\ddot{w} = -w(1+6\varepsilon^2 v_{\varepsilon,\alpha,a}^2(t)+2\gamma^2) - 2\varepsilon^2\dot{w}$$

which we write as

$$\begin{aligned} \dot{w}_1 &= w_2, \\ \dot{w}_2 &= -w_1(1+6\varepsilon^2 v_{\varepsilon,\alpha,a}^2(t)+2\gamma^2) - 2\varepsilon^2 w_2\,. \end{aligned} \tag{4.3.14}$$

Let W_ε be the fundamental solution of (4.3.14). We put

$$\Psi(t) = -2\gamma(t)^2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then for $t \ge s$ we get

$$W_\varepsilon(t)W_\varepsilon(s)^{-1} = \bar{W}_\varepsilon(t)\bar{W}_\varepsilon(s)^{-1} + \int_s^t \bar{W}_\varepsilon(t)\bar{W}_\varepsilon(z)^{-1}\Psi(z)W_\varepsilon(z)W_\varepsilon(s)^{-1}\,dz\,.$$

By putting $U(t) = W_\varepsilon(t)W_\varepsilon(s)^{-1}\,\mathrm{e}^{\varepsilon^2(t-s)}$ we obtain

$$\begin{aligned} \|U(t)\| &\le \left(\sqrt{3}+\delta\right) + \left(\sqrt{3}+\delta\right)\int_s^t \|\Psi(z)\|\|U(z)\|\,dz \\ &= \left(\sqrt{3}+\delta\right) + 2\left(\sqrt{3}+\delta\right)\int_s^t \gamma^2(z)\|U(z)\|\,dz \end{aligned}$$

which gives

$$\|U(t)\| \le \left(\sqrt{3}+\delta\right)\mathrm{e}^{2\left(\sqrt{3}+\delta\right)\int_s^t \gamma^2(z)\,dz}\,.$$

Now if either $t \geq s \gg 1$ or $s \leq t \ll -1$, then $\mathrm{e}^{2\sqrt{3}\int_s^t \gamma^2(z)\,dz}$ is about 1. So then we obtain

$$\|W_\varepsilon(t)W_\varepsilon(s)^{-1}\| \leq K_1\,\mathrm{e}^{-\varepsilon^2(t-s)}$$

with $K_1 \sim \sqrt{3}$ for $s \leq t \in (-\infty,-T_0] \cup [T_0,\infty)$ for $T_0 \gg 1$. Since W_0 satisfies

$$\begin{aligned}\dot{w}_1 &= w_2,\\ \dot{w}_2 &= -w_1(1+2\gamma^2).\end{aligned}$$

we see that

$$W_0(t) = C(t)\begin{pmatrix}1 & 0\\ 0 & 1/2\end{pmatrix},$$

where

$$C(t) = \begin{pmatrix}\cos t - \sin t \tanh t & \sin t + \cos t \tanh t\\ -\sin t - \cos t \tanh t - \sin t \operatorname{sech}^2 t & \cos t - \sin t \tanh t + \cos t \operatorname{sech}^2 t\end{pmatrix}.$$

Then we have

$$\|C(t)\|^2 = \frac{1}{2}\left(4 + \operatorname{sech}^4 t + \operatorname{sech}^2 t\sqrt{8+\operatorname{sech}^4 t}\right) \leq 4, \quad \det C(t) = 2$$

which also imply $\|C(t)^{-1}\| \leq 1$. In summary, we arrive at

$$\|W_\varepsilon(t)W_\varepsilon(s)^{-1}\| \leq K_1\,\mathrm{e}^{-\varepsilon^2(t-s)}$$

with $K_1 \sim \sqrt{3} \times 2 \times \sqrt{3} = 6$ for $s \leq t \in \mathbb{R}$ and $\varepsilon > 0$ small. This is our exponential dichotomy for the linear part of (4.3.10b).

Remark 4.3.2. Note that in general the function $\tilde{f}_\varepsilon(t) = W_\varepsilon(t)\int_{-\infty}^t W_\varepsilon(s)^{-1} f(s)\,ds$ is $O(1/\varepsilon^2)$ for f bounded. But if $f \in L^1(\mathbb{R})$ such an expression is $O(1)$ and we can let $\varepsilon \to 0$. More precisely, set $\tilde{f}_0 := W_0(t)\int_{-\infty}^t W_0(s)^{-1} f(s)\,ds$ and let $\widetilde{T} > 0$ be large. Then $\tilde{f}_\varepsilon(t) = o(1)$ and $\tilde{f}_0(t) = o(1)$ uniformly for all $t \leq -\widetilde{T}$ and ε small. If $t \in [-\widetilde{T},\widetilde{T}]$ then $\tilde{f}_\varepsilon(t) = W_\varepsilon(t)\int_{-\widetilde{T}}^t W_\varepsilon(s)^{-1} f(s)\,ds + W_\varepsilon(t)\int_{-\infty}^{-\widetilde{T}} W_\varepsilon(s)^{-1} f(s)\,ds$. Clearly $W_\varepsilon(t)\int_{-\infty}^{-\widetilde{T}} W_\varepsilon(s)^{-1} f(s)\,ds = o(1)$ and $W_0(t)\int_{-\infty}^{-\widetilde{T}} W_0(s)^{-1} f(s)\,ds = o(1)$ uniformly for all $t \in [-\widetilde{T},\widetilde{T}]$ and ε small. Moreover

$$W_\varepsilon(t)\int_{-\widetilde{T}}^t W_\varepsilon(s)^{-1} f(s)\,ds \to W_0(t)\int_{-\widetilde{T}}^t W_0(s)^{-1} f(s)\,ds$$

uniformly for all $t \in [-\widetilde{T},\widetilde{T}]$ as $\varepsilon \to 0$. Consequently, we obtain $\lim_{\varepsilon\to 0}\tilde{f}_\varepsilon(t) = \tilde{f}_0(t)$ uniformly in any interval $(-\infty,a]$ for $f \in L^1(\mathbb{R})$. If $t \geq \widetilde{T}$ then

$$\tilde{f}_\varepsilon(t) = W_\varepsilon(t)\int_{\widetilde{T}}^t W_\varepsilon(s)^{-1} f(s)\,ds + W_\varepsilon(t)\int_{-\infty}^{\widetilde{T}} W_\varepsilon(s)^{-1} f(s)\,ds.$$

We again deduce that $W_\varepsilon(t)\int_{\widetilde{T}}^t W_\varepsilon(s)^{-1}f(s)\,ds = o(1)$ and $W_0(t)\int_{\widetilde{T}}^t W_0(s)^{-1}f(s)\,ds = o(1)$ uniformly for all $t \geq \widetilde{T}$ and ε small. Next

$$W_\varepsilon(t)\int_{-\infty}^{\widetilde{T}} W_\varepsilon(s)^{-1}f(s)\,ds = W_\varepsilon(t)W_\varepsilon(\widetilde{T})^{-1}\tilde{f}_\varepsilon(\widetilde{T})\,.$$

In summary we obtain $\|\widetilde{f}_\varepsilon\| \leq \left(\sqrt{3}+o(1)\right)\|\widetilde{f}_0\|$. Moreover, when

$$\|f\|_{\tilde{a}} := \sup_{t\leq 0}|f(t)|\,\mathrm{e}^{-\tilde{a}t} < \infty$$

for $\tilde{a} > 0$, $\|\tilde{f}_\varepsilon\|_{\tilde{a}} \leq \frac{K_1}{\tilde{a}}\|f\|_{\tilde{a}}$. So if

$$X_{\tilde{a}} := \left\{ f \in C(-\infty,0] \mid \|f\|_{\tilde{a}} < \infty \right\}$$

and $L_\varepsilon f := \tilde{f}_\varepsilon$, then $L_\varepsilon \in L(X_{\tilde{a}})$. Finally, we can check that $L_\varepsilon \to L_0$ as $\varepsilon \to 0$ in $L(X_{\tilde{a}})$ for $L_0 f = W_0(t)\int_{-\infty}^t W_0^{-1}(s)f(s)\,ds$.

Equation (4.3.10a) has the form

$$\ddot{u} = u(1-6\gamma^2(t)) + h(t), \qquad \dot{u}(0) = 0 \tag{4.3.15}$$

for $h(t) \in C_B(\mathbb{R})$ — the Banach space of bounded and continuous functions on $\mathbb{R}$ endowed with the supremum norm. For this we use the projection

$$\Pi h = \frac{\displaystyle\int_{-\infty}^{\infty} h(s)\dot{\gamma}(s)\,ds}{\displaystyle\int_{-\infty}^{\infty} \dot{\gamma}^2(s)\,ds}\,\dot{\gamma}(t)\,.$$

From Section 4.1, (4.3.15) has a (unique) bounded solution $u = Kh$ if and only if $\Pi h = 0$. We write (4.3.10) in the form

$$\begin{aligned} u(t) &= K(\mathbb{I}-\Pi)\Big(-2\delta\dot{\gamma} - 2\xi\gamma\big[w + v_{\varepsilon,\alpha,a}(t)\big]^2 \\ &\qquad + a_3\cos(t+\alpha) + a_4\sin(t+\alpha)\Big) + O(\varepsilon)\,, \end{aligned} \tag{4.3.16a}$$

$$\begin{aligned} w(t) &= \int_{-\infty}^{t} W_\varepsilon(t)W_\varepsilon(s)^{-1}\Big\{(0, -2\gamma^2 v_{\varepsilon,\alpha,a} - 4\varepsilon^2\gamma u(w+v_{\varepsilon,\alpha,a}) \\ &\qquad -2\varepsilon^4u^2(w+v_{\varepsilon,\alpha,a}) - 6\varepsilon^2w^2v_{\varepsilon,\alpha,a} - 2\varepsilon^2w^3)\Big\}\,ds\,, \end{aligned} \tag{4.3.16b}$$

$$\begin{aligned} &\int_{-\infty}^{\infty}\Big(-2\delta\dot{\gamma} - 2\xi\gamma\big[w + v_{\varepsilon,\alpha,a}\big]^2 \\ &\qquad + a_3\cos(t+\alpha) + a_4\sin(t+\alpha)\Big)\dot{\gamma}(t)\,dt + O(\varepsilon) = 0 \end{aligned} \tag{4.3.16c}$$

for $w=(w_1,w_2)$. Since $v_{\varepsilon,\alpha,a}(t)=v_{0,\alpha,a}(t)+O(\varepsilon)$ by Lemma 4.3.1 and $\gamma\in L^1(\mathbb{R})$, we can consider, according to Remark 4.3.2, (4.3.16b) to be

$$\begin{aligned} w(t) &= \int_{-\infty}^{t} W_\varepsilon(t)W_\varepsilon(s)^{-1}(0,-2\gamma^2 v_{0,\alpha,a})\,ds \\ &\quad - \int_{-\infty}^{t} W_\varepsilon(t)W_\varepsilon(s)^{-1}(0,-6\varepsilon^2 w^2 v_{\varepsilon,\alpha,a}-2\varepsilon^2 w^3)\,ds + o(1)\,. \end{aligned} \tag{4.3.17}$$

We note that

$$z_0(t)=(z_{01}(t),z_{02}(t))=\int_{-\infty}^{t} W_0(t)W_0(s)^{-1}(0,-2\gamma^2 v_{0,\alpha,a}(s))\,ds$$

solves

$$\begin{aligned} \dot z_{01} &= z_{02}, \\ \dot z_{02} &= -z_{01}-2\gamma^2(t)z_{01}-2\gamma^2(t)v_{0,\alpha,a}, \end{aligned} \qquad z_0(-\infty)=0$$

which is the limiting equation for $\varepsilon\to 0$ in (4.3.10b). Since $v_{0,\alpha,a}(t)=c_5(a)\sin(t+\alpha+c_4(a))$, we see that

$$z_{01}(t)=c_5(a)\,\mathrm{e}^{2t}\,\frac{\cos(t+\alpha+c_4(a))-\sin(t+\alpha+c_4(a))}{1+\mathrm{e}^{2t}}\,. \tag{4.3.18}$$

Then, with $s=\alpha+c_4(a)+\pi/4$, we have

$$\begin{aligned} \|z_0\|^2 &= \max_{t\in\mathbb{R}}\left(z_{01}(t)^2+z_{02}(t)^2\right) \\ &= \max_{t\in\mathbb{R}} \frac{2c_5(a)^2\,\mathrm{e}^{4t}}{(1+\mathrm{e}^{2t})^4}[1-2\sin 2(t+s)+4\cos^2(t+s) \\ &\quad +2\,\mathrm{e}^{2t}\left(1-\sin 2(t+s)\right)+\mathrm{e}^{4t}] \\ &\le \max_{t\in\mathbb{R}} \frac{2c_5(a)^2\,\mathrm{e}^{4t}}{(1+\mathrm{e}^{2t})^4}(7+4\,\mathrm{e}^{2t}+\mathrm{e}^{4t}) = \frac{1029}{512}c_5(a)^2. \end{aligned}$$

Further, $\lim_{t\to\infty}\left(z_{01}(t)^2+z_{02}(t)^2\right)=2c_5(a)^2$ so that, finally,

$$\sqrt{2}c_5(a)\le\|z_0\|\le k_1c_5(a)$$

with $k_1=\sqrt{1029/512}\doteq 1.417662$. By using Remark 4.3.2 and (4.3.17), we have

$$\begin{aligned} \|w\| &\le \sqrt{3}\|z_0\|+6(6\|v_{\varepsilon,\alpha,a}\|\|w\|^2+2\|w\|^3)+o(1) \\ &\le \sqrt{3}k_1c_5(a)+36c_5(a)\|w\|^2+12\|w\|^3+o(1)\,. \end{aligned}$$

So if we choose $r_0>0$ so that

$$\begin{aligned} \sqrt{3}k_1c_5(a)+36c_5(a)r_0^2+12r_0^3 &< r_0, \\ 72c_5(a)r_0+36r_0^2 &< 1, \end{aligned} \tag{4.3.19}$$

then for $\varepsilon>0$ small and $\|w\|\le r_0$ we can uniquely solve (4.3.16) using the Banach fixed point theorem 2.2.1 on the ball

$$\left\{(u,w)\in C_B(\mathbb{R})^2 \mid \|u\|\le\tilde{K},\quad \|w\|\le r_0\right\}$$

for a constant

$$\tilde{K}=\|K(\mathbb{I}-\Pi)\|\left(2\delta\|\dot{\gamma}\|+2\xi\|\gamma\|[r_0+c_5(a)]^2+|a_3|+|a_4|\right)+1\,.$$

To find the largest $c_5(a)$ in (4.3.19), we solve

$$\sqrt{3}k_1k_3+36k_2^2k_3+12k_2^3=k_2\,,\quad 72k_2k_3+36k_2^2=1\,,$$

which has a solution

$$k_2=\frac{\sqrt{-3-3\sqrt{3}k_1+\sqrt{3}\sqrt{3+10\sqrt{3}k_1+9k_1^2}}}{6\sqrt{2}}\doteq 0.136179\,,$$

$$k_3=\frac{5+3\sqrt{3}k_1-\sqrt{9+30\sqrt{3}k_1+27k_1^2}}{12\sqrt{-6-6\sqrt{3}k_1+2\sqrt{9+30\sqrt{3}k_1+27k_1^2}}}\doteq 0.0339006\,.$$

So we take $r_0=k_2$, $0<c_5(a)<k_3$. Then (4.3.19) holds. Consequently, we have a bounded solution $w_{\alpha,a,\varepsilon}=(w_{1,\alpha,a,\varepsilon},\dot{w}_{1,\alpha,a,\varepsilon})$ of (4.3.10b). Now we study the limit as $\varepsilon\to 0$. Let $\tilde{w}_{\alpha,a,\varepsilon}$, $\|\tilde{w}_{\alpha,a,\varepsilon}\|\le r_0$ solve

$$\begin{aligned} w(t) = &\int_{-\infty}^t W_\varepsilon(t)W_\varepsilon(s)^{-1}(0,-2\gamma^2v_{0,\alpha,a})\,ds \\ &-\int_{-\infty}^t W_\varepsilon(t)W_\varepsilon(s)^{-1}(0,-6\varepsilon^2w^2v_{\varepsilon,\alpha,a}-2\varepsilon^2w^3)\,ds\,. \end{aligned} \tag{4.3.20}$$

We note that the right-hand side of (4.3.20), denoted $N_{\alpha,a,\varepsilon}(w)$, is a contraction on the ball $\{w\in C_B(\mathbb{R}) \mid \|w\|\le r_0\}$. So by the Banach fixed point theorem 2.2.1, $\tilde{w}_{\alpha,a,\varepsilon}$ exits and satisfies, according to (4.3.17), $\|\tilde{w}_{\alpha,a,\varepsilon}-w_{\alpha,a,\varepsilon}\|=o(1)$ as $\varepsilon\to 0$. Since $\gamma^2\in X_{\tilde{a}}$ for some $\tilde{a}>0$, and $N_{\alpha,a,\varepsilon}:X_{\tilde{a}}\to X_{\tilde{a}}$ is a contraction on any bounded subset, by Remark 4.3.2 we see that $\tilde{w}_{\alpha,a,\varepsilon}\to z_0$ as $\varepsilon\to 0$ in $X_{\tilde{a}}$. So $\tilde{w}_{\alpha,a,\varepsilon}\to z_0$ uniformly on $(-\infty,0]$. Now let us fix an interval $[-n,n]$, $n\in\mathbb{N}$ and take a sequence $\{w_{\alpha,a,\varepsilon_i}\}_{i=0}^\infty$, $\varepsilon_i\to 0$. By the Arzelà-Ascoli theorem 2.1.3, we can suppose that $w_{\alpha,a,\varepsilon_i}\to\tilde{z}$ uniformly on $[-n,n]$. But we already know that $\tilde{z}(t)=z_0(t)$ on $[-n,0]$. Since $\tilde{z}(t)$ satisfies the same ODE on $[-n,n]$ as $z_0(t)$, we get $\tilde{z}(t)=z_0(t)$ also on $[0,n]$. These arguments imply that

$$w_{\alpha,a,\varepsilon}(t) \to z_0(t)$$

for $\varepsilon \to 0$ and uniformly in any compact interval on $\mathbb{R}$. Consequently, the limit bifurcation equation of (4.3.16c) is given by

$$M(\alpha) = \int_{-\infty}^{\infty} \Big(-2\delta\dot{\gamma}(t) - 2\xi\gamma(t)[z_{01}(t) + c_5(a)\sin(t+\alpha+c_4(a))]^2$$
$$+a_3\cos(t+\alpha) + a_4\sin(t+\alpha)\Big)\dot{\gamma}(t)\,dt = -\frac{4}{3}\delta + \pi(a_3\sin\alpha - a_4\cos\alpha)\operatorname{sech}\frac{\pi}{2} = 0\,.$$

The equation $M(\alpha) = 0$ has a simple root if and only if

$$4\delta < 3\pi\sqrt{a_3^2 + a_4^2}\operatorname{sech}\frac{\pi}{2}\,. \tag{4.3.21}$$

From (4.3.8) we derive

$$4(a_1^2 + a_2^2) = 9c_5(a)^6 + 16c_5(a)^2\,. \tag{4.3.22}$$

Since $c_5(a) < k_3$, we get

$$\sqrt{a_1^2 + a_2^2} < k_4 := \frac{\sqrt{9k_3^4 + 16}}{2}k_3 \doteq 0.0678013\,. \tag{4.3.23}$$

So if (4.3.21) holds, then we have a bounded solution for (4.3.4). Using the above method along with an approach from Section 4.1, we can show that it is a transverse homoclinic solution to a small periodic solution with appropriate shift-type dynamics. Finally, we obtain another result.

Theorem 4.3.3. *For any* $(a_1,a_2) \neq (0,0)$ *satisfying* (4.3.23) *there is a unique positive* $c_5(a)$ *solving* (4.3.22)*. Then Eq.* (4.3.4) *has a transverse homoclinic solution for any* $\varepsilon > 0$ *sufficiently small with* $\mu_1 = \varepsilon^3 a_1$*,* $\mu_2 = \varepsilon^2$*,* $\mu_3 = \varepsilon^3 a_2$*,* $\mu_4 = \varepsilon^2 a_3$*,* $\mu_5 = \varepsilon^2 a_4$*, and* δ *satisfying condition* (4.3.21).

Note that if we suppose $\mu_4 = O(\varepsilon^3)$ and $\mu_5 = O(\varepsilon^3)$ in (4.3.4) then we get $M(\alpha) = \frac{4}{3}\delta$, so $M(\alpha) \neq 0$ and we do not get solutions of the desired form.

It is interesting to formulate the conditions in Theorem 4.3.3 in terms of the original parameters as they appear in (4.3.4). The equation $M(\alpha) = 0$, in place of (4.3.21), requires

$$0 < 2\delta\mu_2 < \frac{3}{2}\pi\sqrt{\mu_4^2 + \mu_5^2}\operatorname{sech}\frac{\pi}{2} \tag{4.3.24}$$

while (4.3.23) becomes

$$0 < \sqrt{\mu_1^2 + \mu_3^2} < k_4\mu_2^{3/2}\,. \tag{4.3.25}$$

The condition (4.3.24) is a restriction on the allowed damping relative to forcing in the first equation of (4.3.4). This result could be obtained by ignoring the center part of the problem, i.e. by setting $y = 0$ in the first equation of (4.3.4) and then applying classical Melnikov theory. The effect of the center manifold appears in condition (4.3.25) which imposes a limit on the magnitude of forcing relative to damping in the second equation of (4.3.4).

In this example, the hyperbolic and center parts of the analysis turn out to be separated but this is not always so. For example, if we replace $-2\xi xy^2$ in the first equation with $-2\xi\dot{x}y^2$, the Melnikov function $M(\alpha)$ acquires a contribution from the second equation. Indeed, it has now the form

$$M(\alpha) = -\frac{4}{3}\delta - \xi c_5(a)^2\left[\frac{8}{15} - \frac{2\pi}{3\sinh\pi}\sin 2(\alpha + c_4(a))\right] \\ +\pi\left(a_3\sin\alpha - a_4\cos\alpha\right)\operatorname{sech}\frac{\pi}{2}.$$

By using (4.3.19) we study (4.3.16b) locally as a semilinear equation. In Section 4.3.4, we apply a global approach based on the averaging method [16] (cf Section 2.5.7) in order to study (4.3.5). This improves Theorem 4.3.3.

4.3.3 General Equations

To solve (4.3.1), we shift the time $t \longleftrightarrow t + \alpha$ and substitute

$$x = \gamma + \varepsilon\sum_{i=1}^{d-1}\beta_i u_i + \varepsilon^2 u, \quad y = \varepsilon v,$$
$$\mu_1 = \varepsilon^3\mu_{0,1}, \quad \mu_2 = \varepsilon^2\mu_{0,2}, \quad \mu_0 \neq 0$$

where $\{u_1,\ldots,u_d\}$ is a basis for the vector space of bounded solutions for the linear system $\dot{u} = D_1 f_0(\gamma(t),0)u$ with $u_d = \dot{\gamma}$ and $\mu_0 = (\mu_{0,1},\mu_{0,2})$ is to be determined. We suppose $\mu_{0,2} > 0$. Introducing this change of variables into (4.3.1) yields

$$\dot{u} = D_1 f_0(\gamma,0)u + \frac{1}{2}\sum_{i,j=1}^{d-1} D_{11} f_0(\gamma,0)\beta_i\beta_j u_i u_j \tag{4.3.26a}$$
$$+\mu_{0,2} f_2(\gamma,0,0,t+\alpha) + \frac{1}{2}D_{22} f_0(\gamma,0)vv + O(\varepsilon),$$
$$\dot{v} = D_2 g_0(\gamma,0)v + \frac{\varepsilon^2}{6}D_{222} g_0(\gamma,0)v^3 \tag{4.3.26b}$$
$$+\varepsilon^2\mu_{0,1} g_1(0,0,0,t+\alpha) + \varepsilon^2\mu_{0,2} D_2 g_2(\gamma,0,0)v$$
$$+\phi_0(u,v,\varepsilon,t) + \varepsilon^2\mu_{0,1}\phi_1(u,v,\varepsilon,t) + \varepsilon^2\mu_{0,2}\phi_2(u,v,\varepsilon,t)$$

where

$$\phi_0(u,v,\varepsilon,t) = \frac{1}{\varepsilon} g_0\left(\gamma+\varepsilon\sum_{i=1}^{d-1}\beta_i u_i+\varepsilon^2 u,\varepsilon v\right) - D_2 g_0(\gamma,0)v - \frac{\varepsilon^2}{6} D_{222} g_0(\gamma,0)v^3,$$

$$\begin{aligned}\phi_1(u,v,\varepsilon,t) &= g_1\left(\gamma+\varepsilon\sum_{i=1}^{d-1}\beta_i u_i+\varepsilon^2 u,\varepsilon v,(\varepsilon^3\mu_{0,1},\varepsilon^2\mu_{0,2}),t+\alpha\right)\\ &\quad -g_1(0,0,0,t+\alpha),\end{aligned}$$

$$\phi_2(u,v,\varepsilon,t) = \frac{1}{\varepsilon} g_2\left(\gamma+\varepsilon\sum_{i=1}^{d-1}\beta_i u_i+\varepsilon^2 u,\varepsilon v,(\varepsilon^3\mu_{0,1},\varepsilon^2\mu_{0,2})\right) - D_2 g_2(\gamma,0,0)v.$$

We note that the functions γ and $u_i, i = 1,\ldots,d-1$ have a norm which is dominated by $\mathrm{e}^{-\tilde{a}|t|}$ for some $\tilde{a} > 0$. Using this fact and assumptions (i)–(viii) we have

$$\begin{aligned}\phi_0(u,v,\varepsilon,t) &= O(\varepsilon)\,\mathrm{e}^{-\tilde{a}|t|} + O(\varepsilon^3),\\ \phi_1(u,v,\varepsilon,t) &= O(1)\,\mathrm{e}^{-\tilde{a}|t|} + O(\varepsilon),\\ \phi_2(u,v,\varepsilon,t) &= O(\varepsilon).\end{aligned}$$

We consider the Banach spaces

$$\begin{aligned}X_n &= \left\{x \in C(\mathbb{R},\mathbb{R}^n) \;\middle|\; \sup_{t\in\mathbb{R}}|x(t)| < \infty\right\},\\ Y_n &= \Big\{y \in X_n \;\Big|\; \int_{-\infty}^{\infty}\langle y(t),v(t)\rangle\,dt = 0,\\ &\qquad \text{for every bounded solution } v \text{ to } \dot{v} = -D_1 f_0(\gamma,0)^t v\Big\}\end{aligned}$$

with the supremum norm $||x|| = \sup_{t\in\mathbb{R}}|x(t)|$. Now we recall the following results of Section 4.2.

Lemma 4.3.4. *There exist constants $b > 0$, $B > 0$ independent of ε so that given $\mu_{0,2} > 0$ the variational equation*

$$\dot{v} = \left[D_2 g_0(\gamma(t),0) + \varepsilon^2\mu_{0,2} D_2 g_2(\gamma(t),0,0)\right] v$$

has an exponential dichotomy $(V_\varepsilon,\mathbb{I})$ on $\mathbb{R}$ with constants $(B, b\varepsilon^2\mu_{0,2})$.

Lemma 4.3.5. *Given $h \in Y_n$, the equation $\dot{u} = D_1 f_0(\gamma(t),0)u + h$ has a unique solution $u \in X_n$ satisfying $\langle u(0),u_i(0)\rangle = 0$ for every $i = 1,2,\ldots,d$.*

Lemma 4.3.6. *There exists a continuous projection denoted $\Pi : X_n \to X_n$ so that $\mathscr{R}(\mathbb{I}-\Pi) = Y_n$.*

We define the linear map $\mathscr{K} : Y_n \to X_n$ by $\mathscr{K}h = u$ where h, u are the same as in Lemma 4.3.5. Now, we assume the following conditions:

(ix) For any $\varepsilon > 0$ small and $\alpha \in \mathbb{R}$, there is a $v_{\varepsilon,\alpha} \in X_m$ with $\dot{v}_{\varepsilon,\alpha} \in X_m$ satisfying

$$\dot{v}_{\varepsilon,\alpha}(t) = \left(D_2 g_0(\gamma(t),0) + \varepsilon^2 \mu_{0,2} D_2 g_2(\gamma(t),0,0)\right) v_{\varepsilon,\alpha}(t) + \frac{\varepsilon^2}{6} D_{222} g_0(0,0) v_{\varepsilon,\alpha}(t)^3 + \varepsilon^2 \mu_{0,1} g_1(0,0,0,t+\alpha)$$

along with $\tilde{B} = \sup_{\varepsilon>0,\alpha} \|v_{\varepsilon,\alpha}\| < \infty$. Moreover, $v_{\varepsilon,\alpha}$ is C^1-smooth in $\varepsilon > 0$, α and $\sup_{\varepsilon>0,\alpha} \|\frac{\partial}{\partial\alpha} v_{\varepsilon,\alpha}\| < \infty$. Furthermore, there is a C^1-smooth $v_\alpha \in X_m$ so that $v_{\varepsilon,\alpha} \to v_\alpha$ and $\frac{\partial}{\partial\alpha} v_{\varepsilon,\alpha} \to \frac{\partial}{\partial\alpha} v_\alpha$ as $\varepsilon \to 0_+$ uniformly in any compact interval of $\mathbb{R}$ and uniformly for α as well.

(x) There are constants $\bar{B} > 0$, $\bar{b} > 0$ so that for any $\varepsilon > 0$ small and $\alpha \in \mathbb{R}$, the equation

$$\dot{w}(t) = \left(D_2 g_0(\gamma(t),0) + \varepsilon^2 \mu_{0,2} D_2 g_2(\gamma(t),0,0) + \frac{\varepsilon^2}{2} D_{222} g_0(0,0) v_{\varepsilon,\alpha}(t)^2\right) w(t)$$

has an exponential dichotomy $(W_\varepsilon, \mathbb{I})$ on $\mathbb{R}$ with constants $(\bar{B}, \bar{b}\varepsilon^2)$.

Let $\{v_1, v_2, \ldots, v_d\}$ be a basis of bounded solutions of $\dot{v} = -D_1 f_0(\gamma,0)^t v$. Using the projection Π and the exponential dichotomy W_ε from condition (x), we can rewrite (4.3.26), by changing $v = v_{\varepsilon,\alpha} + w$ in (4.3.26b), as the fixed point problem

$$u = \mathscr{K}(\mathbb{I} - \Pi)\Big(\frac{1}{2} \sum_{i,j=1}^{d-1} D_{11} f_0(\gamma,0) \beta_i \beta_j u_i u_j + \mu_{0,2} f_2(\gamma,0,0,t+\alpha) + \frac{1}{2} D_{22} f_0(\gamma,0)(v_{\varepsilon,\alpha} + w)(v_{\varepsilon,\alpha} + w) + O(\varepsilon)\Big), \tag{4.3.27a}$$

$$\begin{aligned} w(t) = \int_{-\infty}^t W_\varepsilon(t) W_\varepsilon(s)^{-1} \Big\{ &\frac{\varepsilon^2}{6} \left[D_{222} g_0(\gamma(s),0) - D_{222} g_0(0,0)\right] v_{\varepsilon,\alpha}(s)^3 \\ &+ \frac{\varepsilon^2}{6} D_{222} g_0(\gamma(s),0) \left[3 v_{\varepsilon,\alpha}(s) w(s)^2 + w(s)^3\right] \\ &+ \frac{\varepsilon^2}{2} \left[D_{222} g_0(\gamma(s),0) - D_{222} g_0(0,0)\right] v_{\varepsilon,\alpha}(s)^2 w(s) \\ &+ \phi_0(u(s), v_{\varepsilon,\alpha}(s) + w(s), \varepsilon, s) + \varepsilon^2 \mu_{0,1} \phi_1(u(s), v_{\varepsilon,\alpha}(s) + w(s), \varepsilon, s) \\ &+ \varepsilon^2 \mu_{0,2} \phi_2(u(s), v_{\varepsilon,\alpha}(s) + w(s), \varepsilon, s)\Big\} ds, \end{aligned} \tag{4.3.27b}$$

$$\begin{gathered} \int_{-\infty}^{\infty} \Big\langle v_i(t), \frac{1}{2} \sum_{i,j=1}^{d-1} D_{11} f_0(\gamma(t),0) \beta_i \beta_j u_i(t) u_j(t) + \mu_{0,2} f_2(\gamma(t),0,0,t+\alpha) \\ + \frac{1}{2} D_{22} f_0(\gamma(t),0)(v_{\varepsilon,\alpha}(t) + w(t))(v_{\varepsilon,\alpha}(t) + w(t)) + O(\varepsilon)\Big\rangle dt = 0, \\ i = 1, 2, \ldots, d. \end{gathered} \tag{4.3.28}$$

We note that $|\gamma(t)| \leq c\mathrm{e}^{-\tilde{a}|t|}$, $|u_i(t)| \leq c\mathrm{e}^{-\tilde{a}|t|}$, $|v_i(t)| \leq c\mathrm{e}^{-\tilde{a}|t|}$, $i = 1,2,\cdots,d$ for constants $c > 0$, $\tilde{a} > 0$. Moreover, it holds that

$$\int_{-\infty}^{t} \mathrm{e}^{-\bar{b}\varepsilon^2(t-s)}\, ds = \frac{1}{\bar{b}\varepsilon^2}, \qquad \int_{-\infty}^{t} \mathrm{e}^{-\bar{b}\varepsilon^2(t-s)-\tilde{a}|s|}\, ds \leq \int_{-\infty}^{\infty} \mathrm{e}^{-\tilde{a}|s|}\, ds = 2/\tilde{a}\,.$$

Using this we see that (4.3.27b) can be written as

$$w(t) = \frac{\varepsilon^2}{6} \int_{-\infty}^{t} W_\varepsilon(t)W_\varepsilon(s)^{-1} D_{222}g_0(0,0)\left(w(s)^3 + 3w(s)^2 v_{\varepsilon,\alpha}(s)\right) ds + O(\varepsilon)\,.$$

Using the above assumptions and the Banach fixed point theorem 2.2.1 on a ball in $X_n \times X_m$ centered at 0, (4.3.27) has a solution $(u,w) \in X_n \times X_m$ for any sufficiently small ε so that $w = O(\varepsilon)$. Substituting $w = O(\varepsilon)$ and using $v_{\varepsilon,\alpha} \to v_\alpha$, $\frac{\partial}{\partial\alpha} v_{\varepsilon,\alpha} \to \frac{\partial}{\partial\alpha} v_\alpha$ as $\varepsilon \to 0_+$ uniformly in any compact interval of $\mathbb{R}$ and uniformly for α as well we can write (4.3.28) as

$$M_i(\mu_0,\alpha,\beta) + o(1) = 0, \quad i = 1,2,\ldots,d, \tag{4.3.29}$$

where

$$M_i(\mu_0,\alpha,\beta) = \frac{1}{2}\sum_{j,k=1}^{d-1} b_{ijk}\beta_j\beta_k + a_i(\alpha)\mu_{0,2} + \frac{1}{2}\int_{-\infty}^{\infty} \langle v_i(t), D_{22}f_0(\gamma(t),0)v_\alpha(t)^2\rangle\, dt$$

and

$$a_i(\alpha) = \int_{-\infty}^{\infty} \langle v_i(t), f_2(\gamma(t),0,0,t+\alpha)\rangle\, dt, \quad 1 \leq i \leq d;$$

$$b_{ijk} = \int_{-\infty}^{\infty} \langle v_i, D_{11}f_0(\gamma,0)u_j u_k\rangle\, dt, \quad \begin{cases} 1 \leq i \leq d, \\ 1 \leq j,k \leq d-1. \end{cases}$$

We note that $v_\alpha(t)$ depends on μ_0. We put

$$M(\mu_0,\alpha,\beta) = (M_1(\mu_0,\alpha,\beta), M_2(\mu_0,\alpha,\beta), \ldots, M_d(\mu_0,\alpha,\beta))\,.$$

If we suppose (α_0,β_0) are such that $M(\mu_0,\alpha_0,\beta_0) = 0$ and $D_{(\alpha,\beta)}M(\mu_0,\alpha_0,\beta_0)$ is nonsingular then we can solve (4.3.29) by using the implicit function theorem. This gives a bounded solution of (4.3.1). As detected in Section 4.1, we can show that this solution is transversal, i.e. the linearization of (4.3.1) along that solution has an exponential dichotomy on the whole line $\mathbb{R}$. In summary, we get the following result:

Theorem 4.3.7. *Assume that conditions* (i)–(viii) *are satisfied and* (ix)–(x) *hold. If there are* (μ_0,α_0,β_0) *so that* $\mu_{0,2} > 0$, $M(\mu_0,\alpha_0,\beta_0) = 0$ *and* $D_{(\alpha,\beta)}M(\mu_0,\alpha_0,\beta_0)$ *is nonsingular, then for* $\mu_1 = \varepsilon^3\mu_{0,1}$, $\mu_2 = \varepsilon^2\mu_{0,2}$ *with* $\varepsilon > 0$ *small, Equation* (4.3.1) *has a transverse bounded solution with the appropriate shift-type irregular dynamics.*

For $t \geq s$, and using V_ε from Lemma 4.3.4, the equation in condition (x) can be rewritten as

$$w(t) = V_\varepsilon(t)V_\varepsilon(s)^{-1}w(s) + \frac{\varepsilon^2}{2}\int_s^t V_\varepsilon(t)V_\varepsilon(z)^{-1}D_{222}g_0\,(0,0)\,v_{\varepsilon,\alpha}(z)^2 w(z)\,dz\,.$$

This implies

$$|w(t)| \leq B\mathrm{e}^{-b\varepsilon^2\mu_{0,2}(t-s)}\,|w(s)| + \frac{\varepsilon^2}{2}B\tilde{B}^2\|D_{222}g_0(0,0)\|\int_s^t \mathrm{e}^{-b\varepsilon^2\mu_{0,2}(t-z)}\,|w(z)|\,dz$$

which implies

$$|w(t)|\,\mathrm{e}^{b\varepsilon^2\mu_{0,2}(t-s)} \leq B|w(s)| + \frac{\varepsilon^2}{2}B\tilde{B}^2\|D_{222}g_0(0,0)\|\int_s^t \mathrm{e}^{b\varepsilon^2\mu_{0,2}(z-s)}\,|w(z)|\,dz\,.$$

The Gronwall inequality (cf Section 2.5.1 and [11]) gives

$$|w(t)|\,\mathrm{e}^{b\varepsilon^2\mu_{0,2}(t-s)} \leq B\mathrm{e}^{\varepsilon^2 B\tilde{B}^2\|D_{222}g_0(0,0)\|(t-s)/2}\,|w(s)|\,.$$

Consequently, we obtain

$$|w(t)| \leq B\mathrm{e}^{\varepsilon^2(B\tilde{B}\|D_{222}g_0(0,0)\|/2 - b\mu_{0,2})(t-s)}\,|w(s)|\,.$$

Now we see that condition (x) holds provided that

$$\frac{B\tilde{B}^2}{2b}\|D_{222}g_0(0,0)\| < \mu_{0,2}\,.$$

As an application we return to (4.3.4) which we write in the form

$$\begin{aligned}
\ddot{x} &= x - 2x(x^2+\xi y^2) - 2\mu_2\delta\dot{x} + a_3\mu_2\cos t + a_4\mu_2\sin t,\\
\ddot{y} &= -y - 2y(x^2+y^2) - 2\mu_2\dot{y} + a_1\mu_1\cos t + a_2\mu_1\sin t
\end{aligned} \tag{4.3.30}$$

for which we use the usual first order form $x_1 = x$, $x_2 = \dot{x}$, $y_1 = y$, $y_2 = \dot{y}$. That is, we make, as at the beginning of Section 4.3.2, the substitutions $\mu_1 \to a_1\mu_1$, $\mu_2 \to \mu_2$, $\mu_3 \to a_2\mu_1$, $\mu_4 \to a_3\mu_2$, $\mu_5 \to a_4\mu_2$ for some parameters a_i, $i = 1,2,3,4$. Then (4.3.30) becomes

$$\begin{aligned}
\dot{x}_1 &= x_2,\\
\dot{x}_2 &= x_1 - 2x_1(x_1^2+\xi y_1^2) + \mu_2\Big(-2\delta x_2 + a_3\cos t + a_4\sin t\Big),\\
\dot{y}_1 &= y_2,\\
\dot{y}_2 &= -y_1 - 2y_1(x_1^2+y_1^2) - 2\mu_2 y_2 + \mu_1\Big(a_1\cos t + a_4\sin t\Big)
\end{aligned}$$

which is clearly in the form of (4.3.1). We now check the hypotheses of Theorem 4.3.7 for (4.3.30). Conditions (i)–(viii) are easily verified.

In (ix) we write $v_{\varepsilon,\alpha} = (v, \dot{v})$ and then obtain

$$\ddot{v} + (1+2\gamma^2)v + 2\varepsilon^2\mu_{0,2}\dot{v} + 2\varepsilon^2 v^3 = \varepsilon^2\mu_{0,1}[a_1\cos(t+\alpha) + a_2\sin(t+\alpha)]. \quad (4.3.31)$$

Note that this is the second equation in (4.3.5) when $u = 0$ and $\mu_{0,1} = \mu_{0,2} = 1$.

Setting $\mu_{0,1} = \mu_{0,2} = 1$ (since we already have parameters a_i), using the solution $v_{\varepsilon,\alpha,a}(t)$ of (4.3.6) and substituting $v(t) = w(t) + v_{\varepsilon,\alpha,a}(t)$ into (4.3.31), we get

$$\ddot{w} + (1+6\varepsilon^2 v_{\varepsilon,\alpha,a}^2 + 2\gamma^2)w + 2\varepsilon^2\dot{w} + 2\gamma^2 v_{\varepsilon,\alpha,a} + 6\varepsilon^2 w^2 v_{\varepsilon,\alpha,a} + 2\varepsilon^2 w^3 = 0 \quad (4.3.32)$$

which is (4.3.10b) when $u = 0$. Equation (4.3.32) can be rewritten as (4.3.16b) with $u = 0$ and then as (4.3.17). Taking $0 < c_5(a) < k_3$ the conditions of (4.3.19) are satisfied and we obtain the unique solvability of (4.3.32) with solution $w_{\varepsilon,\alpha,a}(t)$ satisfying $\|w_{\varepsilon,\alpha,a}\| \leq r_0$. Consequently, condition (ix) is verified for (4.3.30) with $v_{\varepsilon,\alpha} = (v, \dot{v})$ and $v_\alpha = (\tilde{v}, \dot{\tilde{v}})$ where

$$\begin{aligned} v(t) &= v_{\varepsilon,\alpha,a}(t) + w_{\varepsilon,\alpha,a}(t), \\ \tilde{v}(t) &= a_1\cos(t+\alpha) + a_2\sin(t+\alpha) + z_{01}(t). \end{aligned}$$

Concerning condition (x), we see that the equation from this condition has the form

$$\ddot{w}_1 + (1+2\gamma^2 + 6\varepsilon^2 v^2)w_1 + 2\varepsilon^2\mu_{0,2}\dot{w}_1 = 0$$

with $w_2 = \dot{w}_1$. Again using $\mu_{0,1} = \mu_{0,2} = 1$ and substituting for v we get

$$\ddot{w}_1 + (1+2\gamma^2 + 6\varepsilon^2 v_{\varepsilon,\alpha,a}^2)w_1 + 2\varepsilon^2\dot{w}_1 + 6\varepsilon^2(2v_{\varepsilon,\alpha,a}w_{\varepsilon,\alpha,a} + w_{\varepsilon,\alpha,a}^2)w_1 = 0,$$

which for $t \geq s$ has the form

$$\begin{aligned} w(t) = W_{\varepsilon,\alpha,a}(t)W_{\varepsilon,\alpha,a}(s)^{-1}w(s) - 6\varepsilon^2\int_{-\infty}^t W_{\varepsilon,\alpha,a}(t)W_{\varepsilon,\alpha,a}(s)^{-1}\cdot \\ \left\{\Big(0, (2v_{\varepsilon,\alpha,a}(z)w_{\varepsilon,\alpha,a}(z) + w_{\varepsilon,\alpha,a}(z)^2)w_1(z)\Big)\right\}dz. \end{aligned} \quad (4.3.33)$$

Since $\|v_{\varepsilon,\alpha,a}\| \leq c_5(a) + O(\varepsilon)$ and $\|w_{\varepsilon,\alpha,a}\| \leq r_0$, we get

$$\left|\Big(0, -6\varepsilon^2(2v_{\varepsilon,\alpha,a}(s)w_{\varepsilon,\alpha,a}(s) + w_{\varepsilon,\alpha,a}(s)^2)\Big)\right| \leq \varepsilon^2\theta_\varepsilon,$$

for a constant

$$\theta_\varepsilon = 6(2c_5(a)r_0 + r_0^2) + O(\varepsilon).$$

From (4.3.33) we obtain

$$|w(t)| \leq K_1\,\mathrm{e}^{-\varepsilon^2(t-s)}\,|w(s)| + K_1\varepsilon^2\theta_\varepsilon\int_s^t \mathrm{e}^{-\varepsilon^2(t-z)}\,|w(z)|\,dz$$

which gives

$$|w(t)|\,\mathrm{e}^{\varepsilon^2(t-s)} \leq K_1|w(s)| + K_1\varepsilon^2\theta_\varepsilon \int_s^t \mathrm{e}^{\varepsilon^2(z-s)}\,|w(z)|\,dz\,.$$

The Gronwall inequality again implies

$$|w(t)|\,\mathrm{e}^{\varepsilon^2(t-s)} \leq K_1|w(s)|\,\mathrm{e}^{K_1\varepsilon^2\theta_\varepsilon(t-s)}\,.$$

Since $c_5(a) < k_3$, we see that $K_1\theta_0 < 1$ and then

$$|w(t)| \leq K_1\,\mathrm{e}^{\varepsilon^2(K_1\theta_0-1)(t-s)/2}\,|w(s)|\,.$$

Hence we see that condition (x) is satisfied with $\bar{B} = K_1$ and $\bar{b} = (1-K_1\theta_0)/2$.
In summary, conditions (ix) and (x) are satisfied for (4.3.4).

Remark 4.3.8. The role of resonance is not clear in this section. But it is essential and it is hidden in assumptions (ix) and (x). For simplicity, we explain it again for example (4.3.4) by replacing the forcing terms $\cos t$, $\sin t$ with $\cos\pi t$, $\sin\pi t$, respectively. So we consider the equations

$$\begin{aligned}\ddot{x} &= x - 2x(x^2+\xi y^2) - 2\mu_2\delta\dot{x} + \mu_4\cos\pi(t+\alpha) + \mu_5\sin\pi(t+\alpha)\,,\\ \ddot{y} &= -y - 2y(x^2+y^2) - 2\mu_2\dot{y} + \mu_1\cos\pi(t+\alpha) + \mu_3\sin\pi(t+\alpha)\,.\end{aligned} \tag{4.3.34}$$

Certainly, the linear part of the second equation in (4.3.34) is nonresonant. Then in place of (4.3.6), we get

$$\ddot{\tilde{v}}_{\varepsilon,\alpha,a} = -\tilde{v}_{\varepsilon,\alpha,a} - 2\varepsilon^2\dot{\tilde{v}}_{\varepsilon,\alpha,a} - 2\varepsilon^2\tilde{v}^3_{\varepsilon,\alpha,a} + \varepsilon^2 a_1\cos\pi(t+\alpha) + \varepsilon^2 a_2\sin\pi(t+\alpha)\,.$$

Applying the method of Section 4.3.2, we obtain $\tilde{v}_{\varepsilon,\alpha,a}(t) = O(\varepsilon^2)$ and $\tilde{v}_{\varepsilon,\alpha,a}(t)$ is 2-period. Then (4.3.16b) gives $\tilde{w}_{\varepsilon,\alpha,a}(t) = O(\varepsilon^2)$ without any further restriction, i.e. a_1, a_2 are arbitrary nonzero. Consequently, the corresponding Melnikov function is independent of a_1, a_2. So the hyperbolic and center parts of (4.3.34) are always separated. This is consistent with the method in Section 4.2 for the nonresonant case. In summary, in the nonresonant case, the forcing terms in the center part do not affect the Melnikov function, while in the resonant case the forcing terms in center part do affect it in general.

4.3.4 Averaging Method

When Eq. (4.3.1) satisfies conditions (i)–(viii) the remaining task is to verify conditions (ix) and (x). We note that the equation in (x) is just the linearization of equation (ix) along $v_{\varepsilon,\alpha}(t)$. Consequently, we must study the equation of (ix) and its linearization. For this purpose, we can use also the method of averaging [16] (cf Section 2.5.7). As a concrete illustration of how this can be done we focus on (4.3.31). Using the matrix $C(t)$ from Section 4.3.2, we put

$$v(t) = c_1(t)v_1(t) + c_2(t)v_2(t)\,,$$
$$v_1(t) = \cos t - \sin t \tanh t, \quad v_2(t) = \sin t + \cos t \tanh t$$

into (4.3.31) and set $\mu_{0,1} = \mu_{0,2} = 1$. We get the system

$$\begin{aligned}\dot c_1 &= \varepsilon^2\Big[c_1\dot v_1(t) + c_2\dot v_2(t) + (c_1v_1(t) + c_2v_2(t))^3\\&\quad -\frac{a_1}{2}\cos(t+\alpha) - \frac{a_2}{2}\sin(t+\alpha)\Big]v_2(t),\\\dot c_2 &= \varepsilon^2\Big[-c_1\dot v_1(t) - c_2\dot v_2(t) - (c_1v_1(t) + c_2v_2(t))^3\\&\quad +\frac{a_1}{2}\cos(t+\alpha) + \frac{a_2}{2}\sin t+\alpha)\Big]v_1(t),\end{aligned} \tag{4.3.35}$$

where as usual we put $\dot v(t) = c_1(t)\dot v_1(t) + c_2(t)\dot v_2(t)$. Now we see that

$$v_i(t) \to v_{i,\pm}(t), \quad i = 1,2,$$

being exponentially fast as $t \to \pm\infty$ where

$$v_{1,\pm} = \cos t \mp \sin t, \quad v_{2,\pm}(t) = \sin t \pm \cos t\,.$$

Consequently, Equation (4.3.35) for $t \geq 0$ has the form

$$\begin{aligned}\dot c_1 &= \varepsilon^2\Big\{\Big(c_1\dot v_{1,+}(t) + c_2\dot v_{2,+}(t) + (c_1v_{1,+}(t) + c_2v_{2,+}(t))^3\\&\quad -\frac{a_1}{2}\cos(t+\alpha) - \frac{a_2}{2}\sin(t+\alpha)\Big)v_{2,+}(t) + h_+^1(c_1,c_2,\alpha,t)\Big\},\\\dot c_2 &= \varepsilon^2\Big\{\Big(-c_1\dot v_{1,+}(t) - c_2\dot v_{2,+}(t) - (c_1v_{1,+}(t) + c_2v_{2,+}(t))^3\\&\quad +\frac{a_1}{2}\cos(t+\alpha) + \frac{a_2}{2}\sin(t+\alpha)\Big)v_{1,+}(t) + h_+^2(c_1,c_2,\alpha,t)\Big\}\end{aligned} \tag{4.3.36}$$

while Eq. (4.3.35) for $t \leq 0$ has the form

$$\begin{aligned}\dot c_1 &=\varepsilon^2\Big\{\Big(c_1\dot v_{1,-}(t) + c_2\dot v_{2,-}(t) + (c_1v_{1,-}(t) + c_2v_{2,-}(t))^3\\&\quad -\frac{a_1}{2}\cos(t+\alpha) - \frac{a_2}{2}\sin(t+\alpha)\Big)v_{2,-}(t) + h_-^1(c_1,c_2,\alpha,t)\Big\},\\\dot c_2 &=\varepsilon^2\Big\{\Big(-c_1\dot v_{1,-}(t) - c_2\dot v_{2,-}(t) - (c_1v_{1,-}(t) + c_2v_{2,-}(t))^3\\&\quad +\frac{a_1}{2}\cos(t+\alpha) + \frac{a_2}{2}\sin(t+\alpha)\Big)v_{1,-}(t) + h_-^2(c_1,c_2,\alpha,t)\Big\}\end{aligned} \tag{4.3.37}$$

where $h_\pm^{1,2}(c_1,c_2,\alpha,t) \to 0$, being exponentially fast for $t \to \pm\infty$ and uniformly for $c_{1,2}$ on a bounded set. Now we average Eqs. (4.3.36) and (4.3.37) over $\mathbb{R}_\pm$, respectively, to get for $t \geq 0$ the system

$$\begin{aligned} \dot{c}_1 &= \frac{\varepsilon^2}{4}\Big(-4c_1+6c_1^2c_2+6c_2^3-(a_1+a_2)\cos\alpha+(a_1-a_2)\sin\alpha\Big),\\ \dot{c}_2 &= \frac{\varepsilon^2}{4}\Big(-4c_2-6c_1c_2^2-6c_1^3+(a_1-a_2)\cos\alpha+(a_1+a_2)\sin\alpha\Big), \end{aligned} \tag{4.3.38}$$

while for $t \leq 0$ we obtain the system

$$\begin{aligned} \dot{c}_1 &= \frac{\varepsilon^2}{4}\Big(-4c_1+6c_1^2c_2+6c_2^3+(a_1-a_2)\cos\alpha+(a_1+a_2)\sin\alpha\Big),\\ \dot{c}_2 &= \frac{\varepsilon^2}{4}\Big(-4c_2-6c_1c_2^2-6c_1^3+(a_1+a_2)\cos\alpha+(a_2-a_1)\sin\alpha\Big). \end{aligned} \tag{4.3.39}$$

We put

$$\begin{aligned} A_{1,+} &= -(a_1+a_2)\cos\alpha+(a_1-a_2)\sin\alpha\,,\\ A_{2,+} &= (a_1-a_2)\cos\alpha+(a_1+a_2)\sin\alpha\,,\\ A_{1,-} &= (a_1-a_2)\cos\alpha+(a_1+a_2)\sin\alpha\,,\\ A_{2,-} &= (a_1+a_2)\cos\alpha+(a_2-a_1)\sin\alpha\,. \end{aligned}$$

The systems (4.3.38) and (4.3.39) form one system over $\mathbb{R}$ with a discontinuity at $t=0$. By using arguments of Section 4.3.2 (see (4.3.8)), we observe that the systems

$$\begin{aligned} -4c_1+6c_1^2c_2+6c_2^3+A_{1,\pm} &= 0\,,\\ -4c_2-6c_1c_2^2-6c_1^3+A_{2,\pm} &= 0 \end{aligned} \tag{4.3.40}$$

have unique solutions

$$c_{a,\pm} = (c_{1,a,\pm}, c_{2,a,\pm})\,.$$

Moreover, the eigenvalues of the linearization of (4.3.38), (4.3.39) at $c_{a,\pm}$ are

$$[-4 \pm \imath 6\sqrt{3}(c_{1,a,\pm}^2 + c_{2,a,\pm}^2)]\varepsilon^2/4\,.$$

Consequently, we see that systems (4.3.38), (4.3.39) have unique weakly exponentially attracting equilibria $c_{a,\pm}$, respectively.

Note that for $a=0$ we get $c_{0,\pm}=0$ and then from (4.3.31) $v_{\varepsilon,\alpha}=0$ so the case $a=0$ is trivial. On the other hand, we need $v_{\varepsilon,\alpha}\neq 0$ for the influence of the center part to affect the Melnikov function. For this reason, we assume that $a\neq 0$.

Now if the point $c_{a,-}$ is in the basin of attraction of $c_{a,+}$, then we can construct a solution $c_a(t)$ of (4.3.38), (4.3.39) over $\mathbb{R}$ as follows:

$$c_a(t) = \begin{cases} c_{a,-}, & \text{for} \quad t\leq 0\,,\\ \text{the solution of (4.3.38) starting from } c_{a,-} & \text{for} \quad t\geq 0\,. \end{cases}$$

This solution will generate, according to averaging theory [16] (cf Theorems 2.5.12, 2.5.13), a solution of (4.3.31) satisfying conditions (ix) and (x). We note that averaging theory can be applied to (4.3.36) and (4.3.37) since they are sums of periodic and exponentially fast decaying terms containing t variable. So (4.3.36) and (4.3.37) are KBM-vector fields.

To show that $c_{a,-}$ is in the basin of attraction of $c_{a,+}$ consider the function $H: \mathbb{R}^2 \to \mathbb{R}$ given by

$$H(c_1,c_2) = 3(c_1^2+c_2^2)^2 - 2A_{2,+}c_1 + 2A_{1,+}c_2.$$

For further reference we note that

$$H(c_1,c_2) \le 3(c_1^2+c_2^2)^2 + 2\sqrt{A_{1,+}^2+A_{2,+}^2}\sqrt{c_1^2+c_2^2}, \tag{4.3.41}$$

and if $t \to (c_1(t),c_2(t))$ is a solution of (4.3.38),

$$\begin{aligned}\frac{d}{dt}H(c_1(t),c_2(t)) &= -2\varepsilon^2\left[6(c_1^2+c_2^2)^2 - A_{2,+}c_1 + A_{1,+}c_2\right]\\ &\le -2\varepsilon^2\sqrt{c_1^2+c_2^2}\left[6(c_1^2+c_2^2)^{3/2} - \sqrt{A_{1,+}^2+A_{2,+}^2}\right].\end{aligned} \tag{4.3.42}$$

We define two sets

$$\begin{aligned}D &= \left\{(c_1,c_2)\,\middle|\, c_1^2+c_2^2 < (A_{1,+}^2+A_{2,+}^2)^{1/3}\right\},\\ U &= \left\{(c_1,c_2)\,\middle|\, H(c_1,c_2) < 5(A_{1,+}^2+A_{2,+}^2)^{2/3}\right\}.\end{aligned}$$

Using (4.3.41) it is easy to verify that $D \subset U$. With (4.3.40) we obtain

$$\sqrt{A_{1,+}^2+A_{2,+}^2}\sqrt{c_{1,a,+}^2+c_{2,a,+}^2} \ge A_{2,+}c_{1,a,+} - A_{1,+}c_{2,a,+} = 6(c_{1,a,+}^2+c_{2,a,+}^2)^2$$

from which it follows that $|c_{a,+}|^2 \le (\frac{1}{6})^{2/3}(A_{1,+}^2+A_{2,+}^2)^{1/3}$ so that $c_{a,+} \in U$.

If $t \to (c_1(t),c_2(t))$ is an orbit of (4.3.38) in the complement of $\bar{U}$ then

$$c_1(t)^2 + c_2(t)^2 \ge (A_{1,+}^2+A_{2,+}^2)^{1/3}$$

and it follows from (4.3.42) that

$$\frac{d}{dt}H(c_1(t),c_2(t)) \le -10\varepsilon^2(A_{1,+}^2+A_{2,+}^2)^{2/3}.$$

Thus, $\bar{U}$ is an invariant global attractor. Since the divergence of (4.3.38) is $-2\varepsilon^2$, using Bendixson's criterion 2.5.10, we see that U contains no periodic orbits. Thus by the Poincarè-Bendixson theorem 2.5.9, U is in the basin of attraction for $c_{a,+}$, $c_{a,+}$ is a global attractor and, trivially, $c_{a,-}$ is in the basin of attraction of $c_{a,+}$.

In summary, we get the Melnikov function $M(\alpha)$ of Section 4.3.2 so that Theorem 4.3.3 holds for any $(a_1,a_2) \neq (0,0)$ and we have the following improvement of Theorem 4.3.3.

Theorem 4.3.9. *Equation* (4.3.4) *has a transverse homoclinic solution for any* ξ, *and any small* μ_i, $i = 1,\ldots,5$ *and* δ *satisfying condition* (4.3.24) *and* $(\mu_1, \mu_3) \neq (0,0)$.

Finally, we note that in spite of the fact that the results of Section 4.3.2 are improved in this section, that part is included here since it contains some useful derivations/computations such as the existence of periodic solutions and exponential dichotomies. We note that for general forms of coupled oscillators only local analysis as in Section 4.3.2 can be used to verify assumptions (ix) and (x). As our averaging technique uses the Poincarè-Bendixson theorem and Bendixson's criterion it cannot be used for higher-dimensional systems. In general, the situation depends on the form of the averaged equations.

4.4 Singularly Perturbed and Forced ODEs

4.4.1 Forced Singular ODEs

Consider a singular system of ODEs like

$$\begin{aligned} \varepsilon u' &= f(u,v) + \varepsilon h_1(t,u,v,\varepsilon), \quad u \in \mathbb{R}^n, \quad v \in \mathbb{R}^m, \\ v' &= g(u,v) + \varepsilon h_2(t,u,v,\varepsilon), \quad t \in \mathbb{R}, \quad \varepsilon \in \mathbb{R}, \end{aligned} \tag{4.4.1}$$

under the following conditions:

(a) f, g, h_1, h_2 are C_b^{r+1}-functions in their arguments, $r \geq 2$, defined for $(t,u,v,\varepsilon) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times (-\bar{\varepsilon}, \bar{\varepsilon})$ and their $(r+1)$-derivatives are continuous in u uniformly with respect to (t,v,ε).
(b) $f(0,v) = 0$ for any $v \in \mathbb{R}^m$ and there exists $\delta > 0$ so that for any $v \in \mathbb{R}^m$ and $\lambda(v) \in \sigma(f_x(0,v))$ one has $|\Re\lambda(v)| > \delta > 0$.

Then setting $\varepsilon = 0$ in Eq. (4.4.1) we obtain the so-called *degenerate system*

$$v' = g(0,v), \quad v \in \mathbb{R}^m. \tag{4.4.2}$$

It was shown in [17] that given $T > 0$ the solutions of (4.4.1) are at a $O(\varepsilon)$-distance from the corresponding solutions of (4.4.2), for t in any compact subset of $(0,T]$. This result was improved in [18] leading to a condition similar to the above one about the eigenvalues of $f_u(0,v)$ [19]. Later, a geometric theory of singular systems was developed in [20]. This theory applies to the autonomous case and states, under certain hypotheses, the existence of a *center manifold* for (4.4.1) defined on compact subsets of $\mathbb{R}^m$ on which system (4.4.1) is a regular perturbation of the degenerate system (4.4.2). By means of this theory, a previous result given in [21] was improved in [20], concerning the existence of periodic solutions of (4.4.1). Afterwards geometric theory is used in [22, 23] to study the problem of bifurcation from a heteroclinic orbit of the degenerate system towards a heteroclinic orbit of the overall system (4.4.1). However, since the result of [20] holds in the autonomous case and with some roughness assumptions on system (4.4.2), conclusions in [22,23] are given just in the case of a *transverse heteroclinic orbit*. Later, using different methods, the non-autonomous case together with the homoclinic case have been handled in [24,25]. A result in [25], however, does not contain any conclusion of the smoothness of the bifurcating heteroclinic orbit with respect to the parameter ε, while four classes of differentiability (from C^{r+2} to C^{r-2}) are lost in [24]. Let us mention some related results in this direction. Attractive invariant manifolds of (4.4.1) are studied in [26] when h_1, h_2 are independent of t and $f_u(0,v)$ has all the eigenvalues with negative real parts. The same problem as in [26] is investigated in [27] when h_1, h_2 do depend on t.

4.4.2 Center Manifold Reduction

In this section we apply Theorem 2.5.8 to (4.4.1). Let $\tau = t/\varepsilon$ be the *fast time* and $\dot{}$ denote the derivative with respect to τ. Then (4.4.1) reads:

$$\begin{aligned} \dot{u} &= f(u,v) + \varepsilon h_1(t,u,v,\varepsilon), \\ \dot{v} &= \varepsilon\{g(u,v) + \varepsilon h_2(t,u,v,\varepsilon)\}, \\ \dot{t} &= \varepsilon. \end{aligned} \tag{4.4.3}$$

Take a C^∞-function $\phi : \mathbb{R} \to [0,\bar{\varepsilon}]$ so that $\phi(\varepsilon) = \bar{\varepsilon}$ for $\varepsilon \in (-\frac{\bar{\varepsilon}}{3}, \frac{\bar{\varepsilon}}{3})$, $|\frac{d\phi}{d\varepsilon}| < 2$ and $\operatorname{supp}\phi \subset [-\bar{\varepsilon}, \bar{\varepsilon}]$. It is clear that $\phi \in C_b^{r+1}(\mathbb{R},\mathbb{R})$ since it has a compact support. Then, define $x = u, y = (v,t,\varepsilon\phi(\varepsilon))$ and consider, instead of (4.4.3), the following system

$$\begin{aligned} \dot{x} &= f_u(0,v)x + F(x,y) := A(y)x + F(x,y), \\ \dot{y} &= G(x,y), \end{aligned} \tag{4.4.4}$$

where

$$F(x,y) = F(x,(v,t,\varepsilon)) = f(x,v) - f_u(0,v)x + \varepsilon\phi(\varepsilon)h_1(t,x,v,\varepsilon\phi(\varepsilon)),$$

$$G(x,y) = G(x,(v,t,\varepsilon)) = \varepsilon\phi(\varepsilon)(g(x,v) + \varepsilon\phi(\varepsilon)h_2(t,x,v,\varepsilon\phi(\varepsilon)),1,0).$$

From the fact that the support of $\phi(\varepsilon)$ is a subset of $[-\bar{\varepsilon},\bar{\varepsilon}]$, it follows that $A(y)$, $F(x,y)$, $G(x,y)$ can be considered as C_b^r-functions in $(x,y) \in \mathbb{R}^n \times \mathbb{R}^{m+2}$ and that they satisfy the hypothesis (i) of Section 2.5.5. Moreover one has

$$|F(0,y)| + |F_x(0,y)| \le C|\varepsilon\phi(\varepsilon)| \le C\bar{\varepsilon}^2 < \sigma$$

provided $\bar{\varepsilon} \ll 1$. In the same way we see that $|G(x,y)|, |G_x(x,y)| < \sigma$. As regards the inequality $|G_y(x,y)| < \sigma$, this follows also from the fact that $\sup_{\varepsilon\in\mathbb{R}} |\frac{d}{d\varepsilon}[\varepsilon\phi(\varepsilon)]|$

$\leq \sup_{|\varepsilon|\leq\bar{\varepsilon}} |\varepsilon\phi'(\varepsilon)| + |\phi(\varepsilon)| \leq 3\bar{\varepsilon}$. All the hypotheses of Theorem 2.5.8 are then satisfied and hence the existence of a *global center manifold* for (4.4.4), satisfying the conclusions of Theorem 2.5.8, follows. This center manifold can be represented as:

$$\mathscr{C} = \left\{(\xi,\eta,\alpha,\varepsilon) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times (-\bar{\varepsilon},\bar{\varepsilon}) \mid \xi = H(\eta,\alpha,\varepsilon)\right\}$$

and is invariant under the flow given by (4.4.4). From $\frac{d\varepsilon}{d\tau} = 0$ we obtain that ε is constant, moreover, since any $\varepsilon \in (-\frac{\bar{\varepsilon}^2}{3}, \frac{\bar{\varepsilon}^2}{3})$ can be written as $\frac{\varepsilon}{\bar{\varepsilon}}\phi\left(\frac{\varepsilon}{\bar{\varepsilon}}\right)$, we see that for $|\varepsilon| < \varepsilon_0 = \frac{\bar{\varepsilon}^2}{3}$, such a manifold is invariant for (4.4.3). Any solution of (4.4.3) whose u-component is small must then satisfy (see property (P) of Theorem 2.5.8):

$$u(\tau) = H(y(\tau,\eta,\alpha,\varepsilon)),$$

where $y(\tau,\eta,\alpha,\varepsilon) = (v(\tau,\eta,\alpha,\varepsilon), \varepsilon\tau+\alpha, \varepsilon)$ and $v(\tau) = v(\tau,\eta,\alpha,\varepsilon)$ satisfies

$$\dot{v}(\tau) = \varepsilon\{g(H(v(\tau),\varepsilon\tau+\alpha,\varepsilon),v(\tau)) + \varepsilon h_2(\varepsilon\tau+\alpha, H(v(\tau),\varepsilon\tau+\alpha,\varepsilon),v(\tau),\varepsilon)\}$$

so that $\widetilde{v}(t) = v(t/\varepsilon)$ satisfying

$$\widetilde{v}'(t) = g(H(\widetilde{v}(t),t+\alpha,\varepsilon),\widetilde{v}(t)) + \varepsilon h_2(t+\alpha, H(\widetilde{v}(t),t+\alpha,\varepsilon),\widetilde{v}(t),\varepsilon). \tag{4.4.5}$$

Finally, note that $H(\eta,\alpha,0) = 0$ because of uniqueness. We have then shown the following.

Theorem 4.4.1. *Consider system* (4.4.1) *and assume (a) and (b) hold. Then there exist* $\varepsilon_0, \rho > 0$ *and a* C^r*-function* $H : \mathbb{R}^m \times \mathbb{R} \times (-\varepsilon_0,\varepsilon_0) \to \mathbb{R}$ *so that the following properties hold:*

(i) $\sup_{(\eta,\alpha,\varepsilon)\in\mathbb{R}^m\times\mathbb{R}\times(-\varepsilon_0,\varepsilon_0)} |H(\eta,\alpha,\varepsilon)| \leq \rho.$

(ii) *For any* $\varepsilon \in (-\varepsilon_0,\varepsilon_0)$ *and* $\alpha \in \mathbb{R}$ *the manifold*

$$\mathscr{C}_{\alpha,\varepsilon} = \left\{(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^m \mid \xi = H(\eta,\alpha,\varepsilon)\right\}$$

is invariant for the flow of system (4.4.1)*, with* $t+\alpha$ *instead of* t*, in the sense that if* $(u(\alpha),v(\alpha)) \in \mathscr{C}_{\alpha,\varepsilon}$ *then* $(u(t),v(t)) \in \mathscr{C}_{\alpha,\varepsilon}$ *for any* $t \in \mathbb{R}$.

(iii) *Any solution* $(u(t),v(t))$ *of* (4.4.1)*, with* $t+\alpha$ *instead of* t*, showing that* $||u||_\infty < \rho$*, belongs to* $\mathscr{C}_{\alpha,\varepsilon}$.

As an example of application of this result assume that

(c) The degenerate system (4.4.2) has an orbit $\gamma(t)$ homoclinic to a hyperbolic equilibrium, and the variational system $\dot{v} = g_v(0,\gamma(t))v$ has the unique bounded solution $\dot{\gamma}(t)$ (up to a multiplicative constant).

Then the following theorem holds:

Theorem 4.4.2. *Assume (a), (b), (c) and define*

$$\Delta(\alpha) = \int_{-\infty}^{+\infty} \psi^*(t) \left\{ h_2(t+\alpha,0,\gamma(t),0) - g_u(0,\gamma(t)) f_u(0,\gamma(t))^{-1} h_1(t+\alpha,0,\gamma(t),0) \right\} dt$$

with $\psi^*(t)$ being the unique (up to a multiplicative constant) bounded solution to the adjoint variational system $\dot{v} = -g_v(0,\gamma(t))^* v$. Then, if $\Delta(\alpha)$ has a simple zero at $\alpha = \alpha_0$, there exist $\rho > 0$, $\varepsilon_0 > 0$ so that for $|\varepsilon| < \varepsilon_0$, system (4.4.1) *has a unique solution $(u(t,\varepsilon), v(t,\varepsilon))$ which is C^{r-1} with respect to ε, bounded together with its derivatives (in ε), and satisfying also*:

$$|u(t,\varepsilon)| < \rho \text{ and } \sup_{t\in\mathbb{R}} |u(t,\varepsilon)| + |v(t,\varepsilon) - \gamma(t-\alpha_0)| \to 0 \text{ as } \varepsilon \to 0. \tag{4.4.6}$$

Proof. A solution satisfying (4.4.6) must lie in a manifold $\mathscr{C}_{\alpha,\varepsilon}$ owing to property (iii) of Theorem 4.4.1, hence its v-component must satisfy (4.4.5). The unperturbed system of (4.4.5) is the degenerate system (4.4.2). From regular perturbation theory (see Section 4.1) we obtain the Melnikov function

$$M(\alpha) = \int_{-\infty}^{+\infty} \psi^*(t) \left\{ h_2(t+\alpha,0,\gamma(t),0) + g_u(0,\gamma(t)) H_\varepsilon(\gamma(t),t+\alpha,0) \right\} dt.$$

Taking the derivative with respect to ε at $\varepsilon = 0$ of

$$\begin{aligned} \varepsilon \frac{\partial}{\partial t} H\ (v(t,\eta_0,\alpha,\varepsilon), t+\alpha,\varepsilon) \\ = f(H(v(t,\eta_0,\alpha,\varepsilon),t+\alpha,\varepsilon), v(t,\eta_0,\alpha,\varepsilon)) \\ +\varepsilon h_1(t+\alpha, H(v(t,\eta_0,\alpha,\varepsilon),t+\alpha,\varepsilon), v(t,\eta_0,\alpha,\varepsilon),\varepsilon), \end{aligned}$$

we get (recall $H(\eta,\alpha,0) = 0$)

$$f_u(0,v(t,\eta_0,\alpha,0)) H_\varepsilon(v(t,\eta_0,\alpha,0),t+\alpha,0) + h_1(t+\alpha,0,v(t,\eta_0,\alpha,0),0) = 0. \tag{4.4.7}$$

Now $v(t,\gamma(\alpha),\alpha,0)$ solves (4.4.2) with the condition $v(0) = \gamma(\alpha)$, as a consequence $v(t,\gamma(\alpha),\alpha,0) = \gamma(t)$ and using (4.4.7) we obtain:

$$H_\varepsilon(\gamma(t),t+\alpha,0) = -f_u(0,\gamma(t))^{-1} h_1(t+\alpha,0,\gamma(t),0)\} dt$$

and hence $M(\alpha) = \Delta(\alpha)$. □

Remark 4.4.3. From regular perturbation theory, it follows that the solution, whose existence is stated in Theorem 4.4.2, is C^{r-1} in ε. This improves previous results [24, 25].

As another application of Theorem 4.4.1, the degenerate system (4.4.2) has an orbit heteroclinic to semi-hyperbolic equilibria, but we do not go into details and we refer the readers to [28].

4.4.3 ODEs with Normal and Slow Variables

Only for the reader information, we note in this part an opposite case to (4.4.1) by considering a system

$$\begin{aligned}\dot{x} &= f(x,y) + \varepsilon h(x,y,t,\varepsilon)\,,\\ \dot{y} &= \varepsilon\,(Ay + g(y) + p(x,y,t,\varepsilon) + \varepsilon q(y,t,\varepsilon))\,,\end{aligned} \tag{4.4.8}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\varepsilon > 0$ is sufficiently small, A is an $m \times m$ matrix, and all mappings are smooth, 1-periodic in the time variable $t \in \mathbb{R}$ so that

(i) $f(0,0) = 0$, $g(0) = 0$, $g_x(0) = 0$, $p(0,\cdot,\cdot,\cdot) = 0$.
(ii) The eigenvalues of A and $f_x(0,0)$ lie off the imaginary axis.
(iii) There is a homoclinic solution $\gamma \neq 0$ so that $\lim_{t\to\pm\infty} \gamma(t) = 0$ and $\dot{\gamma}(t) = f(\gamma(t),0)$.

Here g_x, f_x mean derivatives of g and f with respect to x, respectively. The second equation of (4.4.8) has the usual canonical form of the averaging theory (cf Section 2.5.7) in the variable y with $x = 0$, and it is assumed [29] that its averaged equation with $x = 0$ possesses a hyperbolic equilibrium. Hence the homoclinic dynamics of the first equation of (4.4.8) is combined with the dynamics near the slow hyperbolic equilibrium of the averaged second equation of (4.4.8) when $x = 0$. Moreover, the transversality of bounded solutions on $\mathbb{R}$ of (4.4.8) is studied for the sufficiently small parameter $\varepsilon > 0$. Consequently, as a by-product chaotic behavior of (4.4.8) is shown for such ε in [29]. Systems of ODEs with normal and slow variables are investigated also in [30, 31].

Systems like (4.4.8) occur in certain weakly coupled systems. More general ODEs are studied in [32–37], and we refers the readers for further details to these papers.

4.4.4 Homoclinic Hopf Bifurcation

Finally we note that the method of Section 4.4.3 can be applied to systems of ODEs representing an interaction of the homoclinic and Hopf bifurcation, which are given by

$$\begin{aligned}\dot{x} &= f_1(x) + h_1(x,y,\lambda)\,,\\ \dot{y} &= f_2(y,\lambda) + \lambda h_2(x,y,\lambda) + h_3(x,y)\,,\end{aligned} \tag{4.4.9}$$

where $f_1 : \mathbb{R}^n \to \mathbb{R}^n$, $f_2 : \mathbb{R}^3 \to \mathbb{R}^2$, $h_1 : \mathbb{R}^{n+3} \to \mathbb{R}^n$, $h_2 : \mathbb{R}^{n+3} \to \mathbb{R}^2$, $h_3 : \mathbb{R}^{n+2} \to \mathbb{R}^2$ are smooth so that

(i) $f_2(0,\cdot) = 0$, $Df_2(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
(ii) $f_1(0) = 0$ and the eigenvalues of $Df_1(0)$ lie off the imaginary axis.
(iii) There is a homoclinic solution $\gamma \neq 0$ so that $\lim_{t\to\pm\infty} \gamma(t) = 0$ and $\dot{\gamma}(t) = f_1(\gamma(t))$.

(iv) $h_1(\cdot,0,0) = 0$, $h_2(0,\cdot,\cdot) = 0$, $h_3(0,\cdot) = 0$, $h_3(\cdot,0) = 0$.

The system (4.4.9) is an autoparametric system [38–40] consisting of two subsystems: Oscillator and Excited System.The Oscillator which is vibrating according to its nature is given by the second equation of (4.4.9) in the variable y possessing the *Hopf singularity* at $y = 0$ for $\lambda = 0$, $x = 0$ [41]. The Excited System is determined by the first equation of (4.4.9) in the variable x exhibiting a homoclinic structure to the equilibrium $x = 0$ for $\lambda = 0$, $y = 0$. (4.4.9) has for $\lambda = 0$ a semi–trivial solution $x = \gamma$, $y = 0$. Either chaotic or at least periodic dynamics of (4.4.9) near $\gamma \times \{0\}$ for $\lambda \neq 0$ sufficiently small is studied in [42], and we refer the readers to this paper for more details. We note that $x = 0$, $y = 0$ is a nonhyperbolic equilibrium of (4.4.9) for $\lambda = 0$ possessing a homoclinic loop $x = \gamma$, $y = 0$. Related research work is presented in [32, 34, 37, 43].

4.5 Bifurcation from Degenerate Homoclinics

4.5.1 Periodically Forced ODEs with Degenerate Homoclinics

In this section, we consider ODEs of the form

$$\dot{x} = f(x) + h(x,\mu,t), \quad x \in \mathbb{R}^n, \quad \mu \in \mathbb{R}^m \tag{4.5.1}$$

satisfying the following assumptions:

(i) f and h are C^∞ in all arguments.
(ii) $f(0) = 0$ and $h(\cdot,0,\cdot) = 0$.
(iii) The eigenvalues of $Df(0)$ lie off the imaginary axis.
(iv) The unperturbed equation has a homoclinic solution $\gamma \neq 0$ so that $\lim_{t\to\pm\infty} \gamma(t) = 0$ and $\dot{\gamma}(t) = f(\gamma(t))$.
(v) $h(x,\mu,t+1) = h(x,\mu,t)$ for any $t \in \mathbb{R}$.
(vi) The variational linear differential equation

$$\dot{u}(t) = Df(\gamma(t))u(t) \tag{4.5.2}$$

has precisely d, $d \geq 2$ linearly independent solutions bounded on $\mathbb{R}$.

For the unperturbed equation

$$\dot{x} = f(x), \tag{4.5.3}$$

we adopt the standard notation W^s, W^u for the stable and unstable manifolds, respectively, of the origin and $d_s = \dim W^s$, $d_u = \dim W^u$. Since $x = 0$ is a hyperbolic equilibrium, γ must approach the origin along W^s as $t \to +\infty$ and along W^u as $t \to -\infty$. Thus, γ lies on $W^s \cap W^u$. The condition (vi) means that the tangent spaces of W^s and W^u along γ have a d–dimensional intersection.

The case when h is independent of t, $m = 3, d = 2$ is studied in [44] and it is shown that the set of small parameters, for which homoclinics of (4.5.1) exist near γ, forms a Whitney umbrella (cf [45] and Figure 4.2).

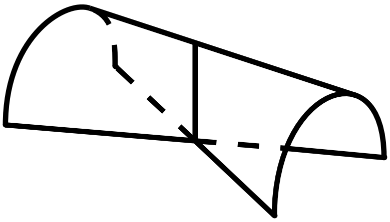

Fig. 4.2 The Whitney umbrella.

Equation (4.5.1) is considered in [46] with $d = 2$ and

$$h(x,\mu,t) = h_1(x,\lambda) + \varepsilon h_2(x,\mu,t), \quad \mu = (\lambda,\varepsilon) \in \mathbb{R}^3 \times \mathbb{R},$$

and it is shown that the set of small parameters, for which homoclinic points of (4.5.1) exist in a small section transverse to γ, is foliated by Whitney umbrellas. Bifurcation results for (4.5.1) are derived from [47] with $m = 1$ and $d = 2$. Bifurcation results in this direction are also established in [1,3–5].

Instead of (4.5.1), we consider

$$\dot{x} = f(x) + h(x,\mu,t+\alpha), \quad x \in \mathbb{R}^n, \quad \mu \in \mathbb{R}^m, \tag{4.5.4}$$

where $\alpha \in S^1 = \mathbb{R}/\mathbb{Z}$ is considered as another global parameter. Here S^1 is the circle.

In this section, we always mean "generically" in the sense that certain transversality (nondegenerate) conditions are satisfied for the studied problems. Those conditions usually are rather involved formulas and their verification is tedious for a concrete example. On the other hand, if one of those transversality conditions fails then we are led to a higher–order degenerate singularity of the studied bifurcation equation with a vague normal form.

We also remark that we focus our attention in this section on describing the set of all small parameters of the above types of (4.5.1) for which homoclinics exist near γ. We do not investigate neither the numbers of those homoclinics nor which kind of bifurcations takes place. But more careful analysis of the bifurcation equations could lead to some results in that direction as [48]. However, their description is outside the scope of this section.

4.5.2 Bifurcation Equation

The bifurcation equation for finding homoclinics of (4.5.4) near γ is derived from Section 4.1.3, so we only recall its form:

$$H(\beta,\alpha,\mu)=(H_1(\beta,\alpha,\mu),\dots,H_d(\beta,\alpha,\mu))=0\,, \tag{4.5.5}$$

where $H : O_1 \times \mathscr{I} \times W_1 \to \mathbb{R}^d$ is smooth for small neighborhoods $0 \in O_1 \subset \mathbb{R}^{d-1}$, $0 \in W_1 \subset \mathbb{R}^m$, a bounded open interval $\mathscr{I} \subset \mathbb{R}$, and

$$H_i(\beta,\alpha,\mu)=\sum_{j=1}^{m}a_{ij}(\alpha)\mu_j+\frac{1}{2}\sum_{j,k=1}^{d-1}b_{ijk}\beta_j\beta_k+\quad \text{h.o.t}\,,$$

$$a_{ij}(\alpha)=-\int_{-\infty}^{\infty}\langle u_i^{\perp}(t),\frac{\partial h}{\partial \mu_j}(\gamma(t),0,t+\alpha)\rangle\,dt\,,$$

$$b_{ijk}=-\int_{-\infty}^{\infty}\langle u_i^{\perp},D^2f(\gamma)u_{d+j}u_{d+k}\rangle\,dt\,.$$

4.5.3 Bifurcation for 2-Parametric Systems

We investigate (4.5.1) in this section for $m=2$ and the condition (vi) holds with $d=2$. Then the bifurcation equation (4.5.5) has the form

$$\begin{aligned} a_{11}(\alpha)\mu_1+a_{12}(\alpha)\mu_2+b_1\beta^2+\quad \text{h.o.t.}=0\\ a_{21}(\alpha)\mu_1+a_{22}(\alpha)\mu_2+b_2\beta^2+\quad \text{h.o.t.}=0\,. \end{aligned} \tag{4.5.6}$$

Since the codimension is 1 of the set of all noninvertible 2×2–matrices in the space of 2×2–matrices (cf Theorem 2.6.2), generically we assume that there is a finite number of $\alpha_1,\dots,\alpha_{l_1}\in S^1$ so that

$$A(\alpha)=\begin{pmatrix} a_{11}(\alpha) & a_{12}(\alpha)\\ a_{21}(\alpha) & a_{22}(\alpha)\end{pmatrix}$$

is noninvertible only for $\alpha=\alpha_1,\dots,\alpha_{l_1}$.

A1. First of all, we study (4.5.6) for α near $\alpha_0\notin\{\alpha_1,\dots,\alpha_{l_1}\}$. Then by applying the implicit function theorem, we obtain from (4.5.6)

$$\mu_1=\mu_1(\alpha,\beta),\quad \mu_2=\mu_2(\alpha,\beta)$$

for α near α_0 and β small. Moreover, (4.5.6) implies

$$\mu_i(\alpha,\beta)=\beta^2\big(\mu_{i1}(\alpha)+\beta d_i(\alpha,\beta)\big),\quad i=1,2\,,$$

where μ_{i1}, d_i, $i=1,2$ are C^∞–smooth. Generically, we have the following possibilities:

A1.1. $\mu_{11}(\alpha_0)\neq 0,\quad \mu_{21}(\alpha_0)\neq 0.$

Theorem 4.5.1. *Generically in the case A1.1, the set of parameters* (α, μ_1, μ_2) *near* $(\alpha_0, 0, 0)$, *for which* (4.5.4) *has a homoclinic near* γ, *is diffeomorphically foliated along the* α*–axis by two curves*

$$\left(\alpha, \tau^2 + \tau^3 e_1(\alpha, \tau), \tau^2\right),$$

where $e_1 \in C^\infty$ *satisfies* $e_1(\alpha_0, 0) \neq 0$ *and* $\tau \in \mathbb{R}$ *is small (Figure* 4.3*).*

Proof. We take

$$\tau = \beta\sqrt{|\mu_{21}(\alpha) + \beta d_2(\alpha, \beta)|}.$$

Then our set has the form

$$\left(\alpha, \tau^2 \mu_{13}(\alpha) + \tau^3 d_3(\alpha, \tau), \tau^2 \operatorname{sgn} \mu_{21}(\alpha_0)\right),$$

where $\mu_{13}, d_3 \in C^\infty$, $\mu_{13}(\alpha_0) \neq 0$ and generically $d_3(\alpha_0, 0) \neq 0$. This set is diffeomorphic to

$$\left(\alpha, \tau^2 + \tau^3 d_3(\alpha, \tau)/\mu_{13}(\alpha), \tau^2\right).$$

The proof is finished. □

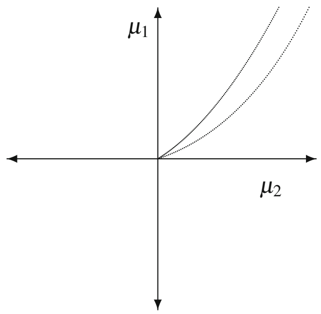

Fig. 4.3 $\mu_{11}(\alpha_0) > 0$, $\mu_{21}(\alpha_0) > 0$.

We note that generically we cannot avoid in the case A1.1 the following situation:

A1.1.1. $\quad \mu_{11}(\alpha_0) \neq 0, \quad \mu_{21}(\alpha_0) \neq 0, \quad e_1(\alpha_0, 0) = 0.$

We note that this case generically occurs only in a finite number of $\alpha_0 \notin \{\alpha_1, \ldots, \alpha_{l_1}\}$.

Theorem 4.5.2. *Generically in the case* A1.1.1, *the set of parameters* (α, μ_1, μ_2) *near* $(\alpha_0, 0, 0)$, *for which* (4.5.4) *has a homoclinic near* γ, *is diffeomorphically foliated along the* α*–axis by two curves*

$$\left(\alpha, \tau^2 + \tau^3(\alpha - \alpha_0) d_4(\alpha, \tau) + d_5(\alpha)\tau^4 + \tau^5 d_6(\alpha, \tau), \tau^2\right), \tag{4.5.7}$$

where $d_4, d_5, d_6 \in C^\infty$ *satisfy* $d_4(\alpha_0, 0) \neq 0$, $d_6(\alpha_0, 0) \neq 0$ *and* $\tau \in \mathbb{R}$ *is small* (*Figure* 4.4).

Proof. The statement of theorem is trivial, since by $e_1(\alpha_0, 0) = 0$, we have

$$e_1(\alpha, \tau) = (\alpha - \alpha_0)d_4(\alpha, \tau) + d_5(\alpha)\tau + \tau^2 d_6(\alpha, \tau).$$

To show the situation in Figure 4.4, we study the intersection of two curves (4.5.7) by solving for small $\tau > 0$ the equation

$$\begin{aligned} &\tau^2 + \tau^3(\alpha - \alpha_0)d_4(\alpha, \tau) + d_5(\alpha)\tau^4 + \tau^5 d_6(\alpha, \tau) \\ =\ &\tau^2 - \tau^3(\alpha - \alpha_0)d_4(\alpha, -\tau) + d_5(\alpha)\tau^4 - \tau^5 d_6(\alpha, -\tau). \\ &(\alpha - \alpha_0)\big(d_4(\alpha, \tau) + d_4(\alpha, -\tau)\big) = -\tau^2\big(d_6(\alpha, \tau) + d_6(\alpha, -\tau)\big). \end{aligned} \tag{4.5.8}$$

By the Whitney theorem 2.6.9, we have

$$d_4(\alpha, \tau) + d_4(\alpha, -\tau) = \tilde{d}_4(\alpha, \tau^2), \quad \tilde{d}_4 \in C^\infty,$$

$$d_6(\alpha, \tau) + d_6(\alpha, -\tau) = \tilde{d}_6(\alpha, \tau^2), \quad \tilde{d}_4 \in C^\infty.$$

Hence (4.5.8) is equivalent to

$$(\alpha - \alpha_0)\tilde{d}_4(\alpha, \tau^2) = -\tau^2 \tilde{d}_6(\alpha, \tau^2). \tag{4.5.9}$$

We can solve τ^2 from (4.5.9) to obtain

$$\tau^2 = \tau_1(\alpha), \quad \tau_1(\alpha_0) = 0, \quad \tau_1'(\alpha_0) \neq 0.$$

Now the situation in Figure 4.4 is clear. □

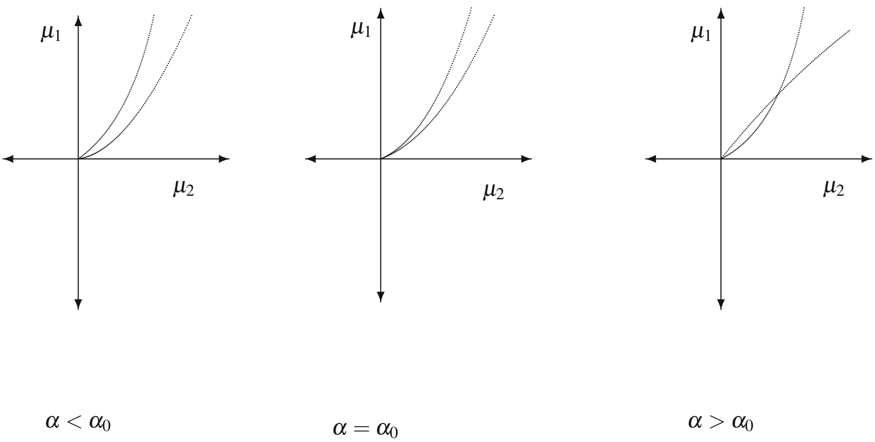

Fig. 4.4 $\mu_{11}(\alpha_0) > 0$, $\mu_{21}(\alpha_0) > 0$, $\tau_1'(\alpha_0) > 0$.

A1.2. $\quad \mu_{11}(\alpha_0) = 0, \quad \mu_{11}'(\alpha_0) \neq 0, \quad \mu_{21}(\alpha_0) \neq 0.$

We note that this case generically occurs only in a finite number of $\alpha_0 \notin \{\alpha_1, \ldots, \alpha_{l_1}\}$.

Theorem 4.5.3. *Generically in the case A1.2, the set of parameters (α, μ_1, μ_2) near $(\alpha_0, 0, 0)$, for which (4.5.4) has a homoclinic near γ, is diffeomorphically foliated along the α–axis by two curves*

$$\left(\alpha, \tau^2(\alpha - \alpha_0) + \tau^3 e_2(\alpha, \tau), \tau^2\right),$$

where $e_2 \in C^\infty$ satisfies $e_2(\alpha_0, 0) \neq 0$ and $\tau \in \mathbb{R}$ is small (Figure 4.5).

Proof. Like in the above proof, our set is equivalent to

$$\left(\alpha, \tau^2 \mu_{13}(\alpha) + \tau^3 d_3(\alpha, \tau), \tau^2\right),$$

where $\mu_{13}, d_3 \in C^\infty$, $\mu_{13}(\alpha_0) = 0$, $\mu_{13}'(\alpha_0) \neq 0$, $d_3(\alpha_0, 0) \neq 0$. Hence we have

$$\left(\alpha, \tau^2(\alpha - \alpha_0)\mu_{14}(\alpha) + \tau^3 d_3(\alpha, \tau), \tau^2\right),$$

where $\mu_{14} \in C^\infty$, $\mu_{14}(\alpha_0) \neq 0$. Consequently, the set is diffeomorphic to

$$\left(\alpha, \tau^2(\alpha - \alpha_0) + \tau^3 d_3(\alpha, \tau)/\mu_{14}(\alpha), \tau^2\right).$$

The proof is finished. $\square$

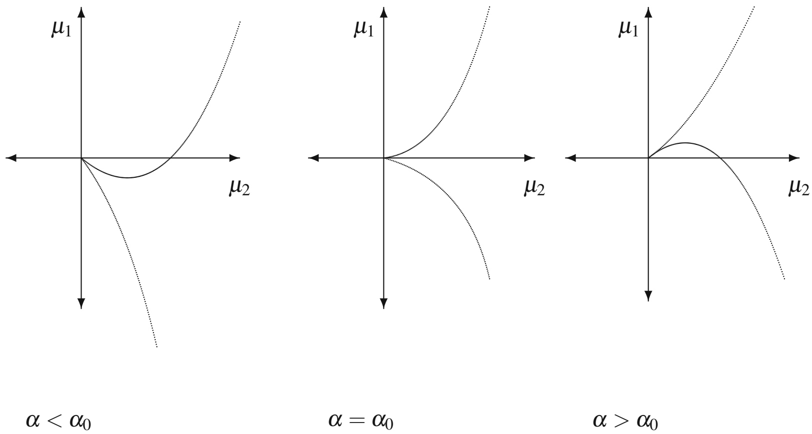

Fig. 4.5 $\mu_{11}'(\alpha_0) > 0$, $\mu_{21}(\alpha_0) > 0$.

A1.3. $\quad \mu_{11}(\alpha_0) \neq 0, \quad \mu_{21}(\alpha_0) = 0, \quad \mu_{21}'(\alpha_0) \neq 0.$

It is clear that this case is the same as A1.2.

A2. The second case is when α is near $\alpha_0 \in \{\alpha_1, \dots, \alpha_{l_1}\}$. So $A(\alpha_0)$ is noninvertible. We can assume

$$a_{11}(\alpha_0) \neq 0, \quad a_{21}(\alpha_0) = 0, \quad a'_{21}(\alpha_0) \neq 0, \quad a_{22}(\alpha_0) = 0, \quad a'_{22}(\alpha_0) \neq 0 .$$

Then we solve

$$\mu_1 = \mu_1(\alpha, \beta, \mu_2)$$

from the first equation of (4.5.6) for α near α_0 and β, μ_2 small. Consequently, by inserting this solution into the second equation of (4.5.6), the bifurcation equation now is reduced to

$$\begin{aligned} Q(\alpha, \beta, \mu_2) = & (\alpha - \alpha_0)\tilde{a}_{21}(\alpha)\mu_1(\alpha, \beta, \mu_2) \\ & + (\alpha - \alpha_0)\tilde{a}_{22}(\alpha)\mu_2 + b_2\beta^2 + \quad \text{h.o.t.} = 0 . \end{aligned} \tag{4.5.10}$$

We note

$$\mu_1(\alpha, \beta, 0) = O(\beta^2), \quad Q(\alpha, \beta, 0) = O(\beta^2), \quad Q(\alpha_0, 0, \mu_2) = O(\mu_2^2) .$$

By using the Malgrange Preparation Theorem 2.6.8, generically (4.5.10) is equivalent to

$$Q_1(\alpha, \beta, \mu_2) = \beta^2 A(\alpha, \beta) + B(\alpha, \beta)\mu_2 + \mu_2^2 = 0, \tag{4.5.11}$$

where $A, B \in C^\infty$ satisfy

$$A(\alpha_0, 0) \neq 0, \quad B(\alpha, \beta) = (\alpha - \alpha_0)B_1(\alpha, \beta) + \beta B_2(\beta),$$

$$B_1, B_2 \in C^\infty, \quad B_1(\alpha_0, 0) \neq 0 .$$

We take

$$\tau = \beta\sqrt{|A(\alpha, \beta)|}, \quad \eta = B(\alpha, \beta) . \tag{4.5.12}$$

Then (4.5.11) is equivalent to

$$\tau^2 \operatorname{sgn} A(\alpha_0, 0) + \eta\mu_2 + \mu_2^2 = 0 . \tag{4.5.13}$$

The discriminant of (4.5.13) is as follows:

$$D(\eta, \tau) = \eta^2 - 4\tau^2 \operatorname{sgn} A(\alpha_0, 0) .$$

We note that

$$\mu_1 = \tilde{\mu}_1(\eta, \tau, \mu_2) = \mu_2 E(\eta, \tau, \mu_2) + \tau^2 F(\eta, \tau) ,$$

where $E, F \in C^\infty$ generically satisfy $E(0,0,0) \neq 0$ and $\frac{\partial E}{\partial \tau}(0,0,0) \neq 0$. Consequently, our set of parameters in the space (η, μ_1, μ_2) near $(0,0,0)$ has the form

$$\big(\eta, \mu_2 E(\eta, \tau, \mu_2) + \tau^2 F(\eta, \tau), \mu_2\big),$$

$$\tau^2 \operatorname{sgn} A(\alpha_0, 0) + \eta\mu_2 + \mu_2^2 = 0 ,$$

where $\tau \in \mathbb{R}$ is small. We consider the following two possibilities.

A2.1. $\operatorname{sgn} A(\alpha_0, 0) = -1$.

In this case, (4.5.13) has the form

$$\tau^2 = \mu_2(\mu_2 + \eta).$$

Hence

$$\tau = \pm\sqrt{\mu_2(\mu_2 + \eta)},$$

where either $\eta \geq 0$, $\mu_2 \geq 0$, $\mu_2 \leq -\eta$ or $\eta \leq 0$, $\mu_2 \leq 0$, $\mu_2 \geq -\eta$. Then

$$\begin{aligned}\mu_2 E(\eta, \tau, \mu_2) + \tau^2 F(\eta, \tau) &= \mu_2 \Big(E\big(\eta, \pm\sqrt{\mu_2(\mu_2 + \eta)}, \mu_2\big) \\ &\quad + (\mu_2 + \eta) F\big(\eta, \pm\sqrt{\mu_2(\mu_2 + \eta)}\big)\Big) \\ &= H_\pm(\eta, \mu_2).\end{aligned}$$

We compute

$$\begin{aligned}\big(H_+(\eta, \mu_2) - H_-(\eta, \mu_2)\big)/\mu_2 &= E\big(\eta, \sqrt{\mu_2(\mu_2 + \eta)}, \mu_2\big) \\ &\quad + (\mu_2 + \eta) F\big(\eta, \sqrt{\mu_2(\mu_2 + \eta)}\big) \\ &\quad - E\big(\eta, -\sqrt{\mu_2(\mu_2 + \eta)}, \mu_2\big) \\ &\quad - (\mu_2 + \eta) F\big(\eta, -\sqrt{\mu_2(\mu_2 + \eta)}\big) \\ &= \Big(\frac{\partial E}{\partial \tau}(\eta, \theta, \mu_2) + (\mu_2 + \eta)\frac{\partial F}{\partial \tau}(\eta, \theta)\Big). \\ &\quad 2\sqrt{\mu_2(\mu_2 + \eta)} \neq 0\end{aligned}$$

for any sufficiently small η and $\mu_2 \neq 0$, $\mu_2 \neq -\eta$. We also note that $H_\pm(\eta, \mu_2) = 0$ for sufficiently small μ_2, η only if $\mu_2 = 0$.

In summary, we obtain the following result.

Theorem 4.5.4. *Generically in the case* A2.1*, the set of parameters* (α, μ_1, μ_2) *near* $(\alpha_0, 0, 0)$*, for which* (4.5.4) *has a homoclinic near* γ *(see* (4.5.12)*), is diffeomorphically foliated along the* η*–axis by four curves*

$$\big(\eta, H_\pm(\eta, \mu_2), \mu_2\big)$$

where either $\eta \geq 0$, $\mu_2 \geq 0$, $\mu_2 \leq -\eta$ *or* $\eta \leq 0$, $\mu_2 \leq 0$, $\mu_2 \geq -\eta$ *(Figure* 4.6*).*

A2.2. $\operatorname{sgn} A(\alpha_0, 0) = 1$.

In this case, (4.5.13) has the form

$$\tau^2 + \eta\mu_2 + \mu_2^2 = 0.$$

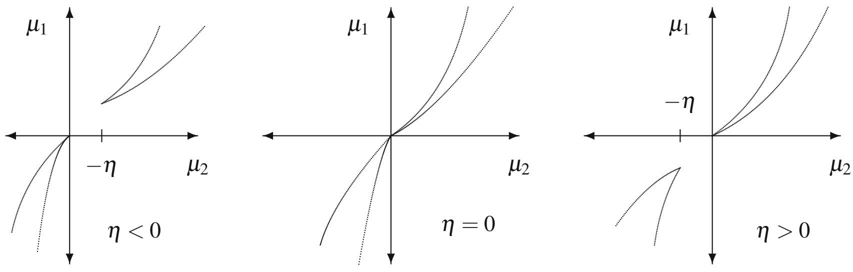

Fig. 4.6 $E(0,0) > 0$.

Hence

$$\tau = \pm\sqrt{-\mu_2(\mu_2+\eta)}$$

where either $\eta \geq 0$, $-\eta \leq \mu_2 \leq 0$ or $\eta \leq 0$, $0 \leq \mu_2 \leq -\eta$. Then

$$\begin{aligned}\mu_2 E(\eta,\tau,\mu_2) + \tau^2 F(\eta,\tau) &= \mu_2\Big(E\big(\eta,\pm\sqrt{-\mu_2(\mu_2+\eta)},\mu_2\big) \\ &\quad -(\mu_2+\eta)F\big(\eta,\pm\sqrt{-\mu_2(\mu_2+\eta)}\big)\Big) \\ &= G_\pm(\eta,\mu_2)\,.\end{aligned}$$

Similarly like the above, we see that $G_+(\eta,\mu_2) \neq G_-(\eta,\mu_2)$ for any sufficiently small η and $\mu_2 \neq 0$, $\mu_2 \neq -\eta$. We also have that $G_\pm(\eta,\mu_2) = 0$ for sufficiently small μ_2, η only if $\mu_2 = 0$. We achieve the following result.

Theorem 4.5.5. *Generically in the case A2.2, the set of parameters* (α,μ_1,μ_2) *near* $(\alpha_0,0,0)$*, for which* (4.5.4) *has a homoclinic near* γ *(see* (4.5.12)*), is diffeomorphically foliated along the* η*–axis by a closed loop*

$$(\eta, H_\pm(\eta,\mu_2),\mu_2)$$

where either $\eta \geq 0$, $-\eta \leq \mu_2 \leq 0$ *or* $\eta \leq 0$, $0 \leq \mu_2 \leq -\eta$. *We note that for* $\eta = 0$ *this is just the point* $(0,0)$ $($*Figure* 4.7$)$.

4.5.4 Bifurcation for 4-Parametric Systems

In this section, we consider the case $m = 4$ and the condition (vi) holds with $d = 2$. Then the bifurcation equation (4.5.5) has the form

$$\begin{aligned}&a_{11}(\alpha)\mu_1 + a_{12}(\alpha)\mu_2 + a_{13}(\alpha)\mu_3 + a_{14}(\alpha)\mu_4 + b_1\beta^2 + \quad \text{h.o.t.} = 0, \\ &a_{21}(\alpha)\mu_1 + a_{22}(\alpha)\mu_2 + a_{13}(\alpha)\mu_3 + a_{24}(\alpha)\mu_4 + b_2\beta^2 + \quad \text{h.o.t.} = 0\,.\end{aligned} \tag{4.5.14}$$

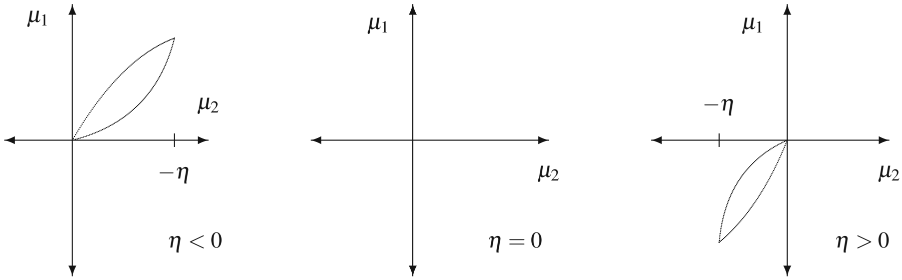

Fig. 4.7 $E(0,0) > 0$.

Since the codimension is 3 of the set of all 2×4–matrices with corank 1 in the space of 2×4–matrices (cf Theorem 2.6.2), generically we may assume the invertibility of the matrix $A(\alpha)$ for any $\alpha \in S^1$. Then by applying the implicit function theorem, we obtain from (4.5.14)

$$\mu_1 = \mu_1(\alpha, \beta, \mu_3, \mu_4), \quad \mu_2 = \mu_2(\alpha, \beta, \mu_3, \mu_4)$$

for $\alpha \in S^1$ and β, μ_3, μ_4 small. Moreover, (4.5.14) implies

$$\mu_i(\alpha, \beta, 0, 0) = O(\beta^2), \quad i = 1, 2.$$

Generically we may assume

$$\left(\frac{\partial^2 \mu_1}{\partial^2 \beta}(\alpha, 0, 0, 0)\right)^2 + \left(\frac{\partial^2 \mu_2}{\partial^2 \beta}(\alpha, 0, 0, 0)\right)^2 \neq 0 \quad \forall, \alpha \in S^1.$$

We take the change of parameters

$$\mu_1 \leftrightarrow A_1(\alpha)\mu_1 + A_2(\alpha)\mu_2, \quad \mu_2 \leftrightarrow -A_2(\alpha)\mu_1 + A_1(\alpha)\mu_2,$$

where

$$A_1(\alpha) = \frac{\partial^2 \mu_1}{\partial^2 \beta}(\alpha, 0, 0, 0) \Big/ \left(\left(\frac{\partial^2 \mu_1}{\partial^2 \beta}(\alpha, 0, 0, 0)\right)^2 + \left(\frac{\partial^2 \mu_2}{\partial^2 \beta}(\alpha, 0, 0, 0)\right)^2\right),$$

$$A_2(\alpha) = \frac{\partial^2 \mu_2}{\partial^2 \beta}(\alpha, 0, 0, 0) \Big/ \left(\left(\frac{\partial^2 \mu_1}{\partial^2 \beta}(\alpha, 0, 0, 0)\right)^2 + \left(\frac{\partial^2 \mu_2}{\partial^2 \beta}(\alpha, 0, 0, 0)\right)^2\right).$$

For these new parameters, we have

$$\frac{\partial^2 \mu_1}{\partial^2 \beta}(\alpha, 0, 0, 0) \neq 0.$$

Then we solve for β small the equation

$$\frac{\partial \mu_1}{\partial \beta}(\alpha,\beta,\mu_3,\mu_4) = 0$$

to obtain $\beta = \tilde{\beta}(\alpha,\mu_3,\mu_4)$, and by replacing β with $\beta + \tilde{\beta}(\alpha,\mu_3,\mu_4)$, we may assume that

$$\mu_1(\alpha,\beta,\mu_3,\mu_4) = \bar{\mu}_1(\alpha,\mu_3,\mu_4) + \beta^2\tilde{\mu}_1(\alpha,\beta,\mu_3,\mu_4)$$

where $\tilde{\mu}_1(\alpha,0,0,0) \neq 0$. Replacing β with $\beta\sqrt{|\tilde{\mu}_1(\alpha,\beta,\mu_3,\mu_4)|}$, we obtain

$$\mu_1(\alpha,\beta,\mu_3,\mu_4) = \bar{\mu}_1(\alpha,\mu_3,\mu_4) \pm \beta^2 .$$

Now we take the change of parameters

$$\mu_1 \leftrightarrow \pm\big(\mu_1 - \bar{\mu}_1(\alpha,\mu_3,\mu_4)\big), \quad \mu_2 \leftrightarrow \mu_2 - \mu_2(\alpha,0,\mu_3,\mu_4) .$$

In this way, we arrive at

$$\mu_1(\alpha,\beta,\mu_3,\mu_4) = \beta^2, \quad \mu_2(\alpha,\beta,\mu_3,\mu_4) = \beta\rho(\alpha,\beta,\mu_3,\mu_4)$$

where $\rho \in C^\infty$ satisfies $\rho(\cdot,0,0,0) = 0$. All the above changes of parameters give a local diffeomorphism

$$\Gamma_1 : S^1 \times \mathscr{O}_1 \to S^1 \times \mathbb{R}^4$$

foliated along S^1, where $\mathscr{O}_1$ is an open neighbourhood of $0 \in \mathbb{R}^4$. Generically we may assume that

$$\Big(\frac{\partial \rho}{\partial \mu_3}(\alpha,0,0,0)\Big)^2 + \Big(\frac{\partial \rho}{\partial \mu_4}(\alpha,0,0,0)\Big)^2 \neq 0, \quad \forall \alpha \in S^1 .$$

We take the change of parameters

$$\mu_3 \leftrightarrow D_1(\alpha)\mu_3 - D_2(\alpha)\mu_4, \quad \mu_4 \leftrightarrow D_2(\alpha)\mu_3 + D_1(\alpha)\mu_4 ,$$

where

$$D_1(\alpha) = \frac{\partial \rho}{\partial \mu_3}(\alpha,0,0,0)\Big/\Big(\Big(\frac{\partial \rho}{\partial \mu_3}(\alpha,0,0,0)\Big)^2 + \Big(\frac{\partial \rho}{\partial \mu_4}(\alpha,0,0,0)\Big)^2\Big),$$

$$D_2(\alpha) = \frac{\partial \rho}{\partial \mu_4}(\alpha,0,0,0)\Big/\Big(\Big(\frac{\partial \rho}{\partial \mu_3}(\alpha,0,0,0)\Big)^2 + \Big(\frac{\partial \rho}{\partial \mu_4}(\alpha,0,0,0)\Big)^2\Big).$$

For these new parameters, we have

$$\frac{\partial \rho}{\partial \mu_3}(\alpha,0,0,0) \neq 0 .$$

Now we split

$$\big(\rho(\alpha,\beta,\mu_3,\mu_4)-\rho(\alpha,0,\mu_3,\mu_4)\big)/\beta=\rho_1(\alpha,\beta^2,\mu_3,\mu_4)+\beta\rho_2(\alpha,\beta^2,\mu_3,\mu_4),$$

where $\rho_i\in C^\infty$, $i=1,2$. For an open neighbourhood $\mathscr{O}_2$ of $0\in\mathbb{R}^4$, we take a local diffeomorphism

$$\Gamma_2:S^1\times\mathscr{O}_2\to S^1\times\mathbb{R}^4$$

given by

$$\begin{aligned}&\Gamma_2(\lambda_5,\lambda_1,\lambda_2,\lambda_3,\lambda_4)\\&=\Big(\lambda_5,\lambda_1,\lambda_2-\lambda_1\rho_1(\lambda_5,\lambda_1,\lambda_3,\lambda_4),\rho(\lambda_5,0,\lambda_3,\lambda_4)+\lambda_1\rho_2(\lambda_5,\lambda_1,\lambda_3,\lambda_4),\lambda_4\Big),\end{aligned}$$

which is foliated along S^1. In summary, we arrive at the following theorem.

Theorem 4.5.6. *Let $d=2$, $m=4$ in* (4.5.1). *Then generically the set of parameters $(\alpha,\mu_1,\mu_2,\mu_3,\mu_4)$ near $(\alpha,0,0,0,0)$, $\alpha\in S^1$, for which* (4.5.4) *has a homoclinic near γ, is diffeomorphically foliated along the α–axis by a surface of the Morin singularity* [49] *given as follows:*

$$(x_1,x_2,x_3)\to(x_1^2,x_1x_2,x_2,x_3)\,. \tag{4.5.15}$$

Proof. It is enough to take the composition of all the above changes of parameters [44, p. 221]. □

Remark 4.5.7. We note that singularity (4.5.15) is just the foliated Whitney umbrella of [46]. Moreover, the foliation along the α–axis is nontrivial. In each α–section, the diffeomorphism between the Morin singularity and the set of small parameters $\mu\in\mathbb{R}^4$ for which (4.5.4) has a homoclinic solution near γ, does depend smoothly on α. This is the main difference between our result and [46]. We do not restrict the existence of homoclinic solutions of (4.5.1) near γ by supposing that they cross a transverse section of γ at $t=0$. We really investigate all possible homoclinic solutions of (4.5.1) geometrically near γ. A similar nontrivial foliation along the α–axis holds for the result of Section 4.5.3. Furthermore, the result of Section 4.5.3 does not follow directly from Section 4.5.4. It is more delicate even for $m=1$, $d=2$ [47]. It seems that the case $m=3$, $d=2$ is more sophisticated than the case of Section 4.5.3. Finally, the result of Section 4.5.4 persists under further perturbations, that is, generically we get the same result for $m\geq 4$ with $d=2$.

4.5.5 Autonomous Perturbations

In this section, we study the case $d\geq 3$ of (4.5.1) with h independent of t. Then the bifurcation equation (4.5.5) is independent of α, so we put $\alpha=0$ in (4.5.5). Moreover, we assume that (4.5.3) is decoupled

$$\begin{gathered}\dot z_j=f_{1,j}(z_j),\quad \dot y=f_2(y),\\ j=1,2,\dots,d-2,\quad x=(z_1,z_2,\dots,z_{d-2},y)\,.\end{gathered} \tag{4.5.16}$$

Hence

$$\gamma = (\gamma_{1,1}, \gamma_{1,2}, \dots, \gamma_{1,d-2}, \gamma_2),$$

and (4.5.2) has the form

$$\dot{u}_j = Df_{1,j}(\gamma_{1,j})u_j, \quad j = 1,2,\dots,d-2, \tag{4.5.17}$$

$$\dot{v} = Df_2(\gamma_2)v. \tag{4.5.18}$$

We suppose the following assumptions:

(H) The variational equations (4.5.17) with $j = 1,2,\dots,d-2$, respectively (4.5.18), have precisely 1, respectively 2, linearly independent solutions bounded on $\mathbb{R}$.

Let

$$W_{ss} = \times_{j=1}^{d-2} \{\gamma_{1,j}(t) \mid t \in \mathbb{R}\} \times \{\gamma_2(t) \mid t \in \mathbb{R}\}$$

be a homoclinic manifold. Theorem 4.1.1 is applicable separately to (4.5.17) and (4.5.18). Then a small transverse section Ψ at $\gamma(0)$ to W_{ss} in $\mathbb{R}^n$ is given, and we study the existence of homoclinic solutions of (4.5.1) crossing Ψ. This leads us to the bifurcation equation (4.5.5) possessing now the form

$$\Omega\mu + \beta^2\omega^* + \quad \text{h.o.t.} = 0, \tag{4.5.19}$$

where $\beta \in \mathbb{R}$ is small, $\omega \in \mathbb{R}^d$ is given and $\Omega : \mathbb{R}^m \to \mathbb{R}^d$ is a matrix. We suppose that $m \geq 2d-1$. Since the codimension is $m-d+1$ of the set of all $d \times m$–matrices with corank 1 in the space of $d \times m$–matrices (cf Theorem 2.6.2), generically we may assume that $\operatorname{rank} \Omega = d$ and so by applying the implicit function theorem to (4.5.19), we obtain

$$\mu_1 = \mu_1(\beta, \mu_2), \quad \mu_2 \in \mathbb{R}^{m-d} \quad \text{is small},$$

where $\mu_1 \in C^\infty$ satisfies $\mu_1(\beta,0) = O(\beta^2)$. Consequently our set has the form

$$\left\{ \big(\mu_1(\beta,\mu_2),\mu_2\big) \mid \beta \in \mathbb{R}, \quad \mu_2 \in \mathbb{R}^{m-d} \quad \text{are small} \right\}.$$

We introduce a mapping $M : \mathscr{O} \to \mathbb{R}^m$ given by

$$M(\beta,\mu_2) = \big(\mu_1(\beta,\mu_2),\mu_2\big),$$

where $\mathscr{O}$ is an open neighbourhood of $0 \in \mathbb{R}^{m-d+1}$. The linearization $DM(0)$ has corank 1. Let $J^1\left(\mathbb{R}^{m-d+1},\mathbb{R}^m\right)$ be the 1–jet bundle (cf Section 2.6), and let S_1 be a submanifold of $J^1\left(\mathbb{R}^{m-d+1},\mathbb{R}^m\right)$ defined by

$$S_1 = \left\{\sigma \in J^1\left(\mathbb{R}^{m-d+1},\mathbb{R}^m\right) \mid \operatorname{corank} \sigma = 1\right\}.$$

Since $m-d+1 \geq d$ and according to Theorem 2.6.3, the codimension is d of the set S_1 in $J^1\left(\mathbb{R}^{m-d+1},\mathbb{R}^m\right)$, by recalling Theorems 2.6.6 and 2.6.7, we can assume that

$$j^1 M \quad \text{intersects transversally} \quad S_1 \quad \text{at} \quad 0\,, \tag{4.5.20}$$

where

$$j^1 M : \mathscr{O} \to J^1\left(\mathbb{R}^{m-d+1}, \mathbb{R}^m\right)$$

is the 1–jet mapping. By applying a result of [49] (see also a proof of [45, Theorem 4.6 on p. 179]), we immediately obtain the following theorem.

Theorem 4.5.8. *Let* $d \geq 3$, $m \geq 2d-1$ *in* (4.5.1) *when* h *is independent of* t. *Suppose* (4.5.16) *and that the assumption (H) holds for* (4.5.17), (4.5.18). *Then generically, when* $\operatorname{rank}\Omega = d$ *and* (4.5.20) *holds, the set of small parameters* $\mu \in \mathbb{R}^m$ *for which* (4.5.1) *has a homoclinic solution crossing* Ψ *is diffeomorphic to a surface of the Morin singularity given by*

$$\left(x_1, x_2, \ldots, x_{m-d+1}\right) \to \left(x_1^2, x_1x_2, x_1x_3, \ldots, x_1x_d, x_2, x_3, \ldots, x_{m-d+1}\right).$$

Remark 4.5.9. 1. Theorem 4.5.8 is valid also for $d = 2$, but then we recover the result of [44] for $m = 3$. We note that the condition $m \geq 2d-1$ is a principal and not a technical restriction. Decoupling of (4.5.3) into (4.5.16) is motivated by examples of [1, 10, 50]: When several oscillators are weakly coupled then (4.5.16) is naturally satisfied. On the other hand, we are not able to find a reasonable result for the case $d \geq 3$ in general (4.5.1) without assuming the decoupling condition (4.5.16).

2. We have a cross-cap singularity [45, p. 179] in Theorem 4.5.8 with $m = 2d-1$.

3. The transversality condition (4.5.20) is the condition on the 2–jet of M at 0 [45, p. 179].

4. Under the assumptions of Theorem 4.5.8, there is a family $\Psi_{\gamma(t)}$ of small transverse sections to W_{ss} at $\gamma(t)$ for any t sufficiently small so that $\Psi_{\gamma(0)} = \Psi$, the family $\Psi_{\gamma(\cdot)}$ represents a tubular neighbourhood of W_{ss} in $\mathbb{R}^n$ near $\gamma(0)$ and the statement of Theorem 4.5.8 holds also for any $\Psi_{\gamma(t)}$.

Finally, we can study more degenerate Morin singularities of M. Let

$$J^k\left(\mathbb{R}^{m-d+1}, \mathbb{R}^m\right), \quad 2 \leq k \in \mathbb{N}$$

be the k–jet bundle, and let

$$j^k M : \mathscr{O} \to J^k\left(\mathbb{R}^{m-d+1}, \mathbb{R}^m\right)$$

be the k–jet mapping. Let S_{1_k} be the contact class in $J^k\left(\mathbb{R}^{m-d+1}, \mathbb{R}^m\right)$ [45, p. 174]. We know by [49] that S_{1_k} is a submanifold of $J^k\left(\mathbb{R}^{m-d+1}, \mathbb{R}^m\right)$ with codimension kd. Let us suppose that $j^k M(0) \in S_{1_k}$. Again by recalling Theorems 2.6.6 and 2.6.7, we can assume that

$$j^k M \quad \text{intersects transversally} \quad S_{1_k} \quad \text{at} \quad 0\,, \tag{4.5.21}$$

provided that $m-d+1 \geq kd$, i.e. $m \geq d(k+1)-1$. Results of [49] give the following theorem.

Theorem 4.5.10. *Let* $d \geq 3$, $m \geq d(k+1)-1$, $2 \leq k \in \mathbb{N}$ *in* (4.5.1) *when* h *is independent of* t. *Suppose* (4.5.16) *and that the assumption (H) holds for* (4.5.17), (4.5.18). *If* $\operatorname{rank}\Omega = d$ *and* $j^k M(0) \in S_{1_k}$ *holds with* (4.5.21) *as well, then the set of small parameters* $\mu \in \mathbb{R}^m$ *for which* (4.5.1) *has a homoclinic solution crossing* Ψ *is diffeomorphic to a surface of the Morin singularity given by*

$$\begin{aligned} y_j &= x_j, \quad 1 \leq j \leq m-d \\ y_{m-d+j} &= \sum_{r=1}^{k} x_{(j-1)k+r} x_{m-d+1}^r, \quad 1 \leq j \leq d-1 \\ y_m &= \sum_{r=1}^{k-1} x_{(d-1)k+r} x_{m-d+1}^r + x_{m-d+1}^{k+1}. \end{aligned}$$

The proof of Theorem 4.5.10 is outside the scope of this book.

4.6 Inflated ODEs

4.6.1 Inflated Carathéodory Type ODEs

Similar to Section 3.5, when we consider an orbit $x(t)$, $t \in \mathbb{R}$ of an ε-inflation of a differential equation $\dot{x} = f(t,x)$, then we deal with a differential inclusion

$$\begin{aligned} &\dot{x}(t) \in f(t,x(t)) + \varepsilon \mathscr{B}_{\mathbb{R}^n} \quad \text{for almost each (f.a.e.) } t \in \mathbb{R}, \\ &x(0) = x_0 . \end{aligned} \tag{4.6.1}$$

Here we suppose that $f : \mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ satisfies Carathéodory type conditions and it is globally Lipschitz continuous function in x (cf [51–53] and Section 2.5.8). We are again not interested in the existence of one solution of (4.6.1), but in the set of all trajectories of (4.6.1). So we consider a single-valued differential equation

$$\begin{aligned} &\dot{x}(t) = f(t,x(t)) + \varepsilon h(t), \quad h(t) \in \mathscr{B}_{\mathbb{R}^n} \text{ f.a.e. } t \in \mathbb{R}, \\ &x(0) = x_0 , \end{aligned} \tag{4.6.2}$$

where $h \in L^\infty(\mathbb{R},\mathbb{R}^n)$ is considered as a parameter. This orbit of (4.6.2) is denoted by $x(h)$. Since f is globally Lipschitz continuous function, this orbit is unique and continuously depends on h. Next, we define an ε-inflated orbit of (4.6.1) given by

$$\mathbf{x}^\varepsilon(x_0)(t) = \Big\{ x(h)(t) \mid h \in L^\infty(\mathbb{R},\mathbb{R}^n),\, h(t) \in \mathscr{B}_{\mathbb{R}^n} \text{ f.a.e. } t \in \mathbb{R} \Big\}.$$

Sets of $\mathbf{x}^\varepsilon(x_0)(t)$ are contractible into themselves to $x_0(t) = \mathbf{x}^0(x_0)(t)$ – the solution of $\dot{x}(t) = f(t,x(t))$ f.a.e. $t \in \mathbb{R}$, $x(0) = x_0$. For $t \neq 0$, the point $x_0(t)$ is in the interior of $\mathbf{x}^\varepsilon(x_0)(t)$. Moreover, $\mathbf{x}^\varepsilon(x_0)(t)$ are compact.

This approach of considering parameterized differential equations (4.6.2) instead of differential inclusions (4.6.1) is used in [53] for investigation of an ε-inflated dynamics near to a hyperbolic equilibrium of a differential equation. More precisely, we construct analogues of the stable and unstable manifolds, which are typical of a single-valued hyperbolic dynamics; moreover, we construct the maximal weakly invariant bounded set and prove that all such sets are graphs of Lipschitz maps.

4.6.2 Inflated Periodic ODEs

In this section we extend the results of Section 3.5.2 to continuous time case, i.e. we start from ODE

$$\dot{x} = h(t,x), \tag{4.6.3}$$

where $h \in C^1(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ satisfies the following hypotheses:

(H1) h is 1-periodic in $t \in \mathbb{R}$. Moreover, (4.6.3) possesses a nonconstant hyperbolic 1-periodic solution $\gamma_0(t)$ along with a homoclinic one $\gamma(t)$ so that $\lim_{t\to\pm\infty}|\gamma(t) - \gamma_0(t)| = 0$. Furthermore, the variational equation $\dot{v} = Dh(t,\gamma(t))v$ has an exponential dichotomy on $\mathbb{R}$.

Let $\phi(t,x)$, $\phi(0,x) = x$ be the evolution operator of (4.6.3). By introducing the Poincarè map $f(x) = \phi(1,x)$ of (4.6.3), diffeomorphism f has a hyperbolic fixed point $x_0 = \gamma_0(0)$ along with a transversal homoclinic orbit $\{x_k^0\}_{k\in\mathbb{Z}}$, $x_k^0 = \gamma(k)$. So Theorem 2.5.4 can be applied to (4.6.3).

Next, we consider a differential inclusion in $\mathbb{R}^n$ of the form

$$\dot{x} \in h(t,x) + q(t,x,\mathscr{B}_{\mathbb{R}^n}), \tag{4.6.4}$$

where $q \in C(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ is a 1-periodic mapping in $t \in \mathbb{R}$, satisfying the following hypotheses:

(H2) There are positive constants λ, Λ so that

$$|q(t,x,p) - q(t,\tilde{x},\tilde{p})| \le \lambda|x-\tilde{x}| + \Lambda|p-\tilde{p}| \quad \text{and} \quad q(t,x,0) = 0$$

for all $t \in \mathbb{R}, x, \tilde{x} \in \mathbb{R}^n, p, \tilde{p} \in \mathscr{B}_{\mathbb{R}^n}$.

We put $\mathscr{L} = L^\infty(\mathbb{R},\mathbb{R}^n)$ with usual supremum norm $\|u\|_\infty = \operatorname{ess\,sup}_{t\in\mathbb{R}}|u(t)|$ and take $u \in \mathscr{B} := \{u \in \mathscr{L} \,|\, \|u\|_\infty \le 1\}$. We remark (see Section 3.5.2) that (4.6.4) is equivalent, i.e. it has the same solution set, to the family of ODE

$$\dot{x} = h(t,x) + q(t,x,u(t)), \qquad u \in \mathscr{B}. \tag{4.6.5}$$

Now we can repeat the arguments of Section 3.5.2. We sketch main steps for the readers' convenience. First we note that (4.6.5) is a continuous time analogy of

(3.5.6). Then we fix $\omega \in \mathbb{N}$ large and for any $\xi \in \mathscr{E}$, $\xi = \{e_j\}_{j\in\mathbb{Z}}$ define a pseudo-orbit $x^\xi \in \mathscr{L}$ as follows for $t \in [2j\omega, \ldots, 2(j+1)\omega)$, $j \in \mathbb{Z}$:

$$x^{\xi}(t) := \begin{cases} \gamma(t-(2j+1)\omega), & \text{for } e_j = 1, \\ \gamma_0(t-(2j+1)\omega), & \text{for } e_j = 0. \end{cases}$$

Following the proof of Lemma 3.5.1 (cf Theorem 4.1.2), we have another result.

Lemma 4.6.1. *There exist $\omega_0 \in \mathbb{N}$ and a constant $c > 0$ so that for any $\xi \in \mathscr{E}$, $u \in \mathscr{L}$, there is a unique solution $w \in W^{1,\infty}(\mathbb{R},\mathbb{R}^n)$ of the linear system*

$$\dot{w} = D_x h(t, x^{\xi}(t))w + u\,.$$

Moreover, w is linear in u and it holds $\|w\|_\infty \le c\|u\|_\infty$.

Following Theorems 3.5.2 and 3.5.3, we get

Theorem 4.6.2. *Assume λ and Λ are sufficiently small. Then there are $\omega_1 > \omega_0$, $\rho_0 > 0$ and $\widetilde{L} > 0$ so that for any $\mathbb{N} \ni \omega \ge \omega_1$ but fixed and for any $\xi \in \mathscr{E}$, $u \in \mathscr{B}$, there is a unique solution $x(u,\xi) \in \mathscr{L}$ of (4.6.5) so that $\|x(u,\xi) - x^\xi\|_\infty \le \rho_0$. Moreover, $\|x(u_1,\xi) - x(u_2,\xi)\|_\infty \le \widetilde{L}\|u_1 - u_2\|_\infty$ for any $\xi \in \mathscr{E}$ and $u_1, u_2 \in \mathscr{B}$. Furthermore, mapping $x : \mathscr{B} \times \mathscr{E} \to L^\infty_{loc}(\mathbb{R},\mathbb{R}^n)$ is continuous, where $L^\infty_{loc}(\mathbb{R},\mathbb{R}^n)$ is the usual topological vector space endowed with a metric*

$$d(u_1,u_2) := \sum_{k\in\mathbb{N}} \frac{\|u_1-u_2\|_{k,\infty}}{2^{|k|+1}(1+\|u_1-u_2\|_{k,\infty})}\,,$$

where $\|\cdot\|_{k,\infty}$ are the supremum norms on $[-k,k]$, $k \in \mathbb{N}$.

Next, it is easy to verify

$$x^{\sigma(\xi)}(t) = x^{\xi}(t+2\omega)\,.$$

Then by the 1-periodicity of (4.6.4) in t and the uniqueness of $x(u,\xi)$, from Theorem 4.6.2, we get

$$x(\widetilde{u},\sigma(\xi))(t) = x(u,\xi)(t+2\omega), \quad \forall t \in \mathbb{R}$$

for $\widetilde{u}(t) := u(t+2\omega)$, i.e. it holds

$$x(u,\xi)(2k\omega) = x\left(\widetilde{\sigma}^k(u), \sigma^k(\xi)\right)(0), \quad \forall k \in \mathbb{Z} \tag{4.6.6}$$

for a shift homeomorphism $\widetilde{\sigma} : \mathscr{B} \to \mathscr{B}$ defined as $\widetilde{\sigma}(u) := \widetilde{u}$.

Let $\varphi_u(t,s,y)$ be the evolution operator of (4.6.5) for $t,s \in \mathbb{R}$, $y \in \mathbb{R}^n$. Here for simplicity we suppose a technical condition that h is also globally Lipschitz continuous function in x. Then clearly

$$x(u,\xi)(2(k+1)\omega) = \varphi_u(2(k+1)\omega, 2k\omega, x(u,\xi)(2k\omega))\,, \quad \forall k \in \mathbb{Z}\,. \tag{4.6.7}$$

So (4.6.6) and (4.6.7) yield

$$x\left(\widetilde{\sigma}^{k+1}(u),\sigma^{k+1}(\xi)\right)(0)=\varphi_u\left(2(k+1)\omega,2k\omega,x\left(\widetilde{\sigma}^{k}(u),\sigma^{k}(\xi)\right)(0)\right),\quad \forall k\in\mathbb{Z},$$

that is,

$$x\left(\widetilde{\sigma}^{k+1}(u),\sigma(\xi)\right)(0)=\varphi_u\left(2(k+1)\omega,2k\omega,x\left(\widetilde{\sigma}^{k}(u),\xi\right)(0)\right),\quad \forall k\in\mathbb{Z}. \tag{4.6.8}$$

Now, introducing the following mappings

$$\Sigma:\mathscr{B}\times\mathscr{E}\times\mathbb{Z}\mapsto\mathscr{B}\times\mathscr{E}\times\mathbb{Z}$$

$$\Sigma(u,\xi,k):=(u,\sigma(\xi),k+1)\,,$$

$$\Phi:\mathscr{B}\times\mathscr{E}\times\mathbb{Z}\mapsto\mathscr{B}\times\mathbb{R}^n\times\mathbb{Z}$$

$$\Phi(u,\xi,k):=\left(u,x\left(\widetilde{\sigma}^{k}(u),\xi\right)(0),k\right)\,,$$

$$F^{2\omega}:\mathscr{B}\times\mathbb{R}^n\times\mathbb{Z}\mapsto\mathscr{B}\times\mathbb{R}^n\times\mathbb{Z}$$

$$F^{2\omega}(u,x,k):=(u,\varphi_u(2(k+1)\omega,2k\omega,x),k+1)\,,$$

and using (4.6.8), we obtain the following analogy of Theorem 3.5.5.

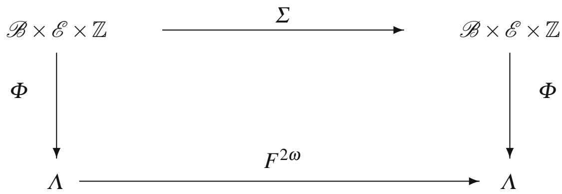

Fig. 4.8 Commutative diagram of inflated deterministic chaos.

Theorem 4.6.3. *The diagram of Figure* 4.8 *is commutative for the set*

$$\Lambda:=\Phi\left(\mathscr{B}\times\mathscr{E}\times\mathbb{Z}\right).$$

Moreover, mappings Σ and Φ are homeomorphisms.

For $u=0$, diagram of Figure 4.8 is again reduced to diagram of Figure 2.1 in Section 2.5.2 with $f(x)=\varphi_0(1,0,x)$ for the 1-time, Poincarè map of (4.6.3). Finally, we can extend very similarly Theorem 3.5.6 to (4.6.4), but we do not write it since that extension is almost identical to Theorem 3.5.6.

4.6.3 Inflated Autonomous ODEs

In general, the situation is different when (4.6.3) is autonomous. Let us consider an ODE

$$\dot{x} = h(x), \tag{4.6.9}$$

where $h \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ satisfies the following assumption:

(A1) (4.6.9) possesses a solution $\gamma(t)$ homoclinic to a hyperbolic equilibrium 0. Moreover, the variational equation $\dot{v} = Dh(\gamma(t))v$ has the only bounded solution $\dot{\gamma}(t)$ on $\mathbb{R}$ up to constant multiplies.

Assumption (A1) means that γ is nondegenerate in the sense that the stable and unstable manifolds of 0 transversally intersect along γ (cf Section 2.5.4 and [7,54]). Moreover, we know from Section 4.1.2 that (A1) implies that the adjoint variational equation $\dot{v} = -Dh(\gamma(t))^* v$ has the only bounded solution $\psi(t)$ on $\mathbb{R}$ up to constant multiplies.

Next, we consider a differential inclusion in $\mathbb{R}^n$ of the form

$$\dot{x} \in h(x) + \varepsilon q(x, \mathscr{B}_{\mathbb{R}^n}) \tag{4.6.10}$$

where $0 \neq \varepsilon \in \mathbb{R}$ is small and $q \in C(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ satisfies the following assumption:

(A2) There are positive constants λ, μ so that

$$|q(x,p) - q(\tilde{x}, \tilde{p})| \leq \lambda |x - \tilde{x}| + \mu |p - \tilde{p}|$$

for all $x, \tilde{x} \in \mathbb{R}^n$, $p, \tilde{p} \in \mathscr{B}_{\mathbb{R}^n}$.

Again (4.6.10) is equivalent to the family of ODEs

$$\dot{x} = h(x) + \varepsilon q(x, u(t)), \qquad u \in \mathscr{B}. \tag{4.6.11}$$

For any fixed $u \in \mathscr{B}$, (4.6.11) is the standard bifurcation problem studied in Section 4.1.3. Consequently, we can state the following result.

Theorem 4.6.4. *There is an $\varepsilon^0 > 0$ so that for any $|\varepsilon| < \varepsilon^0$ and $u \in \mathscr{B}$ there is a unique bounded solution x_u of (4.6.11) with a small amplitude. Next, let us set*

$$M_u(\alpha) := \int_{-\infty}^{\infty} \psi^*(t+\alpha) q(\gamma(t+\alpha), u(t))\, dt. \tag{4.6.12}$$

Then there is an $\varepsilon^0 \geq \varepsilon_0 = \varepsilon_0(u) > 0$ so that for any $0 < |\varepsilon| < \varepsilon_0$ it holds

(i) *If there is an $\alpha_0 \in \mathbb{R}$ so that $M_u(\alpha_0) = 0$ and M_u is strictly monotone at α_0, then there is a unique bounded solution x of (4.6.11) so that*

$$\|x - \gamma(\cdot + \alpha_0)\|_\infty \to 0$$

as $\varepsilon \to 0$, and x is asymptotic to x_u as $|t| \to \infty$. Moreover there is a Smale horseshoe type chaos when u is almost periodic.

(ii) *If M_u is changing the sign over $\mathbb{R}$, then there is a bounded solution x of* (4.6.11) *orbitally near to γ and x is asymptotic to x_u as $|t| \to \infty$. Moreover there is a Smale semi-horseshoe type chaos when u is almost periodic.*

(iii) *If $\inf_{\mathbb{R}} |M_u| > 0$ then there is no bounded solution of* (4.6.11) *near γ and asymptotic to x_u as $|t| \to \infty$.*

Remark 4.6.5. $\mathscr{B}$ contains two disjoint (possible empty) open subsets $\mathscr{B}_1$ and $\mathscr{B}_2$ which are satisfied either of (ii) or (iii) of Theorem 4.6.4.

Example 4.6.6. Let us consider an ε-inflated weakly damped Duffing equation

$$\ddot{x} \in x - 2x^3 + \varepsilon(-\delta\dot{x} + [-1,1])$$

for a $\delta > 0$. Then $\gamma(t) = (\gamma(t), \dot{\gamma}(t))$, $\gamma = \operatorname{sech} t$, $\psi(t) = (-\ddot{\gamma}(t), \dot{\gamma}(t))$, and thus (4.6.12) has the form

$$M_u(\alpha) = \int_{-\infty}^{\infty} \dot{\gamma}(t+\alpha)\,(-\delta\dot{\gamma}(t+\alpha) + u(t))\,dt = -\frac{2}{3}\delta + \int_{-\infty}^{\infty} \dot{\gamma}(t+\alpha)u(t)\,dt.$$

Using

$$|M_u(\alpha)| \geq \frac{2}{3}\delta - \|u\|_\infty \int_{-\infty}^{\infty} |\dot{\gamma}(t+\alpha)|\,dt = \frac{2}{3}\delta - 2\|u\|_\infty,$$

we see that if $\|u\|_\infty < \min\left\{\frac{\delta}{3}, 1\right\}$ then $u \in \mathscr{B}_2$. Particularly, for $\delta > 3$ we get $\mathscr{B} = \mathscr{B}_2$. If $0 < \delta \leq 3$, then we take $u(t) = -\operatorname{sgn} t$. Hence

$$M_{-\operatorname{sgn}}(\alpha) = -\frac{2}{3}\delta - \int_{-\infty}^{\infty} \dot{\gamma}(t+\alpha)\operatorname{sgn} t\,dt = -\frac{2}{3}\delta + 2\operatorname{sech}\alpha.$$

We see that if $\delta = 3$ then $-\operatorname{sgn} t \in \mathscr{B} \setminus (\mathscr{B}_1 \cup \mathscr{B}_2)$ and if $0 < \delta < 3$ then $-\operatorname{sgn} t \in \mathscr{B}_1$. Finally we take $u(t) = \theta\cos t$ for $0 \leq \theta \leq 1$. Hence

$$M_{\theta\cos}(\alpha) = -\frac{2}{3}\delta + \theta \int_{-\infty}^{\infty} \dot{\gamma}(t+\alpha)\cos t\,dt = -\frac{2}{3}\delta - \pi\theta\operatorname{sech}\frac{\pi}{2}\sin\alpha.$$

If $0 < \delta < \frac{3}{2}\pi\operatorname{sech}\frac{\pi}{2} \doteq 1.87806$ then $\theta\cos t \in \mathscr{B}_2$ for $0 \leq \theta < \frac{2\delta}{3\pi}\cosh\frac{\pi}{2}$, $\theta\cos t \in \mathscr{B}_1$ for $1 \geq \theta > \frac{2\delta}{3\pi}\cosh\frac{\pi}{2}$ and $\frac{2\delta}{3\pi}\cosh\frac{\pi}{2}\cos t \in \mathscr{B} \setminus (\mathscr{B}_1 \cup \mathscr{B}_2)$. If $\delta = \frac{3}{2}\pi\operatorname{sech}\frac{\pi}{2}$ then $\theta\cos t \in \mathscr{B}_2$ for $0 \leq \theta < 1$ and $\cos t \in \mathscr{B} \setminus (\mathscr{B}_1 \cup \mathscr{B}_2)$. If $\delta > \frac{3}{2}\pi\operatorname{sech}\frac{\pi}{2}$ then $\theta\cos t \in \mathscr{B}_2$ for $0 \leq \theta \leq 1$. These inequalities are balance between the damping and forcing to either get chaos, or exclude it near the homoclinic solution.

Finally we remark that the inflated chaos could be extended also to the autonomous case (4.6.10) under the assumption

(A3) (4.6.9) possesses a hyperbolic nonconstant periodic solution $x_0(t)$ with a transversal homoclinic point $z \in W^s(x_0) \cap W^u(x_0)$, i.e. $T_zW^s(x_0) \cap T_zW^u(x_0) = \operatorname{span}\{h(z)\}$.

The method of [55] could be used together with our parameterized approach but this is outside scope of this book.

4.7 Nonlinear Diatomic Lattices

4.7.1 Forced and Coupled Nonlinear Lattices

We end this chapter with infinite dimensional ODEs [56, 57]. Let us consider a model of two one-dimensional interacting sublattices of harmonically coupled protons and heavy ions [58–61]. It represents the Bernal-Flower filaments in ice or more complex biological macromolecules in membranes, in which only the degrees of freedom that contribute predominantly to proton mobility have been conserved. In these systems, each proton lies between a pair of "oxygens". The proton part of the Hamiltonian is

$$H_p = \sum_n \frac{1}{2} m\dot{u}_n^2 + U(u_n) + \frac{1}{2} k_1 (u_{n+1} - u_n)^2 ,$$

where u_n denotes the displacement of the nth proton with respect to the center of the oxygen pair and k_1 is the coupling between neighboring protons. Furthermore, $U(u) = \xi_0(1 - u^2/d_0^2)^2$ is the double-well potential with the potential barrier ξ_0, and $2d_0$ is the distance between its two minima. Finally, m is the mass of protons.

Similarly, the oxygen part of the Hamiltonian is

$$H_O = \sum_n \frac{1}{2} M\dot{\rho}_n^2 + \frac{1}{2} M\Omega_0^2 \rho_n^2 + \frac{1}{2} K_1 (\rho_{n+1} - \rho_n)^2 ,$$

where ρ_n is the displacement between two oxygens, M is the mass of oxygens, Ω_0 is the frequency of the optical mode and K_1 is the harmonic coupling between neighboring oxygens.

The last part in the Hamiltonian of the model arises from the dynamical interaction between two sublattices and it is given by

$$H_{int} = \sum_n \chi \rho_n (u_n^2 - d_0^2) ,$$

where χ measures the strength of the coupling. The Hamiltonian of the model is the sum of these three contributions $H = H_p + H_O + H_{int}$.

We are also interested in the influence of external field and damping. For the model studied here, since a spatially homogeneous field is not coupled to the optical motion ρ_n of the oxygens, a force term has to be considered only in the equation of motion of the protons.

In summary, we consider in this section the following coupled infinite chain of oscillators

$$\begin{aligned}\ddot{u}_n+\Gamma_1\dot{u}_n &= \frac{k_1}{m}\big(u_{n+1}-2u_n+u_{n-1}\big)+\frac{4\xi_0}{md_0^2}u_n\Big(1-\frac{u_n^2}{d_0^2}\Big)-2\frac{\chi}{m}\rho_n u_n+\frac{F}{m}\,,\\ \ddot{\rho}_n+\Gamma_2\dot{\rho}_n &= \frac{K_1}{M}\big(\rho_{n+1}-2\rho_n+\rho_{n-1}\big)-\Omega_0^2\rho_n-\frac{\chi}{M}(u_n^2-d_0^2)\,,\end{aligned} \tag{4.7.1}$$

where F is the external force on the protons and Γ_1,Γ_2 are the damping coefficients for the proton and oxygen motions.

We are interested in the existence of homoclinic and chaotic *spatially localized solutions* of (4.7.1). The existence of time periodic spatially localized solutions, the so-called *breathers* are studied in [62–68].

4.7.2 Spatially Localized Chaos

We assume in this section that $\Gamma_1=\varepsilon\delta_1$, $\Gamma_2=\varepsilon\delta_2$, $F/m=\varepsilon f(t)$, $k_1/m=\varepsilon\mu_1$, $K_1/M=\varepsilon\mu_2$, $-2\chi/m=\varepsilon\mu_3$, $-\chi/M=\varepsilon\mu_4$ for a small parameter $\varepsilon>0$, constants $\delta_1\geq 0$, $\delta_2>0$, μ_i, $i=1,2,3,4$ and a C^1-smooth T-periodic function $f(t)$. Putting

$$a^2:=\frac{4\xi_0}{md_0^4}\,,$$

(4.7.1) has the form

$$\begin{aligned}&\ddot{u}_n+\varepsilon\delta_1\dot{u}_n+a^2u_n\big(u_n^2-d_0^2\big)=\varepsilon\mu_1\big(u_{n+1}-2u_n+u_{n-1}\big)+\varepsilon\mu_3\rho_n u_n+\varepsilon f(t)\,,\\ &\ddot{\rho}_n+\varepsilon\delta_2\dot{\rho}_n+\Omega_0^2\rho_n=\varepsilon\mu_2\big(\rho_{n+1}-2\rho_n+\rho_{n-1}\big)+\varepsilon\mu_4(u_n^2-d_0^2)\,.\end{aligned} \tag{4.7.2}$$

We first consider the system

$$\begin{aligned}\ddot{u}+\varepsilon\delta_1\dot{u}+a^2u\big(u^2-d_0^2\big)&=\varepsilon\mu_3\rho u+\varepsilon f(t)\,,\\ \ddot{\rho}+\varepsilon\delta_2\dot{\rho}+\Omega_0^2\rho&=\varepsilon\mu_4(u^2-d_0^2)\,.\end{aligned} \tag{4.7.3}$$

The equation

$$\dot{u}=v,\quad \dot{v}=a^2(d_0^2-u^2)u$$

has a hyperbolic equilibrium $u=v=0$ and centers $u=\pm d_0$, $v=0$ [35]. Furthermore, there are two symmetric homoclinic solutions $(\gamma(t),\dot{\gamma}(t))$ and $(-\gamma(t),-\dot{\gamma}(t))$ for $\gamma(t)=\sqrt{2}d_0\operatorname{sech} ad_0t$. Now we make the change of variable $\rho\leftrightarrow\rho-\frac{\varepsilon\mu_4d_0^2}{\Omega_0^2}$ in (4.7.3) to get

$$\begin{aligned}\ddot{u}+\varepsilon\delta_1\dot{u}+a^2u\big(u^2-d_0^2\big)&=\varepsilon\mu_3\left(\rho-\frac{\varepsilon\mu_4d_0^2}{\Omega_0^2}\right)u+\varepsilon f(t)\,,\\ \ddot{\rho}+\varepsilon\delta_2\dot{\rho}+\Omega_0^2\rho&=\varepsilon\mu_4u^2\,.\end{aligned}$$

To study a small T-periodic solution of the above system, we take its equivalent form

$$\ddot{u}+\varepsilon\delta_1\dot{u}+a^2u\big(u^2-d_0^2\big)=$$
$$\varepsilon\mu_3\left(\frac{\varepsilon\mu_4}{\Omega_\varepsilon}\int_{-\infty}^{t}\mathrm{e}^{-\varepsilon\delta_2(t-s)/2}\sin\Omega_\varepsilon(t-s)u^2(s)\,ds-\frac{\varepsilon\mu_4d_0^2}{\Omega_0^2}\right)u+\varepsilon f(t) \tag{4.7.4}$$

where $\Omega_\varepsilon=\sqrt{\Omega_0^2-\frac{\varepsilon^2\delta_2^2}{4}}$ and $0<\varepsilon<2\Omega_0/\delta_2$. Now it is not difficult to prove for (4.7.4) by using the implicit function theorem the existence of a unique small T-periodic solution $u_\varepsilon(t)=O(\varepsilon)$, $\rho_\varepsilon(t)=O(\varepsilon)$ of (4.7.3) . Then we make in (4.7.2) the change of variables $u_n\leftrightarrow u_n+u_\varepsilon$, $\rho_n\leftrightarrow\rho_n+\rho_\varepsilon$ to get the chain

$$\begin{aligned}
&\dot{u}_n=v_n,\\
&\dot{v}_n+\varepsilon\delta_1v_n-a^2u_nd_0^2+a^2u_n^3+3a^2u_n^2u_\varepsilon+3a^2u_nu_\varepsilon^2\\
&=\varepsilon\mu_1\big(u_{n+1}-2u_n+u_{n-1}\big)+\varepsilon\mu_3\big(\rho_nu_n+\rho_nu_\varepsilon+\rho_\varepsilon u_n\big);\\
&\dot{\rho}_n=\psi_n,\\
&\dot{\psi}_n+\varepsilon\delta_2\psi_n+\Omega_0^2\rho_n=\varepsilon\mu_2\big(\rho_{n+1}-2\rho_n+\rho_{n-1}\big)+\varepsilon\mu_4(u_n^2+2u_\varepsilon u_n)\,.
\end{aligned} \tag{4.7.5}$$

We consider (4.7.5) as an ODE on the Hilbert space

$$H:=\left\{z=\{(u_n,v_n,\rho_n,\psi_n)\}_{n\in\mathbb{Z}}\mid\sum_{n\in\mathbb{Z}}\big(u_n^2+v_n^2+\rho_n^2+\psi_n^2\big)<\infty\right\}$$

with the norm $\|z\|=\sqrt{\sum_{n\in\mathbb{Z}}(u_n^2+v_n^2+\rho_n^2+\psi_n^2)}$. The non-homogeneous linearization of (4.7.5) at $z=0$ has the form

$$\begin{aligned}
&\dot{u}_n=v_n+h_{n1}(t),\\
&\dot{v}_n+\varepsilon\delta_1v_n+u_n(3a^2u_\varepsilon^2-a^2d_0^2-\varepsilon\mu_3\rho_\varepsilon),\\
&-\varepsilon\mu_1\big(u_{n+1}-2u_n+u_{n-1}\big),-\varepsilon\mu_3\rho_nu_\varepsilon=h_{n2}(t);\\
&\dot{\rho}_n=\psi_n+g_{n1}(t),\\
&\dot{\psi}_n+\varepsilon\delta_2\psi_n+\Omega_0^2\rho_n-\varepsilon\mu_2\big(\rho_{n+1}-2\rho_n+\rho_{n-1}\big)-2\varepsilon\mu_4u_\varepsilon u_n=g_{n2}(t),
\end{aligned} \tag{4.7.6}$$

with $w(t)=\big\{(h_{n1}(t),h_{n2}(t),g_{n1}(t),g_{n2}(t))\big\}_{n\in\mathbb{Z}}\in C_b(\mathbb{R},H)$ – the Banach space of all bounded continuous functions from $\mathbb{R}$ to H with the norm $|w|=\sup_{\mathbb{R}}\|w(t)\|$. We look for a solution $z\in C_b(\mathbb{R},H)$ of (4.7.5) for $\varepsilon>0$ small. For this reason, we consider the Hilbert spaces $H_2 := H_1 \times H_1$ and

$$H_1 := \left\{ \{u_n\}_{n\in\mathbb{Z}} \mid \sum_{n\in\mathbb{Z}} u_n^2 < \infty \right\}$$

with the corresponding standard norms and scalar products. We first study the equation

$$\dot{\rho} = \psi + g_1, \quad \dot{\psi} + \varepsilon\delta_2\psi + A_\varepsilon\rho = g_2 \tag{4.7.7}$$

on H_2 for $(g_1, g_2) \in C_b(\mathbb{R}, H_2)$ and

$$A_\varepsilon\rho = \left\{ \Omega_0^2\rho_n - \varepsilon\mu_2(\rho_{n+1} - 2\rho_n + \rho_{n-1}) \right\}_{n\in\mathbb{Z}}.$$

Clearly $A_\varepsilon : H_1 \to H_1$ is symmetrically and positively definite for ε small. Then for any small ε, there is a symmetrically and positively definite $B_\varepsilon : H_1 \to H_1$ so that

$$B_\varepsilon^2 = A_\varepsilon - \frac{\varepsilon^2\delta_2^2}{4}\mathbb{I}.$$

We take the operators $\cos B_\varepsilon t$ and $\sin B_\varepsilon t$ from H_1 to H_1. For any $\rho \in H_1$, we consider the function

$$\phi(t) := |\cos B_\varepsilon t\rho|^2 + |\sin B_\varepsilon t\rho|^2.$$

Then we have

$$\dot{\phi}(t) = -2\langle \cos B_\varepsilon t\rho, B_\varepsilon \sin B_\varepsilon t\rho\rangle + 2\langle \sin B_\varepsilon t\rho, B_\varepsilon \cos B_\varepsilon t\rho\rangle = 0.$$

Hence

$$|\cos B_\varepsilon t\rho|^2 + |\sin B_\varepsilon t\rho|^2 = \rho,$$

and then $\|\cos B_\varepsilon t\| \leq 1$ and $\|\sin B_\varepsilon t\| \leq 1$. Now, the equation

$$\dot{\rho} = \psi, \quad \dot{\psi} + \varepsilon\delta_2\psi + A_\varepsilon\rho = 0 \tag{4.7.8}$$

has the form $\ddot{\rho} + \varepsilon\delta_2\dot{\rho} + A_\varepsilon\rho = 0$ which has the general solution

$$\mathrm{e}^{-\varepsilon\delta_2 t/2}\left[\cos B_\varepsilon t\rho_1 + \sin B_\varepsilon t\rho_2\right]$$

for $\rho_{1,2} \in H_1$. Consequently, the fundamental solution of (4.7.8) has the form

$$V_\varepsilon(t) = \mathrm{e}^{-\varepsilon\delta_2 t/2}W_\varepsilon(t)$$

with uniformly bounded $W_\varepsilon(t)$ for $\varepsilon > 0$ small. Thus, the only bounded solution of (4.7.7) has the form

$$(\rho(t), \psi(t)) = \int_{-\infty}^{t} \mathrm{e}^{-\varepsilon\delta_2(t-s)/2}W_\varepsilon(t-s)\,(g_1(s), g_2(s))\,ds. \tag{4.7.9}$$

Hence

$$|(\rho,\psi)| \le K_1|(g_1,g_2)|/\varepsilon$$

for a constant $K_1 > 0$ independent of $\varepsilon > 0$ small. Furthermore, it is not difficult to see that the linear system

$$\dot{u}_n = v_n + h_{n1}(t), \quad \dot{v}_n + \varepsilon\delta v_n - a^2 d_0^2 u_n = h_{n2}(t)$$

has a unique solution $\{(u_n(t),v_n(t))\}_{n\in\mathbb{Z}} \in C_b(\mathbb{R},H_2)$ so that

$$\big|\{(u_n(t),v_n(t))\}_{n\in\mathbb{Z}}\big| \le K_2\big|\{(h_{n1}(t),h_{n2}(t))\}_{n\in\mathbb{Z}}\big|$$

for a constant $K_2 > 0$ independent of $\varepsilon > 0$ small. Now we turn back to (4.7.6). Summarizing the above arguments, we see, by using the Banach contraction mapping principle 2.2.1 for $\varepsilon > 0$ small, that (4.7.6) has for any $w(t) \in C_b(\mathbb{R},H)$ a unique solution $z \in C_b(\mathbb{R},H)$ so that $|z| \le K_3|w|/\varepsilon$ for a constant $K_3 > 0$ independent of $\varepsilon > 0$ small. Since the system (4.7.6) is T-periodic, we get from Lemma 2.5.5 that (4.7.6) has an exponential dichotomy on $\mathbb{R}$ in the space H for any $\varepsilon > 0$ sufficiently small. Consequently, we get another result.

Theorem 4.7.1. *The T-periodic solution $u_n(t) = u_\varepsilon(t)$, $\rho_n(t) = \rho_\varepsilon(t)$ $\forall n \in \mathbb{Z}$ of* (4.7.2) *is hyperbolic in H for any $\varepsilon > 0$ sufficiently small, i.e. the zero equilibrium of* (4.7.5) *in H is hyperbolic.*

Now we look for more complicated solutions of (4.7.2). For this reason, we shift in (4.7.5) the time $t \leftrightarrow t+\alpha$ to get the system

$$\begin{gathered}\dot{u}_n = v_n\\ \dot{v}_n + \varepsilon\delta_1 v_n - a^2u_nd_0^2 + a^2u_n^3 + 3a^2u_n^2u_\varepsilon(t+\alpha) + 3a^2u_nu_\varepsilon^2(t+\alpha)\\ = \varepsilon\mu_1\big(u_{n+1}-2u_n+u_{n-1}\big) + \varepsilon\mu_3\big(\rho_nu_n+\rho_nu_\varepsilon(t+\alpha)+\rho_\varepsilon(t+\alpha)u_n\big),\\ \dot{\rho}_n = \psi_n\\ \dot{\psi}_n + \varepsilon\delta_2\psi_n + \Omega_0^2\rho_n = \varepsilon\mu_2\big(\rho_{n+1}-2\rho_n+\rho_{n-1}\big)\\ +\ \varepsilon\mu_4(u_n^2+2u_\varepsilon(t+\alpha)u_n).\end{gathered} \tag{4.7.10}$$

We look for a solution of (4.7.10) for $\varepsilon > 0$ small so that $u_n \sim 0$, $v_n \sim 0$ for $n \neq 0$ and $u_0 \sim \gamma$, $v_0 \sim \dot{\gamma}$. Let $(\rho_0,\psi_0) = \{(\rho_n^0,\psi_n^0)\}_{n\in\mathbb{Z}}$ be the unique bounded solution of (4.7.7) for $g_1 = 0$ and $g_2 = \{g_{n2}\}_{n\in\mathbb{Z}}$ with $g_{n2} = 0$ for $n \neq 0$ and $g_{02} = \varepsilon\mu_4(\gamma^2 + 2u_\varepsilon(t+\alpha)\gamma)$. Let us put $u_n^0 = v_n^0 = 0$ for $n \neq 0$ and $u_0^0 = \gamma$, $v_0^0 = \dot{\gamma}$. Now we make in (4.7.10) the change of variables $u_n \leftrightarrow u_n + u_n^0$, $v_n \leftrightarrow v_n + v_n^0$, $\rho_n \leftrightarrow \rho_n + \rho_n^0$, $\psi_n \leftrightarrow \psi_n + \psi_n^0$ to get for $n \neq 0$ the system

$$\begin{aligned}&\dot{u}_n = v_n,\\ &\dot{v}_n + \varepsilon\delta_1 v_n - a^2u_nd_0^2 + a^2u_n^3 + 3a^2u_n^2u_\varepsilon(t+\alpha) + 3a^2u_nu_\varepsilon^2(t+\alpha)\\ &= \varepsilon\mu_1\big(u_{n+1}+u_{n+1}^0-2u_n+u_{n-1}+u_{n-1}^0\big)\\ &\quad + \varepsilon\mu_3\big((\rho_n+\rho_n^0)u_n+(\rho_n+\rho_n^0)u_\varepsilon(t+\alpha)+\rho_\varepsilon(t+\alpha)u_n\big);\end{aligned} \tag{4.7.11}$$

$$\begin{aligned}\dot{\rho}_n &= \psi_n, \\ \dot{\psi}_n + \varepsilon\delta_2\psi_n + \Omega_0^2\rho_n &= \varepsilon\mu_2\big(\rho_{n+1} - 2\rho_n + \rho_{n-1}\big) \\ &\quad + \varepsilon\mu_4(u_n^2 + 2u_\varepsilon(t+\alpha)u_n)\,.\end{aligned}$$

For the mode $n = 0$, we first note that the system

$$\dot{u}_0 = v_0, \quad \dot{v}_0 + a^2(3\gamma^2 - d_0^2)u_0 = h(t)$$

for $h(t) \in C_b(\mathbb{R},\mathbb{R})$ has a solution $(u_0,v_0) \in C_b(\mathbb{R},\mathbb{R}^2)$ (see Section 4.1) if and only if $\int_{-\infty}^{\infty} h(t)\dot{\gamma}(t)\,dt = 0$ and such a solution is unique if $\int_{-\infty}^{\infty} u_0(t)\dot{\gamma}(t)\,dt = 0$. Consequently, for the mode $n = 0$ we get from (4.7.10) the equations

$$\begin{gathered}\dot{u}_0 = v_0, \\ \dot{v}_0 + a^2(3\gamma^2 - d_0^2)u_0 = h(t) - \dot{\gamma}(t)\int_{-\infty}^{\infty} h(t)\dot{\gamma}(t)\,dt \Big/ \int_{-\infty}^{\infty}\dot{\gamma}(t)^2\,dt\,, \\ \int_{-\infty}^{\infty} u_0(t)\dot{\gamma}(t)\,dt = 0; \\ \dot{\rho}_0 = \psi_0 \\ \dot{\psi}_0 + \varepsilon\delta_2\psi_0 + \Omega_0^2\rho_0 = \varepsilon\mu_2\big(\rho_1 - 2\rho_0 + \rho_{-1}\big) \\ + \varepsilon\mu_4(u_0^2 + 2u_0\gamma + 2u_\varepsilon(t+\alpha)u_0)\,,\end{gathered} \tag{4.7.12}$$

and

$$\int_{-\infty}^{\infty} h(t)\dot{\gamma}(t)\,dt = 0 \tag{4.7.13}$$

for

$$\begin{aligned}h(t) = &-a^2(u_0^3 + 3u_0^2\gamma) - \varepsilon\delta_1\dot{\gamma} - 3a^2(u_0+\gamma)^2u_\varepsilon(t+\alpha) - \varepsilon\delta_1 v_0 \\ &- 3a^2(u_0+\gamma)u_\varepsilon^2(t+\alpha) + \varepsilon\mu_1\big(u_1 - 2(u_0+\gamma) + u_{-1}\big) \\ &+ \varepsilon\mu_3\big((\rho_0+\rho_0^0)(u_0+\gamma) + (\rho_0+\rho_0^0)u_\varepsilon(t+\alpha) + \rho_\varepsilon(t+\alpha)(u_0+\gamma)\big)\,.\end{aligned} \tag{4.7.14}$$

Now for $\varepsilon > 0$ small, we can solve (4.7.12) and (4.7.12) to get the solution

$$z = \Big\{\Big(u_n(t), v_n(t), \rho_n(t), \psi_n(t)\Big)\Big\}_{n\in\mathbb{Z}} \in C_b(\mathbb{R},H),$$

so that $z = O(\varepsilon)$. Then we put this z into (4.7.15) to get the function $h_{\varepsilon,\alpha} \in C_b(\mathbb{R},\mathbb{R})$. We note $h_{\varepsilon,\alpha}(t) = O(\varepsilon)$ uniformly for $\varepsilon > 0$ small and $\alpha, t \in \mathbb{R}$. Clearly $h_{\varepsilon,\alpha}(t)$ is T-periodic in α. Then from (4.7.13) we get the bifurcation equation

$$Q(\varepsilon,\alpha) := \frac{1}{\varepsilon}\int_{-\infty}^{\infty} h_{\varepsilon,\alpha}(t)\dot{\gamma}(t)\,dt = 0\,.$$

If we put

$$\lim_{\varepsilon\to 0} u_\varepsilon(t)/\varepsilon = w(t), \quad \lim_{\varepsilon\to 0} \rho_\varepsilon(t)/\varepsilon = \zeta(t),$$

then from (4.7.3) we get

$$\ddot{w} - a^2 d_0^2 w = f(t), \quad \ddot{\zeta} + \Omega_0^2 \zeta = -\mu_4 d_0^2 .$$

Hence $\zeta = -\mu_4 d_0^2/\Omega_0^2$ and

$$w(t) = -\frac{1}{2ad_0}\int_{-\infty}^{t} \mathrm{e}^{-ad_0(t-s)} f(s)\,ds - \frac{1}{2ad_0}\int_{t}^{\infty} \mathrm{e}^{ad_0(t-s)} f(s)\,ds . \tag{4.7.15}$$

Clearly $w(t)$ is T-periodic. Furthermore, since $\gamma(t) \to 0$ as $t \to \pm\infty$ exponentially, from formula (4.7.9) we see that $\lim_{\varepsilon\to 0}(\rho_0, \psi_0)/\varepsilon = \{(\rho_{0n}, \psi_{0n})\}_{n\in\mathbb{Z}}$ with $\rho_{0n} = \psi_{0n} = 0$ for $n \neq 0$ and

$$\ddot{\rho}_{00} + \Omega_0^2 \rho_{00} = \mu_4 \gamma(t)^2 ,$$

i.e. $\rho_{00}(t) = \dfrac{\mu_4}{\Omega_0}\int_{-\infty}^{t} \sin\Omega_0(t-s)\gamma(s)^2\,ds$. In summary, from (4.7.15) we get

$$\begin{aligned} M(\alpha) := Q(0,\alpha) &= \int_{-\infty}^{\infty}\Big[-\delta_1\dot{\gamma}(t) - 3a^2\gamma(t)^2 w(t+\alpha) - 2\mu_1\gamma(t)\Big]\dot{\gamma}(t)\,dt \\ &= -\frac{4}{3}\delta_1 a d_0^3 + a^2\int_{-\infty}^{\infty}\gamma(t)^3\dot{w}(t+\alpha)\,dt . \end{aligned} \tag{4.7.16}$$

Clearly $M(\alpha)$ is T-periodic. We note that similarly we can prove that

$$\lim_{\varepsilon\to 0}\frac{\partial}{\partial\alpha} Q(\varepsilon,\alpha)/\varepsilon = M'(\alpha)$$

uniformly for $\alpha \in \mathbb{R}$. In summary, we get another result.

Theorem 4.7.2. *Let M be given by* (4.7.16). *If there is a simple zero α_0 of M, i.e. $M(\alpha_0) = 0$ and $M'(\alpha_0) \neq 0$, then* (4.7.2) *has for any $\varepsilon > 0$ small a bounded solution $z(t)$ with small u_n, ρ_n for $n \neq 0$ and (u_0, ρ_0) near $(\gamma(t-\alpha_0), 0)$.*

Now, it is not difficult to prove like in the finite-dimensional case (cf Section 4.1) that

$$\left(z(t) - \left\{(u_\varepsilon(t), \dot{u}_\varepsilon(t), \rho_\varepsilon(t), \dot{\rho}_\varepsilon(t))\right\}_{n\in\mathbb{Z}}\right) \to 0$$

is exponentially fast as $t \to \pm\infty$ in H. Moreover, near $z(t)$ we can construct the Smale horseshoe. Consequently, we get in this case the chaos in (4.7.2) with corresponding infinitely many periodic orbits with arbitrarily large periods. This Smale horseshoe of (4.7.2) is spatially localized but not exponentially like in breathers.

To be more concrete, we take

$$f(t) = \Upsilon\cos\omega t$$

for $\Upsilon > 0$. Then (4.7.15) gives

$$w(t) = -\frac{\Upsilon}{\omega^2 + a^2 d_0^2} \cos \omega t \,,$$

and the formula (4.7.16) has now the form

$$M(\alpha) = -\frac{4}{3}\delta_1 a d_0^3 + \frac{\omega \Upsilon \pi \sqrt{2}}{a} \operatorname{sech} \frac{\omega \pi}{2 a d_0} \sin \omega \alpha \,.$$

Consequently, if

$$8\sqrt{2}\delta_1 \xi_0 < 3 m \omega \Upsilon \pi d_0 \operatorname{sech} \frac{\omega d_0 \pi \sqrt{m}}{4\sqrt{\xi_0}} \,, \tag{4.7.17}$$

then $M(\alpha)$ has a simple zero, so (4.7.2) is chaotic for any $\varepsilon > 0$ small. We note that the inequality (4.7.17) gives sufficient conditions between the magnitude of the forcing Υ and the damping δ_1 in order to get chaos in (4.7.2) for $\varepsilon > 0$ small. So chaos is generated by the proton part of (4.7.2). If $\delta_1 = 0$ then (4.7.2) is always chaotic for $f(t) = \Upsilon \cos \omega t$. Furthermore, if $\Gamma_1 > 0, \Gamma_2 > 0$ and $F = 0$, i.e. there is no forcing but damping then it is not difficult to prove that (4.7.1) has no nonconstant periodic solutions in the space H.

Finally, we note that similarly we can study the case when more than one modes are excited. We do not carry out here such computations [64].

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Chapter 5
Chaos in Partial Differential Equations

Functional analytical methods are presented in this chapter to predict chaos for periodically forced PDEs modeling vibrations of beams and depend on parameters.

5.1 Beams on Elastic Bearings

5.1.1 Weakly Nonlinear Beam Equation

This section deals with the beam equation (Figure 5.1)

$$\begin{aligned} &u_{tt} + u_{xxxx} + \varepsilon\delta u_t + \varepsilon\mu h(x, \sqrt{\varepsilon}t) = 0\,, \\ &u_{xx}(0,\cdot) = u_{xx}(\pi/4,\cdot) = 0\,, \\ &u_{xxx}(0,\cdot) = -\varepsilon f(u(0,\cdot)), \quad u_{xxx}(\pi/4,\cdot) = \varepsilon f(u(\pi/4,\cdot)) \end{aligned} \tag{5.1.1}$$

where $\varepsilon > 0$ and μ are sufficiently small parameters, $\delta > 0$ is a constant, $f \in C^2(\mathbb{R})$, $h \in C^2([0,\pi/4] \times \mathbb{R})$ and $h(x,t)$ is 1-periodic in t, provided an associated reduced equation has a homoclinic orbit (cf (5.1.9)). Equation (5.1.1) describes vibrations of a beam resting on two identical bearings with purely elastic responses which are determined by f. The length of the beam is $\pi/4$. Since $\varepsilon > 0$ is small, (5.1.1) is a semilinear, weakly damped, weakly forced and slowly varying problem.

Let us briefly recall some results related to Eq. (5.1.1). The undamped case ($\delta = 0$, $\mu = 0$ and $\varepsilon = 1$) was studied in [1, 2] by using variational methods. In both papers, the problems studied are non-parametric.

The perturbation approach to the beam equation was earlier used in [3]. Recent results in this direction are given in [4, 5]. We note that the problem (5.1.1) is more complicated than the one studied in [3–5], since in those papers the elastic response is distributed continuously along the beam, while in our case it is concentrated just at two end points of the beam. Moreover, the ε-smallness of the restoring force εf at the end points leads to a singularly perturbed problem in studying chaotic orbits

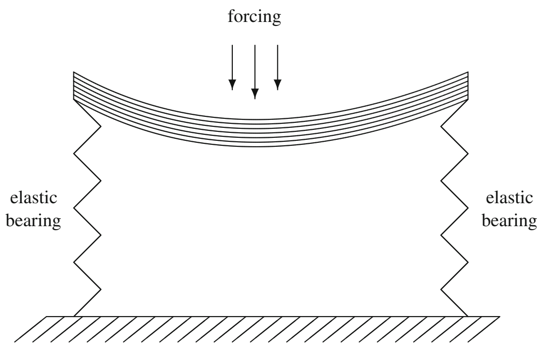

Fig. 5.1 The forced beam resting on two elastic bearings (5.1.1).

of (5.1.1). The existence of homoclinic and chaotic solutions has also been proved in [6–9] for different partial differential equations, with different methods compared with ours.

5.1.2 Setting of the Problem

First of all, we make the linear scale $t \leftrightarrow \sqrt{\varepsilon}t$ in (5.1.1), that is, we take $u(x,t) \leftrightarrow u(x,\sqrt{\varepsilon}t)$ to get the equivalent problem

$$\begin{aligned} &u_{tt} + \frac{1}{\varepsilon}u_{xxxx} + \sqrt{\varepsilon}\delta u_t + \mu h(x,t) = 0\,,\\ &u_{xx}(0,\cdot) = u_{xx}(\pi/4,\cdot) = 0\,,\\ &u_{xxx}(0,\cdot) = -\varepsilon f(u(0,\cdot)), \quad u_{xxx}(\pi/4,\cdot) = \varepsilon f(u(\pi/4,\cdot))\,. \end{aligned} \tag{5.1.2}$$

By a (weak) solution of (5.1.2), we mean any $u(x,t) \in C([0,\pi/4]\times\mathbb{R})$ satisfying the identity

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_0^{\pi/4}\left\{u(x,t)\left[v_{tt}(x,t) + \frac{1}{\varepsilon}v_{xxxx}(x,t) - \sqrt{\varepsilon}\delta v_t(x,t)\right] + \mu h(x,t)v(x,t)\right\}dx\,dt\\ &\quad + \int_{-\infty}^{\infty}\left\{f(u(0,t))v(0,t) + f(u(\pi/4,t))v(\pi/4,t)\right\}dt = 0 \end{aligned} \tag{5.1.3}$$

for any $v(x,t) \in C^\infty([0,\pi/4]\times\mathbb{R})$ so that $v(x,t)$ has a compact support and the following boundary value conditions hold

$$v_{xx}(0,\cdot) = v_{xx}(\pi/4,\cdot) = v_{xxx}(0,\cdot) = v_{xxx}(\pi/4,\cdot) = 0. \qquad (5.1.4)$$

Now, it is well known [2] that there is an orthonormal system of eigenfunctions $\{w_i\}_{i=-1}^{\infty} \in L^2([0,\frac{\pi}{4}])$ of the eigenvalue problem

$$\begin{aligned} &U^{(iv)}(x) = \kappa U(x), \\ &U''(0) = U''(\pi/4) = 0, \quad U'''(0) = U'''(\pi/4) = 0. \end{aligned}$$

As a matter of fact (cf Section 5.1.5), the eigenfunctions $\{w_i\}_{i=-1}^{\infty}$ are uniformly bounded in $C^0([0,\frac{\pi}{4}])$, and setting $\kappa = \mu^4$, the eigenvalues of the above problem satisfy $\mu = \mu_k$, $k = -1,0,1,\dots$ with $\mu_{-1} = \mu_0 = 0$ and $\mu_k = 2(2k+1) + r(k)$, for any $k \in \mathbb{N}$, where $|r(k)| \leq \bar{c}_1 \mathrm{e}^{-\bar{c}_2 k}$ for any $k \geq 1$, for some positive constants $\bar{c}_1$, $\bar{c}_2$. Furthermore, the eigenfunctions $w_{-1}(x)$ and $w_0(x)$ of the zero eigenvalue are:

$$w_{-1}(x) = \frac{2}{\sqrt{\pi}}, \quad w_0(x) = \frac{16}{\pi}\left(x - \frac{\pi}{8}\right)\sqrt{\frac{3}{\pi}}.$$

Thus we seek a solution $u(x,t)$ of (5.1.2) in the form

$$u(x,t) = y_1(t)w_{-1}(x) + y_2(t)w_0(x) + z(x,t)$$

where $z(x,t) \in C\left([0,\frac{\pi}{4}] \times \mathbb{R}\right)$ is orthogonal to the eigenfunctions $w_{-1}(x)$ and $w_0(x)$, satisfying

$$\int_0^{\pi/4} z(x,t)dx = \int_0^{\pi/4} xz(x,t)dx = 0. \qquad (5.1.5)$$

To obtain the equations for $y_1(t)$, $y_2(t)$, and $z(x,t)$ we take $v(x,t) = \phi_1(t)w_{-1}(x) + \phi_2(t)w_0(x) + v_0(x,t)$ in (5.1.3) with $\phi_i \in C^\infty$, $v_0(x,t) \in C^\infty\left([0,\frac{\pi}{4}] \times \mathbb{R}\right)$ with compact supports so that $v_0(x,t)$ satisfies (5.1.4) and is orthogonal to $w_{-1}(x)$ and $w_0(x)$, i.e. it satisfies (5.1.5). Plugging the above expression for $v(x,t)$ into (5.1.3) and using the orthonormality, we arrive at the system of equations

$$\begin{aligned} &\ddot{y}_1(t) + \sqrt{\varepsilon}\delta\dot{y}_1(t) + \frac{2}{\sqrt{\pi}}\mu\int_0^{\pi/4} h(x,t)\,dx \\ &+\frac{2}{\sqrt{\pi}}f\left(\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}y_2(t) + z(0,t)\right) \\ &+\frac{2}{\sqrt{\pi}}f\left(\frac{2}{\sqrt{\pi}}y_1(t) + 2\sqrt{\frac{3}{\pi}}y_2(t) + z(\pi/4,t)\right) = 0, \qquad (5.1.6) \\ &\ddot{y}_2(t) + \sqrt{\varepsilon}\delta\dot{y}_2(t) + \frac{16}{\pi}\sqrt{\frac{3}{\pi}}\mu\int_0^{\pi/4} h(x,t)\left(x - \frac{\pi}{8}\right)dx \\ &-2\sqrt{\frac{3}{\pi}}f\left(\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}y_2(t) + z(0,t)\right) \end{aligned}$$

$$+ 2\sqrt{\frac{3}{\pi}} f\left(\frac{2}{\sqrt{\pi}} y_1(t) + 2\sqrt{\frac{3}{\pi}} y_2(t) + z(\pi/4,t)\right) = 0, \tag{5.1.7}$$

$$\int_{-\infty}^{\infty}\int_0^{\pi/4} \left\{ z(x,t)\left[v_{tt}(x,t) + \frac{1}{\varepsilon} v_{xxxx}(x,t) - \sqrt{\varepsilon}\delta v_t(x,t)\right] + \mu h(x,t) v(x,t)\right\} dx\,dt$$
$$+ \int_{-\infty}^{\infty} \left\{ f(u(0,t))v(0,t) + f(u(\pi/4,t))v(\pi/4,t)\right\} dt = 0 \tag{5.1.8}$$

where we write $v(x,t)$ instead $v_0(x,t)$. Thus, in Eq. (5.1.8), $v(x,t)$ is any function in $C^\infty([0,\frac{\pi}{4}]\times\mathbb{R})$ having compact support so that the conditions (5.1.4), (5.1.5) (with $v(x,t)$ instead of $z(x,t)$) hold. We remark that in this way we have split up the original equation into two parts. Equation (5.1.8) corresponds, in some sense, to Eq. (5.1.1) on a infinite dimensional center manifold, while Eqs. (5.1.6)–(5.1.8) are the equations on a hyperbolic manifold for the unperturbed equation. Since the center manifold is infinitely dimensional, the standard center manifold reduction method (cf Sections 2.5.4, 2.5.5 and [10]) fails for (5.1.1). We use instead a regular singular perturbation method. In fact, the above splitting of Eq. (5.1.1) has also the advantage that the singular part (in ε) is only in the z equation while Eqs. (5.1.6) and (5.1.8) look regular in $\sqrt{\varepsilon}$.

Now we assume that the following conditions hold:

(H1) $f(0) = 0$, $f'(0) < 0$ and the equation $\ddot{x} + f(x) = 0$ has a homoclinic solution $\gamma(t) \neq 0$ that is a nontrivial bounded solution so that $\lim_{t\to\pm\infty} \gamma(t) = 0$;

(H2) let $\gamma_1(t) := \frac{\sqrt{\pi}}{2}\gamma\Big(2\sqrt{\frac{2}{\pi}}t\Big)$. Then the linear equation $\ddot{v} + \frac{24}{\pi} f'\left(\frac{2}{\sqrt{\pi}}\gamma_1(t)\right) v = 0$ has no nontrivial bounded solutions.

Without loss of generality we can also assume that $\ddot{\gamma}(0) \neq \dot{\gamma}(0) = 0$. This implies that $\gamma(t) = \gamma(-t)$ (and then $\gamma_1(t) = \gamma_1(-t)$) since both satisfy the Cauchy problem $\ddot{x} + f(x) = 0$, $x(0) = \gamma(0)$ and $\dot{x}(0) = 0$. Note also that (H1) implies that the system

$$\begin{aligned} &\ddot{y}_1 + \frac{2}{\sqrt{\pi}} f\Big(\frac{2}{\sqrt{\pi}} y_1 - 2\sqrt{\frac{3}{\pi}} y_2\Big) + \frac{2}{\sqrt{\pi}} f\Big(\frac{2}{\sqrt{\pi}} y_1 + 2\sqrt{\frac{3}{\pi}} y_2\Big) = 0,\\ &\ddot{y}_2 - 2\sqrt{\frac{3}{\pi}} f\Big(\frac{2}{\sqrt{\pi}} y_1 - 2\sqrt{\frac{3}{\pi}} y_2\Big) + 2\sqrt{\frac{3}{\pi}} f\Big(\frac{2}{\sqrt{\pi}} y_1 + 2\sqrt{\frac{3}{\pi}} y_2\Big) = 0 \end{aligned} \tag{5.1.9}$$

has a hyperbolic equilibrium $y_1 = y_2 = 0$ with the homoclinic orbit $(\gamma_1(t), 0)$ and that (H2) is equivalent to requiring that the space of bounded solutions of the linear, fourth order system

$$\ddot{y}_1 + \frac{8}{\pi} f'\Big(\frac{2}{\sqrt{\pi}}\gamma_1(t)\Big) y_1 = 0, \quad \ddot{y}_2 + \frac{24}{\pi} f'\Big(\frac{2}{\sqrt{\pi}}\gamma_1(t)\Big) y_2 = 0 \tag{5.1.10}$$

is one-dimensional and spanned by $(y_1(t), \dot{y}_1(t), y_2(t), \dot{y}_2(t)) = (\dot{\gamma}_1(t), \ddot{\gamma}_1(t), 0, 0)$. We look for chaotic solutions of Equations (5.1.6)–(5.1.8) so that the sup-norm of $|y_2(t)| + |z(x,t)|$ on $[0,\frac{\pi}{4}]\times\mathbb{R}$ is small and $y_1(t)$ is orbitally near to $\gamma_1(t)$.

5.1.3 Preliminary Results

We begin our analysis by studying some linear problems associated with Eqs. (5.1.6)–(5.1.8). To start with, let us consider, for $i \in \mathbb{N}$, the following linear non-homogeneous equation

$$\ddot{z}_i(t) + \sqrt{\varepsilon}\delta \dot{z}_i(t) + \frac{1}{\varepsilon}\mu_i^4 z_i(t) = h_i(t)\,, \tag{5.1.11}$$

where $h_i(t)$ belongs to the Banach space $L^\infty(\mathbb{R})$ of bounded measurable functions on $\mathbb{R}$, with norm $\|h_i\|_\infty := \operatorname{ess\,sup}_{t\in\mathbb{R}} |h_i(t)| < \infty$. This equation comes from searching a solution of Eq. (5.1.17) of the form

$$z(x,t) = \sum_{i=1}^{\infty} z_i(t) w_i(x)$$

with $z_i(t) \in W^{2,\infty}(\mathbb{R})$. The only bounded solution of (5.1.11) for $0 < \varepsilon < 2\min_{i\geq 1}\{\frac{\mu_i^2}{\delta}\}$ is given by

$$z_i(t) = L_{i,\varepsilon} h_i := \frac{2\sqrt{\varepsilon}}{\omega_{i,\varepsilon}} \int_{-\infty}^{t} \mathrm{e}^{-\sqrt{\varepsilon}\delta(t-s)/2} \sin\left(\frac{\omega_{i,\varepsilon}}{2\sqrt{\varepsilon}}(t-s)\right) \times h_i(s)\,ds\,, \tag{5.1.12}$$

where $\omega_{i,\varepsilon} = \sqrt{4\mu_i^4 - \varepsilon^2\delta^2}$. Moreover it is easy to see that

$$\|z_i\|_\infty \leq \frac{4}{\delta\mu_i^2}\|h_i\|_\infty\,, \tag{5.1.13}$$

($\|z\|_\infty$ being the sup-norm of $z(t)$) and

$$\|\dot{z}_i\|_\infty \leq \left(\frac{2\sqrt{\varepsilon}}{\mu_i^2} + \frac{2}{\delta\sqrt{\varepsilon}}\right)\|h_i\|_\infty\,, \tag{5.1.14}$$

provided $0 < \varepsilon < \sqrt{3}\min_{i\geq 1}\{\frac{\mu_i^2}{\delta}\}$. Let $h = \{h_i(t)\}_{i=1}^\infty$, $h_i \in L^\infty(\mathbb{R})$ be a sequence of uniformly bounded measurable functions on $\mathbb{R}$, that is, satisfying $\|h\|_\infty := \sup_i \|h_i\|_\infty < \infty$. Consider the function

$$z(x,t) = \sum_{i=1}^{\infty} z_i(t) w_i(x) \tag{5.1.15}$$

where $z_i(t)$ are given by (5.1.12). We put

$$M_1 := \sup\left\{|w_i(x)| \,:\, x \in \left[0, \frac{\pi}{4}\right],\, i \in \mathbb{N}\right\}; \quad M_2 := 4M_1 \sum_{i=1}^{\infty} \frac{1}{\mu_i^2}\,, \tag{5.1.16}$$

with the last series being convergent because of the properties of μ_k, $k \in \mathbb{N}$.

Now, let $H_1(x,t) \in L^\infty([0,\pi/4]\times\mathbb{R})$, $H_2(t), H_3(t) \in L^\infty(\mathbb{R})$ be bounded measurable functions and consider the equation

$$\int_{-\infty}^{\infty}\int_0^{\pi/4}\Big\{z(x,t)\Big[v_{tt}(x,t)+\frac{1}{\varepsilon}v_{xxxx}(x,t)-\sqrt{\varepsilon}\delta v_t(x,t)\Big]+H_1(x,t)v(x,t)\Big]\,dx\,dt$$
$$+\int_{-\infty}^{\infty}\Big\{H_2(t)v(0,t)+H_3(t)v(\pi/4,t)\Big\}\,dt=0 \tag{5.1.17}$$

for any $v(x,t)\in C^\infty([0,\pi/4]\times\mathbb{R})$ so that $v(x,t)$ has compact support and the boundary conditions (5.1.4), (5.1.5) hold. For $i\in\mathbb{N}$ let

$$h_i(t)=-\left(\int_0^{\pi/4}H_1(x,t)w_i(x)\,dx+H_2(t)w_i(0)+H_3(t)w_i(\pi/4)\right) \tag{5.1.18}$$

and take $z_i(t)$, $z(x,t)$ as in (5.1.12), (5.1.15). Note that

$$|h_i(t)|\le M_1\left[\frac{\pi}{4}\|H_1(\cdot,t)\|_\infty+|H_2(t)|+|H_3(t)|\right] \tag{5.1.19}$$

where $\|H_1(\cdot,t)\|_\infty=\sup_{0\le x\le\frac{\pi}{4}}|H_1(x,t)|$ and, similarly,

$$|\dot h_i(t)|\le M_1\left[\frac{\pi}{4}\|H_{1t}(\cdot,t)\|_\infty+|\dot H_2(t)|+|\dot H_3(t)|\right] \tag{5.1.20}$$

provided $\dot H_2(t)$, $\dot H_2(t)$, and the partial derivative of $H_1(x,t)$ with respect to t, $H_{1t}(x,t)$, are bounded measurable functions. Then, we can prove as in [11] that $z(x,t)$ is a solution of Eq. (5.1.17).

Let $m\ge[\varepsilon^{-3/4}]+1$, with $[\varepsilon^{-3/4}]$ being the integer part of $\varepsilon^{-3/4}$. From now on we assume that $0<\varepsilon\le(1/2)^{4/3}$ so that $m\ge 3$. Then, for any $E=\{e_n\}_{n\in\mathbb{Z}}\in\mathscr{E}$, we put

$$\ell_E^\infty=\Big\{\alpha:=\{\alpha_j\}_{j\in\mathbb{Z}}\in\ell^\infty \mid \alpha_j\in\mathbb{R}\quad\text{and}\quad\alpha_j=0\quad\text{if}\quad e_j=0\Big\},$$

with ℓ^∞ being the Banach space of bounded, doubly infinity sequences of real numbers, endowed with the sup-norm. We will also consider a bounded subset of $\mathscr{E}\times\ell^\infty$:

$$X=\Big\{(E,\alpha)\in\mathscr{E}\times\ell^\infty \mid \alpha\in\ell_E^\infty\quad\text{and}\quad\|\alpha\|\le 2\Big\}.$$

Note that X is closed. In fact if $(E_n,\alpha_n)\to(E,\alpha)$ as $n\to\infty$, then, for any fixed $j\in\mathbb{Z}$, we have (with obvious meaning of symbols) $e_j^{(n)}=e_j$ for any $n\in\mathbb{N}$ sufficiently large. Hence $\alpha_j^{(n)}=0$ if $e_j=0$ and n is large enough. Thus $\alpha_j=0$ if $e_j=0$, that is, $(E,\alpha)\in X$.

For any $\xi=(E,\alpha)\in X$ we take the function $\gamma_\xi=\gamma_{(E,\alpha)}\in L^\infty(\mathbb{R})$ defined by

$$\gamma_\xi(t)=\begin{cases}\gamma_1(t-2jm-\alpha_j), & \text{if}\quad(2j-1)m<t\le(2j+1)m\quad\text{and}\quad e_j=1\\ 0, & \text{if}\quad(2j-1)m<t\le(2j+1)m\quad\text{and}\quad e_j=0.\end{cases}$$

For the sake of simplicity we will silently include, in the above definitions, also the end points of the intervals $[(2j-1)m,(2j+1)m]$, $j\in\mathbb{Z}$. We remark that $\gamma_\xi(t)$ has the following properties:

(i) $\gamma_\xi(t)$ is a bounded, piecewise C^2-function, with possible jumps at the points $(2j-1)m$, $j\in\mathbb{Z}$, and satisfies, in any of the intervals $((2j-1)m,(2j+1)m)$, the equation
$$\ddot{x}+\frac{4}{\sqrt{\pi}}f\left(\frac{2}{\sqrt{\pi}}x\right)=0. \tag{5.1.21}$$

(ii) $\gamma_\xi(t)$, $\dot\gamma_\xi(t)$, $\ddot\gamma_\xi(t)$ belong to $L^\infty(\mathbb{R})$ and are bounded uniformly with respect to (ξ,m).

(iii) $\gamma_\xi(t)$, $\dot\gamma_\xi(t)$, $\ddot\gamma_\xi(t)$ are Lipschitz continuous function in $\alpha\in\ell^\infty_E$ uniformly with respect to (E,m). In fact, let $(E,\alpha'),(E,\alpha'')\in X$ and assume that $e_j=1$ (if $e_j=0$ there is nothing to prove). Then, for any $t\in((2j-1)m,(2j+1)m]$ we have, for some $\theta\in\mathbb{R}$:
$$|\gamma_{\xi'}(t)-\gamma_{\xi''}(t)|\le|\dot\gamma_1(\theta)||\alpha'_j-\alpha''_j|\le\sqrt{2}\|\dot\gamma\|_\infty\,\|\alpha'-\alpha''\|. \tag{5.1.22}$$
A similar argument applies to $\dot\gamma_\xi(t)$, whereas we will use point (i) to reduce the study of the Lipschitz continuity of $\ddot\gamma_\xi(t)$ to that of $\gamma_\xi(t)$.

The following result deals with the solvability of Eq. (5.1.17).

Theorem 5.1.1. *For any given functions $H_1(x,t)\in L^\infty([0,\pi/4]\times\mathbb{R})$, $H_2(t),H_3(t)\in L^\infty(\mathbb{R})$ and for $0<\varepsilon<\min_i\{\sqrt{3}\mu_i^2/\delta\}$, Equation* (5.1.17) *has a unique solution $z(x,t)\in C([0,\pi/4]\times\mathbb{R})$ of the form*
$$z(x,t)=\sum_{i=1}^\infty z_i(t)w_i(x)$$
with $z_i(t)\in W^{2,\infty}(\mathbb{R})$. Such a solution satisfies condition (5.1.5)*, moreover if $h_i(t)$ is defined as in* (5.1.18) *the following hold:*

(a) *Assume that there exist positive constants k_1, k_2, α_j and β so that*
$$|h_i(t)|\le k_1+k_2\,\mathrm{e}^{-\beta|t-2jm-\alpha_j|}$$
for any $t\in((2j-1)m,(2j+1)m]$ and $j\in\mathbb{Z}$. Then
$$\|z\|_\infty\le M_2\left[\frac{k_1}{\delta}+\left(\frac{1}{\delta^3}+\frac{2}{\beta}\right)k_2\sqrt{\varepsilon}\right].$$

(b) *Assume that for any $i,j\in\mathbb{Z}$, $h_i(t)\in W^{1,\infty}((2j-1)m,(2j+1)m)$ and that both $h_i(t)$ and $\dot h_i(t)$ satisfy the condition of point (a), then we have*
$$\|z\|_\infty\le M_2\left[5\varepsilon\left(\frac{1}{\delta^5}+1+\frac{1}{\beta}\right)(k_1+k_2)+\frac{2\sqrt{\varepsilon}}{\delta}k_1\right]$$

provided ε satisfies the further estimate $\sqrt{\varepsilon} < 2\delta^2$.

Proof. We only need to prove (a) and (b). Let $(2j-1)m < t \le (2j+1)m$ and $0 < \varepsilon < \min_i\{\sqrt{3}\mu_i^2\delta^{-1}\}$. We have

$$\left|\int_{-\infty}^t \mathrm{e}^{-\sqrt{\varepsilon}\delta(t-s)/2}\sin\frac{\omega_{i,\varepsilon}}{2\sqrt{\varepsilon}}(t-s)\,h_i(s)\,ds\right| \le \int_{-\infty}^t \mathrm{e}^{-\sqrt{\varepsilon}\delta(t-s)/2}[k_1+k_2\varphi(s)]\,ds,$$

where $\varphi(t) = \mathrm{e}^{-\beta|t-2jm-\alpha_j|}$ for $t \in ((2j-1)m,(2j+1)m]$. Then we have

$$\int_{-\infty}^t \mathrm{e}^{-\sqrt{\varepsilon}\delta(t-s)/2}\,ds \le \frac{2}{\sqrt{\varepsilon}\delta},$$

and similarly, using also $t > (2j-1)m$,

$$\int_{-\infty}^{(2j-3)m} \mathrm{e}^{-\sqrt{\varepsilon}\delta(t-s)/2}\,\varphi(s)\,ds \le \int_{-\infty}^{(2j-3)m} \mathrm{e}^{-\sqrt{\varepsilon}\delta(t-s)/2}\,ds \le \frac{2}{\sqrt{\varepsilon}\delta}\,\mathrm{e}^{-\sqrt{\varepsilon}\delta m} < \frac{2}{\delta^3},$$

since $m > \varepsilon^{-3/4}$ and $\theta^2\mathrm{e}^{-\theta} < 1$, when $\theta > 0$. Next,

$$\int_{(2j-3)m}^{(2j-1)m} \mathrm{e}^{-\sqrt{\varepsilon}\delta(t-s)/2}\,\varphi(s)\,ds \le \int_{-m-\alpha_{j-1}}^{m-\alpha_{j-1}} \mathrm{e}^{-\beta|s|}\,ds \le 2\int_0^\infty \mathrm{e}^{-\beta s}\,ds \le 2\beta^{-1},$$

and similarly

$$\int_{(2j-1)m}^{t} \mathrm{e}^{-\sqrt{\varepsilon}\delta(t-s)/2}\,\varphi(s)\,ds \le \int_{-\infty}^{\infty} \mathrm{e}^{-\beta|s|}\,ds \le 2\beta^{-1}.$$

Plugging everything together and using (5.1.12) and $\omega_{i,\varepsilon} \ge \mu_i^2$ since $\varepsilon\delta < \sqrt{3}\mu_i^2$, we obtain

$$\|z_i\|_\infty \le \frac{4}{\mu_i^2}\left[\frac{k_1}{\delta} + k_2\sqrt{\varepsilon}\left(\frac{1}{\delta^3}+\frac{2}{\beta}\right)\right].$$

Thus (a) follows from (5.1.15) and (5.1.16). Now we prove (b). For $(2j-1)m < t \le (2j+1)m$, write

$$\frac{\omega_{i,\varepsilon}}{2\sqrt{\varepsilon}}z_i(t) = \zeta_{i,j} + \tilde{z}_{i,j}(t) \tag{5.1.23}$$

with

$$\zeta_{i,j} = \int_{-\infty}^{(2j-3)m} \mathrm{e}^{-\sqrt{\varepsilon}\delta(t-s)/2}\sin\left(\frac{\omega_{i,\varepsilon}}{2\sqrt{\varepsilon}}(t-s)\right)h_i(s)ds,$$

$$\tilde{z}_{i,j}(t) = \int_{(2j-3)m}^{t} \mathrm{e}^{-\sqrt{\varepsilon}\delta(t-s)/2}\sin\left(\frac{\omega_{i,\varepsilon}}{2\sqrt{\varepsilon}}(t-s)\right)h_i(s)ds.$$

From the proof of point (a) we obtain:

$$|\zeta_{i,j}| \le \frac{2}{\sqrt{\varepsilon}\delta}\,\mathrm{e}^{-\sqrt{\varepsilon}\delta m}(k_1+k_2) \le \frac{10\sqrt{\varepsilon}}{\delta^5}(k_1+k_2) \tag{5.1.24}$$

since $\theta^4 e^{-\theta} \leq (4/e)^4 < 5$. On the other hand, by the same method in the above, we obtain

$$\left|\int_{(2j-3)m}^{t} e^{-\sqrt{\varepsilon}\delta(t-s)/2} \cos\left(\frac{\omega_{i,\varepsilon}}{2\sqrt{\varepsilon}}(t-s)\right) \dot{h}_i(s)\,ds\right| \leq \frac{2}{\sqrt{\varepsilon}\delta}k_1 + \frac{4}{\beta}k_2,$$
$$\left|\int_{(2j-3)m}^{t} e^{-\sqrt{\varepsilon}\delta(t-s)/2} \sin\left(\frac{\omega_{i,\varepsilon}}{2\sqrt{\varepsilon}}(t-s)\right) \dot{h}_i(s)\,ds\right| \leq \frac{2}{\sqrt{\varepsilon}\delta}k_1 + \frac{4}{\beta}k_2. \tag{5.1.25}$$

Then, taking

$$\lambda = \frac{\sqrt{\varepsilon}\delta}{2}, \quad \omega = \frac{\omega_{i,\varepsilon}}{2\sqrt{\varepsilon}}$$

and integrating by parts the function of the s variable

$$e^{-\lambda(t-s)} \sin(\omega(t-s))h_i(s)$$

in the two intervals $[(2j-3)m,(2j-1)m]$, $[(2j-1)m,t]$ and adding the results we get, using also (5.1.25):

$$\left|\int_{(2j-3)m}^{t} e^{-\lambda(t-s)} \sin(\omega(t-s))h_i(s)ds\right| \leq \frac{\omega}{\lambda^2+\omega^2}|h_i(t)|$$
$$+\frac{\lambda+\omega}{\lambda^2+\omega^2}[|h_i((2j-1)m^+)| + |h_i((2j-1)m^-)| + e^{-2\lambda m}|h_i((2j-3)m^+)|]$$
$$+\frac{\lambda+\omega}{\lambda^2+\omega^2}\left[\frac{2}{\sqrt{\varepsilon}\delta}k_1 + \frac{4}{\beta}k_2\right] \leq \frac{\lambda+\omega}{\lambda^2+\omega^2}\left[(3+e^{-2\lambda m})(k_1+k_2) + \frac{2}{\sqrt{\varepsilon}\delta}k_1 + \frac{4}{\beta}k_2\right].$$

Finally, since

$$\frac{\varepsilon\delta + \omega_{i,\varepsilon}}{\omega_{i,\varepsilon}(\varepsilon^2\delta^2 + \omega_{i,\varepsilon}^2)} \leq \frac{\sqrt{2}}{\omega_{i,\varepsilon}\sqrt{\varepsilon^2\delta^2+\omega_{i,\varepsilon}^2}} = \frac{\sqrt{2}}{2\mu_i^2\omega_{i,\varepsilon}} \leq \frac{\sqrt{2}}{2\mu_i^4} \leq \frac{1}{\mu_i^2},$$

we obtain after some algebra:

$$\left|\frac{2\sqrt{\varepsilon}}{\omega_{i,\varepsilon}}\tilde{z}_{i,j}(t)\right| \leq \frac{4\varepsilon}{\mu_i^2}\left[(3+e^{-\sqrt{\varepsilon}\delta m})(k_1+k_2) + \frac{4}{\beta}k_2 + \frac{2}{\sqrt{\varepsilon}\delta}k_1\right].$$

Hence, using (5.1.23), (5.1.24), the assumption $\sqrt{\varepsilon} < 2\delta^2$ and the fact that $e^{-\sqrt{\varepsilon}\delta m} \leq \frac{1}{(\sqrt{\varepsilon}\delta m)^2} < \frac{\sqrt{\varepsilon}}{\delta^2}$:

$$\begin{aligned} \|z_i\|_\infty \leq & \frac{4}{\mu_i^2}\left\{\left[\frac{5}{\delta^5} + 3 + \frac{\sqrt{\varepsilon}}{\delta^2}\right]\varepsilon(k_1+k_2) + \frac{4\varepsilon}{\beta}k_2 + \frac{2\sqrt{\varepsilon}}{\delta}k_1\right\} \\ \leq & \frac{4}{\mu_i^2}\left\{5\varepsilon\left[\frac{1}{\delta^5} + 1 + \frac{1}{\beta}\right](k_1+k_2) + \frac{2\sqrt{\varepsilon}}{\delta}k_1\right\} \end{aligned}.$$

Again, the conclusion follows from (5.1.15) and (5.1.16). The proof is finished. □

In the following we denote by $L_\varepsilon(H_1,H_2,H_3)$ the unique bounded solution of the form (5.1.15) of Eq. (5.1.17) and note that L_ε is a bounded linear map from the space of bounded measurable functions to the space of bounded continuous functions, that is,

$$L_\varepsilon(H_1+\hat{H}_1,H_2+\hat{H}_2,H_3+\hat{H}_2)=L_\varepsilon(H_1,H_2,H_3)+L_\varepsilon(\hat{H}_1,\hat{H}_2,\hat{H}_3).$$

We now study the linear non-homogeneous equation

$$\begin{aligned}&\ddot{x}_1+\frac{8}{\pi}f'\Big(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\Big)x_1=h(t)\,,\\&\dot{x}_1(2jm+\alpha_j)=0,\quad \text{for any } j\in\mathbb{Z} \text{ such that } e_j=1\,.\end{aligned}\tag{5.1.26}$$

Here $h\in L^\infty(\mathbb{R})$, and $x_1(t)$, $\dot{x}_1(t)$ are absolutely continuous functions so that (5.1.26) holds almost everywhere. Let us put

$$a=\sqrt{8|f'(0)|/\pi}\,.$$

Lemma 5.1.2. *There exist positive constants $A,B,C\in\mathbb{R}$ and $m_0\in\mathbb{N}$ so that for any $\xi=(E,\alpha)\in X$, $m\geq m_0$, and $j\in\mathbb{Z}$, there exist linear functionals $\mathscr{L}_{m,\xi,j}:L^\infty(\mathbb{R})\to\mathbb{R}$, so that $\|\mathscr{L}_{m,\xi,j}\|\leq Ae_j\mathrm{e}^{-am}$, with the property that if $h\in L^\infty(\mathbb{R})$ then* (5.1.26) *has a unique C^1 solution $x_1(t,\xi)$ bounded on $\mathbb{R}$ if and only if*

$$\mathscr{L}_{m,\xi,j}h+\int_{(2j-1)m}^{(2j+1)m}\dot{\gamma}_\xi(t)h(t)\,dt=0\tag{5.1.27}$$

for any $j\in\mathbb{Z}$. Moreover, the following properties hold:

(i)
$$\|x_1(\cdot,\xi)\|_\infty\leq B\|h\|_\infty,\quad \|\dot{x}_1(\cdot,\xi)\|_\infty\leq B\|h\|_\infty.\tag{5.1.28}$$

(ii) *Let $x_p(t)$ be the unique bounded solution of equation $\ddot{x}_p+\frac{8}{\pi}f'(0)x_p=h(t)$, then*

$$|x_1(t,\xi)-x_p(t)|\leq C\big(\mathrm{e}^{-am/2}+\mathrm{e}^{-a|t-2jm-\alpha_j|/2}\big)\|h\|_\infty\tag{5.1.29}$$

for $(2j-1)m\leq t\leq(2j+1)m$ and any $j\in\mathbb{Z}$.

(iii) *Let $\xi'=(E,\alpha')$, $\xi''=(E,\alpha'')$ with $\alpha',\alpha''\in\ell^\infty_E$ and ξ be either ξ' or ξ''. Assume that $h(t,\xi)\in L^\infty(\mathbb{R})$ satisfies* (5.1.27). *Then there exists a constant, c_1, independent of ξ, so that the following holds:*

$$\begin{aligned}&\max\{\|x_1(\cdot,\xi')-x_1(\cdot,\xi'')\|_\infty,\|\dot{x}_1(\cdot,\xi')-\dot{x}_1(\cdot,\xi'')\|_\infty\}\\&\leq B\|h(t,\xi')-h(t,\xi'')\|_\infty+c_1\|h(t,\xi'')\|_\infty\|\alpha'-\alpha''\|_\infty.\end{aligned}\tag{5.1.30}$$

Finally, for any $m\geq m_0$, the map $\mathscr{L}_m:X\times L^\infty(\mathbb{R})\to\ell^\infty(\mathbb{R})$ defined as $\mathscr{L}_m(\xi,h)=\{\mathscr{L}_{m,\xi,j}h\}_{j\in\mathbb{Z}}$ is Lipschitz in $\alpha\in\ell^\infty_E$ uniformly with respect to (E,m).

Proof. The equation

$$\ddot{x} + \frac{8}{\pi} f'\left(\frac{2}{\sqrt{\pi}}\gamma_1(t)\right)x = 0 \tag{5.1.31}$$

has a fundamental solution $u(t), v(t)$ with

$$u(0) = 1, \quad \dot{u}(0) = 0, \quad v(0) = 0, \quad \dot{v}(0) = 1\,.$$

Then v is bounded, odd and u is unbounded, even with asymptotic properties:

$$v(t), \dot{v}(t) \sim \mathrm{e}^{-a|t|},\ u(t), \dot{u}(t) \sim \mathrm{e}^{a|t|} \ \text{ as } t \to \pm\infty\,.$$

Note that $\dot{\gamma}_1(t)$ is a solution of (5.1.31) so that $\dot{\gamma}_1(t) \sim \mathrm{e}^{-a|t|}$ and $\dot{\gamma}_1(0) = 0$, $\ddot{\gamma}_1(0) \neq 0$, we get $v(t) = \frac{\dot{\gamma}_1(t)}{\ddot{\gamma}_1(0)}$. Let us pause for a moment to recall some of the properties of the functions $u(t)$, $v(t)$ that will be used later. Equation (5.1.31), or, as a system

$$\dot{u}_1 = u_2, \quad \dot{u}_2 = -\frac{8}{\pi} f'\left(\frac{2}{\sqrt{\pi}}\gamma_1(t)\right)u_1, \tag{5.1.32}$$

has an exponential dichotomy on $\mathbb{R}_+$ and $\mathbb{R}_-$ with exponent a (cf Section 2.5.1). Thus projections P_+, P_- exist so that $\operatorname{rank} P_+ = \operatorname{rank} P_- = 1$ and

$$\begin{aligned}
&\|X(t)P_+X^{-1}(s)\| \le k\mathrm{e}^{-a(t-s)}, && \text{if} \quad 0 \le s \le t,\\
&\|X(t)(\mathbb{I} - P_+)X^{-1}(s)\| \le k\mathrm{e}^{a(t-s)}, && \text{if} \quad 0 \le t \le s,\\
&\|X(t)P_-X^{-1}(s)\| \le k\mathrm{e}^{-a(t-s)}, && \text{if} \quad s \le t \le 0,\\
&\|X(t)(\mathbb{I} - P_-)X^{-1}(s)\| \le k\mathrm{e}^{a(t-s)}, && \text{if} \quad t \le s \le 0
\end{aligned} \tag{5.1.33}$$

where

$$X(t) = \begin{pmatrix} u(t) & v(t) \\ \dot{u}(t) & \dot{v}(t) \end{pmatrix}$$

is the fundamental matrix of (5.1.32) so that $X(0) = \mathbb{I}$. Although P_+ and P_- are not uniquely defined, $\mathscr{R}P_+$ and $\mathscr{N}P_-$ are precisely the one–dimensional vector spaces consisting of all initial conditions one has to assign to the linear system (5.1.32) to obtain solutions bounded on $\mathbb{R}_+$, $\mathbb{R}_-$ respectively. Moreover, any projection possessing $\mathscr{R}P_+$ as range (resp. $\mathscr{N}P_-$ as kernel) satisfies conditions (5.1.33). Now, since $v(t), \dot{v}(t) \to 0$, as $|t| \to \infty$, we see that we can take:

$$P_+\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (\mathbb{I} - P_-)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence the matrix of P_+ and $\mathbb{I} - P_-$ with respect to the canonical basis of $\mathbb{R}^2$ is $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then Eqs. (5.1.33) read:

$$|v(t)\dot{u}(s)|, |v(t)u(s)|, |\dot{v}(t)\dot{u}(s)|, |\dot{v}(t)u(s)| \le k\mathrm{e}^{-a|t-s|} \tag{5.1.34}$$

if $0 \le s \le t$ or $t \le s \le 0$, whereas

$$|u(t)\dot{v}(s)|, |u(t)v(s)|, |\dot{u}(t)\dot{v}(s)|, |\dot{u}(t)v(s)| \le k\mathrm{e}^{-a|t-s|} \tag{5.1.35}$$

if $0 \le t \le s$ or $s \le t \le 0$. Now, let us go back to the proof of the Lemma. We consider Eq. (5.1.26) on $[(2j-1)m, (2j+1)m]$ according to $e_j = 0$ or $e_j = 1$. When $e_j = 0$ (5.1.26) has the general solution

$$\begin{aligned} x_1(t) = & -\frac{1}{2a}\int_{(2j-1)m}^{t} \mathrm{e}^{-a(t-s)}\, h(s)\, ds - \frac{1}{2a}\int_{t}^{(2j+1)m} \mathrm{e}^{a(t-s)}\, h(s)\, ds \\ & + a_j \mathrm{e}^{a(t-(2j+1)m)} + b_j \mathrm{e}^{-a(t-(2j-1)m)} \end{aligned} \tag{5.1.36}$$

with $a_j, b_j \in \mathbb{R}$. When $e_j = 1$ we distinguish between $t \in [2jm + \alpha_j, (2j+1)m]$ and $t \in [(2j-1)m, 2jm + \alpha_j]$. If $t \in [2jm + \alpha_j, (2j+1)m]$ we write the general solution of Equation (5.1.26) with the condition $\dot{x}_1(2jm + \alpha_j) = 0$ as

$$\begin{aligned} x_1(t) = & \int_{2jm+\alpha_j}^{t} v(t-2jm-\alpha_j)u(s-2jm-\alpha_j)h(s)\, ds \\ & + \int_{t}^{(2j+1)m} u(t-2jm-\alpha_j)v(s-2jm-\alpha_j)h(s)\, ds \\ & + a_j^+ u(t-2jm-\alpha_j)/u(m-\alpha_j) \end{aligned} \tag{5.1.37}$$

where $a_j^+ \in \mathbb{R}$. If $t \in [(2j-1)m, 2jm + \alpha_j]$ we take

$$\begin{aligned} x_1(t) = & -\int_{t}^{2jm+\alpha_j} v(t-2jm-\alpha_j)u(s-2jm-\alpha_j)h(s)\, ds \\ & - \int_{(2j-1)m}^{t} u(t-2jm-\alpha_j)v(s-2jm-\alpha_j)h(s)\, ds \\ & + a_j^- u(t-2jm-\alpha_j)/u(-m-\alpha_j) \end{aligned} \tag{5.1.38}$$

where $a_j^- \in \mathbb{R}$. We note that $\dot{x}_1(2jm + \alpha_j) = 0$ in both (5.1.37) and (5.1.38). Thus to obtain a C^1 solution we only need that

$$x_1((2jm+\alpha_j)_-) = x_1((2jm+\alpha_j)_+), \quad \text{for any } j \in \mathbb{Z} \text{ such that } e_j = 1,$$

that is,

$$\int_{(2j-1)m}^{(2j+1)m} v(s-2jm-\alpha_j)h(s)\, ds = \frac{a_j^-}{u(-m-\alpha_j)} - \frac{a_j^+}{u(m-\alpha_j)}. \tag{5.1.39}$$

We note that from Eq. (5.1.36) we get, for any $j \in \mathbb{Z}$:

$$\sup_{(2j-1)m \le t \le (2j+1)m} |x_1(t)| \le |a_j| + |b_j| + \frac{1}{a^2} \operatorname{ess\,sup}_{(2j-1)m \le t \le (2j+1)m} |h(t)| \tag{5.1.40}$$

and

$$\sup_{(2j-1)m\le t\le (2j+1)m} |\dot{x}_1(t)| \le a(|a_j|+|b_j|)+\frac{1}{a}\operatorname{ess\,sup}_{(2j-1)m\le t\le (2j+1)m}|h(t)|. \tag{5.1.41}$$

A similar conclusion also follows (when $e_j = 1$) from (5.1.37) and (5.1.38) using (5.1.34), (5.1.35). Equation (5.1.39) is the compatibility condition where the linear maps $\mathscr{L}_{m,\xi,j}$ come from. For the moment, we forget about these conditions and choose the constants a_j, b_j, a_j^+, a_j^- so that the equalities

$$\begin{aligned} x_1(((2j+1)m)_-) &= x_1((2j+1)m)_+), \quad j\in\mathbb{Z}\\ \dot{x}_1(((2j+1)m)_-) &= \dot{x}_1((2j+1)m)_+), \quad j\in\mathbb{Z} \end{aligned} \tag{5.1.42}$$

are satisfied. According to the values of e_j, e_{j+1} they read

$$\begin{aligned} &a_j - b_{j+1} + b_j\,\mathrm{e}^{-2am} - a_{j+1}\,\mathrm{e}^{-2am}\\ &\quad = \frac{1}{2a}\int_{(2j-1)m}^{(2j+1)m} \mathrm{e}^{-a((2j+1)m-s)}\,h(s)\,ds - \frac{1}{2a}\int_{(2j+1)m}^{(2j+3)m} \mathrm{e}^{a((2j+1)m-s)}\,h(s)\,ds,\\ &a_j + b_{j+1} - b_j\,\mathrm{e}^{-2am} - a_{j+1}\,\mathrm{e}^{-2am}\\ &\quad = -\frac{1}{2a}\int_{(2j-1)m}^{(2j+1)m} \mathrm{e}^{-a((2j+1)m-s)}\,h(s)\,ds - \frac{1}{2a}\int_{(2j+1)m}^{(2j+3)m} \mathrm{e}^{a((2j+1)m-s)}\,h(s)\,ds, \end{aligned} \tag{5.1.43}$$

if $e_j = e_{j+1} = 0$, or

$$\begin{aligned} &a_j - a_{j+1}^- + b_j\,\mathrm{e}^{-2am}\\ &\quad = \frac{1}{2a}\int_{(2j-1)m}^{(2j+1)m} \mathrm{e}^{-a((2j+1)m-s)}\,h(s)\,ds\\ &\qquad - \int_{(2j+1)m}^{2(j+1)m+\alpha_{j+1}} v(-m-\alpha_{j+1})u(s-2(j+1)m-\alpha_{j+1})h(s)\,ds,\\ &a_j - a_{j+1}^-\frac{\dot{u}(-m-\alpha_{j+1})}{au(-m-\alpha_{j+1})} - b_j\,\mathrm{e}^{-2am}\\ &\quad = -\frac{1}{2a}\int_{(2j-1)m}^{(2j+1)m} \mathrm{e}^{-a((2j+1)m-s)}\,h(s)\,ds\\ &\qquad - \frac{1}{a}\int_{(2j+1)m}^{2(j+1)m+\alpha_{j+1}} \dot{v}(-m-\alpha_{j+1})u(s-2(j+1)m-\alpha_{j+1})h(s)\,ds, \end{aligned} \tag{5.1.44}$$

if $e_j = 0$, $e_{j+1} = 1$, or

$$
\begin{aligned}
& a_j^+ - b_{j+1} - a_{j+1}\,\mathrm{e}^{-2am} \\
& \quad = -\frac{1}{2a}\int_{(2j+1)m}^{(2j+3)m} \mathrm{e}^{a((2j+1)m-s)}\,h(s)\,ds \\
& \qquad - \int_{2jm+\alpha_j}^{(2j+1)m} v(m-\alpha_j)u(s-2jm-\alpha_j)h(s)\,ds, \\
& a_j^+ \frac{\dot{u}(m-\alpha_j)}{au(m-\alpha_j)} + b_{j+1} - a_{j+1}\,\mathrm{e}^{-2am} \\
& \quad = -\frac{1}{2a}\int_{(2j+1)m}^{(2j+3)m} \mathrm{e}^{a((2j+1)m-s)}\,h(s)\,ds \\
& \qquad - \frac{1}{a}\int_{2jm+\alpha_j}^{(2j+1)m} \dot{v}(m-\alpha_j)u(s-2jm-\alpha_j)h(s)\,ds,
\end{aligned}
\tag{5.1.45}
$$

if $e_j = 1$, $e_{j+1} = 0$, or

$$
\begin{aligned}
& a_j^+ - a_{j+1}^- \\
& \quad = -\int_{(2j+1)m}^{2(j+1)m+\alpha_{j+1}} v(-m-\alpha_{j+1})u(s-2(j+1)m-\alpha_{j+1})h(s)\,ds \\
& \qquad - \int_{2jm+\alpha_j}^{(2j+1)m} v(m-\alpha_j)u(s-2jm-\alpha_j)h(s)\,ds, \\
& a_j^+ \frac{\dot{u}(m-\alpha_j)}{au(m-\alpha_j)} - a_{j+1}^- \frac{\dot{u}(-m-\alpha_{j+1})}{au(-m-\alpha_{j+1})} \\
& \quad = -\frac{1}{a}\int_{(2j+1)m}^{2(j+1)m+\alpha_{j+1}} \dot{v}(-m-\alpha_{j+1})u(s-2(j+1)m-\alpha_{j+1})h(s)\,ds \\
& \qquad - \frac{1}{a}\int_{2jm+\alpha_j}^{(2j+1)m} \dot{v}(m-\alpha_j)u(s-2jm-\alpha_j)h(s)\,ds,
\end{aligned}
\tag{5.1.46}
$$

if $e_j = e_{j+1} = 1$. We note that when $\xi = (E,\alpha)$ is fixed, for any $j \in \mathbb{Z}$ only one among Equations (5.1.44)–(5.1.46) occurs. We consider these equations as a unique equation for the variable

$$\{(\tilde{a}_j, \tilde{b}_j)\}_{j\in\mathbb{Z}} \in \ell^\infty \times \ell^\infty$$

where $(\tilde{a}_j, \tilde{b}_j) = (a_j, b_j)$ if $e_j = 0$ whereas $(\tilde{a}_j, \tilde{b}_j) = (a_j^-, a_j^+)$ if $e_j = 1$. The left-hand sides of (5.1.44)–(5.1.46) define a linear bounded operator

$$L_{m,\xi} : \ell^\infty \times \ell^\infty \to \ell^\infty \times \ell^\infty, \quad L_{m,\xi}\begin{pmatrix} \{\widetilde{a}_j\} \\ \{\widetilde{b}_j\} \end{pmatrix} = \begin{pmatrix} \{\hat{a}_j\} \\ \{\hat{b}_j\} \end{pmatrix} \tag{5.1.47}$$

where

$$
\begin{aligned}
\hat{a}_j =& (1-e_j)\widetilde{a}_j - [e_{j+1} + (1-e_{j+1})\,\mathrm{e}^{-2am}]\widetilde{a}_{j+1} \\
&+ [e_j + (1-e_j)\,\mathrm{e}^{-2am}]\widetilde{b}_j - (1-e_{j+1})\widetilde{b}_{j+1}\,, \\
\hat{b}_j =& (1-e_j)\widetilde{a}_j - \left[\frac{\dot{u}(-m-\alpha_{j+1})}{au(-m-\alpha_{j+1})} e_{j+1} + (1-e_{j+1})\,\mathrm{e}^{-2am}\right]\widetilde{a}_{j+1} \\
&+ \left[\frac{\dot{u}(m-\alpha_j)}{au(m-\alpha_j)} e_j - (1-e_j)\,\mathrm{e}^{-2am}\right]\widetilde{b}_j + (1-e_{j+1})\widetilde{b}_{j+1}\,.
\end{aligned}
\tag{5.1.48}
$$

Now, since $0 \le 1 - e_j \le 1$, $|\alpha_j| \le 2$, and

$$
\lim_{t\to\pm\infty} \frac{\dot{u}(t)}{au(t)} = \pm 1 \tag{5.1.49}
$$

we see that $m_0 \in \mathbb{N}$ exists so that for any $m \ge m_0$, $\xi \in X$ and $j \in \mathbb{Z}$, we have

$$
|\hat{a}_j| < 3(\|\widetilde{a}\|_\infty + \|\widetilde{b}\|_\infty), \qquad |\hat{b}_j| < 3(\|\widetilde{a}\|_\infty + \|\widetilde{b}\|_\infty)
$$

or $\|L_{m,\xi}\| < 6$. Now, we want to show that for m sufficiently large and any $\xi \in X$, the map $L_{m,\xi} : \ell^\infty \times \ell^\infty \to \ell^\infty \times \ell^\infty$ is invertible. To this end, we claim that when $m \to \infty$, the linear map $L_{m,\xi}$ tends to the map L_E defined as follows:

$$
L_E \begin{pmatrix} \{\widetilde{a}_j\} \\ \{\widetilde{b}_j\} \end{pmatrix} = \begin{pmatrix} \{(1-e_j)\widetilde{a}_j - e_{j+1}\widetilde{a}_{j+1} + e_j\widetilde{b}_j - (1-e_{j+1})\widetilde{b}_{j+1}\} \\ \{(1-e_j)\widetilde{a}_j + e_{j+1}\widetilde{a}_{j+1} + e_j\widetilde{b}_j + (1-e_{j+1})\widetilde{b}_{j+1}\} \end{pmatrix}
$$

in the sense that

$$
\|L_{m,\xi} - L_E\| \to 0 \tag{5.1.50}
$$

as $m \to \infty$ uniformly with respect to $\xi = (E, \alpha) \in X$. In fact,

$$
\begin{aligned}
&(L_{m,\xi} - L_E) \begin{pmatrix} \{\widetilde{a}_j\} \\ \{\widetilde{b}_j\} \end{pmatrix} \\
&= \begin{pmatrix} \{(e_{j+1}-1)\,\mathrm{e}^{-2am}\,\widetilde{a}_{j+1} + (1-e_j)\,\mathrm{e}^{-2am}\,\widetilde{b}_j\} \\ \left\{ \begin{array}{c} \left[\left(\frac{\dot{u}(m-\alpha_j)}{au(m-\alpha_j)} - 1\right) e_j - (1-e_j)\,\mathrm{e}^{-2am}\right]\widetilde{b}_j \\ - \left[\left(\frac{\dot{u}(-m-\alpha_{j+1})}{au(-m-\alpha_{j+1})} + 1\right) e_{j+1} + (1-e_{j+1})\,\mathrm{e}^{-2am}\right]\widetilde{a}_{j+1} \end{array} \right\} \end{pmatrix}.
\end{aligned}
$$

Thus (5.1.50) follows from (5.1.49) and $\|\alpha\| \le 2$. Next, the equation:

$$
L_E \begin{pmatrix} \{\widetilde{a}_j\} \\ \{\widetilde{b}_j\} \end{pmatrix} = \begin{pmatrix} \{\bar{A}_j\} \\ \{\bar{B}_j\} \end{pmatrix}
$$

is equivalent to the infinite dimensional system ($j \in \mathbb{Z}$):

$$\begin{cases} (1-e_j)\widetilde{a}_j + e_j\widetilde{b}_j = \dfrac{\bar{A}_j + \bar{B}_j}{2}, \\ e_{j+1}\widetilde{a}_{j+1} + (1-e_{j+1})\widetilde{b}_{j+1} = \dfrac{\bar{B}_j - \bar{A}_j}{2}. \end{cases}$$

Changing j with $j-1$ we obtain

$$\begin{cases} (1-e_{j-1})\widetilde{a}_{j-1} + e_{j-1}\widetilde{b}_{j-1} = \dfrac{\bar{A}_{j-1} + \bar{B}_{j-1}}{2}, \\ e_j\widetilde{a}_j + (1-e_j)\widetilde{b}_j = \dfrac{\bar{B}_{j-1} - \bar{A}_{j-1}}{2}. \end{cases}$$

Thus, for any $j \in \mathbb{Z}$, $(\widetilde{a}_j, \widetilde{b}_j)$ satisfies

$$\begin{cases} e_j\widetilde{a}_j + (1-e_j)\widetilde{b}_j = \dfrac{\bar{B}_{j-1} - \bar{A}_{j-1}}{2}, \\ (1-e_j)\widetilde{a}_j + e_j\widetilde{b}_j = \dfrac{\bar{A}_j + \bar{B}_j}{2}, \end{cases}$$

which is a linear system in the unknown $(\widetilde{a}_j, \widetilde{b}_j)$ having the solution

$$\widetilde{a}_j = \frac{1}{2}\frac{(1-e_j)(\bar{A}_j + \bar{B}_j) + e_j(\bar{A}_{j-1} - \bar{B}_{j-1})}{1-2e_j},$$
$$\widetilde{b}_j = \frac{1}{2}\frac{(1-e_j)(\bar{B}_{j-1} - \bar{A}_{j-1}) - e_j(\bar{A}_j + \bar{B}_j)}{1-2e_j}.$$

Since e_j is either 0 or 1 we see that $|1-2e_j| = 1$ and then

$$|\widetilde{a}_j|, |\widetilde{b}_j| \le \frac{1}{2}(|\bar{A}_{j-1}| + |\bar{A}_j|) + \frac{1}{2}(|\bar{B}_{j-1}| + |\bar{B}_j|)$$

or

$$\|\widetilde{a}\|_\infty + \|\widetilde{b}\|_\infty \le 2(\|\bar{A}\|_\infty + \|\bar{B}\|_\infty).$$

That is, L_E^{-1} exists and $\|L_E^{-1}\| \le 2$. As a consequence, for any m sufficiently large and $\xi \in X$, $L_{m,\xi}$ has a bounded inverse $L_{m,\xi}^{-1}$ so that, say,

$$\|L_{m,\xi}^{-1}\| \le 3\,. \tag{5.1.51}$$

Thus we can uniquely solve Eqs. (5.1.44)–(5.1.46) for $\tilde{a}_j = \tilde{a}_j(h,\xi)$, $\tilde{b}_j = \tilde{b}_j(h,\xi)$ and a constant $\tilde{c}$ independent of $\xi \in X$ and $m \in \mathbb{N}$ (provided $m \ge m_0$, with m_0 sufficiently large) exists so that

$$|\tilde{a}_j(h,\xi)| \le \tilde{c}\|h\|_\infty, \quad |\tilde{b}_j(h,\xi)| \le \tilde{c}\|h\|_\infty \tag{5.1.52}$$

for any $j \in \mathbb{Z}$. Consequently, the compatibility condition (5.1.39) reads

$$\int_{(2j-1)m}^{(2j+1)m} \dot{\gamma}_\xi(s)h(s)\,ds = -\mathscr{L}_{m,\xi,j}(h) := \ddot{\gamma}_1(0)\left[\frac{a_j^-(h,\xi)}{u(-m-\alpha_j)} - \frac{a_j^+(h,\xi)}{u(m-\alpha_j)}\right]$$

for any $j \in \mathbb{Z}$ so that $e_j = 1$. Since we do not need any compatibility condition when $e_j = 0$, we set

$$\mathscr{L}_{m,\xi,j}(h) = 0 \quad \text{for any } j \in \mathbb{Z} \text{ such that } e_j = 0.$$

Clearly, the existence of a constant $B > 0$ so that Equation (5.1.28) holds, following from Eqs. (5.1.40), (5.1.41) and (5.1.52). Similarly the existence of the constant A as in the statement of the Lemma follows from (5.1.52) together with the fact that $|\alpha_j| \le 2$ for any $j \in \mathbb{Z}$ and $u(t) \sim e^{a|t|}$ as $|t| \to \infty$.

Now we estimate $\bar{v}(t) = x_1(t) - x_p(t)$, $x_p(t)$ being the unique bounded solution of the equation $\ddot{x} + \frac{8}{\pi}f'(0)x = h(t)$. Observe that $\bar{v}(t)$ is a C^1 solution, bounded on $\mathbb{R}$, of the differential equation:

$$\ddot{x} + \frac{8}{\pi}f'(0)x + w(t) = 0$$

where $w(t) = \frac{8}{\pi}\Big(f'\Big(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\Big) - f'(0)\Big)x_1(t)$. Thus

$$\bar{v}(t) = \frac{1}{2a}\int_{-\infty}^{t} e^{-a(t-s)}\,w(s)\,ds + \frac{1}{2a}\int_{t}^{\infty} e^{a(t-s)}\,w(s)\,ds\,.$$

Let $A_1 = 1 + \max_{t\in\mathbb{R}}|\gamma(t)|$ and $N = \max_{x\in[-A_1,A_1]}\{|f'(x)|, |f''(x)|\}$.Then

$$|w(s)| \le \frac{16}{\pi\sqrt{\pi}}BN\|h\|_\infty|\gamma_\xi(s)|$$

and hence

$$|\bar{v}(t)| \le \frac{16BN\|h\|_\infty}{2a\pi\sqrt{\pi}}\left\{\int_{-\infty}^{t} e^{-a(t-s)}\,|\gamma_\xi(s)|\,ds + \int_{t}^{\infty} e^{a(t-s)}\,|\gamma_\xi(s)|\,ds\right\}.$$

So, we consider the integrals

$$I(t,\xi) := \int_{-\infty}^{t} e^{-a(t-s)}\,|\gamma_\xi(s)|\,ds, \quad J(t,\xi) := \int_{t}^{\infty} e^{a(t-s)}\,|\gamma_\xi(s)|\,ds\,.$$

For any $\xi = (E,\alpha) \in X$, $E = \{e_j\}_{j\in\mathbb{Z}} \in \mathscr{E}$, $\alpha := \{\alpha_j\}_{j\in\mathbb{Z}} \in \ell_E^\infty$, let $\widetilde{\xi} = (\widetilde{E},\widetilde{\alpha}) \in X$ be defined as

$$\widetilde{E} := \{e_{-j}\}_{j\in\mathbb{Z}} \in \mathscr{E}, \qquad \widetilde{\alpha} := \{-\alpha_{-j}\}_{j\in\mathbb{Z}} \in \ell_{\widetilde{E}}^\infty.$$

From the definitions of $\gamma_\xi(t)$ and $\gamma_1(t) = \gamma_1(-t)$ we see that $\gamma_\xi(t) = \gamma_{\widetilde{\xi}}(-t)$ for any $t \in \mathbb{R}$, $t \neq (2j-1)m$, $j \in \mathbb{Z}$, and then

$$J(t,\xi) = \int_{-\infty}^{-t} \mathrm{e}^{a(t+s)} |\gamma_\xi(-s)|\, ds = \int_{-\infty}^{-t} \mathrm{e}^{-a(-t-s)} |\gamma_{\widetilde{\xi}}(s)|\, ds = I(-t,\widetilde{\xi}).$$

Thus we see that it is enough to estimate $I(t,\xi)$. Let $(2j-1)m < t \leq (2j+1)m$. We have

$$\int_{-\infty}^{(2j-3)m} \mathrm{e}^{-a(t-s)} |\gamma_\xi(s)|\, ds \leq \frac{A_1}{a}\, \mathrm{e}^{-2am} < \frac{A_1}{a}\, \mathrm{e}^{-am/2}\,.$$

Next, we estimate

$$\int_{(2j-3)m}^{(2j-1)m} \mathrm{e}^{-a(t-s)} |\gamma_\xi(s)|\, ds, \qquad \int_{(2j-1)m}^{t} \mathrm{e}^{-a(t-s)} |\gamma_\xi(s)|\, ds\,.$$

Since $\gamma_\xi(t) = 0$ if $(2i-1)m < t \leq (2i+1)m$ and $e_i = 0$ we see that we can assume that $e_{j-1} = e_j = 1$ and

$$\gamma_\xi(t) = \begin{cases} \gamma_1(t - 2(j-1)m - \alpha_{j-1}), & \text{if } (2j-3)m < t \leq (2j-1)m, \\ \gamma_1(t - 2jm - \alpha_j), & \text{if } (2j-1)m < t \leq (2j+1)m. \end{cases}$$

Now, let $A_2 > 0$ be such that

$$\max\Big\{|\gamma_1(t)|, |\dot{\gamma}_1(t)|, |\ddot{\gamma}_1(t)|\Big\} \leq A_2\, \mathrm{e}^{-a|t|}\,. \tag{5.1.53}$$

Then

$$\int_{(2j-3)m}^{(2j-1)m} \mathrm{e}^{-a(t-s)} |\gamma_\xi(s)|\, ds \leq \int_{(2j-3)m}^{(2j-1)m} \mathrm{e}^{-a(t-s)} |\gamma_1(s - 2(j-1)m - \alpha_{j-1})| ds$$

$$\leq A_2 \int_{(2j-3)m}^{(2j-1)m} \mathrm{e}^{-a(t-s)}\, \mathrm{e}^{-a|s-2(j-1)m-\alpha_{j-1}|}\, ds$$

$$\leq A_2 \int_{2(j-1)m+\alpha_{j-1}}^{(2j-1)m} \mathrm{e}^{-a(t-s)}\, \mathrm{e}^{-a(s-2(j-1)m-\alpha_{j-1})}\, ds + A_2 \int_{(2j-3)m}^{2(j-1)m+\alpha_{j-1}} \mathrm{e}^{-a(t-s)}\, ds$$

$$\leq A_2\, \mathrm{e}^{-a(m-2)}(m+2) + \frac{A_2}{a}\, \mathrm{e}^{-a(m-2)} \leq \frac{A_2(\mathrm{e}^{4a}+1)}{a}\, \mathrm{e}^{-a(m-2)/2}\,.$$

Finally, if $(2j-1)m < t \leq 2jm + \alpha_j$ we have:

$$\int_{(2j-1)m}^{t} \mathrm{e}^{-a(t-s)} |\gamma_\xi(s)|\, ds \leq A_2 \int_{(2j-1)m}^{t} \mathrm{e}^{-a(t-s)}\, \mathrm{e}^{a(s-2jm-\alpha_j)}\, ds$$

$$\leq \frac{A_2}{2a}\, \mathrm{e}^{-a|t-2jm-\alpha_j|} \leq \frac{A_2}{2a}\, \mathrm{e}^{-a|t-2jm-\alpha_j|/2}$$

whereas if $2jm+\alpha_j < t \leq (2j+1)m$

$$\int_{(2j-1)m}^{t} e^{-a(t-s)}\,|\gamma_\xi(s)|\,ds$$

$$\leq A_2 \int_{(2j-1)m}^{t} e^{-a(t-s)}\,e^{-a|s-2jm-\alpha_j|}\,ds$$

$$\leq A_2 \int_{(2j-1)m}^{2jm+\alpha_j} e^{-a(t-s)}\,e^{-a(2jm+\alpha_j-s)}\,ds + A_2 \int_{2jm+\alpha_j}^{t} e^{-a(t-s)}\,e^{-a(s-2jm-\alpha_j)}\,ds$$

$$\leq \frac{A_2}{2a}\,e^{-a(t-2jm-\alpha_j)} + A_2\,e^{-a(t-2jm-\alpha_j)}(t-2jm-\alpha_j) \leq \frac{3A_2}{2a}\,e^{-a(t-2jm-\alpha_j)/2},$$

since $a\theta\,e^{-a\theta} \leq e^{-a\theta/2}$ for any $\theta \geq 0$. The fact that inequality (5.1.29) holds in the closed interval $[(2j-1)m,(2j+1)m]$ follows from continuity. We now prove (iii). Let $w(t) \in C^\infty(\mathbb{R})$ be a smooth function so that $\operatorname{supp} w \in (-1,1)$ and $w'(0) = 1$ and set

$$\hat{x}_1(t) = x_1(t,\xi') - x_1(t,\xi'') + e_j\dot{x}_1(2jm+\alpha_j',\xi'')w(t-2jm-\alpha_j')$$

if $(2j-1)m < t \leq (2j+1)m$ and $j \in \mathbb{Z}$. Note that $\hat{x}_1(t)$ is a bounded C^1-function on $\mathbb{R}$ that satisfies, in any interval $((2j-1)m,(2j+1)m]$, the equation:

$$\ddot{x}_1 + \frac{8}{\pi} f'(\frac{2}{\sqrt{\pi}}\gamma_{\xi'}(t))x_1$$

$$= h(t,\xi') - h(t,\xi'')$$

$$+\frac{8}{\pi}\left[f'\left(\frac{2}{\sqrt{\pi}}\gamma_{\xi''}(t)\right) - f'\left(\frac{2}{\sqrt{\pi}}\gamma_{\xi'}(t)\right)\right] x_1(t,\xi'')$$

$$-e_j\dot{x}_1(2jm+\alpha_j',\xi'')\left[\ddot{w}(t-2jm-\alpha_j') + \frac{8}{\pi}f'\left(\frac{2}{\sqrt{\pi}}\gamma_{\xi'}(t)\right)w(t-2jm-\alpha_j')\right]$$

together with $\dot{x}_1(2jm+\alpha_j') = 0$ when $e_j = 1$. Thus, because of (i) and (5.1.22),

$$\max\{\|x_1(\cdot,\xi') - x_1(t,\xi'')\|_\infty, \|\dot{x}_1(\cdot,\xi') - \dot{x}_1(t,\xi'')\|_\infty\}$$

$$\leq B\|h(\cdot,\xi') - h(\cdot,\xi'')\|_\infty$$

$$+\tilde{B}\sup_{j\in\mathbb{Z}}|e_j\dot{x}_1(2jm+\alpha_j',\xi'')| + \frac{16B^2N}{\pi\sqrt{\pi}}\|h(\cdot,\xi'')\|_\infty\|\gamma_{\xi'} - \gamma_{\xi''}\|_\infty$$

$$\leq B\|h(\cdot,\xi') - h(\cdot,\xi'')\|_\infty + \tilde{B}\sup_{j\in\mathbb{Z}}|e_j\dot{x}_1(2jm+\alpha_j',\xi'')|$$

$$+B_1\|h(\cdot,\xi'')\|_\infty\,\|\alpha' - \alpha''\| \tag{5.1.54}$$

for some choice of the positive constants B_1 and $\tilde{B}$. On the other hand, when $e_j = 1$, we have, since $\dot{x}_1(2jm + \alpha''_j, \xi'') = 0$,

$$\dot{x}_1(2jm + \alpha'_j, \xi'') = \int_{2jm+\alpha''_j}^{2jm+\alpha'_j} \ddot{x}_1(t, \xi'')dt$$

$$= \int_{2jm+\alpha''_j}^{2jm+\alpha'_j} \left(h(t, \xi'') - \frac{8}{\pi} f'\left(\frac{2}{\sqrt{\pi}} \gamma_{\xi''}(t) \right) x_1(t, \xi'') \right) dt$$

and hence

$$|\dot{x}_1(2jm + \alpha'_j, \xi'')| \le \left[1 + \frac{8B}{\pi} |f'(0)| \right] \|h(\cdot, \xi'')\|_\infty |\alpha'_j - \alpha''_j|$$

$$+ \frac{8B}{\pi} \|h(\cdot, \xi'')\|_\infty \int_0^{\alpha'_j - \alpha''_j} \left| f'\left(\frac{2}{\sqrt{\pi}} \gamma_1(t) \right) - f'(0) \right| dt$$

$$\le \left\{ 1 + \frac{8B}{\pi} [|f'(0)| + A_1 N] \right\} \|h(\cdot, \xi'')\|_\infty |\alpha'_j - \alpha''_j|. \quad (5.1.55)$$

Then (iii) follows from (5.1.54), (5.1.55). Finally, the proof of Lipschitz continuity of the map $\mathscr{L}_m$ with respect to α is given in Section 5.1.6. □

Now we consider the equation

$$\ddot{x}_2 + \frac{24}{\pi} f'\left(\frac{2}{\sqrt{\pi}} \gamma_\xi(t) \right) x_2 = h \in L^\infty(\mathbb{R}) \quad (5.1.56)$$

and prove the following.

Lemma 5.1.3. *There exist positive constants $B_1, C_1 \in \mathbb{R}$ and $m_1 \in \mathbb{N}$, so that for any $\xi = (E, \alpha) \in X$ and $m \ge m_1$, Equation* (5.1.56) *has a unique C^1 solution $x_2(t, \xi)$ which is bounded on $\mathbb{R}$ and satisfies*

$$\|x_2(\cdot, \xi)\|_\infty \le B_1 \|h\|_\infty, \quad \|\dot{x}_2(\cdot, \xi)\|_\infty \le B_1 \|h\|_\infty. \quad (5.1.57)$$

Moreover the following properties hold:

(i) *Let $z_p(t)$ be the unique bounded solution of equation $\ddot{z}_p + \frac{24}{\pi} f'(0) z_p = h(t)$, then*

$$|x_2(t, \xi) - z_p(t)| \le C_1 \left(e^{-am/2} + e^{-a|t-2jm-\alpha_j|/2} \right) \|h\|_\infty \quad (5.1.58)$$

for $(2j-1)m \le t \le (2j+1)m$ and any $j \in \mathbb{Z}$.

(ii) *Let $\xi' = (E, \alpha')$, $\xi'' = (E, \alpha'')$ with $\alpha', \alpha'' \in \ell_E^\infty$ and ξ be either ξ' or ξ''. Assume that $h(t, \xi) \in L^\infty(\mathbb{R})$. Then there exists a constant, $\hat{c}_1$, independent of ξ, so that the following holds:*

$$\max\left\{\|x_2(\cdot,\xi')-x_2(\cdot,\xi'')\|_\infty,\|\dot{x}_2(\cdot,\xi')-\dot{x}_2(\cdot,\xi'')\|_\infty\right\}$$
$$\leq B_1\|h(t,\xi')-h(t,\xi'')\|_\infty+\hat{c}_1\|h(t,\xi'')\|_\infty\|\alpha'-\alpha''\|_\infty. \quad (5.1.59)$$

Proof. Since the proof is very similar to that of Lemma 5.1.2 (actually simpler) we only sketch it emphasizing the differences. Because of assumption (H2), the homogeneous equation associated with (5.1.56) has an exponential dichotomy on $\mathbb{R}$, that is, there exists a projection P of rank one so that the fundamental system $X(t)$ of (5.1.56) satisfies:

$$\|X(t)PX^{-1}(s)\|\leq k\mathrm{e}^{-b(t-s)}, \qquad \text{for any } s\leq t,$$
$$\|X(t)(\mathbb{I}-P)X^{-1}(s)\|\leq k\mathrm{e}^{-b(t-s)}, \qquad \text{for any } t\leq s$$

where $b=\sqrt{\frac{24}{\pi}|f'(0)|}$. Let $v_0\in\mathscr{R}P$, $u_0\in\mathscr{N}P$ be unitary vectors, and set

$$\begin{pmatrix}u(t)\\ \dot{u}(t)\end{pmatrix}:=X(t)u_0, \qquad \begin{pmatrix}v(t)\\ \dot{v}(t)\end{pmatrix}:=X(t)v_0.$$

Then it can be proved that (5.1.34) holds for any $t\leq s$ whereas (5.1.35) holds for any $s\leq t$. Now, when $e_j=0$ Equation (5.1.36), with b instead of a, gives the solution to (5.1.56) but now, since when $e_j=1$ we do not impose the condition $\dot{x}_2(2jm+\alpha_j)=0$, we do not need to split the interval $[(2j-1)m,2(j+1)m]$ into two parts and the general solution of (5.1.56) can be written as:

$$\begin{aligned}x_1(t)=&\int_{(2j-1)m}^{t}v(t-2jm)u(s-2jm)h(s)\,ds\\&+\int_{t}^{(2j+1)m}u(t-2jm)v(s-2jm)h(s)\,ds\\&+a_ju(t-2jm)/u(-m)+b_jv(t-2jm)/v(m).\end{aligned}$$

It is easy to see that $x_1(t)$ belongs to $L^\infty(\mathbb{R})$ and is C^1 in any open interval $((2j-1)m,(2j+1)m)$. Thus we obtain a unique bounded C^1 solution of Eq. (5.1.56) provided we show that Eq. (5.1.42) can be uniquely solved. This fact and the properties (i), (ii) are proved in the proof of Lemma 5.1.2 and so we omit it. □

In order to apply Lemma 5.1.2, we consider the set

$$\mathscr{S}_{m,\xi}:=\left\{h\in L^\infty(\mathbb{R})\mid \mathscr{L}_{m,\xi,j}h+\int_{(2j-1)m}^{(2j+1)m}\dot{\gamma}_\xi(t)h(t)\,dt=0 \quad \text{for any} \quad j\in\mathbb{Z}\right\}.$$

Note that if $\xi=0$ (i.e. $(E,\alpha)=(0,0)$) then $\mathscr{S}_{m,\xi}=L^\infty(\mathbb{R})$. Then we construct a projection $Q_{m,\xi}:L^\infty(\mathbb{R})\to\mathscr{S}_{m,\xi}$ as follows. If $\xi=0$ we set $Q_{m,\xi}=\mathbb{I}$, whereas if $\xi\neq 0$ (and hence $E\neq 0$) we proceed in the following way. For any $c=\{c_i\}_{i\in\mathbb{Z}}\in\ell^\infty_E$, we put

$$\gamma_c(t) = c_j \dot{\gamma}_\xi(t) \quad \text{for} \quad (2j-1)m < t \le (2j+1)m\,.$$

We recall that $\ell_E^\infty := \Big\{ c = \{c_i\}_{i\in\mathbb{Z}} \in \ell^\infty \mid c_i = 0 \quad \text{for} \quad e_i = 0 \Big\}$. Hence $\gamma_c \in L^\infty(\mathbb{R})$ and

$$|\gamma_c(t)| \le \|c\|_\infty |\dot{\gamma}_\xi(t)| \le \|c\|_\infty \|\dot{\gamma}_1\|_\infty.$$

For any $h \in L^\infty(\mathbb{R})$ we take $h_c = h - \gamma_c$ and consider the system of equations

$$\mathscr{L}_{m,\xi,j} h_c + \int_{(2j-1)m}^{(2j+1)m} \dot{\gamma}_\xi(t) h_c(t)\, dt = 0, \quad j \in \mathbb{Z}. \tag{5.1.60}$$

Our purpose is to determine a solution $c \in \ell_E^\infty$ of the above system. Note that when $e_j = 0$, one has $\mathscr{L}_{m,\xi,j} = 0$, $\gamma_\xi(t) = 0$ and then the above equation is trivially satisfied regardless of the value of c_j. This is the reason why we take $c_j = 0$ when $e_j = 0$. On the other hand, since $\gamma_\xi(t) = 0$ in $((2j-1)m, (2j+1)m]$ when $e_j = 0$, the value of c_j does not matter to defining $\gamma_c(t)$ in this interval. We can write (5.1.60) as

$$[\mathscr{M}_{m,\xi} + \mathscr{L}_{m,\xi} G_{m,\xi}]c = [\mathscr{L}_{m,\xi} + N_{m,\xi}]h \tag{5.1.61}$$

where

$$\mathscr{L}_{m,\xi} h = \{\mathscr{L}_{m,\xi,j} h\}_{j\in\mathbb{Z}} \in \ell_E^\infty, \quad \mathscr{M}_{m,\xi} c = \left\{ c_j \int_{(2j-1)m}^{(2j+1)m} \dot{\gamma}_\xi^2(t) dt \right\}_{j\in\mathbb{Z}} \in \ell_E^\infty,$$

$$G_{m,\xi} c = \gamma_c(t) = \sum_{j\in\mathbb{Z}} c_j \dot{\gamma}_\xi(t) \chi_{((2j-1)m,(2j+1)m]}(t) \in L^\infty(\mathbb{R}),$$

$$N_{m,\xi} h = \left\{ \int_{(2j-1)m}^{(2j+1)m} \dot{\gamma}_\xi(t) h(t) dt \right\}_{j\in\mathbb{Z}} \in \ell_E^\infty.$$

Note that for any fixed $E \in \mathscr{E}$, both sides of Eq. (5.1.61) are elements of ℓ_E^∞.

Now, we have already observed that $\|G_{m,\xi} c\|_\infty \le \|\dot{\gamma}_1\|_\infty \cdot \|c\|_\infty$, moreover, from Lemma 5.1.2 it follows that $\|\mathscr{L}_{m,\xi} h\|_\infty \le A\,\mathrm{e}^{-am}\|h\|_\infty$. Hence

$$\|\mathscr{L}_{m,\xi} G_{m,\xi} c\|_\infty \le A\,\mathrm{e}^{-am}\|\dot{\gamma}\|_\infty \cdot \|c\|_\infty\,. \tag{5.1.62}$$

Next, setting

$$\tilde{A}_1 = \int_{-\infty}^{\infty} |\dot{\gamma}_1(t)|\, dt > 0, \quad \tilde{A}_2 = \int_{-\infty}^{\infty} \dot{\gamma}_1(t)^2\, dt > 0$$

we have, for m sufficiently large, and any $j \in \mathbb{Z}$, with $e_j = 1$

$$\frac{\tilde{A}_2}{2} \le \left| \int_{(2j-1)m}^{(2j+1)m} \dot{\gamma}_\xi(t)^2 dt \right| = \left| \int_{-m-\alpha_j}^{m-\alpha_j} \dot{\gamma}_1(t)^2 dt \right| \le \tilde{A}_2$$

since $|\alpha_j| \le 2$ for any $j \in \mathbb{Z}$. Thus $\mathscr{M}_{m,\xi} : \ell_E^\infty \to \ell_E^\infty$ is a bounded linear map ($\|\mathscr{M}_{m,\xi}\| \le \tilde{A}_2$) which is invertible and it is easy to see that its inverse $\mathscr{M}_{m,\xi}^{-1}$ satisfies:

$$\frac{1}{\tilde{A}_2} \leq \|\mathscr{M}_{m,\xi}^{-1}\| \leq \frac{2}{\tilde{A}_2}$$

provided $m \in \mathbb{N}$ is sufficiently large. Thus, using also (5.1.62) we see that $[\mathscr{M}_{m,\xi} + \mathscr{L}_{m,\xi} G_{m,\xi}]^{-1}$ exists and is bounded uniformly with respect to (ξ, m) provided m is large enough. Finally:

$$\|N_{m,\xi}h\| = \sup_{j\in\mathbb{Z}} \left| e_j \int_{-m}^{m} \dot{\gamma}_1(t-\alpha_j)h(t+2jm)dt \right| \leq \tilde{A}_1 \|h\| \tag{5.1.63}$$

and hence Equation (5.1.61) has the unique solution, linear with h

$$c(m,\xi)h = \left[\mathscr{M}_{m,\xi} + \mathscr{L}_{m,\xi} G_{m,\xi}\right]^{-1} \left[\mathscr{L}_{m,\xi} + N_{m,\xi}\right] h \in \ell_E^\infty$$

and the linear map $h \mapsto c(m,\xi)h$ is a bounded linear map from $L^\infty(\mathbb{R})$ into ℓ_E^∞ with bound independent of (m,ξ) (of course with $m \geq \bar{m}$ sufficiently large). We set

$$P_{m,\xi}h = \gamma_{c(m,\xi)h}, \quad Q_{m,\xi} = \mathbb{I} - P_{m,\xi}.$$

Obviously we mean that $c(m,0) = 0$ for any $m \in \mathbb{N}$ so that $P_{m,0} = 0$ and $Q_{m,0} = \mathbb{I}$. We have the following:

Theorem 5.1.4. $P_{m,\xi} : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ *is a projection on* $L^\infty(\mathbb{R})$ *which is uniformly bounded with respect to* (m,ξ) *and Lipschitz in* $\alpha \in \ell_E^\infty$ *uniformly with respect to* (m,E)*. That is, a constant L, independent of* (m,E)*, exists such that* $\|P_{m,(E,\alpha)} - P_{m,(E,\alpha')}\| \leq L\|\alpha - \alpha'\|$ *for any* $m \geq \bar{m}$ *and* $(E,\alpha),(E,\alpha') \in X$*. Furthermore*

$$|[P_{m,\xi}h](t)| \leq |c(m,\xi)| \|h\|_\infty |\dot{\gamma}_\xi(t)| \tag{5.1.64}$$

and $P_{m,\xi}h = 0$ *if and only if*

$$[\mathscr{L}_{m,\xi} + N_{m,\xi}]h = 0. \tag{5.1.65}$$

Proof. Since there is nothing to prove when $\xi = 0$ we assume $\xi \neq 0$. The fact that $P_{m,\xi}$ is bounded uniformly with respect to (m,ξ) and actually satisfies (5.1.64) has already been proved. We now prove the last statement: the equation $P_{m,\xi}h = 0$ holds if and only if $\gamma_{c(m,\xi)h} = 0$, that is, if and only if $h = h_{c(m,\xi)h}$. Thus (5.1.65) follows because $c(m,\xi)h$ satisfies Eq. (5.1.60). On the contrary, if h satisfies (5.1.65), we have $c(m,\xi)h = 0$ because of uniqueness and then $P_{m,\xi}h = 0$. We can now prove that $P_{m,\xi}$ is a projection. In fact, we have $P_{m,\xi}[Q_{m,\xi}h] = P_{m,\xi}[h - P_{m,\xi}h] = 0$ because $h - P_{m,\xi}h = h - \gamma_{c(m,\xi)h}$ satisfies (5.1.65). Thus $P_{m,\xi} = P_{m,\xi}^2$. Finally we prove the Lipschitz continuity of $P_{m,\xi}$. First we prove that

$$(\xi,h) \mapsto N_{m,\xi}h = \left\{ e_j \int_{-m}^{m} \dot{\gamma}_1(t-\alpha_j)h(t+2jm)dt \right\}_{j\in\mathbb{Z}}$$

from $X \times L^\infty$ into ℓ^∞_E, is Lipschitz continuous function in α uniformly with respect to (m,E). In fact, for $\tau'',\tau' \in \mathbb{R}$ with $|\tau''|,|\tau'| \le 2$, we have, using $\ddot\gamma_1(t) = \frac{4}{\sqrt{\pi}} f(\frac{2}{\sqrt{\pi}}\gamma_1(t))$, $|f'(x)| \le N$ and (5.1.53):

$$\begin{aligned}
&\left|\int_{-m}^{m}[\dot\gamma_1(t-\tau'')-\dot\gamma_1(t-\tau')]h(t+2jm)dt\right| \\
&\le \int_{-m}^{m}\int_0^1 |\ddot\gamma_1(t-\theta\tau''-(1-\theta)\tau')|d\theta\, dt\, \|h\|_\infty|\tau''-\tau'| \\
&\le \frac{8N}{\pi}\int_{-m}^{m}\int_0^1 |\gamma_1(t-\theta\tau''-(1-\theta)\tau')|d\theta\, dt\, \|h\|_\infty|\tau''-\tau'| \\
&\le \frac{8N}{\pi}\int_{-m}^{m}\int_0^1 A_2\,\mathrm{e}^{-a|t-\theta\tau''-(1-\theta)\tau'|}d\theta\, dt\, \|h\|_\infty|\tau''-\tau'| \\
&\le \frac{8N}{\pi}\int_{-m}^{m}\int_0^1 A_2\,\mathrm{e}^{-a(|t|-2)}d\theta\, dt\, \|h\|_\infty|\tau''-\tau'| \le \frac{16NA_2\,\mathrm{e}^{2a}}{a\pi}\|h\|_\infty|\tau''-\tau'|\,.
\end{aligned}$$

Similarly we can prove that the bounded linear maps $\mathscr{M}_{m,\xi} : \ell^\infty_E \to \ell^\infty_E$ and $G_{m,\xi} : \ell^\infty_E \to L^\infty$ are Lipschitz continuous function in α uniformly with respect to (E,m). Then the inverse $[\mathscr{M}_{m,\xi} + \mathscr{L}_{m,\xi}G_{m,\xi}]^{-1}$ has the same property and the same holds for the solution $c(m,\xi)h$ of Eq. (5.1.61). Finally, let $\xi' = (E,\alpha'), \xi'' = (E,\alpha'') \in X$. Then for any $t \in ((2j-1)m,(2j+1)m]$ we have

$$[P_{m,\xi'}h - P_{m,\xi''}h](t) = \dot\gamma_{\xi''}(t)[c_j(m,\xi')h - c_j(m,\xi'')h] + [\dot\gamma_{\xi'}(t) - \dot\gamma_{\xi''}(t)]c_j(m,\xi')h$$

and hence $P_{m,\xi}$ is Lipschitz continuous function in α uniformly with respect to (E,m), so are $c(m,\xi)$ and $\dot\gamma_\xi(t)$ and both are bounded uniformly with respect to (ξ,m). The proof is complete. □

Remark 5.1.5. (a) Obviously $Q_{m,\xi}$ is also Lipschitz continuous function in α, uniformly with respect to (m,E) and, using $P_{m,\xi}Q_{m,\xi} = 0$, we see that the equation

$$\mathscr{L}_{m,\xi,j}Q_{m,\xi}h + \int_{(2j-1)m}^{(2j+1)m}\dot\gamma_\xi(t)[Q_{m,\xi}h](t)dt = 0$$

holds for any $j \in \mathbb{Z}$. That is, $Q_{m,\xi}$ is a projection from $L^\infty(\mathbb{R})$ onto $\mathscr{S}_{m,\xi}$ which is bounded uniformly with respect to (ξ,m), so is $P_{m,\xi}$.

(b) It follows from the arguments in Section 5.1.6 that $\mathscr{L}_{m,\xi}$ is not differentiable in α. Hence $P_{m,\xi}$ and $Q_{m,\xi}$ are also not differentiable in α. So the Lipschitz continuity of these maps is their best smoothness in α.

(c) If $h(t) = \dot\gamma_\xi(t)$ and $c_j = e_j$ for any $j \in \mathbb{Z}$, we have $h_c(t) = \dot\gamma_\xi(t) - \dot\gamma_\xi(t) = 0$ and then (5.1.60) is satisfied. Thus, because of uniqueness, $P_{m,\xi}\dot\gamma_\xi = \dot\gamma_\xi$ or

$$Q_{m,\xi}\dot\gamma_\xi = 0. \tag{5.1.66}$$

5.1.4 Chaotic Solutions

We look for solutions of Eqs. (5.1.6)–(5.1.8), for which, the sup-norms of $y_1(t) - \gamma_\xi(t)$, $y_2(t)$ and $z(x,t)$ are small. Since the function $\gamma_\xi(t)$ has small jumps at the points $t = (2j-1)m$, $j \in \mathbb{Z}$, we introduce a function $v_\xi(t) \in L^\infty(\mathbb{R})$ which has small norm, so that

$$\Gamma_\xi(t) = \gamma_\xi(t) + v_\xi(t)$$

is C^1. As an example, we can take the function:

$$v_\xi(t) = \frac{p_j}{4m^2}\big(t-(2j-1)m\big)^2 + \frac{q_j}{8m^3}\big(t-(2j-1)m\big)^3$$

if $(2j-1)m < t \le (2j+1)m$, $j \in \mathbb{Z}$, where

$$\begin{aligned} p_j &= 3\big(\gamma_\xi(((2j+1)m)_+) - \gamma_\xi(((2j+1)m)_-)\big) \\ &\quad + 2m\big(\dot\gamma_\xi(((2j+1)m)_-) - \dot\gamma_\xi(((2j+1)m)_+)\big)\,, \\ q_j &= 2m\big(\dot\gamma_\xi(((2j+1)m)_+) - \dot\gamma_\xi(((2j+1)m)_-)\big) \\ &\quad + 2\big(\gamma_\xi(((2j+1)m)_-) - \gamma_\xi(((2j+1)m)_+)\big)\,. \end{aligned}$$

Again, we will silently include, in the definition of $v_\xi(t)$ and $\Gamma_\xi(t)$, also the end points of the intervals $[(2j-1)m, (2j+1)m]$ as we did for the function $\gamma_\xi(t)$. Next, from (5.1.53) we obtain, for any $j \in \mathbb{Z}$:

$$\max\Big\{|\gamma_\xi(((2j+1)m)_\pm)|, |\dot\gamma_\xi(((2j+1)m)_\pm)|\Big\} \le A_2\,\mathrm{e}^{2a}\,\mathrm{e}^{-am} = \bar{A}_2\,\mathrm{e}^{-am}$$

where $\bar{A}_2 = A_2\,\mathrm{e}^{2a}$. As a consequence, we get

$$\begin{aligned} \|v_\xi\|_\infty &\le (10+8m)\bar{A}_2\,\mathrm{e}^{-am}, \\ \|\dot v_\xi\|_\infty &\le (12+10m)\bar{A}_2\,\mathrm{e}^{-am}/m, \\ \|\ddot v_\xi\|_\infty &\le (9+8m)\bar{A}_2\,\mathrm{e}^{-am}/m^2\,, \end{aligned} \tag{5.1.67}$$

or, since $0 < \varepsilon \le 2^{-4/3}$ (and hence $m > \varepsilon^{-3/4} \ge 2$):

$$\|v_\xi\|_\infty < \frac{12\bar{A}_2}{a^{7/3}}\varepsilon, \quad \|\dot v_\xi\|_\infty < \frac{6\bar{A}_2}{a^{4/3}}\varepsilon, \quad \|\ddot v_\xi\|_\infty < \frac{6\bar{A}_2}{a}\varepsilon^{3/2}\,. \tag{5.1.68}$$

Note that to obtain the inequalities (5.1.68) from (5.1.67) we have used the fact that for $\lambda > 0$, and $\theta > 0$ we have $\theta^\lambda\,\mathrm{e}^{-\theta} \le (\lambda/\mathrm{e})^\lambda$ and $\left(\frac{4}{3\mathrm{e}}\right)^{4/3} < \frac{2}{5}$, $\frac{1}{\mathrm{e}} < \frac{1}{2}$, $\left(\frac{7}{3\mathrm{e}}\right)^{7/3} < 1$. Let $\Lambda = \max\left\{\frac{12\mathrm{e}^{2a}}{a^{7/3}}, \frac{6\mathrm{e}^{2a}}{a^{4/3}}, \frac{6\mathrm{e}^{2a}}{a}, \mathrm{e}^{2a}\right\}$, then:

$$\|v_\xi\|_\infty \le \Lambda A_2\varepsilon, \quad \|\dot v_\xi\|_\infty \le \Lambda A_2\varepsilon, \quad \|\ddot v_\xi\|_\infty \le \Lambda A_2\varepsilon^{3/2}\,. \tag{5.1.69}$$

For reasons that will be clearer later, we now prove that the functions $v_\xi(t)$, $\dot v_\xi(t)$ and $\ddot v_\xi(t)$ are Lipschitz continuous funcfions in α, uniformly with respect to (E,m) and that the Lipschitz constant is of the order $O(\varepsilon)$ as $\varepsilon \to 0$, uniformly with respect to (E,m). So, let $\xi' = (E,\alpha'), \xi'' = (E,\alpha'') \in X$. For any $t \in ((2j-1),(2j+1)m]$ we have (with obvious meaning of symbols):

$$|v_{\xi'}(t) - v_{\xi''}(t)| \le |p'_j - p''_j| + |q'_j - q''_j|$$

$$|\dot v_{\xi'}(t) - \dot v_{\xi''}(t)| \le \frac{2|p'_j - p''_j| + 3|q'_j - q''_j|}{2m}$$

$$|\ddot v_{\xi'}(t) - \ddot v_{\xi''}(t)| \le \frac{|p'_j - p''_j| + 3|q'_j - q''_j|}{2m^2}.$$

Thus it is enough to estimate $|p'_j - p''_j|$ and $|q'_j - q''_j|$. Assume $e_j = 1$, then

$$\gamma_\xi(((2j+1)m)_-) = \gamma_1(m - \alpha_j)$$

and hence, using (5.1.53) and $|\alpha'_j|, |\alpha''_j| \le 2$ (recall that $\bar A_2 = A_2 \mathrm{e}^{2a}$),

$$|\gamma_{\xi'}(((2j+1)m)_-) - \gamma_{\xi''}(((2j+1)m)_-)| \le \bar A_2 \mathrm{e}^{-am} |\alpha'_j - \alpha''_j|.$$

Similarly, if $e_{j+1} = 1$,

$$|\gamma_{\xi'}(((2j+1)m)_+) - \gamma_{\xi''}(((2j+1)m)_+)| \le \bar A_2 \mathrm{e}^{-am} |\alpha'_{j+1} - \alpha''_{j+1}|.$$

On the other hand, if, say, $e_j = 0$ then $\gamma_\xi(((2j+1)m)_-) = 0$, $\alpha'_j = \alpha''_j = 0$ and the same conclusion holds. Thus we get, for any $j \in \mathbb{Z}$ (recall that $m > 3$):

$$|p'_j - p''_j| \le (6+4m)\bar A_2 \mathrm{e}^{-am} \|\alpha' - \alpha''\| < 6m\bar A_2 \mathrm{e}^{-am} \|\alpha' - \alpha''\|$$

and similarly,

$$|q'_j - q''_j| \le (4+4m)\bar A_2 \mathrm{e}^{-am} \|\alpha' - \alpha''\| < 6m\bar A_2 \mathrm{e}^{-am} \|\alpha' - \alpha''\|.$$

Hence, like for (5.1.69), we see that the following holds:

$$\begin{aligned} \|v_{\xi'} - v_{\xi''}\|_\infty &< A_2 \Lambda \varepsilon \|\alpha' - \alpha''\|, \\ \|\dot v_{\xi'} - \dot v_{\xi''}\|_\infty &< A_2 \Lambda \varepsilon \|\alpha' - \alpha''\|, \\ \|\ddot v_{\xi'} - \ddot v_{\xi''}\|_\infty &< A_2 \Lambda \varepsilon^{3/2} \|\alpha' - \alpha''\| \end{aligned} \tag{5.1.70}$$

which is what we want to prove. Now we replace $y_1(t)$ with $y_1(t) + \Gamma_\xi(t)$ in (5.1.6)–(5.1.8) and project the right-hand side of the differential equation for the new $y_1(t)$ to $\mathscr{S}_{m,\xi}$. Since $\gamma_\xi(t)$ satisfies (5.1.21) and $Q_{m,\xi}\dot\gamma_\xi(t) = 0$ (see (5.1.66)), we obtain:

$$
\begin{aligned}
&\ddot{y}_1(t)+\frac{8}{\pi}f'\left(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\right)y_1(t)\\
&= -Q_{m,\xi}\Big\{\sqrt{\varepsilon}\delta\dot{y}_1(t)+\frac{2}{\sqrt{\pi}}\mu\int_0^{\pi/4}h(x,t)\,dx\\
&\quad+\frac{2}{\sqrt{\pi}}f\left(\frac{2}{\sqrt{\pi}}[y_1(t)+\Gamma_\xi(t)]-2\sqrt{\frac{3}{\pi}}y_2(t)+z(0,t)\right)-\frac{4}{\sqrt{\pi}}f\left(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\right)\\
&\quad+\frac{2}{\sqrt{\pi}}f\left(\frac{2}{\sqrt{\pi}}[y_1(t)+\Gamma_\xi(t)]+2\sqrt{\frac{3}{\pi}}y_2(t)+z(\frac{\pi}{4},t)\right)\\
&\quad-\frac{8}{\pi}f'\left(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\right)y_1(t)+\sqrt{\varepsilon}\delta\dot{v}_\xi(t)+\ddot{v}_\xi(t)\Big\},
\end{aligned}
\tag{5.1.71}
$$

$$
\begin{aligned}
&\ddot{y}_2(t)+\frac{24}{\pi}f'\left(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\right)y_2(t)\\
&= -\Big\{\sqrt{\varepsilon}\delta\dot{y}_2(t)+\frac{16}{\pi}\sqrt{\frac{3}{\pi}}\mu\int_0^{\pi/4}h(x,t)\Big(x-\frac{\pi}{8}\Big)\,dx\\
&\quad-2\sqrt{\frac{3}{\pi}}f\Big(\frac{2}{\sqrt{\pi}}[y_1(t)+\Gamma_\xi(t)]-2\sqrt{\frac{3}{\pi}}y_2(t)+z(0,t)\Big)\\
&\quad+2\sqrt{\frac{3}{\pi}}f\Big(\frac{2}{\sqrt{\pi}}[y_1(t)+\Gamma_\xi(t)]+2\sqrt{\frac{3}{\pi}}y_2(t)+z(\pi/4,t)\Big)\\
&\quad-\frac{24}{\pi}f'(\frac{2}{\sqrt{\pi}}\gamma_\xi(t))y_2(t)\Big\},
\end{aligned}
\tag{5.1.72}
$$

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\int_0^{\pi/4}\Big\{z(x,t)\Big[v_{tt}(x,t)+\frac{1}{\varepsilon}v_{xxxx}(x,t)-\sqrt{\varepsilon}\delta v_t(x,t)\Big]+\mu h(x,t)v(x,t)\Big\}\,dx\,dt\\
&+\int_{-\infty}^{\infty}\Big\{f(u(0,t))v(0,t)+f(u(\pi/4,t))v(\pi/4,t)\Big\}\,dt=0,
\end{aligned}
\tag{5.1.73}
$$

in (5.1.73) we write $u(x,t)$ for $\frac{2}{\sqrt{\pi}}[y_1(t)+\Gamma_\xi(t)]+y_2(t)w_0(x)+z(x,t)$.

Let $C_b^1(\mathbb{R})$ be the space of C^1 functions bounded together with their first derivative on $\mathbb{R}$. To make notations simpler we define the Banach spaces Y_1 and Y_2 as the space $C_b^1(\mathbb{R})$ endowed with the norms

$$
\|y_1\|=\frac{2}{\sqrt{\pi}}\sup_{t\in\mathbb{R}}\{|y_1(t)|,|\dot{y}_1(t)|\},\quad \|y_2\|=2\sqrt{\frac{3}{\pi}}\sup_{t\in\mathbb{R}}\{|y_2(t)|,|\dot{y}_2(t)|\},
$$

respectively. Unless otherwise specified, $y_1(t)$, $\hat{y}_1(t)$, resp. $y_2(t)$, $\hat{y}_2(t)$ will denote functions in Y_1, resp. Y_2 and the norm in $Y_1 \times Y_2$ will be $\|y_1\| + \|y_2\|$. Next, let $\rho > 0$ be a fixed positive number, $y_1(t) \in Y_1$, $y_2(t) \in Y_2$ and $z(x,t) \in C_b^0([0, \frac{\pi}{4}] \times \mathbb{R})$ be such that $\|y_1\| + \|y_2\| + \|z\|_\infty \leq \rho$. For any fixed choice of such functions we set:

$$
\begin{aligned}
H_1(x,t) &= \mu h(x,t),\\
H_2(t,\xi) &= f\left(\frac{2}{\sqrt{\pi}}[y_1(t)+\Gamma_\xi(t)] - 2\sqrt{\frac{3}{\pi}}y_2(t)+z(0,t)\right) - f\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right)\\
&\quad - f'\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right)\left[\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}y_2(t)+z(0,t)\right],\\
H_3(t,\xi) &= f\left(\frac{2}{\sqrt{\pi}}\left[y_1(t)+\Gamma_\xi(t)\right] + 2\sqrt{\frac{3}{\pi}}y_2(t)+z\left(\frac{\pi}{4},t\right)\right) - f\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right)\\
&\quad - f'\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right)\left[\frac{2}{\sqrt{\pi}}y_1(t) + 2\sqrt{\frac{3}{\pi}}y_2(t)+z\left(\frac{\pi}{4},t\right)\right],\\
\hat{H}_2(t,\xi) &= f\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right)\\
&\quad + \left[f'\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right) - f'(0)\right]\left[\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}y_2(t)+z(0,t)\right],\\
\hat{H}_3(t,\xi) &= f\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right) \qquad (5.1.74)\\
&\quad + \left[f'\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right) - f'(0)\right]\left[\frac{2}{\sqrt{\pi}}y_1(t) + 2\sqrt{\frac{3}{\pi}}y_2(t)+z\left(\frac{\pi}{4},t\right)\right],\\
\tilde{H}_{21}(t) &= f'(0)\left[\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}y_2(t)\right], \qquad \tilde{H}_{22}(t) = f'(0)z(0,t),\\
\tilde{H}_{31}(t) &= f'(0)\left[\frac{2}{\sqrt{\pi}}y_1(t) + 2\sqrt{\frac{3}{\pi}}y_2(t)\right], \qquad \tilde{H}_{32}(t) = f'(0)z\left(\frac{\pi}{4},t\right),\\
\hat{H}_{20}(t,\xi) &= \hat{H}_{30}(t,\xi) = f\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right) - f(0),\\
\hat{H}_{21}(t,\xi) &= \left[f'\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right) - f'(0)\right]\left[\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}y_2(t)+z(0,t)\right],\\
\hat{H}_{31}(t,\xi) &= \left[f'\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right) - f'(0)\right]\left[\frac{2}{\sqrt{\pi}}y_1(t) + 2\sqrt{\frac{3}{\pi}}y_2(t)+z(\pi/4,t)\right].
\end{aligned}
$$

Let us continue to denote with N an upper bound for $f'(x)$ and $f''(x)$ in a neighborhood of $\gamma(t)$. We have the following result.

Lemma 5.1.6. *There exist positive constant* k_3 *and a function* $\tilde{\Delta}(\rho) > 0$ *with* $\lim_{\rho\to 0}\tilde{\Delta}(\rho) = 0$*, so that if* $\|y_1\| + \|y_2\| + \|z\|_\infty \le \rho \le 1$*,* $E \in \mathscr{E}$ *and* $\alpha', \alpha'' \in \ell^\infty_E$ *the following hold*

$$|H_k(t,\xi') - H_k(t,\xi'')| \le \rho\tilde{\Delta}(\rho)\left[\varepsilon + \mathrm{e}^{-a|t-2jm|}\right]\|\alpha' - \alpha''\|, \quad k = 2,3,$$
$$|\hat{H}_{k1}(t,\xi') - \hat{H}_{k1}(t,\xi'')| \le k_3\rho\left[\varepsilon + \mathrm{e}^{-a|t-2jm|}\right]\|\alpha' - \alpha''\|, \quad k = 2,3$$

where $\xi' = (E,\alpha')$ *and* $\xi'' = (E,\alpha'')$ *and* $t \in ((2j-1)m,(2j+1)m]$*. Furthermore,* $\hat{H}_{20}(t,\xi') - \hat{H}_{20}(t,\xi'') = \hat{H}_{30}(t,\xi') - \hat{H}_{30}(t,\xi'')$ *can be written as the sum of two piecewise* C^1*-functions* $H_{01}(t) + H_{02}(t)$*, so that*

$$|H_{01}(t)| \le k_3\varepsilon\|\alpha' - \alpha''\|,$$
$$|H_{02}(t)| \le k_3\,\mathrm{e}^{-a|t-2jm|}\,\|\alpha' - \alpha''\|,$$
$$|\dot{H}_{02}(t)| \le k_3\,\mathrm{e}^{-a|t-2jm|}\,\|\alpha' - \alpha''\|$$

where $\xi' = (E,\alpha')$*,* $\xi'' = (E,\alpha'')$ *and* $t \in ((2j-1)m,(2j+1)m]$*.*

Proof. Let $e_j = 1$. Then, for any $t \in ((2j-1)m,(2j+1)m]$, we have

$$\begin{aligned}|\Gamma_{\xi'}(t) - \Gamma_{\xi''}(t)| &\le \left[|\dot{\gamma}_1(t-2jm-\theta\alpha'_j-(1-\theta)\alpha''_j)| + A_2\Lambda\varepsilon\right]\|\alpha'-\alpha''\|\\ &\le \left[A_2\Lambda\varepsilon + \bar{A}_2\,\mathrm{e}^{-a|t-2jm|}\right]\|\alpha'-\alpha''\|.\end{aligned}$$

Obviously a similar conclusion holds when $e_j = 0$ since in this case we have $\Gamma_\xi(t) = v_\xi(t)$ for any $t \in ((2j-1)m,(2j+1)m]$. Next, for any $x \in \mathbb{R}$ we have $|x + \frac{2}{\sqrt{\pi}}\Gamma_\xi(t)| \le |x| + \frac{2}{\sqrt{\pi}}\|v_\xi\|_\infty + \|\gamma\|_\infty \le |x| + \frac{2}{\sqrt{\pi}}\Lambda A_2\varepsilon + A_1$. Thus, for any (y_1,y_2,z) $|y_1| + |y_2| + |z| \le \rho$ and $\xi \in X$, the functions $f^{(k)}(y_1 + \Gamma_\xi(t) + y_2 + z)$, $k = 0,1,2$ are bounded. Since

$$\begin{aligned}H_2(t,\xi') - H_2(t,\xi'') = &\int_{\frac{2}{\sqrt{\pi}}\Gamma_{\xi''}(t)}^{\frac{2}{\sqrt{\pi}}\Gamma_{\xi'}(t)} f'\left(\frac{2}{\sqrt{\pi}}y_1(t) + \theta - 2\sqrt{\frac{3}{\pi}}y_2(t) + z(0,t)\right)\\ &-f'(\theta) - f''(\theta)\left[\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}y_2(t) + z(0,t)\right]d\theta\\ =&\int_{\frac{2}{\sqrt{\pi}}\Gamma_{\xi''}(t)}^{\frac{2}{\sqrt{\pi}}\Gamma_{\xi'}(t)}\int_0^1 f''\left(\theta + \sigma\left[\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}y_2(t) + z(0,t)\right]\right)\\ &-f''(\theta)d\sigma d\theta\left[\frac{2}{\sqrt{\pi}}y_1(t) - 2\sqrt{\frac{3}{\pi}}y_2(t) + z(0,t)\right],\end{aligned}$$

we obtain:

$$\begin{aligned}|H_2(t,\xi')-H_2(t,\xi'')| &\leq \frac{2}{\sqrt{\pi}}\rho\Delta_0(\rho)|\Gamma_{\xi'}(t)-\Gamma_{\xi''}(t)| \\ &\leq \rho\widetilde{\Delta}(\rho)\left[\varepsilon+\mathrm{e}^{-a|t-2jm|}\right]\|\alpha'-\alpha''\|\end{aligned}$$

where $\Delta_0(\rho) := \sup_{\{|y|\leq\rho\ ;|x|\leq A_1\}}|f''(x+y)-f''(x)| \to 0$ as $\rho\to 0$ and $\widetilde{\Delta}(\rho) = \frac{2}{\sqrt{\pi}}A_2\Lambda\Delta_0(\rho)$. Similarly,

$$|H_3(t,\xi')-H_3(t,\xi'')| \leq \rho\widetilde{\Delta}(\rho)\left[\varepsilon+\mathrm{e}^{-a|t-2jm|}\right]\|\alpha'-\alpha''\|$$

whereas for $k=2,3$ we get:

$$\begin{aligned}|\hat{H}_{k1}(t,\xi')-\hat{H}_{k1}(t,\xi'')| &\leq \frac{2N}{\sqrt{\pi}}\rho|\Gamma_{\xi'}(t)-\Gamma_{\xi''}(t)| \\ &\leq \frac{2A_2\Lambda N}{\sqrt{\pi}}\rho\left[\varepsilon+\mathrm{e}^{-a|t-2jm|}\right]\|\alpha'-\alpha''\|.\end{aligned}$$

The first part of the Lemma then follows. For the second we write:

$$\hat{H}_{20}(t,\xi')-\hat{H}_{20}(t,\xi'') = H_{01}(t)+H_{02}(t)$$

where

$$H_{01}(t) = f\left(\frac{2}{\sqrt{\pi}}\Gamma_{\xi'}(t)\right)-f\left(\frac{2}{\sqrt{\pi}}\gamma_{\xi'}(t)\right)-f\left(\frac{2}{\sqrt{\pi}}\Gamma_{\xi''}(t)\right)+f\left(\frac{2}{\sqrt{\pi}}\gamma_{\xi''}(t)\right),$$

$$H_{02}(t) = f\left(\frac{2}{\sqrt{\pi}}\gamma_{\xi'}(t)\right)-f\left(\frac{2}{\sqrt{\pi}}\gamma_{\xi''}(t)\right).$$

Then, using (5.1.22) and (5.1.70), we have

$$\begin{aligned}&|H_{01}(t)| \\ &\leq \left|f\left(\frac{2}{\sqrt{\pi}}\Gamma_{\xi'}(t)\right)-f\left(\frac{2}{\sqrt{\pi}}[\gamma_{\xi'}(t)+v_{\xi''}(t)]\right)\right| \\ &\quad+\left|f\left(\frac{2}{\sqrt{\pi}}\left[\gamma_{\xi'}(t)+v_{\xi''}(t)\right]\right)-f\left(\frac{2}{\sqrt{\pi}}\gamma_{\xi'}(t)\right)-f\left(\frac{2}{\sqrt{\pi}}\left[\gamma_{\xi''}(t)+v_{\xi''}(t)\right]\right)\right. \\ &\quad\left.+f\left(\frac{2}{\sqrt{\pi}}\gamma_{\xi''}(t)\right)\right| \leq \frac{2}{\sqrt{\pi}}NA_2\Lambda\varepsilon\|\alpha'-\alpha''\| \\ &\quad+\int_0^{\frac{2}{\sqrt{\pi}}v_{\xi''}(t)}\left|f'\left(\frac{2}{\sqrt{\pi}}\gamma_{\xi'}(t)+\theta\right)-f'\left(\frac{2}{\sqrt{\pi}}\gamma_{\xi''}(t)+\theta\right)\right|d\theta \\ &\leq \frac{2}{\sqrt{\pi}}NA_2\Lambda\varepsilon\left(1+\frac{2\sqrt{2}}{\sqrt{\pi}}\|\dot{\gamma}\|_\infty\right)\|\alpha'-\alpha''\| \leq k_3\varepsilon\|\alpha'-\alpha''\|.\end{aligned}$$

Finally, for any $t\in((2j-1)m,(2j+1)m]$, $j\in\mathbb{Z}$, with $e_j=1$, we have

$$H_{02}(t) = f\left(\frac{2}{\sqrt{\pi}}\gamma_1(t-2jm-\alpha_j')\right) - f\left(\frac{2}{\sqrt{\pi}}\gamma_1(t-2jm-\alpha_j'')\right)$$
$$= \frac{2}{\sqrt{\pi}}\int_{\alpha_j'}^{\alpha_j''} f'\left(\frac{2}{\sqrt{\pi}}\gamma_1(t-2jm-\theta)\right)\dot{\gamma}_1(t-2jm-\theta)d\theta.$$

Thus

$$|H_{02}(t)| \le \frac{2\bar{A}_2 N}{\sqrt{\pi}} \mathrm{e}^{-a|t-2jm|}|\alpha_j' - \alpha_j''| \le k_3 \mathrm{e}^{-a|t-2jm|}\|\alpha' - \alpha''\|$$

and similarly, differentiating with respect to t, we have

$$|\dot{H}_{02}(t)| \le \frac{2\bar{A}_2 N}{\sqrt{\pi}}\left(1+\frac{2}{\sqrt{\pi}}\bar{A}_2\right)\mathrm{e}^{-a|t-2jm|}|\alpha_j' - \alpha_j''| \le k_3 \mathrm{e}^{-a|t-2jm|}\|\alpha' - \alpha''\|.$$

The proof is complete. □

Now, consider the unique solution, whose existence is stated in Theorem 5.1.1, of Eq. (5.1.73) with $\hat{z}(x,t)$ instead of $z(x,t)$ and $\frac{2}{\sqrt{\pi}}[y_1(t)+\Gamma_\xi(t)]+y_2(t)w_0(x)+z(x,t)$ instead of $u(x,t)$:

$$\hat{z}(x,t) = F_1(z,y_1,y_2,\xi,\mu,\varepsilon) + L_{1\varepsilon}(y_1,y_2) + L_{2\varepsilon}(z)$$

where

$$F_1(z,y_1,y_2,\xi,\mu,\varepsilon) := L_\varepsilon(H_1,H_2,H_3) + L_\varepsilon(0,\hat{H}_2,\hat{H}_3),$$
$$L_{1\varepsilon}(y_1,y_2) := L_\varepsilon(0,\tilde{H}_{21},\tilde{H}_{31}),$$
$$L_{2\varepsilon}(z) := L_\varepsilon(0,\tilde{H}_{22},\tilde{H}_{32}).$$

We are thinking of $F_1(z,y_1,y_2,\xi,\mu,\varepsilon)$ as a map from

$$C_b^0([0,\pi/4]\times\mathbb{R})\times Y_1\times Y_2\times X\times\mathbb{R}\times\mathbb{R}_+ \to C_b^0([0,\pi/4]\times\mathbb{R}).$$

We will need the following result.

Lemma 5.1.7. *For any fixed, small,* $\varepsilon > 0$, $L_{1\varepsilon} : Y_1 \times Y_2 \to C_b^0([0,\pi/4]\times\mathbb{R})$ *and* $L_{2\varepsilon} : C_b^0([0,\pi/4]\times\mathbb{R}) \to C_b^0([0,\pi/4]\times\mathbb{R})$ *are bounded linear maps whose norms satisfy:*

$$\|L_{1\varepsilon}\| \le 2M_1M_2|f'(0)|\delta^{-1}, \quad \|L_{2\varepsilon}\| \le 2M_1M_2|f'(0)|\delta^{-1}. \tag{5.1.75}$$

Moreover a function $\Delta(\rho) > 0$ *exists so that* $\lim_{\rho\to 0}\Delta(\rho) = 0$ *and for* $\|y_1\|+\|y_2\|+\|z\| \le \rho$, $\|\tilde{y}_1\|+\|\tilde{y}_2\|+\|\tilde{z}\| \le \rho$ *the following hold:*

(i)

$$\| F_1 (z,y_1,y_2,\mu,\xi,\varepsilon)\|_\infty$$
$$\le \frac{\pi}{2}M_1M_2\sqrt{\varepsilon}|\mu|(\sqrt{\varepsilon}\|h\|_\infty + \delta^{-1}\|h_t\|_\infty)$$

$$+\frac{4}{\sqrt{\pi}}A_2M_1M_2N\Lambda\left[5\left(\frac{1}{\delta^5}+1+\frac{1}{a}\right)\left(\frac{1}{\Lambda}+\varepsilon\right)+2\delta^{-1}\sqrt{\varepsilon}\right]\varepsilon$$
$$+2M_1M_2\Big[\delta^{-1}\Delta(\rho)+\frac{2}{\sqrt{\pi}}A_2N\sqrt{\varepsilon}\Big(\frac{1}{\delta^3}+\frac{2}{a}+\frac{\Lambda}{\delta}\sqrt{\varepsilon}\Big)\Big]\cdot$$
$$(\|y_1\|+\|y_2\|+\|z\|_\infty)\,. \tag{5.1.76}$$

(ii) *for any* $\xi'=(E,\alpha'),\xi''=(E,\alpha'')\in X$, μ', μ'', *we have*

$$\|F_1(z,y_1,y_2,\mu',\xi',\varepsilon)-F_1(\tilde z,\tilde y_1,\tilde y_2,\mu'',\xi'',\varepsilon)\|_\infty$$
$$\le\frac{\pi}{4}M_1M_2\sqrt{\varepsilon}\left[5\sqrt{\varepsilon}\left(\frac{1}{\delta^5}+1+\frac{1}{a}\right)+2\delta^{-1}\right](\|h\|_\infty+\|h_t\|_\infty)|\mu'-\mu''|$$
$$+2M_1M_2\Big[\delta^{-1}\Delta(\rho)+\frac{2}{\sqrt{\pi}}A_2N\sqrt{\varepsilon}\Big(\frac{1}{\delta^3}+\frac{2}{a}+\frac{\Lambda}{\delta}\sqrt{\varepsilon}\Big)\Big]\cdot$$
$$(\|y_1-\tilde y_1\|+\|y_2-\tilde y_2\|+\|z-\tilde z\|_\infty)$$
$$+4k_3M_1M_2\left(\frac{\sqrt{\varepsilon}}{\delta}+\frac{1}{\delta^3}+\frac{2}{a}\right)\rho\sqrt{\varepsilon}\|\alpha'-\alpha''\|$$
$$+10k_3M_1M_2\varepsilon\left(\frac{1}{5\delta}+\frac{1}{\delta^5}+1+\frac{1}{a}\right)\|\alpha'-\alpha''\|\,, \tag{5.1.77}$$

with k_3 *being the positive constant of Lemma 5.1.6.*

Proof. By following the above estimates, it is easy to derive (5.1.75) along with the estimate

$$\|L_\varepsilon(H_1,H_2,H_3)\|_\infty\le\frac{M_1M_2\pi}{2}\sqrt{\varepsilon}|\mu|(\sqrt{\varepsilon}\|h\|_\infty+\delta^{-1}\|h_t\|_\infty)$$
$$+2M_1M_2\delta^{-1}\Delta(\rho)(\|y_1\|+\|y_2\|+\|z\|_\infty) \tag{5.1.78}$$

where

$$\Delta(\rho)=\sup_{\substack{|y_1|+|y_2|+|z|\le\rho\\ -\infty<t<\infty}}\left|f'\left(y_1+\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)+y_2+z\right)-f'\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right)\right|\to 0$$

as $\rho\to 0$ (cf [11, Lemma 2, Eq. (3.17), (3.20)] for more details). Since $f(0)=0$ we have $\hat H_2(t,\xi)=\hat H_{20}(t,\xi)+\hat H_{21}(t,\xi)$ and $\hat H_3(t,\xi)=\hat H_{30}(t,\xi)+\hat H_{31}(t,\xi)$, $\hat H_{ij}(t)$ defined in (5.1.75). Now, $\hat H_{20}(t,\xi)\in C_b^1(\mathbb{R})$ and the following inequalities hold (see also (5.1.53)):

$$|\hat H_{20}(t,\xi)|\le\frac{2N}{\sqrt{\pi}}|\Gamma_\xi(t)|\le\frac{2}{\sqrt{\pi}}A_2N\left[\Lambda\varepsilon+\mathrm{e}^{-a|t-2jm-\alpha_j|}\right],$$
$$|\dot{\hat H}_{20}(t,\xi)|\le\frac{2N}{\sqrt{\pi}}|\dot\Gamma_\xi(t)|\le\frac{2}{\sqrt{\pi}}A_2N\left[\Lambda\varepsilon+\mathrm{e}^{-a|t-2jm-\alpha_j|}\right]$$

for $(2j-1)m < t \le (2j+1)m,\ j \in \mathbb{Z}$. Hence, from Theorem 5.1.1–(b), (5.1.19), (5.1.20) we get

$$\|L_\varepsilon(0,\hat{H}_{20},\hat{H}_{30})\|_\infty \le \frac{4}{\sqrt{\pi}} A_2 M_1 M_2 N \Lambda \left[5\left(\frac{1}{\delta^5}+1+\frac{1}{a}\right)\left(\frac{1}{\Lambda}+\varepsilon\right)+\frac{2}{\delta}\sqrt{\varepsilon}\right]\varepsilon \tag{5.1.79}$$

Next,

$$\begin{aligned}|\hat{H}_{21}(t,\xi)| &\le \frac{2N}{\sqrt{\pi}}|\Gamma_\xi(t)|[\|y_1\|+\|y_2\|+\|z\|_\infty]\\ &\le \frac{2}{\sqrt{\pi}}A_2N[\Lambda\varepsilon+\mathrm{e}^{-a|t-2jm-\alpha_j|}][\|y_1\|+\|y_2\|+\|z\|_\infty],\\ |\hat{H}_{31}(t,\xi)| &\le \frac{2N}{\sqrt{\pi}}|\Gamma_\xi(t)|[\|y_1\|+\|y_2\|+\|z\|_\infty]\\ &\le \frac{2}{\sqrt{\pi}}A_2N[\Lambda\varepsilon+\mathrm{e}^{-a|t-2jm-\alpha_j|}][\|y_1\|+\|y_2\|+\|z\|_\infty].\end{aligned}$$

Thus, from Theorem 5.1.1(a) and (5.1.19) we obtain:

$$\|L_\varepsilon(0,\hat{H}_{21},\hat{H}_{31})\|_\infty \le \frac{4}{\sqrt{\pi}} M_1M_2A_2N\sqrt{\varepsilon}\left(\frac{1}{\delta^3}+\frac{2}{a}+\Lambda\frac{\sqrt{\varepsilon}}{\delta}\right)[\|y_1\|+\|y_2\|+\|z\|_\infty] \tag{5.1.80}$$

and (5.1.76) follows from (5.1.78), (5.1.79), and (5.1.80). Finally, we prove (5.1.77). Using arguments similar to the above we see that

$$\begin{aligned}&\| F_1(z,y_1,y_2,\mu'',\xi'',\varepsilon)-F_1(\tilde{z},\tilde{y}_1,\tilde{y}_2,\mu'',\xi'',\varepsilon)\|_\infty\\ &\le 2M_1M_2\left[\delta^{-1}\Delta(\rho)+\frac{2}{\sqrt{\pi}}A_2N\sqrt{\varepsilon}\left(\frac{1}{\delta^3}+\frac{2}{a}+\frac{\Lambda}{\delta}\sqrt{\varepsilon}\right)\right]\\ &\quad\cdot[\|y_1-\tilde{y}_1\|+\|y_2-\tilde{y}_2\|+\|z-\tilde{z}\|_\infty].\end{aligned}$$

Next,

$$\begin{aligned}&F_1\ (z,y_1,y_2,\mu',\xi',\varepsilon)-\ F_1(z,y_1,y_2,\mu'',\xi'',\varepsilon)\\ &\quad= L_\varepsilon((\mu'-\mu'')h,0,0)+\ L_\varepsilon(0,H_2(\cdot,\xi')-H_2(\cdot,\xi''),H_3(\cdot,\xi')-H_3(\cdot,\xi''))\\ &\qquad+\ L_\varepsilon(0,\hat{H}_{20}(\cdot,\xi')-\hat{H}_{20}(\cdot,\xi''),\hat{H}_{30}(\cdot,\xi')-\hat{H}_{30}(\cdot,\xi''))\\ &\qquad+\ L_\varepsilon(0,\hat{H}_{21}(\cdot,\xi')-\hat{H}_{21}(\cdot,\xi''),\hat{H}_{31}(\cdot,\xi')-\hat{H}_{31}(\cdot,\xi''))\end{aligned}$$

and hence, from Lemma 5.1.6, Theorem 5.1.1, (5.1.19) and (5.1.20) we obtain:

$$\begin{aligned}&\| F_1\ (z,y_1,y_2,\mu',\xi',\varepsilon)-F_1(z,y_1,y_2,\mu'',\xi'',\varepsilon)\|_\infty\\ &\le\ 4k_3M_1M_2\left(\frac{\sqrt{\varepsilon}}{\delta}+\frac{1}{\delta^3}+\frac{2}{a}\right)\rho\sqrt{\varepsilon}\|\alpha'-\alpha''\|\end{aligned}$$

$$+\frac{\pi}{4}M_1M_2\sqrt{\varepsilon}\left[5\sqrt{\varepsilon}\left(\frac{1}{\delta^5}+1\right)+\frac{2}{\delta}\right](\|h\|_\infty+\|h_t\|_\infty)|\mu'-\mu''|$$
$$+10k_3M_1M_2\varepsilon\left(\frac{1}{5\delta}+\frac{1}{\delta^5}+1+\frac{1}{a}\right)\|\alpha'-\alpha''\|.$$

(5.1.77) then follows from the above two estimates. The proof is complete. □

Now, for given $(y_1(t),y_2(t),z(x,t))\in Y_1\times Y_2\times C_b^0([0,\frac{\pi}{4}]\times\mathbb{R})$, we denote with $(\hat{y}_1(t),\hat{y}_2(t))$ the unique solution of

$$\begin{aligned}\ddot{\hat{y}}_1(t)+\frac{8}{\pi}f'\left(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\right)\hat{y}_1(t)&=g_1(t),\\ \ddot{\hat{y}}_2(t)+\frac{24}{\pi}f'\left(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\right)\hat{y}_2(t)&=g_2(t)\end{aligned}\tag{5.1.81}$$

where $g_1(t)$, $g_2(t)$ are the right-hand sides of Eqs. (5.1.71), (5.1.72), that satisfy $\dot{\hat{y}}_1(2jm+\alpha_j)=0$ for any $j\in\mathbb{Z}$ so that $e_j=1$. These solutions exist because of Lemmas 5.1.2 and 5.1.3, moreover

$$\|\hat{y}_1\|\le B\|g_1\|,\quad \|\hat{y}_2\|\le B_1\|g_2\|\tag{5.1.82}$$

where B and B_1 have been defined in Lemma 5.1.2 and Lemma 5.1.3. Note that in the above formulas the norm on the left is the norm in Y_1 (resp. Y_2), while $\|g_1\|=\frac{2}{\sqrt{\pi}}\sup_{t\in\mathbb{R}}|g_1(t)|$ and $\|g_2\|=2\sqrt{\frac{3}{\pi}}\sup_{t\in\mathbb{R}}|g_2(t)|$. Let

$$g_{11}(t)=g_1(t)+Q_{m,\xi}\left\{\frac{2}{\sqrt{\pi}}f'\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right)[z(0,t)+z(\pi/4,t)]\right\},$$
$$g_{21}(t)=g_2(t)+2\sqrt{\frac{3}{\pi}}f'\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right)[z(\pi/4,t)-z(0,t)].$$

Then $(\hat{y}_1(t),\hat{y}_2(t))$ can be written as

$$\hat{y}_1(t)=\hat{y}_{11}(t)+\hat{y}_{10}(t),\quad \hat{y}_2(t)=\hat{y}_{21}(t)+\hat{y}_{20}(t)$$

where $(\hat{y}_{11}(t),\hat{y}_{21}(t))\in Y_1\times Y_2$ is the unique bounded solution of

$$\begin{aligned}\ddot{\hat{y}}_{11}(t)+\frac{8}{\pi}f'\left(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\right)\hat{y}_{11}(t)&=g_{11}(t),\\ \ddot{\hat{y}}_{21}(t)+\frac{24}{\pi}f'\left(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\right)\hat{y}_{21}(t)&=g_{21}(t)\end{aligned}\tag{5.1.83}$$

that satisfies $\dot{\hat{y}}_{11}(2jm+\alpha_j)=0$ for any $j\in\mathbb{Z}$ so that $e_j=1$, and $(\hat{y}_{10}(t),\hat{y}_{20}(t))\in Y_1\times Y_2$ is the unique bounded solution of

$$\ddot{\hat{y}}_{10}(t) + \frac{8}{\pi} f'\left(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\right)\hat{y}_{10}(t) = -Q_{m,\xi}\left[\frac{2}{\sqrt{\pi}} f'\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right)[z(0,t)+z(\pi/4,t)]\right],$$

$$\ddot{\hat{y}}_{20}(t) + \frac{24}{\pi} f'\left(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\right)\hat{y}_{20}(t) = -2\sqrt{\frac{3}{\pi}} f'\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right)[z(\pi/4,t)-z(0,t)] \tag{5.1.84}$$

that satisfies $\dot{\hat{y}}_{10}(2jm+\alpha_j) = 0$ for any $j \in \mathbb{Z}$ so that $e_j = 1$. We set

$$F_2(z,y_1,y_2,\xi,\mu,\varepsilon) = (\hat{y}_{11},\hat{y}_{21}) \in Y_1 \times Y_2, \quad Lz = (\hat{y}_{10},\hat{y}_{20}).$$

Then we have the following result:

Lemma 5.1.8. $L : C_b^0([0,\pi/4]\times\mathbb{R}) \to Y_1 \times Y_2$ *is a bounded linear map. Moreover, positive constants* k_6 *and* k_7 *and a function* $\overline{\Delta}(\rho,\varepsilon) > 0$ *exist so that* $\lim_{(\rho,\varepsilon)\to(0,0)} \overline{\Delta}(\rho,\varepsilon) = 0$ *and for* $\|y_1\|+\|y_2\|+\|z\| \le \rho$, $\|\widetilde{y}_1\|+\|\widetilde{y}_2\|+\|\widetilde{z}\| \le \rho$ *the following hold:*

(i)

$$\|F_2(z,y_1,y_2,\xi,\mu,\varepsilon)\| \le \bar{\Delta}(\rho,\varepsilon)[\|y_1\|+\|y_2\|+\|z\|_\infty] + k_6|\mu| + k_7\varepsilon. \tag{5.1.85}$$

(ii) *For any* $\xi = (E,\alpha)$, $\widetilde{\xi} = (E,\widetilde{\alpha}) \in X$, μ, $\widetilde{\mu}$, *we have*

$$\begin{aligned} \| F_2 &(z,y_1,y_2,\xi,\mu,\varepsilon) - F_2(\widetilde{z},\widetilde{y}_1,\widetilde{y}_2,\widetilde{\xi},\widetilde{\mu},\varepsilon)\| \\ &\le \bar{\Delta}(\rho,\varepsilon)[\|y_1-\widetilde{y}_1\|+\|y_2-\widetilde{y}_2\|+\|z-\widetilde{z}\|_\infty] \\ &\quad + [\rho\bar{\Delta}(\rho,\varepsilon)+k_6|\mu|+k_7\varepsilon]\|\alpha-\widetilde{\alpha}\|_\infty + k_6|\mu-\widetilde{\mu}|. \end{aligned} \tag{5.1.86}$$

Proof. First we note that from Remark 5.1.5 (a) the existence follows of a constant $A_4 > 0$ so that $\|Q_{m,\xi}\| \le A_4$ and $\|Q_{m,\xi'} - Q_{m,\xi''}\| \le A_4\|\alpha'-\alpha''\|$ for any m sufficiently large and any $\xi,\xi',\xi'' \in X$ with $\xi' = (E,\alpha')$, $\xi'' = (E,\alpha'')$. Then, L is obviously linear and from (5.1.82) it easily follows that

$$\|\hat{y}_{10}\| + \|\hat{y}_{20}\| \le \frac{8N(A_4B+3B_1)}{\pi}\|z\|_\infty,$$

that is, L is bounded and

$$\|L\| \le \frac{8N(A_4B+3B_1)}{\pi}.$$

Next, it is easy to see that

$$\begin{aligned} \|g_{11}\| \le A_4\Big\{&\sqrt{\varepsilon}\delta\|y_1\| + |\mu|\|h\|_\infty + \frac{2\Lambda A_2}{\sqrt{\pi}}(1+\delta)\varepsilon^{3/2} + \frac{16\Lambda A_2N}{\pi\sqrt{\pi}}\varepsilon \\ &+ \frac{16\Lambda A_2N}{\pi\sqrt{\pi}}\varepsilon\|y_1\| + \frac{8}{\pi}\Delta(\rho)[\|y_1\|+\|y_2\|+\|z\|_\infty]\Big\} \\ \le \frac{1}{2B}&\{\bar{\Delta}(\rho,\varepsilon)[\|y_1\|+\|y_2\|+\|z\|_\infty] + k_6|\mu| + 2k_7\varepsilon\} \end{aligned}$$

where $\bar{\Delta}(\rho,\varepsilon)\to 0$ as $\rho+|\varepsilon|\to 0$ and k_6, k_7 are suitably chosen. Similarly

$$\|g_{21}\| \le \frac{1}{2B_1}\{\bar{\Delta}(\rho,\varepsilon)[\|y_1\|+\|y_2\|+\|z\|_\infty]+k_6|\mu|\}.$$

Thus (5.1.85) follows from (5.1.81).

To prove (5.1.86), let $(z(x,t),y_1(t),y_2(t),\xi,\mu)$, $(\widetilde{z}(x,t),\widetilde{y}_1(t),\widetilde{y}_2(t),\widetilde{\xi},\widetilde{\mu})$ be in the statement of the theorem and write $g_{11}(t,z(0,t),z(\frac{\pi}{4},t),y_1(t),y_2(t),\xi,\mu,\varepsilon)$ for $g_{11}(t)$ and $\widetilde{g}_{11}(t)$ for $g_{11}(t,\widetilde{z}(0,t),\widetilde{z}(\pi/4,t),\widetilde{y}_1(t),\widetilde{y}_2(t),\widetilde{\xi},\widetilde{\mu},\varepsilon)$. From Lemma 5.1.2-(iii) and Lemma 5.1.3-(ii) we know that

$$\begin{aligned}&\| F_2(z,y_1,y_2,\xi,\mu,\varepsilon)-F_2(\widetilde{z},\widetilde{y}_1,\widetilde{y}_2,\widetilde{\xi},\widetilde{\mu},\varepsilon)\|\\ &\le B\|g_{11}-\widetilde{g}_{11}\|+B_1\|g_{21}-\widetilde{g}_{21}\|+[c_1\|g_{11}\|+\hat{c}_1\|g_{21}\|]\,\|\alpha-\widetilde{\alpha}\|\end{aligned}$$

where

$$\widetilde{g}_{21}(t)=g_{21}(t,\widetilde{z}(0,t),\widetilde{z}(\pi/4,t),\widetilde{y}_1(t),\widetilde{y}_2(t),\widetilde{\xi},\widetilde{\mu},\varepsilon).$$

Now we have

$$g_{11}(t)-\widetilde{g}_{11}(t)=G_{11}(t)+\widetilde{G}_{11}(t)$$

where

$$\begin{aligned}G_{11}(t)&=g_{11}(t,z(0,t),z(\pi/4,t),y_1(t),y_2(t),\xi,\mu,\varepsilon)\\ &\quad -g_{11}(t,z(0,t),z(\pi/4,t),y_1(t),y_2(t),\widetilde{\xi},\widetilde{\mu},\varepsilon),\\ \widetilde{G}_{11}(t)&=g_{11}(t,z(0,t),z(\pi/4,t),y_1(t),y_2(t),\widetilde{\xi},\widetilde{\mu},\varepsilon)\\ &\quad -g_{11}(t,\widetilde{z}(0,t),\widetilde{z}(\pi/4,t),\widetilde{y}_1(t),\widetilde{y}_2(t),\widetilde{\xi},\widetilde{\mu},\varepsilon).\end{aligned}$$

An argument similar to the above shows that

$$\|\widetilde{G}_{11}\|\le \frac{1}{2B}\bar{\Delta}(\rho,\varepsilon)[\|y_1-\widetilde{y}_1\|+\|y_2-\widetilde{y}_2\|+\|z-\widetilde{z}\|_\infty].$$

On the other hand, since

$$\begin{aligned}g_{11}(t)=-Q_{m,\xi}\Big\{&\sqrt{\varepsilon}\delta\dot{y}_1(t)+\frac{2}{\sqrt{\pi}}\mu\int_0^{\pi/4}h(x,t)\,dx+\frac{2}{\sqrt{\pi}}[H_2(t,\xi)+H_3(t,\xi)]\\ &-\frac{8}{\pi}\Big[f'\Big(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\Big)-f'\Big(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\Big)\Big]y_1(t)+\sqrt{\varepsilon}\delta\dot{v}_\xi(t)+\ddot{v}_\xi(t)\\ &+\frac{4}{\sqrt{\pi}}\Big[f\Big(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\Big)-f\Big(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\Big)\Big]\Big\}\end{aligned}$$

we have, using also the estimate for $H_{01}(t)$ given in the proof of Lemma 5.1.6,

$$\|G_{11}\|\le \frac{1}{2A_4B}\|Q_{m,\xi}-Q_{m,\widetilde{\xi}}\|\Big\{\bar{\Delta}(\rho,\varepsilon)[\|y_1\|+\|y_2\|+\|z\|_\infty]+k_6|\mu|+2k_7\varepsilon\Big\}$$

$$+\frac{1}{2B}\Big\{k_6|\mu-\widetilde{\mu}|+k_8[\rho\widetilde{\Delta}(\rho)(1+\varepsilon)+\varepsilon(1+\rho)]\|\alpha-\widetilde{\alpha}\|\Big\}$$

$$\leq\frac{1}{2B}\Big[\rho\bar{\Delta}(\rho,\varepsilon)+k_6|\mu|+2k_7\varepsilon+k_8\Big(\rho\widetilde{\Delta}(\rho)(1+\varepsilon)+\varepsilon(1+\rho)\Big)\Big]\|\alpha-\widetilde{\alpha}\|$$

$$+\frac{1}{2B}k_6|\mu-\widetilde{\mu}|$$

and then

$$\|g_{11}-\widetilde{g}_{11}\|\leq\frac{1}{2B}\Big\{\bar{\Delta}(\rho,\varepsilon)[\|y_1-\widetilde{y}_1\|+\|y_2-\widetilde{y}_2\|+\|z-\widetilde{z}\|_\infty]+k_6|\mu-\widetilde{\mu}|$$

$$+\Big[\rho\bar{\Delta}(\rho,\varepsilon)+k_6|\mu|+2k_7\varepsilon+k_8\Big(\rho\widetilde{\Delta}(\rho)(1+\varepsilon)+\varepsilon(1+\rho)\Big)\Big]\|\alpha-\widetilde{\alpha}\|\Big\}.$$

Similarly

$$\|g_{21}-\widetilde{g}_{21}\|\leq\frac{1}{2B_1}\Big\{\bar{\Delta}(\rho,\varepsilon)[\|y_1-\widetilde{y}_1\|+\|y_2-\widetilde{y}_2\|+\|z-\widetilde{z}\|_\infty]$$

$$+k_8\Big(\rho\widetilde{\Delta}(\rho)(1+\varepsilon)+\varepsilon(1+\rho)\Big)\|\alpha-\widetilde{\alpha}\|+k_6|\mu-\widetilde{\mu}|\Big\},$$

hence, (5.1.86) follows from (5.1.30), (5.1.59) and (5.1.81) provided $\varepsilon>0$ and $\rho>0$ are sufficiently small. The proof is complete. □

Our goal is to prove that the map $(z(x,t),y_1(t),y_2(t))\mapsto(\hat{z}(x,t),\hat{y}_1(t),\hat{y}_2(t))$ has a unique fixed point which is then a solution of Eqs. (5.1.71)–(5.1.73). To this end, we will make use of the following result, whose proof is omitted since it is a slight modification of Lemma 3 in [11].

Lemma 5.1.9. *Let Z, Y be Banach spaces, $B_{Z\times Y}(\rho)$ be the closed ball centered at zero and of radius ρ, S be a set of parameters, $M\subset S\times(0,\bar{\sigma}]$, and $F:B_{Z\times Y}(\rho)\times M\times[-\bar{\mu},\bar{\mu}]\times(0,\bar{\sigma}]\to Z\times Y$ be a map defined as:*

$$F(z,y,\kappa,\mu,\sigma)=\begin{pmatrix}F_1(z,y,\kappa,\mu,\sigma)+L_{1\sigma}y+L_{2\sigma}z\\ F_2(z,y,\kappa,\mu,\sigma)+Lz\end{pmatrix},$$

with $L_{1\sigma}:Y\to Z$, $L_{2\sigma}:Z\to Z$ and $L:Z\to Y$ being uniformly bounded linear maps for $\sigma>0$ small. Assume that a constant C and a function $\Delta(\rho,\mu,\sigma)$ exist so that $\lim_{(\rho,\mu,\sigma)\to(0,0,0)}\Delta(\rho,\mu,\sigma)=0$, and

$$\begin{aligned}
&\|F_1(z,y,\kappa,\mu,\sigma)\|\leq C(|\mu|+\sigma)\sigma+\Delta(\rho,\mu,\sigma)(\|z\|+\|y\|),\\
&\|F_2(z,y,\kappa,\mu,\sigma)\|\leq C|\mu|+\Delta(\rho,\mu,\sigma)(\|z\|+\|y\|),\\
&\|L_{1\sigma}F_2(z,y,\kappa,\mu,\sigma)\|\leq C(|\mu|+\sigma)\sigma+\Delta(\rho,\mu,\sigma)(\|z\|+\|y\|)\\
&\|F_i(z_2,y_2,\kappa,\mu,\sigma)-F_i(z_1,y_1,\kappa,\mu,\sigma)\|\leq\Delta(\rho,\mu,\sigma)(\|z_2-z_1\|+\|y_2-y_1\|)
\end{aligned}\tag{5.1.87}$$

when $\|z\| + \|y\| < \rho$, $\|z_1\| + \|y_1\| < \rho$, *and* $\|z_2\| + \|y_2\| < \rho$. *If there are* $0 < \lambda < 1$ *and* $\bar{\sigma}_0 > 0$ *so that*

$$\|L_{1\sigma}L + L_{2\sigma}\| < \lambda$$

for any $0 < \sigma \le \bar{\sigma}_0$, *then there exist* $\mu_0 > 0$, $\sigma_0 > 0$, $\rho_1 > 0$ *and* $\rho_2 > 0$ *so that for* $|\mu| \le \mu_0$, $\kappa \in M$, *and* $0 < \sigma \le \sigma_0$, F *has a unique fixed point* $(z(\mu,\sigma,\kappa), y(\mu,\sigma,\kappa)) \in B_Z(\rho_1) \times B_Y(\rho_2)$. *Moreover,*

$$\|z(\mu,\sigma,\kappa)\| + \|y(\mu,\sigma,\kappa)\| \le C_1(|\mu| + \sigma) \tag{5.1.88}$$

for some positive constant C_1 *independent of* (μ,σ,κ), *and*

$$\|z(\mu,\sigma,\kappa)\|/(|\mu| + \sigma) \to 0$$

uniformly with respect to κ, *as* $(\mu,\sigma) \to (0,0)$, $\sigma > 0$. *Finally,* $(z(\mu,\sigma,\kappa), y(\mu,\sigma,\kappa))$ *is* C^r, $r \ge 0$, *in* (μ,σ) *if* $F(z,y,\kappa,\mu,\sigma)$ *is* C^r *in* (z,y,μ,σ).

We apply Lemma 5.1.9 with $\sigma = \sqrt{\varepsilon} \le \bar{\sigma} = (1/2)^{2/3}$, $S = X \times \mathbb{N}$, $\kappa = (\xi, m, \sigma) \in M := X \times \{(m,\sigma) \in \mathbb{N} \times (0,\bar{\sigma}) : m \ge [\sigma^{-3/2}] + 1\}$ and

$$\begin{aligned}
&F_1(z,y_1,y_2,\xi,\mu,\sigma) = L_\varepsilon(H_1,H_2,H_3) + L_\varepsilon(0,\hat{H}_2,\hat{H}_3),\\
&F_2(z,y_1,y_2,\xi,\mu,\varepsilon) = (\hat{y}_{11},\hat{y}_{21}),\\
&L_{1\sigma}(y_1,y_2) := L_{1\varepsilon}(y_1,y_2) = L_\varepsilon(0,\tilde{H}_{21},\tilde{H}_{31}),\\
&L_{2\sigma}z := L_{2\varepsilon}z = L_\varepsilon(0,\tilde{H}_{22},\tilde{H}_{32}),\\
&Lz = (\hat{y}_{10},\hat{y}_{20})
\end{aligned}$$

where $H_i(t)$, $\hat{H}_i(t)$ and $\tilde{H}_{ij}(t)$ have been defined in (5.1.75). We get the following result.

Theorem 5.1.10. *Assume that the conditions* (H1)–(H2) *hold and that* $\delta > 0$ *is a fixed positive number so that*

(H3) $2M_1M_2|f'(0)| < \delta$.

Let $\Gamma > 0$ *be fixed. Then there exist positive numbers* $\rho_1 > 0$, $\rho_2 > 0$, $\varepsilon_0 > 0$, *and* $\mu_0 > 0$ *so that for any* $\xi \in X$, $0 < \varepsilon < \varepsilon_0$, $|\mu| < \mu_0$, $m > \varepsilon^{-3/4}$ *and* $\varepsilon \le \Gamma|\mu|$, *the integro-differential system* (5.1.71)–(5.1.73) *has a unique bounded solution*

$$(z(x,t,\mu,\varepsilon,\delta,\xi,m), \quad y_1(t,\mu,\varepsilon,\delta,\xi,m), \quad y_2(t,\mu,\varepsilon,\delta,\xi,m))$$

so that

$$\|z(x,t,\mu,\varepsilon,\delta,\xi,m)\|_\infty < \rho_1, \quad \|y_1(t,\mu,\varepsilon,\delta,\xi,m)\| + \|y_2(t,\mu,\varepsilon,\delta,\xi,m)\| < \rho_2.$$

Moreover

$$\|z(\cdot,\cdot,\mu,\varepsilon,\delta,\xi,m)\|_\infty + \|y_1(\cdot,\mu,\varepsilon,\delta,\xi,m)\| + \|y_2(\cdot,\mu,\varepsilon,\delta,\xi,m)\| \le \tilde{C}_1(|\mu| + \sqrt{\varepsilon})$$

for some positive constant $\tilde{C}_1$ *independent of* (μ,ε,ξ)*, and*

$$\|z(\cdot,\cdot,\mu,\varepsilon,\delta,\xi,m)\|_\infty/(|\mu|+\sqrt{\varepsilon})\to 0$$

uniformly with respect to (ξ,m)*, as* $(\mu,\varepsilon)\to(0,0)$*,* $\varepsilon>0$*. Finally,*

$$z(\cdot,\cdot,\mu,\varepsilon,\delta,\xi,m),\quad y_1(\cdot,\mu,\varepsilon,\delta,\xi,m),\quad y_2(\cdot,\mu,\varepsilon,\delta,\xi,m)$$

are Lipschitz in α *uniformly with respect to* (E,m) *and the Lipschitz constants are* $O(\sqrt{\varepsilon}+|\mu|)$ *for* y_1*,* y_2 *and* $o(\sqrt{\varepsilon}+|\mu|)$ *for* z*.*

Proof. We shall prove that the assumptions of Lemma 5.1.9 are satisfied. Of course, we take $Z=C_b^0([0,\pi/4]\times\mathbb{R})$, $Y=Y_1\times Y_2$ as Banach spaces, $S=X\times\mathbb{N}$ and $M=\{(\xi,m,\sigma)\mid \xi\in X, m\in\mathbb{N}, m>\sigma^{-3/2}\}$. The fact that $L_{1\sigma}=L_{1\varepsilon}$ and $L_{2\sigma}=L_{2\varepsilon}$ are bounded linear maps, as well as the fact that $\hat{z}=F_1(z,y_1,y_2,\xi,\mu,\varepsilon)$ satisfies the first and fourth conditions in (5.1.87) follow from Lemma 5.1.7. Similarly the facts that $L:Z\to Y$ is a bounded linear map and $F_2(z,y_1,y_2,\xi,\mu,\varepsilon)$ satisfies the second and fourth inequalities in (5.1.87) follow from Lemma 5.1.8 (see (5.1.85), (5.1.86)) and the assumption $\varepsilon\le\Gamma|\mu|$. Thus, in order to apply Lemma 5.1.9, we only need to prove that

$$\|L_{1\varepsilon}(\hat{y}_{11},\hat{y}_{21})\|_\infty\le C(|\mu|+\sqrt{\varepsilon})\sqrt{\varepsilon}+\Delta(\rho,\mu,\sqrt{\varepsilon})(\|z\|_\infty+\|y_1\|+\|y_2\|)\quad (5.1.89)$$

and that

$$\|(L_{1\varepsilon}L+L_{2\varepsilon})z\|_\infty\le\lambda\|z\|_\infty \quad (5.1.90)$$

for any $\varepsilon>0$ small enough and some $\lambda\in(0,1)$. First we prove (5.1.89). We have

$$L_{1\varepsilon}(\hat{y}_{11},\hat{y}_{21})=$$

$$L_\varepsilon\left(0,f'(0)\left[\frac{2}{\sqrt{\pi}}\hat{y}_{11}(t)-2\sqrt{\frac{3}{\pi}}\hat{y}_{21}(t)\right],f'(0)\left[\frac{2}{\sqrt{\pi}}\hat{y}_{11}(t)+2\sqrt{\frac{3}{\pi}}\hat{y}_{21}(t)\right]\right).$$

Now, from (5.1.85), (5.1.86) and the definition of the norms in Y_1, Y_2, we see that $\hat{y}_{11}(t)$ and $\hat{y}_{21}(t)$ are bounded together with their first derivatives. Thus, using Theorem 5.1.1(b), (5.1.85), (5.1.86), and assumption (H3) we get:

$$\begin{aligned}\|L_{1\varepsilon}(\hat{y}_{11},\hat{y}_{21})\|_\infty &\le 2M_1M_2|f'(0)|\left[5\varepsilon\left(\frac{1}{\delta^5}+1\right)+\frac{2}{\delta}\sqrt{\varepsilon}\right]\cdot[\|\hat{y}_{11}\|+\|\hat{y}_{21}\|]\\ &\le\sqrt{\varepsilon}\left[5\sqrt{\varepsilon}\left(\frac{1}{\delta^4}+\delta\right)+2\right]\cdot[\|\hat{y}_{11}\|+\|\hat{y}_{21}\|]\\ &\le\tilde{c}_1\sqrt{\varepsilon}[\Delta(\rho)+\sqrt{\varepsilon}(\delta+\sqrt{\varepsilon})](\|y_1\|+\|y_2\|+\|z\|_\infty)\\ &\quad+\tilde{c}_2\sqrt{\varepsilon}(|\mu|\|h\|_\infty+\varepsilon(\delta\sqrt{\varepsilon}+2))\end{aligned}$$

for some suitable choice of the positive constants $\tilde{c}_1$ and $\tilde{c}_2$ (possibly dependent on δ). Thus (5.1.89) follows. Now, we look at $L_{1\varepsilon}Lz$. We have

$$L_{1\varepsilon}Lz = L_\varepsilon\left(0, f'(0)\left[\frac{2}{\sqrt{\pi}}\hat{y}_{10}(t) - 2\sqrt{\frac{3}{\pi}}\hat{y}_{20}(t)\right], f'(0)\left[\frac{2}{\sqrt{\pi}}\hat{y}_{10}(t) + 2\sqrt{\frac{3}{\pi}}\hat{y}_{20}(t)\right]\right)$$

where $(\hat{y}_{10}(t), \hat{y}_{20}(t)) \in Y_1 \times Y_2$ is the unique bounded solution of Equation (5.1.84) that satisfies $\dot{\hat{y}}_{10}(2jm + \alpha_j) = 0$ for any $j \in \mathbb{Z}$ so that $e_j = 1$. Let $(\hat{y}_{12}(t), \hat{y}_{22}(t)) \in Y_1 \times Y_2$ be the unique bounded solution of

$$\ddot{\hat{y}}_{12}(t) + \frac{8}{\pi}f'(0)\hat{y}_{12}(t) = -Q_{m,\xi}\left\{\frac{2}{\sqrt{\pi}}f'\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right)[z(0,t) + z(\pi/4,t)]\right\},$$
$$\ddot{\hat{y}}_{22}(t) + \frac{24}{\pi}f'(0)\hat{y}_{22}(t) = -2\sqrt{\frac{3}{\pi}}f'\left(\frac{2}{\sqrt{\pi}}\Gamma_\xi(t)\right)[z(\pi/4,t) - z(0,t)]$$

and $(\hat{y}_{13}(t), \hat{y}_{23}(t)) \in Y_1 \times Y_2$ be the unique bounded solution of

$$\ddot{\hat{y}}_{13}(t) + \frac{8}{\pi}f'(0)\hat{y}_{13}(t) = -\frac{2}{\sqrt{\pi}}f'(0)[z(0,t) + z(\pi/4,t)]$$
$$\ddot{\hat{y}}_{23}(t) + \frac{24}{\pi}f'(0)\hat{y}_{23}(t) = -2\sqrt{\frac{3}{\pi}}f'(0)[z(\pi/4,t) - z(0,t)].$$

We set

$$\begin{aligned}
\bar{H}_{23}(t) &= f'(0)\left[\frac{2}{\sqrt{\pi}}(\hat{y}_{10}(t) - \hat{y}_{12}(t)) - 2\sqrt{\frac{3}{\pi}}(\hat{y}_{20}(t) - \hat{y}_{22}(t))\right],\\
\tilde{H}_{23}(t) &= f'(0)\left[\frac{2}{\sqrt{\pi}}(\hat{y}_{12}(t) - \hat{y}_{13}(t)) - 2\sqrt{\frac{3}{\pi}}(\hat{y}_{22}(t) - \hat{y}_{23}(t))\right],\\
\hat{H}_{23}(t) &= f'(0)\left[\frac{2}{\sqrt{\pi}}\hat{y}_{13}(t) - 2\sqrt{\frac{3}{\pi}}\hat{y}_{23}(t)\right],\\
\bar{H}_{33}(t) &= f'(0)\left[\frac{2}{\sqrt{\pi}}(\hat{y}_{10}(t) - \hat{y}_{12}(t)) + 2\sqrt{\frac{3}{\pi}}(\hat{y}_{20}(t) - \hat{y}_{22}(t))\right],\\
\tilde{H}_{33}(t) &= f'(0)\left[\frac{2}{\sqrt{\pi}}(\hat{y}_{12}(t) - \hat{y}_{13}(t)) + 2\sqrt{\frac{3}{\pi}}(\hat{y}_{22}(t) - \hat{y}_{23}(t))\right],\\
\hat{H}_{33}(t) &= f'(0)\left[\frac{2}{\sqrt{\pi}}\hat{y}_{13}(t) + 2\sqrt{\frac{3}{\pi}}\hat{y}_{23}(t)\right]
\end{aligned}$$

and note that

$$L_{1\varepsilon}Lz = L_\varepsilon\left(0, \bar{H}_{23}(t), \bar{H}_{33}(t)\right) + L_\varepsilon(0, \tilde{H}_{23}(t), \tilde{H}_{33}(t)) + L_\varepsilon(0, \hat{H}_{23}(t), \hat{H}_{33}(t)).$$

We know from [11, above equation (3.39)] that

$$\begin{aligned}
\|L_\varepsilon(0, \hat{H}_{23}(t), \hat{H}_{33}(t))\|_\infty &\le 8M_1M_2|f'(0)|\sqrt{\varepsilon}\left(2a\delta^{-1} + \sqrt{\varepsilon}\right)\|z\|_\infty\\
&\le 4\sqrt{\varepsilon}(2a + \delta\sqrt{\varepsilon})\|z\|_\infty.
\end{aligned}$$

Then from Lemma 5.1.2–(ii) and Lemma 5.1.3–(i) we obtain:

$$|\hat{y}_{10}(t) - \hat{y}_{12}(t)| \leq \frac{4A_4NC}{\sqrt{\pi}} \left(\mathrm{e}^{-am/2} + \mathrm{e}^{-a|t-2jm-\alpha_j|/2}\right) \|z\|_\infty,$$

$$|\hat{y}_{20}(t) - \hat{y}_{22}(t)| \leq 4NC_1\sqrt{\frac{3}{\pi}} \left(\mathrm{e}^{-am/2} + \mathrm{e}^{-a|t-2jm-\alpha_j|/2}\right) \|z\|_\infty$$

for $t \in ((2j-1)m, (2j+1)m]$, whereas Lemma 5.1.2–(iii) and Lemma 5.1.3–(ii) with $E = \{0\}$ and $\alpha' = \alpha'' = 0$, give:

$$\|\hat{y}_{12} - \hat{y}_{13}\| \leq \frac{8B}{\pi}(A_4+1)N\|z\|_\infty, \quad \|\hat{y}_{22} - \hat{y}_{23}\| \leq \frac{48B_1}{\pi}N\|z\|_\infty,$$

with the norms of the left-hand sides being in Y_1 and Y_2 respectively. Thus, Theorem 5.1.1–(a) implies, after some algebra:

$$\|L_\varepsilon(0, \bar{H}_{23}, \bar{H}_{33})\|_\infty \leq \frac{8N}{\pi}(A_4C + 3C_1)\left(\frac{1}{(2a^2)^{1/3}} + \frac{1}{\delta^2} + \frac{4\delta}{a}\right)\sqrt{\varepsilon}\|z\|_\infty$$

using the inequality $\mathrm{e}^{-am/2} < \sqrt{\varepsilon}\left(\frac{1}{2a^2}\right)^{1/3}$ that follows from $\left(\frac{am}{2}\right)^{2/3}\mathrm{e}^{-\frac{am}{2}} < \frac{1}{2}$ and $m \geq \varepsilon^{-3/4}$. Next, applying again Theorem 5.1.1(b) with $k_2 = 0$ (and hence letting β tend to $+\infty$) gives:

$$\|L_\varepsilon(0, \tilde{H}_{23}, \tilde{H}_{33})\|_\infty \leq \frac{8N}{\pi}(B(A_4+1) + 6B_1)\sqrt{\varepsilon}\left[5\sqrt{\varepsilon}\left(\frac{1}{\delta^4} + \delta\right) + 2\right]\|z\|_\infty.$$

Plugging everything together we obtain

$$\|L_{1\varepsilon}L\| \leq K\sqrt{\varepsilon}$$

where K is a positive constant depending only on δ. Thus, using (5.1.75) we get

$$\|L_{1\varepsilon}L + L_{2\varepsilon}\| \leq 2M_1M_2\delta^{-1}|f'(0)| + K\sqrt{\varepsilon}$$

and then, from assumption (H3), we see that $\varepsilon_0 > 0$ exists so that for any $\varepsilon \in (0, \varepsilon_0)$, (5.1.90) holds. Since the assumptions of Lemma 5.1.9 are satisfied we obtain a solution of Equations (5.1.71)–(5.1.73) provided $0 < \varepsilon < \varepsilon_0$, $|\mu| < \mu_0$ and $\varepsilon \leq \Gamma|\mu|$.

Finally, we prove that this solution satisfies the Lipschitz condition in $\alpha \in \ell_E^\infty$ as stated in the Theorem. Let $\xi' = (E, \alpha') \in X$, $\xi'' = (E, \alpha'') \in X$ and set

$$y_1'(t) = y_1(t, \mu, \varepsilon, \delta, \xi', m), \quad y_1''(t) = y_1(t, \mu, \varepsilon, \delta, \xi'', m),$$
$$y_2'(t) = y_2(t, \mu, \varepsilon, \delta, \xi', m), \quad y_2''(t) = y_2(t, \mu, \varepsilon, \delta, \xi'', m),$$
$$z'(x,t) = z(x, t, \mu, \varepsilon, \delta, \xi', m),\ z''(x,t) = z(x, t, \mu, \varepsilon, \delta, \xi'', m).$$

Then $(z(x,t), y_1(t), y_2(t)) = (z'(x,t) - z''(x,t), y_1'(t) - y_1''(t), y_2'(t) - y_2''(t))$ is a fixed point of the map

$$\begin{aligned} z(x,t) &= F_1\left(z(x,t)+z''(x,t),y_1(t)+y_1''(t),y_2(t)+y_1''(t),\xi',\mu,\varepsilon\right) \\ &\quad -F_1\left(z''(x,t),y_1''(t),y_2''(t),\xi'',\mu,\varepsilon\right)+L_{1\varepsilon}(y_1(t),y_2(t))+L_{2\varepsilon}z(x,t,), \\ (y_1(t),y_2(t)) &= F_2(z(x,t)+z''(x,t),y_1(t)+y_1''(t),y_2(t)+y_1''(t),\xi',\mu,\varepsilon) \\ &\quad -F_2(z''(x,t),y_1''(t),y_2''(t),\xi'',\mu,\varepsilon)+Lz(x,t)\,. \end{aligned} \tag{5.1.91}$$

From (5.1.86) we obtain

$$\begin{aligned} &\| F_2(z(x,t)+z''(x,t),y_1(t)+y_1''(t),y_2(t)+y_1''(t),\xi',\mu,\varepsilon) \\ &-F_2(z''(x,t),y_1''(t),y_2''(t),\xi'',\mu,\varepsilon)\| \\ &\le \bar{\Delta}(\rho,\varepsilon)(\|y_1\|+\|y_2\|+\|z\|_\infty)+k_4(|\mu|+\varepsilon+\rho\bar{\Delta}(\rho,\varepsilon))\|\alpha'-\alpha''\| \end{aligned} \tag{5.1.92}$$

where $\bar{\Delta}(\rho,\varepsilon)\to 0$ as $\rho+\varepsilon\to 0^+$ and $k_4>0$ is a suitable constant. Thus, using Theorem 5.1.1(b) (with $k_2=0$ and $\beta=+\infty$) we see that a positive constant k_5 exists so that

$$\begin{aligned} &\| L_{1\varepsilon}(F_2(z(x,t)+z''(x,t),y_1(t)+y_1''(t),y_2(t)+y_1''(t),\xi',\mu,\varepsilon) \\ &\quad - F_2(z''(x,t),y_1''(t),y_2''(t),\xi'',\mu,\varepsilon))\|_\infty \\ &\le k_5\sqrt{\varepsilon}(|\mu|+\varepsilon+\rho\bar{\Delta}(\rho,\varepsilon))\|\alpha'-\alpha''\|+k_5\sqrt{\varepsilon}\bar{\Delta}(\rho,\varepsilon)(\|y\|+\|z\|_\infty) \end{aligned} \tag{5.1.93}$$

for $\|y\|=\|y_1\|+\|y_2\|$. Now we replace $(y_1(t),y_2(t))$ in $L_{1\varepsilon}(y_1(t),y_2(t))$ in the first equation in (5.1.91) with the fixed point of the second equation in (5.1.91). Using Lemma 5.1.7, Lemma 5.1.8, (5.1.92) and (5.1.93), we get

$$\begin{aligned} &\|z\|_\infty \le \Delta_2(\rho,\varepsilon)(\|y\|+\|z\|_\infty)+k_9\sqrt{\varepsilon}(\sqrt{\varepsilon}+\rho+|\mu|)\|\alpha'-\alpha''\|+\lambda\|z\|_\infty\,, \\ &\|y\| \le \Delta_1(\rho,\varepsilon)(\|y\|+\|z\|_\infty)+k_4(\rho\bar{\Delta}(\rho,\varepsilon)+|\mu|+\varepsilon)\|\alpha'-\alpha''\|+\|L\|\|z\|_\infty \end{aligned} \tag{5.1.94}$$

where $\Delta_1(\rho,\varepsilon),\Delta_2(\rho,\varepsilon)\to 0$ as $\rho+\varepsilon\to 0^+$ and k_9 is a positive constant. From (5.1.88) we know that $\rho=O(\sqrt{\varepsilon}+|\mu|)$. Thus, if ε is sufficiently small, we can solve the first inequality in (5.1.94) for $\|z\|_\infty$ and get:

$$\|z\|_\infty \le \bar{\Delta}_2(\rho,\varepsilon)\|y\|+\sqrt{\varepsilon}O(|\mu|+\sqrt{\varepsilon})\|\alpha'-\alpha''\| \tag{5.1.95}$$

for $\bar{\Delta}_2(\rho,\varepsilon)\to 0$ as $\rho+\varepsilon\to 0^+$. Then we plug this estimate of $\|z\|_\infty$ into the second inequality in (5.1.94) and get:

$$\|y\| \le O(|\mu|+\sqrt{\varepsilon})\|\alpha'-\alpha''\|\,.$$

Finally, we plug again this estimate into (5.1.95) and obtain

$$\|z\|_\infty \le o(\sqrt{\varepsilon}+|\mu|)\|\alpha'-\alpha''\|.$$

The proof is complete. □

In order to find a bounded solution, near $\gamma_\xi(t)$, of Eqs. (5.1.6)–(5.1.8) we need to show that the equation

$$\begin{aligned}
&G(\xi,\varepsilon,\mu,\delta,m)\\
&:= P_{m,\xi}\Big\{\sqrt{\varepsilon}\delta\dot{y}_1(t,\mu,\varepsilon,\delta,\xi,m)+\frac{2}{\sqrt{\pi}}\mu\int_0^{\pi/4}h(x,t)\,dx\\
&\quad+\sqrt{\varepsilon}\delta\dot{\gamma}_\xi(t)+\frac{2}{\sqrt{\pi}}f\Big(\frac{2}{\sqrt{\pi}}[y_1(t,\mu,\varepsilon,\delta,\xi,m)+\Gamma_\xi(t)]\\
&\quad-2\sqrt{\frac{3}{\pi}}y_2(t,\mu,\varepsilon,\delta,\xi,m)+z(0,t,\mu,\varepsilon,\delta,\xi,m)\Big)\\
&\quad+\frac{2}{\sqrt{\pi}}f\Big(\frac{2}{\sqrt{\pi}}[y_1(t,\mu,\varepsilon,\delta,\xi,m)+\Gamma_\xi(t)]\\
&\quad+2\sqrt{\frac{3}{\pi}}y_2(t,\mu,\varepsilon,\delta,\xi,m)+z(\frac{\pi}{4},t,\mu,\varepsilon,\delta,\xi,m)\Big)\\
&\quad-\frac{4}{\sqrt{\pi}}f\left(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\right)-\frac{8}{\pi}f'\left(\frac{2}{\sqrt{\pi}}\gamma_\xi(t)\right)y_1(t,\mu,\varepsilon,\delta,\xi,m)\\
&\quad+\sqrt{\varepsilon}\delta\dot{v}_\xi(t)+\ddot{v}_\xi(t)\Big\}=0
\end{aligned}$$

can be solved for some values of the parameters. From Theorem 5.1.10, we know that

$$\begin{aligned}
\|y_1(t,\mu,\varepsilon,\delta,\xi,m)\| &= O(|\mu|+\sqrt{\varepsilon}),\\
\|y_2(t,\mu,\varepsilon,\delta,\xi,m)\| &= O(|\mu|+\sqrt{\varepsilon}),\\
\|z(x,t,\mu,\varepsilon,\delta,\xi,m)\|_\infty &= o(|\mu|+\sqrt{\varepsilon}),\\
\|y_1(t,\mu,\varepsilon,\delta,\xi',m)-y_1(t,\mu,\varepsilon,\delta,\xi'',m)\| &\le O(|\mu|+\sqrt{\varepsilon})\|\alpha'-\alpha''\|,\\
\|y_2(t,\mu,\varepsilon,\delta,\xi',m)-y_2(t,\mu,\varepsilon,\delta,\xi'',m)\| &\le O(|\mu|+\sqrt{\varepsilon})\|\alpha'-\alpha''\|,\\
\|z(x,t,\mu,\varepsilon,\delta,\xi',m)-z(x,t,\mu,\varepsilon,\delta,\xi'',m)\|_\infty &\le o(|\mu|+\sqrt{\varepsilon})\|\alpha'-\alpha''\|
\end{aligned}\tag{5.1.96}$$

where $\xi=(E,\alpha)$, $\xi'=(E,\alpha')$, $\xi''=(E,\alpha'')$, and $O(|\mu|+\sqrt{\varepsilon})$, $o(|\mu|+\sqrt{\varepsilon})$ are uniform with respect to (ξ,m). Thus, we set $\mu=\sqrt{\varepsilon}\eta$, where η belongs to a compact subset of $\mathbb{R}\setminus\{0\}$ where the condition $\Gamma|\eta|\ge\varepsilon$ is satisfied (possibly taking ε smaller). By multiplying the equation $G(\xi,\varepsilon,\sqrt{\varepsilon}\eta,\delta,m)=0$ by $\varepsilon^{-1/2}$, we obtain the equation:

$$\widetilde{B}(\xi,\varepsilon,\eta,\delta,m):=P_{m,\xi}\left\{\delta\dot{\gamma}_\xi(t)+\frac{2}{\sqrt{\pi}}\eta\int_0^{\pi/4}h(x,t)\,dx+r(t,\xi,\varepsilon,\eta,\delta,m)\right\}=0\tag{5.1.97}$$

where $\widetilde{B}(\xi,\varepsilon,\eta,\delta,m)=\varepsilon^{-1/2}G(\xi,\varepsilon,\sqrt{\varepsilon}\eta,\delta,m)$. Using (5.1.70) and (5.1.97) we see that

$$\begin{aligned}&\|r(t,\xi,\varepsilon,\eta,\delta,m)\|_\infty = o(1),\\&\|r(t,\xi',\varepsilon,\eta,\delta,m)-r(t,\xi'',\varepsilon,\eta,\delta,m)\|_\infty \le o(1)\|\alpha'-\alpha''\|\end{aligned} \tag{5.1.98}$$

as $\varepsilon \to 0^+$ uniformly with respect to (ξ,η,m). Let

$$M_\eta(\alpha) = \delta\int_{-\infty}^{\infty}\dot\gamma_1(s)^2\,ds + \frac{2}{\sqrt{\pi}}\eta\int_{-\infty}^{\infty}\int_0^{\pi/4}\dot\gamma_1(s)h(x,s+\alpha)\,dx\,ds \tag{5.1.99}$$

and consider the space $\mathscr{C} = C^0\big([0,\pi/4]\times\mathbb{R},\mathbb{R}\big)$ endowed with the metric $d_{\mathscr{C}}$ given by

$$d_{\mathscr{C}}(u_1,u_2) = \sum_{n\in\mathbb{N}} 2^{-|n|}\max_{[0,\pi/4]\times[-n,n]}|u_1(x,t)-u_2(x,t)|\,.$$

Finally we define a (weak) solution of (5.1.1) to be any $u(x,t)\in C([0,\pi/4]\times\mathbb{R})$ satisfying the identity

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_0^{\pi/4}\Big\{u(x,t)\Big[v_{tt}(x,t)+v_{xxxx}(x,t)-\varepsilon\delta v_t(x,t)\Big]+\varepsilon\mu h(x,\sqrt{\varepsilon}t)v(x,t)\Big\}\,dx\,dt\\&+\varepsilon\int_{-\infty}^{\infty}\Big\{f(u(0,t))v(0,t)+f(u(\pi/4,t))v(\pi/4,t)\Big\}\,dt = 0\end{aligned} \tag{5.1.100}$$

for any $v(x,t)\in C^\infty([0,\pi/4]\times\mathbb{R})$ so that $v(x,t)$ has a compact support and satisfies boundary value conditions (5.1.4). Now we have the following result.

Theorem 5.1.11. *Let $f(x)\in C^2(\mathbb{R})$ and $h(x,t)=h(x,t+1)\in C^2([0,\pi/4]\times\mathbb{R})$ be so that (H1), (H2) hold. Let $\delta>0$ be a fixed positive number that satisfies (H3). Then, if $\eta_0\neq 0$ can be chosen in such a way that the equation $M_\eta(\alpha)=0$ for $\eta=\eta_0$, has a simple root $\alpha_0\in[0,1]$, there exist $\bar\varepsilon>0$, $\bar\eta>0$ so that for any $\varepsilon\in(0,\bar\varepsilon]$, $\mu=\sqrt{\varepsilon}\eta$ with $|\eta-\eta_0|\le\bar\eta$ and $m>\varepsilon^{-3/4}$, $m\in\mathbb{N}$, there is a continuous map $\Pi:\mathscr{E}\to C^0([0,\pi/4]\times\mathbb{R},\mathbb{R})$ so that $\Pi(E)=u_E(x,t)$ is a weak solution of Equation* (5.1.1). *Moreover, $\Pi:\mathscr{E}\to\Pi(\mathscr{E})$ is a homeomorphism satisfying*

$$\Pi(\sigma(E))(x,t) = \Pi(E)(x,t+(2m/\sqrt{\varepsilon}))$$

with $\sigma:\mathscr{E}\to\mathscr{E}$ being the Bernoulli shift. Consequently, the Smale horseshoe can be embedded into the dynamics of (5.1.1).

Proof. We will prove that Eq. (5.1.97) can be solved for any $\xi\in X$ and ε, μ and η as in the statement of the theorem. Of course, there is nothing to prove if $\xi=0$ since $P_{m,0}=0$. Thus we assume $E\neq 0$ and recall (see Theorem 5.1.4) that $P_{m,\xi}h=0$ is equivalent to $[N_{m,\xi}+\mathscr{L}_{m,\xi}]h=0$. So, we solve the equation

$$[N_{m,\xi}+\mathscr{L}_{m,\xi}]\left\{\delta\dot\gamma_\xi(t)+\frac{2}{\sqrt{\pi}}\eta\int_0^{\pi/4}h(x,t)\,dx+r(t,\xi,\varepsilon,\eta,\delta,m)\right\}=0. \tag{5.1.101}$$

From (5.1.22) and (5.1.98) we know that the term in braces in the above equation is Lipschitz continuous function in $\alpha\in\ell^\infty_E$ uniformly with respect to (E,ε,η,m).

But $\|\mathscr{L}_{m,\xi}\| \le A\,\mathrm{e}^{-am} < \frac{2A}{5a^{4/3}}\varepsilon$ (having used again $\theta^{4/3}\mathrm{e}^{-\theta} < \frac{2}{5}$) and in Section 5.1.6 that follows, we will see that a positive constant $\tilde{A}$ exists so that $\|\mathscr{L}_{m,\xi'} - \mathscr{L}_{m,\xi''}\| \le \tilde{A}\,\mathrm{e}^{-am}\|\alpha' - \alpha''\| < \frac{2\tilde{A}}{5a^{4/3}}\varepsilon\|\alpha' - \alpha''\|$ for any $\xi' = (E,\alpha')$, $\xi'' = (E,\alpha'')$. As a consequence the function of ξ

$$\mathscr{L}_{m,\xi}\left\{\delta\dot{\gamma}_{\xi}(t) + \frac{2}{\sqrt{\pi}}\eta\int_0^{\pi/4} h(x,t)\,dx + r(t,\xi,\varepsilon,\eta,\delta,m)\right\}$$

is Lipschitz in $\alpha \in \ell_E^\infty$, with a $O(\varepsilon)$ Lipschitz constant which can be taken independently of (E,η,m). Next we consider

$$N_{m,\xi}\left\{\delta\dot{\gamma}_{\xi}(t) + \frac{2}{\sqrt{\pi}}\eta\int_0^{\pi/4} h(x,t)\,dx + r(t,\xi,\varepsilon,\eta,\delta,m)\right\}.$$

From the proof of Theorem 5.1.4 we know that $\xi \mapsto \|N_{m,\xi}\|$ is bounded uniformly with respect to (ξ,m) (see (5.1.63)) and Lipschitz continuous function in $\alpha \in \ell_E^\infty$ uniformly in (E,m) (actually we proved that $\|N_{m,\xi'} - N_{m,\xi''}\| \le \frac{16\bar{A}_2 N}{a\pi}\|\alpha' - \alpha''\|$). So, using (5.1.98) we see that $N_{m,\xi}r(t,\xi,\varepsilon,\eta,\delta,m)$ is Lipschitz continuous function in $\alpha \in \ell_E^\infty$ uniformly in (E,m,η) and the Lipschitz constant tends to 0 as $\varepsilon \to 0$. Finally, we consider the map from ℓ_E^∞ into itself:

$$\alpha \mapsto N_{m,(E,\alpha)}\left\{\delta\dot{\gamma}_{(E,\alpha)}(t) + \frac{2}{\sqrt{\pi}}\eta\int_0^{\pi/4} h(x,t)\,dx\right\} - \widetilde{\mathscr{M}_\eta}(\alpha) \in \ell_E^\infty \qquad (5.1.102)$$

where

$$\widetilde{\mathscr{M}_\eta}(\alpha) = \left\{e_j M_\eta(\alpha_j)\right\}_{j\in\mathbb{Z}}.$$

It is easy to see that the j–th component of the map (5.1.102) is given by the sum of the following two terms:

$$-e_j\int_{-\infty}^{-m-\alpha_j}\dot{\gamma}_1(t)\left[\delta\dot{\gamma}_1(t) + \frac{2}{\sqrt{\pi}}\eta\int_0^{\pi/4} h(x,t+\alpha_j)\,dx\right]dt,$$

$$-e_j\int_{m+\alpha_j}^{\infty}\dot{\gamma}_1(t)\left[\delta\dot{\gamma}_1(t) + \frac{2}{\sqrt{\pi}}\eta\int_0^{\pi/4} h(x,t+\alpha_j)\,dx\right]dt$$

and that the above functions are Lipschitz continuous function in α uniformly in (η,m,j) and with a $O(\varepsilon)$ Lipschitz constant, provided η belongs to a compact domain and ε is small. In fact, we have, for example, using also (5.1.53):

$$\left|\int_{-\infty}^{-m-\alpha_j'}\dot{\gamma}_1(t)\int_0^{\pi/4} h(x,t+\alpha_j')\,dx\,dt - \int_{-\infty}^{-m-\alpha_j''}\dot{\gamma}_1(t)\int_0^{\pi/4} h(x,t+\alpha_j'')\,dx\,dt\right|$$

$$\le \left|\int_{-m-\alpha_j''}^{-m-\alpha_j'}\dot{\gamma}_1(t)\int_0^{\pi/4} h(x,t+\alpha_j')\,dx\,dt\right|$$

$$+\left|\int_{-\infty}^{-m-\alpha_j''}\dot{\gamma}_1(t)\int_0^{\pi/4}[h(x,t+\alpha_j')-h(x,t+\alpha_j'')]\,dx\,dt\right|$$
$$\leq \frac{\bar{A}_2\pi}{4a}\,\mathrm{e}^{-am}[\|h\|_\infty|\mathrm{e}^{a\alpha_j'}-\mathrm{e}^{a\alpha_j''}|+\|h_t\|_\infty|\alpha_j'-\alpha_j''|]$$
$$= O(\varepsilon)[\|h\|_\infty+\|h_t\|_\infty]\|\alpha'-\alpha''\|\,.$$

A similar argument applies to the other quantities. Next, it is easy to see that the map $\widetilde{\mathscr{M}_\eta}:\ell_E^\infty\to\ell_E^\infty$ is C^1 in (α,η), and its derivative, with respect to α at the point $(\{e_j\alpha_0\}_{j\in\mathbb{Z}},\eta_0)\in\ell_E^\infty\times\mathbb{R}$, is given by:

$$\alpha\mapsto\{M_{\eta_0}'(\alpha_0)\alpha_j\}_{j\in\mathbb{Z}}=\mathscr{M}_{\eta_0}'(\alpha_0)\alpha.$$

As a matter of fact, we have:

$$\tilde{M}_\eta(\alpha)-\tilde{M}_\eta(\alpha_0)-\tilde{M}_\eta'(\alpha_0)(\alpha-\alpha_0)=o(\|\alpha-\alpha_0\|)$$

uniformly with respect to (η,E). So, we write (5.1.101) as a fixed point equation in ℓ_E^∞:

$$\alpha=\alpha-\widetilde{\mathscr{M}_{\eta_0}'}(\alpha_0)^{-1}\widetilde{\mathscr{M}_\eta}(\alpha)-\widetilde{\mathscr{M}_{\eta_0}'}(\alpha_0)^{-1}R(\xi,\varepsilon,\eta,\delta)$$

where $R(\xi,\varepsilon,\eta,\delta)$ is Lipschitz continuous function in $\alpha\in\ell_E^\infty$ with a $o(1)$ constant independent of (E,m,η). Moreover, the map $(\alpha,\eta)\mapsto\alpha-\widetilde{\mathscr{M}_{\eta_0}'}(\alpha_0)^{-1}\widetilde{\mathscr{M}_\eta}(\alpha)$ is C^1 and its $\alpha-$derivative vanishes at $\alpha=\alpha_0$ and $\eta=\eta_0$. Thus, from the uniform contraction principle 2.2.1 it follows the existence of $\bar{\varepsilon}>0$ and $\bar{\eta}>0$ so that for any $\varepsilon\in(0,\bar{\varepsilon}]$, $|\eta-\eta_0|\leq\bar{\eta}$ and $m>\varepsilon^{-3/4}$, $m\in\mathbb{N}$, the map

$$\alpha\mapsto\alpha-\widetilde{\mathscr{M}_{\eta_0}'}(\alpha_0)^{-1}\widetilde{\mathscr{M}_\eta}(\alpha)-\widetilde{\mathscr{M}_{\eta_0}'}(\alpha_0)^{-1}R(\xi,\varepsilon,\eta,\delta)$$

has a unique fixed point $\alpha=\alpha(E,m,\eta,\delta,\varepsilon)$ that tends to α_0 as $\varepsilon\to0$ and $\eta\to\eta_0$, uniformly with respect to (E,m). This implies that for any $\varepsilon\in(0,\varepsilon_0]$, $|\eta-\eta_0|\leq\bar{\eta}$ and $m>\varepsilon^{-3/4}$ the function

$$\begin{aligned}u_E(x,t):=&\left[y_1(\sqrt{\varepsilon}t,\sqrt{\varepsilon}\eta,\varepsilon,\delta,E,\alpha(E,m,\eta,\delta,\varepsilon))+\varGamma_\xi(t)\right]w_{-1}(x)\\&+y_2(\sqrt{\varepsilon}t,\sqrt{\varepsilon}\eta,\varepsilon,\delta,E,\alpha(E,m,\eta,\delta,\varepsilon))w_0(x)\\&+z(x,\sqrt{\varepsilon}t,\sqrt{\varepsilon}\eta,\varepsilon,\delta,E,\alpha(E,m,\eta,\delta,\varepsilon))\end{aligned}$$

is a solution of (5.1.101) near $\gamma_E(t)$ defined as

$$\gamma_E(t)=\begin{cases}\gamma\Big(2\sqrt{\dfrac{2}{\pi}}(\sqrt{\varepsilon}t-2jm-\alpha_0)\Big), \text{ for } & (2j-1)m<\sqrt{\varepsilon}t\leq(2j+1)m\\ & \text{and}\quad e_j=1\,,\\ 0, \qquad \text{for} & (2j-1)m<\sqrt{\varepsilon}t\leq(2j+1)m\\ & \text{and}\quad e_j=0\,.\end{cases}$$

Since $u_E(x, 2jm\varepsilon^{-1/2})$ is near to $u = 0$ if $e_j = 0$ or to $u = \gamma\Big(-2\sqrt{\frac{2}{\pi}}\alpha_0\Big) \neq 0$ if $e_j = 1$, we see that for $\bar{\varepsilon}$ sufficiently small, the map $\Pi : E \to u_E$ is one-to-one and the choice of E determines the oscillatory properties of $u_E(x,t)$ near $\gamma(t)$. Moreover, $u_E(x,t)$ is the unique solution of (5.1.101) that satisfies the above oscillatory property and can be written as a totally convergent series:

$$u_E(x,t) = \sum_{i=-1}^{\infty} u_{i,E}(t) w_i(x).$$

Let $\sigma : \mathscr{E} \to \mathscr{E}$ be the shift map defined by $\sigma(\{e_j\}_{j\in\mathbb{Z}}) = \{e_{j+1}\}_{j\in\mathbb{Z}}$. Then $u_{\sigma(E)}(x,t)$ has the same oscillatory properties between $u = 0$ and $u = \gamma\Big(-2\sqrt{\frac{2}{\pi}}\alpha_0\Big) \neq 0$ as $u_E(x, t + 2m\varepsilon^{-1/2})$. But we have

$$u_E(x, t + 2m\varepsilon^{-1/2}) = \sum_{i=-1}^{\infty} u_{i,E}(t + 2m\varepsilon^{-1/2}) w_i(x)$$

and the series is again totally convergent. Thus, because of the uniqueness, we obtain:

$$u_{\sigma(E)}(x,t) = u_E(x, t + 2m/\sqrt{\varepsilon}).$$

We now prove the continuity of Π, with respect to the topologies on $\mathscr{E}$ and $\mathscr{C} = C\big([0,\pi/4] \times \mathbb{R}, \mathbb{R}\big)$ induced by the metrics $d_{\mathscr{E}}$ and $d_{\mathscr{C}}$. First, we observe that Theorem 5.1.1 implies the existence of a positive constant c_0 so that for any $E \in \mathscr{E}$, the components $u_{i,E}(t)$ of $u_E(x,t)$ satisfy:

$$\|u_{i,E}\|_\infty \leq c_0/(\mu_i^2 + 1), \quad \|\dot{u}_{i,E}\|_\infty \leq c_0 \tag{5.1.103}$$

with c_0 being a suitable constant (see (5.1.13), (5.1.14)). Hence, for any $R > 0$ there exists $n_0 \in \mathbb{N}$ so that, for any $E \in \mathscr{E}$, we have

$$\|u_E(x,t) - \sum_{i=-1}^{n_0} u_{i,E}(t) w_i(x)\|_\infty \leq 1/R.$$

Now, let $\{E_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathscr{E}$. From (5.1.103) and the Arzelà-Ascoli theorem 2.1.3 the existence follows of a subsequence $\{j_k^{(-1)}\}$ of $\{j_k^{(-2)} := k\}$ so that $u_{-1,E_{j_k^{(-1)}}}(t)$ converges uniformly in any interval $[-n,n]$. Then another application of the Arzelà-Ascoli theorem 2.1.3 implies the existence of a subsequence $\{j_k^{(0)}\}$ of $\{j_k^{(-1)}\}$ so that $u_{0,E_{j_k^{(0)}}}(t)$ converges uniformly in any interval $[-n,n]$. Proceeding in this way, for any $i = -1, 0, 1\ldots$, we construct a subsequence $\{j_k^{(i)}\}$ of $\{j_k^{(i-1)}\}$ so that $u_{i,E_{j_k^{(i)}}}(t)$ converges uniformly in any interval $[-n,n]$. Then, we use Cantor diagonal procedure to see that for any $i = -1, 0, 1\ldots$ the sequence $u_{i,E_{j_k^{(k)}}}(t)$ converges uniformly in any interval $[-n,n]$. Now, let E_{j_n} be a subsequence of E_j so that for any $i = -1, 0, \ldots$, $u_{i,E_{jn}}(t)$ converges to a continuous function $u_i(t)$ uniformly on any compact subset of $\mathbb{R}$. We have just proved that the set of such subsequences is not empty. From (5.1.103) we obtain $\|u_i\|_\infty \le c_0/(\mu_i^2+1)$ and hence the series $\sum_{i=-1}^{\infty} u_i(t)w_i(x)$ is totally convergent and defines a continuous function $u(x,t)$. Moreover, for $(x,t) \in [0, \frac{\pi}{4}] \times [-n,n]$ and any $R > 0$, we have

$$\left|u_{E_{j_k}}(x,t) - u(x,t)\right| \le \left|u_{E_{j_k}}(x,t) - \sum_{i=-1}^{n_0} u_{i,E_{j_k}}(t)w_i(x)\right| \\ + M_1 \sum_{i=-1}^{n_0} \left|u_{i,E_{j_k}}(t) - u_i(t)\right| + \left|u(x,t) - \sum_{i=-1}^{n_0} u_i(t)w_i(x)\right|.$$

So,

$$\overline{\lim}_{k\to\infty} |u_{E_{j_k}}(x,t) - u(x,t)| \le 2/R.$$

As a consequence, $u_{E_{jn}}(x,t) \to u(x,t)$ uniformly on compact sets. Thus the following statement holds:

for any given sequence $\{E_j\}_{j\in\mathbb{N}}$ *in* $\mathscr{E}$ *there exists a subsequence* $\{E_{j_k}\}_{k\in\mathbb{N}}$ *so that* $\{u_{E_{j_k}}(x,t)\}_{k\in\mathbb{N}}$ *converges uniformly on compact sets to a continuous function*

$$u(x,t) = \sum_{i=-1}^{\infty} u_{i,E}(t)w_i(x)$$

with the series being totally convergent and $u(x,t)$ *being a weak solution of* (5.1.1).

Now, assume that Π is not continuous. Then $E, E_j \in \mathscr{E}$, $j \in \mathbb{N}$ exist so that $d_{\mathscr{E}}(E_j, E) \to 0$, as $j \to \infty$ but $d_{\mathscr{C}}(u_{E_j}, u_E)$ is greater than a positive number for any $j \in \mathbb{N}$. Passing to a subsequence, if necessary, we can assume that $u_{E_j}(x,t)$ converges uniformly on compact sets to a weak solution $\hat{u}(x,t)$ of (5.1.1). Then, for any $(x,t) \in [0, \frac{\pi}{4}] \times \mathbb{R}$, we have

$$|\hat{u}(x,t) - \gamma_E(t)| \le |u_{E_{jn}}(x,t) - \hat{u}(x,t)| + |u_{E_{jn}}(x,t) - \gamma_{E_{jn}}(t)| + |\gamma_{E_{jn}}(t) - \gamma_E(t)|$$

and hence, passing to the limit for $n \to \infty$:

$$|\hat{u}(x,t) - \gamma_E(t)| \le \sup_n \|u_{E_{jn}} - \gamma_{E_{jn}}\|_\infty + \overline{\lim}_{n\to\infty} |\gamma_{E_{jn}}(t) - \gamma_E(t)|.$$

But, since $d_{\mathscr{E}}(E_j, E) \to 0$ we see that for $n > \overline{n}(\varepsilon, t)$ we have $\gamma_{E_{jn}}(t) = \gamma_E(t)$. So $\hat{u}(x,t)$ is orbitally close to $\gamma_E(t)$ and then, because of uniqueness,

$$\hat{u}(x,t) = u_E(x,t) = \Pi(E)$$

contradicting the assumption that Π was not continuous. The proof is complete. □

Remark 5.1.12. (a) If (H2) fails so that linear equation (5.1.10) has a two-dimensional space of bounded solutions on $\mathbb{R}$, then we can perform again the above procedure but we get a two-dimensional mapping like (5.1.99) of the form $M_\eta(\alpha,\beta)$, $(\alpha,\beta)\in\mathbb{R}^2$ (cf Section 4.1.3) and the existence of a simple root of function $M_\eta(\alpha,\beta)$ implies a result similar to Theorem 5.1.11.

(b) Assuming also that f is odd, i.e. $f(-y)=-f(y)$, then we get the additional homoclinic orbit $(0,\gamma_2(t)) := \left(0,\frac{1}{2}\sqrt{\frac{\pi}{3}}\gamma\left(2\sqrt{\frac{6}{\pi}}t\right)\right)$ for (5.1.9) and we can repeat the above approach by assuming the non-degeneracy of $\gamma_2(t)$ as in (H2). We get in this way another chaotic solutions of (5.1.1) when the corresponding mapping like (5.1.99) has a simple root. We do not perform here such computations.

(c) If we consider in (5.1.1) the time scale 1, i.e. we have $h(x,t)$ in (5.1.1), then (5.1.2) becomes a rapidly oscillating perturbed problem. So we should arrive at an exponentially small bifurcation problem [12, 13].

5.1.5 Useful Numerical Estimates

To get more information on condition (H3), we give in this section a numerical estimate of the constants M_1 and M_2 (see (5.1.16)). For this purpose, we recall [2]

$$w_k(x)=\frac{4}{\sqrt{\pi}W_k}\left[\cosh(\mu_k x)+\cos(\mu_k x)-\frac{\cosh\xi_k-\cos\xi_k}{\sinh\xi_k-\sin\xi_k}\left(\sinh(\mu_k x)+\sin(\mu_k x)\right)\right], \tag{5.1.104}$$

where $\xi_k=\mu_k\pi/4$ are determined by the equation $\cos\xi_k\cosh\xi_k=1$ and the constants W_k are given by the formula

$$W_k=\cosh\xi_k+\cos\xi_k-\frac{\cosh\xi_k-\cos\xi_k}{\sinh\xi_k-\sin\xi_k}\left(\sinh\xi_k+\sin\xi_k\right). \tag{5.1.105}$$

We first evaluate W_k. Numerically we find $\xi_1 \doteq 4.73004075$. Moreover, $0<\xi_1<\xi_2<\cdots$ and so $\cosh\xi_1<\cosh\xi_2<\cdots$. Since $\xi_k\sim\pi(2k+1)/2$ and $\cos(\pi(2k+1)/2)=0$, we get

$$|\sin\theta_k|\cdot|\xi_k-\pi(2k+1)/2|=|\cos\xi_k-\cos(\pi(2k+1)/2)|=\frac{1}{\cosh\xi_k}\leq 2\,\mathrm{e}^{-\xi_k}$$

for a $\theta_k\in(\xi_k,\pi(2k+1)/2)$. But we have

$$1\geq|\sin\xi_k|=\sqrt{1-\cos^2\xi_k}\geq\sqrt{1-\cos^2\xi_1}\doteq 0.999844212\,,$$

since $0<\cos\xi_k=\operatorname{sech}\xi_k\leq\operatorname{sech}\xi_1=\cos\xi_1$. Next, we can easily see that in fact $(4k-1)\pi/2<\xi_{2k-1}$, $\xi_{2k}<(4k+1)\pi/2$ and function $\cos x$ is positive in intervals $(\xi_k,\pi(2k+1)/2)$ for any $k\in\mathbb{N}$. So function $\sin x$ is increasing in these intervals, and it is positive on $[\xi_{2k},(4k+1)\pi/2]$ and negative on $[(4k-1)\pi/2,\xi_{2k-1}]$. Hence

$\sin\xi_{2k} = \sqrt{1-\cos^2\xi_{2k}}$. Using also $\cosh\xi_k = \frac{1}{\cos\xi_k}$ and $\sinh\xi_k = \sqrt{\cosh^2\xi_k - 1}$ form (5.1.105) we derive $W_{2k} = -2$. Similarly, from $\sin\xi_{2k-1} < 0$, $k \in \mathbb{N}$ we derive $\sin\xi_{2k-1} = -\sqrt{1-\cos^2\xi_{2k-1}}$ and then $W_{2k-1} = 2$. Consequently, $|W_k| = 2$ for any $n \in \mathbb{N}$. Next, (5.1.104) implies

$$\begin{aligned}
&|w_k(x)| \\
&\leq \frac{2}{\sqrt{\pi}}\left(\left|\cosh(\mu_k x) - \frac{\cosh\xi_k\sinh(\mu_k x)}{\sinh\xi_k - \sin\xi_k}\right| + 1 + \cos\xi_k\frac{\sinh\xi_k}{\sinh\xi_k - 1} + \frac{\cosh\xi_k}{\sinh\xi_k - 1}\right) \\
&\leq \frac{2}{\sqrt{\pi}}\left(\frac{\sinh\left(\mu_k\left(\frac{\pi}{4} - x\right)\right) + 2\cosh\xi_k + \cos\xi_k\sinh\xi_k}{\sinh\xi_k - 1} + 1\right) \\
&\leq \frac{2}{\sqrt{\pi}}\left(\frac{\sinh\xi_1 + 2\cosh\xi_1 + \cos\xi_1\sinh\xi_1}{\sinh\xi_1 - 1} + 1\right) \doteq 4.5949831827\,.
\end{aligned}$$

Hence $M_1 \leq 4.594983183$. Now we estimate M_2. From the above arguments we deduce $|\sin\theta_k| \geq |\sin\xi_k| \geq |\sin\xi_1| \doteq 0.999844212$. This gives

$$|\xi_k - \pi(2k+1)/2| \leq \frac{2}{|\sin\xi_1|}\,\mathrm{e}^{-\xi_1} \doteq 0.017654973\,.$$

So we obtain $\xi_k \geq \frac{\pi(2k+1)}{2} - 0.017654973 \geq \pi k$. Consequently, we arrive at

$$|\xi_k - \pi(2k+1)/2| \leq \frac{2}{|\sin\xi_1|}\,\mathrm{e}^{-\xi_k} \leq \frac{2}{|\sin\xi_1|}\,\mathrm{e}^{-\pi k} \leq c\frac{\pi}{4}\,\mathrm{e}^{-\pi k} \tag{5.1.106}$$

for $c \doteq 2.546875863$. Furthermore, since $\xi_k \geq \xi_1 > 4$, we have

$$\left|\frac{1}{\xi_k^2} - \frac{4}{\pi^2(2k+1)^2}\right| = 2\left|\frac{1}{\xi_k} + \frac{2}{\pi(2k+1)}\right| \cdot \left|\frac{\xi_k - \pi(2k+1)/2}{\xi_k\pi(2k+1)}\right| \leq \frac{3}{16|\sin\xi_1|}\,\mathrm{e}^{-\pi k}\,.$$

Hence, we arrive at

$$\begin{aligned}
\sum_{k=7}^{\infty}\left|\frac{1}{\xi_k^2} - \frac{4}{\pi^2(2k+1)^2}\right| &\leq \sum_{k=7}^{\infty}\frac{3}{16|\sin\xi_1|}\,\mathrm{e}^{-\pi k} = \frac{3}{16|\sin\xi_1|}\frac{\mathrm{e}^{-7\pi}}{1-\mathrm{e}^{-\pi}} \\
&\doteq 5.51594097\cdot 10^{-11}\,.
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{k=1}^{\infty} 1/\xi_k^2 \\
&\quad\leq \sum_{k=1}^{6} 1/\xi_k^2 + \sum_{k=7}^{\infty}\left|\frac{1}{\xi_k^2} - \frac{4}{\pi^2(2k+1)^2}\right| + \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} - \frac{4}{\pi^2}\sum_{k=0}^{6}\frac{1}{(2k+1)^2} \\
&\quad\leq \sum_{k=1}^{6} 1/\xi_k^2 + \frac{3}{16|\sin\xi_1|}\frac{\mathrm{e}^{-7\pi}}{1-\mathrm{e}^{-\pi}} + \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=0}^{6}\frac{1}{(2k+1)^2} \doteq 0.09438295\,.
\end{aligned}$$

This implies $M_2 = \frac{\pi^2}{4} M_1 \sum_{k=1}^{\infty} 1/\xi_k^2 \leq 1.07008241$. In summariy, we see that condition (H3) holds if

$$9.8340213469 \cdot |f'(0)| < \delta .$$

Finally, we note that $w_k(x)$ and $w_k\left(\frac{\pi}{4} - x\right)$ solve the same eigenvalue problem

$$u_{xxxx}(x) = \mu_k u(x), \quad u_{xx}(0) = u_{xx}(\pi/4) = u_{xxx}(0) = u_{xxx}(\pi/4) = 0 .$$

Since $\{w_k \mid k \in \mathbb{N}\}$ is an orthonormal system in $L^2([0,\pi/4])$, we see that $w_k(x) = \pm w_k\left(\frac{\pi}{4} - x\right)$. But $w_k(\pi/4) = 4/\sqrt{\pi}$ and $w_k(0) = 4/\sqrt{\pi}$ when k is odd, and $w_k(0) = -4/\sqrt{\pi}$ when k is even. So $w_{2k}\left(\frac{\pi}{4} - x\right) = -w_{2k}(x)$ and $w_{2k-1}\left(\frac{\pi}{4} - x\right) = w_{2k-1}(x)$, $\forall k \in \mathbb{N}$.

5.1.6 Lipschitz Continuity

Here we prove the Lipschitz continuity property of the linear map $\mathscr{L}_{m,\xi} : L^\infty(\mathbb{R}) \to \ell^\infty$ defined as

$$\mathscr{L}_{m,\xi}(h) = \{\mathscr{L}_{m,\xi,j}(h)\}_{j\in\mathbb{Z}}$$

with respect to α uniformly in $E \in \mathscr{E}$ and $m \geq m_0$. We start with the family of linear maps $L_{m,\xi} : \ell^\infty \times \ell^\infty \to \ell^\infty \times \ell^\infty$ defined as

$$L_{m,\xi}(\widetilde{a},\widetilde{b}) = \{L_{m,\xi,j}(\widetilde{a},\widetilde{b})\}_{j\in\mathbb{Z}}$$

where $\widetilde{a} = \{\widetilde{a}_j\}_{j\in\mathbb{Z}}$, $\widetilde{b} = \{\widetilde{b}_j\}_{j\in\mathbb{Z}}$ and prove that it is Lipschitz continuous function in α uniformly withy respect to (E,m), $E \in \mathscr{E}$ and $m \geq m_0$.

As in the proof of Lemma 5.1.2, $u(t)$ denotes the (unbounded) solution of $\ddot{x} + \frac{8}{\pi} f'\left(\frac{2}{\sqrt{\pi}}\gamma_1(t)\right)x = 0$ so that $u(0) = 1$ and $\dot{u}(0) = 0$. For simplicity we also set: $\hat{u}(t) = \frac{\dot{u}(t)}{au(t)}$ and note that $\hat{u}(t)$ is uniformly continuous in $\mathbb{R}$ since $\lim_{t\to\pm\infty} \hat{u}(t) = \pm 1$ (see (5.1.49)). Moreover we have

$$\frac{d}{dt}\left(\frac{\dot{u}(t)}{au(t)}\right) = \frac{\ddot{u}(t)}{au(t)} - \frac{1}{a}\left(\frac{\dot{u}(t)}{u(t)}\right)^2 = -\frac{8}{a\pi} f'\left(\frac{2}{\sqrt{\pi}}\gamma_1(t)\right) - a\left(\frac{\dot{u}(t)}{au(t)}\right)^2 \to 0$$

as $t \to \pm\infty$. Hence $\frac{d\hat{u}}{dt}(t)$ is also uniformly continuous in $\mathbb{R}$. As a matter of fact, $\hat{u}(t)$ is Lipschitz continuous function with constant, say, $\tilde{\Lambda}$, since $\frac{d\hat{u}}{dt}(t)$ is bounded on $\mathbb{R}$.

Now, let $\xi = (E,\alpha)$, $\xi' = (E,\alpha')$ be elements of X and consider the difference $L_{m,\xi} - L_{m,\xi'}$. From (5.1.47), (5.1.48) we see that for any $\widetilde{a} = \{\widetilde{a}_j\}_{j\in\mathbb{Z}}$, $\widetilde{b} = \{\widetilde{b}_j\}_{j\in\mathbb{Z}}$, we have

$$[L_{m,\xi} - L_{m,\xi'}]\begin{pmatrix}\widetilde{a}\\ \widetilde{b}\end{pmatrix} = \begin{pmatrix}0\\ \widetilde{B}\end{pmatrix} \tag{5.1.107}$$

with $\widetilde{B} = \{\widetilde{B}_j\}_{j\in\mathbb{Z}}$ and

$$\widetilde{B}_j = [\hat{u}(-m-\alpha'_{j+1}) - \hat{u}(-m-\alpha_{j+1})]e_{j+1}\widetilde{a}_{j+1} + [\hat{u}(m-\alpha_j) - \hat{u}(m-\alpha'_j)]e_j\widetilde{b}_j. \tag{5.1.108}$$

Then we have, using the Lipschitz continuity of $\hat{u}(t)$:

$$\begin{aligned}|\widetilde{B}_j| \leq & |\hat{u}(m-\alpha_j) - \hat{u}(m-\alpha'_j)||\widetilde{b}_j| + |\hat{u}(-m-\alpha'_{j+1}) - \hat{u}(-m-\alpha_{j+1})||\widetilde{a}_{j+1}| \\ \leq & \tilde{\Lambda}|\alpha_j - \alpha'_j||\widetilde{b}_j| + \tilde{\Lambda}|\alpha_{j+1} - \alpha'_{j+1}||\widetilde{a}_{j+1}| \leq \tilde{\Lambda}\|\alpha - \alpha'\|_\infty[\|\widetilde{a}\|_\infty + \|\widetilde{b}\|_\infty].\end{aligned}$$

As a consequence,

$$\|L_{m,\xi} - L_{m,\xi'}\|_\infty \leq \tilde{\Lambda}\|\alpha - \alpha'\|_\infty \tag{5.1.109}$$

uniformly with respect to (E,m), $E \in \mathscr{E}$ and $m \geq m_0$. Then the same conclusion holds for the inverse map $L^{-1}_{m,\xi}$. In fact, from $L^{-1}_{m,\xi} - L^{-1}_{m,\xi'} = L^{-1}_{m,\xi}[L_{m,\xi'} - L_{m,\xi}]L^{-1}_{m,\xi'}$ we obtain $\|L^{-1}_{m,\xi} - L^{-1}_{m,\xi'}\| \leq 9\widetilde{\Lambda}\|\alpha - \alpha'\|$, since $\|L^{-1}_{m,\xi}\| \leq 3$ (see (5.1.51)). Now,

$$\mathscr{L}_{m,\xi,j}(h) = -e_j\ddot{\gamma}_1(0)\left[\frac{\widetilde{a}_j}{u(-m-\alpha_j)} - \frac{\widetilde{b}_j}{u(m-\alpha_j)}\right]$$

where $(\widetilde{a},\widetilde{b})$ is obtained by solving the equation $L_{m,\xi}(\widetilde{a},\widetilde{b}) = (A_\xi h, B_\xi h)$ and $A_\xi h$, $B_\xi h$ are the linear (in $h \in L^\infty(\mathbb{R})$) maps defined by the right-hand sides of Equations (5.1.44)–(5.1.46):

$$\begin{aligned}A_\xi h &= \left\{(1-e_j)C_j - (1-e_{j+1})\hat{C}_{j+1} - e_jD_j(\alpha_j) - e_{j+1}\hat{D}_{j+1}(\alpha_{j+1})\right\}_{j\in\mathbb{Z}}, \\ B_\xi h &= \left\{-(1-e_j)C_j - (1-e_{j+1})\hat{C}_{j+1} - e_jF_j(\alpha_j) - e_{j+1}\hat{F}_{j+1}(\alpha_{j+1})\right\}_{j\in\mathbb{Z}},\end{aligned}$$

where

$$C_j = \frac{1}{2a}\int_{(2j-1)m}^{(2j+1)m} \mathrm{e}^{-a((2j+1)m-s)}\,h(s)ds,$$

$$\hat{C}_j = \frac{1}{2a}\int_{(2j-1)m}^{(2j+1)m} \mathrm{e}^{a((2j-1)m-s)}\,h(s)ds,$$

$$D_j(\alpha) = \int_{2jm+\alpha}^{(2j+1)m} v(m-\alpha)u(s-2jm-\alpha)h(s)ds,$$

$$\hat{D}_j(\alpha) = \int_{(2j-1)m}^{2jm+\alpha} v(-m-\alpha)u(s-2jm-\alpha)h(s)ds,$$

$$F_j(\alpha) = \frac{1}{a}\int_{2jm+\alpha}^{(2j+1)m} \dot{v}(m-\alpha)u(s-2jm-\alpha)h(s)ds$$

$$\hat{F}_j(\alpha) = \frac{1}{a}\int_{(2j-1)m}^{2jm+\alpha} \dot{v}(-m-\alpha)u(s-2jm-\alpha)h(s)ds.$$

So, if we prove that the linear map $h \mapsto (A_\xi h, B_\xi h)$ is bounded uniformly with respect to $\xi \in X$ and Lipschitz continuous function in α uniformly with respect to

(E,m), we get that $\mathscr{L}_{m,\xi}(h)$ is Lipschitz continuous function in α uniformly with respect to (E,m) and that the Lipschitz constant is $O(\mathrm{e}^{-am}) = O(\varepsilon)$ as $\varepsilon \to 0$ uniformly with respect to (E,m). Now, the fact that $A_\xi h$, $B_\xi h$ are bounded uniformly with respect to $\xi \in X$ easily follows from

$$\begin{gathered}\max\left\{|C_j|,|\hat{C}_j|\right\} \leq \tfrac{1}{2a^2}\|h\|_\infty,\\ \max\left\{|D_j(\alpha)|,|\hat{D}_j(\alpha)|,|F_j(\alpha)|,|\hat{F}_j(\alpha)|\right\} \leq \tfrac{k}{a}\|h\|_\infty.\end{gathered} \tag{5.1.110}$$

Then it is enough to study the Lipschitz continuity of the maps

$$\begin{gathered}(\xi,h) \mapsto \left\{D_j(\alpha_j)e_j\right\}_{j\in\mathbb{Z}},\ \ (\xi,h) \mapsto \left\{\hat{D}_j(\alpha_j)e_j)\right\}_{j\in\mathbb{Z}},\\ (\xi,h) \mapsto \left\{F_j(\alpha_j)e_j\right\}_{j\in\mathbb{Z}},\ \ (\xi,h) \mapsto \left\{\hat{F}_j(\alpha_j)e_j)\right\}_{j\in\mathbb{Z}},\end{gathered} \tag{5.1.111}$$

with respect to α. Writing $D_j(\alpha,m)$, $\hat{D}_j(\alpha,m)$, etc. to emphasize dependence on m we see that

$$\hat{D}_j(\alpha,m) = -D_{-j}(\alpha,-m), \qquad \hat{F}_j(\alpha,m) = -F_{-j}(\alpha,-m).$$

Thus we only need to look at $D_j(\alpha)$ and $F_j(\alpha)$. We focus our attention on the map $(\xi,h) \mapsto \{D_j(\alpha_j)e_j\}$, $\xi = (E,\alpha)$, $F_j(\alpha)$ being handled similarly. First, we look at the difference $D_j(\tau'') - D_j(\tau')$, where $\tau',\tau'' \in \mathbb{R}$, $\tau'' \geq \tau'$ and $|\tau'|,|\tau''| \leq 2$. We see that $D_j(\tau'') - D_j(\tau')$ equals:

$$\begin{aligned}&\int_{2jm+\tau''}^{(2j+1)m} \left[v(m-\tau'')u(s-2jm-\tau'') - v(m-\tau')u(s-2jm-\tau')\right] h(s)ds\\ &\quad -\int_{2jm+\tau'}^{2jm+\tau''} v(m-\tau')u(s-2jm-\tau')h(s)ds\,.\end{aligned}$$

Then (5.1.34) implies

$$\left|\int_{2jm+\tau'}^{2jm+\tau''} v(m-\tau')u(s-2jm-\tau')h(s)ds\right| \leq k\|h\|_\infty|\tau''-\tau'|\,.$$

Similarly, we get

$$\begin{aligned}&\left|\int_{2jm+\tau''}^{(2j+1)m} \left[v(m-\tau'')u(s-2jm-\tau'') - v(m-\tau')u(s-2jm-\tau')\right] h(s)ds\right|\\ &= \left|\int_{2jm+\tau''}^{(2j+1)m} \left(\int_{\tau'}^{\tau''} \left[\dot{v}(m-\tau)u(s-2jm-\tau) - v(m-\tau)\dot{u}(s-2jm-\tau)\right]d\tau\right)\right.\\ &\qquad \left.\cdot h(s)ds\right|\\ &\leq \frac{2k}{a}\|h\|_\infty|\tau''-\tau'|\,.\end{aligned}$$

Consequently, we obtain

$$|D_j(\tau'') - D_j(\tau')| \leq \left(\frac{2k}{a} + k\right) \|h\|_\infty |\tau'' - \tau'| .$$

Thus $(\xi, h) \mapsto \{D_j(\alpha_j)e_j\}_{j\in\mathbb{Z}}$ is Lipschitz continuous function in α with the constant $\frac{2k}{a} + k$ independent of (E, m). Similarly we can prove the global Lipschitz continuity in α of $F_j(\alpha)$. This completes the proof of the uniform Lipschitz continuity in α of $\mathscr{L}_{m,\xi}(h)$. Note that when $h \in L^\infty$, the maps in (5.1.111) are not differentiable in α.

5.2 Infinite Dimensional Non-Resonant Systems

5.2.1 Buckled Elastic Beam

To motivate the ideas of this section consider the partial differential equation

$$\ddot{u} = -u'''' - P_0 u'' + \left[\int_0^\pi u'(s)^2\, ds\right] u'' - 2\mu_2 \dot{u} + \mu_1 \cos\omega_0 t \tag{5.2.1}$$

where P_0, μ_1, μ_2, ω_0 are constants and u is a real valued function of two variables $t \in \mathbb{R}$, $x \in [0, \pi]$, subject to the boundary conditions

$$u(0,t) = u(\pi,t) = u''(0,t) = u''(\pi,t) = 0 .$$

In (5.2.1), a superior dot denotes differentiation with respect to t and prime differentiation with respect to x. This is a model for oscillations of an elastic beam with a compressive axial load P_0 (Figure 5.2). When P_0 is sufficiently large, (5.2.1) can exhibit chaotic behavior. The first work on this was done in [3]. Some more recent work on the full equation is in [4, 14]. An undamped buckled beam is investigated in [15] to show Arnold diffusion type motions. We will discuss some of them in more detail when we return to this problem in Section 5.2.6.

In (5.2.1) substitute $u(x,t) = \sum_{k=1}^{\infty} u_k(t) \sin kx$, multiply by $\sin nx$ and integrate from 0 to π. This yields the infinite set of ordinary differential equations

$$\ddot{u}_n = n^2(P_0 - n^2)u_n - \frac{\pi}{2} n^2 \left[\sum_{k=1}^{\infty} k^2 u_k^2\right] u_n - 2\mu_2 \dot{u}_n + 2\mu_1 \left[\frac{1-(-1)^n}{\pi n}\right] \cos\omega_0 t,$$
$$n = 1, 2, \ldots .$$

We see that the linear parts of these equations are uncoupled and the equations are divided into two types. The system of equations defined by $1 \leq n^2 < P_0$ has a hyperbolic equilibrium in origin whereas for the system of equations satisfying $n^2 \geq P_0$, this equilibrium is a center. For simplicity let us assume $1 < P_0 < 4$. Then

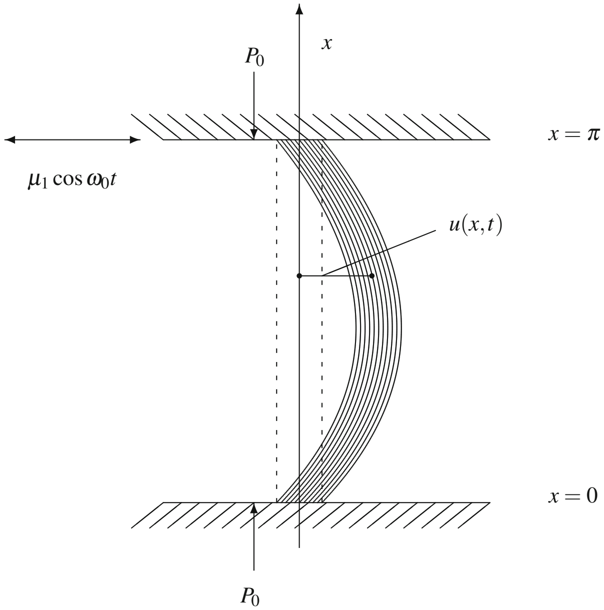

Fig. 5.2 The forced buckled beam (5.2.1).

only the equation with $n = 1$ is hyperbolic while the system of remaining equations has a center. To emphasize this let us define $p = u_1$ and $q_n = u_{n+1}$, $n = 1, 2, \ldots$. The preceding equations now take the form

$$\ddot{p} = a^2 p - \frac{\pi}{2}\left[p^2 + \sum_{k=1}^{\infty}(k+1)^2 q_k^2\right] p - 2\mu_2 \dot{p} + \frac{4}{\pi}\mu_1 \cos \omega_0 t, \quad (5.2.2a)$$

$$\ddot{q}_n = -\omega_n^2 q_n - \frac{\pi}{2}(n+1)^2 \left[p^2 + \sum_{k=1}^{\infty}(k+1)^2 q_k^2\right] q_n$$
$$-2\mu_2 \dot{q}_n + 2\mu_1 \left[\frac{1-(-1)^{n+1}}{\pi(n+1)}\right] \cos \omega_0 t, \quad (5.2.2b)$$
$$n = 1, 2, \ldots$$

where we have defined $a^2 = P_0 - 1$ and $\omega_n^2 = (n+1)^2\left[(n+1)^2 - P_0\right]$. In (5.2.2) we project onto the hyperbolic subspace by setting $q = 0$ in (5.2.2a) to obtain what we shall call the *reduced equation*. In our example this is

$$\ddot{p} = a^2 p - \frac{\pi}{2} p^3 - 2\mu_2 \dot{p} + \frac{4}{\pi}\mu_1 \cos \omega_0 t. \quad (5.2.3)$$

We see that this is the forced, damped Duffing equation with negative stiffness for which standard theory yields chaotic dynamics (cf Section 4.1). The purpose of this section is to show that the chaotic dynamics of (5.2.3) are, in some sense, shadowed in the dynamics of the full equation (5.2.2). To put our example in the first order form we define $x = (p, \dot{p})$ and

$$y = (q_1, \dot{q}_1/\omega_1, q_2, \dot{q}_2/\omega_2, \ldots).$$

Equations (5.2.2 a and b) now become

$$\begin{aligned}
\dot{x}_1 &= x_2\,, \\
\dot{x}_2 &= a^2 x_1 - \frac{\pi}{2}\left[x_1^2 + \sum_{k=1}^{\infty}(k+1)^2 y_{2k-1}^2\right] x_1 \\
&\quad -2\mu_2 x_2 + \frac{4}{\pi}\mu_1 \cos\omega_0 t\,,
\end{aligned} \tag{5.2.4a}$$

$$\begin{aligned}
\dot{y}_{2n-1} &= \omega_n y_{2n}\,, \\
\dot{y}_{2n} &= -\omega_n y_{2n-1} - \frac{\pi}{2}\frac{(n+1)^2}{\omega_n}\left[x_1^2 + \sum_{k=1}^{\infty}(k+1)^2 y_{2k-1}^2\right] y_{2n-1} \\
&\quad -2\mu_2 y_{2n} + 2\mu_1\left[\frac{1-(-1)^{n+1}}{\pi(n+1)\omega_n}\right]\cos\omega_0 t.
\end{aligned} \tag{5.2.4b}$$

For these equations we define the Hilbert space

$$\mathbb{Y} = \left\{ y = \{y_n\}_{n=1}^{\infty} \;\middle|\; y_n \in \mathbb{R}, \qquad \sum_{n=1}^{\infty} \omega_n^2 (y_{2n-1}^2 + y_{2n}^2) < \infty \right\}$$

with inner product $\langle u, v\rangle = \sum_{n=1}^{\infty} \omega_n^2 (u_{2n-1}v_{2n-1} + u_{2n}v_{2n})$. By a weak solution to (5.2.4) we mean a pair of functions $x_0 : \mathbb{R} \to \mathbb{R}^2$, $y_0 : \mathbb{R} \to \mathbb{Y}$ so that x_0 is differentiable and y_0 has a derivative $\dot{y}_0 \to \ell^2$, which satisfy (5.2.4a) pointwise in $\mathbb{R}^2$, (5.2.4b) pointwise in ℓ^2. Note that in this case we have

$$(u_1, u_2, \ldots) = (x, p_1, p_2, \ldots), \quad x^2 + \sum_{n=1}^{\infty} \omega_n^2 p_n^2 < \infty,$$

$$(\dot{u}_1, \dot{u}_2, \ldots) = (\dot{x}, \dot{p}_1, \dot{p}_2 \ldots) \in \ell^2$$

so that for the original differential equation (5.2.1), $u \in H^2(0,\pi) \cap H_0^1(0,\pi)$ and $\dot{u} \in L^2(0,\pi)$. This is discussed in [5]. In the next section we will formulate an abstract problem for which the hypotheses will consist of the essential features of (5.2.4). We have already mentioned one of them: when y is set equal to zero in (5.2.4a) the resulting equation is the transverse perturbation of an autonomous equation with a homoclinic solution. To see another important property we linearize (5.2.4b) in origin which yields the system of equations

$$\begin{aligned}
\dot{v}_{2n-1} &= \omega_n v_{2n}, \\
\dot{v}_{2n} &= -\omega_n v_{2n-1} - 2\mu_2 v_{2n}, \quad n \in \mathbb{N}.
\end{aligned} \tag{5.2.5}$$

Note that for each n we get a pair of equations uncoupled from the others and for $|\mu_2| < \omega_n$ we have a fundamental solution for (v_{2n-1}, v_{2n}) given by

$$V_n(t) = \begin{bmatrix} \cos\tilde{\omega}_n t + \frac{\mu_2}{\tilde{\omega}_n}\sin\tilde{\omega}_n t & \frac{\omega_n}{\tilde{\omega}_n}\sin\tilde{\omega}_n t \\ -\frac{\omega_n}{\tilde{\omega}_n}\sin\tilde{\omega}_n t & \cos\tilde{\omega}_n t - \frac{\mu_2}{\tilde{\omega}_n}\sin\tilde{\omega}_n t \end{bmatrix} e^{-\mu_2 t}$$

where $\tilde{\omega}_n = \sqrt{\omega_n^2 - \mu_2^2}$. This solution has the properties $V_n(0) = \mathbb{I}$ and

$$|V_n(t)V_n(s)^{-1}| = |V_n(t)V_n(-s)| = |V_n(t-s)| \le K e^{\mu_2(s-t)},$$

where $K > 0$ is independent of n. Using the sequence $\{V_n\}_{n=1}^{\infty}$ we can define a group $\{V_{\mu_2}(t)\}$ of bounded operators from $\mathbb{Y}$ to $\mathbb{Y}$ by

$$\begin{bmatrix} (V_{\mu_2}(t)y)_{2n-1} \\ (V_{\mu_2}(t)y)_{2n} \end{bmatrix} = V_n(t)\begin{bmatrix} y_{2n-1} \\ y_{2n} \end{bmatrix}.$$

Then $|V_{\mu_2}(t)V_{\mu_2}(s)^{-1}| \le K e^{\mu_2(s-t)}$. For $y^0 \in \mathbb{Y}$, $y(t) = V_{\mu_2}(t)y^0$ is the weak solution to (5.2.5) satisfying $y(0) = y^0$. If we retain the forcing term from (5.2.4b) we obtain the system of nonhomogeneous variational equations

$$\begin{aligned} \dot{v}_{2n-1} &= \omega_n v_{2n}\,, \\ \dot{v}_{2n} &= -\omega_n v_{2n-1} - 2\mu_2 v_{2n} + \mu_1 \nu_n \cos\omega_0 t \end{aligned}$$

where $\nu_n = \dfrac{2[1-(-1)^{n+1}]}{\pi(n+1)\omega_n}$. Here we encounter the question of resonance. In the nonresonant case, i.e. $\omega_n \neq \omega_0$, the precedent has a particular solution in $\mathbb{Y}$ with components given by

$$\begin{bmatrix} v_{2n-1}(t) \\ v_{2n}(t) \end{bmatrix} = \frac{\mu_1 \nu_n}{(\omega_n^2-\omega_0^2)^2 + 4\mu_2^2\omega_0^2}\begin{bmatrix} \omega_n(\omega_n^2-\omega_0^2)\cos\omega_0 t + 2\mu_2\omega_0\omega_n\sin\omega_0 t \\ -\omega_0(\omega_n^2-\omega_0^2)\sin\omega_0 t + 2\mu_2\omega_0^2\cos\omega_0 t \end{bmatrix}.$$

We make the existence of such a solution a separate hypothesis.

Finally, we mention other work on chaos in partial differential equations. For the complex Ginzburg-Landau equation in the near nonlinear Schrödinger regime, i.e. perturbed nonlinear Schrödinger equation, existence of homoclinic orbits is proved in [7, 16, 17], and existence of chaos is shown in [8, 18] under generic conditions. For perturbed sine-Gordon equation, existence of chaos and chaos cascade around a homoclinic tube was proved in [19–21]. For the reaction-diffusion equation, entropy study on the complexity of attractor is conducted in [22–24]. Chaotic oscillations of a linear wave equation with nonlinear boundary conditions are shown in [25]. The development of chaos and its controlling for PDEs is summarized in [26, 27].

5.2.2 Abstract Problem

Using the example in the preceding section as a model we now develop an abstract theory. Let $\mathbb{Y}$ and $\mathbb{H}$ be separable real Hilbert spaces with $\mathbb{Y} \subset \mathbb{H}$. We now consider differential equations of the form

$$\begin{aligned} \dot{x} &= f(x,y,\mu,t) = f_0(x,y) + \mu_1 f_1(x,y,\mu,t) + \mu_2 f_2(x,y,\mu,t)\,, \\ \dot{y} &= g(x,y,\mu,t) = Ay + g_0(x,y) + \mu_1 \nu \cos\omega_0 t + \mu_2 g_2(x,y,\mu) \end{aligned} \tag{5.2.6}$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{Y}$, $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$, $\nu \in \mathbb{Y}$. We make the following assumptions of (5.2.6):

(H1) $A \in L(\mathbb{Y}, \mathbb{H})$.
(H2) $f_0 \in C^4(\mathbb{R}^n \times \mathbb{Y}, \mathbb{R}^n)$, $f_1, f_2 \in C^4(\mathbb{R}^n \times \mathbb{Y} \times \mathbb{R}^2 \times \mathbb{R}, \mathbb{R}^n)$, $g_0 \in C^4(\mathbb{R}^n \times \mathbb{Y}, \mathbb{Y})$ and $g_2 \in C^4(\mathbb{R}^n \times \mathbb{Y} \times \mathbb{R}^2, \mathbb{Y})$.
(H3) f_1 and f_2 are periodic in t with period $T = 2\pi/\omega_0$.
(H4) $f_0(0,0) = 0$ and $D_2 f_0(x,0) = 0$.
(H5) The eigenvalues of $D_1 f_0(0,0)$ lie off the imaginary axis.
(H6) The equation $\dot{x} = f_0(x,0)$ has a nontrivial solution homoclinic to $x = 0$.
(H7) $g_0(x,0) = g_2(x,0,\mu) = 0$, $D_{12} g_0(0,0) = 0$ and $D_{22} g_0(x,0) = 0$.
(H8) There are constants $K > 0$, $\delta > 0$ and $b > 0$ so that when $0 \le |\mu_2| \le \delta$ the variational equation $\dot{v} = \big(A + \mu_2 D_2 g_2(0,0,0)\big) v$ has a group $\{V_{\mu_2}(t)\}$ of bounded evolution operators from $\mathbb{Y}$ to $\mathbb{Y}$ satisfying $|V_{\mu_2}(t) V_{\mu_2}(s)^{-1}| \le K \mathrm{e}^{b\mu_2(s-t)}$.
(H9) There is a constant $K > 0$ so that the nonhomogeneous variational equation $\dot{v} = [A + \mu_2 D_2 g_2(0,0,0)]\, v + \mu_1 \nu \cos\omega_0 t$ has a particular solution $\psi : \mathbb{R} \to \mathbb{Y}$ satisfying $|\psi(t)| \le K |\mu_1| |\nu|$.

By a weak solution to (5.2.6) we mean a pair of continuous functions $x_0 : \mathbb{R} \to \mathbb{R}^n$, $y_0 : \mathbb{R} \to \mathbb{Y}$ so that x_0 is differentiable and y_0 has a derivative $\dot{y}_0 : \mathbb{R} \to \mathbb{H}$, which satisfy (5.2.6) pointwise in $\mathbb{H}$. By (H8) we mean that $V_{\mu_2}(s)^{-1} = V_{\mu_2}(-s)$, $V_{\mu_2}(s) \circ V_{\mu_2}(t) = V_{\mu_2}(s+t)$, $V_{\mu_2}(0) = \mathbb{I}$ and that for $y_0 \in \mathbb{Y}$, $y(t) = V_{\mu_2}(t) y_0$ is the weak solution to $\dot{v} = [A + \mu_2 D_2 g_2(0,0,0)]\, v$ satisfying $y(0) = y_0$.

5.2.3 Chaos on the Hyperbolic Subspace

The reduced system of equations for (5.2.6) is

$$\dot{x} = f(x,0,\mu,t) = f_0(x,0) + \mu_1 f_1(x,0,\mu,t) + \mu_2 f_2(x,0,\mu,t) \tag{5.2.7}$$

with $x \in \mathbb{R}^n$. By (H6), (5.2.7) has a nontrivial homoclinic solution γ when $\mu = 0$. The variational equation along γ is the linear equation $\dot{u} = D_1 f_0(\gamma, 0) u$ and its adjoint variational equation

$$\dot{v} = -D_1 f_0(\gamma,0)^* v\,. \tag{5.2.8}$$

By repeating arguments of Section 4.2.2, we have the following result (cf Theorem 4.2.1).

Theorem 5.2.1. *Let M be as in* (4.2.6) *or* (4.2.7) *and suppose μ_0, α_0, β_0 are such that $M(\mu_0,\alpha_0,\beta_0)=0$ and $D_{(\alpha,\beta)}M(\mu_0,\alpha_0,\beta_0)$ is nonsingular. Then there exists an interval $J=(0,\xi_0]$ so that for each $\xi\in J$ the equation $\dot{x}=f(x,0,\xi\mu_0,t)$ has a homoclinic solution γ_ξ to a small hyperbolic periodic solution. Furthermore, γ_ξ depends continuously on ξ, $\lim_{\xi\to 0}\gamma_\xi(t)=\gamma(t-\alpha_0)$ (or $=\gamma_{\beta_0}(t-\alpha_0)$, respectively) uniformly in t and the variational equation along γ_ξ has an exponential dichotomy on $\mathbb{R}$.*

Then we can show chaos for the differential equation $\dot{x}=f(x,0,\xi\mu_0,t)$. For this, first, for any $m\in\mathbb{N}$, $\xi\in J$ and $E=\{e_j\}_{j\in\mathbb{Z}}\in\mathscr{E}$ (cf Section 2.5.2) define the function $\gamma_{\xi,E,m}\in L^\infty(\mathbb{R},\mathbb{R}^n)$ by

$$\gamma_{\xi,E,m}(t)=\begin{cases}\gamma_\xi(t-2jmT), & \text{if}\quad (2j-1)mT<t\le(2j+1)mT \quad\text{and}\quad e_j=1,\\ 0, & \text{if}\quad (2j-1)mT<t\le(2j+1)mT \quad\text{and}\quad e_j=0.\end{cases}$$

Now following arguments of Sections 3.5.2 and 5.1.4, we obtain the following version of Smale-Birkhoff homoclinic theorem 2.5.4.

Theorem 5.2.2. (a) *Let μ_0, α_0, β_0, ξ_0 be as in Theorem 5.2.1. Fix $\xi\in(0,\xi_0]$ and let γ_ξ be obtained from Theorem 5.2.1. Then there exist an $\varepsilon_0>0$ and a function $\varepsilon\to M(\varepsilon)\in\mathbb{N}$ so that given ε with $0<\varepsilon\le\varepsilon_0$ and a positive integer $m\ge M(\varepsilon)$ the equation $\dot{x}=f(x,0,\xi\mu_0,t)$ has for each $E\in\mathscr{E}$ a unique solution $t\to x_E(t)$ satisfying*

$$|x_E(t)-\gamma_{\xi,E,m}(t)|\le\varepsilon,\quad \forall t\in\mathbb{R}.$$

(b) *x_E depends continuously on E and $x_E(t+2mT)=x_{\sigma(E)}(t)$ where σ is the Bernoulli shift on $\mathscr{E}$.*

(c) *The correspondence $\phi(E)=x_E(0)$ is a homeomorphism of $\mathscr{E}$ onto the compact subset Λ of $\mathbb{R}^n$ given by*

$$\Lambda:=\{x_E(0)\mid E\in\mathscr{E}\}$$

for which the $2m$th iterate F^{2m} of the period map F of (5.2.7) *is invariant and satisfies $F^{2m}\circ\phi=\phi\circ\sigma$.*

Theorem 5.2.2 asserts that the following diagram is commutative.

$$\begin{array}{ccc}\mathscr{E} & \xrightarrow{\quad\sigma\quad} & \mathscr{E}\\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\phi}\\ \Lambda & \xrightarrow[\quad F^{2m}\quad]{} & \Lambda\end{array}$$

This means that $F^{2m}:\Lambda\mapsto\Lambda$ has the same dynamics on Λ as the Bernoulli shift σ on $\mathscr{E}$. Consequently, F^{2m} is chaotic on Λ, so (5.2.7) is also chaotic. This construction is sometimes referred to as embedding a Smale horseshoe in the flow of the differential equation.

5.2.4 Chaos in the Full Equation

Since the homoclinic orbit γ_ξ obtained in Section 5.2.3 is hyperbolic the variational equation $\dot{u} = D_1 f(\gamma_\xi, 0, \xi\mu_0, t)u$ has an exponential dichotomy on $\mathbb{R}$ with constant K_ξ. Now, by Section 4.2.3, K_ξ tends to infinity as $\xi \to 0$. For this reason we consider the following modification of (5.2.6)

$$\begin{aligned} \dot{x} &= f(x,y,\mu,\lambda,t) := f(x,\lambda y,\mu,t)\,, \\ \dot{y} &= g(x,y,\mu,\lambda,t) := Ay + g_0(x,y) + \lambda\mu_1 \nu\cos\omega_0 t + \mu_2 g_2(x,y,\mu) \end{aligned} \tag{5.2.9}$$

for a parameter $\lambda \in [0,1]$. Now let $(\mu_0, \alpha_0, \beta_0)$ with $\mu_{0,2} \neq 0$ and γ_ξ be as in Theorem 5.2.1. Following the arguments of Section 4.2.3, we obtain a constant $\bar{\xi}_0$ and for each $\xi \in (0, \bar{\xi}_0]$ a homoclinic orbit

$$\Gamma(\lambda,\xi)(t) = \big(\Gamma_1(\lambda,\xi)(t), \Gamma_2(\lambda,\xi)(t)\big)$$

for (5.2.9) with $\mu = \xi\mu_0$ so that

$$\begin{aligned} \Gamma_1(\lambda,\xi)(t) \to \gamma(t-\alpha_0) \quad &(\text{or} \to \gamma_{\beta_0}(t-\alpha_0), \text{ respectively})\,, \\ \text{and} \quad &\Gamma_2(\lambda,\xi)(t) \to 0 \end{aligned}$$

as $\xi \to 0$ uniformly for $\lambda \in [0,1]$. Moreover, we have $\Gamma(0,\xi) = (\gamma_\xi, 0)$ and $\Gamma(1,\xi)$ is a homoclinic solution for (5.2.6). The linearization of (5.2.9) with $\mu = \xi\mu_0$ along $\Gamma(\lambda,\xi)(t)$ has an exponential dichotomy on $\mathbb{R}$ with dichotomy constants uniformly with respect to $0 \le \lambda \le 1$ and fixed ξ. Analogous to the construction in Section 5.2.3, for each $E \in \mathscr{E}$, $\xi \in (0, \bar{\xi}_0]$ and $m \in \mathbb{N}$ we construct from $\Gamma(\lambda,\xi)$ a corresponding

$$\Gamma_E(\lambda,\xi,m) = (\Gamma_{1,E}(\lambda,\xi,m), \Gamma_{2,E}(\lambda,\xi,m))\,.$$

Similarly, from γ_ξ we obtain $\gamma_{\xi,E,m}$. Then we have $\Gamma_{1,E}(0,\xi,m) = \gamma_{\xi,E,m}$ and also $\Gamma_{2,E}(0,\xi,m) = 0$. Using the uniform exponential dichotomy, following Sections 3.5.2 and 5.1.4, we now obtain the following extension of Theorem 5.2.2.

Theorem 5.2.3. (a) *Let μ_0, α_0, β_0 be as in Theorem 5.2.1 with $\mu_{0,2} \neq 0$. Fix $\xi \in (0, \bar{\xi}_0]$ and let $\Gamma(\lambda,\xi,m)(t)$ be obtained above. Then there exist an $\bar{\varepsilon}_0 > 0$ and a function $\varepsilon \to \bar{M}(\varepsilon) \in \mathbb{N}$ so that given ε with $0 < \varepsilon \le \bar{\varepsilon}_0$ and a positive integer $m \ge \bar{M}(\varepsilon)$ Eq.* (5.2.9) *with $\mu = \xi\mu_0$ has for each $E \in \mathscr{E}$ a unique weak solution $t \to \big(x_{E,\lambda}(t), y_{E,\lambda}(t)\big)$ satisfying*

$$|x_{E,\lambda}(t) - \Gamma_{1,E}(\lambda,\xi,m)(t)| + |y_{E,\lambda}(t) - \Gamma_{2,E}(\lambda,\xi,m)(t)| \le \varepsilon \quad \forall t \in \mathbb{R}\,.$$

(b) *The functions* $\big(x_{E,\lambda}(t), y_{E,\lambda}(t)\big)$ *depend continuously on E,* λ *and we also have* $x_{E,\lambda}(t+2mT) = x_{\sigma(E),\lambda}(t)$, $y_{E,\lambda}(t+2mT) = y_{\sigma(E),\lambda}(t)$.

(c) *The correspondence* $\phi_\lambda(E) = \big(x_{E,\lambda}(0), y_{E,\lambda}(0)\big)$ *is a homeomorphism of* $\mathscr{E}$ *onto the compact subset* Λ_λ *of* $\mathbb{R}^n \times \mathbb{Y}$ *given by*

$$\Lambda_\lambda := \big\{ \big(x_{E,\lambda}(0), y_{E,\lambda}(0)\big) \mid E \in \mathscr{E} \big\}$$

for which the 2*mth iterate* F_λ^{2m} *of the period map* F_λ *of* (5.2.9) *is invariant and satisfies* $F_\lambda^{2m} \circ \phi_\lambda = \phi_\lambda \circ \sigma$.

(d) $(x_{E,0}(t), y_{E,0}(t)) = (x_E(t), 0)$ *and* $\phi_0 = \phi$ *where* ϕ *is as in Theorem* 5.2.2.

In summary, we obtain the following main result.

Theorem 5.2.4. *Suppose (H*1*)–(H*9*) hold. Let M be as in* (4.2.6) *or* (4.2.7) *and suppose* $(\mu_0, \alpha_0, \beta_0)$ *are such that* $M(\mu_0, \alpha_0, \beta_0) = 0$ *and* $D_{(\alpha,\beta)}M(\mu_0, \alpha_0, \beta_0)$ *is nonsingular. Then there exists* $\bar{\xi}_0 > 0$ *so that if* $0 < \xi \leq \bar{\xi}_0$*, if the parameters in* (5.2.6) *are given by* $\mu = \xi\mu_0$*, and* $\mu_{0,2} \neq 0$ *then there exists a homeomorphism,* ϕ_1*, of* $\mathscr{E}$ *onto a compact subset of* $\mathbb{R}^n \times \mathbb{Y}$ *for which the* 2*mth iterate,* F_1^{2m}*, of the period map* F_1 *of* (5.2.6) *is invariant and satisfies* $F_1^{2m} \circ \phi_1 = \phi_1 \circ \sigma$*. Here* $m \in \mathbb{N}$ *is sufficiently large.*

We might paraphrase Theorem 5.2.4, loosely, say, the Smale horseshoe embedded in the flow of the reduced equation (5.2.7) is shadowed by a horseshoe in the full equation (5.2.6).

5.2.5 Applications to Vibrating Elastic Beams

We now return to the example in Section 5.2.1 and apply our theory to the problem of vibrating elastic beams. We shall consider a number of different cases and generalizations. In each case our procedure will be:

(i) Use a Galerkin expansion to convert the partial differential equation to an infinite set of ordinary differential equations as (5.2.6).
(ii) Truncate the equation to get the finite problem(5.2.7).
(iii) Apply Theorem 5.2.2 to getting a Smale horseshoe for the finite problem. For this we must verify (H1) through (H6).
(iv) Use Theorem 5.2.4 to lift the horseshoe to the flow of the original partial differential differential equation. This requires (H7)–(H9).

5.2.6 Planer Motion with One Buckled Mode

The boundary value problem for planer deflections of an elastic beam with a compressive axial load P_0 and pinned ends is

$$\ddot{u} = -u'''' - P_0 u'' + \left[\int_0^\pi u'(s)^2\, ds\right] u'' - 2\mu_2 \dot{u} + \mu_1 \cos \omega_0 t,$$
$$u(0,t) = u(\pi,t) = u''(0,t) = u''(\pi,t) = 0$$

where $u(x,t)$ is the transverse deflection at a distance x from one end at time t. We consider the μ_i terms as perturbations. Our first step is to consider the linearized, unperturbed problem. We compute the eigenvalues in origin to be $\lambda_n = n^2(n^2 - P_0)$ with corresponding eigenfunctions $\varphi_n(x) = \sin nx$ for $n = 1, 2, \ldots$. For small P_0 the origin is a center. As P_0 is increased the first bifurcation occurs at $P_0 = 1$, the first *Euler buckling load*. The corresponding eigenfunction, $\varphi_1(x) = \sin x$, is referred to as the first *buckled mode*. The second bifurcation occurs at $P_0 = 4$. Thus, the simplest case, which we now consider, consists of $1 < P_0 < 4$. In the first equation we define

$$a^2 = \lambda_1 = P_0 - 1\,.$$

The eigenvalues for the center modes, or unbuckled modes, provide the frequencies used in (5.2.6) as we define

$$\omega_{n-1}^2 = \lambda_n = n^2[n^2 - P_0], \quad n = 2, 3, \ldots.$$

We now use the eigenfunctions for the Galerkin expansion $u(x,t) = \sum_{k=1}^{\infty} u_k(t) \sin kx$ and obtain the system of equations

$$\begin{aligned} \ddot{u}_n = n^2(P_0 - n^2)u_n - \frac{\pi}{2} n^2 \left[\sum_{k=1}^{\infty} k^2 u_k^2\right] u_n \\ -2\mu_2 \dot{u}_n + 2\mu_1 \left[\frac{1-(-1)^n}{\pi n}\right] \cos \omega_0 t, \quad n = 1, 2, \ldots. \end{aligned} \tag{5.2.10}$$

To obtain a first order system as in (5.2.6) we define

$$x = (u_1, \dot{u}_1), \quad y = (u_2, \dot{u}_2/\omega_1, u_3, \dot{u}_3/\omega_2, \ldots).$$

The reduced equations are

$$\begin{aligned} \dot{x}_1 &= x_2\,, \\ \dot{x}_2 &= a^2 x_1 - \frac{\pi}{2} x_1^3 - 2\mu_2 x_2 + \frac{4}{\pi} \mu_1 \cos \omega_0 t \end{aligned} \tag{5.2.11}$$

obtained by setting $y = 0$ in the hyperbolic part. When $\mu = 0$, (5.2.11) has a homoclinic solution given by $\gamma = (r, \dot{r})$ where $r(t) = (2a/\sqrt{\pi})\,\mathrm{sech}\, at$. Equation (5.2.8) becomes

$$\dot{v}_1 = -(a^2 - \frac{3\pi}{2} r^2) v_2, \qquad \dot{v}_2 = -v_1$$

with solution $(v_1, v_2) = (-\ddot{r}, \dot{r})$. We have $d = 1$ so the variable β does not appear, M is a scalar function, and the function $M = M_1$ becomes

$$M(\alpha) = \left[\frac{8\omega_0}{\sqrt{\pi}} \sin \omega_0 \alpha \operatorname{sech} \frac{\pi\omega_0}{2a}\right] \mu_1 - \left(\frac{16a^3}{3\pi}\right) \mu_2.$$

Thus, the conditions $M(\mu_0, \alpha_0) = 0$, $(\partial M/\partial \alpha)(\mu_0, \alpha_0) \neq 0$ are satisfied for all μ_0 so that $\left|\frac{\mu_{0,2}}{\mu_{0,1}}\right| < \frac{3\sqrt{\pi}\omega_0}{2a^3} \operatorname{sech} \frac{\pi\omega_0}{2a}$. Now we check condition (H9) which, for the present problem, requires us to consider the equation

$$\dot{v}_{2n-1} = \omega_n v_{2n},$$
$$\dot{v}_{2n} = -\omega_n v_{2n-1} - 2\mu_2 v_{2n} + \mu_1 \nu_n \cos \omega_0 t$$

where $\nu_n = \frac{2[1-(-1)^{n-1}]}{\pi(n+1)\omega_n}$. This system has a particular solution in $\mathbb{Y}$ with components given by

$$\begin{bmatrix} v_{2n-1}(t) \\ v_{2n}(t) \end{bmatrix} = \frac{\mu_1 \nu_n}{(\omega_n^2 - \omega_0^2)^2 + 4\mu_2^2 \omega_0^2} \begin{bmatrix} \omega_n(\omega_n^2 - \omega_0^2) \cos \omega_0 t + 2\mu_2 \omega_0 \omega_n \sin \omega_0 t \\ -\omega_0(\omega_n^2 - \omega_0^2) \sin \omega_0 t + 2\mu_2 \omega_0^2 \cos \omega_0 t \end{bmatrix}.$$

From this we see that (H9) is satisfied whenever $\omega_0 \neq \omega_n$ for all n.

We note that while the conditions $M(\alpha) = 0$, $M'(\alpha) \neq 0$ can be satisfied with $\mu_2 = 0$, $\alpha = 0$ we require $\mu_2 \neq 0$ in Section 5.2.4 where we use a weak exponential dichotomy to lift the full equation. Thus, we obtain the following result using Theorem 5.2.4.

Theorem 5.2.5. *If $\omega_0 \neq \omega_n$ for all n then whenever μ_0 satisfies $\mu_{0,1} \neq 0$ and*

$$0 < \left|\frac{\mu_{0,2}}{\mu_{0,1}}\right| < \frac{3\sqrt{\pi}\omega_0}{2a^3} \operatorname{sech} \frac{\pi\omega_0}{2a}, \tag{5.2.12}$$

there exists a corresponding $\bar{\xi}_0 > 0$ so that if $0 < \xi \leq \bar{\xi}_0$, if the parameters in (5.2.10) *are given by $\mu = \xi\mu_0$ then there exists a compact subset of $\mathbb{R}^2 \times \mathbb{Y}$ on which the 2mth iterate, F^{2m}, of the period map F of* (5.2.10) *is invariant and conjugate to the Bernoulli shift on $\mathscr{E}$. Here $m \in \mathbb{N}$ is sufficiently large.*

These results are stated in terms of the Galerkin equations (5.2.10) but they can be transferred back to the original partial differential equation. In this case we get a Bernoulli shift embedded in $\left[H_0^1(0,\pi) \cap H^2(0,\pi)\right] \times L^2(0,\pi)$. This is discussed in [5]. In the μ_1-μ_2 plane we get from the condition (5.2.12) four small open wedge-shaped regions of parameter values for which the partial differential equation exhibits chaos (Figure 5.3). These regions are bounded by the lines $\mu_1/\mu_2 = \pm\frac{3\sqrt{\pi}\omega_0}{2a^3} \operatorname{sech} \frac{\pi\omega_0}{2a}$ and $\mu_2 = 0$.

It is interesting to look at some history of this problem. The first work was done in [28] in which the author started with the PDE and carried out the Galerkin expansion but restricted his analysis to the reduced equation (5.2.11). The significance of that work is that it introduced the idea of Melnikov analysis. In subsequent work [3], the results are extended to infinite dimension but the Galerkin approach is abandoned in favor of nonlinear semigroup techniques directly in infinite dimensions. In our

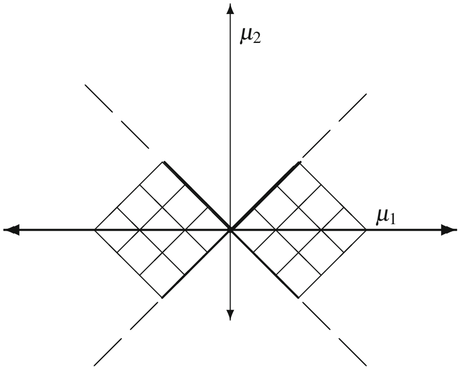

Fig. 5.3 The chaotic open wedge-shaped region of (5.2.10) in $\mathbb{R}^2$.

section we go back to the original, simpler analysis of the reduced equation and then show that the results apply to the original PDE. Some advantages of this are that the Galerkin projection is a technique familiar to many engineers and physicists and, also, we are able to utilize our general Melnikov results in Section 5.2.3. This is illustrated further in the generalizations to follow. We note that Equation (5.2.10) was treated also in [4].

5.2.7 Nonplaner Symmetric Beams

Let us consider a beam with symmetric cross section, pinned ends and compressive axial load P_0 and assume now that the beam is not constrained to defect in a plane. If $u(x,t)$ and $w(x,t)$ denote the transverse defections at position x and time t we obtain the following boundary value problem.

$$\begin{aligned}
\ddot{u} &= -u'''' - P_0 u'' + \left[\int_0^\pi \left(u'(s)^2 + w'(s)^2\right) ds\right] u'' \\
&\quad -2\mu_2 \dot{u}\cos\eta + \mu_1 \cos\zeta \cos\omega_0 t\,, \\
\ddot{w} &= -w'''' - P_0 w'' + \left[\int_0^\pi \left(u'(s)^2 + w'(s)^2\right) ds\right] w'' \\
&\quad -2\mu_2 \dot{w}\sin\eta + \mu_1 \sin\zeta \cos\omega_0 t\,, \\
u(0,t) &= u(\pi,t) = u''(0,t) = u''(\pi,t) = w(0,t) \\
&= w(\pi,t) = w''(0,t) = w''(\pi,t) = 0
\end{aligned}$$

where η, ζ are constants. The parameters μ_1, μ_2 represent the coefficients of, respectively, total transverse forcing and total viscous damping. These effects are distributed between the two directions of motion. The quantity $\tan\zeta$ represents the ratio of forcing in the u-direction to forcing in the w-direction while $\tan\eta$ plays the same role in the damping. We suppose $\eta,\zeta \in (0,\pi/2)$ in order to avoid certain degeneracies. In these equations we use the Galerkin expansions

$$u(x,t)=\sum_{k=1}^{\infty} u_k(t)\sin kx\,, \quad w(x,t)=\sum_{k=1}^{\infty} w_k(t)\sin kx$$

and proceed as before. This yields the system of equations

$$\begin{aligned}
\ddot{u}_n &= n^2(P_0-n^2)u_n-\frac{\pi}{2}n^2\left[\sum_{k=1}^{\infty} k^2(u_k^2+w_k^2)\right]u_n \\
&\quad -2\mu_2\dot{u}_n\cos\eta+2\mu_1\cos\zeta\left[\frac{1-(-1)^n}{\pi n}\right]\cos\omega_0 t\,, \\
\ddot{w}_n &= n^2(P_0-n^2)w_n-\frac{\pi}{2}n^2\left[\sum_{k=1}^{\infty} k^2(u_k^2+w_k^2)\right]w_n \\
&\quad -2\mu_2\dot{w}_n\sin\eta+2\mu_1\sin\zeta\left[\frac{1-(-1)^n}{\pi n}\right]\cos\omega_0 t\,.
\end{aligned} \tag{5.2.13}$$

As before, we assume $1<P_0<4$ and define $a^2=P_0-1$, $\omega_{n-1}^2=n(n^2-P_0)$, $n=2,3,\ldots$. Equations (5.2.13) take the form of (5.2.6) when we define $x=(u_1,\dot{u}_1,w_1,\dot{w}_1)$ and $y=(u_2,\dot{u}_2/\omega_1,w_2,\dot{w}_2/\omega_1,u_3,\dot{u}_3/\omega_2,w_3,\dot{w}_3/\omega_2,\ldots)$. The reduced equations are

$$\begin{aligned}
\dot{x}_1 &= x_2\,, \\
\dot{x}_2 &= a^2x_1-\frac{\pi}{2}(x_1^2+x_3^2)x_1-2\mu_2x_2\cos\eta+\frac{4}{\pi}\mu_1\cos\zeta\cos\omega_0 t\,, \\
\dot{x}_3 &= x_4\,, \\
\dot{x}_4 &= a^2x_3-\frac{\pi}{2}(x_1^2+x_3^2)x_3-2\mu_2x_4\sin\eta+\frac{4}{\pi}\mu_1\sin\zeta\cos\omega_0 t\,.
\end{aligned}$$

When $\mu=0$ we have a two-dimensional homoclinic manifold given by $\gamma_\beta=(r\cos\beta,\dot{r}\cos\beta,r\sin\beta,\dot{r}\sin\beta)$ where, as before, $r(t)=(2a/\sqrt{\pi})\,\mathrm{sech}\,at$ and β is a parameter. The adjoint equations (5.2.8) take the form

$$\begin{aligned}
\dot{v}_1 &= \left[-a^2+\frac{\pi}{2}(3r^2\cos^2\beta+r^2\sin^2\beta)\right]v_2+\left(\pi r^2\sin\beta\cos\beta\right)v_4\,, \\
\dot{v}_2 &= -v_1\,, \\
\dot{v}_3 &= \left(\pi r^2\sin\beta\cos\beta\right)v_2+\left[-a^2+\frac{\pi}{2}(r^2\cos^2\beta+3r^2\sin^2\beta)\right]v_4\,, \\
\dot{v}_4 &= -v_3\,.
\end{aligned}$$

A one-parameter family of bounded solutions to these equations is given by

$$v_{\beta 1} = (-\dot{r}\sin\beta, r\sin\beta, \dot{r}\cos\beta, -r\cos\beta),$$
$$v_{\beta 2} = (-\ddot{r}\cos\beta, \dot{r}\cos\beta, -\ddot{r}\sin\beta, \dot{r}\sin\beta) \tag{5.2.14}$$

and the function, M, as in (4.2.7) becomes

$$M_1(\mu,\alpha,\beta) = \left[\frac{8}{\sqrt{\pi}}\sin(\beta-\zeta)\cos\omega_0\alpha\,\mathrm{sech}\,\frac{\pi\omega_0}{2a}\right]\mu_1,$$

$$M_2(\mu,\alpha,\beta) = \left[\frac{8\omega_0}{\sqrt{\pi}}\cos(\beta-\zeta)\sin\omega_0\alpha\,\mathrm{sech}\,\frac{\pi\omega_0}{2a}\right]\mu_1 - \left[\frac{16a^3(\cos\eta\cos^2\beta+\sin\eta\sin^2\beta)}{3\pi}\right]\mu_2.$$

Next, the conditions $M(\mu_0,\alpha_0,\beta_0)=0$, $D_{(\alpha,\beta)}M(\mu_0,\alpha_0,\beta_0)$ nonsingular are satisfied in two different cases. Of course, we suppose $\mu_{0,1}\neq 0$, $\mu_{0,2}\neq 0$ and then put $\lambda_0 = \frac{\mu_{0,2}}{\mu_{0,1}}$. We have the following two cases:

Case 1. We can choose either $\beta_0=\zeta$ and then look for a simple root of the equation

$$\lambda_0 = m_1\sin\omega_0\alpha, \tag{5.2.15}$$

or $\beta_0=\zeta+\pi$ and look for a simple root of the equation

$$\lambda_0 = -m_1\sin\omega_0\alpha \tag{5.2.16}$$

for

$$m_1 = \frac{3\sqrt{\pi}\omega_0}{2a^2(\cos\eta\cos^2\zeta+\sin\eta\sin^2\zeta)}\,\mathrm{sech}\,\frac{\pi\omega_0}{2a}.$$

Supposing under the condition

$$0<|\lambda_0|<m_1, \tag{5.2.17}$$

there is a simple root α_0 of (5.2.15). Similarly, (5.2.16) has also a simple root $-\alpha_0$. According to the formulas (5.2.14) for v_{β_1} and v_{β_2}, these simple roots (ζ,α_0) and $(\zeta+\pi,-\alpha_0)$ give two different solutions of (5.2.13).

Case 2. We begin from choosing $\omega_0\alpha_0=(2k_0+1)\frac{\pi}{2}$ for $k_0\in\{0,1\}$ and then we look for a simple root $\beta_0\neq\zeta+k\pi$, $\forall k\in\mathbb{Z}$ of

$$\lambda_0 = (-1)^{k_0}\Phi(\beta) \tag{5.2.18}$$

where

$$\Phi(\beta) = \frac{3\omega_0\sqrt{\pi}}{2a^3}\frac{\cos(\beta-\zeta)}{\cos\eta\cos^2\beta+\sin\eta\sin^2\beta}\,\mathrm{sech}\,\frac{\pi\omega_0}{2a}.$$

Let $m_2 = \max_{\beta\in\mathbb{R}} \Phi(\beta)$. A computation of the constant m_2 is discussed in [29]. Since $\Phi(\beta+\pi) = -\Phi(\beta)$, the range of Φ is the closed interval $[-m_2, m_2]$. We now split this case into two parts:

Part 2A). For $\eta = \pi/4$ we get $\Phi(\beta) = m_1\cos(\beta-\zeta)$, along with $m_2 = m_1 = \dfrac{3\omega_0\sqrt{\pi}}{\sqrt{2}a^3}\operatorname{sech}\dfrac{\pi\omega_0}{2a}$. Equation (5.2.18) has now the form

$$(-1)^{k_0}\frac{3\omega_0\sqrt{\pi}}{\sqrt{2}a^3}\operatorname{sech}\frac{\pi\omega_0}{2a}\cos(\beta-\zeta) = \lambda_0\,,$$

so under condition (5.2.17), there is a simple root β_0 different from $\zeta + k\pi$, $\forall k\in\mathbb{Z}$. This holds for both cases $k_0\in\{0,1\}$ so we have two different solutions of (5.2.13). In addition, the results of Case 1 still apply here. Thus, in this situation, we have in the μ_1-μ_2 plane four wedged-shaped regions of parameter values bounded by $\mu_2/\mu_1 = \pm m_1$, $\mu_2 = 0$ for which the partial differential equation exhibits chaos. Particularly, (5.2.13) has four distinct homoclinic solutions, two from Case 1, two from Case 2A. These regions are labeled *II* in Figure 5.4. In this case there are no regions labeled *I*.

Part 2B). For $\eta \neq \pi/4$ we get $\Phi'(\zeta)\neq 0$, so $m_1 < m_2$. Certainly for the solvability of (5.2.18) we need $|\lambda_0| \le m_2$. Now we claim:

Lemma 5.2.6. *If*

$$\lambda_0 \in (-m_2, m_2)\setminus\{\pm m_1, 0\}\,, \tag{5.2.19}$$

then Eq. (5.2.18) *has a simple root* $\beta_0\in[0,2\pi]\setminus\{\zeta,\zeta+\pi\}$.

Proof. Assume to the contrary that (5.2.18) has no simple roots for a $\lambda_0\in(-m_2, m_2)\setminus\{\pm m_1, 0\}$. Then there are $0\le\beta_1<\beta_2\le 2\pi$ so that

$$\Phi(\beta_{1,2}) = (-1)^{k_0}\lambda_0,\quad \Phi'(\beta_{1,2}) = 0,\quad \Phi''(\beta_{1,2}) = 0\,. \tag{5.2.20}$$

Note that $\beta_{1,2}\neq\zeta+k\pi$ and $\beta_{1,2}\neq\zeta+\frac{2k+1}{2}\pi$, $\forall k\in\{0,1\}$. After some calculation we derive from (5.2.20) that $\cos 2\beta_{1,2}\neq 0$, $\sin 2\beta_{1,2}\neq 0$ and that (5.2.20) is equivalent to

$$\begin{aligned}\frac{\cos(\beta_{1,2}-\zeta)}{\cos\eta\cos^2\beta_{1,2}+\sin\eta\sin^2\beta_{1,2}} &= \frac{\sin(\beta_{1,2}-\zeta)}{(\cos\eta-\sin\eta)\sin 2\beta_{1,2}}\\ = \frac{\cos(\beta_{1,2}-\zeta)}{2(\cos\eta-\sin\eta)\cos 2\beta_{1,2}} &= (-1)^{k_0}\frac{2a^3}{3\omega_0\sqrt{\pi}}\cosh\frac{\pi\omega_0}{2a}\lambda_0\,.\end{aligned} \tag{5.2.21}$$

From (5.2.21) we derive

$$\cos 2\beta_{1,2} = \frac{\cos\eta+\sin\eta}{3(\cos\eta-\sin\eta)},\quad 2\tan(\beta_{1,2}-\zeta) = \tan 2\beta_{1,2}\,. \tag{5.2.22}$$

Hence

$$\beta_2 \in \{\pi - \beta_1, \pi + \beta_1, 2\pi - \beta_1\} .$$

If $\beta_2 = \pi - \beta_1$ then from $2\tan(\beta_2 - \zeta) = \tan 2\beta_2$ we get $2\tan(\beta_1 + \zeta) = \tan 2\beta_1$, but $2\tan(\beta_1 - \zeta) = \tan 2\beta_1$, so $\tan(\beta_1 + \zeta) = \tan(\beta_1 - \zeta)$, i.e. $\zeta = k\pi/2$, $k \in \{0,1\}$. This contradicts $\zeta \in (0,\pi/2)$. If $\beta_2 = \pi + \beta_1$ then

$$(-1)^{k_0}\lambda_0 = \Phi(\beta_2) = \Phi(\beta_1 + \pi) = -\Phi(\beta_1) = (-1)^{k_0+1}\lambda_0$$

which implies $\lambda_0 = 0$, a contradiction. If $\beta_2 = 2\pi - \beta_1$ then again we derive $\tan(\beta_1 + \zeta) = \tan(\beta_1 - \zeta)$, so that $\zeta = k\pi/2$, $k \in \{0,1\}$, a contradiction to $\zeta \in (0,\pi/2)$. The proof is finished. □

Note that $\beta_0 \in \{\zeta, \zeta + \pi\}$ for the Case 1, while $\beta_0 \in [0,2\pi) \setminus \{\zeta, \zeta + \pi\}$ for the Case 2. Lemma 5.2.6 can be applied to both cases $\alpha_0 = \frac{\pi}{2\omega_0}(2k_0 + 1)$, $k_0 \in \{0,1\}$, so Part 2B yields, in the μ_1-μ_2 plane, four wedge-shaped regions of parameter values bounded by $\mu_2/\mu_1 = \pm m_2$, $\mu_2/\mu_1 = \pm m_1$, $\mu_2 = 0$ for which (5.2.13) has two different homoclinic solutions. These regions are labeled I in Figure 5.4. Note that we have four different solutions of (5.2.13) in regions labeled II, since there Case 1 can be also applied (see (5.2.15) and (5.2.16)). This completes the analysis of the Melnikov function. We now check about resonance. Because in the present problem all coupling terms are nonlinear, the linear equation in (H9) consists in two copies of the system of equations in the preceding example. This yields the following result obtained from Theorem 5.2.4.

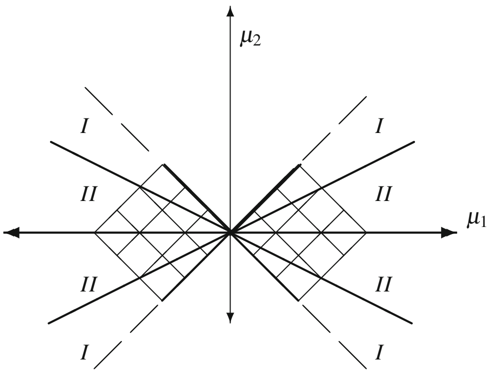

Fig. 5.4 The chaotic wedge-shaped regions of (5.2.13) in $\mathbb{R}^2$.

Theorem 5.2.7. *Suppose* $\omega_0 \neq \omega_n$ *for all* n *and let* m_1, m_2 *be as above.*

(a) *If* $m_0 \neq 0$ *satisfies one but not both of* $|m_0| < m_i$ *then if* $\mu_{0,2}/\mu_{0,1} = m_0$ *there exists a corresponding* $\bar{\xi}_0 > 0$ *so that if* $0 < \xi \leq \bar{\xi}_0$*, if the parameters in* (5.2.13) *are given by* $\mu = \xi\mu_0$ *then there exist two homoclinic orbits which can be used to construct a compact subset of* $\mathbb{R}^4 \times \mathbb{Y}$ *on which the* $2m$*th iterate,* F^{2m}*, of the period* *map F of* (5.2.13) *is invariant and conjugate to the Bernoulli shift on* $\mathscr{E}$. *Here* $m \in \mathbb{N}$ *is sufficiently large.*

(b) *If* $m_0 \neq 0$ *satisfies each of* $|m_0| < m_i$ *then there are four homoclinic orbits as in (i).*

In summary, we obtain eight open small wedge-shaped regions of parameter values in the μ_1-μ_2 plane bounded by the lines $\mu_2/\mu_1 = \pm m_1$, $\mu_2/\mu_1 = \pm m_2$ and $\mu_2 = 0$ with $m_1 \leq m_2$ for which the partial differential equation exhibits chaos (Figure 5.4). In the regions labeled I there are two homoclinics while in regions II there exist four. It is interesting to note that in this case, by adjusting the parameters η and ζ, it is possible to make the size of the wedge arbitrarily close to filling the μ_1-μ_2 plane.

5.2.8 Nonplaner Nonsymmetric Beams

For the case of a nonsymmetric beam with nonplaner motion we have the boundary value problem

$$\begin{aligned}\ddot{u} &= -u'''' - P_0 u'' + \left[\int_0^\pi \left(u'(s)^2 + w'(s)^2\right) ds\right] u'' \\ &\quad -2\mu_2 \dot{u} \cos\eta + \mu_1 \cos\zeta \cos\omega_0 t\,, \\ \ddot{w} &= -R^2 w'''' - P_0 w'' + \left[\int_0^\pi \left(u'(s)^2 + w'(s)^2\right) ds\right] w'' \\ &\quad -2\mu_2 \dot{w} \sin\eta + \mu_1 \sin\zeta \cos\omega_0 t\,,\end{aligned}$$

$$\begin{aligned}u\ (0,t) &= u(\pi,t) = u''(0,t) = u''(\pi,t) \\ &= w(0,t) = w(\pi,t) = w''(0,t) = w''(\pi,t) = 0\end{aligned}$$

where R^2 is constant representing the stiffness ratio for the two directions. We assume that $R > 1$ which amounts to choosing w as the direction with stiffer cross-section. Note that $R = 1$ reduces to Section 5.2.7. As before we assume that η, $\zeta \in (0, \pi/2)$. The Galerkin expansion becomes

$$\begin{aligned}\ddot{u}_n =&\ n^2(P_0 - n^2)u_n - \frac{\pi}{2}n^2 \left[\sum_{k=1}^{\infty} k^2(u_k^2 + w_k^2)\right] u_n \\ &- 2\mu_2 \dot{u}_n \cos\eta + 2\mu_1 \cos\zeta \left[\frac{1-(-1)^n}{\pi n}\right] \cos\omega_0 t\,, \\ \ddot{w}_n =&\ n^2(P_0 - n^2 R^2)w_n - \frac{\pi}{2}n^2 \left[\sum_{k=1}^{\infty} k^2(u_k^2 + w_k^2)\right] w_n \\ &- 2\mu_2 \dot{w}_n \sin\eta + 2\mu_1 \sin\zeta \left[\frac{1-(-1)^n}{\pi n}\right] \cos\omega_0 t\,.\end{aligned} \tag{5.2.23}$$

If P_0 is increased only enough to give one buckled mode, necessarily in the u direction, the problem reduces to Section 5.2.6. We shall assume here the next simplest case consisting of one buckled mode in each direction which occurs when $1 < P_0 < 4$ and $R^2 < P_0 < 4R^2$. Note that this requires $R < 2$ and we assume that $R^2 < P_0 < 4$. If the stiffness ratio is too high there will be multiple buckled in the u (soft) direction before occurrence of the first buckled mode in the w (stiff) direction. We define

$$a_1^2 = P_0 - 1, \qquad \omega_{n-1,1}^2 = n^2[(n^2 - P_0], \quad n = 2,3,\dots;$$
$$a_2^2 = P_0 - R^2, \qquad \omega_{n-1,2}^2 = n^2[n^2R^2 - P_0], \quad n = 2,3,\dots.$$

We put (5.2.23) in the form of (5.2.6) by defining

$$x = (u_1, \dot{u}_1, w_1, \dot{w}_1),$$

$$y = (u_2, \dot{u}_2/\omega_{1,1}, w_2, \dot{w}_2/\omega_{1,2}, u_3, \dot{u}_3/\omega_{2,1}, w_3, \dot{w}_3/\omega_{2,2}, \dots).$$

The reduced equations are

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = a_1^2 x_1 - \frac{\pi}{2}(x_1^2 + x_3^2)x_1 - 2\mu_2 x_2 \cos\eta + \frac{4}{\pi}\mu_1 \cos\zeta \cos\omega_0 t,$$
$$\dot{x}_3 = x_4,$$
$$\dot{x}_4 = a_2^2 x_3 - \frac{\pi}{2}(x_1^2 + x_3^2)x_3 - 2\mu_2 x_4 \sin\eta + \frac{4}{\pi}\mu_1 \sin\zeta \cos\omega_0 t.$$

For the unperturbed equations we have two homoclinic solutions given by

$$\gamma_1 = (r_1, \dot{r}_1, 0, 0), \qquad \gamma_2 = (0, 0, r_2, \dot{r}_2)$$

where $r_1(t) = (2a_1/\sqrt{\pi})\operatorname{sech} a_1 t$ and $r_2(t) = (2a_2/\sqrt{\pi})\operatorname{sech} a_2 t$. Using γ_1 the adjoint equations (5.2.8) become

$$\dot{v}_1 = \left(-a_1^2 + \frac{3\pi}{2}r_1^2\right)v_2, \quad \dot{v}_2 = -v_1,$$
$$\dot{v}_3 = \left(-a_2^2 + \frac{\pi}{2}r_1^2\right)v_4, \quad \dot{v}_4 = -v_3.$$

The essential issue here is to determine the space of bounded solutions to these equations. We can write these in the form

$$\ddot{v}_2 = \left(a_1^2 - \frac{3\pi}{2}r_1^2\right)v_2, \qquad \ddot{v}_4 = \left(a_2^2 - \frac{\pi}{2}r_1^2\right)v_4.$$

The v_2 equation has a one-dimensional space of bounded solutions spanned by the solution $v_2 = \dot{r}_1$, obtained from $\dot{\gamma}_1$. For the v_4 equation we have the following result.

Lemma 5.2.8. *Let* $\kappa > 0$. *The equation*

$$\ddot{v} + (-\lambda + \kappa \operatorname{sech}^2 t)v = 0$$

has a bounded solution if and only if there exists an integer M *so that*

$$\lambda = \frac{1}{4}\left(\sqrt{4\kappa+1} - 4M - 1\right)^2 \quad \text{for} \quad 0 \le M < \frac{1}{4}\left(\sqrt{4\kappa+1} - 1\right)$$

$$\text{or} \quad \lambda = \frac{1}{4}\left(\sqrt{4\kappa+1} - 4M - 3\right)^2 \quad \text{for} \quad 0 \le M < \frac{1}{4}\left(\sqrt{4\kappa+1} - 3\right).$$

The idea for the proof of this lemma is to express the solution as the product of a power of $\operatorname{sech} t$ and a hypergeometric function with argument $-\sinh^2 t$. The condition for the existence of a bounded solution is that the hypergeometric series terminate and the resulting polynomial is of sufficiently small degree. The details for this have been worked out in Appendix of [30]. See also Sections 23, 25 of [31].

Applying Lemma 5.2.8 to the equation for v_4 we find that the condition for a bounded solution is $a_1 = a_2$ which is ruled out by the assumption of $R > 1$. Hence, the system of equations for v has a one-dimensional space of bounded solutions spanned by $v = (-\ddot{r}_1, \dot{r}, 0, 0)$ and the Melnikov function (4.2.6) is

$$M(\alpha) = \left[\frac{8\omega_0 \cos\zeta}{\sqrt{\pi}} \sin\omega_0\alpha \operatorname{sech}\frac{\pi\omega_0}{2a_1}\right]\mu_1 - \left(\frac{16a_1^3\cos\eta}{3\pi}\right)\mu_2.$$

The non-resonance hypothesis follows as in the previous examples which leads, in the present case, to the following result obtained from Theorem 5.2.4.

Theorem 5.2.9. *If* $\omega_0 \neq \omega_{n,i}$ *for all* n *and for* $i = 1, 2$, *then whenever* μ_0 *satisfies* $\mu_{0,1} \neq 0$ *and*

$$0 < \left|\frac{\mu_{0,2}}{\mu_{0,1}}\right| < \frac{3\sqrt{\pi}\,\omega_0\cos\zeta}{2a_1^3\cos\eta}\operatorname{sech}\frac{\pi\omega_0}{2a_1}$$

there exists a corresponding $\bar{\xi}_0 > 0$ *so that if* $0 < \xi \le \bar{\xi}_0$, *if the parameters in* (5.2.23) *are given by* $\mu = \xi\mu_0$ *then there exists a compact subset of* $\mathbb{R}^4 \times \mathbb{Y}$ *on which the* $2m$*th iterate,* F^{2m}, *of the period map* F *of* (5.2.23) *is invariant and conjugate to the Bernoulli shift on* $\mathscr{E}$. *Here* $m \in \mathbb{N}$ *is sufficiently large.*

Replacing γ_1 with γ_2 yields the following analogous result.

Theorem 5.2.10. *If* $\omega_0 \neq \omega_{n,i}$ *for all* n *and for* $i = 1, 2$, *then whenever* μ_0 *satisfies* $\mu_{0,1} \neq 0$ *and*

$$0 < \left|\frac{\mu_{0,2}}{\mu_{0,1}}\right| < \frac{3\sqrt{\pi}\,\omega_0\sin\zeta}{2a_2^3\sin\eta}\operatorname{sech}\frac{\pi\omega_0}{2a_2}$$

there exists a corresponding $\bar{\xi}_0 > 0$ *so that if* $0 < \xi \le \bar{\xi}_0$, *if the parameters in* (5.2.23) *are given by* $\mu = \xi\mu_0$ *then there exists a compact subset of* $\mathbb{R}^4 \times \mathbb{Y}$ *on which the* $2m$*th iterate,* F^{2m}, *of the period map* F *of* (5.2.23) *is invariant and conjugate to the Bernoulli shift on* $\mathscr{E}$. *Here* $m \in \mathbb{N}$ *is sufficiently large.*

In the μ_1-μ_2 plane in this case we get a diagram as in Figure 5.4. For parameter values in the regions labeled *I* there is one homoclinic orbit while for those in *II* there are two.

5.2.9 Multiple Buckled Modes

One has to consider the situation where the axial load, P_0, is increased sufficiently to produce multiple buckled modes. We will look at the case of a beam constrained to planer motion. The calculations for the non-planer case are similar. We return to the boundary value problem of Section 5.2.6 and use the same Galerkin equations

$$\begin{aligned} \ddot{u}_n = n^2(P_0 - n^2)u_n - \tfrac{\pi}{2}n^2 \left[\sum_{k=1}^{\infty} k^2 u_k^2\right] u_n \\ -2\mu_2\dot{u}_n + 2\mu_1 \left[\frac{1-(-1)^n}{\pi n}\right] \cos\omega_0 t, \quad n = 1,2,\dots. \end{aligned} \tag{5.2.24}$$

In the present case we assume that there exists an integer N so that $N^2 < P_0 < (N+1)^2$. We then define

$$\begin{aligned} a_n^2 &= n^2(P_0 - n^2), \quad \text{for } n = 1,2,\dots,N; \\ \omega_{n-N}^2 &= n^2(n^2 - P_0), \quad \text{for } n = N+1, N+2, \dots \end{aligned}$$

and put (5.2.24) in the form of (5.2.6) by defining

$$\begin{aligned} x &= (u_1, \dot{u}_1, u_2, \dot{u}_2, \dots, u_N, \dot{u}_N), \\ y &= (u_{N+1}, \dot{u}_{N+1}/\omega_1, u_{N+2}, \dot{u}_{N+2}/\omega_2, \dots). \end{aligned}$$

A truncated version of the resulting equations with $N = 2$ was studied in [30]. The reduced equations are

$$\left.\begin{aligned} \dot{x}_{2n-1} &= x_{2n} \\ \dot{x}_{2n} &= a_n^2 x_{2n-1} - \frac{\pi n^2}{2}\left(\sum_{k=1}^{N} k^2 x_{2k-1}^2\right) x_{2n-1} \\ &\quad -2\mu_2 x_{2n} + 2\mu_1 \left[\tfrac{1-(-1)^n}{\pi n}\right] \cos\omega_0 t \end{aligned}\right\} n = 1,2,\dots,N.$$

When $\mu = 0$ we have N homoclinic solutions given by

$$\gamma_m = (0,\dots,0, \underbrace{r_m, \dot{r}_m}_{2m-1,2m}, 0,\dots,0), \quad m = 1,2,\dots,N$$

where $r_m(t) = (2a_m/m^2\sqrt{\pi})\,\mathrm{sech}\, a_m t$ and the adjoint equation (5.2.8) along γ_m is

$$\left.\begin{aligned} \dot{v}_{2n-1} &= \left(-a_n^2 + \frac{\pi m^2 n^2}{2} r_m^2\right) v_{2n}\,, \\ \dot{v}_{2n} &= -v_{2n-1}\,, \end{aligned}\right\} n \neq m$$

$$\begin{aligned} \dot{v}_{2m-1} &= \left(-a_m^2 + \frac{3\pi m^4}{2} r_m^2\right) v_{2m}\,, \\ \dot{v}_{2m} &= -v_{2m-1}\,. \end{aligned}$$

For the distinguished equation we have the bounded solution $v_{2m-1} = -\ddot{r}_m$, $v_{2m} = \dot{r}_m$ while for the equations with $n \neq m$ we must solve

$$\frac{d^2 v_{2n}}{dx^2} = \left(\frac{a_n^2}{a_m^2} - \frac{2n^2}{m^2}\operatorname{sech}^2 x\right) v_{2n}.$$

Using Lemma 5.2.8 we find that this last equation has a bounded solution if and only if there is an integer M so that one of the following conditions holds:

$$\frac{n^2(P_0 - n^2)}{m^2(P_0 - m^2)} = \frac{1}{4}\left[\sqrt{\frac{8n^2}{m^2} + 1} - 4M - 1\right]^2 \tag{5.2.25a}$$

$$\text{for } 0 \le M < \frac{1}{4}\left(\sqrt{\frac{8n^2}{m^2} + 1} - 1\right),$$

$$\frac{n^2(P_0 - n^2)}{m^2(P_0 - m^2)} = \frac{1}{4}\left[\sqrt{\frac{8n^2}{m^2} + 1} - 4M - 3\right]^2 \tag{5.2.25b}$$

$$\text{for } 0 \le M < \frac{1}{4}\left(\sqrt{\frac{8n^2}{m^2} + 1} - 3\right).$$

If, for some fixed m, none of the equations in (5.2.25 a and b) is satisfied for $n \neq m$ we can proceed much as in Section 5.2.6 since then the adjoint equation obtained from γ_m has a one-dimensional space of bounded solutions spanned by

$$v = (0, \ldots, 0, \underbrace{-\ddot{r}_m, \dot{r}_m}_{2m-1,2m}, 0, \ldots, 0).$$

One complication has been introduced by our assumption in the original partial differential equation that the transverse-applied load is uniform in x. This assumption causes the μ_1 terms to drop out in (5.2.24) for n even which prohibits nonsingular solutions of $M(\alpha) = 0$ as can be seen by examining Section 5.2.6. For this reason, we must choose m odd. Theorem 5.2.4 now yields the following result.

Theorem 5.2.11. *Let m be an odd integer,* $1 \le m \le N$, *and suppose* P_0 *is chosen so that none of the equations in* (5.2.25 a and b) *is satisfied. If* $\omega_0 \neq \omega_n$ *for all n, then* *whenever μ_0 satisfies $\mu_{0,1} \neq 0$ and*

$$0 < \left| \frac{\mu_{0,2}}{\mu_{0,1}} \right| < \frac{3m\sqrt{\pi}\,\omega_0}{2a_m^3} \operatorname{sech} \frac{\pi\omega_0}{2a_m}$$

there exists a corresponding $\bar{\xi}_0 > 0$ so that if $0 < \xi \leq \bar{\xi}_0$, if the parameters in (5.2.24) *are given by $\mu = \xi\mu_0$ then there exists a compact subset of $\mathbb{R}^{2N} \times \mathbb{Y}$ on which the 2kth iterate, F^{2k}, of the period map F of* (5.2.24) *is invariant and conjugate to the Bernoulli shift on $\mathscr{E}$. Here $k \in \mathbb{N}$ is sufficiently large.*

We can simplify the preceding results by finding cases where the equations in (5.2.25) can never have a solution. The following is a helpful result along these lines.

Lemma 5.2.12. *The equations in* (5.2.25) *can never be satisfied for $n < m \leq N$.*

Proof. For (5.2.25a) we have $\frac{1}{4}\left(\sqrt{8n^2/m^2+1}-1\right) < \frac{1}{2}$ so we have only one equation to consider with $M = 0$. But then we have, first, $\frac{n^2(P_0-n^2)}{m^2(P_0-m^2)} > \frac{n^2}{m^2}$, and also

$$\frac{1}{4}\left[\sqrt{\frac{8n^2}{m^2}+1}-1\right]^2 - \frac{n^2}{m^2} = \frac{2\frac{n^2}{m^2}\left(\frac{n^2}{m^2}-1\right)}{2\frac{n^2}{m^2}+1+\sqrt{\frac{8n^2}{m^2}+1}} < 0$$

so that Equation (5.2.25a) has no solution for any P_0. Next we note that when $n < m$, we have $\frac{1}{4}\left(\sqrt{8n^2/m^2+1}-3\right) < 0$ so that there are no equations for (5.2.25b). □

When $m = N$ the preceding result will eliminate any restriction, obtained from (5.2.25), on P_0. This fact was shown with a different technique in [4] where they used a more general transverse forcing term which allowed for the possibility of a μ_2 term for each n in (5.2.24) and, hence, also for each n in the reduced equation. They then take $m = N$. Since, for our specific form of loading, we must have m odd we have the following result.

Theorem 5.2.13. *Let N and P_0 be as for* (5.2.24) *and suppose one of the following holds:*

(i) *N is odd and $m = N$.*
(ii) *N is even, $N \geq 4$, $m = N-1$ and*

$$P_0 \neq \frac{4N^2-(N-1)^2\left[\sqrt{9N^2-2N+1}-3(N-1)\right]^2}{4N^2-\left[\sqrt{9N^2-2N+1}-3(N-1)\right]^2}.$$

(iii) *$N = 2$, $m = 1$ and*

$$P_0 \neq \frac{37+5\sqrt{33}}{16}, \quad P_0 \neq \frac{55+9\sqrt{33}}{16}.$$

Suppose in addition that $\omega_n \neq \omega_0$ for all n. Then whenever μ_0 satisfies $\mu_{0,1} \neq 0$ and

$$0 < \left|\frac{\mu_{0,2}}{\mu_{0,1}}\right| < \frac{3m\sqrt{\pi}\,\omega_0}{2a_m^3}\,\mathrm{sech}\,\frac{\pi\omega_0}{2a_m}$$

there exists a corresponding $\bar{\xi}_0 > 0$ so that if $0 < \xi \leq \bar{\xi}_0$, if the parameters in (5.2.24) *are given by $\mu = \xi\mu_0$ then there exists a compact subset of $\mathbb{R}^{2N} \times \mathbb{Y}$ on which the* $2k$*th iterate, F^{2k}, of the period map F of* (5.2.24) *is invariant and conjugate to the Bernoulli shift on $\mathscr{E}$. Here $k \in \mathbb{N}$ is sufficiently large.*

Proof. The result is obtained by using γ_m and proceeding as in Section 5.2.6. This is valid as long as Equations (5.2.25) have no solutions for $n \neq m$ so it remains to show that this is true in each case. If (i) holds we can use Lemma 5.2.12.

If $m = N-1$ then, using Lemma 5.2.12, we need check only $n = N$. Define

$$f_a(N) = \frac{1}{4}\left(\sqrt{\frac{8N^2}{(N-1)^2}+1}-1\right),$$

$$f_b(N) = \frac{1}{4}\left(\sqrt{\frac{8N^2}{(N-1)^2}+1}-3\right).$$

Then (5.2.25a) must be checked for integers $M \in [0, f_a(N))$ and (5.2.25b) for integers $M \in [0, f_b(N))$.

In case (ii) we have $N \geq 4$ which implies $1/2 < f_a(N) \leq (\sqrt{137}-3)/12 < 1$ so we need consider only $M = 0$. In this case we solve

$$\frac{N^2(P_0-N^2)}{(N-1)^2[P_0-(N-1)^2]} = 4f_a(N)^2$$

for P_0 to get

$$P_0 = \frac{N^4-4f_a(N)^2(N-1)^4}{N^2-4f_a(N)^2(N-1)^2} = \frac{(N-1)^2}{2}\left[1-2\frac{N^2}{(N-1)^2}-\sqrt{\frac{8N^2}{(N-1)^2}+1}\right].$$

But this value is negative and can be discarded. Similarly, we have, for $N \geq 4$, $0 < f_b(N) \leq (\sqrt{137}-9)/12 < 1$, so in (5.2.25b) we need also consider only $M = 0$. Here we get

$$P_0 = \frac{N^4-4f_b(N)^2(N-1)^4}{N^2-4f_b(N)^2(N-1)^2} = \frac{4N^4-(N-1)^2\left[\sqrt{9N^2-2N+1}-3(N-1)\right]^2}{4N^2-\left[\sqrt{9N^2-2N+1}-3(N-1)\right]^2}.$$

Next, we consider (iii) where $N = 2$, $m = 1$. Since $2 > f_a(2) = (\sqrt{33}-1)/4 > 1$ we must consider $M = 0$ and $M = 1$ in (5.2.25a). When $M = 0$ we get the value $P_0 = -(7+\sqrt{33})/2 < 0$ which can be discarded while for $M = 1$ we have $P_0 = (37+5\sqrt{33})/16$. Finally, $0 < f_b(2) = (\sqrt{33}-3)/4 < 1$, so only $M = 0$ must be considered in (5.2.25b) and this yields $P_0 = (55+9\sqrt{33})/16$. □

5.3 Periodically Forced Compressed Beam

5.3.1 Resonant Compressed Equation

This section is a continuation of Section 5.2, and it is devoted to the study of a system modelling a compressed beam with friction subjected to a small periodic forcing. Particularly we are interested in the existence of chaotic patterns. The model is described by the following PDE

$$u_{tt} + u_{xxxx} + \gamma u_{xx} - \kappa u_{xx} f\left(\int_0^\pi u_x^2(\xi,t)\,d\xi\right) = \varepsilon(\nu h(x,\sqrt{\varepsilon}t) - \delta u_t), \tag{5.3.1}$$

$$u(0,t) = u(\pi,t) = 0 = u_{xx}(0,t) = u_{xx}(\pi,t) \tag{5.3.2}$$

where $u(x,t) \in \mathbb{R}$ is the transverse deflection of the axis of the beam; $\gamma > 0$ is an external load, $\kappa > 0$ is a ratio indicating the external rigidity and $\delta > 0$ is the damping, ε and ν are small parameters, the function $h(x,t)$ represents the periodic (in time) forcing distributed along the whole beam. We assume that $h \in L^\infty\left(\mathbb{R}, L^2([0,\pi])\right)$ is a 1-periodic function of t with $\left\|\int_0^\pi h(x,\cdot)^2\,dx\right\|_\infty = 1$. Therefore $\varepsilon\nu$ represents the strength of the forcing.

Section 5.2 discusses Equation (5.3.1) when the external load γ is not resonant and $\kappa \in \mathbb{R}$ is fixed. Here we discuss the complementary case. Precisely we assume that γ is slightly larger than the i-th eigenvalue of the unperturbed problem: $\gamma = i^2 + \varepsilon\sigma^2$, where $i \in \mathbb{N}$ is fixed, $\varepsilon > 0$ and $\sigma \in (0,1]$. Therefore we will also assume that $\kappa = \varepsilon k$, so that the contribution given from the stress due to the external rigidity, does not drive the system too far away from the resonance.

5.3.2 Formulation of Weak Solutions

It is easily observed that the unperturbed problem

$$u_{xxxx} + \gamma u_{xx} = 0,$$

$$u(0,t) = u(\pi,t) = 0 = u_{xx}(0,t) = u_{xx}(\pi,t)$$

admits $\{j^2 \mid j \in \mathbb{N}\}$ as set of eigenvalues and that the corresponding eigenfunctions $\sqrt{\frac{2}{\pi}}\sin(jx)$, where $j \in \mathbb{N}$, form an orthonormal system in $L^2([0,\pi])$ which generates the space $H_0^2([0,\pi])$. First of all we make the linear scale $t \leftrightarrow \sqrt{\varepsilon}t$. Then Eqs. (5.3.1), (5.3.2) read:

$$u_{tt} + \frac{1}{\varepsilon}[u_{xxxx} + (i^2 + \varepsilon\sigma^2)u_{xx}] - kf\left(\int_0^\pi u_x^2(\xi,t)d\xi\right)u_{xx} = \nu h(x,t) - \sqrt{\varepsilon}\delta u_t,$$

$$u(0,t) = u(\pi,t) = 0 = u_{xx}(0,t) = u_{xx}(\pi,t). \tag{5.3.3}$$

We want to solve (5.3.3) in a *weak* form, that is, we look for a function $u \in L^\infty\left(\mathbb{R}, H_0^2([0,\pi])\right) \subset L^\infty([0,\pi]\times\mathbb{R})$ so that

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_0^\pi \Bigg\{u(x,t)\Bigg(&\Psi_{tt} + \frac{1}{\varepsilon}\left[\Psi_{xxxx} + (i^2+\varepsilon\sigma^2)\Psi_{xx}\right] \\ -kf\left(\int_0^\pi u_x^2(\xi,t)d\xi\right)\Psi_{xx} - \sqrt{\varepsilon}\delta\Psi_t\Bigg) &- \nu\Psi(x,t)h(x,t)\Bigg\}dxdt = 0\end{aligned} \tag{5.3.4}$$

for any $\Psi(x,t) \in C^\infty([0,\pi]\times\mathbb{R})$ with compact support so that

$$\Psi(0,t) = \Psi(\pi,t) = \Psi_{xx}(0,t) = \Psi_{xx}(\pi,t) = 0.$$

5.3.3 Chaotic Solutions

In this section, the existence of chaotic solutions is studied for (5.3.1). To start with, note that we can expand the function $u(x,t) \in L^\infty\left(\mathbb{R}, H_0^2([0,\pi])\right)$ as follows:

$$u(x,t) = \sqrt{\frac{2}{\pi}}\left[\sum_{0<l<i}\phi_l(t)\sin(lx) + y(t)\sin(ix) + \sum_{j>i} z_j(t)\sin(jx)\right],$$

where $\phi_l(t), y(t), z_j(t) \in L^\infty(\mathbb{R})$, the expansion holding in $H_0^2([0,\pi])$. Similarly we write:

$$\Psi(x,t) = \sqrt{\frac{2}{\pi}}\left[\sum_{l=1}^{i-1}\psi_l(t)\sin(lx) + \psi_i(t)\sin(ix) + \sum_{j=i+1}^{\infty}\psi_j(t)\sin(jx)\right],$$

where, for any $k \geq 1$, $\psi_k(t) \in C_0^\infty(\mathbb{R})$, the space of C^∞-functions on $\mathbb{R}$ having compact supports. Plugging the above expression for $u(x,t)$ and $\Psi(x,t)$ into (5.3.4) and using the orthonormality, we arrive at the system of equations for the components $(\phi_l(t), y(t), z_j(t))$ of $u(x,t)$

$$\ddot{\phi}_l(t) - \frac{i^2 - l^2 + \varepsilon\sigma^2}{\varepsilon} l^2 \phi_l(t) + kl^2 f\Big(\sum_{0<l<i} l^2 \phi_l(t)^2 + i^2 y^2(t) + \sum_{j>i} j^2 z_j(t)^2\Big)\phi_l(t)$$

$$+\sqrt{\varepsilon}\delta\dot{\phi}_l(t) - \nu\sqrt{\frac{2}{\pi}}\int_0^{\pi} h(x,t)\sin(lx)\,dx = 0, \tag{5.3.5}$$

$$\ddot{y}(t) - \sigma^2 i^2 y(t) + ki^2 f\Big(\sum_{0<l<i} l^2 \phi_l^2(t) + i^2 y^2(t) + \sum_{j>i} j^2 z_j^2(t)\Big) y(t) + \sqrt{\varepsilon}\delta\dot{y}(t)$$

$$-\nu\sqrt{\frac{2}{\pi}}\int_0^{\pi} h(x,t)\sin(ix)\,dx = 0 \tag{5.3.6}$$

$$\ddot{z}_j(t) + \frac{j^2 - i^2 - \varepsilon\sigma^2}{\varepsilon} j^2 z_j(t) + kj^2 f\Big(\sum_{0<l<i} l^2 \phi_l(t)^2 + i^2 y^2(t) + \sum_{j>i} j^2 z_j(t)^2\Big) z_j(t)$$

$$+\sqrt{\varepsilon}\delta\dot{z}_j(t) - \nu\sqrt{\frac{2}{\pi}}\int_0^{\pi} h(x,t)\sin(jx)\,dx = 0 \tag{5.3.7}$$

where $0 < l < i < j$. In this way we have decomposed the problem along three submanifolds: a strongly hyperbolic second order problem in $\mathbb{R}^{i-1}$, a hyperbolic second order problem in $\mathbb{R}$, and a second order problem in an infinite dimensional center manifold. We assume that $f(x)$ satisfies the following hypotheses:

(F1) The function $f \in C([0,\infty),[0,\infty)) \cap C^2((0,\infty),[0,\infty))$. Moreover we assume that the following conditions hold:

$$f(0) = 0, \quad \limsup_{x\to 0^+} |xf'(x^2)| < \infty, \quad \limsup_{x\to 0^+} |x^3 f''(x^2)| < \infty.$$

(F2) The equation

$$\ddot{y} - \sigma^2 y + kf(y^2)y = 0 \tag{5.3.8}$$

has a positive homoclinic solution that is a C^2-solution $\gamma(t) > 0$ so that $\lim_{|t|\to\infty} \gamma(t) = \lim_{|t|\to\infty} \dot{\gamma}(t) = 0$.

Remark 5.3.1. (a) Observe that $\gamma_i(t) = \gamma(it)/i$ solves the equation

$$\ddot{y} - i^2\sigma^2 y + ki^2 f(i^2 y^2) y = 0 \tag{5.3.9}$$

for any $i \in \mathbb{N} \setminus \{0\}$. That is, $\gamma_i(t)$ is a solution of the equation obtained from (5.3.6) taking $\phi_l(t) = 0$, $z_j(t) = 0$ and $\varepsilon = \nu = 0$. We will refer to Eq. (5.3.9) as the *unperturbed problem*.

(b) Equation (5.3.8) has the energy function

$$E(y,\dot{y}) = \dot{y}^2 + \int_0^{y^2} \left(kf(s) - \sigma^2\right) ds$$

which is even in both y and $\dot{y}$. Since $\lim_{t\to\infty} \gamma(t) = 0$, we see that $\dot{\gamma}(t) = 0$ has a solution t_0. It is easy to prove [32] that this solution is unique. Hence we can assume that $t_0 = 0$ and then $\gamma(t) = \gamma(-t)$ because of uniqueness. Thus $\gamma(t)$ has either a positive maximum or a negative minimum at the point $t = 0$. Since $-\gamma(t)$ satisfies Eq. (5.3.8) as $\gamma(t)$ does, we see that the assumption $\gamma(t) > 0$ is not restrictive. Then, $\gamma(t)$ is increasing on $(-\infty, 0]$ and decreasing on $[0,\infty)$. As a consequence, $0 \le \gamma(t) \le M := \gamma(0)$. Since the energy function $E(y,\dot{y})$ is constant along $(\gamma(t),\dot{\gamma}(t))$ and $\dot{\gamma}(0) = 0$ we get

$$\int_0^{M^2} \left(kf(s) - \sigma^2\right) ds = 0$$

(note that $\lim_{t\to\infty} E(\gamma(t),\dot{\gamma}(t)) = E(0,0) = 0$) and

$$\int_0^{x^2} \left(kf(s) - \sigma^2\right) ds < 0$$

for $0 < x < M$. Finally $kf(M^2) \neq \sigma^2$, since, otherwise $x = M$ would be a fixed point of Equation (5.3.8). As a matter of fact, we have $kf(M^2) > \sigma^2$, since the function $\int_0^{x^2} \left(kf(s) - \sigma^2\right) ds$ passes from negative values to 0 when $x \to M^-$ and then its derivative at $x = M$ must be nonnegative. As a consequence, assumption (F2) implies that the following condition holds:

(F2') There exists $M > 0$ so that $\int_0^{x^2} \left[kf(s) - \sigma^2\right] ds < 0$ for any $0 < x < M$ and $\int_0^{M^2} \left[kf(s) - \sigma^2\right] ds = 0$. Moreover $kf(M^2) > \sigma^2$.

On the other hand, if condition (F2') holds then the solution $\gamma(t)$ of (5.3.8), $\gamma(0) = M$ and $\dot{\gamma}(0) = 0$, satisfies $0 < \gamma(t) < M$ for any $t \neq 0$, and is homoclinic to the (hyperbolic) fixed point $x = 0$, $\dot{x} = 0$ of (5.3.8). Thus the two conditions (F2) and (F2') are equivalent. Finally we observe that the curve $(\gamma(t),\dot{\gamma}(t))$ is contained in the sector $\{(y,\dot{y}) \mid y \ge 0 \text{ and } |\dot{y}| \le \sigma y\}$, that is, $|\dot{\gamma}(t)| \le \sigma\gamma(t)$ for any $t \in \mathbb{R}$.

(c) Since we look for solutions close to the homoclinic orbit, in fact, it is enough that f is defined just for $0 \le x \le M^2 + 1$.

(b) Assumption (F1) is satisfied in particular if we take any function $f(x)$ of the form $f(x) = g(x^\alpha)$, where $\alpha \ge \frac{1}{2}$ and $g(x) \in C^2\left([0,\infty),[0,\infty)\right)$ is a positive function so that $g(0) = 0$.

We see that (5.3.5), (5.3.6) and (5.3.7) are similar to (5.1.6), (5.1.8) and (5.1.8). So we can repeat arguments of Section 5.1, i.e. we can apply a Lyapunov-Schmidt reduction method like for the system of (5.1.6), (5.1.8) and (5.1.8) to deriving a Melnikov function for (5.3.1), (5.3.2). We do not go into details, and we refer the readers to [33], we only here recall the following notations (cf Section 5.1.3). For any $E = \{e_j\}_{j\in\mathbb{Z}} \in \mathscr{E}$, we put

$$\ell_E^\infty = \Big\{\alpha := \{\alpha_j\}_{j\in\mathbb{Z}} \in \ell^\infty(\mathbb{R}) \mid \alpha_j \in \mathbb{R} \text{ and } \alpha_j = 0 \text{ if } e_j = 0\Big\},$$

with $\ell^\infty(\mathbb{R})$ being the Banach space of bounded, doubly infinity sequences of real numbers, endowed with the sup-norm. For any $(E,\alpha) \in \mathscr{E} \times \ell_E^\infty$ we take the function $\gamma_{(E,\alpha)} \in L^\infty(\mathbb{R})$ defined as

$$\gamma_{(E,\alpha)}(t) = \begin{cases} \gamma(t-2jm-\alpha_j) & \text{if } (2j-1)m < t \le (2j+1)m \text{ and } e_j = 1 \\ 0 & \text{if } (2j-1)m < t \le (2j+1)m \text{ and } e_j = 0 . \end{cases}$$

Now we can state the following main result proved in [33].

Theorem 5.3.2. *Assume that the conditions (F1) and (F2) are satisfied, and that* $h \in L^\infty(\mathbb{R}, L^2([0,\pi]))$ *is* 1*-periodic with respect to* t *and* $\left\| \int_0^\pi h(x,\cdot)^2\,dx \right\|_\infty = 1$. *Assume, further, that* $\mu_0 \in \mathbb{R}$ *exists so that the function*

$$\bar{M}(\tau) := \delta \int_{-\infty}^{\infty} \dot{\gamma}(t)^2 dt - \mu_0 \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \int_0^\pi \dot{\gamma}(t) h(x,(t+\tau)/i) \sin(ix)\,dx\,dt$$

has a simple zero at $\tau = \tau_0 \in [0,1]$, *that is,* $\bar{M}(\tau_0) = 0$ *and* $\bar{M}'(\tau_0) \neq 0$. *Then there exist* $\bar{\rho} > 0$, $\bar{\varepsilon} > 0$ *and* $\bar{\mu} > 0$ *so that for any* $0 < \varepsilon < \bar{\varepsilon}$, $|\mu - \mu_0| \le \bar{\mu}$ *and* $m > \varepsilon^{-3/4}$, *with* $m = ki$ *and* $k \in \mathbb{N}$, *there is a continuous function* $\alpha_{\varepsilon,\mu,m} : \mathscr{E} \to \ell^\infty(\mathbb{R})$ *so that* $\alpha_{\varepsilon,\mu,m}(E) \in \ell_E^\infty$ *and a continuous map* $\Pi_{\varepsilon,\mu,m} : \mathscr{E} \to L^\infty\left(\mathbb{R}, H_0^2([0,\pi])\right)$ *so that*

$$u_E(x,t,\varepsilon) := i^{-1} \Pi_{\varepsilon,\mu,m}(E)(x, i\sqrt{\varepsilon}t)$$

is a weak solution of (5.3.1) with $\nu = \sqrt{\varepsilon}\mu$ *that satisfies*

$$\text{ess sup}_{t\in\mathbb{R}} \left\| iu_E(x,t,\varepsilon) - \sqrt{\frac{2}{\pi}} \gamma_{(E,\alpha_{\varepsilon,\mu,m}(E))}(i\sqrt{\varepsilon}t) \sin(ix) \right\|_{H_0^2([0,\pi])} \le \bar{\rho}$$

where $\|\cdot\|_{H_0^2([0,\pi])}$ *is the norm in* $H_0^2([0,\pi])$. *Moreover, the map* $\Pi_{\varepsilon,\mu,m} : \mathscr{E} \to \Pi(\mathscr{E})$ *is a homeomorphism satisfying*

$$\Pi_{\varepsilon,\mu,m}(\sigma(E))(x,t) = \Pi_{\varepsilon,\mu,m}(E)\,(x,t+2m) .$$

Hence $u_{\sigma(E)}(x,t,\varepsilon) = u_E(x,t+2k/\sqrt{\varepsilon},\varepsilon)$.

Finally we note that from (F1) it follows that:

$$\lim_{x\to 0^+} xf'(x) = \lim_{x\to 0} x^2 f'(x^2) = 0, \quad \lim_{x\to 0^+} x^2 f''(x) = \lim_{x\to 0} x^4 f''(x^2) = 0 .$$

Hence the function $xf(x^2)$ is C^1 on $\mathbb{R}$ and its second derivative is bounded on $K \setminus \{0\}$, with K being any fixed compact subset of $\mathbb{R}$. In fact, for $x \neq 0$, we have

$$\frac{d}{dx}[xf(x^2)] = 2x^2 f'(x^2) + f(x^2) \to 0 = \frac{d}{dx}[xf(x^2)]_{|x=0}$$

as $x \to 0$. Thus $\frac{d}{dx}[xf(x^2)]$ is continuous in $\mathbb{R}$. Next

$$\frac{d^2}{dx^2}[xf(x^2)] = 6xf'(x^2) + 4x^3 f''(x^2)$$

is bounded on $K \setminus \{0\}$ for any given compact subset K of $\mathbb{R}$ because of assumption (F1).

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Chapter 6
Chaos in Discontinuous Differential Equations

This chapter is devoted to proving chaos for periodically perturbed piecewise smooth ODEs. We study two cases: firstly, when the homoclinic orbit of the unperturbed piecewise smooth ODE transversally crosses the discontinuity surface, and secondly, when a part of homoclinic orbit is sliding on the discontinuity surface.

6.1 Transversal Homoclinic Bifurcation

6.1.1 Discontinuous Differential Equations

DDEs occur in several situations such as in mechanical systems with dry frictions or with impacts or in control theory, electronics, economics, medicine and biology [1–8]. Recently attempts have been made to extend the theory of chaos to differential equations with discontinuous right-hand sides. For examples, planar discontinuous differential equations are investigated in [9, 10], piecewise linear three-dimensional discontinuous differential equations are investigated in [11, 12] and weakly discontinuous systems are studied in [13–15]. Melnikov type analysis is also presented for DDEs in [16–21]. An overview of some aspects of chaotic dynamics in hybrid systems is given in [22]. A survey of controlling chaotic differential equations is presented in [23]. The switchability of flows of general DDEs is discussed in [24–26]. Planar discontinuous differential equations are investigated in [27, 28] using analytic and numeric approaches. Periodic and almost periodic solutions of DDEs are considered in [29–33].

In [34] bifurcations of bounded solutions from homoclinic orbits are investigated for time perturbed discontinuous differential equations in any finite dimensional space. We anticipated that under the conditions of [34] not only the existence of bounded solutions on $\mathbb{R}$, but also chaotic solutions could occur. The purpose of this section is to justify this conjecture about the existence of chaotic solutions. To handle this kind of problem one has to face the new problem that stable and unstable manifolds may only be Lipschitz in the state variable, even if they are possibly smooth with respect to parameters. So it is not clear what the notion of transverse intersection of invariant manifolds would be.

6.1.2 Setting of the Problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set in $\mathbb{R}^n$ and $G(z)$ be a C^r-function on $\bar{\Omega}$, with $r \geq 2$. We set

$$\Omega_\pm = \{z \in \Omega \mid \pm G(z) > 0\}, \quad \Omega_0 := \{z \in \Omega \mid G(z) = 0\}.$$

Let $f_\pm(z) \in C_b^r(\bar{\Omega}_\pm)$ and $g \in C_b^r(\mathbb{R} \times \bar{\Omega} \times \mathbb{R})$, i.e. $f_\pm$ and g have uniformly bounded derivatives up to the r-th order on $\bar{\Omega}_\pm$ and $\mathbb{R} \times \bar{\Omega} \times \mathbb{R}$, respectively. We also assume that the r-th order derivatives of $f_\pm$ and g are uniformly continuous. Let $\varepsilon_0 \in (0,1)$. Throughout this section ε will denote a real parameter so that $|\varepsilon| \leq \varepsilon_0$. Particularly ε is bounded.

Remark 6.1.1. For technical purposes, we C_b^r-smoothly extend $f_\pm$ on $\mathbb{R}^n$, g on $\mathbb{R}^{n+2}$ and $\gamma_\pm$, γ_0 on $\mathbb{R}$ in such a way that

$$\sup\{|f_\pm(z)| \mid z \in \mathbb{R}^n\} \leq 2\sup\{|f_\pm(z)| \mid z \in \bar{\Omega}_\pm\},$$
$$\sup\{|g(t,z,\varepsilon)| \mid (t,z,\varepsilon) \in \mathbb{R}^{n+2}\} \leq 2\sup\{|g(t,z,\varepsilon)| \mid t \in \mathbb{R}, z \in \bar{\Omega}, |\varepsilon| \leq \varepsilon_0\}.$$

We also assume that up to the r-th order all the derivatives of the extended $f_\pm$ and g are uniformly continuous and continue to keep the same notations for extended mappings and functions.

We say that a function $z(t)$ is a solution of the equation

$$\dot{z} = f_\pm(z) + \varepsilon g(t,z,\varepsilon), \quad z \in \bar{\Omega}_\pm, \tag{6.1.1}$$

if it is continuous, piecewise C^1 satisfies Eq. (6.1.1) on $\Omega_\pm$ and, moreover, the following holds: if for some t_0 we have $z(t_0) \in \Omega_0$, then there exists $r > 0$ so that for any $t \in (t_0 - r, t_0 + r)$ with $t \neq t_0$, we have $z(t) \in \Omega_- \cup \Omega_+$. Moreover, if, for example, $z(t) \in \Omega_-$ for any $t \in (t_0 - r, t_0)$, then the left derivative of $z(t)$ at $t = t_0$ satisfies: $\dot{z}(t_0^-) = f_-(z(t_0)) + \varepsilon g(t_0, z(t_0), \varepsilon)$; similarly, if $z(t) \in \Omega_-$ for any $t \in (t_0, t_0 + r)$, then $\dot{z}(t_0^+) = f_-(z(t_0)) + \varepsilon g(t_0, z(t_0), \varepsilon)$. A similar meaning is assumed when $z(t) \in \Omega_+$ for either $t \in (t_0 - r, t_0)$ or $t \in (t_0, t_0 + r)$. Note that since $z(t) \notin \Omega_0$ for $t \in (t_0 - r, t_0 + r) \setminus \{t_0\}$ we have either $z(t) \in \Omega_-$ or $z(t) \in \Omega_+$ when $t \in (t_0 - r, t_0)$ or $t \in (t_0, t_0 + r)$.

We assume (Figure 6.1) that

(H1) For $\varepsilon = 0$ Eq. (6.1.1) has the hyperbolic equilibrium $x = 0 \in \Omega_-$ and a continuous (not necessarily C^1) solution $\gamma(t)$ which is homoclinic to $x = 0$ and consists of three branches

$$\gamma(t) = \begin{cases} \gamma_-(t), \text{ if } t \le -\bar{T}, \\ \gamma_0(t), \text{ if } -\bar{T} \le t \le \bar{T}, \\ \gamma_+(t), \text{ if } t \ge \bar{T}, \end{cases}$$

where $\gamma_\pm(t) \in \Omega_-$ for $|t| > \bar{T}$, $\gamma_0(t) \in \Omega_+$ for $|t| < \bar{T}$ and

$$\gamma_-(-\bar{T}) = \gamma_0(-\bar{T}) \in \Omega_0, \quad \gamma_+(\bar{T}) = \gamma_0(\bar{T}) \in \Omega_0.$$

(H2) It results: $G'(\gamma(-\bar{T}))f_\pm(\gamma(-\bar{T})) > 0$ and $G'(\gamma(\bar{T}))f_\pm(\gamma(\bar{T})) < 0$.

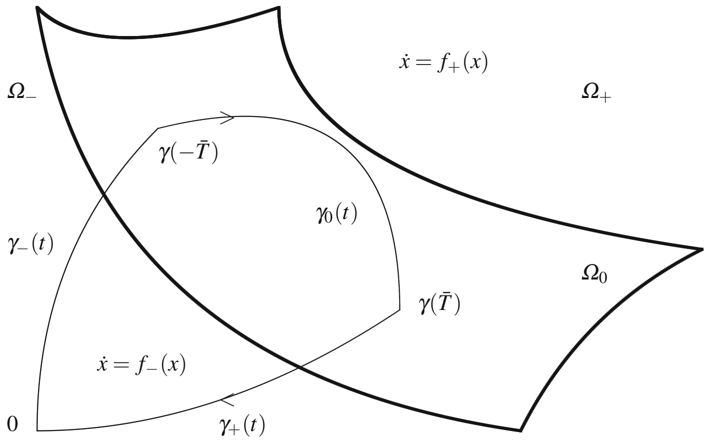


Fig. 6.1 Transversal homoclinic cycle $\gamma(t)$ of $\dot{x} = f_\pm(x)$.

According to (H1) and because of roughness of exponential dichotomies the linear systems $\dot{x} = f'_-(\gamma_-(t))x$ and $\dot{x} = f'_-(\gamma_+(t))x$ have exponential dichotomies on $(-\infty, -\bar{T}]$ and $[\bar{T}, \infty)$ respectively, that is, projections $P_\pm : \mathbb{R}^n \to \mathbb{R}^n$ and positive numbers $k \ge 1$ and $\delta > 0$ exist so that the following hold:

$$\begin{aligned} \|X_-(t)P_-X_-^{-1}(s)\| &\le k\mathrm{e}^{-\delta(t-s)}, && \text{if } s \le t \le -\bar{T}, \\ \|X_-(t)(\mathbb{I}-P_-)X_-^{-1}(s)\| &\le k\mathrm{e}^{\delta(t-s)}, && \text{if } t \le s \le -\bar{T}, \\ \|X_+(t)P_+X_+^{-1}(s)\| &\le k\mathrm{e}^{-\delta(t-s)}, && \text{if } \bar{T} \le s \le t, \\ \|X_+(t)(\mathbb{I}-P_+)X_+^{-1}(s)\| &\le k\mathrm{e}^{\delta(t-s)}, && \text{if } \bar{T} \le t \le s, \end{aligned} \tag{6.1.2}$$

where $X_-(t)$ and $X_+(t)$ are the fundamental matrices of the linear systems $\dot{x} = f'_-(\gamma_-(t))x$ and $\dot{x} = f'_-(\gamma_+(t))x$, respectively, so that $X_-(-\bar{T}) = X_+(\bar{T}) = \mathbb{I}$. Later in this section we will need to extend the validity of (6.1.2) to a larger set of values of s,t. So, let us take, for example, $u(t) = X_+(t)(\mathbb{I}-P_+)X_+^{-1}(s)$, with $\bar{T} \le s \le t \le s+2$. Then,

$$u(t) = u(s) + \int_s^t f'_-(\gamma_+(\tau))u(\tau)\, d\tau$$

and hence (using also $|u(s)| \le k$ (see (6.1.2))

$$|u(t)| \le k + K_- \int_s^t |u(\tau)|\, d\tau$$

where $K_- = \sup\{f'_-(\gamma_+(t)) \mid t \ge \bar{T}\}$. From Gronwall inequality (cf Section 2.5.1) we obtain:

$$|X_+(t)(\mathbb{I} - P_+)X_+^{-1}(s)| \le k\mathrm{e}^{K_-(t-s)} \le \hat{k}\mathrm{e}^{\delta(t-s)}, \quad \text{if } \bar{T} \le s \le t \le s+2,$$

where, for example, $\hat{k} = k\max\{1, \mathrm{e}^{2(K_- - \delta)}\}$. By similar arguments we prove that possibly replacing k with a larger value:

$$\begin{aligned}
&\|X_-(t)P_-X_-^{-1}(s)\| \le k\mathrm{e}^{-\delta(t-s)}, && \text{if } s-2 \le s,t \le -\bar{T},\\
&\|X_-(t)(\mathbb{I} - P_-)X_-^{-1}(s)\| \le k\mathrm{e}^{\delta(t-s)}, && \text{if } t-2 \le s,t \le -\bar{T},\\
&\|X_+(t)P_+X_+^{-1}(s)\| \le k\mathrm{e}^{-\delta(t-s)}, && \text{if } \bar{T} \le s,t \le t+2,\\
&\|X_+(t)(\mathbb{I} - P_+)X_+^{-1}(s)\| \le k\mathrm{e}^{\delta(t-s)}, && \text{if } \bar{T} \le s,t \le s+2.
\end{aligned} \tag{6.1.3}$$

We now state our third assumption. It is a kind of nondegeneracy condition of the homoclinic orbit $\gamma(t)$ with respect to $\dot{x} = f_\pm(x)$, that reduces to the known notion of nondegeneracy in the smooth case [35, 36]. This is discussed in more detail in Section 6.1.3.

Let $R_0 : \mathbb{R}^n \to \mathbb{R}^n$ be the projection onto $\mathscr{N}G'(\gamma(\bar{T}))$ along the direction of $\dot{\gamma}_0(\bar{T})$, i. e.

$$R_0 w = w - \frac{G'(\gamma(\bar{T}))w}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})}\dot{\gamma}_0(\bar{T})$$

and $X_0(t)$ be the fundamental solution of the linear system $\dot{z} = f'_+(\gamma_0(t))z$, $-\bar{T} \le t \le \bar{T}$, satisfying $X_0(-\bar{T}) = \mathbb{I}$. Then let

$$\mathscr{S}' = \mathscr{N}P_- \cap \mathscr{N}G'(\gamma(-\bar{T})) \quad \text{and} \quad \mathscr{S}'' = \mathscr{R}P_+ \cap \mathscr{N}G'(\gamma(\bar{T}))\,.$$

Since $\dot{\gamma}_-(-\bar{T}) \notin \mathscr{N}G'(\gamma(-\bar{T}))$, $\dim \mathscr{N}G'(\gamma(-\bar{T})) = n-1$ and $\dot{\gamma}_-(-\bar{T}) \in \mathscr{N}P_-$, we have $\dim[\mathscr{N}P_- + \mathscr{N}G'(\gamma(-\bar{T}))] = n$ and hence:

$$\begin{aligned}
\dim \mathscr{S}' &= \dim[\mathscr{N}P_- \cap \mathscr{N}G'(\gamma(-\bar{T}))]\\
&= \dim \mathscr{N}P_- + \dim \mathscr{N}G'(\gamma(-\bar{T})) - n = \dim \mathscr{N}P_- - 1\,.
\end{aligned}$$

Similarly, from $\dot{\gamma}_+(\bar{T}) \notin \mathscr{N}G'(\gamma(\bar{T}))$, $\dot{\gamma}_+(\bar{T}) \in \mathscr{R}P_+$ and $\dim \mathscr{N}G'(\gamma(\bar{T})) = n-1$, we see that

$$\begin{aligned}
\dim \mathscr{S}'' &= \dim[\mathscr{R}P_+ \cap \mathscr{N}G'(\gamma(\bar{T}))]\\
&= \dim \mathscr{R}P_+ + \dim \mathscr{N}G'(\gamma(\bar{T})) - n = \dim \mathscr{R}P_+ - 1\,.
\end{aligned}$$

We assume that the following condition holds:

(H3) $\mathscr{S}'' + R_0[X_0(\bar{T})\mathscr{S}']$ has codimension 1 in $\mathscr{R}R_0$.

Lemma 6.1.2. *From (H3), the linear subspaces $\mathscr{S}''$ and $\mathscr{S}''' = R_0[X_0(\bar{T})\mathscr{S}']$ intersect transversally in $\mathscr{R}R_0$. Moreover, we have* $\dim \mathscr{S}''' = \dim \mathscr{S}'$.

Proof. We have $\dim \mathscr{S}''' \le \dim \mathscr{S}' = \dim \mathscr{N}P_- - 1$. Moreover from (H3) we get $\dim[\mathscr{S}'' + \mathscr{S}'''] = n-2$, and then:

$$\begin{aligned}\dim\left[\mathscr{S}''\cap\mathscr{S}'''\right] &= \dim\mathscr{S}'' + \dim\mathscr{S}''' - \dim\left[\mathscr{S}''+\mathscr{S}'''\right]\\ &\le \dim\mathscr{R}P_+ - 1 + \dim\mathscr{N}P_- - 1 - (n-2) = \dim\mathscr{R}P_+ + \dim\mathscr{N}P_- - n = 0\,.\end{aligned}$$

So the inequality is an equality and $\dim \mathscr{S}''' = \dim \mathscr{S}'$. The proof is finished. □

According to Lemma 6.1.2, we have a unitary vector $\psi \in \mathscr{R}R_0$ so that

$$\mathbb{R}^n = \operatorname{span}\{\psi\} \oplus \mathscr{N}R_0 \oplus \mathscr{S}'' \oplus \mathscr{S}''' \tag{6.1.4}$$

and

$$\langle \psi, v\rangle = 0, \quad \text{for any } v \in \mathscr{S}'' \oplus \mathscr{S}'''. \tag{6.1.5}$$

The main result of this section is the following:

Theorem 6.1.3. *Assume that $f_\pm(z)$ and $g(t,z,\varepsilon)$ are C^2–functions with bounded derivatives and that their second order derivatives are uniformly continuous. Let conditions (H1), (H2) and (H3) hold. Then there exists a C^2-function $\mathscr{M}(\alpha)$ of the real variable α so that if $\mathscr{M}(\alpha^0) = 0$ and $\mathscr{M}'(\alpha^0) \neq 0$ for some $\alpha^0 \in \mathbb{R}$, then the following hold: there exist $\rho > 0$, $\widetilde{c}_1 > 0$ and $\widetilde{\varepsilon} > 0$ so that for any $0 \neq \varepsilon \in (-\widetilde{\varepsilon}, \widetilde{\varepsilon})$, there exists $\nu_\varepsilon \in (0, |\varepsilon|)$ (cf* (6.1.91)*) so that for any increasing sequence $\mathscr{T} = \{T_m\}_{m\in\mathbb{Z}}$ that satisfies*

$$T_{m+1} - T_m > \bar{T} + 1 - 2\delta^{-1}\ln|\varepsilon| \textit{ for any } m \in \mathbb{Z}$$

along with the following recurrence condition

$$|g(t+T_{2m}, z, 0) - g(t,z,0)| < \nu_\varepsilon \quad \textit{for any } (t,z,m) \in \mathbb{R}^{n+1}\times\mathbb{Z}, \tag{6.1.6}$$

there exist unique sequences $\hat{\alpha} = \{\hat{\alpha}_m\}_{m\in\mathbb{Z}}, \hat{\beta} = \{\hat{\beta}_m\}_{m\in\mathbb{Z}} \in \ell^\infty(\mathbb{R})$ (depending on $\mathscr{T}$ and ε, i.e. $\hat{\alpha} = \hat{\alpha}_{\mathscr{T}}(\varepsilon)$, $\hat{\beta} = \hat{\beta}_{\mathscr{T}}(\varepsilon)$) so that $\sup_{m\in\mathbb{Z}}|\hat{\alpha}_m - \alpha^0| < \widetilde{c}_1|\varepsilon|$, $\sup_{m\in\mathbb{Z}}|\hat{\beta}_m - \alpha^0| < \widetilde{c}_1|\varepsilon|$ and a unique solution $z(t,\mathscr{T},\varepsilon)$ of Eq. (6.1.1) *satisfying*

$$\begin{aligned}&\sup_{t\in[T_{2m-1}+\hat{\beta}_{m-1}, T_{2m}-\bar{T}+\hat{\alpha}_m]}|z(t) - \gamma_-(t - T_{2m} - \hat{\alpha}_m)| < \rho,\\ &\sup_{t\in[T_{2m}-\bar{T}+\hat{\alpha}_m, T_{2m}+\bar{T}+\hat{\beta}_m]}|z(t) - \gamma_0(t - T_{2m} - \hat{\alpha}_m)| < \rho,\\ &\sup_{t\in[T_{2m}+\bar{T}+\hat{\beta}_m, T_{2m+1}+\hat{\beta}_m]}|z(t) - \gamma_+(t - T_{2m} - \hat{\beta}_m)| < \rho.\end{aligned} \tag{6.1.7}$$

We conclude this section with a remark on the projections of the dichotomies of the systems $\dot{x} = f'(\gamma_{\pm}(t))x$ on $[\bar{T},\infty)$ and $(-\infty,-\bar{T}]$:

$$P_{\pm}(t) = X_{\pm}(\pm t)P_{\pm}X_{\pm}^{-1}(\pm t). \tag{6.1.8}$$

Let P_0 be the projection of the dichotomy of the linear system $\dot{x} = f'(0)x$ on $\mathbb{R}$. We have (see Lemma 2.5.1) $\lim_{t\to\infty} \|P_{\pm}(t) - P_0\| = 0$. Thus $T > \bar{T}$ exists so that

$$\mathscr{N}P_{+}(t') \oplus \mathscr{R}P_{-}(t'') = \mathbb{R}^n \quad \text{for any } t',t'' \geq T. \tag{6.1.9}$$

We prove that a positive constant $\widetilde{c}$ exists so that

$$\max\{|x_+|,|x_-|\} \leq \widetilde{c}|x_+ + x_-| \quad \forall (x_+,x_-) \in \mathscr{N}P_{+}(t') \times \mathscr{R}P_{-}(t''). \tag{6.1.10}$$

Since it is clear that $|x_+ + x_-| \leq 2\max\{|x_+|,|x_-|\}$ we get, then, that the two norms $|x_+ + x_-|$ and $\max\{|x_+|,|x_-|\}$ are equivalent. To prove the statement (6.1.10) take $0 < \upsilon < 1/2$ and fix $T > \bar{T}$ so that for any $t',t'' \geq T > \bar{T}$ we have

$$\|P_0 - P_+(t')\| \leq \upsilon, \quad \|P_0 - P_-(t'')\| \leq \upsilon\,.$$

Next consider a linear mapping $A_\upsilon : \mathbb{R}^n \mapsto \mathbb{R}^n$ given by

$$A_\upsilon z := (\mathbb{I} - P_+(t'))z + P_-(t'')z\,.$$

Note that

$$A_\upsilon z = z - \left[(P_+(t') - P_0) + (P_0 - P_-(t'')\right] z\,.$$

Since $\|(P_+(t') - P_0) + (P_0 - P_-(t'')\| \leq 2\upsilon < 1$, A_υ is invertible and

$$\|A_\upsilon\| \leq 1 + 2\upsilon, \quad \|A_\upsilon^{-1}\| \leq 1/(1-2\upsilon)\,.$$

So for any $x \in \mathbb{R}^n$ there is a unique $z \in \mathbb{R}^n$ so that

$$x = A_\upsilon z = x_+ + x_-$$

where $x_+ = (\mathbb{I} - P_+(t'))z \in \mathscr{N}P_+(t')$ and $x_- = P_-(t'')z \in \mathscr{R}P_-(t'')$. Then

$$|x_+| \leq \|\mathbb{I} - P_+(t')\||z| \leq \|\mathbb{I} - P_+(t')\|\|A_\upsilon^{-1}\||x| \leq \frac{\|\mathbb{I} - P_0\| + \upsilon}{1-2\upsilon}|x|\,,$$
$$|x_-| \leq \|P_-(t'')\||z| \leq \|P_-(t'')\|\|A_\upsilon^{-1}\||x| \leq \frac{\|P_0\| + \upsilon}{1-2\upsilon}|x|\,.$$

This proves (6.1.10) with, for example,

$$\widetilde{c} = \frac{\max\{\|\mathbb{I} - P_0\| + \upsilon, \|P_0\| + \upsilon\|\}}{1-2\upsilon} \leq \frac{1 + \|P_0\| + \upsilon}{1-2\upsilon} \leq 2(1 + \|P_0\|)$$

for $\upsilon \leq \frac{1+\|P_0\|}{1+4(1+\|P_0\|)} < \frac{1}{2}$.

6.1.3 Geometric Interpretation of Nondegeneracy Condition

Now we present a geometric meaning of condition (H3). For any $x \in \Omega_0$ near $\gamma(-\bar{T})$ we consider the solution $\phi_-(t,x)$ of $\dot{x} = f_-(x)$ and the solution $\phi_0(t,x)$ of $\dot{x} = f_+(x)$ so that $\phi_-(-\bar{T},x) = \phi_0(-\bar{T},x) = x$, respectively. Similarly, for any $\widetilde{x} \in \Omega_0$ near $\gamma(\bar{T})$ we take a solution $\phi_+(t,\widetilde{x})$ of $\dot{x} = f_-(x)$ so that $\phi_+(\bar{T},\widetilde{x}) = \widetilde{x}$.

By the implicit function theorem, for any $x \in \Omega_0$ near $x_0 := \gamma(-\bar{T})$ there is a unique time $\tau(x)$ so that

$$G(\phi_0(\tau(x),x)) = 0, \quad \tau(x_0) = \bar{T}\,. \tag{6.1.11}$$

In summary, for any $x \in \Omega_0$ near x_0, we have constructed a solution $\phi(t,x)$ of $\dot{x} = f_\pm(x)$ defined as

$$\phi(t,x) = \begin{cases} \phi_-(t,x), & \text{for } t \le -\bar{T}\,, \\ \phi_0(t,x), & \text{for } -\bar{T} \le t \le \tau(x)\,, \\ \phi_+(t-\tau(x)+\bar{T}, \phi_0(\tau(x),x)), & \text{for } \tau(x) \le t\,. \end{cases}$$

We recall the following properties of the function $\phi(t,x)$:

$$\begin{aligned} &\phi_-(t,\gamma(-\bar{T})) = \gamma_-(t), \text{ for } \ t \le -\bar{T}, \\ &\phi_0(t,\gamma(-\bar{T})) = \gamma_0(t), \ \text{ for } \ -\bar{T} \le t \le \bar{T}, \\ &\phi_+(t,\gamma(\bar{T})) = \gamma_+(t), \ \text{ for } \ t \ge \bar{T}, \\ &\phi_0(\tau(x),x) \in \Omega_0, \quad \text{ for any } \ x \in \Omega_0 \text{ (near } \gamma(-\bar{T})) \end{aligned} \tag{6.1.12}$$

and note that from (6.1.12) we get, for any $\eta \in \mathscr{N}G'(\gamma(\bar{T}))$:

$$\left[\frac{\partial \phi_0}{\partial x}(\bar{T},x_0) + \dot{\phi}_0(\bar{T},x_0)\tau'(x_0)\right]\eta \in \mathscr{N}G'(\gamma(\bar{T})) = \mathscr{R}R_0\,. \tag{6.1.13}$$

We are interested in the linearization $\widetilde{\phi}(t) := \frac{\partial \phi}{\partial x}(t,x_0)\eta$ of $\phi(t,x)$ at $x = x_0$ along $\eta \in \mathscr{N}G'(\gamma(-\bar{T})) = T_{\gamma(-\bar{T})}\Omega_0$ that is using $\phi_\pm(\pm\bar{T},x) = x$, $\phi_0(-\bar{T},x) = x$ and (6.1.12):

$$\widetilde{\phi}(t) = \begin{cases} X_-(t)\eta, & t \le -\bar{T}\,, \\ X_0(t)\eta, & -\bar{T} \le t \le \bar{T}\,, \\ \tilde{X}_+(t)\eta, & \bar{T} < t\,, \end{cases}$$

where

$$\begin{aligned} \tilde{X}_+(t) =& \frac{\partial \phi_+}{\partial x}(t,\gamma(\bar{T}))\left[\dot{\phi}_0(\bar{T},x_0)\tau'(x_0) + \frac{\partial \phi_0}{\partial x}(\bar{T},x_0)\right] - \dot{\phi}_+(t,x_0)\tau'(x_0) \\ =& X_+(t)\left[(\dot{\gamma}_0(\bar{T}) - \dot{\gamma}_+(\bar{T}))\tau'(x_0) + X_0(\bar{T})\right]. \end{aligned} \tag{6.1.14}$$

Next, differentiating (6.1.11) we get $G'(\gamma(\bar{T}))\left[\frac{\partial\phi_0}{\partial x}(\bar{T},x_0)+\dot{\phi}_0(\bar{T},x_0)\tau'(x_0)\right]=0$, that is,

$$G'(\gamma(\bar{T}))\left[X_0(\bar{T})+\dot{\gamma}_0(\bar{T})\tau'(x_0)\right]=0.$$

As a consequence, we have, for any $\eta\in\mathscr{N}G'(\gamma(-\bar{T}))$:

$$\tau'(x_0)\eta=-\frac{G'(\gamma(\bar{T}))X_0(\bar{T})\eta}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})}.$$

Plugging everything together and using the definition of R_0, we finally arrive at:

$$\left[\frac{\partial\phi_0}{\partial x}(\bar{T},x_0)+\dot{\phi}_0(\bar{T},x_0)\tau'(x_0)\right]\eta=[X_0(\bar{T})+\dot{\gamma}_0(\bar{T})\tau'(x_0)]\eta=R_0X_0(\bar{T})\eta$$

and

$$\tilde{X}_+(t)\eta=X_+(t)[R_0X_0(\bar{T})\eta-\dot{\gamma}_+(\bar{T})\tau'(x_0)\eta].$$

Now, if $\widetilde{\phi}(t)$ is bounded on $\mathbb{R}$ we need $\eta\in\mathscr{N}P_-$ and hence, being $\eta\in\mathscr{N}G'(\gamma(-\bar{T}))$, we need $\eta\in\mathscr{S}'$. Moreover, since $\dot{\gamma}_+(\bar{T})\in\mathscr{R}P_+$ we see that $\tilde{X}_+(t)\eta$ is bounded on $\mathbb{R}_+$ if and only if so is $X_+(t)R_0X_0(\bar{T})\eta$, i.e. $R_0X_0(\bar{T})\eta\in\mathscr{R}P_+$. But $R_0X_0(\bar{T})\eta\in R_0X_0(\bar{T})\mathscr{S}'\subset\mathscr{R}R_0$. Hence assumption (H3) implies that $R_0X_0(\bar{T})\eta\in(\mathscr{R}R_0\cap\mathscr{R}P_+)\cap R_0X_0(\bar{T})\mathscr{S}'=\mathscr{S}''\cap\mathscr{S}'''=\{0\}$ as we proved in Lemma 6.1.2. In summary we derive the following result.

Theorem 6.1.4. *Condition (H3) is equivalent to, say, that $\widetilde{\phi}(t)$ is bounded if and only if it is equal to zero. This corresponds to some nondegenerate condition on $\gamma(t)$ with respect to $\dot{x}=f_\pm(x)$.*

For the smooth case, i.e. when $f_-(x)=f_+(x)=f(x)\in C^r(\Omega)$, we have $\dot{\gamma}_0(\bar{T})=\dot{\gamma}_+(\bar{T})$ and hence $\widetilde{\phi}(t)=X(t)\eta$ where $X(t)$ is the fundamental matrix of the variational equation $\dot{x}=f'(\gamma(t))x$ along $\gamma(t)$ with $X(-\bar{T})=\mathbb{I}$. Note that $\eta\in T_{\gamma(-\bar{T})}\Omega_0$ and $T_{\gamma(-\bar{T})}\Omega_0$ is a transversal section to the homoclinic solution $\gamma(t)$ at $\gamma(-\bar{T})$. So in the smooth case, Theorem 6.1.4 states that condition (H3) is equivalent to the property that the only bounded solutions of the variational equation $\dot{x}=f'(\gamma(t))x$ are multiples of $\dot{\gamma}(t)$. Hence in the smooth case, condition (H3) is just the well-known *nondegeneracy condition* of $\gamma(t)$ (cf [35]).

Finally, we observe that (6.1.14) can be written as

$$\tilde{X}_+(t)=X_+(t)[\mathbb{I}+S]X_0(\bar{T})$$

where S is the so called *transition matrix* S [8, 13, 14, 19] and is given by

$$Sw:=\left(\dot{\gamma}_+(\bar{T})-\dot{\gamma}_0(\bar{T})\right)\frac{G'(\gamma(\bar{T}))w}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})}=\left(\dot{\gamma}_+(\bar{T})-\dot{\gamma}_0(\bar{T})\right)\frac{((R_0-\mathbb{I})w,\dot{\gamma}_0(\bar{T}))}{\|\dot{\gamma}_0(\bar{T})\|^2}$$

with the last equality following easily from the definition of R_0, where $(\cdot,\cdot)$ is a scalar product on $\mathbb{R}^n$ with the corresponding norm $\|\cdot\|$.

6.1.4 Orbits Close to the Lower Homoclinic Branches

Let $\rho > 0$ be sufficiently small, $\alpha, \beta \in \mathbb{R}$ so that $|\beta - \alpha| < \min\{1, 2\bar{T}\}$, and $\ell_T^\infty(\mathbb{R})$ be the space of doubly infinite sequences $\{T_m\}_{m\in\mathbb{Z}}$ so that $T_{m+1} - T_m \geq T + 1$ where T is chosen so that (6.1.9) holds. Note that $T_m - T_0 \geq mT$ if m is positive and $T_m - T_0 \leq mT$ if m is negative.

In this section we show how to construct solutions $z_m^-(t)$ and $z_m^+(t)$ of (6.1.1) in the intervals $[T_{2m-1} + \alpha - 1, T_{2m} - \bar{T} + \alpha]$ and $[T_{2m} + \bar{T} + \beta, T_{2m+1} + \beta + 1]$ respectively, in such a way that

$$\begin{aligned} &\sup\nolimits_{t\in[T_{2m-1}-1, T_{2m}-\bar{T}]} |z_m^-(t+\alpha) - \gamma_-(t - T_{2m})| < \rho, \\ &\sup\nolimits_{t\in[T_{2m}+\bar{T}, T_{2m+1}+1]} |z_m^+(t+\beta) - \gamma_+(t - T_{2m})| < \rho\,. \end{aligned} \tag{6.1.15}$$

Note that $T_{2m-1} + \alpha - 1 < T_{2m} - \bar{T} + \alpha < T_{2m} + \bar{T} + \beta < T_{2m+1} + \beta + 1$. We show how to construct $z_m^-(t)$ for $t \in [T_{2m-1} + \alpha - 1, T_{2m} - \bar{T} + \alpha]$, the construction of $z_m^+(t)$ for $t \in [T_{2m} + \bar{T} + \beta, T_{2m+1} + \beta + 1]$ is similar. Let

$$\begin{aligned} I_m^- &:= [T_{2m-1} - 1, T_{2m} - \bar{T}], \quad I_m^+ := [T_{2m} + \bar{T}, T_{2m+1} + 1], \\ I_{m,\alpha}^- &:= [T_{2m-1} + \alpha - 1, T_{2m} - \bar{T} + \alpha], \\ I_{m,\beta}^+ &:= [T_{2m} + \bar{T} + \beta, T_{2m+1} + \beta + 1] \end{aligned} \tag{6.1.16}$$

and set, for $t \in I_m^-$

$$x(t) = z_m^-(t+\alpha) - \gamma_-(t - T_{2m})$$

and

$$\begin{aligned} h_m^-(t, x, \alpha, \varepsilon) = {} & f_-(x + \gamma_-(t - T_{2m})) - f_-(\gamma_-(t - T_{2m})) \\ & - f_-'(\gamma_-(t - T_{2m}))x + \varepsilon g(t + \alpha, x + \gamma_-(t - T_{2m}), \varepsilon). \end{aligned} \tag{6.1.17}$$

Then $z_m^-(t)$ satisfies Eq. (6.1.1) for $t \in I_{m,\alpha}^-$ together with (6.1.15) if and only if $x(t)$ is a solution, in I_m^-, of the equation

$$\dot{x} - f_-'(\gamma_-(t - T_{2m}))x = h_m^-(t, x, \alpha, \varepsilon), \tag{6.1.18}$$

so that $\sup_{t\in I_m^-} |x(t)| < \rho$.

Remark 6.1.5. According to Remark 6.1.1, we see that up to the r-th order all derivatives of $h_m^-(t, x, \alpha, \varepsilon)$ with respect to (x, α, ε) are bounded and uniformly continuous in (x, α, ε) uniformly with respect to $t \in I_m^-$ and $m \in \mathbb{Z}$. This statement easily follows from the fact that for $t \leq -\bar{T}$, one has $h_m^-(t + T_{2m}, x, \alpha, \varepsilon) = f_-(x + \gamma_-(t)) - f_-(\gamma_-(t)) - f_-'(\gamma_-(t))x + \varepsilon g(t + T_{2m} + \alpha, x + \gamma_-(t), \varepsilon)$ and the conclusion holds as far as $f(x)$ and $g(t + T_{2m} + \alpha, x + \gamma_-(t), \varepsilon)$ are concerned.

We will need the following Lemma [37, 38]:

Lemma 6.1.6. *Let the linear system $\dot{x} = A(t)x$ have an exponential dichotomy on $(-\infty, -\bar{T}]$ with projection P, and let $X(t)$ be its fundamental matrix so that $X(-\bar{T}) = \mathbb{I}$. Set $P(t) := X(t)PX^{-1}(t)$. Then for any continuous function $h(t) \in C^0([-T, -\bar{T}])$, $\xi_- \in \mathscr{N}P$ and $\varphi_- \in \mathscr{R}P(-T)$, the linear non homogeneous system*

$$\dot{x} = A(t)x + h(t) \tag{6.1.19}$$

has a unique solution $x(t)$ so that

$$(\mathbb{I} - P)x(-\bar{T}) = \xi_-, \quad P(-T)x(-T) = \varphi_- \tag{6.1.20}$$

and this solution satisfies

$$\begin{aligned} x(t) = X(t)\xi_- + X(t)PX^{-1}(-T)\varphi_- + \int_{-T}^{t} X(t)PX^{-1}(s)h(s)ds \\ - \int_{t}^{\bar{T}} X(t)(\mathbb{I} - P)X^{-1}(s)h(s)ds. \end{aligned} \tag{6.1.21}$$

Proof. We can directly verify that (6.1.21) solves (6.1.19) and it satisfies (6.1.20) as well. Next, if $h = 0$, $\xi_- = 0$ and $\varphi_- = 0$, then (6.1.19) implies $x(t) = X(t)x_0$ for some x_0, while (6.1.20) gives $(\mathbb{I} - P)x_0 = 0$ and $X(-T)Px_0 = 0$. Since $X(-T)$ is invertible, we obtain $x_0 = 0$, which yields to the uniqueness of $x(t)$. The proof is finished. □

Remark 6.1.7. From (6.1.2) and (6.1.21) we immediately obtain the following estimate for $|x(t)|$:

$$\sup_{-T \le t \le -\bar{T}} |x(t)| \le k \left[|\xi_-| + |\varphi_-| + 2\delta^{-1} \sup_{-T \le t \le -\bar{T}} |h(t)| \right]. \tag{6.1.22}$$

We apply Lemma 6.1.6 and Remark 6.1.7 with $A(t) = f_-'(\gamma_-(t - T_{2m}))$ in the interval I_m^- (instead of $[-T, -\bar{T}]$). Note that the fundamental matrix $X(t)$ and the projection P of the dichotomy on $(-\infty, T_{2m} - \bar{T}]$ of the linear system $\dot{x} = f_-'(\gamma_-(t - T_{2m}))x$ are $X_-(t - T_{2m})$ and P_-, respectively. Thus, in the notation of (6.1.8) and Lemma 6.1.6 we have

$$\begin{aligned} P_{-,m} : &= P(T_{2m-1} - 1) = X_-(T_{2m-1} - T_{2m} - 1)P_- X_-^{-1}(T_{2m-1} - T_{2m} - 1) \\ &= P_-(T_{2m} - T_{2m-1} + 1). \end{aligned}$$

Set:

$$\|x\|_{I_m^-} = \sup_{t \in I_m^-} |x(t)|.$$

Then a trivial application of Lemma 6.1.6 and (6.1.22) gives the following

Corollary 6.1.8. *Let $h(t) \in C^0(I_m^-)$, $\xi_- \in \mathscr{N}P_-$ and $\varphi_- \in \mathscr{R}P_{-,m}$. Then the linear nonhomogeneous system*

$$\dot{x} = f'_-(\gamma_-(t-T_{2m}))x + h(t)$$

has a unique solution $x(t) \in C^1(I_m^-)$ *so that*

$$(\mathbb{I} - P_-)x(T_{2m} - \bar{T}) = \xi_-, \quad P_{-,m}x(T_{2m-1} - 1) = \varphi_- . \tag{6.1.23}$$

Moreover this solution satisfies (*see* (6.1.22))

$$\|x(t)\|_{I_m^-} \le k\left[|\xi_-| + |\varphi_-| + 2\delta^{-1}\|h(t)\|_{I_m^-}\right] \tag{6.1.24}$$

and

$$\begin{aligned} x(t) = {} & X_-(t-T_{2m})\xi_- + X_-(t-T_{2m})P_-X_-^{-1}(T_{2m-1} - 1 - T_{2m})\varphi_- \\ & + \int_{T_{2m-1}-1}^{t} X_-(t-T_{2m})P_-X_-^{-1}(s-T_{2m})h(s)ds \\ & - \int_{t}^{T_{2m}-\bar{T}} X_-(t-T_{2m})(\mathbb{I}-P_-)X_-^{-1}(s-T_{2m})h(s)ds. \end{aligned} \tag{6.1.25}$$

Using Corollary 6.1.8 we define a map from $C^0(I_m^-) \times \mathscr{N}P_- \times \mathscr{R}P_{-,m} \times \mathbb{R}^2$ into $C^0(I_m^-)$ as

$$(x(t), \xi_-, \varphi_-, \alpha, \varepsilon) \mapsto \hat{x}(t) \tag{6.1.26}$$

where $y(t) = \hat{x}(t)$ is the unique solution given by Corollary 6.1.8 of the equation

$$\dot{y}(t) - f'_-(\gamma_-(t-T_{2m}))y(t) = h_m^-(t, x(t), \alpha, \varepsilon)$$

that satisfies conditions (6.1.23). We observe that the map

$$(x(t), \alpha, \varepsilon) \mapsto h_m^-(t, x(t), \alpha, \varepsilon)$$

is a C^r map from $C^0(I_m^-) \times \mathbb{R}^2$ into $C^0(I_m^-)$ [39] and hence, from (6.1.25) we see that so is the map (6.1.26) from $C^0(I_m^-) \times \mathscr{N}P_- \times \mathscr{R}P_{-,m} \times \mathbb{R}^2$ into $C^0(I_m^-)$. Next, from (6.1.17) we obtain immediately:

$$\|h_m^-(\cdot, x, \alpha, \varepsilon)\| \le \Delta_-(|x|)|x| + N|\varepsilon| \tag{6.1.27}$$

where

$$\Delta_-(r) = \sup\left\{\left|f'_-(x+\gamma_-(t)) - f'_-(\gamma_-(t))\right| \mid t \le -\bar{T}, |x| \le r\right\}$$

is an increasing function so that $\Delta_-(0) = 0$ and

$$N = \sup\left\{|g(t,z,\varepsilon)| \mid (t,z,\varepsilon) \in \mathbb{R}^{n+2}\right\}$$

and hence, using (6.1.24) we get:

$$\|\hat{x}\|_{I_m^-} \le k\left[|\xi_-| + |\varphi_-| + 2\delta^{-1}\Delta_-(\|x\|_{I_m^-})\|x\|_{I_m^-} + 2\delta^{-1}N|\varepsilon|\right]. \tag{6.1.28}$$

Similarly, for fixed $(\xi_-, \varphi_-, \alpha, \varepsilon) \in \mathscr{N}P_- \times \mathscr{R}P_{-,m} \times \mathbb{R}^2$ and $x_1(t), x_2(t) \in C^0(I_m^-)$ we see that

$$\|\hat{x}_2 - \hat{x}_1\|_{I_m^-} \leq 2k\delta^{-1}\left[\Delta_-(\bar{r}) + N'|\varepsilon|\right] \|x_2 - x_1\|_{I_m^-} \tag{6.1.29}$$

where $\bar{r} = \max\{\|x_1\|_{I_m^-}, \|x_2\|_{I_m^-}\}$ and

$$N' = \sup\left\{\left|\frac{\partial g}{\partial x}(t,z,\varepsilon)\right| \mid (t,z,\varepsilon) \in \mathbb{R}^{n+2}\right\}.$$

Thus if $\rho > 0$, $|\xi_-|$, $|\varphi_-|$ and $|\varepsilon|$ are sufficiently small, the map (6.1.26) is a C^r−contraction in the ball of center $x(t) = 0$ and radius ρ in $C^0(I_m^-)$, which is uniform with respect to the other parameters $(\xi_-, \varphi_-, \alpha, \varepsilon)$ and $m \in \mathbb{Z}$. Hence we obtain the following:

Theorem 6.1.9. *Take on (H1), (H2) and let* $(\xi_-, \varphi_-, \alpha, \varepsilon) \in \mathscr{N}P_- \times \mathscr{R}P_{-,m} \times \mathbb{R}^2$, $\rho > 0$ *be such that* $2k\left[|\xi_-| + |\varphi_-| + 2\delta^{-1}N|\varepsilon|\right] \leq \rho$ *and* $4k\delta^{-1}\left[\Delta_-(\rho) + N'|\varepsilon|\right] < 1$. *Then, for* $t \in I_m^-$, *Eq.* (6.1.18) *has a unique bounded solution* $x_m^-(t) = x_m^-(t, \xi_-, \varphi_-, \alpha, \varepsilon)$ *which is* C^r *in the parameters* $(\xi_-, \varphi_-, \alpha, \varepsilon)$ *and* $m \in \mathbb{Z}$, *and satisfies*

$$\|x_m^-(\cdot, \xi_-, \varphi_-, \alpha, \varepsilon)\|_{I_m^-} \leq 2k\left[|\xi_-| + |\varphi_-| + 2\delta^{-1}N|\varepsilon|\right] \leq \rho \tag{6.1.30}$$

together with

$$(\mathbb{I} - P_-)x_m^-(T_{2m} - \bar{T}) = \xi_-, \qquad P_{-,m}x_m^-(T_{2m-1} - 1) = \varphi_-.$$

Moreover the derivatives of $x_m^-(t, \xi_-, \varphi_-, \alpha, \varepsilon)$ *with respect to* $(\xi_-, \varphi_-, \alpha, \varepsilon)$ *are also bounded in* I_m^- *uniformly with respect to* $(\xi_-, \varphi_-, \alpha, \varepsilon)$ *and* $m \in \mathbb{Z}$ *and they are uniformly continuous in* $(\xi_-, \varphi_-, \alpha, \varepsilon)$ *uniformly with respect to* m *and* $t \in I_m^-$.

Proof. Only the last part of the statement needs to be proved. We know that $x_m^-(t, \xi_-, \varphi_-, \alpha, \varepsilon)$ is the unique fixed point of the map given by the right-hand side of Eq. (6.1.25) with $h_m^-(t, x(t), \alpha, \varepsilon)$ instead of $h(t)$. Since $\xi_- \in \mathscr{N}P_-$ we have $|X_-(t - T_{2m})\xi_-| = |X_-(t - T_{2m})(\mathbb{I} - P_-)X_-(-\bar{T})\xi_-| \leq k\mathrm{e}^{\delta(t - T_{2m} - \bar{T})}|\xi_-| \leq k|\xi_-|$ for any $t \in I_m^-$. A similar argument shows that $|X_-(t - T_{2m})P_-X_-^{-1}(T_{2m-1} - 1 - T_{2m})\varphi_-| \leq k|\varphi_-|$ for any $t \in I_m^-$. As a consequence, the right-hand side of (6.1.25) consists of a bounded linear map in (ξ_-, φ_-), with bound independent of $m \in \mathbb{Z}$, and the nonlinear map from $C_b^0(I_m^-) \times \mathbb{R} \times \mathbb{R}$:

$$(x(\cdot), \alpha, \varepsilon) \mapsto \int_{T_{2m-1}-1}^{t} X_-(t - T_{2m})P_-X_-^{-1}(s - T_{2m})h_m^-(s, x(s), \alpha, \varepsilon)ds$$
$$- \int_{t}^{T_{2m} - \bar{T}} X_-(t - T_{2m})(\mathbb{I} - P_-)X_-^{-1}(s - T_{2m})h_m^-(s, x(s), \alpha, \varepsilon)ds$$

whose derivatives up to the r-th order are bounded and uniformly continuous in (x, α, ε) uniformly with respect to m because of the properties of $h_m^-(t, x, \alpha, \varepsilon)$ (see Remark 6.1.5 and 6.1.2). The proof is complete. □

We are now ready to prove the main result of this section:

Theorem 6.1.10. *Take on (H1), (H2) and let* $(\xi_-,\varphi_-,\alpha,\varepsilon)\in \mathscr{N}P_-\times\mathscr{R}P_{-,m}\times\mathbb{R}^2$, $\rho>0$ *be such that* $2k\left[|\xi_-|+|\varphi_-|+2\delta^{-1}N|\varepsilon|\right]\le\rho$ *and* $4k\delta^{-1}\left[\Delta_-(\rho)+N'|\varepsilon|\right]<1$. *Then, for* $t\in I^-_{m,\alpha}$, *equation* $\dot z=f_-(z)+\varepsilon g(t,z,\varepsilon)$ *has a unique bounded solution* $z^-_m(t)=z^-_m(t,\xi_-,\varphi_-,\alpha,\varepsilon)$ *which is* C^r *in the parameters* $(\xi_-,\varphi_-,\alpha,\varepsilon)$ *and satisfies*

$$\|z^-_m(\cdot+\alpha,\xi_-,\varphi_-,\alpha,\varepsilon)-\gamma_-(\cdot-T_{2m})\|_{I^-_m}\le 2k\left[|\xi_-|+|\varphi_-|+2\delta^{-1}N|\varepsilon|\right]\le\rho \tag{6.1.31}$$

together with

$$(\mathbb{I}-P_-)[z^-_m(T_{2m}-\bar T+\alpha)-\gamma_-(-\bar T)]=\xi_-,$$
$$P_{-,m}[z^-_m(T_{2m-1}+\alpha-1)-\gamma_-(T_{2m-1}-T_{2m}-1)]=\varphi_-.$$

Moreover $x^-_m(t):=z^-_m(t+\alpha,\xi_-,\varphi_-,\alpha,\varepsilon)-\gamma_-(t-T_{2m})$ *is the unique fixed point of the map (6.1.25) and* $z^-_m(t,\xi_-,\varphi_-,\alpha,\varepsilon)$ *and its derivatives with respect to* $(\xi_-,\varphi_-,\alpha,\varepsilon)$ *are also bounded in* I^-_m *uniformly with respect to* $(\xi_-,\varphi_-,\alpha,\varepsilon)$ *and* $m\in\mathbb{Z}$, *uniformly continuous in* $(\xi_-,\varphi_-,\alpha,\varepsilon)$ *uniformly with respect to* (t,m) *with* $t\in I^-_m$, $m\in\mathbb{Z}$ *and satisfy:*

$$\frac{\partial z^-_m}{\partial\xi_-}(t+\alpha,0,0,\alpha,0)=X_-(t-T_{2m})(\mathbb{I}-P_-),$$
$$\frac{\partial z^-_m}{\partial\varphi_-}(t+\alpha,0,0,\alpha,0)\varphi_-=X_-(t-T_{2m})P_-X_-^{-1}(T_{2m-1}-T_{2m}-1)\varphi_-$$
$$\frac{\partial z^-_m}{\partial\varepsilon}(t+\alpha,0,0,\alpha,0) \tag{6.1.32}$$
$$=\int_{T_{2m-1}-1}^{t}X_-(t-T_{2m})P_-X_-^{-1}(s-T_{2m})g(s+\alpha,\gamma_-(s-T_{2m}),0)ds$$
$$-\int_t^{T_{2m}-\bar T}X_-(t-T_{2m})\left(\mathbb{I}-P_-\right)X_-^{-1}(s-T_{2m})g(s+\alpha,\gamma_-(s-T_{2m}),0)ds$$

Proof. Setting $x(t):=z^-_m(t+\alpha)-\gamma_-(t-T_{2m})$ the existence of $z^-_m(t,\xi_-,\varphi_-,\alpha,\varepsilon)$ satisfying (6.1.31) follows from Theorem 6.1.9. Thus we only need to prove (6.1.32). From (6.1.28) we see that $x^-_m(t,0,0,\alpha,0)=0$ and then differentiating equation (6.1.25) with $x^-_m(t,\xi_-,\varphi_-,\alpha,\varepsilon)$ instead of $x(t)$ and $h^-_m(t,x^-_m(t,\xi_-,\varphi_-,\alpha,\varepsilon),\alpha,\varepsilon)$ instead of $h(t)$ we see that

$$\frac{\partial z^-_m}{\partial\xi_-}(t+\alpha,0,0,\alpha,0)\xi_-=\frac{\partial x^-_m}{\partial\xi_-}(t,0,0,\alpha,0)\xi_-=X_-(t-T_{2m})\xi_-.$$

Similarly we obtain the rest of (6.1.32). □

Remark 6.1.11. The function $z^-_m(t)=z^-_m(t,\xi_-,\varphi_-,\alpha,\varepsilon)$ is a bounded solution of Eq. (6.1.1) in the interval $I^-_{m,\alpha}$ as long as it remains in Ω_- for $t\in I^-_{m,\alpha}$, and satisfies (6.1.31). However in order that $z_m^-(t) \in \Omega_-$ for $t \in I_{m,\alpha}^-$ it is sufficient that $G(z_m^-(T_{2m} - \bar{T} + \alpha)) = 0$. This follows directly from (H2) and (6.1.31).

Next, let

$$\begin{aligned}
\Delta_+(r) &:= \sup\left\{\left|f_-'(x+\gamma_+(t)) - f_-'(\gamma_+(t))\right| \mid \bar{T} \le t, |x| \le r\right\}, \\
P_{+,m} &:= P_+(T_{2m+1} - T_{2m} + 1) \\
&= X_+(T_{2m+1} - T_{2m} + 1) P_+ X_+(T_{2m+1} - T_{2m} + 1)^{-1}, \qquad (6.1.33) \\
h_m^+(t,x,\beta,\varepsilon) &= f_-(x+\gamma_+(t-T_{2m})) - f_-(\gamma_+(t-T_{2m})) \\
&\quad - f_-'(\gamma_+(t-T_{2m}))x + \varepsilon g(t+\beta, x+\gamma_+(t-T_{2m}), \varepsilon).
\end{aligned}$$

By an almost identical argument we show the following:

Theorem 6.1.12. *Take on (H1), (H2) and let $(\xi_+, \varphi_+, \beta, \varepsilon) \in \mathscr{R}P_+ \times \mathscr{N}P_{+,m} \times \mathbb{R}^2$ and $\rho > 0$ be such that $2k\left[|\xi_+| + |\varphi_+| + 2\delta^{-1}N|\varepsilon|\right] \le \rho$ and $4k\delta^{-1}\left[\Delta_+(\rho) + N'|\varepsilon|\right] < 1$. Then, for $t \in I_{m,\beta}^+$, equation $\dot{z} = f_+(z) + \varepsilon g(t,z,\varepsilon)$ has a unique bounded solution $z_m^+(t) = z_m^+(t,\xi_+,\varphi_+,\beta,\varepsilon)$ which is C^r in the parameters $(\xi_+,\varphi_+,\beta,\varepsilon)$ and satisfies*

$$\|z_m^+(\cdot+\beta,\xi_+,\varphi_+,\beta,\varepsilon) - \gamma_+(\cdot - T_{2m})\|_{I_m^+} \le 2k\left[|\xi_+| + |\varphi_+| + 2\delta^{-1}N|\varepsilon|\right] \le \rho \qquad (6.1.34)$$

together with

$$\begin{aligned}
P_+[z_m^+(T_{2m} + \bar{T} + \beta) - \gamma_+(\bar{T})] &= \xi_+, \\
(\mathbb{I} - P_{+,m})[z_m^+(T_{2m+1} + \beta + 1) - \gamma_+(T_{2m+1} - T_{2m} + 1)] &= \varphi_+.
\end{aligned}$$

Moreover $x_m^+(t) := z_m^+(t+\beta,\xi_+,\varphi_+,\beta,\varepsilon) - \gamma_+(t - T_{2m})$ is the unique fixed point of the map

$$\begin{aligned}
(x(t),\xi_+,\varphi_+,\ \beta,\varepsilon) \mapsto \\
& X_+(t-T_{2m})\xi_+ + X_+(t-T_{2m})(\mathbb{I} - P_+)X_+^{-1}(T_{2m+1} - T_{2m} + 1)\varphi_+ \\
&+ \int_{T_{2m}+\bar{T}}^t X_+(t-T_{2m})P_+X_+^{-1}(s-T_{2m})h_m^+(s,x(s),\beta,\varepsilon)ds \\
&- \int_t^{T_{2m+1}+1} X_+(t-T_{2m})(\mathbb{I} - P_+)X_+^{-1}(s-T_{2m})h_m^+(s,x(s),\beta,\varepsilon)ds,
\end{aligned} \qquad (6.1.35)$$

and $z_m^+(t,\xi_+,\varphi_+,\beta,\varepsilon)$ and its derivatives with respect to $(\xi_+,\varphi_+,\beta,\varepsilon)$ are also bounded in I_m^+ uniformly with respect to $(\xi_+,\varphi_+,\beta,\varepsilon)$ and $m \in \mathbb{Z}$, uniformly continuous in $(\xi_+,\varphi_+,\beta,\varepsilon)$ uniformly with respect to (t,m) with $t \in I_m^+$, $m \in \mathbb{Z}$ and satisfy:

$$\frac{\partial z_m^+}{\partial \xi_+}(t+\beta,0,0,\beta,0) = X_+(t-T_{2m})P_+$$

$$\frac{\partial z_m^+}{\partial \varphi_+}(t+\beta,0,0,\beta,0)\varphi_+ = X_+(t-T_{2m})(\mathbb{I}-P_+)X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi_+$$

$$\begin{aligned}&\frac{\partial z_m^+}{\partial \varepsilon}(t+\beta,0,0,\beta,0)\\ &\quad = \int_{T_{2m}+\bar{T}}^{t} X_+(t-T_{2m})P_+X_+^{-1}(s-T_{2m})g(s+\beta,\gamma_+(s-T_{2m}),0)ds\\ &\qquad - \int_{t}^{T_{2m+1}+1} X_+(t-T_{2m})(\mathbb{I}-P_+)X_+^{-1}(s-T_{2m})g(s+\beta,\gamma_+(s-T_{2m}),0)ds.\end{aligned} \tag{6.1.36}$$

Remark 6.1.13. Note that $z_m^-(t,\xi_-,\varphi_-,\alpha,\varepsilon)$ (resp. $z_m^+(t,\xi_+,\varphi_+,\alpha,\varepsilon)$) depends on m by means of T_{2m-1} and T_{2m} (resp. T_{2m} and T_{2m+1}). Consequently, we may also write $x^-(t,\xi_-,\varphi_-,\alpha,\varepsilon,T_{2m},T_{2m-1})$, $x^+(t,\xi_+,\varphi_+,\alpha,\varepsilon,T_{2m},T_{2m+1})$ instead of $x_m^-(t,\xi_-,\varphi_-,\alpha,\varepsilon)$, $x_m^+(t,\xi_+,\varphi_+,\alpha,\varepsilon)$ and say that $x^-(t,\xi_-,\varphi_-,\alpha,\varepsilon,T_{2m},T_{2m-1})$, $x^+(t,\xi_+,\varphi_+,\alpha,\varepsilon,T_{2m},T_{2m+1})$, respectively, is uniformly continuous with respect to $(\xi_-,\varphi_-,\alpha,\varepsilon)$, resp. $(\xi_+,\varphi_+,\beta,\varepsilon)$, uniformly with respect to T_{2m},T_{2m-1}, resp. T_{2m},T_{2m+1}, and $t\in I_m^-$, (resp. $t\in I_m^+$).

6.1.5 Orbits Close to the Upper Homoclinic Branch

Theorem 6.1.14. *Take on (H1), (H2). Then there exist positive constants c, ε_0 and $\widetilde{\rho}_0$ so that for any $\alpha,\beta,\varepsilon\in\mathbb{R}$ and $\bar{\xi}\in\mathbb{R}^n$ so that $|\beta-\alpha|<\min\{1,2\bar{T}\}$, $|\varepsilon|\le\varepsilon_0$ and $|\bar{\xi}-\gamma_0(-\bar{T})|<\widetilde{\rho}_0$, there exists a unique solution $z_m^0(t)=z_m^0(t,\bar{\xi},\alpha,\beta,\varepsilon)$ of equation $\dot{z}=f_+(z)+\varepsilon g(t,z,\varepsilon)$, for $t\in[T_{2m}-\bar{T}+\alpha,T_{2m}+\bar{T}+\beta]$ so that*

$$z_m^0(T_{2m}-\bar{T}+\alpha)=\bar{\xi}$$

and

$$\|z_m^0(t)-\gamma_0(t-T_{2m}-\alpha)\|_{[T_{2m}-\bar{T}+\alpha,T_{2m}+\bar{T}+\beta]}\le c[|\bar{\xi}-\gamma_0(-\bar{T})|+2N\delta^{-1}|\varepsilon|]\,. \tag{6.1.37}$$

Moreover $z_m^0(t,\bar{\xi},\alpha,\beta,\varepsilon)$ and its derivatives with respect to $(\bar{\xi},\alpha,\beta,\varepsilon)$ are bounded in $[T_{2m}-\bar{T}+\alpha,T_{2m}+\bar{T}+\beta]$ uniformly with respect to $m\in\mathbb{Z}$, uniformly continuous in $(\bar{\xi},\alpha,\beta,\varepsilon)$, uniformly with respect to $t\in[T_{2m}-\bar{T}+\alpha,T_{2m}+\bar{T}+\beta]$, $m\in\mathbb{Z}$, and have the following properties:

(i) $x_m^0(t)=z_m^0(t+\alpha,\bar{\xi},\alpha,\beta,\varepsilon)-\gamma_0(t-T_{2m})$ *is a fixed point of the map*

$$\begin{aligned}x(t)\mapsto{}& X_0(t-T_{2m})\left[\bar{\xi}-\gamma_0(-\bar{T})\right]\\ &+\int_{T_{2m}-\bar{T}}^{t}X_0(t-T_{2m})X_0^{-1}(s-T_{2m})h_m^0(s,x(s),\alpha,\varepsilon)ds\end{aligned} \tag{6.1.38}$$

where

$$h_m^0(t,x,\alpha,\varepsilon) = f_+(x+\gamma_0(t-T_{2m})) - f_+(\gamma_0(t-T_{2m})) - f'_+(\gamma_0(t-T_{2m}))x$$
$$+\varepsilon g(t+\alpha, x+\gamma_0(t-T_{2m}),\varepsilon).$$

(ii) *The following equalities hold:*

$$\begin{aligned}
&\frac{\partial z_m^0}{\partial \alpha}(t,\gamma_0(\bar{T}),\alpha,\beta,0) = -\dot{\gamma}_0(t-T_{2m}-\alpha),\\
&\frac{\partial z_m^0}{\partial \beta}(t,\gamma_0(-\bar{T}),\alpha,\beta,0) = 0,\\
&\frac{\partial z_m^0}{\partial \bar{\xi}}(t,\gamma_0(-\bar{T}),\alpha,\beta,0) = X_0(t-T_{2m}-\alpha),\\
&\frac{\partial z_m^0}{\partial \varepsilon}(t+\alpha,\gamma_0(-\bar{T}),\alpha,\beta,0)\\
&\quad = \int_{T_{2m}-\bar{T}}^{t} X_0(t-T_{2m})X_0^{-1}(s-T_{2m})g(s+\alpha,\gamma_0(s-T_{2m}),0)\,ds.
\end{aligned} \tag{6.1.39}$$

Proof. The statement concerning the existence of the solution $z_m^0(t) = z_m^0(t,\xi,\alpha,\beta,\varepsilon)$ from which (6.1.37) holds, follows from the continuous dependence on the data. Moreover the fact that $x_m^0(t)$ is a fixed point of the map (6.1.38) follows from the variation of constants formula. The boundedness and continuity properties of $z_m^0(t,\xi,\alpha,\beta,\varepsilon)$ follow from the similar properties of $h_m^0(t,x,\alpha,\varepsilon)$ as in Theorems 6.1.10, 6.1.12. Then, because of uniqueness of fixed points we also get:

$$z_m^0(t,\gamma_0(-\bar{T}),\alpha,\beta,0) = \gamma_0(t-T_{2m}-\alpha)$$

from which the first two equalities of point (ii) easily follow. Differentiating (6.1.38) with respect to ξ, ε respectively and using the fact that $h_m^0(t,x,\alpha,0)$ is of the second order in x, we derive the other two equalities in (ii). □

Note that if

$$c[\widetilde{\rho}_0 + 2N\delta^{-1}\varepsilon_0] < \rho$$

from (6.1.37) we obtain:

$$\sup\{|z_m^0(t+\alpha) - \gamma_0(t-T_{2m})| \mid t \in [T_{2m}-\bar{T}, T_{2m}+\bar{T}+\beta-\alpha]\} < \rho. \tag{6.1.40}$$

Remark 6.1.15. Note that $z_m^0(t,\bar{\xi},\alpha,\beta,\varepsilon))$ depends on m by means of T_{2m}. Thus we may also write $z^0(t,\bar{\xi},\alpha,\beta,\varepsilon,T_{2m})$ instead of $z_m^0(t,\bar{\xi},\alpha,\beta,\varepsilon)$ and say that $z^0(t,\bar{\xi},\alpha,\beta,\varepsilon,T_{2m})$ is uniformly continuous in $(\bar{\xi},\alpha,\beta,\varepsilon)$ uniformly with respect to T_{2m} and $t \in [T_{2m}-\bar{T}+\alpha, T_{2m}+\bar{T}+\beta]$.

6.1.6 Bifurcation Equation

Let $\varepsilon_0 > 0$, $\widetilde{\rho}_0 > 0$ and $c > 0$ be constants as in Theorem 6.1.14, $C := \max\{c, 2k\}$, $\chi < 1$ a positive constant that will be specified and fixed below and $\rho_0 \leq c\widetilde{\rho}_0$ be the largest positive number satisfying

$$4k\delta^{-1}\left[\Delta_{\pm}(\rho_0) + \frac{N'\delta}{2NC}\rho_0\right] \leq 1.$$

Next, let $0 < \rho < \rho_0$ and $\varepsilon_\rho := \min\left\{\frac{\rho\delta}{2CN}, \varepsilon_0\right\}$. For any $\alpha = \{\alpha_m\}_{m\in\mathbb{Z}} \in \ell^\infty(\mathbb{R})$ and $\varepsilon \in (-\varepsilon_\rho, \varepsilon_\rho)$ we set

$$\begin{aligned}
&\ell^\infty_{\rho,\alpha,\varepsilon} := \Big\{\theta := \big\{(\varphi_m^-, \varphi_m^+, \xi_m^-, \xi_m^+, \bar{\xi}_m, \beta_m)\big\}_{m\in\mathbb{Z}} \in \ell^\infty(\mathbb{R}^{5n+1}) : \\
&(\varphi_m^-, \varphi_m^+, \xi_m^-, \xi_m^+, \bar{\xi}_m, \beta_m) \in \mathscr{R}P_{-,m} \times \mathscr{N}P_{+,m} \times \mathscr{N}P_- \times \mathscr{R}P_+ \times \mathbb{R}^{n+1}, \\
&2k\left[|\xi_m^\pm| + |\varphi_m^\pm| + 2\delta^{-1}N|\varepsilon|\right] < \rho, \quad c[|\bar{\xi}_m - \gamma_0(-\bar{T})| + 2N\delta^{-1}|\varepsilon|] < \rho, \\
&\sup_{m\in\mathbb{Z}} |\alpha_{m+1} - \beta_m| < \chi\Big\}
\end{aligned}$$

and

$$\ell^\infty_\rho = \Big\{(\theta, \alpha, \varepsilon) \in \ell^\infty_{\rho,\alpha,\varepsilon} \times \ell^\infty(\mathbb{R}) \times (-\varepsilon_\rho, \varepsilon_\rho) : \alpha \in \ell^\infty_\chi\Big\}$$

where

$$\ell^\infty_\chi = \Big\{\alpha \in \ell^\infty(\mathbb{R}) : \sup_{m\in\mathbb{Z}} |\alpha_m - \alpha_{m-1}| < \chi\Big\}.$$

Note that because of the choice of ρ, ε_ρ, $\ell^\infty_{\rho,\alpha,\varepsilon}$, ℓ^∞_ρ and ℓ^∞_χ are open nonempty subsets of

$$\ell^\infty(\mathscr{R}P_{-,m} \times \mathscr{N}P_{+,m} \times \mathscr{N}P_- \times \mathscr{R}P_+ \times \mathbb{R}^n \times \mathbb{R}),$$

$$\ell^\infty(\mathscr{R}P_{-,m} \times \mathscr{N}P_{+,m} \times \mathscr{N}P_- \times \mathscr{R}P_+ \times \mathbb{R}^n \times \mathbb{R}) \times \ell^\infty(\mathbb{R}) \times (-\varepsilon_\rho, \varepsilon_\rho)$$

and $\ell^\infty(\mathbb{R})$, respectively. In $\ell^\infty_{\rho,\alpha,\varepsilon}$ we take the norm

$$\begin{aligned}
\|\theta\| &= \left\|\big\{(\varphi_m^-, \varphi_m^+, \xi_m^-, \xi_m^+, \bar{\xi}_m, \beta_m)\big\}_{m\in\mathbb{Z}}\right\| \\
&= \sup_{m\in\mathbb{Z}} \max\big\{|\varphi_m^- + \varphi_m^+|, |\xi_m^-|, |\xi_m^+|, |\bar{\xi}_m|, |\beta_m|\big\}.
\end{aligned}$$

Let $\mathscr{T} = \{T_m\}_{m\in\mathbb{Z}}$ be given as in Section 6.1.4 and take $(\theta, \alpha, \varepsilon) \in \ell^\infty_\rho$. In this section we want to find such conditions that system (6.1.1) has a solution $z(t)$ defined on $\mathbb{R}$ so that any $m \in \mathbb{Z}$ satisfies:

$$\begin{aligned}
&\|z(t) - \gamma_-(t - T_{2m} - \alpha_m)\|_{\widetilde{I}_m^-} < \rho, \\
&\|z(t) - \gamma_0(t - T_{2m} - \alpha_m)\|_{\widetilde{I}_m^0} < \rho, \\
&\|z(t) - \gamma_+(t - T_{2m} - \beta_m)\|_{\widetilde{I}_m^+} < \rho
\end{aligned}$$

where $\widetilde{I}_m^- = [T_{2m-1}+\alpha_m-1, T_{2m}-\bar{T}+\alpha_m]$, $I_m^0 = [T_{2m}-\bar{T}+\alpha_m, T_{2m}+\bar{T}+\beta_m]$ and $\widetilde{I}_m^+ = [T_{2m}+\bar{T}+\beta_m, T_{2m+1}+\beta_m]$.

We note that for any $(\theta,\alpha,\varepsilon)\in\ell_\rho^\infty$ assumptions of Theorems 6.1.10, 6.1.12 and 6.1.14 are satisfied. Indeed we have

$$4k\delta^{-1}\left[\Delta_\pm(\rho)+N'|\varepsilon|\right] < 4k\delta^{-1}\left[\Delta_\pm(\rho)+N'\varepsilon_\rho\right] < 4k\delta^{-1}\left[\Delta_\pm(\rho_0)+\frac{N'\delta}{2NC}\rho_0\right] \le 1$$

along with $|\varepsilon|<\varepsilon_0$ and

$$|\bar{\xi}-\gamma_0(-\bar{T})| < \frac{\rho}{c} < \frac{\rho_0}{c} \le \widetilde{\rho}_0 .$$

So according to the previous sections and because of uniqueness of the solutions $z_m^+(t,\xi_m^+,\varphi_m^+,\beta_m,\varepsilon)$, $z_m^-(t,\xi_m^-,\varphi_m^-,\alpha_m,\varepsilon)$ and $z_m^0(t,\bar{\xi}_m,\alpha_m,\beta_m,\varepsilon)$ we see that such a solution can be found if and only if we are able to solve the infinite set of equations $(m\in\mathbb{Z})$:

$$\begin{cases} z_m^+(T_{2m+1}+\beta_m,\xi_m^+,\varphi_m^+,\beta_m,\varepsilon) - z_{m+1}^-(T_{2m+1}+\beta_m,\xi_{m+1}^-,\varphi_{m+1}^-,\alpha_{m+1},\varepsilon) = 0,\\ z_m^0(T_{2m}-\bar{T}+\alpha_m,\bar{\xi}_m,\alpha_m,\beta_m,\varepsilon) - z_m^-(T_{2m}-\bar{T}+\alpha_m,\xi_m^-,\varphi_m^-,\alpha_m,\varepsilon) = 0,\\ z_m^0(T_{2m}+\bar{T}+\beta_m,\bar{\xi}_m,\alpha_m,\beta_m,\varepsilon) - z_m^+(T_{2m}+\bar{T}+\beta_m,\xi_m^+,\varphi_m^+,\beta_m,\varepsilon) = 0,\\ G(z_m^-(T_{2m}-\bar{T}+\alpha_m,\xi_m^-,\varphi_m^-,\alpha_m,\varepsilon)) = 0,\\ G(z_m^0(T_{2m}+\bar{T}+\beta_m,\bar{\xi}_m,\alpha_m,\beta_m,\varepsilon)) = 0,\\ G(z_m^+(T_{2m}+\bar{T}+\beta_m,\xi_m^+,\varphi_m^+,\beta_m,\varepsilon)) = 0\ . \end{cases} \tag{6.1.41}$$

Since $T_{2m+1}+\alpha_{m+1}-1 < T_{2m+1}+\beta_m$, system (6.1.41) is well posed. Note that from Theorem 6.1.14, the second of the above equations reads:

$$\bar{\xi}_m = z_m^-(T_{2m}-\bar{T}+\alpha_m,\xi_m^-,\varphi_m^-,\alpha_m,\varepsilon)$$

and gives the sequence $\{\bar{\xi}_m\}_{m\in\mathbb{Z}}$ in terms of the sequences $\{\xi_m^-\}_{m\in\mathbb{Z}}$, $\{\varphi_m^-\}_{m\in\mathbb{Z}}$, $\{\alpha_m\}_{m\in\mathbb{Z}}$, and ε. Moreover, if ρ is sufficiently small, $z_m^0(T_{2m}+\bar{T}+\beta_m,\bar{\xi}_m,\alpha_m,\beta_m,\varepsilon)$ is close to $\gamma_0(\bar{T}+\beta_m-\alpha_m)$, while $z_m^+(T_{2m}+\bar{T}+\beta_m,\xi_m^+,\varphi_m^+,\beta_m,\varepsilon)$ is close to $\gamma_+(\bar{T})=\gamma_0(\bar{T})$. So there is a positive constant $\chi<\min\{1,2\bar{T}\}$ so that the 5th and the 6th equations in (6.1.41) imply that the 3rd equation is equivalent to

$$R_0\left[z_m^0(T_{2m}+\bar{T}+\beta_m,\bar{\xi}_m,\alpha_m,\beta_m,\varepsilon) - z_m^+(T_{2m}+\bar{T}+\beta_m,\xi_m^+,\varphi_m^+,\beta_m,\varepsilon)\right] = 0$$

where $R_0:\mathbb{R}^n\to\mathbb{R}^n$ is the projection defined in Section 6.1.2. From now on, we fix such a χ. Here we use the fact $|\beta_m-\alpha_m|<2\chi$ for any $m\in\mathbb{Z}$, so $\gamma_0(\bar{T}+\beta_m-\alpha_m)$ and $\gamma_0(\bar{T})$ are sufficiently close for χ is small enough uniformly for any $m\in\mathbb{Z}$.

Let

$$\ell_1^\infty = \ell^\infty(\mathbb{R}^n\times\mathbb{R}^n\times\mathscr{R}R_0\times\mathbb{R}\times\mathbb{R}\times\mathbb{R})$$

with the norm

$$\sup_{m\in\mathbb{Z}} \max\left\{|a_m|,|b_m|,|c_m|,|d_m|,|e_m|,|f_m|\right\}$$

for $\{(a_m,b_m,c_m,d_m,e_m,f_m)\}_{m\in\mathbb{Z}} \in \ell_1^\infty$. We define a map $\mathscr{G}_{\mathscr{T}} \in C^r\left(\ell_\rho^\infty, \ell_1^\infty\right)$ as

$$\mathscr{G}_{\mathscr{T}}(\theta,\alpha,\varepsilon) = \mathscr{G}_{\mathscr{T}}(\{(\varphi_m^-,\varphi_m^+,\xi_m^-,\xi_m^+,\bar{\xi}_m,\beta_m)\}_{m\in\mathbb{Z}},\{\alpha_m\}_{m\in\mathbb{Z}},\varepsilon) :=$$

$$\left\{\begin{array}{c} z_m^+(T_{2m+1}+\beta_m,\xi_m^+,\varphi_m^+,\beta_m,\varepsilon) - z_{m+1}^-(T_{2m+1}+\beta_m,\xi_{m+1}^-,\varphi_{m+1}^-,\alpha_{m+1},\varepsilon) \\ \bar{\xi}_m - z_m^-(T_{2m}-\bar{T}+\alpha_m,\xi_m^-,\varphi_m^-,\alpha_m,\varepsilon) \\ R_0[z_m^0(T_{2m}+\bar{T}+\beta_m,\bar{\xi}_m,\alpha_m,\beta_m,\varepsilon) - z_m^+(T_{2m}+\bar{T}+\beta_m,\xi_m^+,\varphi_m^+,\beta_m,\varepsilon)] \\ G(z_m^-(T_{2m}-\bar{T}+\alpha_m,\xi_m^-,\varphi_m^-,\alpha_m,\varepsilon)) \\ G(z_m^0(T_{2m}+\bar{T}+\beta_m,\bar{\xi}_m,\alpha_m,\beta_m,\varepsilon)) \\ G(z_m^+(T_{2m}+\bar{T}+\beta_m,\xi_m^+,\varphi_m^+,\beta_m,\varepsilon)) \end{array}\right\}_{m\in\mathbb{Z}}$$

so that Eq. (6.1.41) reads

$$\mathscr{G}_{\mathscr{T}}(\theta,\alpha,\varepsilon) = 0. \tag{6.1.42}$$

Before giving our main result we state few properties of the map $\mathscr{G}_{\mathscr{T}}$. First, from [39] it follows that $\mathscr{G}_{\mathscr{T}}$ is C^r and has bounded derivatives. More precisely, from the continuity properties of the solutions $z_m^+(t,\xi_m^+,\varphi_m^+,\beta_m,\varepsilon)$, $z_m^-(t,\xi_m^-,\varphi_m^-,\alpha_m,\varepsilon)$, and $z_m^0(t,\bar{\xi}_m,\alpha_m,\beta_m,\varepsilon)$ we see that $\mathscr{G}_{\mathscr{T}}(\theta,\alpha,\varepsilon)$ and its derivatives are bounded and uniformly continuous in $(\theta,\alpha,\varepsilon)$ uniformly with respect to $\mathscr{T} \in \ell_T^\infty(\mathbb{R})$. Next, for any $\alpha \in \ell_\chi^\infty$, we set:

$$\theta_\alpha = \{(0,0,0,0,\gamma_0(-\bar{T}),\alpha_m)\}_{m\in\mathbb{Z}}.$$

From (6.1.31), (6.1.34), (6.1.37), and $G(\gamma_\pm(\pm\bar{T}))=0$, $\gamma_\pm(\pm\bar{T})=\gamma_0(\pm\bar{T})$, we get

$$\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0) = \left\{\begin{array}{c} \gamma_+(T_{2m+1}-T_{2m}) - \gamma_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1}) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{array}\right\}_{m\in\mathbb{Z}}.$$

Now, for $t \geq T$ we have

$$|\gamma_+(t)| \leq \int_t^\infty |\dot{\gamma}_+(s)|ds \leq \int_t^\infty k\,\mathrm{e}^{-\delta(s-\bar{T})}\,|\dot{\gamma}_+(\bar{T})|ds = k\delta^{-1}\,\mathrm{e}^{-\delta(t-\bar{T})}\,|\dot{\gamma}_+(\bar{T})|$$

and similarly

$$|\gamma_-(t)| \leq k\delta^{-1}\,\mathrm{e}^{\delta(t+\bar{T})}\,|\dot{\gamma}_-(-\bar{T})|$$

for any $t \leq -\bar{T}$. Thus

$$\begin{aligned}
&|\gamma_+(T_{2m+1}-T_{2m})-\gamma_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})|\\
&\quad\leq k\delta^{-1}\,\mathrm{e}^{-\delta(T_{2m+1}-T_{2m}-\bar{T})}\,|\dot{\gamma}_+(\bar{T})|+k\delta^{-1}\,\mathrm{e}^{\delta(T_{2m+1}-T_{2m+2}+\bar{T}+1)}\,|\dot{\gamma}_-(-\bar{T})|\\
&\quad\leq 2k\delta^{-1}\,\mathrm{e}^{-\delta(T-\bar{T})}\max\{|\dot{\gamma}_-(-\bar{T})|,|\dot{\gamma}_+(\bar{T})|\},
\end{aligned}$$

that is,

$$\|\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\|\leq 2k\delta^{-1}\,\mathrm{e}^{-\delta(T-\bar{T})}\max\{|\dot{\gamma}_-(-\bar{T})|,|\dot{\gamma}_+(\bar{T})|\}. \tag{6.1.43}$$

Similarly we get:

$$\frac{d}{d\alpha}[\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)]\widetilde{\alpha}=\left\{\begin{array}{c}\dot{\gamma}_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})(\widetilde{\alpha}_{m+1}-\widetilde{\alpha}_m)\\0\\0\\0\\0\\0\end{array}\right\}_{m\in\mathbb{Z}}$$

and hence

$$\left\|\frac{d}{d\alpha}[\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)]\right\|\leq 2k\delta^{-1}\,\mathrm{e}^{-\delta(T-\bar{T})}\,|\dot{\gamma}_-(-\bar{T})|. \tag{6.1.44}$$

Next, from Theorems 6.1.10, 6.1.12, 6.1.14, the equality $R_0\dot{\gamma}_0(\bar{T})=0$ and the identities

$$\begin{aligned}
P_-X_-^{-1}(T_{2m-1}-T_{2m}-1)\varphi_m^- &= X_-^{-1}(T_{2m-1}-T_{2m}-1)\varphi_m^-,\\
(\mathbb{I}-P_+)X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi_m^+ &= X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi_m^+
\end{aligned} \tag{6.1.45}$$

(that follow from $\varphi_m^-\in\mathscr{R}P_{-,m}$, $\varphi_m^+\in\mathscr{N}P_{+,m}$), we see that the derivative $D_1\mathscr{G}_{\mathscr{T}}$ of $\mathscr{G}_{\mathscr{T}}$ with respect to $\theta\in\ell^\infty_{\rho,\alpha,\varepsilon}$ at the point $(\theta_\alpha,\alpha,0)$ is given by

$$D_1\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\theta=\left\{\begin{array}{c}\mathscr{L}_\alpha(\varphi_{m+1}^-,\varphi_m^+,\xi_{m+1}^-,\xi_m^+,\bar{\xi}_m,\beta_m)\\ \bar{\xi}_m-\xi_m^--X_-^{-1}(T_{2m-1}-T_{2m}-1)\varphi_m^-\\ R_0[X_0(\bar{T})\bar{\xi}_m-\xi_m^+-X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi_m^+]\\ G'(\gamma_0(-\bar{T}))\cdot[\xi_m^-+X_-^{-1}(T_{2m-1}-T_{2m}-1)\varphi_m^-]\\ G'(\gamma_0(\bar{T}))\cdot[X_0(\bar{T})\bar{\xi}_m+\dot{\gamma}_0(\bar{T})\beta_m]\\ G'(\gamma_+(\bar{T}))\cdot[\xi_m^++X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi_m^+]\end{array}\right\}_{m\in\mathbb{Z}}$$

where, we recall $\theta=\{(\varphi_m^-,\varphi_m^+,\xi_m^-,\xi_m^+,\bar{\xi}_m,\beta_m)\}_{m\in\mathbb{Z}}$, and

$$\begin{aligned}
&\mathscr{L}_\alpha(\varphi_{m+1}^-,\varphi_m^+,\xi_{m+1}^-,\xi_m^+,\bar{\xi}_m,\beta_m)\\
&\quad = X_+(T_{2m+1}-T_{2m})\xi_m^+\\
&\qquad -X_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})\xi_{m+1}^- - \dot{\gamma}_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})\beta_m\\
&\qquad +X_+(T_{2m+1}-T_{2m})(\mathbb{I}-P_+)X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi_m^+\\
&\qquad -X_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})P_-X_-^{-1}(T_{2m+1}-T_{2m+2}-1)\varphi_{m+1}^-\,.
\end{aligned}$$

Then, using again (6.1.45) we obtain:

$$\begin{aligned}
&\left|X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi_m^+\right| \le k\mathrm{e}^{-\delta(T_{2m+1}-T_{2m}-\bar{T}+1)}\,|\varphi_m^+| \le k\mathrm{e}^{-\delta(T-\bar{T}+2)}\,|\varphi_m^+|\\
&\left|X_-^{-1}(T_{2m-1}-T_{2m}-1)\varphi_m^-\right| \le k\mathrm{e}^{-\delta(T_{2m}-T_{2m-1}+1-\bar{T})}\,|\varphi_m^-| \le k\mathrm{e}^{-\delta(T-\bar{T}+2)}\,|\varphi_m^-|\,.
\end{aligned} \tag{6.1.46}$$

Moreover,

$$|X_+(T_{2m+1}-T_{2m})\xi_m^+| = |X_+(T_{2m+1}-T_{2m})P_+X_+^{-1}(\bar{T})\xi_m^+| \le k\mathrm{e}^{-\delta(T-\bar{T}+1)}\,|\xi_m^+| \tag{6.1.47}$$

and, since $|\alpha_m-\alpha_{m+1}|<1$ implies that $T_{2m+2}-T_{2m+1}-\alpha_m+\alpha_{m+1}\ge T>\bar{T}$:

$$\begin{aligned}
&|X_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})\xi_{m+1}^-|\\
&\quad = |X_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})(\mathbb{I}-P_-)X_-^{-1}(-\bar{T})\xi_{m+1}^-|\\
&\quad \le k\mathrm{e}^{-\delta(T-\bar{T})}\,|\xi_{m+1}^-|,\\
&\qquad |\dot{\gamma}_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})| \le k\mathrm{e}^{-\delta(T-\bar{T})}\,|\dot{\gamma}_-(-\bar{T})|
\end{aligned} \tag{6.1.48}$$

for any $m\in\mathbb{Z}$. Next,

$$\begin{aligned}
&X_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})P_-X_-^{-1}(T_{2m+1}-T_{2m+2}-1)\varphi_{m+1}^-\\
&\quad \in \mathscr{R}P_-(T_{2m+2}-T_{2m+1}-\alpha_m+\alpha_{m+1}),\\
&X_+(T_{2m+1}-T_{2m})(\mathbb{I}-P_+)X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi_m^+\\
&\quad \in \mathscr{N}P_+(T_{2m+1}-T_{2m}),
\end{aligned}$$

and (see (6.1.9))

$$\mathscr{N}P_+(T_{2m+1}-T_{2m})\oplus\mathscr{R}P_-(T_{2m+2}-T_{2m+1}-\alpha_m+\alpha_{m+1})=\mathbb{R}^n.$$

Hence the linear map

$$\begin{aligned}
\mathscr{L}_{\alpha,m}:(\varphi_{m+1}^-,\varphi_m^+)\mapsto\ & X_+(T_{2m+1}-T_{2m})(\mathbb{I}-P_+)X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi_m^+\\
&-X_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})P_-X_-^{-1}(T_{2m+1}-T_{2m+2}-1)\varphi_{m+1}^-
\end{aligned}$$

is a linear isomorphism from $\mathscr{R}P_{-,m+1}\oplus\mathscr{N}P_{+,m}=\mathbb{R}^n$ into $\mathscr{N}P_+(T_{2m+1}-T_{2m})\oplus\mathscr{R}P_-(T_{2m+2}-T_{2m+1}-\alpha_m+\alpha_{m+1})=\mathbb{R}^n$ whose inverse is given by:

$$\mathscr{L}_{\alpha,m}^{-1} : (\tilde{\varphi}_{m+1}^{-}, \tilde{\varphi}_m^{+}) \mapsto X_+(T_{2m+1} - T_{2m} + 1)(\mathbb{I} - P_+)X_+^{-1}(T_{2m+1} - T_{2m})\tilde{\varphi}_m^{+}$$
$$-X_-(T_{2m+1} - T_{2m+2} - 1)P_-X_-^{-1}(T_{2m+1} - T_{2m+2} + \alpha_m - \alpha_{m+1})\tilde{\varphi}_{m+1}^{-} .$$

Note that (see (6.1.3)):

$$|X_-(T_{2m+1} - T_{2m+2} - 1)P_-X_-^{-1}(T_{2m+1} - T_{2m+2} + \alpha_m - \alpha_{m+1})\tilde{\varphi}_{m+1}^{-}|$$
$$\leq k\,\mathrm{e}^{\delta(1+\alpha_m-\alpha_{m+1})}\,|\tilde{\varphi}_{m+1}^{-}| \leq k\,\mathrm{e}^{\delta(1+\chi)}\,|\tilde{\varphi}_{m+1}^{-}|;$$
$$|X_+(T_{2m+1} - T_{2m} + 1)(\mathbb{I} - P_+)X_+^{-1}(T_{2m+1} - T_{2m})\tilde{\varphi}_{m+1}^{-}| \leq k\,\mathrm{e}^{\delta}\,|\tilde{\varphi}_{m+1}^{-}|$$

and

$$\frac{\partial}{\partial\alpha}\mathscr{L}_{\alpha,m}(\varphi_{m+1}^{-}, \varphi_m^{+})\alpha = -f_-'\left(\gamma_-(T_{2m+1} - T_{2m+2} + \alpha_m - \alpha_{m+1})\right)\cdot$$
$$X_-(T_{2m+1} - T_{2m+2} + \alpha_m - \alpha_{m+1})P_-X_-^{-1}(T_{2m+1} - T_{2m+2} - 1)\varphi_{m+1}^{-}(\alpha_m - \alpha_{m+1}).$$

Thus we obtain (see also (6.1.10)):

$$|\mathscr{L}_{\alpha,m}(\varphi_{m+1}^{-}, \varphi_m^{+})| \leq k\,\mathrm{e}^{-\delta}\,|\varphi_m^{+}| + k\,\mathrm{e}^{-\delta(1-\chi)}\,|\varphi_{m+1}^{-}| \leq k\widetilde{c}|\varphi_m^{+} + \varphi_{m+1}^{-}|,$$
$$|\mathscr{L}_{\alpha,m}^{-1}(\widetilde{\varphi}_{m+1}^{-}, \varphi_m^{+})| \leq k\,\mathrm{e}^{\delta}\,|\varphi_m^{+}| + k\,\mathrm{e}^{\delta(1+\chi)}\,|\widetilde{\varphi}_{m+1}^{-}| \leq k\widetilde{c}\,\mathrm{e}^{2\delta}\,|\varphi_m^{+} + \varphi_{m+1}^{-}|,$$
$$\left|\frac{\partial}{\partial\alpha}\mathscr{L}_{\alpha,m}(\varphi_{m+1}^{-}, \varphi_m^{+})| \leq 2N_-k|\varphi_{m+1}^{-}\right|$$

for $N_- := \sup_{x\in\mathbb{R}^n}|f_-(x)|$. So, using also $\frac{\partial}{\partial\alpha}\mathscr{L}_{\alpha,m}^{-1} = \mathscr{L}_{\alpha,m}^{-1} \circ \frac{\partial}{\partial\alpha}\mathscr{L}_{\alpha,m} \circ \mathscr{L}_{\alpha,m}^{-1}$:

$$\|\mathscr{L}_{\alpha,m}\| \leq k\widetilde{c} \quad \text{and} \quad \|\mathscr{L}_{\alpha,m}^{-1}\| \leq k\widetilde{c}\,\mathrm{e}^{2\delta},$$
$$\left\|\frac{\partial}{\partial\alpha}\mathscr{L}_{\alpha,m}\right\| \leq 2N_-k \quad \text{and} \quad \left\|\frac{\partial}{\partial\alpha}\mathscr{L}_{\alpha,m}^{-1}\right\| \leq 2N_-k^3\widetilde{c}^2\,\mathrm{e}^{4\delta}.$$

Next, using (6.1.47), (6.1.48):

$$\left|\mathscr{L}_\alpha(\varphi_{m+1}^{-}, \varphi_m^{+}, \xi_{m+1}^{-}, \xi_m^{+}, \bar{\xi}_m, \beta_m) - \mathscr{L}_{\alpha,m}(\varphi_{m+1}^{-}, \varphi_m^{+})\right|$$
$$\leq k\,\mathrm{e}^{-\delta(T-\bar{T})}(2 + |\dot{\gamma}_-(-\bar{T})|)\|\theta\| \tag{6.1.49}$$

(recall $\theta = \{(\varphi_m^{-}, \varphi_m^{+}, \xi_m^{-}, \xi_m^{+}, \bar{\xi}_m, \beta_m)\}_{m\in\mathbb{Z}}$). We define $\mathscr{H}_\alpha : \ell_{\rho,\alpha,\varepsilon}^{\infty} \to \ell_1^{\infty}$ as

$$\mathscr{H}_\alpha\theta = \left\{\begin{array}{c} \mathscr{L}_{\alpha,m}(\varphi_{m+1}^{-}, \varphi_m^{+}) \\ \bar{\xi}_m - \xi_m^{-} \\ R_0[X_0(\bar{T})\bar{\xi}_m - \xi_m^{+}] \\ G'(\gamma_0(-\bar{T}))\xi_m^{-} \\ G'(\gamma_0(\bar{T}))\cdot[X_0(\bar{T})\bar{\xi}_m + \dot{\gamma}_0(\bar{T})\beta_m] \\ G'(\gamma_+(\bar{T}))\cdot\xi_m^{+} \end{array}\right\}_{m\in\mathbb{Z}}.$$

Clearly

$$\frac{\partial}{\partial\alpha}\mathscr{H}_\alpha\theta = \left\{\begin{matrix} \frac{\partial}{\partial\alpha}\mathscr{L}_{\alpha,m}(\varphi_{m+1}^-,\varphi_m^+) \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix}\right\}_{m\in\mathbb{Z}}$$

and so

$$\left\|\frac{\partial}{\partial\alpha}\mathscr{H}_\alpha\right\| \le 2N_-k. \tag{6.1.50}$$

Next, note that

$$[D_1\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)-\mathscr{H}_\alpha]\theta$$

$$= \left\{\begin{matrix} \mathscr{L}_\alpha(\varphi_{m+1}^-,\varphi_m^+,\xi_{m+1}^-,\xi_m^+,\bar{\xi}_m,\beta_m)-\mathscr{L}_{\alpha,m}(\varphi_{m+1}^-,\varphi_m^+) \\ -X_-^{-1}(T_{2m-1}-T_{2m}-1)\varphi_m^- \\ -R_0X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi_m^+ \\ G'(\gamma_0(-\bar{T}))X_-^{-1}(T_{2m-1}-T_{2m}-1)\varphi_m^- \\ 0 \\ G'(\gamma_+(\bar{T}))X_+^{-1}(T_{2m+1}-T_{2m}+1)\varphi_m^+ \end{matrix}\right\}_{m\in\mathbb{Z}}. \tag{6.1.51}$$

Hence, from (6.1.46) and (6.1.49), we get

$$\|D_1\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)-\mathscr{H}_\alpha\| \le \widetilde{c}_3k\,\mathrm{e}^{-\delta(T-\bar{T})} \tag{6.1.52}$$

where

$$\widetilde{c}_3 := \max\left\{2+|\dot{\gamma}_-(-\bar{T})|,\|R_0\|\,\mathrm{e}^{-2\delta},|G'(\gamma_0(-\bar{T}))|\,\mathrm{e}^{-2\delta},|G'(\gamma_+(\bar{T}))|\,\mathrm{e}^{-2\delta}\right\}.$$

Next, given $\{(a_m,b_m,c_m,d_m,e_m,f_m)\}_{m\in\mathbb{Z}} \in \ell_1^\infty$ we want to solve the linear equation

$$\mathscr{H}_\alpha\theta = \left\{\begin{matrix} a_m \\ b_m \\ c_m \\ d_m \\ e_m \\ f_m \end{matrix}\right\}_{m\in\mathbb{Z}} \tag{6.1.53}$$

that is the set of equations:

$$\begin{cases} \mathscr{L}_{\alpha,m}(\varphi_{m+1}^{-},\varphi_m^{+}) = a_m, \\ \bar{\xi}_m - \xi_m^{-} = b_m, \\ R_0[X_0(\bar{T})\bar{\xi}_m - \xi_m^{+}] = c_m, \\ G'(\gamma_0(-\bar{T}))\xi_m^{-} = d_m, \\ G'(\gamma_0(\bar{T}))\cdot[X_0(\bar{T})\bar{\xi}_m + \dot{\gamma}_0(\bar{T})\beta_m] = e_m, \\ G'(\gamma_+(\bar{T}))\cdot\xi_m^{+} = f_m\,. \end{cases} \tag{6.1.54}$$

To solve (6.1.54) we write:

$$\begin{gathered} \xi_m^{-} = \eta_m^{\perp} + \mu_m^{-}\dot{\gamma}_-(-\bar{T})\,, \\ \xi_m^{+} = \zeta_m^{\perp} + \mu_m^{+}\dot{\gamma}_+(\bar{T}), \quad m \in \mathbb{Z}\,, \\ \{\eta_m^{\perp}\}_{m\in\mathbb{Z}} \in \ell^\infty(\mathscr{S}'), \quad \{\zeta_m^{\perp}\}_{m\in\mathbb{Z}} \in \ell^\infty(\mathscr{S}''), \quad \{\mu_m^{\pm}\}_{m\in\mathbb{Z}} \in \ell^\infty(\mathbb{R})\,, \end{gathered} \tag{6.1.55}$$

and plug (6.1.55) into (6.1.54). We obtain

$$\begin{gathered} (\varphi_{m+1}^{-},\varphi_m^{+}) = \mathscr{L}_{\alpha,m}^{-1}a_m\,, \\ \mu_m^{-} = \frac{d_m}{G'(\gamma_-(-\bar{T}))\dot{\gamma}_-(-\bar{T})}\,, \\ \mu_m^{+} = \frac{f_m}{G'(\gamma_+(\bar{T}))\dot{\gamma}_+(\bar{T})}\,, \\ \bar{\xi}_m = \eta_m^{\perp} + \mu_m^{-}\dot{\gamma}_-(-\bar{T}) + b_m\,, \\ \beta_m = \frac{e_m - G'(\gamma_0(\bar{T}))X_0(\bar{T})\bar{\xi}_m}{G'(\gamma_0(\bar{T}))\dot{\gamma}_0(\bar{T})}\,, \\ R_0X_0(\bar{T})\eta_m^{\perp} - \zeta_m^{\perp} = c_m - \mu_m^{-}R_0X_0(\bar{T})\dot{\gamma}_-(-\bar{T}) - R_0X_0(\bar{T})b_m + \mu_m^{+}R_0\dot{\gamma}_+(\bar{T})\,. \end{gathered} \tag{6.1.56}$$

Now we denote by $\Pi : \mathscr{R}R_0 \to \mathscr{S}'' \oplus \mathscr{S}''' \subset \mathscr{R}R_0$ the orthogonal projection onto $\mathscr{S}'' \oplus \mathscr{S}'''$ along $\mathrm{span}\{\psi\}$ (recall that $\psi \in \mathscr{R}R_0 = \mathscr{N}G'(\gamma(T))$ is a unitary vector so that (6.1.4) and (6.1.5) hold). In other words:

$$(\mathbb{I} - \Pi)w = \langle \psi, w\rangle \psi \tag{6.1.57}$$

for any $w \in \mathscr{R}R_0$. Assumption (H3) implies that the linear mapping $\mathscr{S}'' \oplus \mathscr{S}' \mapsto \mathscr{S}'' \oplus \mathscr{S}''' = \mathscr{R}\Pi$ defined as $(\zeta^{\perp},\eta^{\perp}) \to -\zeta^{\perp} + R_0X_0(\bar{T})\eta^{\perp}$ is invertible. So in order to solve (6.1.56), we need to suppose

$$\{(a_m,b_m,c_m,d_m,e_m,f_m)\}_{m\in\mathbb{Z}} \in \ell^\infty(\mathscr{S}^{iv})\,,$$

where

$$\mathscr{S}^{iv} = \Big\{(a,b,c,d,e,f) \in \mathbb{R}^{2n} \times \mathscr{R}R_0 \times \mathbb{R}^3 : (\mathbb{I} - \Pi)L(a,b,c,d,e,f) = 0\Big\}$$

and $L:\mathbb{R}^n\times\mathbb{R}^n\times\mathscr{R}R_0\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathscr{R}R_0$ is the linear map given by:

$$\begin{aligned}L(a,b,c,d,e,f)=c-&\frac{d}{G'(\gamma_-(-\bar T))\dot\gamma_-(-\bar T)}R_0X_0(\bar T)\dot\gamma_-(-\bar T)\\&-R_0X_0(\bar T)b+\frac{f}{G'(\gamma_+(\bar T))\dot\gamma_+(\bar T)}R_0\dot\gamma_+(\bar T)\,.\end{aligned}\tag{6.1.58}$$

Note that $\mathscr{S}^{iv}$ is a codimension 1 linear subspace of $\mathbb{R}^{2n}\times\mathscr{R}R_0\times\mathbb{R}^3$. Hence $\widetilde\psi\in\mathbb{R}^{2n}\times\mathscr{R}R_0\times\mathbb{R}^3$ exists so that

$$\operatorname{span}\{\widetilde\psi\}\oplus\mathscr{S}^{iv}=\mathbb{R}^{2n}\times\mathscr{R}R_0\times\mathbb{R}^3\,.$$

Of course, to be more precisely, we can take $\widetilde\psi$ so that $\langle\widetilde\psi,v\rangle=0$ for any $v\in\mathscr{S}^{iv}$, where $\langle\cdot,\cdot\rangle$ is the usual scalar product on $\mathbb{R}^{3n+3}$. To construct such a $\widetilde\psi$ we note that from (6.1.57), it follows that $(\mathbb{I}-\Pi)Lv=\langle\psi,Lv\rangle\psi=\langle L^*\psi,v\rangle\psi$, where we take the natural restriction of $\langle\cdot,\cdot\rangle$ onto $\mathscr{R}R_0\subset\mathbb{R}^n$. Thus $v=(a,b,c,d,e,f)\in\mathscr{S}^{iv}$ if and only if $\langle L^*\psi,v\rangle=0$ or $v\in\{L^*\psi\}^\perp$ and we can take

$$\widetilde\psi=L^*\psi/|L^*\psi|\,.$$

Let $\widetilde\Pi:\mathbb{R}^{2n}\times\mathscr{R}R_0\times\mathbb{R}^3\to\mathscr{S}^{iv}$ be the orthogonal projection onto $\mathscr{S}^{iv}$ along $\operatorname{span}\{\widetilde\psi\}$. Then

$$(\mathbb{I}-\widetilde\Pi)v=\langle\widetilde\psi,v\rangle\widetilde\psi=\frac{\langle L^*\psi,v\rangle}{|L^*\psi|}\widetilde\psi=\frac{\langle\psi,Lv\rangle}{|L^*\psi|}\widetilde\psi.$$

We set

$$\ell^\infty_\psi=\ell^\infty(\operatorname{span}\{\widetilde\psi\})\subset\ell^\infty_1\,.$$

Let $\Pi_\psi:\ell^\infty_1\to\ell^\infty(\mathscr{S}^{iv})$ be the projection onto $\ell^\infty(\mathscr{S}^{iv})$ along ℓ^∞_ψ given by

$$\Pi_\psi\left(\{(a_m,b_m,c_m,d_m,e_m,f_m)\}_{m\in\mathbb{Z}}\right)=\left\{\widetilde\Pi(a_m,b_m,c_m,d_m,e_m,f_m)\right\}_{m\in\mathbb{Z}}\,.$$

In summary, we see from (6.1.56) that there is a continuous inverse $\mathscr{H}_\alpha^{-1}:\ell^\infty(\mathscr{S}^{iv})\mapsto\ell^\infty_2$, where

$$\begin{aligned}\ell^\infty_2=\Big\{&\{(\varphi_m^-,\varphi_m^+,\xi_m^-,\xi_m^+,\bar\xi_m,\beta_m)\}_{m\in\mathbb{Z}}\in\ell^\infty\left(\mathbb{R}^{5n+1}\right):\\&(\varphi_m^-,\varphi_m^+,\xi_m^-,\xi_m^+,\bar\xi_m,\beta_m)\in\mathscr{R}P_{-,m}\times\mathscr{N}P_{+,m}\times\mathscr{N}P_-\times\mathscr{R}P_+\times\mathbb{R}^{n+1},\forall m\in\mathbb{Z}\Big\}.\end{aligned}$$

Note that from (6.1.56) it easily follows that $\|\mathscr{H}_\alpha^{-1}\|$ and $\left\|\frac{\partial}{\partial\alpha}\mathscr{H}_\alpha^{-1}\right\|\le\left\|\frac{\partial}{\partial\alpha}\mathscr{H}_\alpha\right\|\|\mathscr{H}_\alpha^{-1}\|^2$ are uniformly bounded with respect to α.

Finally, we define projections onto $\mathscr{R}G'(\gamma(\bar T))$ and $\mathscr{R}G'(\gamma(-\bar T))$, respectively, as

$$
\begin{aligned}
(\mathbb{I}-R_+)w &= \frac{G'(\gamma(\bar T))w}{G'(\gamma(\bar T))\dot\gamma_+(\bar T)}\dot\gamma_+(\bar T)\\
(\mathbb{I}-R_-)w &= \frac{G'(\gamma(-\bar T))w}{G'(\gamma(-\bar T))\dot\gamma_-(-\bar T)}\dot\gamma_-(-\bar T).
\end{aligned}
\tag{6.1.59}
$$

Note that R_+ is the projection onto $\mathscr{N}G'(\gamma(\bar T))$ along $\dot\gamma_+(\bar T)$ whereas R_- is the projection onto $\mathscr{N}G'(\gamma(-\bar T))$ along $\dot\gamma_-(-\bar T)$. First, we observe that for any $w\in\mathbb{R}^n$ we have $[\mathbb{I}-P_+]R_+P_+=0$, since $\dot\gamma_+(\bar T)\in\mathscr{R}P_+$. So $R_+P_+=P_+R_+P_+$ and then for any $w\in\mathbb{R}^n$ we have $R_+P_+w\in\mathscr{R}P_+\cap\mathscr{R}R_+=\mathscr{S}''$. As a consequence, we see that $\psi^*R_+P_+w=0$ for any $w\in\mathbb{R}^n$ (see (6.1.4)). Similarly we see that $P_-R_-[\mathbb{I}-P_-]=0$, hence $R_-[\mathbb{I}-P_-]w\in\mathscr{N}P_-\cap\mathscr{R}R_-=\mathscr{N}P_-\cap\mathscr{N}G'(\gamma(-T))=\mathscr{S}'$ for any $w\in\mathbb{R}^n$. As a consequence, we get $\psi^*R_0X_0(T)R_-[\mathbb{I}-P_-]w=0$ for any $w\in\mathbb{R}^n$ since $R_0X_0(T)R_-(\mathbb{I}-P_+)w\in R_0X_0(T)\mathscr{S}'$. Consequently we arrive at

$$
P_+^*R_+^*\psi=0,\quad \left(\mathbb{I}-P_-^*\right)R_-^*X_0(\bar T)^*R_0^*\psi=0\,. \tag{6.1.60}
$$

Next we set:

$$
\psi(t)=\begin{cases}
X_-^{-1*}(t)R_-^*X_0(\bar T)^*R_0^*\psi, & \text{if } t\le-\bar T\,,\\
X_0^{-1*}(t)X_0(\bar T)^*R_0^*\psi, & \text{if } -\bar T<t\le\bar T\,,\\
X_+^{-1*}(t)R_+^*\psi, & \text{if } t>\bar T,
\end{cases} \tag{6.1.61}
$$

and

$$
\mathscr{M}(\alpha)=\int_{-\infty}^{\infty}\psi^*(t)g(t+\alpha,\gamma(t),0)dt. \tag{6.1.62}
$$

Using (6.1.60), we easily obtain:

$$
\begin{aligned}
&|\psi(t)|\le\|X_+^{-1*}(t)(\mathbb{I}-P_+^*)X_+^*(\bar T)\||R_+^*\psi|\le k\|R_+\|\,\mathrm{e}^{-\delta(t-\bar T)} && \text{if } t\ge\bar T,\\
&|\psi(t)|\le k\|R_0X_0(\bar T)R_-\|\,\mathrm{e}^{\delta(t+\bar T)} && \text{if } t\le-\bar T.
\end{aligned}
\tag{6.1.63}
$$

Thus $\mathscr{M}(\alpha)$ is a well defined C^2 function because of Lebesgue theorem. We are now ready to state the following result.

Theorem 6.1.16. *Assume that $f_\pm(z)$ and $g(t,z,\varepsilon)$ are C^r-functions with bounded derivatives and that their r-order derivatives are uniformly continuous. Assume, moreover, that conditions (H1), (H2) and (H3) hold.*

Then given $c_0>0$ there exist constants $\rho_0>0$, $\chi>0$ and $c_1>0$ so that for any $0<\rho<\rho_0$, there is $\bar\varepsilon_\rho>0$ so that for any ε, $0<|\varepsilon|<\bar\varepsilon_\rho$, for any increasing sequence $\mathscr{T}=\{T_m\}_{m\in\mathbb{Z}}\subset\mathbb{R}$ with $T_m-T_{m-1}>\bar T+1-2\delta^{-1}\ln|\varepsilon|$ so that

$$
\mathscr{M}\left(T_{2m}+\alpha_m^0\right)=0\ \forall m\in\mathbb{Z} \text{ and } \inf_{m\in\mathbb{Z}}|\mathscr{M}'\left(T_{2m}+\alpha_m^0\right)|>c_0 \tag{6.1.64}
$$

for some $\alpha_0=\{\alpha_m^0\}_{m\in\mathbb{Z}}\in\ell_\chi^\infty$, there exist unique sequences $\{\hat\alpha_m\}_{m\in\mathbb{Z}}=\big\{\hat\alpha_m(\mathscr{T},\varepsilon)\big\}_{m\in\mathbb{Z}}\in\ell_\chi^\infty(\mathbb{R})$ and $\{\hat\beta_m\}_{m\in\mathbb{Z}}=\{\hat\beta_m(\mathscr{T},\varepsilon)\}_{m\in\mathbb{Z}}\in\ell^\infty(\mathbb{R})$ with $|\hat\alpha_m(\mathscr{T},\varepsilon)-\alpha_m^0|<$

$c_1|\varepsilon|$ and $|\hat\beta_m(\mathscr{T},\varepsilon)-\alpha_m^0|<c_1|\varepsilon|$ $\forall m\in\mathbb{Z}$, and a unique bounded solution $z(t)=z(\mathscr{T},\varepsilon)(t)$ of system (6.1.1) *so that*

$$\begin{aligned}&\sup_{t\in[T_{2m-1}+\hat\beta_{m-1},T_{2m}-\bar T+\hat\alpha_m]}|z(t)-\gamma_-(t-T_{2m}-\hat\alpha_m)|<\rho,\\&\sup_{t\in[T_{2m}-\bar T+\hat\alpha_m,T_{2m}+\bar T+\hat\beta_m]}|z(t)-\gamma_0(t-T_{2m}-\hat\alpha_m)|<\rho,\\&\sup_{t\in[T_{2m}+\bar T+\hat\beta_m,T_{2m+1}+\hat\beta_m]}|z(t)-\gamma_+(t-T_{2m}-\hat\beta_m)|<\rho\end{aligned}$$

for any $m\in\mathbb{Z}$ (cf (6.1.7)*). Hence $z(t)$ is orbitally close to $\gamma(t)$ in the sense that* $\operatorname{dist}(z(t),\Gamma)<\rho$ *where $\Gamma=\{\gamma(t)\mid t\in\mathbb{R}\}$ is the orbit of $\gamma(t)$.*

Proof. If ρ and $\bar\varepsilon_\rho<\varepsilon_\rho$ are sufficiently small then, for $t\in I^-_{m,\alpha}$, the solution $z(t)$ we look for must satisfy $z(t)=z_m^-(t,\xi_-,\varphi_-,\alpha,\varepsilon)$ for some value of the parameters $(\xi_-,\varphi_-,\alpha,\varepsilon)$ and similarly in the other intervals $[T_{2m}-\bar T+\alpha,T_{2m}+\bar T+\beta]$ and $I^+_{m,\beta}$. So, we solve Eq. (6.1.42) for $(\theta,\alpha)\in\ell^\infty_{\rho,\alpha,\varepsilon}\times\ell^\infty_\chi$ in terms of $\mathscr{T}$ and $\varepsilon\in(-\bar\varepsilon_\rho,\bar\varepsilon_\rho)$. Set

$$\begin{aligned}&\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon)\\&=\mathscr{G}_{\mathscr{T}}(\theta,\alpha,\varepsilon)-\mathscr{H}_\alpha(\theta-\theta_\alpha)\\&=\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\\&\quad+[\mathscr{G}_{\mathscr{T}}(\theta,\alpha,0)-\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)-D_1\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)(\theta-\theta_\alpha)]\\&\quad+(D_1\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)-\mathscr{H}_\alpha)(\theta-\theta_\alpha)+\varepsilon\int_0^1D_3\mathscr{G}_{\mathscr{T}}(\theta,\alpha,\tau\varepsilon)\,d\tau\end{aligned}$$

where $D_3\mathscr{G}_{\mathscr{T}}(\theta,\alpha,\varepsilon)$ denotes the derivative of $\mathscr{G}_{\mathscr{T}}$ with respect to ε. It is easy to see that

$$\begin{gathered}\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)=\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon),\quad D_1\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon)=D_1\mathscr{G}_{\mathscr{T}}(\theta,\alpha,\varepsilon)-\mathscr{H}_\alpha,\\D_1\mathscr{F}_{\mathscr{T}}(\theta_1,\alpha,\varepsilon)-D_1\mathscr{F}_{\mathscr{T}}(\theta_2,\alpha,\varepsilon)=D_1\mathscr{G}_{\mathscr{T}}(\theta_1,\alpha,\varepsilon)-D_1\mathscr{G}_{\mathscr{T}}(\theta_2,\alpha,\varepsilon),\\D_2\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon)=D_2\mathscr{G}_{\mathscr{T}}(\theta,\alpha,\varepsilon)-\frac{\partial\mathscr{H}_\alpha}{\partial\alpha}(\theta-\theta_\alpha)-\mathscr{H}_\alpha\frac{\partial\theta_\alpha}{\partial\alpha}.\end{gathered}\tag{6.1.65}$$

For simplicity we also set:

$$\mu=\mathrm{e}^{-\delta(T-\bar T)}.$$

From the definition of $\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon)$ we see that Eq. (6.1.42) has the form

$$\theta-\theta_\alpha+\mathscr{H}_\alpha^{-1}\Pi_\psi\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon)=0,\tag{6.1.66}$$

and

$$\left(\mathbb{I}-\Pi_\psi\right)\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon)=0.\tag{6.1.67}$$

We denote with $c^{(1)}_{\mathscr{G}}$, resp. $c^{(2)}_{\mathscr{G}}$, upper bounds for the norms of the first order, resp. second order, derivatives of $\mathscr{G}_{\mathscr{T}}(\theta,\alpha,\varepsilon)$, in ℓ^∞_ρ. Thus for example,

$$c_{\mathscr{G}}^{(1)} = \sup_{(\theta,\alpha,\varepsilon)\in\ell_\rho^\infty} \{\|D_1\mathscr{G}_{\mathscr{T}}(\theta,\alpha,\varepsilon)\|, \|D_2\mathscr{G}_{\mathscr{T}}(\theta,\alpha,\varepsilon)\|, \|D_3\mathscr{G}_{\mathscr{T}}(\theta,\alpha,\varepsilon)\|\}$$

and $c_{\mathscr{G}}^{(2)}$ is similar. Then

$$\begin{aligned}&\mathscr{G}_{\mathscr{T}}(\theta,\alpha,0) - \mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0) - D_1\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)(\theta-\theta_\alpha)\\ &\quad = \int_0^1 \left(D_1\mathscr{G}_{\mathscr{T}}(\tau\theta+(1-\tau)\theta_\alpha,\alpha,0) - D_1\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\right) d\tau(\theta-\theta_\alpha)\\ &\quad = \eta(\theta,\theta_\alpha,\alpha)(\theta-\theta_\alpha),\end{aligned}$$

where

$$\|\eta(\theta,\theta_\alpha,\alpha)\| \le c_{\mathscr{G}}^{(2)}\|\theta-\theta_\alpha\|.$$

Hence, since

$$\begin{aligned}\mathscr{F}_{\mathscr{T}}\ &(\theta,\alpha,\varepsilon) - \mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)\\ &= \int_0^1 [D_1\mathscr{F}_{\mathscr{T}}(\tau\theta+(1-\tau)\theta_\alpha,\alpha,\varepsilon)]\, d\tau(\theta-\theta_\alpha)\\ &= \int_0^1 [D_1\mathscr{F}_{\mathscr{T}}(\tau\theta+(1-\tau)\theta_\alpha,\alpha,\varepsilon) - D_1\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)] d\tau(\theta-\theta_\alpha)\\ &\quad + D_1\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)(\theta-\theta_\alpha)\\ &= \int_0^1 [D_1\mathscr{G}_{\mathscr{T}}(\tau\theta+(1-\tau)\theta_\alpha,\alpha,\varepsilon) - D_1\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)] d\tau(\theta-\theta_\alpha)\\ &\quad + [D_1\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon) - \mathscr{H}_\alpha](\theta-\theta_\alpha)\end{aligned} \tag{6.1.68}$$

(see also (6.1.65)) we derive, using also (6.1.52) (recall $\mu = \mathrm{e}^{-\delta(T-\bar{T})}$)

$$\|\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon) - \mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)\| \le \frac{1}{2}c_{\mathscr{G}}^{(2)}\|\theta-\theta_\alpha\|^2 + (k\widetilde{c}_3\mu + c_{\mathscr{G}}^{(2)}|\varepsilon|)\|\theta-\theta_\alpha\| \tag{6.1.69}$$

and (see also (6.1.43), (6.1.65))

$$\|\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon)\| \le \frac{c_{\mathscr{G}}^{(2)}}{2}\|\theta-\theta_\alpha\|^2 + (k\widetilde{c}_3\mu + c_{\mathscr{G}}^{(2)}|\varepsilon|)\|\theta-\theta_\alpha\| + c_{\mathscr{G}}^{(1)}|\varepsilon| + c_\gamma\mu \tag{6.1.70}$$

where $c_\gamma = 2k\delta^{-1}\max\{|\dot{\gamma}_-(-\bar{T})|, |\dot{\gamma}_+(\bar{T})|\}$. Note that c_γ, $c_{\mathscr{G}}^{(1)}$, $c_{\mathscr{G}}^{(2)}$ and $\widetilde{c}_3$ do not depend on $(\alpha,\mathscr{T},\varepsilon) \in \ell_\chi^\infty \times \ell_{\mathscr{T}}^\infty(\mathbb{R}) \times \mathbb{R}$. Next, from (6.1.50), (6.1.52) and (6.1.65) we get

$$\begin{aligned}&\|D_1\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\| \le k\widetilde{c}_3\mu,\\ &\|D_1\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon) - D_1\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)\| \le c_{\mathscr{G}}^{(2)}\|\theta-\theta_\alpha\|,\\ &\|D_2\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon) - D_2\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)\| \le \left(c_{\mathscr{G}}^{(2)} + 2kN_-\right)\|\theta-\theta_\alpha\|.\end{aligned} \tag{6.1.71}$$

From (6.1.70) and (6.1.71) we conclude that

$$\lim_{(\theta,\varepsilon,\mu)\to(\theta_\alpha,0,0)} \mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon) = 0, \qquad \lim_{(\theta,\varepsilon,\mu)\to(\theta_\alpha,0,0)} D_1\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon) = 0$$

uniformly with respect to α. Thus, if $\bar{\rho}_0 > 0$, $\mu_0 > 0$ and $0 < \bar{\varepsilon}_0 \le \varepsilon_\rho$ are sufficiently small and $0 < \mu < \mu_0$, $|\varepsilon| < \bar{\varepsilon}_0$, from the implicit function theorem the existence follows of a unique solution $\theta = \theta_{\mathscr{T}}(\alpha,\varepsilon)$ of (6.1.66) which is defined for any $\alpha \in \ell^\infty_\chi$, $|\varepsilon| < \bar{\varepsilon}_0$, $0 < \mu \le \mu_0$ and $\mathscr{T} = \{T_m\}_{m\in\mathbb{Z}}$ so that $T_{m+1} - T_m > T + 1$ where $T - \bar{T} = -\delta^{-1}\ln\mu$. Moreover $\theta_{\mathscr{T}}(\alpha,\varepsilon)$ satisfies

$$\sup_{\alpha,\mathscr{T},\varepsilon} \|\theta_{\mathscr{T}}(\alpha,\varepsilon) - \theta_\alpha\| < \bar{\rho}_0 \tag{6.1.72}$$

with the sup being taken over all α, $\mathscr{T}$ and ε satisfying the above conditions. Next, using (6.1.66) with $\theta_{\mathscr{T}}(\alpha,\varepsilon)$ instead of θ and (6.1.70), we see that:

$$\begin{aligned}&\|\theta_{\mathscr{T}}(\alpha,\varepsilon) - \theta_\alpha\| \le \|\mathscr{H}_\alpha^{-1}\Pi_\psi\|\|\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)\| \le \\ &\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|\left(\frac{c_{\mathscr{G}}^{(2)}}{2}\|\theta_{\mathscr{T}}(\alpha,\varepsilon) - \theta_\alpha\|^2 + (k\widetilde{c}_3\mu + c_{\mathscr{G}}^{(2)}|\varepsilon|)\|\theta_{\mathscr{T}}(\alpha,\varepsilon) - \theta_\alpha\| \right.\\ &\qquad\qquad \left. + c_{\mathscr{G}}^{(1)}|\varepsilon| + c_\gamma\mu\right).\end{aligned}$$

Hence if $\bar{\rho}_0$, μ_0 and ε_0 are so small that

$$\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|[c_{\mathscr{G}}^{(2)}(\bar{\rho}_0 + 2\varepsilon_0) + 2k\widetilde{c}_3\mu_0] < 1 \tag{6.1.73}$$

we obtain:

$$\|\theta_{\mathscr{T}}(\alpha,\varepsilon) - \theta_\alpha\| \le 2\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|(c_\gamma\mu + c_{\mathscr{G}}^{(1)}|\varepsilon|). \tag{6.1.74}$$

Note that since $\widetilde{\Pi}$ is an orthogonal projection, it is enough to choose μ_0, ε_0 and $\bar{\rho}_0$ in such a way that $c_{\mathscr{G}}^{(2)}(\bar{\rho}_0 + 2\varepsilon_0) + 2k\widetilde{c}_3\mu_0 < \|\mathscr{H}_\alpha^{-1}\|^{-1}$. Moreover, plugging (6.1.74) into (6.1.69) we obtain

$$\begin{aligned}\|\mathscr{F}_{\mathscr{T}}\ &(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon) - \mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)\| \\ &\le 2c_{\mathscr{G}}^{(2)}\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|^2(c_\gamma\mu + c_{\mathscr{G}}^{(1)}|\varepsilon|)^2 \\ &\quad + 2(k\widetilde{c}_3\mu + c_{\mathscr{G}}^{(2)}|\varepsilon|)\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|(c_\gamma\mu + c_{\mathscr{G}}^{(1)}|\varepsilon|) \le \Lambda_1(\mu + |\varepsilon|)^2\end{aligned} \tag{6.1.75}$$

where $\Lambda_1 > 0$ is independent of $(\mathscr{T},\alpha,\mu,\varepsilon)$. For example:

$$\Lambda_1 = 2\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|\max\{c_\gamma, c_{\mathscr{G}}^{(1)}, c_{\mathscr{G}}^{(2)}, k\widetilde{c}_3\}^2\left[\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|c_{\mathscr{G}}^{(2)} + 1\right].$$

Next, differentiating the equality

$$\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha+\mathscr{H}_\alpha^{-1}\Pi_\psi\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)=0$$

with respect to α we obtain:

$$\begin{aligned}\frac{\partial}{\partial\alpha}[\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha] &= -\mathscr{H}_\alpha^{-1}\Pi_\psi\tfrac{\partial}{\partial\alpha}\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)\\ &\quad-\left[\frac{\partial}{\partial\alpha}\mathscr{H}_\alpha^{-1}\Pi_\psi\right]\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)\\ &= -\mathscr{H}_\alpha^{-1}\Pi_\psi\Big\{\tfrac{\partial}{\partial\alpha}[\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)-\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)]\\ &\quad+\frac{\partial}{\partial\alpha}[\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)-\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,0)]+\frac{\partial}{\partial\alpha}\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\Big\}\\ &\quad-\left[\frac{\partial}{\partial\alpha}\mathscr{H}_\alpha^{-1}\Pi_\psi\right]\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon).\end{aligned}\tag{6.1.76}$$

Then note that

$$\begin{aligned}&\frac{\partial}{\partial\alpha}\ [\ \mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)-\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)]\\ &\quad=\frac{\partial}{\partial\alpha}\int_0^1 D_1\mathscr{F}_{\mathscr{T}}(\tau\theta_{\mathscr{T}}(\alpha,\varepsilon)+(1-\tau)\theta_\alpha,\alpha,\varepsilon)\,d\tau(\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha)\\ &\quad=\Big\{\int_0^1 D_1^2\mathscr{F}_{\mathscr{T}}(\tau\theta_{\mathscr{T}}(\alpha,\varepsilon)+(1-\tau)\theta_\alpha,\alpha,\varepsilon)\frac{\partial}{\partial\alpha}[\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha]\tau\,d\tau\\ &\qquad+\int_0^1 D_1^2\mathscr{F}_{\mathscr{T}}(\tau\theta_{\mathscr{T}}(\alpha,\varepsilon)+(1-\tau)\theta_\alpha,\alpha,\varepsilon)\frac{d}{d\alpha}\theta_\alpha\,d\tau\\ &\qquad+\int_0^1 D_1D_2\mathscr{F}_{\mathscr{T}}(\tau\theta_{\mathscr{T}}(\alpha,\varepsilon)+(1-\tau)\theta_\alpha,\alpha,\varepsilon)\,d\tau\Big\}(\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha)\\ &\qquad+\int_0^1 D_1\mathscr{F}_{\mathscr{T}}(\tau\theta_{\mathscr{T}}(\alpha,\varepsilon)+(1-\tau)\theta_\alpha,\alpha,\varepsilon)\,d\tau\frac{\partial}{\partial\alpha}[\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha].\end{aligned}\tag{6.1.77}$$

First we derive

$$\begin{aligned}&\left\|\int_0^1 D_1^2\mathscr{F}_{\mathscr{T}}(\tau\theta_{\mathscr{T}}(\alpha,\varepsilon)+(1-\tau)\theta_\alpha,\alpha,\varepsilon)\frac{\partial}{\partial\alpha}[\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha]\tau\,d\tau\right\|\\ &\quad\le\int_0^1 c_{\mathscr{G}}^{(2)}\tau\,d\tau\left\|\frac{\partial}{\partial\alpha}[\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha]\right\|=\frac{1}{2}c_{\mathscr{G}}^{(2)}\left\|\frac{\partial}{\partial\alpha}[\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha]\right\|.\end{aligned}$$

Next, from (6.1.71) we obtain

$$\left\|\int_0^1 D_1\mathscr{F}_{\mathscr{T}}(\tau\theta_{\mathscr{T}}(\alpha,\varepsilon)+(1-\tau)\theta_\alpha,\alpha,\varepsilon)\,d\tau\frac{\partial}{\partial\alpha}[\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha]\right\|$$

$$\leq\Big(\int_0^1\|D_1\mathscr{F}_{\mathscr{T}}(\tau\theta_{\mathscr{T}}(\alpha,\varepsilon)+(1-\tau)\theta_\alpha,\alpha,\varepsilon)-D_1\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)]\|\,d\tau$$

$$+\|D_1\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)-D_1\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\|+\|D_1\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\|\Big)$$

$$\left\|\frac{\partial}{\partial\alpha}[\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha]\right\|$$

$$\leq\Big(\int_0^1 c_{\mathscr{G}}^{(2)}\|\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha\|\,\tau\,d\tau+c_{\mathscr{G}}^{(2)}|\varepsilon|+k\widetilde{c}_3\mu\Big)\left\|\frac{\partial}{\partial\alpha}[\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha]\right\|$$

$$\leq\left(c_{\mathscr{G}}^{(2)}\left(\frac{1}{2}\|\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha\|+|\varepsilon|\right)+k\widetilde{c}_3\mu\right)\left\|\frac{\partial}{\partial\alpha}[\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha]\right\|.$$

Finally, using (6.1.50), (6.1.72) and (6.1.74), the identity

$$\frac{d\theta_\alpha}{d\alpha}=(0,0,0,0,0,\mathbb{I})\tag{6.1.78}$$

and $D_1D_2\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon)=D_1D_2\mathscr{G}_{\mathscr{T}}(\theta,\alpha,\varepsilon)-\frac{\partial\mathscr{H}_\alpha}{\partial\alpha}$, we conclude

$$\begin{aligned}&\left\|\frac{\partial}{\partial\alpha}[\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)-\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)]\right\|\\&\quad\leq[c_{\mathscr{G}}^{(2)}(\bar\rho_0+\varepsilon_0)+k\widetilde{c}_3\mu_0]\left\|\frac{\partial}{\partial\alpha}[\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha]\right\|\\&\qquad+4\left(c_{\mathscr{G}}^{(2)}+kN_-\right)\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|\left(c_\gamma\mu+c_{\mathscr{G}}^{(1)}|\varepsilon|\right).\end{aligned}\tag{6.1.79}$$

Similarly, we obtain

$$\begin{aligned}&\left\|\frac{\partial}{\partial\alpha}[\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)-\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,0)]\right\|=|\varepsilon|\\&\left\|\frac{\partial}{\partial\alpha}\int_0^1 D_3\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\tau\varepsilon)\,d\tau\right\|\leq 2c_{\mathscr{G}}^{(2)}|\varepsilon|.\end{aligned}\tag{6.1.80}$$

Now, since

$$\left\|\frac{\partial}{\partial\alpha}\mathscr{H}_\alpha^{-1}\Pi_\psi\right\|\leq\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|^2\left\|\frac{\partial}{\partial\alpha}\mathscr{H}_\alpha\right\|\leq 2kN_-\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|^2,$$

we derive, using also (6.1.75), (6.1.43):

$$\begin{aligned}
&\left\|\left[\frac{\partial}{\partial\alpha}\mathscr{H}_\alpha^{-1}\Pi_\psi\right]\mathscr{F}_\mathscr{T}(\theta_\mathscr{T}(\alpha,\varepsilon),\alpha,\varepsilon)\right\| \\
&\leq 2kN_-\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|^2 \\
&\quad\cdot\Big\{\|\mathscr{F}_\mathscr{T}(\theta_\mathscr{T}(\alpha,\varepsilon),\alpha,\varepsilon)-\mathscr{F}_\mathscr{T}(\theta_\alpha,\alpha,\varepsilon)\|+\|\mathscr{G}_\mathscr{T}(\theta_\alpha,\alpha,\varepsilon)\|\Big\} \\
&\leq 2kN_-\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|^2\left[\Lambda_1(\mu+|\varepsilon|)^2+c_\gamma\mu+c_\mathscr{G}^{(1)}|\varepsilon|\right].
\end{aligned} \tag{6.1.81}$$

Plugging (6.1.79), (6.1.80), (6.1.81) into (6.1.76) and assuming, instead of (6.1.73), that

$$2\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|[c_\mathscr{G}^{(2)}(\bar\rho_0+\bar\varepsilon_0)+k\widetilde{c}_3\mu_0]\leq 1$$

we obtain

$$\begin{aligned}
&\left\|\frac{\partial}{\partial\alpha}[\theta_\mathscr{T}(\alpha,\varepsilon)-\theta_\alpha]\right\| \\
&\leq 2\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|\Big\{4\left(c_\mathscr{G}^{(2)}+kN_-\right)\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|\left(c_\gamma\mu+c_\mathscr{G}^{(1)}|\varepsilon|\right) \\
&\quad+2c_\mathscr{G}^{(2)}|\varepsilon|+c_\gamma\mu+2kN_-\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|\left[\Lambda_1(\mu+|\varepsilon|)^2+c_\gamma\mu+c_\mathscr{G}^{(1)}|\varepsilon|\right]\Big\} \\
&\leq \Lambda_2(\mu+|\varepsilon|),
\end{aligned} \tag{6.1.82}$$

where Λ_2 is a positive constant that does not depend on $(\mathscr{T},\alpha,\mu,\varepsilon)$. We now take

$$\mu=\varepsilon^2$$

that is $T=\bar T-2\delta^{-1}\ln|\varepsilon|$. Note that from (6.1.74), we get:

$$\|\theta_\mathscr{T}(\alpha,\varepsilon)-\theta_\alpha\|\leq 2\|\mathscr{H}_\alpha^{-1}\Pi_\psi\|(c_\gamma|\varepsilon|+c_\mathscr{G}^{(1)})|\varepsilon|. \tag{6.1.83}$$

Then, if we can solve the equation $\left(\mathbb{I}-\Pi_\psi\right)\mathscr{F}_\mathscr{T}(\theta_\mathscr{T}(\alpha,\varepsilon),\alpha,\varepsilon)=0$ for $\alpha=\alpha_\mathscr{T}(\varepsilon)=\{\alpha_{m,\mathscr{T}}(\varepsilon)\}_{m\in\mathbb{Z}}$ and define $z_{m,\mathscr{T}}^{\pm}(t,\varepsilon)$, $z_{m,\mathscr{T}}^0(t,\varepsilon)$ as $z_m^+(t,\xi_m^+,\varphi_m^+,\beta_m,\varepsilon)$, $z_m^-(t,\xi_m^-,\varphi_m^-,\alpha_m,\varepsilon)$ and $z_m^0(t,\bar\xi_m,\alpha_m,\beta_m,\varepsilon)$, with

$$\theta_\mathscr{T}(\varepsilon)=\theta_\mathscr{T}(\alpha_\mathscr{T}(\varepsilon),\varepsilon)$$

instead of $\theta=\left\{(\varphi_m^-,\varphi_m^+,\xi_m^-,\xi_m^+,\bar\xi_m,\beta_m)\right\}_{m\mathbb{Z}}$ and with $\mu=\varepsilon^2$, we see that condition (6.1.7) follows from (6.1.31), (6.1.34) and (6.1.40) provided $|\varepsilon|<\varepsilon_\rho$, taking ε_ρ smaller if necessary. Thus to complete the proof of Theorem 6.1.16 we only need to show that the equation

$$(\mathbb{I}-\Pi_\psi)\mathscr{F}_\mathscr{T}(\theta_\mathscr{T}(\alpha,\varepsilon),\alpha,\varepsilon)=0$$

can be solved for α in terms of $\varepsilon\in(-\varepsilon_\rho,\varepsilon_\rho)$ and $\mathscr{T}$ satisfying the conditions of Theorem 6.1.16. Now, from (6.1.83) we see that

$$\lim_{\varepsilon\to 0}(\mathbb{I}-\Pi_\psi)\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)=\lim_{\varepsilon\to 0}(\mathbb{I}-\Pi_\psi)\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)=0$$

uniformly with respect to $(\alpha,\mathscr{T})$ (recall that, see (6.1.43), $\|\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\|\le c_\gamma\mu=c_\gamma\varepsilon^2$). Hence we are led to prove that the bifurcation function

$$\frac{1}{\varepsilon}(\mathbb{I}-\Pi_\psi)\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)=0 \tag{6.1.84}$$

can be solved for α in terms of $\varepsilon\in(-\varepsilon_\rho,\varepsilon_\rho)$, $\varepsilon\neq 0$, and $\mathscr{T}$ satisfying the conditions of Theorem 6.1.16. We observe that, with $\mu=\varepsilon^2$, (6.1.75) reads:

$$\|\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)-\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)\|\le\Lambda_1(1+|\varepsilon|)^2\varepsilon^2.$$

Hence, using also (6.1.65) and (6.1.43) with $\mu=\mathrm{e}^{-\delta(T-\bar{T})}=\varepsilon^2$:

$$\begin{aligned}B_{\mathscr{T}}(\alpha,\varepsilon)&=\frac{1}{\varepsilon}\left(\mathbb{I}-\Pi_\psi\right)\left\{\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)+O(\varepsilon^2)\right\}\\&=\frac{1}{\varepsilon}\left(\mathbb{I}-\Pi_\psi\right)[\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)-\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)]+O(\varepsilon)\\&=\left(\mathbb{I}-\Pi_\psi\right)D_3\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)+O(\varepsilon)\end{aligned}$$

where $O(\varepsilon)$ is uniform with respect to $(\mathscr{T},\alpha)$. Now we look at:

$$D_1B_{\mathscr{T}}(\alpha,\varepsilon)=\frac{1}{\varepsilon}(\mathbb{I}-\Pi_\psi)\frac{\partial}{\partial\alpha}\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)\,. \tag{6.1.85}$$

Subtracting

$$\begin{aligned}&\left(D_1^2\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\frac{d\theta_\alpha}{d\alpha}+D_1D_2\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\right)(\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha)\\&\quad=\frac{d}{d\alpha}[D_1\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,0)](\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha)\end{aligned}$$

from both sides of (6.1.77) and using the uniform continuity of $D_1^2\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon)$, $D_1D_2\mathscr{F}_{\mathscr{T}}(\theta,\alpha,\varepsilon)$ in $(\theta,\alpha,\varepsilon)$, uniformly with respect to $\mathscr{T}$ we see that:

$$\begin{aligned}&\Big\|\frac{\partial}{\partial\alpha}\mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon)-\frac{\partial}{\partial\alpha}\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)\\&\quad-\left(D_1^2\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\frac{d\theta_\alpha}{d\alpha}+D_1D_2\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\right)(\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha)\Big\|\\&\le\left(\left(c_{\mathscr{G}}^{(2)}\|\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha\|+|\varepsilon|\right)+k\widetilde{c}_3\varepsilon^2\right)\left\|\frac{\partial}{\partial\alpha}(\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha)\right\|\\&\quad+\eta(\|\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha\|+|\varepsilon|)\|\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha\|\end{aligned}$$

where $\eta(r) \to 0$ as $r \to 0$, uniformly with respect to $(\mathscr{T}, \alpha, \varepsilon)$, So, using (6.1.83) and (6.1.82) with $\mu = \varepsilon^2$ we obtain:

$$\begin{aligned}&\frac{\partial}{\partial\alpha}\ \mathscr{F}_{\mathscr{T}}(\theta_{\mathscr{T}}(\alpha,\varepsilon),\alpha,\varepsilon) - \frac{\partial}{\partial\alpha}\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)\\ &\quad -\frac{d}{d\alpha}[D_1\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,0)](\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha) = o(\varepsilon)\end{aligned} \tag{6.1.86}$$

uniformly with respect to $(\alpha, \mathscr{T})$. So, plugging (6.1.86) into (6.1.85), using (6.1.65) and (6.1.44) with $\mu = \mathrm{e}^{-\delta(T-\bar{T})} = \varepsilon^2$, we obtain:

$$\begin{aligned}D_1B_{\mathscr{T}}(\alpha,\varepsilon) &= (\mathbb{I}-\Pi_\psi)\frac{\partial}{\partial\alpha}\frac{\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,\varepsilon)-\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,0)}{\varepsilon}\\ &\quad + (\mathbb{I}-\Pi_\psi)\left\{\varepsilon^{-1}\frac{d}{d\alpha}[D_1\mathscr{F}_{\mathscr{T}}(\theta_\alpha,\alpha,0)][\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha]\right\} + o(1)\\ &= \frac{d}{d\alpha}(\mathbb{I}-\Pi_\psi)D_3\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\\ &\quad + (\mathbb{I}-\Pi_\psi)\left\{\varepsilon^{-1}\frac{d}{d\alpha}[D_1\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)-\mathscr{H}_\alpha][\theta_{\mathscr{T}}(\alpha,\varepsilon)-\theta_\alpha]\right\}\\ &\quad + o(1)\end{aligned}$$

with $o(1)$ being uniform with respect to α. But, differentiating (6.1.51) we see that

$$\frac{d}{d\alpha}(D_1\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)-\mathscr{H}_\alpha) = \{(\mathscr{L}_m^\alpha,0,0,0,0,0)\}_{m\in\mathbb{Z}}$$

where

$$\begin{aligned}\mathscr{L}_m^\alpha(\widetilde{\alpha})(\theta) &= \mathscr{L}_m^\alpha(\widetilde{\alpha})(\varphi_{m+1}^-,\varphi_m^+,\xi_{m+1}^-,\xi_m^+,\bar{\xi}_m,\beta_m)\\ &= \left[\dot{X}_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})(\widetilde{\alpha}_{m+1}-\widetilde{\alpha}_m)\right]\xi_{m+1}^-\\ &\quad +[\ddot{\gamma}_-(T_{2m+1}-T_{2m+2}+\alpha_m-\alpha_{m+1})(\widetilde{\alpha}_{m+1}-\widetilde{\alpha}_m)]\beta_m\\ &\le 2N_-k\delta^{-1}(\delta+|\dot{\gamma}_-(-\bar{T})|)\mu\|\theta\|\,\|\widetilde{\alpha}\| = O(\varepsilon^2)\|\theta\|\|\widetilde{\alpha}\|\end{aligned}$$

and hence

$$\left\|\frac{d}{d\alpha}[D_1\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)-\mathscr{H}_\alpha]\right\| = O(\varepsilon^2).$$

In summary, we obtain:

$$D_1B_{\mathscr{T}}(\alpha,\varepsilon) = \frac{d}{d\alpha}[(\mathbb{I}-\Pi_\psi)D_3\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)] + o(1) \tag{6.1.87}$$

uniformly with respect to α and $\mathscr{T}$. We have then

$$\lim_{\varepsilon\to 0} B_{\mathscr{T}}(\alpha,\varepsilon) = (\mathbb{I}-\Pi_\psi)D_3\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0) = \frac{1}{|L^*\psi|}\langle\psi, LD_3\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\rangle\widetilde{\psi},$$

$$\lim_{\varepsilon\to 0} D_1B_{\mathscr{T}}(\alpha,\varepsilon) = \frac{d}{d\alpha}\frac{1}{|L^*\psi|}\langle\psi, LD_3\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\rangle\widetilde{\psi},$$

uniformly with respect to α and $\mathscr{T}$ (recall that L has been defined in (6.1.58)). To conclude the proof of Theorem 6.1.16 we evaluate $\langle\psi, LD_3\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\rangle$. We have:

$$D_3\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0) =$$

$$\left\{\begin{array}{c}
\frac{\partial z_m^+}{\partial\varepsilon}(T_{2m+1}+\alpha_m,0,0,\alpha_m,0) - \frac{\partial z_{m+1}^-}{\partial\varepsilon}(T_{2m+1}+\alpha_m,0,0,\alpha_{m+1},0) \\
-\frac{\partial z_m^-}{\partial\varepsilon}(T_{2m}-\bar{T}+\alpha_m,0,0,\alpha_m,0) \\
R_0\left[\frac{\partial z_m^0}{\partial\varepsilon}(T_{2m}+\bar{T}+\alpha_m,\gamma_0(-\bar{T}),\alpha_m,\alpha_m,0) - \frac{\partial z_m^+}{\partial\varepsilon}(T_{2m}+\bar{T}+\alpha_m,0,0,\alpha_m,0)\right] \\
G'(\gamma(-\bar{T}))\frac{\partial z_m^-}{\partial\varepsilon}(T_{2m}-\bar{T}+\alpha_m,0,0,\alpha_m,0) \\
G'(\gamma(\bar{T}))\frac{\partial z_m^0}{\partial\varepsilon}(T_{2m}+\bar{T}+\alpha_m,\gamma_0(-\bar{T}),\alpha_m,\alpha_m,0) \\
G'(\gamma(\bar{T}))\frac{\partial z_m^+}{\partial\varepsilon}(T_{2m}+\bar{T}+\alpha_m,0,0,\alpha_m,0)
\end{array}\right\}_{m\in\mathbb{Z}}.$$

Thus:

$$\begin{aligned}
&LD_{\ \varepsilon}\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0) \\
&= R_0\Big\{\frac{\partial z_m^0}{\partial\varepsilon}(T_{2m}+\bar{T}+\alpha_m,\gamma_0(-\bar{T}),\alpha_m,\alpha_m,0) \\
&\quad -\frac{\partial z_m^+}{\partial\varepsilon}(T_{2m}+\bar{T}+\alpha_m,0,0,\alpha_m,0) \\
&\quad -\frac{G'(\gamma(-\bar{T}))\frac{\partial z_m^-}{\partial\varepsilon}(T_{2m}-\bar{T}+\alpha_m,0,0,\alpha_m,0)}{G'(\gamma(-\bar{T}))\dot{\gamma}_-(-\bar{T})}X_0(\bar{T})\dot{\gamma}_-(-\bar{T}) \\
&\quad +X_0(\bar{T})\frac{\partial z_m^-}{\partial\varepsilon}(T_{2m}-\bar{T}+\alpha_m,0,0,\alpha_m,0) \\
&\quad +\frac{G'(\gamma(\bar{T}))\frac{\partial z_m^+}{\partial\varepsilon}(T_{2m}+\bar{T}+\alpha_m,0,0,\alpha_m,0)}{G'(\gamma(\bar{T}))\dot{\gamma}_+(\bar{T})}\dot{\gamma}_+(\bar{T})\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= R_0\Big\{\frac{\partial z_m^0}{\partial \varepsilon}(T_{2m}+\bar T+\alpha_m,\gamma_0(-\bar T),\alpha_m,\alpha_m,0)\\
&\quad +X_0(\bar T)R_-\frac{\partial z_m^-}{\partial \varepsilon}(T_{2m}-\bar T+\alpha_m,0,0,\alpha_m,0)\\
&\quad -R_+\frac{\partial z_m^+}{\partial \varepsilon}(T_{2m}+\bar T+\alpha_m,0,0,\alpha_m,0)\dot\gamma_+(\bar T)\Big\}\\
&= R_0\Big\{\frac{\partial z_m^0}{\partial \varepsilon}(T_{2m}+\bar T+\alpha_m,\gamma_0(-\bar T),\alpha_m,\alpha_m,0)\\
&\quad +X_0(\bar T)R_-\frac{\partial z_m^-}{\partial \varepsilon}(T_{2m}-\bar T+\alpha_m,0,0,\alpha_m,0)\Big\}\\
&\quad -R_+\frac{\partial z_m^+}{\partial \varepsilon}(T_{2m}+\bar T+\alpha_m,0,0,\alpha_m,0)\dot\gamma_+(\bar T)
\end{aligned}
$$

since $\mathscr{R}R_+\subset\mathscr{R}R_0$. Next from Eqs. (6.1.32), (6.1.36), (6.1.39) we get:

$$
\begin{aligned}
&\frac{\partial z_m^0}{\partial \varepsilon}(T_{2m}+\bar T+\alpha_m,\gamma_0(-\bar T),\alpha_m,\alpha_m,0)\\
&\quad =\int_{-\bar T}^{\bar T} X_0(\bar T)X_0^{-1}(t)g(t+T_{2m}+\alpha_m,\gamma_0(t),0)dt\,,\\
&\frac{\partial z_m^-}{\partial \varepsilon}(T_{2m}-\bar T+\alpha_m,0,0,\alpha_m,0)\\
&\quad =\int_{T_{2m-1}-T_{2m}-1}^{-\bar T} P_-X_-^{-1}(t)g(t+T_{2m}+\alpha_m,\gamma_-(t),0)dt\,,\\
&\frac{\partial z_m^+}{\partial \varepsilon}(T_{2m}+\bar T+\alpha_m,0,0,\alpha_m,0)\\
&\quad =-\int_{\bar T}^{T_{2m+1}-T_{2m}+1}(\mathbb{I}-P_+)X_+^{-1}(t)g(t+T_{2m}+\alpha_m,\gamma_+(t),0)dt\,.
\end{aligned}
\tag{6.1.88}
$$

As a consequence, using also (6.1.60), we get:

$$
\begin{aligned}
&\langle\psi,LD_3\mathscr{G}_{\mathscr{T}}(\theta_\alpha,\alpha,0)\rangle\\
&=\psi^*\Big[\int_{T_{2m-1}-T_{2m}-1}^{-\bar T} R_0X_0(\bar T)R_-P_-X_-^{-1}(t)g(t+T_{2m}+\alpha_m,\gamma_-(t),0)dt\\
&\quad +\int_{-\bar T}^{\bar T} R_0X_0(\bar T)X_0^{-1}(t)g(t+T_{2m}+\alpha_m,\gamma_0(t),0)dt\\
&\quad +\int_{\bar T}^{T_{2m+1}-T_{2m}+1} R_+(\mathbb{I}-P_+)X_+^{-1}(t)g(t+T_{2m}+\alpha_m,\gamma_+(t),0)dt\Big]\\
&=\int_{T_{2m-1}-T_{2m}-1}^{T_{2m+1}-T_{2m}+1}\psi^*(t)g(t+T_{2m}+\alpha_m,\gamma(t),0)dt\\
&=\int_{-\infty}^{\infty}\psi^*(t)g(t+T_{2m}+\alpha_m,\gamma(t),0)dt+O(\mathrm{e}^{-\delta(T+1)})\\
&=\int_{-\infty}^{\infty}\psi^*(t)g(t+T_{2m}+\alpha_m,\gamma(t),0)dt+O(\varepsilon^2)
\end{aligned}
\tag{6.1.89}
$$

where $\psi(t)$ has been defined in (6.1.61). Thus we prove that

$$B_{\mathscr{T}}(\alpha,\varepsilon)=\frac{1}{|L^*\psi|}\{\mathscr{M}(\alpha_m+T_{2m})\widetilde{\psi}\}_{m\in\mathbb{Z}}+O(\varepsilon),$$
$$D_1B_{\mathscr{T}}(\alpha,\varepsilon)=\frac{1}{|L^*\psi|}\{\mathscr{M}'(\alpha_m+T_{2m})\widetilde{\psi}\}_{m\in\mathbb{Z}}+o(1),$$

where $O(\varepsilon)$ and $o(1)$ are uniform with respect to α and $\mathscr{T}$. Now assume that $\mathscr{T}=\{T_m\}_{m\in\mathbb{Z}}$ and $\alpha_0=\{\alpha_m^0\}_{m\in\mathbb{Z}}$ satisfy the assumptions of Theorem 6.1.16. We have:

$$\lim_{\varepsilon\to 0}B_{\mathscr{T}}(\alpha_0,\varepsilon)=0,$$
$$\lim_{\varepsilon\to 0}D_1B_{\mathscr{T}}(\alpha_0,\varepsilon)=\frac{1}{|L^*\psi|}\{\mathscr{M}'(\alpha_m^0+T_{2m})\widetilde{\psi}\}_{m\in\mathbb{Z}}$$

uniformly with respect to $\mathscr{T}$. That is $\|D_1B_{\mathscr{T}}(\alpha_0,\varepsilon)\|>\frac{c_0}{2|L^*\psi|}$ provided $|\varepsilon|$ is sufficiently small. From the implicit function theorem we deduce the existence of $0<\bar{\varepsilon}_\rho<\varepsilon_0$ so that for any $0\neq\varepsilon\in(-\bar{\varepsilon}_\rho,\bar{\varepsilon}_\rho)$ and any sequence $\mathscr{T}=\{T_m\}_{m\in\mathbb{Z}}$ that satisfy the assumption of Theorem 6.1.16 there exists a unique sequence $\alpha(\mathscr{T},\varepsilon)=\{\alpha_m(\mathscr{T},\varepsilon)\}_{m\in\mathbb{Z}}\in\ell^\infty_\chi$ so that $\alpha(\mathscr{T},0)=\alpha_0$ and

$$B_{\mathscr{T}}(\alpha(\mathscr{T},\varepsilon),\varepsilon)=0.$$

Taking $\theta_{\mathscr{T}}(\varepsilon)=\theta_{\mathscr{T}}(\alpha(\mathscr{T},\varepsilon),\varepsilon)$ and

$$z(t)=\begin{cases} z^-_{m,\mathscr{T}}(t,\varepsilon), & \text{if } t\in[T_{2m-1}+\beta_{m-1,\mathscr{T}}(\varepsilon),T_{2m}-\bar{T}+\alpha_{m,\mathscr{T}}(\varepsilon)],\\ z^0_{m,\mathscr{T}}(t,\varepsilon), & \text{if } t\in[T_{2m}-\bar{T}+\alpha_{m,\mathscr{T}}(\varepsilon),T_{2m}+\bar{T}+\beta_{m,\mathscr{T}}(\varepsilon)],\\ z^+_{m,\mathscr{T}}(t,\varepsilon), & \text{if } t\in[T_{2m}+\bar{T}+\beta_{m,\mathscr{T}}(\varepsilon),T_{2m+1}+\beta_{m,\mathscr{T}}(\varepsilon)],\end{cases}$$

we see that $z(t)$ satisfies the conclusion of Theorem 6.1.16 with $\hat{\alpha}_m=\alpha_m(\mathscr{T},\varepsilon)$ and $\hat{\beta}_m=\beta_m(\alpha(\mathscr{T},\varepsilon),\varepsilon)$. The proof is complete. □

Remark 6.1.17. Functions $\mathscr{M},\mathscr{M}':\mathbb{R}\to\mathbb{R}$ are bounded.

Remark 6.1.18. Following the above arguments, we can consider also cases when $\bar{m}\in\mathbb{Z}$ exists so that either $T_j=-\infty\ \forall j\leq 2\bar{m}-1$ or $T_j=\infty\ \forall j\geq 2\bar{m}+1$. Then Theorem 6.1.16 is obviously modified (see (6.1.97), (6.1.98) and (6.1.99) below).

Remark 6.1.19. Here we emphasize that during the proof of Theorem 6.1.16, we only use the fact that f and g are C^2 with bounded and uniformly continuous derivatives. We should need higher derivatives if α_0 is a degenerate root of $\mathscr{M}_{\mathscr{T}}(\alpha)=\{\mathscr{M}(T_{2m}+\alpha_m)\}_{m\in\mathbb{Z}}$, when condition (6.1.64) fails.

We are now able to give the proof of Theorem 6.1.3. First we show the following preparatory results.

Lemma 6.1.20. *For any $\varepsilon\neq 0$ there exists $|\varepsilon|>\nu_\varepsilon>0$ so that if a sequence $\mathscr{T}=\{T_m\}_{m\in\mathbb{Z}}$ satisfies* (6.1.6) *then also it holds*

$$|D_1 g(t+T_{2m},z,0) - D_1 g(t,z,0)| < |\varepsilon| \tag{6.1.90}$$

for any $(t,z,m) \in \mathbb{R}^{n+1} \times \mathbb{Z}$.

Proof. Let $\varepsilon \neq 0$. Take $n_\varepsilon \in \mathbb{N}$ and $\nu_\varepsilon > 0$ as

$$n_\varepsilon = 2\left[\frac{\|D_{11}g\|}{|\varepsilon|}\right] + 1, \quad \nu_\varepsilon := \frac{|\varepsilon|}{4n_\varepsilon} \tag{6.1.91}$$

and let $\mathscr{T} = \{T_m\}_{m\in\mathbb{Z}}$ be a sequence satisfying (6.1.6). Then we derive [40]:

$$\begin{aligned}
&|D_1 g(t+T_{2m},z,0) - D_1 g(t,z,0)| \\
&\le \left|D_1 g(t+T_{2m},z,0) - n_\varepsilon\left[g\left(t+T_{2m}+\frac{1}{n_\varepsilon},z,0\right) - g(t+T_{2m},z,0)\right]\right| \\
&\quad + \left|D_1 g(t,z,0) - n_\varepsilon\left[g\left(t+\frac{1}{n_\varepsilon},z,0\right) - g(t,z,0)\right]\right| \\
&\quad + n_\varepsilon\left|g\left(t+T_{2m}+\frac{1}{n_\varepsilon},z,0\right) - g\left(t+\frac{1}{n_\varepsilon},z,0\right)\right| \\
&\quad + n_\varepsilon\left|g\left(t+T_{2m},z,0\right) - g(t,z,0)\right| \\
&\le n_\varepsilon \int_0^{1/n_\varepsilon} |D_1 g\left(t+T_{2m}+\eta,z,0\right) - D_1 g(t+T_{2m},z,0)|\, d\eta \\
&\quad + n_\varepsilon \int_0^{1/n_\varepsilon} |D_1 g\left(t+\eta,z,0\right) - D_1 g(t,z,0)|\, d\eta + 2n_\varepsilon \nu_\varepsilon \\
&\le \frac{\|D_{11}g\|}{n_\varepsilon} + 2n_\varepsilon\nu_\varepsilon < |\varepsilon|.
\end{aligned}$$

The proof of Lemma 6.1.20 is complete. □

Lemma 6.1.21. *If* $\varepsilon \neq 0$ *is sufficiently small then for any given sequence* $\{T_m\}_{m\in\mathbb{Z}}$ *with the properties of Lemma* 6.1.20*, a sequence* $\{\alpha_m^0\}_{m\in\mathbb{Z}} \in \ell_\chi^\infty$ *exists satisfying* (6.1.64) *for some* $c_0 > 0$.

Proof. Let $|\mathscr{M}'(\alpha^0)| = 4c_0$. We have:

$$\mathscr{M}(T_{2m}+\alpha) = \mathscr{M}(\alpha) + \int_{-\infty}^{\infty} \psi^*(t)[g(t+T_{2m}+\alpha,\gamma(t),0) - g(t+\alpha,\gamma(t),0)]dt$$

and hence:

$$|\mathscr{M}(T_{2m}+\alpha) - \mathscr{M}(\alpha)| \le |\varepsilon| \int_{-\infty}^{\infty} |\psi^*(t)|dt \le 2K\delta^{-1}|\varepsilon|$$

since $|\psi^*(t)| \le K\mathrm{e}^{-\delta|t|}$ for some $K \ge 1$ (see (6.1.63)). Similarly, from (6.1.90) we get

$$|\mathscr{M}'(T_{2m}+\alpha) - \mathscr{M}'(\alpha)| \le 2K\delta^{-1}|\varepsilon|\,.$$

Let $\chi/2 > \delta_1 > 0$ be so small that $\mathscr{M}(\alpha^0 - \delta_1)\mathscr{M}(\alpha^0 + \delta_1) < 0$ and $|\mathscr{M}'(\alpha)| \ge 2c_0$ for $\alpha \in [\alpha^0 - \delta_1, \alpha^0 + \delta_1]$. Then, there is an $\widetilde{\varepsilon}_0 > 0$ so that for $0 < |\varepsilon| < \widetilde{\varepsilon}_0$ and for any $m \in \mathbb{Z}$ the equation $\mathscr{M}(T_{2m}+\alpha)=0$ has a unique solution $\alpha_m^0=\alpha(T_{2m}) \in (\alpha^0-\delta_1, \alpha^0+\delta_1)$ along with $|\mathscr{M}'(T_{2m}+\alpha)| \geq c_0$ for $\alpha \in [\alpha^0-\delta_1, \alpha^0+\delta_1]$. The proof of Lemma 6.1.20 is complete. $\square$

Now we proceed with the proof of Theorem 6.1.3. Using Lemma 6.1.21, assumptions of Theorem 6.1.16 are verified and consequently, we obtain sequences $\{\hat{\alpha}_{m,\mathscr{T}}(\varepsilon)\}$, $\{\hat{\beta}_{m,\mathscr{T}}(\varepsilon)\}$, and a unique solution $z(t)$ of Eq. (6.1.1) that satisfies (6.1.7). To prove that $\sup_{m\in\mathbb{Z}} |\hat{\alpha}_{m,\mathscr{T}}(\varepsilon)-\alpha^0| < c_1|\varepsilon|$ and $\sup_{m\in\mathbb{Z}} |\hat{\beta}_{m,\mathscr{T}}(\varepsilon)-\alpha^0| < c_1|\varepsilon|$ assume for simplicity that $\mathscr{M}'(\alpha^0)=4c_0$ (a similar argument applies when $\mathscr{M}'(\alpha^0)=-4c_0$). Then we have, since $\mathscr{M}'(T_{2m}+\alpha)>c_0$ for any $\alpha \in [\alpha_0-\delta_1, \alpha_0+\delta_1]$:

$$2K\delta^{-1}|\varepsilon| \geq \left| \int_{\alpha^0}^{\alpha_m^0} \mathscr{M}'(T_{2m}+\tau)d\tau \right| \geq c_0|\alpha_m^0-\alpha^0|,$$

hence

$$|\hat{\alpha}_{m,\mathscr{T}}(\varepsilon)-\alpha^0| \leq |\hat{\alpha}_m(\mathscr{T},\varepsilon)-\alpha_m^0|+|\alpha_m^0-\alpha^0| \leq c_1|\varepsilon|+\frac{2K|\varepsilon|}{\delta c_0}=\widetilde{c}_1|\varepsilon|.$$

Similarly we get (possibly changing $\widetilde{c}_1$): $|\hat{\beta}_{m,\mathscr{T}}(\varepsilon)-\alpha^0| \leq \widetilde{c}_1|\varepsilon|$. The proof of Theorem 6.1.3 is complete.

Remark 6.1.22. By (6.1.91), we get $\nu_\varepsilon \sim \varepsilon^2$ in Theorem 6.1.3.

6.1.7 Chaotic Behaviour

Set (cf Section 2.5.2)

$$\hat{\mathscr{E}} := \{e \in \mathscr{E} \mid \inf\{m \in \mathbb{Z} \mid e_m=1\}=-\infty, \sup\{m \in \mathbb{Z} \mid e_m=1\}=\infty\},$$

$$\mathscr{E}_+ := \{e \in \mathscr{E} \mid \inf\{m \in \mathbb{Z} \mid e_m=1\}>-\infty, \sup\{m \in \mathbb{Z} \mid e_m=1\}=\infty\},$$

$$\mathscr{E}_- := \{e \in \mathscr{E} \mid \inf\{m \in \mathbb{Z} \mid e_m=1\}=-\infty, \sup\{m \in \mathbb{Z} \mid e_m=1\}<\infty\},$$

$$\mathscr{E}_0 := \{e \in \mathscr{E} \mid \inf\{m \in \mathbb{Z} \mid e_m=1\}>-\infty, \sup\{m \in \mathbb{Z} \mid e_m=1\}<\infty\}.$$

Note that $\hat{\mathscr{E}}$, $\mathscr{E}_-$, $\mathscr{E}_+$, $\mathscr{E}_0$ are invariant under the Bernoulli shift. In this section we suppose for simplicity that assumptions of Theorem 6.1.16 are satisfied with a technical condition $\|\alpha_0\| < \chi/2$, i.e the following holds:

(C) For any $\varepsilon \neq 0$ sufficiently small there is a sequence $\mathscr{T}=\{T_m\}_{m\in\mathbb{Z}}$ so that $T_{m+1}-T_m > \bar{T}+1-2\delta^{-1}\ln|\varepsilon|$ along with the existence of an $\alpha_0=\{\alpha_m^0\}_{m\in\mathbb{Z}} \in \ell_\chi^\infty$ with $\|\alpha_0\| < \chi/2$, satisfying (6.1.64).

Let $\mathscr{T}=\{T_m\}_{m\in\mathbb{Z}}$ be as in assumption (C). Assume, first, that $e \in \hat{\mathscr{E}}$. Let $\{n_m^e\}_{m\in\mathbb{Z}}$ be a fixed increasing doubly-infinite sequence of integers so that $e_k=1$ if and only if $k = n_m^e$. We define sequences $\mathscr{T}^e = \{T_m^e\}_{m\in\mathbb{Z}}$ and $\alpha_0^e = \{\alpha_m^{0e}\}_{m\in\mathbb{Z}}$ as

$$T_m^e := \begin{cases} T_{2n_k^e}, & \text{if } m = 2k, \\ T_{2n_k^e-1}, & \text{if } m = 2k-1, \end{cases} \tag{6.1.92}$$

and similarly

$$\alpha_m^{0e} := \alpha_{n_m^e}^0 \, . \tag{6.1.93}$$

Note that $T_{m+1}^e - T_m^e > \bar{T} + 1 - 2\delta^{-1}\ln|\varepsilon|$ for any $m \in \mathbb{Z}$ and $\mathscr{M}_{\mathscr{T}^e}(\alpha)$ has a simple zero α_0^e, i.e. (6.1.64) holds with exchanges $\mathscr{T}^e \leftrightarrow \mathscr{T}$ and $\alpha_0^e \leftrightarrow \alpha_0$. Since $|\alpha_{m+1}^{0e} - \alpha_m^{0e}| < \chi$ for any $m \in \mathbb{Z}$, assumptions of Theorem 6.1.16 are satisfied by $\mathscr{M}_{\mathscr{T}^e}(\alpha)$, $\mathscr{T}^e$ and α_0^e. Let $z(t) = z(t, \mathscr{T}^e)$ be the corresponding solution of Eq. (6.1.1) whose existence is stated in Theorem 6.1.16. Then $z(t)$ satisfies

$$\begin{aligned} &\sup\nolimits_{t\in[T_{2m-1}^e+\beta_{m-1}^e, T_{2m}^e-\bar{T}+\alpha_m^e]} |z(t) - \gamma_-(t - T_{2m}^e - \alpha_m^e)| < \rho, \\ &\sup\nolimits_{t\in[T_{2m}^e-\bar{T}+\alpha_m^e, T_{2m}^e+\bar{T}+\beta_m^e]} |z(t) - \gamma_0(t - T_{2m}^e - \alpha_m^e)| < \rho, \\ &\sup\nolimits_{t\in[T_{2m}^e+\bar{T}+\beta_m^e, T_{2m+1}^e+\beta_m^e]} |z(t) - \gamma_+(t - T_{2m}^e - \beta_m^e)| < \rho \,, \end{aligned} \tag{6.1.94}$$

where the sequences $\alpha^e = \{\alpha_m^e\}_{m\in\mathbb{Z}}$ and $\beta^e = \{\beta_m^e\}_{m\in\mathbb{Z}}$ are determined as in Theorem 6.1.16 (note here we remove hats for notational simplicity).

Now, consider the sequence $\widetilde{n}_m^e := n_{m+1}^e$ instead of n_m^e and denote with $\widetilde{\mathscr{T}^e}$, $\widetilde{\alpha}^e$, $\widetilde{\beta}^e$ and $\widetilde{\alpha}_0^e$ the corresponding sequences:

$$\widetilde{T}_m^e = T_{m+2}^e, \quad \widetilde{\alpha}_m^e = \alpha_{m+1}^e, \quad \widetilde{\beta}_m^e = \beta_{m+1}^e, \quad \widetilde{\alpha}_m^{0e} = \alpha_{m+1}^{0e}. \tag{6.1.95}$$

Then $\mathscr{M}_{\widetilde{\mathscr{T}^e}}(\alpha)$ has a simple zero $\widetilde{\alpha}_0^e$ and Theorem 6.1.16 is applicable. But clearly $\widetilde{z}(t) := z(t, \widetilde{\mathscr{T}^e})$ satisfies the same set of estimates (6.1.94) and hence, because of uniqueness, $z(t, \widetilde{\mathscr{T}^e}) = z(t, \mathscr{T}^e)$ depends only on e and $\mathscr{T}$ (and not on the choice of n_m^e). So we will write $z(t, \mathscr{T}, e)$ instead of $z(t, \mathscr{T}^e)$.

Now, assume that $e_j = 1$. Then $j = n_m^e$ for some $m \in \mathbb{Z}$ and (6.1.94) gives, provided $|\varepsilon|$ is sufficiently small,

$$\begin{aligned} |z(T_{2j}) - \gamma_0(-\alpha_j^0)| &\le |z(T_{2j}) - \gamma_0(-\alpha_m^e)| + |\gamma_0(-\alpha_m^e) - \gamma_0(-\alpha_j^0)| \\ &< \rho + \sup_{t\in\mathbb{R}} |\dot{\gamma}_0(t)|\,|\alpha_m^e - \alpha_j^0| \\ &< \rho + c_1|\varepsilon| \sup_{t\in\mathbb{R}} |\dot{\gamma}_0(t)| < \frac{3}{2}\rho \end{aligned}$$

since $T_{2m}^e = T_{2j}$. On the other hand, if $e_j = 0$, let $m \in \mathbb{Z}$ be such that $n_m^e < j < n_{m+1}^e$. Then $n_{m+1}^e - 1 \ge j \ge n_m^e + 1$ and so

$$\begin{aligned} T_{2j} - T_{2n_m^e} - \bar{T} - \beta_m^e &\geq T_{2n_m^e+2} - T_{2n_m^e} - \bar{T} - \|\alpha_0\| - c_1|\varepsilon| \\ &\geq \bar{T} + 2 - 4\delta^{-1}\ln|\varepsilon| - \|\alpha_0\| - c_1|\varepsilon| \\ &> 0 \end{aligned}$$

and

$$\begin{aligned} T_{2m+1}^e + \beta_m^e - T_{2j} &\geq T_{2n_{m+1}^e-1} - T_{2n_{m+1}^e-2} - \|\alpha_0\| - c_1|\varepsilon| \\ &\geq \bar{T} + 1 - 2\delta^{-1}\ln|\varepsilon| - \|\alpha_0\| - c_1|\varepsilon| \\ &> 0 \end{aligned}$$

for $0 < |\varepsilon| \ll 1$. Consequently, we have $T_{2j} \in \left[T_{2m}^e + \bar{T} + \beta_m^e, T_{2m+1}^e + \beta_m^e\right]$, and using (6.1.94), we get

$$|z(T_{2j})| \leq |z(T_{2j}) - \gamma_+(T_{2j} - T_{2n_m^e} - \beta_m^e)| + |\gamma_+(T_{2j} - T_{2n_m^e} - \beta_m^e)| < \frac{3}{2}\rho$$

since $T_{2j} - T_{2n_m^e} - \beta_m^e \geq T_{2n_m^e+2} - T_{2n_m^e} - \|\alpha_0\| - c_1|\varepsilon| > 2\bar{T} + 2 - 4\delta^{-1}\ln|\varepsilon| - \|\alpha_0\| - c_1|\varepsilon| \gg 1$ for $0 < |\varepsilon| \ll 1$, and thus $\left|\gamma_+(T_{2j} - T_{2n_m^e} - \beta_m^e)\right| < \rho/2$. So $z(t, \mathscr{T}, e)$ has the following property

$$\begin{aligned} &|z(T_{2j}) - \gamma_0(-\alpha_j^0)| < \frac{3}{2}\rho, \text{ if } e_j = 1\,, \\ &|z(T_{2j})| < \frac{3}{2}\rho, \qquad\qquad \text{if } e_j = 0\,. \end{aligned} \tag{6.1.96}$$

Next, assume that $e \in \mathscr{E}_+$ and let again $\{n_m^e\}_{m\in\mathbb{Z}}$ be a fixed increasing sequence of integers so that $e_k = 1$ if and only if $k = n_m^e$ and $\lim_{m\to\infty} n_m^e = \infty$. Corresponding to this sequence, we define $\mathscr{T}^e$ as in (6.1.92) and then we obtain α^e and β^e as in (6.1.94) with the difference that $T_m^e = -\infty$ and $\alpha_m^e = \beta_m^e = 0$ for any $m < 2\bar{m}$ where $\bar{m}$ is such that $e_{n_{\bar{m}}^e} = 1$ and $e_j = 0$ for any $j < n_{\bar{m}}^e$. According to this choice, by Remark 6.1.18, we obtain a solution $z(t) = z(t, \mathscr{T}^e)$ of Eq. (6.1.1) that satisfies (6.1.94) when $m > \bar{m}$ whereas for $m = \bar{m}$ it satisfies:

$$\begin{aligned} &\sup\nolimits_{t\in(-\infty, T_{2\bar{m}}^e - \bar{T} + \alpha_{\bar{m}}^e]} |z(t) - \gamma_-(t - T_{2\bar{m}}^e - \alpha_{\bar{m}}^e)| < \rho, \\ &\sup\nolimits_{t\in[T_{2\bar{m}}^e - \bar{T} + \alpha_{\bar{m}}^e, T_{2\bar{m}}^e + \bar{T} + \beta_{\bar{m}}^e]} |z(t) - \gamma_0(t - T_{2\bar{m}}^e - \alpha_{\bar{m}}^e)| < \rho, \\ &\sup\nolimits_{t\in[T_{2\bar{m}}^e + \bar{T} + \beta_{\bar{m}}^e, T_{2\bar{m}+1}^e + \beta_{\bar{m}}^e]} |z(t) - \gamma_+(t - T_{2\bar{m}}^e - \beta_{\bar{m}}^e)| < \rho. \end{aligned} \tag{6.1.97}$$

Note, then, that if we take, as in the previous case, $\widetilde{n}_m^e = n_{m+1}^e$ and $\widetilde{\mathscr{T}^e}$, $\widetilde{\alpha}^e$, $\widetilde{\beta}^e$ as in (6.1.95), then (6.1.94) holds with $\widetilde{\mathscr{T}^e}$ instead $\mathscr{T}^e$, provided $m > \bar{m} - 1$ whereas (6.1.97) holds with $\widetilde{T}_{2(\bar{m}-1)}^e$ and $\widetilde{T}_{2\bar{m}-1}^e$ instead of $T_{2\bar{m}}^e$ and $T_{2\bar{m}+1}^e$ respectively. So in this case we can also see that $z(t, \mathscr{T}^e) = z(t, \mathscr{T}, e)$ depends only on $(\mathscr{T}, e)$ and not on the choice of the sequence n_m^e. Moreover, (6.1.96) holds also in this case. In fact if either $e_j = 1$ or $e_j = 0$ and there exists $m \in \mathbb{Z}$ so that $n_m^e < j < n_{m+1}^e$ the same proof as before goes through. If, instead, $e_j = 0$ and $j < n_{\bar{m}}^e$, then the estimate

$|z(T_{2j})| < \frac{3}{2}\rho$ follows from the first estimate in (6.1.97) since $2j \leq 2n_{\bar{m}}^e - 2$ and then $T_{2j}^e - T_{2\bar{m}}^e - \alpha_{\bar{m}}^e \leq T_{2n_{\bar{m}}^e-2}^e - T_{2n_{\bar{m}}^e} + \|\alpha_0\| + c_1|\varepsilon| \leq -2\bar{T} - 2 - 4\delta^{-1}\ln|\varepsilon| + \|\alpha_0\| + c_1|\varepsilon| \ll 0$ for $0 < |\varepsilon| \ll 1$.

Similarly, if $e \in \mathscr{E}_-$ then by Remark 6.1.18, we obtain a solution $z(t) = z(t, \mathscr{T}^e)$ of Eq. (6.1.1) that satisfies (6.1.94) when $m < \bar{m}$ whereas for $m = \bar{m}$ it satisfies

$$\begin{aligned}
&\sup\nolimits_{t\in(T_{2\bar{m}-1}^e, T_{2\bar{m}}^e - \bar{T} + \alpha_{\bar{m}}^e]} |z(t) - \gamma_-(t - T_{2\bar{m}}^e - \alpha_{\bar{m}}^e)| < \rho, \\
&\sup\nolimits_{t\in[T_{2\bar{m}}^e - \bar{T} + \alpha_{\bar{m}}^e, T_{2\bar{m}}^e + \bar{T} + \beta_{\bar{m}}^e]} |z(t) - \gamma_0(t - T_{2\bar{m}}^e - \alpha_{\bar{m}}^e)| < \rho, \\
&\sup\nolimits_{t\in[T_{2\bar{m}}^e + \bar{T} + \beta_{\bar{m}}^e, \infty)} |z(t) - \gamma_+(t - T_{2\bar{m}}^e - \beta_{\bar{m}}^e)| < \rho.
\end{aligned} \tag{6.1.98}$$

From an argument similar to the previous one (in this case, we can take, for example, $\widetilde{n}_m^e = n_{m-1}^e$) we see that $z(t, \mathscr{T}^e) = z(t, \mathscr{T}, e)$ depends only on $(\mathscr{T}, e)$ and not on the choice of the sequence n_m^e and (6.1.96) holds.

Next, assume that $e \in \mathscr{E}_0$ with $e \neq 0$. Then there are $\bar{m}_- < \bar{m}_+$ so that $e_k = 0$ if either $k < n_{m_-}^e$ or $k > n_{m_+}^e$ and Eq. (6.1.1) has a unique solution $z(t, \mathscr{T}^e)$ so that (6.1.94) holds when $\bar{m}_- < m < \bar{m}_+$ whereas when either $m = m_-$ or $m = m_+$ it satisfies

$$\begin{aligned}
&\sup\nolimits_{t\in(-\infty, T_{2\bar{m}_-}^e - \bar{T} + \alpha_{\bar{m}_-}^e]} |z(t) - \gamma_-(t - T_{2\bar{m}_-}^e - \alpha_{\bar{m}_-}^e)| < \rho, \\
&\sup\nolimits_{t\in[T_{2\bar{m}_-}^e - \bar{T} + \alpha_{\bar{m}_-}^e, T_{2\bar{m}_-}^e + \bar{T} + \beta_{\bar{m}_-}^e]} |z(t) - \gamma_0(t - T_{2\bar{m}_-}^e - \alpha_{\bar{m}}^e)| < \rho, \\
&\sup\nolimits_{t\in[T_{2\bar{m}_-}^e + \bar{T} + \beta_{\bar{m}_-}^e, T_{2\bar{m}_-+1}^e + \beta_{\bar{m}_-}^e]} |z(t) - \gamma_+(t - T_{2\bar{m}_-}^e - \beta_{\bar{m}_-}^e)| < \rho, \\
&\sup\nolimits_{t\in(T_{2\bar{m}_+-1}^e, T_{2\bar{m}_+}^e - \bar{T} + \alpha_{\bar{m}_+}^e]} |z(t) - \gamma_-(t - T_{2\bar{m}_+}^e - \alpha_{\bar{m}_+}^e)| < \rho, \\
&\sup\nolimits_{t\in[T_{2\bar{m}_+}^e - \bar{T} + \alpha_{\bar{m}_+}^e, T_{2\bar{m}_+}^e + \bar{T} + \beta_{\bar{m}_+}^e]} |z(t) - \gamma_0(t - T_{2\bar{m}_+}^e - \alpha_{\bar{m}_+}^e)| < \rho, \\
&\sup\nolimits_{t\in[T_{2\bar{m}_+}^e + \bar{T} + \beta_{\bar{m}_+}^e, \infty)} |z(t) - \gamma_+(t - T_{2\bar{m}_+}^e - \beta_{\bar{m}_+}^e)| < \rho.
\end{aligned} \tag{6.1.99}$$

Moreover $z(t, \mathscr{T}^e) = z(t, \mathscr{T}, e)$ depends only on $(\mathscr{T}, e)$ and not on the choice of n_m^e and (6.1.96) holds.

Finally, if $e = 0$, that is $e_k = 0$ for any $k \in \mathbb{Z}$, by we define $z(t, \mathscr{T}, 0) = u(t)$ as the unique bounded solution of (6.1.1) so that

$$\sup_{t\in\mathbb{R}} |u(t)| < \rho. \tag{6.1.100}$$

The existence and uniqueness of $u(t)$ follow from the standard regular perturbation theory (see [41–44], Remark 4.1.7). Now we are able to prove the following theorem:

Theorem 6.1.23. *Let assumptions (H1), (H2), (H3) and (C) be satisfied. Then there exists* $\bar{\rho} > 0$ *so that for any* $0 < \rho < \bar{\rho}$ *there exists* $\varepsilon_0 > 0$ *so that for any* $\varepsilon \neq 0$, $|\varepsilon| < \varepsilon_0$ *and for any* $e \in \mathscr{E}$, *Eq.* (6.1.1) *has a unique solution* $z(t, \mathscr{T}, e, \varepsilon)$ *that satisfies one among* (6.1.94), (6.1.97), (6.1.98) *or* (6.1.99) *and consequently* (6.1.96). *Moreover, setting* $\mathscr{T}^{(k)} := \{T_{n+2k}\}_{n\in\mathbb{Z}}$, *we have*

$$z(t,\mathscr{T}^{(k+1)},\sigma(e),\varepsilon) = z(t,\mathscr{T}^{(k)},e,\varepsilon) \tag{6.1.101}$$

for any $t\in\mathbb{R}$ and $e\in\mathscr{E}$.

Proof. We only need to prove that (6.1.101) holds. Since $z(t,\mathscr{T},e,\varepsilon)$ does not depend on the choice of $\{n_m^e\}_{m\in\mathbb{Z}}$ we see that we can take $n_m^{\sigma(e)} = n_m^e - 1$ and then, setting $\mathscr{T}' = \{T_{m+2}\}_{m\in\mathbb{Z}}$, we have, if $m=2k$:

$$T_{2k}'^{\sigma(e)} = T_{2n_k^{\sigma(e)}+2} = T_{2n_k^e} = T_{2k}^e$$

and, if $m=2k-1$:

$$T_{2k-1}'^{\sigma(e)} = T_{2n_k^{\sigma(e)}+1} = T_{2n_k^e-1} = T_{2k-1}^e$$

that is

$$\mathscr{T}'^{\sigma(e)} = \mathscr{T}^e. \tag{6.1.102}$$

Hence we see that, for any $t\in\mathbb{R}$ and any $e\in\mathscr{E}$, the following holds

$$z(t,\mathscr{T}',\sigma(e),\varepsilon) = z(t,\mathscr{T},e,\varepsilon). \tag{6.1.103}$$

Now, from the definition of $\mathscr{T}^{(k)}$ we see that $\mathscr{T}^{(k+1)} = \mathscr{T}^{(k)\prime}$, thus (6.1.101) follows from (6.1.103). The proof is complete. □

Now, we define $F_k:\mathbb{R}^n\to\mathbb{R}^n$ so that $F_k(\xi)$ is the value at time $T_{2(k+1)}$ of the solution $z(t)$ of Eq. (6.1.1) so that $z(T_{2k})=\xi$:

$$\dot z = f_\pm(z) + \varepsilon g(t,z,\varepsilon), \quad z(T_{2k}) = \xi \tag{6.1.104}$$

and let $\Phi_k(e) := z(T_{2k},\mathscr{T}^{(k)},e,\varepsilon)$. Then we have:

$$\begin{aligned}\Phi_{k+1}\circ\sigma(e) &= z(T_{2(k+1)},\mathscr{T}^{(k+1)},\sigma(e),\varepsilon) = z(T_{2(k+1)},\mathscr{T}^{(k)},e,\varepsilon)\\ &= F_k(z(T_{2k},\mathscr{T}^{(k)},e,\varepsilon)) = F_k\circ\Phi_k(e).\end{aligned} \tag{6.1.105}$$

Note that (6.1.105) can be stated in the following way. Let

$$\mathscr{S}_k = \left\{(z(T_{2k},\mathscr{T}^{(k)},e,\varepsilon)\mid e\in\mathscr{E}\right\}, \quad k\in\mathbb{Z}.$$

Although F_k may not be defined in the whole $\mathbb{R}^n$, for sure it is defined in the set $\mathscr{S}_k$. It is standard to prove (see [36], Section 3.5) that $\mathscr{S}_k$ are compact in $\mathbb{R}^n$ and $\Phi_k:\mathscr{E}\mapsto\mathscr{S}_k$ are continuous and clearly onto. Moreover, by (6.1.105), all $F_k:\mathscr{S}_k\to\mathscr{S}_{k+1}$ are homeomorphisms.

Remark 6.1.24. Here we silently suppose that F_k are defined. We can do that since we can modify (6.1.1) outside of a neighbourhood of the homoclinic orbit.

Next, let $e,e'\in\mathscr{E}$ be two different sequences in $\mathscr{E}$. Then there exists $j\in\mathbb{Z}$ so that, for example, $e_j'=0$ and $e_j=1$. From $[-\chi/2,\chi/2]\subset[-\bar T,\bar T]$ and (6.1.96) we see that

$$\begin{aligned}&\left|z(T_{2j},\mathscr{T},e,\varepsilon)-z(T_{2j},\mathscr{T},e',\varepsilon)\right|\\&\quad\geq\left|\gamma_0(-\alpha_j^0)\right|-\left|z(T_{2j},\mathscr{T},e,\varepsilon)-\gamma_0(-\alpha_j^0)\right|-\left|z(T_{2j},\mathscr{T},e',\varepsilon)\right|\\&\quad\geq\left|\gamma_0(-\alpha_m^0)\right|-3\rho\geq\min_{t\in[-\bar{T},\bar{T}]}|\gamma_0(t)|-3\rho>0\end{aligned}$$

provided ρ is sufficiently small. As a consequence, $z(T_{2j},\mathscr{T},e,\varepsilon)\neq z(T_{2j},\mathscr{T},e',\varepsilon)$ and, since both are solutions of the same Eq. (6.1.1):

$$z(t,\mathscr{T},e,\varepsilon)\neq z(t,\mathscr{T},e',\varepsilon) \tag{6.1.106}$$

for any $t\in\mathbb{R}$. Thus we have proved that the map $e\mapsto z(t,\mathscr{T},e,\varepsilon)$ is one-to-one. Hence if $\Phi_k(e)=\Phi_k(e')$ then $e=e'$ since otherwise:

$$\Phi_k(e)=z(T_{2k},\mathscr{T}^{(k)},e,\varepsilon)\neq z(T_{2k},\mathscr{T}^{(k)},e',\varepsilon)=\Phi_k(e').$$

So $\Phi_k:\mathscr{E}\to\mathscr{S}_k$ is one-to-one and a homeomorphism for any $k\in\mathbb{Z}$. In summary, we get another result.

Theorem 6.1.25. *Assume that (H1), (H2), (H3) and (C) hold. Then for any $\varepsilon\neq 0$ sufficiently small, the following diagrams commute:*

$$\begin{array}{ccc}\mathscr{E} & \xrightarrow{\quad\sigma\quad} & \mathscr{E}\\ {\scriptstyle\Phi_k}\downarrow & & \downarrow{\scriptstyle\Phi_{k+1}}\\ \mathscr{S}_k & \xrightarrow[\quad F_k\quad]{} & \mathscr{S}_{k+1}\end{array}$$

for all $k\in\mathbb{Z}$. Moreover, all Φ_k are homeomorphisms.

Sequences of 2-dimensional maps are also studied in [45].

Remark 6.1.26. We improve (6.1.94) as follows. First, assume that $e_j=1$, and $e_{j+1}=0$. The cases $e_j=0$, $e_{j+1}=1$ and $e_j=e_{j+1}=1$ can be similarly handled. Then, if $j=n_k^e$, we have $n_{k+1}^e>n_k^e+1$ and then if

$$t\in[T_{2n_k^e+1}+\beta_k^e,T_{2n_{k+1}^e-1}+\beta_k^e]=\bigcup_{j=2n_k^e+1}^{2(n_{k+1}^e-1)}[T_j+\beta_k^e,T_{j+1}+\beta_k^e],$$

we have $t\in[T_{2k}^e+\bar{T}+\beta_k^e,T_{2k+1}^e+\beta_k^e]$ and

$$t-T_{2k}^e-\beta_k^e\in[T_{2n_k^e+1}-T_{2n_k^e},T_{2n_{k+1}^e-1}-T_{2n_k^e}]\subset(\bar{T}+1-2\delta^{-1}\ln|\varepsilon|,\infty)$$

and hence if ε is small enough that $|\gamma_-(t)|<\rho$ for any $t\geq\bar{T}+1-2\delta^{-1}\ln|\varepsilon|$, by (6.1.94) we get:

$$\sup_{t\in[T_j+\beta_k^e,T_{j+1}+\beta_k^e]}|z(t)-u(t)|<3\rho$$

for any $j \in \{2n_k^e+1,...,2(n_{k+1}^e-1)\}$. On the other hand,

$$\sup_{t\in[T_{2n_k^e-1}+\beta_{k-1}^e, T_{2n_k^e}+\bar{T}+\beta_k^e]} |z(t)-\gamma(t-T_{2n_k^e}-\alpha_k^e)| < \rho,$$
$$\sup_{t\in[T_{2n_k^e}+\bar{T}+\beta_k^e, T_{2n_k^e+1}+\beta_k^e]} |z(t)-\gamma(t-T_{2n_k^e}-\beta_k^e)| < \rho .$$

In summary, we can roughly state that for $t \in [T_{2j-1}, T_{2j+1}]$ the solution $z(t)$ is close either to the homoclinic orbit $\gamma(t)$ or to the bounded solution according to $e_j = 1$ or $e_j = 0$.

6.1.8 Almost and Quasiperiodic Cases

In this section we assume that $g(t,x,\varepsilon)$ is almost periodic in t uniformly in (x,ε), that is, the following holds:

(H4) For any $\nu > 0$ there exists $L_\nu > 0$ so that in any interval of a length greater than L_ν there exists T_ν which is an *almost period* for ν satisfying:

$$|g(t+T_\nu,x,\varepsilon)-g(t,x,\varepsilon)| < \nu$$

for any $(t,x,\varepsilon) \in \mathbb{R}^{n+2}$.

Note that under (H4), function $\mathscr{M}(\alpha)$ is almost periodic in α. In this section we suppose the existence of a simple zero α^0 of $\mathscr{M}(\alpha)$. Then following the arguments of the proof of Theorem 6.1.3 we see that for any $\varepsilon \neq 0$ sufficiently small there is a sequence $\mathscr{T}^\varepsilon = \{T_m^\varepsilon\}_{m\in\mathbb{Z}}$ so that $T_{m+1}^\varepsilon - T_m^\varepsilon > \bar{T}+1+4|\alpha^0|-2\delta^{-1}\ln|\varepsilon|$ along with the existence of $\alpha^\varepsilon = \{\alpha_m^\varepsilon\}_{m\in\mathbb{Z}} \in \ell^\infty$ with $\|\alpha^\varepsilon\| \leq 2|\alpha^0|$, satisfying $\mathscr{M}(T_{2m}^\varepsilon+\alpha_m^\varepsilon) = 0$ for any $m \in \mathbb{Z}$ and $\inf_{m\in\mathbb{Z}} |\mathscr{M}'(T_{2m}^\varepsilon+\alpha_m^\varepsilon)| > c_0$ for some $c_0 > 0$. Then taking $T_{2m} = T_{2m}^\varepsilon+\alpha_m^\varepsilon$, $T_{2m-1} = T_{2m-1}^\varepsilon$ and $\alpha_0 = 0$, assumption (C) is satisfied. So applying Theorem 6.1.25, system (6.1.1) is chaotic for any $\varepsilon \neq 0$ small. In summary we obtain the following theorem.

Theorem 6.1.27. *Assume that (H1)–(H4) hold and that the almost periodic Melnikov function $\mathscr{M}(\alpha)$ has a simple zero. Then system* (6.1.1) *is chaotic for any $\varepsilon \neq 0$ sufficiently small.*

Next, it is well known (see [41–44], Remark 4.1.7) that near the hyperbolic equilibrium $x = 0$ of the equation $\dot{x} = f_-(x)$ there exists a unique almost periodic solution of $\dot{x} = f_-(x)+\varepsilon g(t,x,\varepsilon)$. More precisely, given $\rho > 0$ there exists $\bar{\varepsilon} > 0$ so that for any $|\varepsilon| < \bar{\varepsilon}$ equation $\dot{x} = f_-(x)+\varepsilon g(t,x,\varepsilon)$ has a solution $u(t) = u(t,\varepsilon)$ so that $|u(t)| < \rho$ for any $t \in \mathbb{R}$ and it is almost periodic with common almost periods as $g(t,x,\varepsilon)$, i.e. assumption (H4) holds in addition with

$$|u(t+T_\nu)-u(t)| < \hat{c}_0\nu \quad \forall t \in \mathbb{R}$$

for a positive constant $\hat{c}_0$. Note that $u(t)$ is a bounded solution of $\dot{x} = f_-(x) + \varepsilon g(t,x,\varepsilon)$ mentioned in (6.1.100). Thus the conclusion of Remark 6.1.26 holds with the further property that $u(t)$ is almost periodic.

Results of this section generalize the deterministic chaos of [42–44, 46] to the discontinuous almost periodic system (6.1.1).

Finally, if $g(t,x,\varepsilon)$ is quasiperiodic in t the following holds:

(H5) $g(t,x,\varepsilon) = q(\omega_1 t,\dots,\omega_m t,x,\varepsilon)$ for $\omega_1,\dots,\omega_m \in \mathbb{R}$ with $q \in C_b^r(\mathbb{R}^{m+n+1},\mathbb{R}^n)$ and $q(\theta_1,\dots,\theta_m,x,\varepsilon)$ is 1-periodic in each θ_i, $i = 1,2,\dots,m$. Moreover, ω_i, $i = 1,2,\dots,m$ are linearly independent of $\mathbb{Z}$, i.e. if $\sum_{i=1}^m l_i\omega_i = 0$, $l_i \in \mathbb{Z}$, $i = 1,2,\dots,m$, then $l_i = 0$, $i = 1,2,\cdots,m$.

Then $g(t,x,\varepsilon)$ satisfies assumption (H4) [40, 42] and hence the conclusion of Theorem 6.1.27 holds.

6.1.9 Periodic Case

Here we assume that $g(t+p,z,\varepsilon) = g(t,z,\varepsilon)$ that is $g(t,z,\varepsilon)$ is p-periodic. Then $\mathscr{M}(\alpha)$ is also p-periodic. We suppose the existence of a simple zero α^0 of $\mathscr{M}(\alpha)$. Then Theorem 6.1.3 is applicable with $T_m = mT$ and $2T = rp$ for $r \gg 1$, $r \in \mathbb{N}$. So

$$T_m^e = \begin{cases} 2n_k^e T, & \text{if } m = 2k, \\ (2n_k^e - 1)T, & \text{if } m = 2k-1. \end{cases}$$

Since we can take $n_m^{\sigma(e)} = n_m^e - 1$ we see that

$$T_m^{\sigma(e)} = \left.\begin{cases} 2n_k^e T - 2T, & \text{if } m = 2k \\ (2n_k^e - 1)T - 2T, & \text{if } m = 2k-1 \end{cases}\right\} = T_m^e - 2T$$

for any $m \in \mathbb{Z}$. Again we denote with $z(t) = z(t,\mathscr{T},e)$ the solution of equation (6.1.1) corresponding to the sequence $\mathscr{T}^e$. Then $Z(t) := z(t+2T)$ satisfies the equation

$$\dot{z} = f_\pm(z) + \varepsilon g(t,z,\varepsilon)$$

together with the estimates:

$$\begin{aligned} &\sup_{t\in[T_{2m-1}^{\sigma(e)}+\beta_{m-1}^e,\, T_{2m}^{\sigma(e)}-\bar{T}+\alpha_m^e]} |Z(t) - \gamma_-(t - T_{2m}^{\sigma(e)} - \alpha_m^e)| < \rho, \\ &\sup_{t\in[T_{2m}^{\sigma(e)}-\bar{T}+\alpha_m^e,\, T_{2m}^{\sigma(e)}+\bar{T}+\beta_m^e]} |Z(t) - \gamma_0(t - T_{2m}^{\sigma(e)} - \alpha_m^e)| < \rho, \\ &\sup_{t\in[T_{2m}^{\sigma(e)}+\bar{T}+\beta_m^e,\, T_{2m-1}^{\sigma(e)}+\beta_m^e]} |Z(t) - \gamma_+(t - T_{2m}^{\sigma(e)} - \beta_m^e)| < \rho. \end{aligned} \tag{6.1.107}$$

Thus, because of uniqueness:

$$\alpha(\mathscr{T}^e,\varepsilon) = \alpha(\mathscr{T}^{\sigma(e)},\varepsilon) \in \ell^\infty(\mathbb{R}), \quad \beta(\mathscr{T}^e,\varepsilon) = \beta(\mathscr{T}^{\sigma(e)},\varepsilon) \in \ell^\infty(\mathbb{R})$$

and $z(t+2T,\mathscr{T},e,\varepsilon)=z(t,\mathscr{T},\sigma(e),\varepsilon)$. Thus, using (6.1.101) and recalling that $T_k=kT$:

$$z(T_{2(k+1)},\mathscr{T}^{(k+1)},e,\varepsilon)=z(T_{2k},\mathscr{T}^{(k+1)},\sigma(e),\varepsilon)=z(T_{2k},\mathscr{T}^{(k)},e,\varepsilon),$$

that is, we see that

$$\Phi_k(e)=\Phi(e), \quad \mathscr{S}_k=\mathscr{S}$$

are independent of k. Similarly, because of uniqueness and periodicity, the value at the time $T_{2(k+1)}=2(k+1)T$ of the solution of (6.1.104) is the same as the value at time $2T$ of the solution of

$$\dot{z}=f_\pm(z)+\varepsilon g(t,z,\varepsilon), \quad z(0)=\xi,$$

that is, also $F_k(\xi)=F(\xi)$ are independent of k and we have:

$$\Phi\circ\sigma=F\circ\Phi.$$

In summary we arrive at the following result.

Theorem 6.1.28. *Assume that $g(t+p,z,\varepsilon)=g(t,z,\varepsilon)$, that is, $g(t,z,\varepsilon)$ is p-periodic. If $\varepsilon\neq 0$ is sufficiently small and there is a simple zero α^0 of $\mathscr{M}(\alpha)$ then the following diagram commutes:*

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{\quad\sigma\quad} & \mathscr{E} \\ \Phi\downarrow & & \downarrow\Phi \\ \mathscr{S} & \xrightarrow[\quad F\quad]{} & \mathscr{S} \end{array}$$

for any $\mathbb{N}\ni r\gg 1$. Here $F=\varphi_\varepsilon^r=\varphi_\varepsilon\circ\ldots\circ\varphi_\varepsilon$ (r times) is the rth iterate of the p-period map φ_ε of (6.1.1).

Theorem 6.1.28 generalizes the deterministic chaos of Section 2.5.2 [36, 47] to the discontinuous periodic system (6.1.1).

6.1.10 Piecewise Smooth Planar Systems

In this section we apply the theory developed in the previous parts to a two–dimensional system ($x,y\in\mathbb{R}$)

$$\begin{aligned} \dot{x}&=P^\pm(x,y),\\ \dot{y}&=Q^\pm(x,y), \end{aligned} \tag{6.1.108}$$

where $+$ holds if $(x,y)\in\Omega_+=\{(x,y)\mid G(x,y)>0\}$ and $-$ when $(x,y)\in\Omega_-=\{(x,y)\mid G(x,y)<0\}$. We will construct an explicit expression for $\mathscr{M}(\alpha)$ showing that it extends to the discontinuous case, the usual Melnikov function, thus validating the name of Melnikov function we have given to $\mathscr{M}(\alpha)$. Let us write the homoclinic orbit

$$\gamma(t) = \begin{cases} \gamma_-(t), & \text{if } t \le -\bar{T}, \\ \gamma_0(t), & \text{if } -\bar{T} \le t \le \bar{T}, \\ \gamma_+(t), & \text{if } \bar{T} \le t, \end{cases}$$

as

$$\gamma_\pm(t) = \begin{pmatrix} u_\pm(t) \\ v_\pm(t) \end{pmatrix} \in \bar{\Omega}_-, \quad \gamma_0(t) = \begin{pmatrix} u_0(t) \\ v_0(t) \end{pmatrix} \in \bar{\Omega}_+.$$

Then let

$$a_\pm(t) = P_x^-(u_\pm(t), v_\pm(t)) + Q_y^-(u_\pm(t), v_\pm(t)),$$
$$a_0(t) = P_x^+(u_0(t), v_0(t)) + Q_y^+(u_0(t), v_0(t))$$

be the trace of the Jacobian matrix of the linearization of (6.1.108) along $(u_\pm(t), v_\pm(t))$ and $(u_0(t), v_0(t))$ respectively, and

$$a(t) := \begin{cases} a_-(t), & \text{if } t < -\bar{T}, \\ a_0(t), & \text{if } -\bar{T} \le t \le \bar{T}, \\ a_+(t), & \text{if } t > \bar{T}. \end{cases}$$

Then $\begin{pmatrix} \dot{v}_\pm(t) \\ -\dot{u}_\pm(t) \end{pmatrix} \mathrm{e}^{-\int_{\pm\bar{T}}^t a_\pm(\tau)d\tau}$ satisfy the adjoint variational system:

$$\dot{x} = -P_x^-(\gamma_\pm(t))x - Q_x^-(\gamma_\pm(t))y,$$
$$\dot{y} = -P_y^-(\gamma_\pm(t))x - Q_y^-(\gamma_\pm(t))y$$

and similarly $\begin{pmatrix} \dot{v}_0(t) \\ -\dot{u}_0(t) \end{pmatrix} \mathrm{e}^{-\int_{-\bar{T}}^t a_0(\tau)d\tau}$ satisfies the adjoint system:

$$\dot{x} = -P_x^+(\gamma_0(t))x - Q_x^+(\gamma_0(t))y,$$
$$\dot{y} = -P_y^+(\gamma_0(t))x - Q_y^+(\gamma_0(t))y.$$

As a consequence,

$$\begin{pmatrix} \dot{v}_\pm(t) \\ -\dot{u}_\pm(t) \end{pmatrix} \mathrm{e}^{-\int_{\pm\bar{T}}^t a_\pm(\tau)d\tau} = X_\pm^*(t)^{-1} \begin{pmatrix} \dot{v}_\pm(\pm\bar{T}) \\ -\dot{u}_\pm(\pm\bar{T}) \end{pmatrix}$$

and

$$\begin{pmatrix} \dot{v}_0(t) \\ -\dot{u}_0(t) \end{pmatrix} \mathrm{e}^{-\int_{-\bar{T}}^t a_0(\tau)d\tau} = X_0^*(t)^{-1} \begin{pmatrix} \dot{v}_0(-\bar{T}) \\ -\dot{u}_0(-\bar{T}) \end{pmatrix}.$$

Next, since the system is two-dimensional, we have

$$\operatorname{span}\{\psi\} = \mathscr{R}R_0 = \operatorname{span}\left\{ \begin{pmatrix} G_y(\gamma(\bar{T})) \\ -G_x(\gamma(\bar{T})) \end{pmatrix} \right\}.$$

So we take:

$$\psi = \frac{1}{|G'(\gamma(\bar{T}))|} \begin{pmatrix} G_y(\gamma(\bar{T})) \\ -G_x(\gamma(\bar{T})) \end{pmatrix}.$$

Let $\{e_1, e_2\}$ be the canonical basis of $\mathbb{R}^2$. According to the definition of $R_\pm$, R_0 we have

$$\begin{aligned}
R_+e_1 &= e_1 - \frac{G_x(\gamma(\bar{T}))}{G'(\gamma(\bar{T}))\dot{\gamma}_+(\bar{T})}\dot{\gamma}_+(\bar{T}), \\
R_+e_2 &= e_2 - \frac{G_y(\gamma(\bar{T}))}{G'(\gamma(\bar{T}))\dot{\gamma}_+(\bar{T})}\dot{\gamma}_+(\bar{T}), \\
R_-e_1 &= e_1 - \frac{G_x(\gamma(-\bar{T}))}{G'(\gamma(-\bar{T}))\dot{\gamma}_-(-\bar{T})}\dot{\gamma}_-(-\bar{T}), \\
R_-e_2 &= e_2 - \frac{G_y(\gamma(-\bar{T}))}{G'(\gamma(-\bar{T}))\dot{\gamma}_-(-\bar{T})}\dot{\gamma}_-(-\bar{T}), \\
R_0e_1 &= e_1 - \frac{G_x(\gamma(\bar{T}))}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})}\dot{\gamma}_0(\bar{T}), \\
R_0e_2 &= e_2 - \frac{G_y(\gamma(\bar{T}))}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})}\dot{\gamma}_0(\bar{T})
\end{aligned}$$

and then (here $\mathscr{M}_L$ denotes the matrix of the linear map L with respect to the basis $\{e_1, e_2\}$ of $\mathbb{R}^2$)

$$\begin{aligned}
\mathscr{M}_{R_+^*} &= \frac{1}{G'(\gamma(\bar{T}))\dot{\gamma}_+(\bar{T})} \begin{pmatrix} \dot{v}_+(\bar{T}) \\ -\dot{u}_+(\bar{T}) \end{pmatrix} \big(G_y(\gamma(\bar{T})) \ \ -G_x(\gamma(\bar{T})) \big), \\
\mathscr{M}_{R_-^*} &= \frac{1}{G'(\gamma(-\bar{T}))\dot{\gamma}_-(-\bar{T})} \begin{pmatrix} \dot{v}_-(-\bar{T}) \\ -\dot{u}_-(-\bar{T}) \end{pmatrix} \big(G_y(\gamma(-\bar{T})) \ \ -G_x(\gamma(-\bar{T})) \big), \\
\mathscr{M}_{R_0^*} &= \frac{1}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})} \begin{pmatrix} \dot{v}_0(\bar{T}) \\ -\dot{u}_0(\bar{T}) \end{pmatrix} \big(G_y(\gamma(\bar{T})) \ \ -G_x(\gamma(\bar{T})) \big).
\end{aligned}$$

As a consequence,

$$\begin{aligned}
& X_-^*(t)^{-1}R_-^*X_0^*(\bar{T})R_0^*\psi \\
&= \frac{|G'(\gamma(\bar{T}))|}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})} X_-^*(t)^{-1}R_-^*X_0^*(\bar{T}) \begin{pmatrix} \dot{v}_0(\bar{T}) \\ -\dot{u}_0(\bar{T}) \end{pmatrix} \\
&= \frac{|G'(\gamma(\bar{T}))|}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})} X_-^*(t)^{-1}R_-^* \begin{pmatrix} \dot{v}_0(-\bar{T}) \\ -\dot{u}_0(-\bar{T}) \end{pmatrix} \mathrm{e}^{\int_{-\bar{T}}^{\bar{T}} a_0(\tau)d\tau}
\end{aligned}$$

$$= \frac{|G'(\gamma(\bar{T}))|}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})} \frac{G'(\gamma(-\bar{T}))\dot{\gamma}_0(-\bar{T})}{G'(\gamma(-\bar{T}))\dot{\gamma}_-(-\bar{T})} \begin{pmatrix} \dot{v}_-(t) \\ -\dot{u}_-(t) \end{pmatrix} \mathrm{e}^{\int_t^{\bar{T}} a(\tau)d\tau}$$

for any $t \le -\bar{T}$. Similarly, for $-\bar{T} \le t \le \bar{T}$ we have:

$$X_0^*(t)^{-1} X_0^*(\bar{T}) R_0^* \psi = \frac{|G'(\gamma(\bar{T}))|}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})} \begin{pmatrix} \dot{v}_0(t) \\ -\dot{u}_0(t) \end{pmatrix} \mathrm{e}^{\int_t^{\bar{T}} a_0(\tau)d\tau}$$

and

$$X_+^*(t)^{-1} R_+^* \psi = \frac{|G'(\gamma(\bar{T}))|}{G'(\gamma(\bar{T}))\dot{\gamma}_+(\bar{T})} \begin{pmatrix} \dot{v}_+(t) \\ -\dot{u}_+(t) \end{pmatrix} \mathrm{e}^{\int_t^{\bar{T}} a_+(\tau)d\tau}$$

for $t \ge \bar{T}$. Putting everything together we obtain

$$\begin{aligned} \mathscr{M}(\alpha) = {} & \frac{|G'(\gamma(\bar{T}))|}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})} \\ & \cdot \Bigg\{ \frac{G'(\gamma(-\bar{T}))\dot{\gamma}_0(-\bar{T})}{G'(\gamma(-\bar{T}))\dot{\gamma}_-(-\bar{T})} \int_{-\infty}^{-\bar{T}} \begin{pmatrix} \dot{v}_-(t) \\ -\dot{u}_-(t) \end{pmatrix} g(t+\alpha,\gamma(t),0)\, \mathrm{e}^{-\int_{\bar{T}}^t a(\tau)d\tau}\, dt \\ & + \int_{-\bar{T}}^{\bar{T}} \begin{pmatrix} \dot{v}_0(t) \\ -\dot{u}_0(t) \end{pmatrix} g(t+\alpha,\gamma(t),0)\, \mathrm{e}^{-\int_{\bar{T}}^t a_0(\tau)d\tau}\, dt \\ & + \frac{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})}{G'(\gamma(\bar{T}))\dot{\gamma}_+(\bar{T})} \int_{\bar{T}}^{\infty} \begin{pmatrix} \dot{v}_+(t) \\ -\dot{u}_+(t) \end{pmatrix} g(t+\alpha,\gamma(t),0)\, \mathrm{e}^{-\int_{\bar{T}}^t a_+(\tau)d\tau}\, dt \Bigg\} \end{aligned}$$

that can be written as:

$$\begin{aligned} \mathscr{M}(\alpha) = {} & -\frac{|G'(\gamma(\bar{T}))|}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})} \\ & \cdot \Bigg\{ \frac{G'(\gamma(-\bar{T}))\dot{\gamma}_0(-\bar{T})}{G'(\gamma(-\bar{T}))\dot{\gamma}_-(-\bar{T})} \int_{-\infty}^{-\bar{T}} f_-(\gamma(t)) \wedge g(t+\alpha,\gamma(t),0)\, \mathrm{e}^{-\int_{\bar{T}}^t a(\tau)d\tau}\, dt \\ & + \int_{-\bar{T}}^{\bar{T}} f_+(\gamma(t)) \wedge g(t+\alpha,\gamma(t),0)\, \mathrm{e}^{-\int_{\bar{T}}^t a(\tau)d\tau}\, dt \qquad (6.1.109) \\ & + \frac{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})}{G'(\gamma(\bar{T}))\dot{\gamma}_+(\bar{T})} \int_{\bar{T}}^{\infty} f_-(\gamma(t)) \wedge g(t+\alpha,\gamma(t),0)\, \mathrm{e}^{-\int_{\bar{T}}^t a(\tau)d\tau}\, dt \Bigg\}, \end{aligned}$$

where

$$f_\pm(x,y) = \begin{pmatrix} P^\pm(x,y) \\ Q^\pm(x,y) \end{pmatrix}.$$

Note that we can write:

$$\mathscr{M}(\alpha) = -\frac{|G'(\gamma(\bar{T}))|\,\mathrm{e}^{\int_0^{\bar{T}} a_0(\tau)d\tau}}{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})}\left\{\int_{-\infty}^{\infty} f(\gamma(t)) \wedge g(t+\alpha,\gamma(t),0)\,\mathrm{e}^{-\int_0^t a(\tau)d\tau}\,dt + \delta_- + \delta_+\right\}$$

where

$$\delta_- = \left(\frac{G'(\gamma(-\bar{T}))\dot{\gamma}_0(-\bar{T})}{G'(\gamma(-\bar{T}))\dot{\gamma}_-(-\bar{T})} - 1\right)\int_{-\infty}^{-\bar{T}} f_-(\gamma(t)) \wedge g(t+\alpha,\gamma(t),0)\,\mathrm{e}^{-\int_0^t a(\tau)d\tau}\,dt,$$

$$\delta_+ = \left(\frac{G'(\gamma(\bar{T}))\dot{\gamma}_0(\bar{T})}{G'(\gamma(\bar{T}))\dot{\gamma}_+(\bar{T})} - 1\right)\int_{\bar{T}}^{\infty} f_-(\gamma(t)) \wedge g(t+\alpha,\gamma(t),0)\,\mathrm{e}^{-\int_0^t a(\tau)d\tau}\,dt.$$

Remark that the extra terms $\delta_\pm$ vanish in cases $\dot{\gamma}_0(-\bar{T}) = \dot{\gamma}_-(-\bar{T})$ and $\dot{\gamma}_0(\bar{T}) = \dot{\gamma}_+(\bar{T})$. Thus $\mathscr{M}(\alpha)$ extends the usual Melnikov function (cf Section 4.1) to the discontinuous case.

6.1.11 3D Quasiperiodic Piecewise Linear Systems

In in section, we consider the example

$$\dot{x} = \begin{cases} Ax + \varepsilon\left(g_1 \sin\omega_1 t + g_2 \sin\omega_2 t\right), & \text{for } \widetilde{a}\cdot x < d,\\ Ax + b + \varepsilon\left(g_1 \sin\omega_1 t + g_2 \sin\omega_2 t\right), & \text{for } \widetilde{a}\cdot x > d \end{cases} \tag{6.1.110}$$

of a quasiperiodically perturbed piecewise linear 3-dimensional differential equation. Here $d > 0$, $\omega_{1,2} > 0$, $\widetilde{a}, x, g_{1,2} \in \mathbb{R}^3$, $\widetilde{a}\cdot x$ is the scalar product in $\mathbb{R}^3$. Moreover, we consider system (6.1.110) under the following assumptions

(i) A is a 3×3-matrix with semi-simple eigenvalues, $\lambda_1, \lambda_2 > 0 > \lambda_3$ and with the corresponding eigenvectors, e_1, e_2, e_3.
(ii) Let $b = \sum_{i=1}^{3} b_i e_i$ and $a_i := \widetilde{a}\cdot e_i$, $i = 1,2,3$. Then $a_1, b_3 \geq 0$, $a_2, a_3 > 0$ and $b_1, b_2 < 0$.

Remark 6.1.29. Certainly we can study more general systems

$$\dot{x} = \begin{cases} Ax + \varepsilon \sum_{k=1}^{m} g_k \sin\omega_k t, & \text{for } \widetilde{a}\cdot x < d,\\ Ax + b + \varepsilon \sum_{k=1}^{m} g_k \sin\omega_k t, & \text{for } \widetilde{a}\cdot x > d \end{cases}$$

but for simplicity we concentrate on (6.1.110) in this section.

If either $g_1 = 0$, $g_2 = 0$ or the ratio $\frac{\omega_1}{\omega_2}$ is rational, then we get the periodic case studied in [34]. Theorem 6.1.28, however, improves the result in [34] in the sense that here we obtain chaotic behaviour of the solutions. Thus, we focus here on the case

(iii) $g_1 \neq 0$, $g_2 \neq 0$ and ω_1/ω_2 is irrational.

Given the vectors in $\mathbb{R}^3$: $x=\sum_{i=1}^3 x_i e_i$ and $y=\sum_{i=1}^3 y_i e_i$ we define

$$\langle x,y\rangle=\sum_{i=1}^3 x_i y_i.$$

Then $\langle x,y\rangle$ is a scalar product in $\mathbb{R}^3$ that makes $\{e_1,e_2,e_3\}$ an orthonormal basis of $\mathbb{R}^3$. From now on we will write also (x_1,x_2,x_3) for the vector $x=\sum_{i=1}^3 x_i e_i$ and hence we identify e_1,e_2,e_3 with $(1,0,0),(0,1,0),(0,0,1)$ respectively.

Writing $x=\sum_{i=1}^3 x_i e_i$ and $g_j=\sum_{i=1}^3 g_{ji}e_i$, $j=1,2$, (6.1.110) has the form

$$\dot{x}_i=\begin{cases}\lambda_i x_i+\varepsilon\left(g_{1i}\sin\omega_1 t+g_{2i}\sin\omega_2 t\right), & \text{for}\quad \langle a,x\rangle<d\,,\\ \lambda_i x_i+b_i+\varepsilon\left(g_{1i}\sin\omega_1 t+g_{2i}\sin\omega_2 t\right), & \text{for}\quad \langle a,x\rangle>d\,,\end{cases}\tag{6.1.111}$$

$i=1,2,3$, where $a=\sum_{i=1}^3 a_i e_i$. Hence $G(x)=\langle a,x\rangle-d=\sum_{j=1}^3 a_j x_j-d$ and thus

$$\begin{aligned}\Omega_-&=\Big\{(x_1,x_2,x_3)\in\mathbb{R}^3\mid \sum_{i=1}^3 a_i x_i<d\Big\},\\ \Omega_+&=\Big\{(x_1,x_2,x_3)\in\mathbb{R}^3\mid \sum_{i=1}^3 a_i x_i>d\Big\}.\end{aligned}$$

Theorem 6.1.30. *If conditions* (i)–(ii) *and the next ones*

$$a_3b_3(\mathrm{e}^{2\lambda_3\bar{T}}-1)=d\lambda_3,\quad \sum_{j=1}^2\frac{a_jb_j}{\lambda_j}(\mathrm{e}^{-2\lambda_j\bar{T}}-1)=d\tag{6.1.112}$$

hold, then system

$$\dot{x}_i=\begin{cases}\lambda_i x_i, & \textit{for}\quad \langle a,x\rangle<d\,,\\ \lambda_i x_i+b_i, & \textit{for}\quad \langle a,x\rangle>d\,,\end{cases}\tag{6.1.113}$$

$i=1,2,3$, *has a homoclinic orbit to* $x=0$*:*

$$\gamma(t)=\begin{cases}\gamma_-(t), & \text{if } t\le-\bar{T},\\ \gamma_0(t), & \text{if } -T\le t\le\bar{T},\\ \gamma_+(t), & \text{if } t\ge\bar{T},\end{cases}$$

where

$$\begin{aligned}\gamma_-(t)&=\left(\mathrm{e}^{\lambda_1(t+\bar{T})}\left(\mathrm{e}^{-2\lambda_1\bar{T}}-1\right)\frac{b_1}{\lambda_1},\mathrm{e}^{\lambda_2(t+\bar{T})}\left(\mathrm{e}^{-2\lambda_2\bar{T}}-1\right)\frac{b_2}{\lambda_2},0\right),\\ \gamma_0(t)&=\left(\left(\mathrm{e}^{\lambda_1(t-\bar{T})}-1\right)\frac{b_1}{\lambda_1},\left(\mathrm{e}^{\lambda_2(t-\bar{T})}-1\right)\frac{b_2}{\lambda_2},\left(\mathrm{e}^{\lambda_3(t+\bar{T})}-1\right)\frac{b_3}{\lambda_3}\right),\\ \gamma_+(t)&=\left(0,0,\frac{d}{a_3}\mathrm{e}^{\lambda_3(t-\bar{T})}\right),\end{aligned}$$

and conditions (H1), (H2) and (H3) are satisfied.

Proof. With a view to constructing the homoclinic solution $\gamma(t)$ of system (6.1.113), we describe the local stable and unstable manifolds of the fixed point $(x_1,x_2,x_3) = (0,0,0) \in \Omega_-$: the local unstable manifold of the origin is

$$\mathscr{W}^s_{loc}(0) = \{(x_1,x_2,0) \mid x_1,x_2 \in \mathbb{R},\ a_1x_1 + a_2x_2 < d\}$$

and the local stable manifold is

$$\mathscr{W}^u_{loc}(0) = \{(0,0,x_3) \mid x_3 \in \mathbb{R},\ a_3x_3 < d\}\,.$$

Thus it must be:

$$\gamma_-(t) = \begin{pmatrix} c_1\,\mathrm{e}^{\lambda_1 t} \\ c_2\,\mathrm{e}^{\lambda_2 t} \\ 0 \end{pmatrix}, \quad \gamma_+(t) = \begin{pmatrix} 0 \\ 0 \\ c_3\,\mathrm{e}^{\lambda_3 t} \end{pmatrix}$$

as long as $\gamma_-(t), \gamma_+(t) \in \Omega_-$. Note that, if $c_1,c_2,c_3 \geq 0$ then, because of (ii), the scalar product $\langle a, \gamma_-(t)\rangle$ (resp. $\langle a, \gamma_+(t)\rangle$) is increasing (resp. decreasing) and hence $\gamma_-(t) \in \Omega_-$ for $t < -\bar{T}$ and $\gamma_+(t) \in \Omega_-$ for $t > \bar{T}$ together with $\gamma_-(-\bar{T}), \gamma_+(\bar{T}) \in \partial\Omega_-$ if and only if $\langle a, \gamma_-(-\bar{T})\rangle = \langle a, \gamma_+(\bar{T})\rangle = d$, that is, if the following conditions on the non-negative numbers $\bar{T}$, d, c_1, c_2, c_3 hold

$$a_1c_1\,\mathrm{e}^{-\lambda_1\bar{T}} + a_2c_2\,\mathrm{e}^{-\lambda_2\bar{T}} = d, \quad a_3c_3\,\mathrm{e}^{\lambda_3\bar{T}} = d. \tag{6.1.114}$$

Next we have to choose $c_1 \geq 0$, $c_2 \geq 0$ and $c_3 \geq 0$ in such a way that the solution $\gamma_0(t)$ of system (6.1.113) with $\gamma_0(-\bar{T}) = \gamma_-(-\bar{T})$ belongs to Ω_+ for $-\bar{T} < t < \bar{T}$ and satisfies $\gamma_0(\bar{T}) = \gamma_+(\bar{T})$. Now, it is easy to see that if the solution of (6.1.113) belongs to Ω_+ and satisfies $\gamma_0(-\bar{T}) = \gamma_-(-\bar{T})$, then it must be

$$\gamma_0(t) = \begin{pmatrix} \lambda_1^{-1}[\mathrm{e}^{\lambda_1 t}(b_1\,\mathrm{e}^{\lambda_1\bar{T}} + c_1\lambda_1) - b_1] \\ \lambda_2^{-1}[\mathrm{e}^{\lambda_2 t}(b_2\,\mathrm{e}^{\lambda_2\bar{T}} + c_2\lambda_2) - b_2] \\ b_3\lambda_3^{-1}(\mathrm{e}^{\lambda_3(t+\bar{T})} - 1) \end{pmatrix}.$$

Hence the condition $\gamma_0(\bar{T}) = \gamma_+(\bar{T})$ is equivalent to:

$$\begin{aligned} c_1\lambda_1 &= -2b_1\sinh(\lambda_1\bar{T}), \\ c_2\lambda_2 &= -2b_2\sinh(\lambda_2\bar{T}), \\ c_3\lambda_3 &= 2b_3\sinh(\lambda_3\bar{T})\,. \end{aligned} \tag{6.1.115}$$

Plugging these values of c_1,c_2,c_3 into (6.1.114) (note that $c_1,c_2,c_3 > 0$) we obtain (6.1.112) on $\bar{T}$, d. We assume that conditions (6.1.112) are satisfied and show that in this case, $\gamma_0(t) \in \Omega_+$ for any $t \in (-\bar{T}, \bar{T})$. To this end we consider the function:

$$\phi(t) := G(\gamma_0(t)) = \sum_{j=1}^{2} \frac{a_j b_j}{\lambda_j}(\mathrm{e}^{\lambda_j(t-\bar{T})} - 1) + \frac{a_3 b_3}{\lambda_3}(\mathrm{e}^{\lambda_3(t+\bar{T})} - 1) - d.$$

We derive

$$\phi''(t) = \sum_{j=1}^{2} a_j b_j \lambda_i \mathrm{e}^{\lambda_j(t-\bar{T})} + a_3 b_3 \lambda_3 \mathrm{e}^{\lambda_3(t+\bar{T})}.$$

From assumptions (ii) and (6.1.112) we see that

$$\phi(-\bar{T}) = \phi(\bar{T}) = 0, \quad \phi''(t) < 0 \quad \text{for any} \quad t \in \mathbb{R}.$$

Hence we obtain

$$\phi(t) > 0 \quad \text{on} \quad (-\bar{T}, \bar{T})$$

that gives $\gamma_0(t) \in \Omega_+$ for $-\bar{T} < t < \bar{T}$. Moreover, from $\phi(-\bar{T}) = 0$ and $\phi''(t) < 0$, we also get $\phi'(-\bar{T}) > 0$ and similarly $\phi'(\bar{T}) < 0$, that is,

$$\sum_{j=1}^{2} a_j b_j \mathrm{e}^{-2\lambda_j \bar{T}} + a_3 b_3 > 0, \quad \sum_{j=1}^{2} a_j b_j + a_3 b_3 \mathrm{e}^{2\lambda_3 \bar{T}} < 0. \tag{6.1.116}$$

Condition (H1) is verified. Now we verify (H2) by checking the inequalities:

$$G'(\gamma(-\bar{T})) f_{\pm}(\gamma(-\bar{T})) > 0 \quad \text{and} \quad G'(\gamma(\bar{T})) f_{\pm}(\gamma(\bar{T})) < 0$$

that in this case read:

$$\begin{aligned} &\sum_{j=1}^{2} a_j b_j (\mathrm{e}^{-2\lambda_j \bar{T}} - 1) > 0, \qquad &&\sum_{j=1}^{2} a_j b_j \mathrm{e}^{-2\lambda_j \bar{T}} + a_3 b_3 > 0, \\ &\sum_{j=1}^{2} a_j b_j + a_3 b_3 \mathrm{e}^{2\lambda_3 \bar{T}} < 0, &&d\lambda_3 < 0. \end{aligned} \tag{6.1.117}$$

The first and the fourth inequalities come immediately from assumptions (i)–(ii); the second and the third ones from (6.1.116). So (H2) also holds. Next we verify condition (H3). First we note that $\nabla G(x) = a$, for any $x \in \mathbb{R}^3$, and

$$P_+ = P_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.1.118}$$

Then, since $\mathscr{N}[G'(\gamma(\bar{T}))] = \{a\}^{\perp}$ and $a_3 > 0$, we get

$$\mathscr{S}'' = \mathscr{N}[G'(\gamma(\bar{T}))] \bigcap \mathscr{R} P_+ = \{0\}.$$

Similarly, since $\mathscr{N} P_- = \operatorname{span}\{e_1, e_2\}$ and $\mathscr{N} G'(\gamma(-\bar{T})) = \{a\}^{\perp}$, we obtain

$$\mathscr{S}' = \operatorname{span}\{(a_2, -a_1, 0)\}.$$

Next, from (6.1.113), we see that

$$X_0(t) = X_-(t) = \begin{pmatrix} e^{\lambda_1(t+\bar{T})} & 0 & 0 \\ 0 & e^{\lambda_2(t+\bar{T})} & 0 \\ 0 & 0 & e^{\lambda_3(t+\bar{T})} \end{pmatrix} \tag{6.1.119}$$

and

$$X_+(t) = \begin{pmatrix} e^{\lambda_1(t-\bar{T})} & 0 & 0 \\ 0 & e^{\lambda_2(t-\bar{T})} & 0 \\ 0 & 0 & e^{\lambda_3(t-\bar{T})} \end{pmatrix}. \tag{6.1.120}$$

Hence

$$X_0(\bar{T})\mathscr{S}' = \operatorname{span}\{w_0\} \quad \text{with} \quad w_0 := \begin{pmatrix} a_2\, e^{2\lambda_1\bar{T}} \\ -a_1\, e^{2\lambda_2\bar{T}} \\ 0 \end{pmatrix}. \tag{6.1.121}$$

Since $\nabla G(x) = a$, we have:

$$\begin{aligned} R_0 w &= w - \frac{\langle a, w\rangle}{\langle a, \dot{\gamma}_0(\bar{T})\rangle}\dot{\gamma}_0(\bar{T})\,, \\ R_+ w &= w - \frac{\langle a, w\rangle}{\langle a, \dot{\gamma}_+(\bar{T})\rangle}\dot{\gamma}_+(\bar{T})\,, \\ R_- w &= w - \frac{\langle a, w\rangle}{\langle a, \dot{\gamma}_-(-\bar{T})\rangle}\dot{\gamma}_-(-\bar{T})\,. \end{aligned} \tag{6.1.122}$$

As a consequence,

$$R_0 w_0 = w_0 - \frac{\langle a, w_0\rangle}{\langle a, \dot{\gamma}_0(\bar{T})\rangle}\dot{\gamma}_0(\bar{T}) \neq 0 \tag{6.1.123}$$

since from (ii) it follows that w_0 is not parallel to $\dot{\gamma}_0(\bar{T}) = \begin{pmatrix} b_1 & b_2 & b_3\, e^{2\lambda_3\bar{T}} \end{pmatrix}^*$. Thus we get $\mathscr{S}''' = R_0 X_0(\bar{T})\mathscr{S}' \neq \{0\}$ that is $\dim \mathscr{S}''' = 1$ and condition (H3) is satisfied. The proof is completed. □

We start with construction of $\psi(t)$: Since $\mathscr{S}'' = \{0\}$, we see that ψ is such that $\{\psi\}^\perp = \operatorname{span}\{a, R_0 w_0\}$. From (6.1.123) it is easy to see that $\langle a, R_0 w_0\rangle = 0$, hence we can take:

$$\psi = a \wedge R_0 w_0\,,$$

where $\wedge$ denotes the cross product.

First we construct $\psi(t)$ *for* $t \leq -\bar{T}$: Since: $(\mathbb{I} - P_-^*)R_-^* X_0^*(\bar{T})R_0^*\psi = 0$, we can compute $P_-^* R_-^* X_0^*(\bar{T})R_0^*\psi$ instead of $R_-^* X_0^*(\bar{T})R_0^*\psi$, with the first one being simpler. We recall that $R_0 w = w$ for any $w \in \{a\}^\perp$ and $R_0\dot{\gamma}_0(\bar{T}) = 0$. Thus the eigenvalues of R_0 are 0 (simple) and 1 (double). The same conclusion holds for $R_\pm$. As a consequence, we obtain:

$$\operatorname{trace} R_0 = \operatorname{trace} R_+ = \operatorname{trace} R_- = 2. \tag{6.1.124}$$

We also remark that

$$\dot\gamma_0(\bar T) = \begin{pmatrix} b_1 \\ b_2 \\ b_3\,\mathrm{e}^{2\lambda_3\bar T}\end{pmatrix}, \quad \dot\gamma_+(\bar T) = \begin{pmatrix} 0 \\ 0 \\ \dfrac{d\lambda_3}{a_3}\end{pmatrix}, \quad \dot\gamma_-(-\bar T) = \begin{pmatrix} b_1(\mathrm{e}^{-2\lambda_1\bar T}-1) \\ b_2(\mathrm{e}^{-2\lambda_2\bar T}-1) \\ 0\end{pmatrix}$$

and, using (6.1.119)

$$X_0(\bar T) = \begin{pmatrix} \mathrm{e}^{2\lambda_1\bar T} & 0 & 0 \\ 0 & \mathrm{e}^{2\lambda_2\bar T} & 0 \\ 0 & 0 & \mathrm{e}^{2\lambda_3\bar T}\end{pmatrix}.$$

Hence we get:

$$P_-^* R_-^* X_0^*(\bar T) R_0^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A_{13} & A_{23} & A_{33}\end{pmatrix},$$

where

$$A_3 = (A_{13}, A_{23}, A_{33})^* = R_0 X_0(\bar T) R_- e_3 \tag{6.1.125}$$

is the third column of the matrix $R_0 X_0(\bar T) R_-$. Thus

$$P_-^* R_-^* X_0^*(\bar T) R_0^* \psi = \begin{pmatrix} 0 \\ 0 \\ \langle A_3, \psi\rangle\end{pmatrix}.$$

Since $\psi = a \wedge R_0 w_0$ we get, using (6.1.61) and (6.1.119):

$$\psi(t) = \mathrm{e}^{-\lambda_3(t+\bar T)}\langle A_3, a\wedge R_0 w_0\rangle e_3$$

for $t \le -\bar T$. Note that

$$\langle A_3, \psi\rangle = \det\begin{pmatrix} A_{13} & a_1 & (R_0 w_0)_1 \\ A_{23} & a_2 & (R_0 w_0)_2 \\ A_{33} & a_3 & (R_0 w_0)_3\end{pmatrix} = \det(A_3\ a\ R_0 w_0)$$

where $(R_0 w_0)_j$ is the j-th component of $R_0 w_0$ and that $A_3 = R_0[X_0(\bar T) R_- e_3] \in \mathscr{R} R_0 = \{a\}^\perp$ so both A_3 and $R_0 w_0$ belong to $\operatorname{span}\{a\}^\perp$, but of course this does not mean they are parallel. The computation of the vector A_3 is really messy even in an example as simple as this, so we don't proceed further with its computation now, but will do it later when we fix some particular values of the parameters.

Next, we look at the expression of $\psi(t)$ for $-\bar{T} < t \leq \bar{T}$: Since the linear system $\dot{x} = Ax$ is autonomous, and $X_0(-\bar{T}) = \mathbb{I}$, we have $X_0^{-1*}(t)X_0^*(\bar{T}) = X_0^*(-t)$. Next, to compute $R_0^*\psi$ we make use of the following identity.

Lemma 6.1.31. *For a given $3\times 3-$matrix M, it holds*

$$(Mu)\wedge v + u\wedge (Mv) - (\operatorname{trace} M)\, u\wedge v = -M^*(u\wedge v) \tag{6.1.126}$$

for any $u, v \in \mathbb{R}^3$.

Proof. Indeed, taking the scalar product with a vector $w \in \mathbb{R}^3$, (6.1.126) is equivalent to

$$\det(Mw,u,v) + \det(w,Mu,v) + \det(w,u,Mv) = (\operatorname{trace} \mathrm{M}) \det(w,u,v)\,. \tag{6.1.127}$$

To prove (6.1.127), we note that the map from $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$ to $\mathbb{R}$ given by $(w,u,v) \mapsto \det(Mw,u,v) + \det(w,Mu,v) + \det(w,u,Mv) \in \mathbb{R}$ is multilinear and alternating. Thus there exists $\kappa \in \mathbb{R}$ so that

$$\det(Mw,u,v) + \det(w,Mu,v) + \det(w,u,Mv) = \kappa \det(w,u,v).$$

Taking $w = e_1$, $u = e_2$ and $v = e_3$ we see that $\kappa = \operatorname{trace} M$ and (6.1.127) is proved. The proof of Lemma 6.1.31 is completed. □

We apply (6.1.126) with $M = R_0$, $u = a$ and $v = R_0w_0$. We get, since $\operatorname{trace} R_0 = 2$:

$$-R_0^*\psi = -R_0^*[a\wedge R_0w_0] = R_0a\wedge R_0w_0 + a\wedge [R_0R_0w_0] - 2a\wedge R_0w_0$$
$$= R_0a\wedge R_0w_0 - a\wedge R_0w_0 = -(\mathbb{I} - R_0)a\wedge R_0w_0 = -\frac{|a|^2}{\langle a, \dot{\gamma}_0(\bar{T})\rangle}\dot{\gamma}_0(\bar{T})\wedge R_0w_0$$

and then

$$\psi(t) = \frac{|a|^2}{\langle a, \dot{\gamma}_0(\bar{T})\rangle} X_0(-t)[\dot{\gamma}_0(\bar{T})\wedge R_0w_0]$$

for $-\bar{T} < t \leq \bar{T}$, since $X_0^*(t) = X_0(t)$.

Finally we compute $\psi(t)$ when $t > \bar{T}$: Applying again (6.1.126) with $M = R_+$, $u = a$ and $v = R_0w_0$. We get:

$$-R_+^*\psi = -R_+^*(a\wedge R_0w_0) = (R_+a)\wedge (R_0w_0) + a\wedge (R_+R_0w_0) - 2a\wedge R_0w_0$$

since $\operatorname{trace} R_+ = 2$. Now, we have:

$$(R_+a)\wedge (R_0w_0) = a\wedge R_0w_0 - \frac{|a|^2}{\langle a, \dot{\gamma}_+(\bar{T})\rangle}\dot{\gamma}_+(\bar{T})\wedge R_0w_0, \quad R_+R_0w_0 = R_0w_0$$

since $R_0w_0 \in \mathscr{R}Q = \mathscr{R}R_+$ and R_+ is a projection. Thus:

$$R_+^*\psi = R_+^*(a\wedge R_0w_0) = \frac{|a|^2}{\langle a, \dot{\gamma}_+(\bar{T})\rangle}\dot{\gamma}_+(\bar{T})\wedge R_0w_0 = \frac{|a|^2}{a_3} e_3\wedge R_0w_0$$

and

$$\psi(t) = \frac{|a|^2}{a_3} X_+^{-1}(t)[e_3 \wedge R_0 w_0]$$

for $t > \bar{T}$, since $X_+^*(t) = X_+(t)$. In summary, we conclude with the following result.

Theorem 6.1.32. *Let assumptions* (i)–(ii) *hold and suppose* (6.1.112) *is satisfied. Then the function* $\psi(t)$ *of* (6.1.61) *for the system* (6.1.113) *reads*

$$\psi(t) = \begin{cases} \mathrm{e}^{-\lambda_3(t+\bar{T})} \langle A_3, a \wedge R_0 w_0 \rangle e_3, & \text{if } t \leq -\bar{T}, \\ \frac{|a|^2}{\langle a, \dot{\gamma}_0(\bar{T}) \rangle} X_0(-t)[\dot{\gamma}_0(\bar{T}) \wedge R_0 w_0], & \text{if } -\bar{T} < t \leq \bar{T}, \\ \frac{|a|^2}{a_3} X_+^{-1}(t)[e_3 \wedge R_0 w_0], & \text{if } t > \bar{T} \end{cases} \tag{6.1.128}$$

where $X_0(t)$*,* $X_+(t)$*,* w_0*,* R_0*,* A_3 *are given by* (6.1.119)*,* (6.1.120)*,* (6.1.121)*,* (6.1.122)*,* (6.1.125)*, respectively.*

So we are in position to apply Theorem 6.1.16. Writing $g_j = (g_{j1}, g_{j2}, g_{j3})^*$, $j = 1, 2$, we get the Melnikov function (6.1.62)

$$\begin{aligned} \mathscr{M}(\alpha) &= \int_{-\infty}^{\infty} [\sin \omega_1 (t+\alpha) \psi^*(t) g_1 + \sin \omega_2 (t+\alpha) \psi^*(t) g_2]\, dt \\ &= \sin(\alpha\omega_1) \int_{-\infty}^{\infty} \cos(\omega_1 t) \psi^*(t) g_1\, dt + \cos(\alpha\omega_1) \int_{-\infty}^{\infty} \sin(\omega_1 t) \psi^*(t) g_1\, dt \\ &\quad + \sin(\alpha\omega_2) \int_{-\infty}^{\infty} \cos(\omega_2 t) \psi^*(t) g_2\, dt + \cos(\alpha\omega_2) \int_{-\infty}^{\infty} \sin(\omega_2 t) \psi^*(t) g_2\, dt \\ &= A_1(\omega_1) \sin(\omega_1 \alpha + \varpi_1(\omega_1)) + A_2(\omega_2) \sin(\omega_2 \alpha + \varpi_2(\omega_2)) \end{aligned}$$

where

$$A_i(\omega_i) := \sqrt{\left(\int_{-\infty}^{\infty} \cos \omega_i t\, \psi^*(t) g_i\, dt \right)^2 + \left(\int_{-\infty}^{\infty} \sin \omega_i t\, \psi^*(t) g_i\, dt \right)^2}$$

for $i = 1, 2$. Now we consider the following two possibilities:

1. Either $A_1(\omega_1) \neq 0, A_2(\omega_2) = 0$ or $A_1(\omega_1) = 0, A_2(\omega_2) \neq 0$. Then $\mathscr{M}(\alpha)$ has the simple zero $\alpha_0 = -\varpi_i(\omega_i)/\omega_i$, $i = 1, 2$, respectively.
2. $A_1(\omega_1) \neq 0$ and $A_2(\omega_2) \neq 0$. Let $s_i := \operatorname{sgn} A_i(\omega_i) \in \{-1, 1\}$, $i = 1, 2$. Then $s_1 \omega_1 A_1(\omega_1) + s_2 \omega_2 A_2(\omega_2) = \omega_1 |A_1(\omega_1)| + \omega_2 |A_2(\omega_2)| > 0$. Since $\cos \frac{1-s_i}{2} \pi = s_i$ and $\sin \frac{1-s_i}{2} \pi = 0$ for $i = 1, 2$, and ω_1/ω_2 is irrational, from [40] the existence follows from α_0 (as a matter of fact infinitely many α_0 exists) so that $\omega_i \alpha_0 + \varpi_i(\omega_i)$ are near to $\frac{1-s_i}{2} \pi$ modulo 2π, $i = 1, 2$, and $\mathscr{M}(\alpha_0) = 0$ while

$$\mathscr{M}'(\alpha_0) \geq \frac{s_1 \omega_1 A_1(\omega_1) + s_2 \omega_2 A_2(\omega_2)}{2} > 0\,.$$

Hence also in this case we have a simple zero of $\mathscr{M}(\alpha)$.

Consequently if $A_1(\omega_1)$ and $A_1(\omega_1)$ do not vanish simultaneously, Theorem 6.1.27 applies and we conclude that (6.1.110) behaves chaotically for any $\varepsilon \neq 0$ sufficiently small. Next, we note that $A_i(\omega_i) \neq 0$ if and only if

$$\Phi_i(\omega_i) := \int_{-\infty}^{\infty} \mathrm{e}^{-\omega_i t \imath} \psi^*(t) g_i \, dt \neq 0 . \tag{6.1.129}$$

Since $\psi(t) \neq 0$, Plancherel Theorem (cf Section 2.1) ensures that

$$V(\omega) := \int_{-\infty}^{\infty} \mathrm{e}^{-\omega t \imath} \psi(t) \, dt \neq 0. \tag{6.1.130}$$

Note that $\Phi_i(\omega) = V(\omega)^* g_i$. So condition (6.1.129) is equivalent to the non-orthogonality of $V(\omega)$ to g_i. Furthermore, it is not difficult to observe that $\Phi_i(\omega)$ are analytic for $\omega > 0$. Indeed, we have $|\psi(t)| \leq k\mathrm{e}^{-\delta|t|}$, for some positive constants k and δ, and for $\omega, \eta \in \mathbb{R}$ we have: $\sin((\omega + \imath\eta)x) = \sin(\omega x)\mathrm{e}^{-\eta x} + \imath \mathrm{e}^{-\imath\omega x} \sinh \eta x$. Thus the function

$$\int_{-\infty}^{\infty} \sin(zt) \psi^*(t) g_i \, dt$$

is holomorphic in the strip $\{\omega + \imath\eta \in \mathbb{C} \mid |\eta| < \delta\}$. A similar argument works with $\cos(zt)$ instead of $\sin(zt)$. Consequently, when functions $\Phi_i(\omega)$ are not identically zero, they have at most countable many positive zeroes with possible accumulations at $+\infty$ (cf Section 2.6.5). In summary, we get the following result.

Theorem 6.1.33. *Let assumptions* (i)–(iii) *hold and suppose* (6.1.112) *holds. When both $\Phi_1(\omega)$ and $\Phi_2(\omega)$ are not identically zero, there is at most a countable set $\{\widetilde{\omega}_j\} \subset (0,\infty)$ with possible accumulating point at $+\infty$ so that if $\omega_1, \omega_2 \in (0,\infty) \setminus \{\widetilde{\omega}_j\}$ then system* (6.1.110) *is chaotic for any $\varepsilon \neq 0$ sufficiently small.*

Since in general, the above formulas are rather difficult to find the solution, now we consider the following concrete examples.

Example 6.1.34. We take

$$\begin{gathered} a_1 = 0, \quad a_2 = a_3 = 1, \quad \lambda_1 = 2, \quad \lambda_2 = 1, \quad \lambda_3 = -1, \\ b_1 = b_2 = -1, \quad b_3 = 1, \quad d = 3/4 . \end{gathered} \tag{6.1.131}$$

Then (6.1.112) is satisfied with $\bar{T} = \ln 2$. With these parameters values we have:

$$R_0 w_0 = w_0 = 16 e_1, \quad \dot{\gamma}_0(\bar{T}) = -e_1 - e_2 + \frac{1}{4} e_3 .$$

Thus,

$$\dot{\gamma}_0(\bar{T}) \wedge R_0 w_0 = 4 e_2 + 16 e_3$$

and we get

$$\psi(t) = \begin{cases} -\dfrac{64}{3}[\mathrm{e}^{-t} e_2 + \mathrm{e}^t e_3], & \text{for } -\ln 2 < t \le \ln 2, \\ 64\,\mathrm{e}^{-t} e_2, & \text{for } t > \ln 2, \end{cases}$$

since

$$X_0(t) = X_-(t) = \begin{pmatrix} 4\,\mathrm{e}^{2t} & 0 & 0 \\ 0 & 2\,\mathrm{e}^t & 0 \\ 0 & 0 & \dfrac{1}{2}\,\mathrm{e}^{-t} \end{pmatrix}, \quad X_+(t) = \begin{pmatrix} \dfrac{1}{4}\,\mathrm{e}^{2t} & 0 & 0 \\ 0 & \dfrac{1}{2}\,\mathrm{e}^t & 0 \\ 0 & 0 & 2\,\mathrm{e}^{-t} \end{pmatrix}.$$

Finally we compute $\psi(t)$ for $t \le -\ln 2$. First we need to know A_3 which is the third column of $R_0 X_0(\bar{T}) R_-$ that is

$$A_3 = R_0 X_0(\bar{T}) R_- e_3.$$

We have $R_- e_3 = -\frac{5}{4} e_1 - e_2 + e_3$, then $X_0(\bar{T}) R_- e_3 = -20 e_1 - 4 e_2 + \frac{1}{4} e_3$ and thus $A_3 = -15 e_1 + e_2 - e_3$. As a consequence,

$$\langle A_3, a \wedge R_0 w_0 \rangle = \det \begin{pmatrix} -15 & 0 & 16 \\ 1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} = 32$$

and $\psi(t) = 64\,\mathrm{e}^t e_3$ for $t \le -\ln 2$. We conclude that (see (6.1.128))

$$\psi(t) = \begin{cases} 64\,\mathrm{e}^t e_3, & \text{for } t \le -\ln 2, \\ -\dfrac{64}{3}[\mathrm{e}^{-t} e_2 + \mathrm{e}^t e_3], & \text{for } -\ln 2 < t \le \ln 2, \\ 64\,\mathrm{e}^{-t} e_2, & \text{for } t > \ln 2. \end{cases}$$

Putting this formula of $\psi(t)$ into (6.1.130), we finally obtain

$$V(\omega) = -\frac{256 \sin(\omega \ln 2)}{3(\omega^2+1)} \left[\omega(e_2+e_3) + \imath(e_2-e_3)\right].$$

Then from $\Phi_i(\omega) = V(\omega)^* g_i$, we have:

$$\Phi_i(\omega) = -\frac{256 \sin(\omega \ln 2)}{3(\omega^2+1)} \Big(\omega(g_{i2}+g_{i3}) + \imath(g_{i2}-g_{i3})\Big). \tag{6.1.132}$$

So for the parameters (6.1.131), $\Phi_i(\omega)$ is identically zero if and only if $g_{i2} = g_{i3} = 0$. Otherwise, it has only the simple positive zeroes $\widetilde{\omega}_j = \pi j/\ln 2$, $j \in \mathbb{N}$. In consequence of Theorem 6.1.33 we get the following.

Corollary 6.1.35. *Consider* (6.1.110) *with parameters* (6.1.131) *and* (iii) *holds. If either $g_{i2} \ne 0$ or $g_{i3} \ne 0$ for some $i \in \{1,2\}$ and $\omega_1, \omega_2 \ne \pi j/\ln 2$, $\forall j \in \mathbb{N}$ then system* (6.1.110) *is chaotic for any $\varepsilon \ne 0$ small.*

Example 6.1.36. On the other hand, for the following set of parameters

$$\begin{aligned} &a_1 = a_2 = a_3 = 1, \quad b_1 = b_2 = -1, \quad b_3 = 13/8, \\ &\lambda_1 = 2, \quad \lambda_2 = 1, \quad \lambda_3 = -1, \quad d = 39/32, \end{aligned} \tag{6.1.133}$$

we get $\bar{T} = \ln 2$ and (see (6.1.128))

$$\psi(t) = \begin{cases} \frac{1344}{17} \mathrm{e}^t e_3, & \text{for} \quad t \le -\ln 2, \\ -\frac{16}{17}(13\mathrm{e}^{-2t} e_1 + 26\mathrm{e}^{-t} e_2 + 20\mathrm{e}^t e_3), & \text{for} \quad -\ln 2 < t \le \ln 2, \\ \frac{48}{17}(49\mathrm{e}^{-2t} e_1 + 18\mathrm{e}^{-t} e_2), & \text{for} \quad t > \ln 2. \end{cases}$$

Then

$$\Phi_i(\omega) = \frac{2^{6-\imath\omega}(13 \cdot 4^{\imath\omega} - 10)(g_{i1} + 2g_{i2} + g_{i1}\omega^2 + g_{i2}\omega^2 - 2g_{i3} + \omega^2 g_{i3} - \imath(g_{i2} + 3g_{i3})\omega)}{17(\omega - \imath)(\omega - 2\imath)(1 - \imath\omega)}$$

for $i = 1,2$. Clearly, for the parameters (6.1.133), $\Phi_i(\omega)$ is not identically zero. If $g_{i2} \neq -3g_{i3}$ then $\Phi_i(\omega)$ has no positive roots. If $g_{i2} = -3g_{i3}$ then $\Phi_i(\omega)$ has the only positive root $\omega_{i1} = \sqrt{\frac{g_{i1}-8g_{i3}}{2g_{i3}-g_{i1}}}$ provided $\frac{g_{i1}-8g_{i3}}{2g_{i3}-g_{i1}} > 0$. In consequence of Theorem 6.1.33 we obtain the following.

Corollary 6.1.37. *Consider* (6.1.110) *with parameters* (6.1.133) *and* (iii) *holds. If one of the following conditions is satisfied*

- $g_{i2} \neq -3g_{i3}$,
- $g_{i2} = -3g_{i3}$, $g_{i1} = 2g_{i3} \neq 0$,
- $g_{i2} = -3g_{i3}$, $g_{i1} \neq 2g_{i3}$, $\dfrac{g_{i1} - 8g_{i3}}{2g_{i3} - g_{i1}} < 0$,
- $g_{i2} = -3g_{i3}$, $g_{i1} \neq 2g_{i3}$, $\dfrac{g_{i1} - 8g_{i3}}{2g_{i3} - g_{i1}} > 0$ *and* $\omega_i \neq \sqrt{\dfrac{g_{i1} - 8g_{i3}}{2g_{i3} - g_{i1}}}$,

for some $i \in \{1,2\}$ *then system* (6.1.110) *is chaotic for any* $\varepsilon \neq 0$ *small.*

Remark 6.1.38. Parameters (6.1.131) and (6.1.133) give Examples 6.1.34 and 6.1.36 for which $\Phi_i(\omega)$ is either identically zero, or has infinitely many positive roots, or has no positive roots, or has finitely many positive roots.

Remark 6.1.39. If $\Phi_1(\omega_1) = 0$ and $\Phi_2(\omega_2) = 0$ then $\mathscr{M}(\alpha)$ is identically zero and a *second-order Melnikov function* must be derived as in Section 4.1.4. But those computations should be very awkward for (6.1.110), so we omit them.

Finally when $g_1 \neq 0$, $g_2 \neq 0$ and ω_1/ω_2 is rational, we get a different situation. For instance, consider Example 6.1.34 with $\omega_1 = 1$, $\omega_2 = 3$ and $g_{i2} = g_{i3}$, $i = 1,2$. Thus (6.1.110) is 2π-periodic and

$$\mathscr{M}(\alpha) = \Phi_1(1)\sin\alpha + \Phi_2(3)\sin 3\alpha = \sin\alpha - \frac{1}{3}\sin 3\alpha = \frac{4}{3}\sin^3\alpha$$

provided $\Phi_1(1) = 1$ and $\Phi_2(3) = -\frac{1}{3}$. From (6.1.132) we derive

$$g_{12} + g_{13} = -\frac{3}{128\sin(\ln 2)}, \quad g_{22} + g_{23} = \frac{5}{384\sin(3\ln 2)}.$$

Then the Melnikov function is $\mathscr{M}(\alpha) = \frac{4}{3}\sin^3\alpha$ and it has only the zero $\alpha_0 = 0$ in $[-\pi,\pi]$ which is not simple but has Brouwer index 1 (cf Section 2.2.4). So Theorem 6.1.27 is not applicable, but we still get a chaos for (6.1.110) with $\varepsilon \neq 0$ small as in Remark 3.1.9 [15].

6.1.12 Multiple Transversal Crossings

The above results can be extended to cases when homoclinics are transversally passing through several discontinuity manifolds. More precisely, let $\Omega \subset \mathbb{R}^n$ be a bounded open set in $\mathbb{R}^n$ and $G_j(z)$, $j = 1,...,p$ be C^r-functions on $\bar{\Omega}$, with $r \geq 2$. We set $S_j = \{z \in \Omega \mid G_j(z) = 0\}$, and

$$\Omega \setminus \bigcup_{j=1}^{p} S_j := \bigcup_{i=0}^{q} \Omega_i$$

with Ω_i being the connected components of $\Omega \setminus \cup_{j=1}^{p} S_j$. Let $f_i(z) \in C_b^r(\mathbb{R}^n)$ and $g_i(t,z,\varepsilon) \in C_b^r(\mathbb{R}^{n+2})$, i.e. $f_i(z)$ and $g_i(t,z,\varepsilon)$ have uniformly bounded derivatives up to the r-th order on $\mathbb{R}^n$ and $\mathbb{R}^{n+2}$, respectively. We also assume that the r-th order derivatives of $f_i(z)$ and $g_i(t,z,\varepsilon)$ are uniformly continuous. We set

$$f(z) := f_i(z), \qquad g(t,z,\varepsilon) := g_i(t,z,\varepsilon) \quad \text{if } z \in \Omega_i$$

and

$$G(z) := \Pi_{j=1}^{p} G_j(z).$$

Definition 6.1.40. We say that a piecewise C^1-function $z(t)$ is a solution of the equation

$$\dot{z} = f(z) + \varepsilon g(t,z,\varepsilon), \quad z \in \bar{\Omega}, \tag{6.1.134}$$

if it satisfies Eq. (6.1.134) when $z(t) \in \Omega_i$, and moreover, the following holds: if for some t_* we have $z(t_*) \in S_j$, then $z(t_*) \notin S_l$ for any $l \neq j$ and there exists $r > 0$ so that for any $t \in (t_* - r, t_* + r)$ with $t \neq t_*$, we have $z(t) \in \cup_{i=0}^{q} \Omega_i$. Moreover, if, for example, $z(t) \in \Omega_i$ for any $t \in (t_* - r, t_*)$, then the left derivative of $z(t)$ at $t = t_*$ satisfies: $\dot{z}(t_*^-) = f_i(z(t_*)) + \varepsilon g_i(t_*, z(t_*), \varepsilon)$; similarly, if $z(t) \in \Omega_i$ for any $t \in (t_*, t_* + r)$, then $\dot{z}(t_*^+) = f_i(z(t_*)) + \varepsilon g_i(t_*, z(t_*), \varepsilon)$.

Remark 6.1.41. Since $z(t) \in \cup_{i=0}^{q}\Omega_i$ for any $t \in (t_* - r, t_* + r) \setminus \{t_*\}$ there exist two indices $i = i'_j, i''_j$ so that $z(t) \in \Omega_{i'_j}$ when $t \in (t_* - r, t_*)$ and $z(t) \in \Omega_{i''_j}$ for $t \in (t_*, t_* + r)$. Moreover, since $z(t) \notin \cup_{j=1}^{p} S_j$, for any $t \in (t_* - r, t_*) \cup (t_*, t_* + r)$, $z(t) \in \cup_{j=1}^{p} S_j$ only for t in a discrete increasing subset $\{t_j\}$ of $\mathbb{R}$ with possible accumulation points at $\pm\infty$. Moreover $z(t) \in C^{r+1}(\mathbb{R} \setminus \{t_j\})$.

We assume (Figure 6.2) that:

(H1) For $\varepsilon = 0$ Eq. (6.1.134) has the hyperbolic equilibrium $x = 0 \in \Omega_0$ and a continuous, piecewise C^1-solution $\gamma(t) \in \Omega$ which is homoclinic to $x = 0$ and consists of three branches

$$\gamma(t) = \begin{cases} \gamma_-(t), & \text{if } t \leq -\bar{T}, \\ \gamma_0(t), & \text{if } -\bar{T} \leq t \leq \bar{T}, \\ \gamma_+(t), & \text{if } t \geq \bar{T}, \end{cases}$$

where $\gamma_\pm(t) \in \Omega_0$ for $|t| > \bar{T}$, $\gamma_0(t) \in \Omega$ for $|t| < \bar{T}$ and

$$\gamma_-(-\bar{T}) = \gamma_0(-\bar{T}) \in \partial\Omega_0, \quad \gamma_+(\bar{T}) = \gamma_0(\bar{T}) \in \partial\Omega_0 .$$

(H2) At any point $t_* \in \mathbb{R}$ so that $\gamma(t_*) \in S_j$, we have

$$G'(\gamma(t_*))f_{i'_j}(\gamma(t_*)) \cdot G'(\gamma(t_*))f_{i''_j}(\gamma(t_*)) > 0 ,$$

where i'_j, i''_j are the two indices defined in Remark 6.1.41.

Let t_* be such that $\gamma(t_*) \in S_j$ for some j. Then (H2) means that both $\dot{\gamma}(t_*^+)$ and $\dot{\gamma}(t_*^-)$ are transverse to S_j at the point $\gamma(t_*)$. Next, since $\gamma(t) \in \Omega_0$ for $|t| \geq \bar{T}$, it follows that $\gamma_0(t)$ intersect $\cup_{i=1}^{p} S_i$ only a finite number of times denoted by $-\bar{T} = t_0 < t_1 < \cdots < t_{N-1} < t_N = \bar{T}$. In summary $\gamma(t) \in \cup_{i=1}^{p} S_i$ if and only if $t \in \{-\bar{T} = t_0 < t_1 < \cdots < t_{N-1} < t_N = \bar{T}\}$ and $\gamma(t)$ is continuous, piecewise C^1 in $\mathbb{R}$ and has left and right derivatives at the points $t = t_i$, $i = 0, \ldots, N$. Next for $l = 0, \ldots, N-1$, we define i_l, j_l so that $\gamma_0(t) \in \Omega_{i_l}$ for any $t \in (t_l, t_{l+1})$ and $\gamma_0(t_l) \in S_{j_l}$, $\gamma_0(t_N) \in S_{j_N}$. Thus, with reference to the notation of Remark 6.1.41, we have $i'_{j_l} = i_{l-1}$ and $i''_{j_l} = i_l$.

For $l = 1, \ldots, N$ let $X_l(t)$, $t \in [t_{l-1}, t_l]$ be the fundamental matrix solution of $\dot{x} = f'_{i_{l-1}}(\gamma_0(t))x$ with $X_l(t_{l-1}) = \mathbb{I}$. The transition matrix $\mathscr{S}_l : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$\mathscr{S}_l w := w + \left[\dot{\gamma}_0(t_l^+) - \dot{\gamma}_0(t_l^-)\right] \frac{G'(\gamma_0(t_l))w}{G'(\gamma_0(t_l))\dot{\gamma}_0(t_l^-)} \tag{6.1.135}$$

for $l = 1, \ldots, N-1$. It is easy to see that all $\mathscr{S}_l$ are invertible. Finally we define the fundamental matrix solution of the variational equation of (6.1.1) along $\gamma_0(t)$ at $\varepsilon = 0$ as follows:

$$X_0(t) := X_l(t)\mathscr{S}_{l-1}X_{l-1}(t_{l-1})\mathscr{S}_{l-2}\cdots\mathscr{S}_1 X_1(t_1) \quad \text{for} \quad t \in [t_{l-1}, t_l)$$

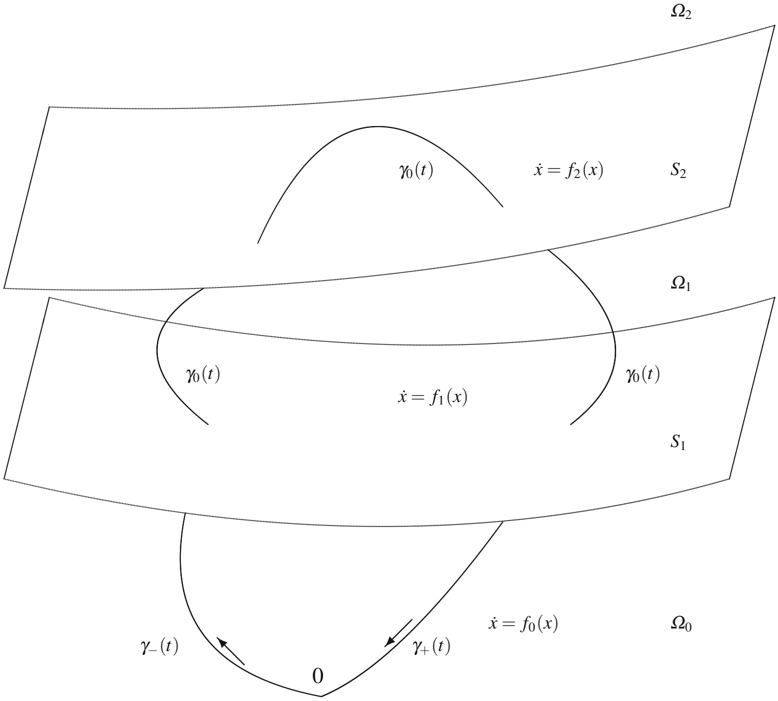


Fig. 6.2 Homoclinic orbit $\gamma(t)$ transversally crosses discontinuity manifolds S_1 and S_2. It may cross $S_{1,2}$ several but finite times before eventually getting in Ω_0.

and $l = 2,\ldots,N$, where we have $X_0(t) = X_1(t)$ on $[t_0,t_1)$. Note that $X_0(t)$ solves the following impulsive linear matrix differential Cauchy problem

$$\dot{X}_0(t) = Df(\gamma_0(t))X_0(t),$$

$$X_0(t_l^+) = \mathscr{S}_l X_0(t_l^-), \quad l = 1,\ldots,N-1, \quad X_0(-\bar{T}) = \mathbb{I}$$

for $t \in [-\bar{T},\bar{T}]$. Now we can repeat the above arguments over (6.1.134) by introducing (6.1.61), (6.1.62) and then restate Theorem 6.1.16 and the other above results [48].

6.2 Sliding Homoclinic Bifurcation

6.2.1 Higher Dimensional Sliding Homoclinics

In [34] the problem of bifurcations from homoclinic orbits is studied whereas in Section 6.1 chaotic behaviour of solutions is proved for time perturbed discontinuous differential equations in a finite dimensional space, when the homoclinic orbits of the unperturbed problem crosses transversally the discontinuity manifold. Thus, it is natural to argue if a similar behaviour arises also when sliding homoclinic orbits are concerned. The purpose of this section is to give an affirmative answer to this question. It has been observed in Section 6.1 that one of the problems we have to face studying discontinuous differential equations, is the loss of smoothness of invariant manifolds, a problem persisting also in the sliding case. Moreover in the sliding case the additional problem arises, that is, during the *sliding time* the system should be considered only on the discontinuity manifold, thus reducing the dimension of the system. However, we show in this section that the method used in Section 6.1 to prove chaotic behavior can be arranged to handle the case of sliding homoclinic orbits, leading to a similar conclusion.

Typical examples of sliding motions are in relay controllers, impact oscillators and stick-slip friction systems where the stick motion corresponds to sliding. Many non-smooth models can be found in [6,7,11,14,27,28,49–54]. Sliding homoclinic solutions to pseudo-saddles (saddles lying on discontinuity curves/lines) of planar DDEs are studied in [6,51] both numerically and analytically. A theoretical discussion on sliding homoclinic solutions to saddles of planar DDEs is presented in [6]. However, we have not found any concrete example in literature with a sliding homoclinic orbit to a saddle, except in [28] where an example is given with two discontinuity lines. In our opinion the reason why it is so difficult to find examples is because when the discontinuity manifold is linear, the DDE must be nonlinear in the open subset the equilibrium point belongs to and this makes computations harder. Of course, one can imagine a linear system of ODE with a sliding homoclinic orbit to a nonlinear discontinuity manifold. But one can reduce to the linear discontinuity manifold (and then to a nonlinear equation) by a simple change of variables, and for computational reasons, it is better to work with linear discontinuity manifolds. For this reason we investigate examples of DDEs exhibiting sliding chaotic behaviour in consequence of Theorem 6.2.5 in Sections 6.2.2 and 6.2.3.

Now we go into details. Let $\mathbb{R}^n = \mathbb{R}\times\mathbb{R}^{n-1}$ with corresponding projections $P_z : \mathbb{R}^n \to \mathbb{R}$ and $P_y : \mathbb{R}^n \to \mathbb{R}^{n-1}$. For $x\in\mathbb{R}^n$ we write $x=(z,y)\in\mathbb{R}\times\mathbb{R}^{n-1}$. Consider a discontinuous system in $\mathbb{R}^n$ with a small parameter such as:

$$\dot{x} = f(x) + \varepsilon g(t,x,\varepsilon)\,, \tag{6.2.1}$$

where

$$f(x) = \begin{cases} f_+(z,y) & \text{for,} \quad z>0\,,\\ f_-(z,y) & \text{for,} \quad z<0\,,\end{cases}$$

with $f_\pm : \Omega \to \mathbb{R}^n$, $f_\pm \in C_b^r(\Omega)$ and $g : \mathbb{R}\times\Omega\times\mathbb{R} \to \mathbb{R}^n$, $g\in C_b^r(\mathbb{R}\times\Omega\times\mathbb{R})$, with Ω being a bounded open subset of $\mathbb{R}^n$ that has nonempty intersection with the hyperplane $z=0$. Note that we allow the possibility that $f_+(0,y)\neq f_-(0,y)$. We also assume that the r-th order derivatives of $f_\pm(x)$ and $g(t,x,\varepsilon)$ are uniformly continuous. We set

$$\Omega_\pm = \{x = (z,y) \in \Omega \mid \pm z > 0\}, \quad \Omega_0 = \{x = (z,y) \in \Omega \mid z = 0\}.$$

By putting

$$f_\pm = (h_\pm(z,y), k_\pm(z,y)),$$

where $h_\pm : \Omega \to \mathbb{R}$ and $k_\pm : \Omega \to \mathbb{R}^{n-1}$, we assume that

(H1) For any $(0,y) \in \Omega_0$ it results:

$$h_-(0,y) - h_+(0,y) > 0. \tag{6.2.2}$$

Then we set (see [8, Eq. (2.12)]))

$$H(y) := V(y)\frac{k_+(0,y) - k_-(0,y)}{2} + \frac{k_+(0,y) + k_-(0,y)}{2},$$

where

$$V(y) := \frac{h_+(0,y) + h_-(0,y)}{h_-(0,y) - h_+(0,y)},$$

and for $(0,y) \in \Omega_0$, we consider the equation

$$\dot{y} = H(y). \tag{6.2.3}$$

Note that $H(y)$ has the following symmetric form with respect to indices $\pm$:

$$H(y) = \frac{h_-(0,y)k_+(0,y) - h_+(0,y)k_-(0,y)}{h_-(0,y) - h_+(0,y)}.$$

We suppose that

(H2) The unperturbed equation $\dot{x} = f_-(x)$ has a hyperbolic fixed point $x_0 \in \Omega_-$ and two solutions $\gamma_\pm(t)$, defined respectively for $t \geq \bar{T}$ and $t \leq -\bar{T}$, so that $\lim_{t\to\pm\infty} \gamma_\pm(t) = x_0$ and $\gamma_\pm(\pm\bar{T}) \in \Omega_0$.

(H3) Equation (6.2.3) has a solution $y_0(t)$, $(0, y_0(t)) \in \Omega_0$ for $-\bar{T} \leq t \leq \bar{T}$ so that

$$\gamma_-(-\bar{T}) = \gamma_0(-\bar{T}), \quad \gamma_+(\bar{T}) = \gamma_0(\bar{T})$$

where $\gamma_0(t) = (0, y_0(t))$, and the following hold:
$h_+(\gamma_0(t)) < 0$ for any $t \in [-\bar{T}, \bar{T}]$;
$h_-(\gamma_0(t)) > 0$ for any $t \in [-\bar{T}, \bar{T})$;
$h_-(\gamma_0(\bar{T})) = 0$ and $k_-(\gamma_0(\bar{T}))$ is not orthogonal to $\nabla_y h_-(\gamma_0(\bar{T})) \neq 0$. Here $\nabla_y h_-(\gamma_0(\bar{T}))$ is the gradient of $h_-(0,y)$ at the point $\gamma_0(\bar{T}) \in \Omega_0$.

Remark 6.2.1. 1. Note that the assumption that system (6.2.1) has a discontinuity on the hyperplane $z = 0$ is made only for sake of simplicity. We could have assumed, instead, that the singularity was at a hypersurface $x_1 = \varphi(x_2, \ldots, x_n)$ since we can reduce to our hypothesis by the simple change of variables:

$$y = (x_2, \ldots, x_n), \quad z = x_1 - \varphi(x_2, \ldots, x_n).$$

2. It will result from the argument given in the next sections that we may as well consider the case

$$g(x) = \begin{cases} g_+(t,z,y), & \text{for} \quad z > 0\,, \\ g_-(t,z,y), & \text{for} \quad z < 0\,, \end{cases}$$

with $g_\pm : \mathbb{R} \times \Omega \times \mathbb{R} \to \mathbb{R}^n$, $g_\pm \in C_b^r(\mathbb{R} \times \Omega \times \mathbb{R})$. However, for simplicity, we will continue to assume that $g \in C_b^r(\mathbb{R} \times \Omega \times \mathbb{R})$.

Remark 6.2.2. From (H3) it follows that $h_-^{-1}(0)$ is a submanifold $\mathscr{K}$ of Ω_0 of codimension 1 near the point $\gamma_0(\bar{T})$ (here we consider the restriction $h_- : \Omega_0 \to \mathbb{R}$). Moreover, since $V(y_0(\bar{T})) = -1$, we get

$$H(y_0(\bar{T})) = k_-(\gamma_0(\bar{T})),$$

so $\dot{\gamma}_0(\bar{T}) = (0, H(y_0(\bar{T}))) = (0, k_-(\gamma_0(\bar{T}))) = f_-(\gamma_0(\bar{T}))$. Thus condition (H3) means that $\dot{\gamma}_0(\bar{T})$ is transverse to $\mathscr{K}$ in Ω_0. Next, from (H3) it follows immediately that

$$\nabla_y h_-(0, y_0(\bar{T}))\dot{y}_0(\bar{T}) < 0.$$

Note that $\nabla_y h_-(0, y_0(t))\dot{y}_0(t) = h'_-(\gamma_0(t))\dot{\gamma}_0(t)$ for $t \in [-\bar{T}, \bar{T}]$. Finally, for the validity of the results of this section, it is enough that condition (H1) holds for y in a neigbourhood of $y_0(t)$, $-\bar{T} \le t \le \bar{T}$.

We set:

$$\gamma(t) = \begin{cases} \gamma_-(t), & \text{if } t \le -\bar{T}, \\ \gamma_0(t), & \text{if } -\bar{T} \le t \le \bar{T}, \\ \gamma_+(t), & \text{if } t \ge \bar{T} \end{cases}$$

and will refer to $\gamma(t)$ as the *sliding homoclinic* solution of (6.2.1) when $\varepsilon = 0$ (Figure 6.3).

We note that $\gamma(t)$ is C^1-smooth also at $t = \bar{T}$. In fact from $h_-(0, y_0(\bar{T})) = h_-(\gamma(\bar{T})) = 0$ we obtain $V(y_0(\bar{T})) = -1$ and then:

$$\dot{\gamma}_+(\bar{T}) = f_-(\gamma(\bar{T})) = \begin{pmatrix} h_-(\gamma(\bar{T})) \\ k_-(\gamma(\bar{T})) \end{pmatrix} = \begin{pmatrix} 0 \\ k_-(\gamma(\bar{T})) \end{pmatrix} = \begin{pmatrix} 0 \\ H(y_0(\bar{T})) \end{pmatrix} = \dot{\gamma}_0(\bar{T}).$$

Recalling $x = (z, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we set

$$f_\pm(x) + \varepsilon g(t, x, \varepsilon) = (h_\pm(t, z, y, \varepsilon), k_\pm(t, z, y, \varepsilon)).$$

and

$$H(t, y, \varepsilon) := \frac{h_-(t,0,y,\varepsilon)k_+(t,0,y,\varepsilon) - h_+(t,0,y,\varepsilon)k_-(t,0,y,\varepsilon)}{h_-(t,0,y,\varepsilon) - h_+(t,0,y,\varepsilon)}\,.$$

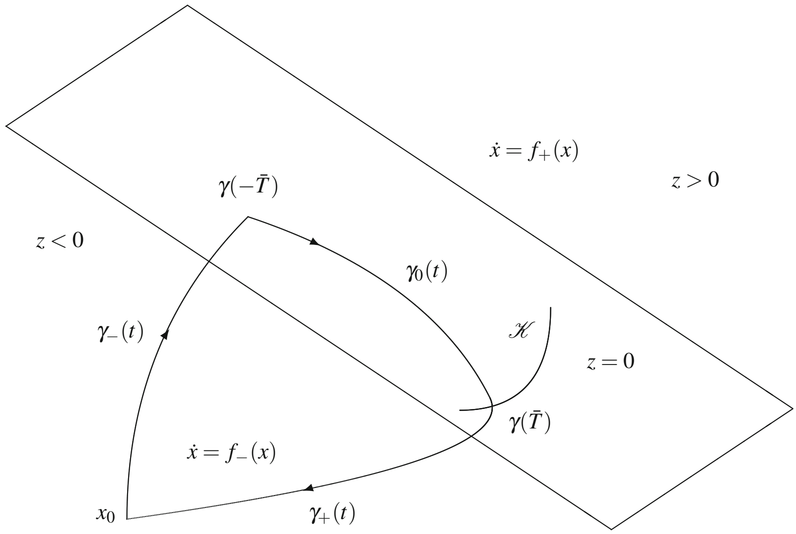


Fig. 6.3 A homoclinic sliding orbit $\gamma(t)$ of (6.2.1) with $\varepsilon = 0$ to the hyperbolic equilibrium $x = x_0$.

Note that $h_-(t,0,y,\varepsilon) - h_+(t,0,y,\varepsilon) = h_-(0,y) - h_+(0,y) > 0$ for any $y \in \Omega_0$ by (6.2.2). So $H(t,y,\varepsilon)$ is well defined. We are interested in the chaotic dynamics of (6.2.1) near $\gamma(t)$ for $\varepsilon \neq 0$ small.

Definition 6.2.3. By a *sliding* solution $x(t)$ of (6.2.1) we mean a function $x : \mathbb{R} \to \mathbb{R}^n$ for which the following hold:

There exists an increasing sequence $\{\widetilde{T}_m\}$ (possibly finite or with $m \leq m_0 \in \mathbb{Z}$, or $m \geq m_0 \in \mathbb{Z}$, with $m_0 \in \mathbb{Z}$, or $m \in \mathbb{Z}$) so that $x(t)$ is C^1-smooth for any $t \in \mathbb{R} \setminus \{\widetilde{T}_{2m}\}$ and possesses right and left derivatives at $t = \widetilde{T}_{2m}$. If $t \in (\widetilde{T}_{2m-1}, \widetilde{T}_{2m})$ then $x(t) \in \Omega_-$ and satisfies the equation $\dot{x} = f_-(x) + \varepsilon g(t,x,\varepsilon)$. If $t \in (\widetilde{T}_{2m}, \widetilde{T}_{2m+1})$ then $x(t) = (0,y(t)) \in \Omega_0$ and $y(t)$ satisfies the equation $\dot{y} = H(t,y,\varepsilon)$. At $t = \widetilde{T}_{2m+1}$ the equation $h_-(\widetilde{T}_{2m+1}, 0, y(\widetilde{T}_{2m+1}), \varepsilon) = 0$ is satisfied.

Since x_0 is a hyperbolic fixed point of $\dot{x} = f_-(x)$, the linear system $\dot{x} = f'_-(\gamma_+(t))x$ has an exponential dichotomy on $[\bar{T}, \infty)$ with projection P_+, and denotes by $X_+(t)$ its fundamental matrix with $X_+(\bar{T}) = \mathbb{I}$. Similarly the equation $\dot{x} = f'_-(\gamma_-(t))x$ has an exponential dichotomy on $(-\infty, -\bar{T}]$ with projection P_-, and denotes by $X_-(t)$ its fundamental matrix with $X_-(-\bar{T}) = \mathbb{I}$. Let

$$\mathscr{S}' := \mathscr{N}P_- \bigcap P_y(\mathbb{R}^n) = \{y \in \mathbb{R}^{n-1} \mid (0,y) \in \mathscr{N}P_-\} \subset \mathbb{R}^{n-1}.$$

Note that $\dim \mathscr{S}' = \dim \mathscr{N}P_- - 1$. Next we define projections Q and R as follows:

$Q : \mathbb{R}^n \to \mathbb{R}^n$ is the projection on $\mathbb{R}^n$ with $\mathscr{R}Q = \{0\} \times \mathbb{R}^{n-1}$ and $\mathscr{N}Q = \operatorname{span}\{\dot{\gamma}_-(-\bar{T})\}$,

$R : \mathscr{R}P_y \to \mathscr{R}P_y$ is the projection on $\mathscr{R}P_y$ so that $\mathscr{R}R = \mathscr{N}\nabla_y h_-(0, y_0(\bar{T}))$ and $\mathscr{N}R = \text{span}\{\dot{y}_0(\bar{T})\}$.

Let $Y_0(t)$ be the fundamental solution of $\dot{y} = H'(y_0(t))y$, with $Y_0(-\bar{T}) = \mathbb{I}$. Since $\dim \mathscr{S}' = \dim \mathscr{N}P_- - 1$, it is obvious that $\dim \begin{pmatrix} 0 \\ RY_0(\bar{T})\mathscr{S}' \end{pmatrix} \leq \dim \mathscr{N}P_- - 1$. Then

$$\begin{aligned} 0 &\leq \dim \left[\begin{pmatrix} 0 \\ RY_0(\bar{T})\mathscr{S}' \end{pmatrix} \cap \mathscr{R}P_+ \right] \\ &= \dim \begin{pmatrix} 0 \\ RY_0(\bar{T})\mathscr{S}' \end{pmatrix} + \dim \mathscr{R}P_+ - \dim \left[\begin{pmatrix} 0 \\ RY_0(\bar{T})\mathscr{S}' \end{pmatrix} + \mathscr{R}P_+ \right] \\ &\leq \dim \mathscr{N}P_- - 1 + \dim \mathscr{R}P_+ - \dim \left[\begin{pmatrix} 0 \\ RY_0(\bar{T})\mathscr{S}' \end{pmatrix} + \mathscr{R}P_+ \right] \\ &= n - 1 - \dim \left[\begin{pmatrix} 0 \\ RY_0(\bar{T})\mathscr{S}' \end{pmatrix} + \mathscr{R}P_+ \right] \end{aligned} \tag{6.2.4}$$

since $\dim \mathscr{R}P_+ + \dim \mathscr{N}P_- = n$. As a consequence,

$$\dim \left[\begin{pmatrix} 0 \\ RY_0(\bar{T})\mathscr{S}' \end{pmatrix} + \mathscr{R}P_+ \right] \leq n - 1.$$

Our next assumption is as follows:

(H4) $\begin{pmatrix} 0 \\ RY_0(\bar{T})\mathscr{S}' \end{pmatrix} + \mathscr{R}P_+$ has codimension 1 in $\mathbb{R}^n$.

It follows from (H4) that a unitary vector $\psi \in \mathbb{R}^n$ exists so that

$$\{\psi\}^\perp = \begin{pmatrix} 0 \\ RY_0(\bar{T})(\mathscr{S}') \end{pmatrix} + \mathscr{R}P_+.$$

Using this vector we define the function

$$\psi(t) = \begin{cases} X_-^{-1}(t)^* P_-^* Q^* P_y^* Y_0(\bar{T})^* R^* P_y \psi, & \text{for} t \leq -\bar{T}, \\ P_y^* Y_0^{-1}(t)^* Y_0(\bar{T})^* P_y \psi, & \\ -\dfrac{k_+(0, y_0(t)) + k_-(0, y_0(t))}{h_+(0, y_0(t)) - h_-(0, y_0(t))} P_z^* Y_0^{-1}(t)^* Y_0(\bar{T})^* P_y \psi, & \text{for } -\bar{T} < t \leq \bar{T}, \\ X_+^{-1}(t)^* (\mathbb{I} - P_+^*) \psi, & \text{for } t > \bar{T}. \end{cases}$$

Set

$$\mathscr{M}(\alpha) := \int_{-\infty}^{\infty} \psi^*(t) g(t + \alpha, \gamma(t), 0) dt.$$

Remark 6.2.4. (i) Since $\dot y_0(\bar T) = Y_0(\bar T)\dot y_0(-\bar T)$, we get $RY_0(\bar T)\dot y_0(-\bar T) = 0$. But (H4) and (6.2.4) imply that $\dim RY_0(\bar T)\mathscr{S}' = \dim \mathscr{N}P_- - 1 = \dim Y_0(\bar T)\mathscr{S}' = \dim \mathscr{S}'$. Then $RY_0(\bar T) : \mathscr{S}' \to RY_0(\bar T)\mathscr{S}'$ is an isomorphism and hence $\dot y_0(-\bar T) \notin \mathscr{S}'$. This means that $\dot\gamma_0(-\bar T)$ transversally crosses the unstable manifold W_0^u of $\dot x = f_-(x)$ at $\gamma_0(-\bar T)$. Consequently recalling also (6.2.4), assumption (H4) is a kind of nondegeneracy and transversality condition as well.
(ii) If $\dim \mathscr{N}P_- = n-1$ and $\dot y_0(-\bar T) \notin \mathscr{S}'$, then $\mathscr{R}P_+ = \operatorname{span}\{\dot\gamma_+(\bar T)\} = \operatorname{span}\{\dot\gamma_0(\bar T)\}$ and $RY_0(\bar T) : \mathscr{S}' \to RY_0(\bar T)\mathscr{S}'$ is 1 : 1. As a consequence, $\begin{pmatrix} 0 \\ RY_0(\bar T)\mathscr{S}' \end{pmatrix} \cap \mathscr{R}P_+ = \{0\}$ and all the inequalities in (6.2.4) are equalities. Consequently, if $\dim \mathscr{N}P_- = n-1$ then $\dot y_0(-\bar T) \notin \mathscr{S}'$ if and only if (H4) holds. Moreover, we get $\psi = e_1 = (1,0,...,0)$ and $P_y\psi = 0$. Hence

$$\psi(t) = \begin{cases} 0, & \text{for } t \le \bar T, \\ X_+^{-1}(t)^*(\mathbb{I} - P_+^*)\psi, & \text{for } t > \bar T \end{cases} \tag{6.2.5}$$

and

$$\mathscr{M}(\alpha) = \int_{\bar T}^{\infty} \psi^*(t) g(t+\alpha, \gamma(t), 0)\,dt. \tag{6.2.6}$$

Formula (6.2.6) corresponds to formula [27, (2.45)] for the planar case, that is, the Melnikov function contains only the $\gamma_+(t)$ part of $\gamma(t)$.

We recall that $g(t,x,\varepsilon)$ is quasiperiodic in t, if hypothesis (H5) of Section 6.1.8 holds. Now we can directly follow the method of Section 6.1 so we omit details and we refer the readers to [55]. Here we state the following result:

Theorem 6.2.5. *Assume that* (*H*1)–(*H*4) *and* (*H*5) *of Section 6.1.8 hold. If* $\mathscr{M}$ *has a simple zero* α_0*, i.e.* $\mathscr{M}(\alpha_0) = 0$ *and* $\mathscr{M}'(\alpha_0) \ne 0$*, then for any* $\varepsilon \ne 0$ *sufficiently small, there are sequences* $\{T_k^\varepsilon\}_{k\in\mathbb{Z}} \subset \mathbb{R}$, $\{\mathscr{S}_k^\varepsilon\}_{k\in\mathbb{Z}}$, $\{\Phi_k^\varepsilon\}_{k\in\mathbb{Z}}$ *so that*

(a) $\inf_{k\in\mathbb{Z}}(T_{k+1}^\varepsilon - T_k^\varepsilon) \to \infty$ *as* $\varepsilon \to 0$,
(b) $\mathscr{S}_k^\varepsilon \subset \mathbb{R}^n$ *are compact,*
(c) $\Phi_k^\varepsilon : \mathscr{E} \mapsto \mathscr{S}_k^\varepsilon$ *are homeomorphisms,*
(d) *Let* $F_k^\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ *be defined so that* $F_k^\varepsilon(\xi)$ *is the value at time* $T_{2(k+1)}^\varepsilon$ *of the solution of Eq.* (6.2.1) *so that* $z(T_{2k}^\varepsilon) = \xi$*. Then the following diagrams commute:*

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{\quad\sigma\quad} & \mathscr{E} \\ \Big\downarrow{\scriptstyle \Phi_k^\varepsilon} & & \Big\downarrow{\scriptstyle \Phi_{k+1}^\varepsilon} \\ \mathscr{S}_k^\varepsilon & \xrightarrow[\quad F_k^\varepsilon\quad]{} & \mathscr{S}_{k+1}^\varepsilon \end{array}$$

for all $k \in \mathbb{Z}$*. If, in addition,* $g(t,z,\varepsilon)$ *is p-periodic in t then* $F^\varepsilon = \varphi_\varepsilon^{r_\varepsilon} = \varphi_\varepsilon \circ \ldots \circ \varphi_\varepsilon = F_k^\varepsilon$ *(*r_ε *times) is the* r_ε*th iterate of the p-period map* φ_ε *of* (6.2.1) *for some large* $r_\varepsilon \in \mathbb{N}$, $\mathscr{S}^\varepsilon = \mathscr{S}_k^\varepsilon$ *and* $\Phi^\varepsilon = \Phi_k^\varepsilon$*, that is, in the periodic case the above diagram is independent of k.*

Here we recall Remark 6.1.24. Finally, Theorem 6.2.5 generalizes results of [43, 44, 46, 47] to the DDE (6.2.1) (cf Section 4.1).

6.2.2 Planar Sliding Homoclinics

First, we apply our theory to the planar discontinuous system

$$\begin{aligned} \dot{z} &= f_+(z) + \varepsilon g(z,t,\varepsilon) \quad \text{for} \quad y > 1\,, \\ \dot{z} &= f_-(z) + \varepsilon g(z,t,\varepsilon) \quad \text{for} \quad y < 1 \end{aligned} \tag{6.2.7}$$

where $z = (x,y) \in \mathbb{R}^2$, $f_\pm$, g are C^3-smooth and g is 1-periodic in t. Here we set

$$q_\pm(z,t,\varepsilon) = f_\pm(z) + \varepsilon g(z,t,\varepsilon)\,.$$

On $y = 1$ (cf (6.2.3)), we consider the system

$$\begin{aligned} \dot{x} = &\frac{q_{+2}(x,1,t,\varepsilon)}{q_{+2}(x,1,t,\varepsilon) - q_{-2}(x,1,t,\varepsilon)} q_{+1}(x,1,t,\varepsilon) \\ &+ \frac{q_{-2}(x,1,t,\varepsilon)}{q_{-2}(x,1,t,\varepsilon) - q_{+2}(x,1,t,\varepsilon)} q_{-1}(x,1,t,\varepsilon)\,, \end{aligned}$$

where $q_\pm = (q_{\pm 1}, q_{\pm 2})$. We suppose the following conditions hold:

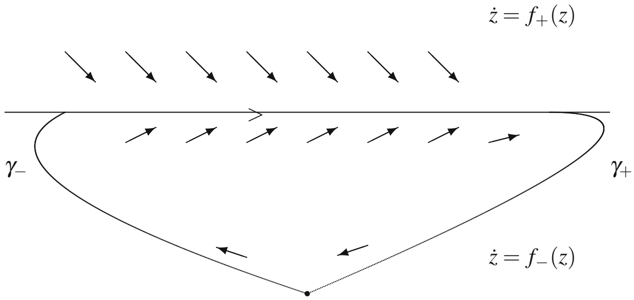


Fig. 6.4 A planar homoclinic sliding on the line of discontinuity.

(i) $f_-(0) = 0$ and $Df_-(0)$ has no eigenvalues on the imaginary axis.
(ii) There are two solutions $\gamma_-(s)$, $\gamma_+(s)$ of $\dot{z} = f_-(z)$, $y \leq 1$ defined on $\mathbb{R}_- = (-\infty, 0]$, $\mathbb{R}_+ = [0, +\infty)$, respectively, so that $\lim_{s\to\pm\infty} \gamma_\pm(s) = 0$ and $\gamma_\pm(s) = (x_\pm(s), y_\pm(s))$ with $y_\pm(0) = 1$, $x_-(0) < x_+(0)$. Moreover, $f_\pm(z) = (f_{\pm 1}(z), f_{\pm 2}(z))$ with $f_{\pm 1}(x,1) > 0$, $f_{+2}(x,1) < 0$ for $x_-(0) \leq x \leq x_+(0)$. Furthermore, $f_{-2}(x,1) > 0$ for $x_-(0) \leq x < x_+(0)$, $f_{-2}(x_+(0),1) = 0$ and $\partial_x f_{-2}(x_+(0),1) < 0$.

Assumptions (i) and (ii) mean that (6.2.7) for $\varepsilon = 0$ has a sliding homoclinic solution γ, created by $\gamma_\pm$, to a hyperbolic equilibrium 0 (Figure 6.4). Now we have a case of Remark 6.2.4-(ii), so we can use the formulas (6.2.5)–(6.2.6) to derive:

$$\mathscr{M}(\alpha) = \int_0^{+\infty} \psi(s)^* g(\gamma_+(s), \alpha + s, 0)\, ds \tag{6.2.8}$$

where $\psi(t)$ is a basis of a space of bounded solutions on $\mathbb{R}_+$ of the adjoint variational system $\dot{w} = -Df_-^*(\gamma_+(s))w$. By Theorem 6.2.5, we arrive at the following result.

Theorem 6.2.6. *If there is a simple root of* $\mathscr{M}$ *given by* (6.2.8)*, then* (6.2.7) *is chaotic with* $\varepsilon \neq 0$ *small.*

As a concrete example we consider

$$\begin{aligned} &\dot{y} = z, \quad \dot{z} = y - \frac{1}{2}y^3 + yz, && \text{for } z < \mathrm{e}^{-\frac{4\sqrt{3}\pi}{9}}, \\ &\dot{y} = z, \quad \dot{z} = y - \frac{1}{2}y^3 + (y-q)z && \text{for } z > \mathrm{e}^{-\frac{4\sqrt{3}\pi}{9}} \end{aligned} \tag{6.2.9}$$

that have a sliding homoclinic orbit to a saddle $(0,0)$ for any $q \geq 6.947$. Indeed, we start from (6.2.12) with $\beta = 1/2$. Note the phase portrait of (6.2.9) looks like Figure 6.4. Then we get $\tau = \sqrt{3}/2$ (cf (6.2.16)), $\Omega_\tau = \mathrm{e}^{-\frac{4\sqrt{3}\pi}{9}}$ (cf (6.2.18)) and $y_+(\bar{T}) = \sqrt{2 + 2\mathrm{e}^{-\frac{4\sqrt{3}\pi}{9}}}$ (cf (6.2.19)). The segment

$$\left\{ \left(y, \mathrm{e}^{-\frac{4\sqrt{3}\pi}{9}} \right) \in \mathbb{R}^2 \mid 0 \leq y \leq y_+(\bar{T}) \right\}$$

is attractive from above for (6.2.9), if

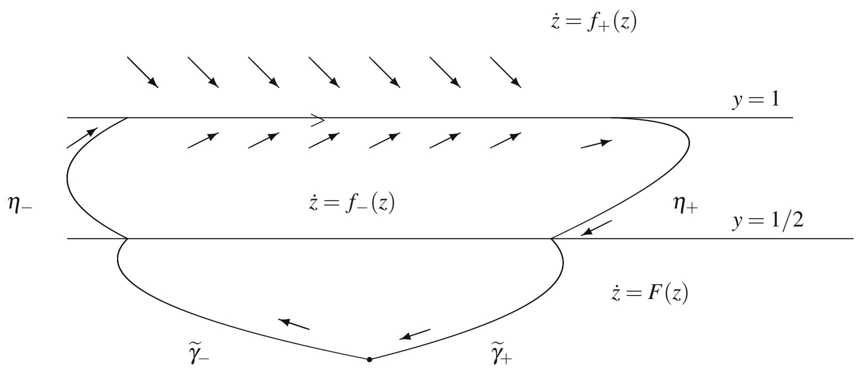


Fig. 6.5 A planar homoclinic sliding on the line of discontinuity with transversal crossing of another discontinuity line.

$$q > \max_{y\in[0,y_+(\bar{T})]} \frac{1}{\Omega_\tau}\left(y - \frac{1}{2}y^3 + \Omega_\tau y\right) = \frac{2\sqrt{6}}{9\Omega_\tau}(1+\Omega_\tau)^{3/2} \doteq 6.94609\,.$$

Hence we could take $q \geq 6.947$. Next we may also add its periodic perturbation

$$\begin{aligned} \dot{y} &= z, \quad \dot{z} = y - \frac{1}{2}y^3 + yz + \varepsilon\cos\omega t, \qquad &&\text{for } z < \mathrm{e}^{-\frac{4\sqrt{3}\pi}{9}}\,, \\ \dot{y} &= z, \quad \dot{z} = y - \frac{1}{2}y^3 + (y-q)z + \varepsilon\cos\omega t \qquad &&\text{for } z > \mathrm{e}^{-\frac{4\sqrt{3}\pi}{9}}\,. \end{aligned} \tag{6.2.10}$$

Then the Melnikov function is the same as in Section 6.2.3, and we could apply Theorem 6.2.8 with $F(1/2) \doteq 0.00228$ and $D(1/2) \doteq 25.3974$. Consequently, if either $0 < \omega < 0.0022$ or $\omega > 25.3975$ then (6.2.10) is chaotic.

The above approach to (6.2.7) can be generalized [28, 48] to cases when homoclinic orbit $\gamma(s)$ transversally crosses another curves of discontinuity. For simplicity, we suppose that such a discontinuity in (6.2.7) occurs at the level $y = 1/2$, i.e. we deal with the system

$$\begin{aligned} \dot{z} &= f_+(z) + \varepsilon g(z,t,\varepsilon), \quad &&\text{for } y > 1\,, \\ \dot{z} &= f_-(z) + \varepsilon g(z,t,\varepsilon), \quad &&\text{for } 1/2 < y < 1\,, \\ \dot{z} &= F(z) + \varepsilon g(z,t,\varepsilon), \quad &&\text{for } y < 1/2 \end{aligned} \tag{6.2.11}$$

where $z = (x,y) \in \mathbb{R}^2$, $f_\pm$, F, g are C^3-smooth and g is 1-periodic in t. We suppose the following conditions hold:

(a) $F(0) = 0$ and $DF(0)$ has no eigenvalues on the imaginary axis.
(b) There are two solutions η_-, η_+ of $\dot{z} = f_-(z)$, $1/2 \leq y \leq 1$ defined on $[a_-,0]$, $[0,a_+]$, $a_- < 0 < a_+$, respectively, so that $\eta_\pm(s) = (\widetilde{x}_\pm(s), \widetilde{y}_\pm(s))$ with $\widetilde{y}_\pm(0) = 1$, $\widetilde{y}_\pm(a_\pm) = 1/2$, $\widetilde{x}_-(0) < \widetilde{x}_+(0)$, $\widetilde{x}_-(a_-) < \widetilde{x}_+(a_+)$. Moreover, $f_\pm(z) = (f_{\pm 1}(z), f_{\pm 2}(z))$ with $f_{\pm 1}(x,1) > 0$, $f_{+2}(x,1) < 0$ for $\widetilde{x}_-(0) \leq x \leq \widetilde{x}_+(0)$. Furthermore, $f_{-2}(x,1) > 0$ for $\widetilde{x}_-(0) \leq x < \widetilde{x}_+(0)$, $f_{-2}(\widetilde{x}_+(0),1) = 0$ and $\partial_x f_{-2}(\widetilde{x}_+(0),1) < 0$. Finally, we suppose that $f_{-2}(\eta_-(a_-)) > 0$ and $f_{-2}(\eta_+(a_+)) < 0$.
(c) There are two solutions $\widetilde{\gamma}_-(s)$, $\widetilde{\gamma}_+(s)$ of $\dot{z} = F(z)$, $y \leq 1/2$ defined on $\mathbb{R}_- = (-\infty,0]$, $\mathbb{R}_+ = [0,+\infty)$, respectively, so that $\lim_{s\to\pm\infty} \widetilde{\gamma}_\pm(s) = 0$ and $\widetilde{\gamma}_\pm(0) = \eta_\pm(a_\pm)$. Moreover, $F(z) = (F_1(z), F_2(z))$ with $F_2(\widetilde{\gamma}_-(0)) > 0$ and $F_2(\widetilde{\gamma}_+(0)) < 0$.

Again, assumptions (a), (b) and (c) imply that (6.2.11) for $\varepsilon = 0$ has a sliding homoclinic solution $\widetilde{\gamma}$, created by $\eta_\pm$ and $\widetilde{\gamma}_\pm$, to a hyperbolic equilibrium 0 (Figure 6.5). We do not make further computations for (6.2.11), instead, we refer to [28, 48] for more details.

6.2.3 Three-Dimensional Sliding Homoclinics

This section is devoted to a construction of a concrete example (cf (6.2.20), (6.2.21), (6.2.23)) of (6.2.1) to which the above theory is applied. Then we proceed with a more particular perturbation (cf Theorems 6.2.8, 6.2.9). In order to construct our example, we start from [56]

$$\begin{aligned} \dot{z} &= y - \beta y^3 + yz, \\ \dot{y} &= z \end{aligned} \tag{6.2.12}$$

for $\beta > 1/8$. Then $(0,0)$ is hyperbolic and $(1/\sqrt{\beta},0)$ is an unstable focus. Since $(0,0)$ is hyperbolic it has one-dimensional stable and unstable manifolds. In the following we first show that these two manifolds have the structure depicted in Figure 6.6 where the stable manifold is tangent (and the unstable manifold is transverse) to the horizontal straight line. Performing the transformation $u = 1 - \beta y^2$, $y > 0$, $v = z$ we get

$$\begin{aligned} \dot{u} &= -2\beta v \frac{\sqrt{1-u}}{\sqrt{\beta}}, \quad z < 1, \\ \dot{v} &= (u+v)\frac{\sqrt{1-u}}{\sqrt{\beta}}. \end{aligned} \tag{6.2.13}$$

Note that $(0,0)$ corresponds to $(1,0)$ and $(1/\sqrt{\beta},0)$ to $(0,0)$. Let $' = \frac{d}{d\theta}$ and consider the linear system

$$\begin{aligned} u' &= -2\beta v, \\ v' &= u + v, \\ u(0) &= 1, \quad v(0) = 0 \end{aligned} \tag{6.2.14}$$

whose solution has the form

$$\begin{aligned} u_\tau(\theta) &= \mathrm{e}^{\theta/2}\cos(\tau\theta) - \frac{1}{2\tau}\mathrm{e}^{\theta/2}\sin(\tau\theta), \\ v_\tau(\theta) &= \frac{1}{\tau}\mathrm{e}^{\theta/2}\sin(\tau\theta) \end{aligned} \tag{6.2.15}$$

with

$$\tau = \frac{\sqrt{8\beta - 1}}{2}, \tag{6.2.16}$$

and so $\beta = \frac{4\tau^2+1}{8}$. Note that

$$u_\tau'(\theta) = -2\beta v_\tau(\theta) = -2\beta\frac{1}{\tau}\mathrm{e}^{\theta/2}\sin(\tau\theta)$$

has the opposite sign to $\sin(\tau\theta)$ thus $u_\tau(\theta) \le u_\tau(0) = 1$ for any $\theta \in \left(-\frac{\pi}{\tau}, \frac{\pi}{\tau}\right)$. On the other hand, if $\tau\theta \le -\pi$ we have

$$u_\tau(\theta) \le \mathrm{e}^{-\frac{\pi}{2\tau}}\frac{\sqrt{4\tau^2+1}}{2\tau} = \mathrm{e}^{-\frac{\pi}{2\tau}}\sqrt{1 + \left(\frac{1}{2\tau}\right)^2} < 1$$

since $e^{\pi s} > \sqrt{1+s^2}$ for any $s > 0$. As a consequence, $(u_\tau(\theta), v_\tau(\theta))$ is tangent to the line $u = 1$ from the left at $\theta = 0$ for $\theta \in (-\infty, \theta_\tau^+)$. Here $\theta_\tau^+ > 0$ is the least positive value so that $u_\tau(\theta) = 1$. Next, let θ_τ^- be the greatest negative value for which $v'_\tau(\theta) = 0$ and $v_\tau(\theta) > 0$. Then θ_τ^- solves the following system:

$$\cos(\tau\theta_\tau^-) + \frac{1}{2\tau}\sin(\tau\theta_\tau^-) = 0\,, \quad \sin(\tau\theta_\tau^-) > 0$$

so

$$\tau\theta_\tau^- = -\arctan 2\tau - \pi\,.$$

Given $\bar{T} > 0$ (we will fix it later) we consider the solution $\theta^-(t)$ of the equation:

$$\dot{\theta} = \sqrt{\frac{1-u_\tau(\theta)}{\beta}}, \quad \theta(\bar{T}) = \theta_\tau^-\,. \tag{6.2.17}$$

Separating variables we see that

$$\int_{\theta_\tau^-}^{\theta^-(t)} \frac{d\theta}{\sqrt{1-u_\tau(\theta)}} = \frac{t-\bar{T}}{\sqrt{\beta}}$$

or

$$\theta^-(t) = \Theta_-^{-1}\left(\frac{t-\bar{T}}{\sqrt{\beta}}\right), \quad \Theta_-(\theta) = \int_{\theta_\tau^-}^{\theta} \frac{d\theta}{\sqrt{1-u_\tau(\theta)}}\,.$$

From (6.2.14) we easily see that

$$1 - u_\tau(\theta) = \beta\theta^2 + o(\theta^2)$$

as $\theta \to 0$. As a consequence, $\Theta_-(\theta)$ is an increasing function that tends to $+\infty$ as $\theta \to 0$. Thus $\theta^-(t)$ is increasing and tends to 0 as $t \to \infty$. Moreover, since $u_\tau(\theta) < 1$ for $\theta < 0$ we also see that $\theta(t) \to -\infty$ as $t \to -\infty$. Summarizing $\theta^-(t)$ is an increasing function defined on $(-\infty, \infty)$, taking values on $(-\infty, 0)$, $\theta^-(\bar{T}) = \theta_\tau^-$. Setting

$$y_+(t) = \sqrt{\frac{1-u_\tau(\theta^-(t))}{\beta}}, \quad z_+(t) = v_\tau(\theta^-(t))$$

we see that $(z_+(t), y_+(t))$ is a solution of Eq. (6.2.12) so that

$$\lim_{t\to\infty}(z_+(t), y_+(t)) = \lim_{\theta\to 0}\left(v_\tau(\theta), \sqrt{\frac{1-u_\tau(\theta)}{\beta}}\right) = (0,0),$$

$$\lim_{t\to-\infty}(z_+(t), y_+(t)) = \lim_{\theta\to-\infty}\left(v_\tau(\theta), \sqrt{\frac{1-u_\tau(\theta)}{\beta}}\right) = \left(0, \sqrt{\frac{1}{\beta}}\right),$$

that is, $(z_+(t), y_+(t))$ is a heteroclinic connection from $\left(0, \sqrt{\frac{1}{\beta}}\right)$ to $(0,0)$. Next, we know that θ_τ^- is the greatest negative value so that $v'(\theta) = 0$ and $v(\theta) > 0$. This means that at $t = \bar{T}$ we have

$$z_+(\bar{T}) = v_\tau(\theta_\tau^-) := \Omega_\tau > 0, \quad \dot{z}_+(\bar{T}) = 0$$

and these two conditions are not satisfied when $t > \bar{T}$. Note that:

$$\Omega_\tau = \frac{1}{\tau} e^{\theta_\tau^-/2} \sin(\tau\theta_\tau^-) = 2 e^{\theta_\tau^-/2} \sqrt{\frac{1}{1+4\tau^2}} = e^{\theta_\tau^-/2} \sqrt{\frac{1}{2\beta}}, \tag{6.2.18}$$

moreover:

$$y_+(\bar{T}) = \sqrt{\frac{1-u_\tau(\theta_\tau^-)}{\beta}} = \sqrt{\frac{1+v_\tau(\theta_\tau^-)}{\beta}} = \sqrt{\frac{1+\Omega_\tau}{\beta}}. \tag{6.2.19}$$

Now we consider the solution $(z_-(t), y_-(t))$ of Eq. (6.2.12) that belongs to the unstable manifold of the saddle $(0,0)$. Since $(z_-(t), y_-(t)) \to (0,0)$ as $t \to -\infty$ it follows that we have to look for a solution $(u(t), v(t))$ of (6.2.13) so that $(u(t), v(t)) \to (1,0)$ as $t \to -\infty$. Thus we consider again Eq. (6.2.14) with $\theta \in (0, \theta_\tau^+)$. Thus $\theta = \theta^+(t)$ is again a solution of

$$\dot{\theta} = \sqrt{\frac{1-u_\tau(\theta)}{\beta}}$$

with the initial condition $\theta(0) = \theta_\tau^+$. So we obtain:

$$\int_{\theta_\tau^+}^{\theta^+(t)} \frac{d\theta}{\sqrt{1-u_\tau(\theta)}} = \frac{t}{\sqrt{\beta}}$$

that is

$$\theta^+(t) = \Theta_+^{-1}\left(\frac{t}{\sqrt{\beta}}\right), \quad \Theta_+(\theta) = \int_{\theta_\tau^+}^{\theta} \frac{d\theta}{\sqrt{1-u_\tau(\theta)}}.$$

Obviously $\Theta_+(\theta)$ is an increasing function and since $\theta \in (0, \theta_\tau^+)$, $\Theta_+(\theta) < 0$ for $0 \le \theta < \theta_\tau^+$. Arguing as before we see that $\lim_{\theta \to 0} \Theta_+(\theta) = -\infty$ and hence $\lim_{t \to -\infty} \theta^+(t) = 0$. For $t \in (-\infty, 0]$ (and hence $\theta^+(t) \in (0, \theta_\tau^+]$) we set:

$$y_-(t) = \sqrt{\frac{1-u_\tau(\theta^+(t))}{\beta}}, \quad z_-(t) = v_\tau(\theta^+(t))$$

and note that the following hold:

$$y_-(0) = 0, \quad z_-(0) = v_\tau(\theta_\tau^+),$$

$$\lim_{t\to-\infty}(z_-(t),y_-(t)) = \lim_{\theta\to 0}\left(v_\tau(\theta),\sqrt{\frac{1-u_\tau(\theta)}{\beta}}\right) = (0,0)\,.$$

Now, since $u_\tau(\theta) < 1$ for $\theta \in (0,\theta_\tau^+)$, we see that $u_\tau'(\theta_\tau^+) \geq 0$ and then $z_-(0) = v_\tau(\theta_\tau^+) \leq 0$. But it must be $z_-(0) = v_\tau(\theta_\tau^+) < 0$ otherwise $(z_-(0),y_-(0)) = (0,0)$ because of uniqueness. Next $(z_-(t),y_-(t))$ belongs to the unstable manifold of the equilibrium $(0,0)$ and $y_-(t) > 0$ for any $t \in (-\infty,0)$, thus

$$\frac{(\dot z_-(t),\dot y_-(t))}{\sqrt{\dot z_-(t)^2+\dot y_-(t)^2}} \to v_-$$

as $t \to -\infty$, with v_- being the eigenvector of the positive eigenvalue of the linearization of Equation (6.2.12) at $(0,0)$, i.e.

$$\begin{aligned}\dot z &= y\,,\\ \dot y &= z\end{aligned}$$

having a positive second component. Hence $z_-(t) = \dot y_-(t)$ is eventually positive for $t \to -\infty$. Thus the curve $(z_-(t),y_-(t))$ has to pass from the first quadrant to the fourth one and this can be realized only by passing above the line $z = z_+(\bar T)$ because otherwise it would intersect the curve $(z_+(t),y_+(t))$. As a consequence, $t_0 < 0$ must exist so that $z_-(t_0) = z_+(\bar T) = \Omega_\tau$ and $z_-(t) < \Omega_\tau$ for any $t < t_0$. We set

$$\bar y_\tau := \sqrt{\frac{1-u_\tau(\theta_\tau^+(t_0))}{\beta}}\,.$$

Shifting time we can suppose without loss of generality that $t_0 = -\bar T$. Thus we have found solutions $\tilde\gamma_\pm(t) = (z_\pm(t),y_\pm(t))$ of (6.2.12) so that

$$\begin{aligned}&\tilde\gamma_-(t) \to (0,0), &&\text{as } t \to -\infty,\\ &\tilde\gamma_+(t) \to (0,0), &&\text{as } t \to +\infty,\\ &z_-(-\bar T) = z_+(\bar T) = \Omega_\tau\ .\end{aligned}$$

The graphs of the above-mentioned invariant manifolds of (6.2.12) and the line $z = \Omega_\tau$ in the right half-plane for $\beta = 37/8$, i.e. $\tau = 3$, are given in Figure 6.6.

We are now able to construct our example. We take

$$\begin{aligned}\dot z &= y_1 - \beta y_1^3 + z y_1 + y_2^2,\\ \dot y_1 &= z,\\ \dot y_2 &= y_2(1+z)\end{aligned} \tag{6.2.20}$$

for $z < \Omega_\tau$ and

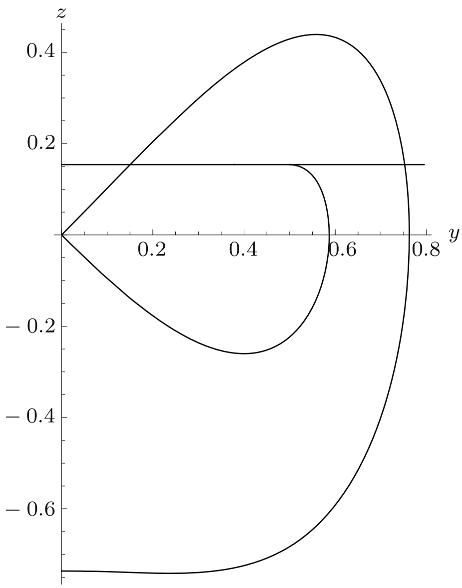


Fig. 6.6 The stable and unstable manifolds of system (6.2.12).

$$\begin{aligned} \dot{z} &= -z, \\ \dot{y}_1 &= 0, \\ \dot{y}_2 &= 0 \end{aligned} \tag{6.2.21}$$

when $z > \Omega_\tau$, that is, we take:

$$\begin{aligned} f_+(z,y_1,y_2) &= \begin{pmatrix} -z \\ 0 \\ 0 \end{pmatrix} \quad \text{for } z > \Omega_\tau, \\ f_-(z,y_1,y_2) &= \begin{pmatrix} y_1 - \beta y_1^3 + zy_1 + y_2^2 \\ z \\ y_2(1+z) \end{pmatrix} \quad \text{for } z < \Omega_\tau\,. \end{aligned} \tag{6.2.22}$$

Then

$$h_-(\Omega_\tau,y_1,y_2) = y_1 - \beta y_1^3 + \Omega_\tau y_1 + y_2^2, \quad h_+(\Omega_\tau,y_1,y_2) = -\Omega_\tau$$

and

$$H(y_1,y_2) = \frac{\Omega_\tau}{y_1 - \beta y_1^3 + \Omega_\tau(y_1+1) + y_2^2} \begin{pmatrix} \Omega_\tau \\ y_2(1+\Omega_\tau) \end{pmatrix}.$$

We note that

$$h_-(\Omega_\tau,y_1,y_2) - h_+(\Omega_\tau,y_1,y_2) = y_1(1 - \beta y_1^2 + \Omega_\tau) + y_2^2 + \Omega_\tau > 0$$

if $0 \le y_1 \le \sqrt{\frac{1+\Omega_\tau}{\beta}}$. Then we take the solution $\tilde{y}_1(t)$ of

$$\dot{y}_1 = \frac{\Omega_\tau^2}{y_1 - \beta y_1^3 + \Omega_\tau(y_1+1)}$$

so that $y_1(0) = \bar{y}_\tau$ and let $\tilde{T}$ be such that $y_1(\tilde{T}) = \sqrt{\frac{1+\Omega_\tau}{\beta}}$. Note that according to the previous remark, $h_-(\Omega_\tau, y_1, y_2) - h_+(\Omega_\tau, y_1, y_2) > 0$ in a neighborhood of $\tilde{y}_1(t)$, $0 \le t \le \tilde{T}$. Thus we are in position to apply Remark 6.2.2. Now we define $\bar{T} = \frac{\tilde{T}}{2}$ and set

$$\gamma_0(t) = (\Omega_\tau, \tilde{y}_1(t+\bar{T}), 0), \quad \gamma_-(t) = (\tilde{\gamma}_-(t), 0), \quad \gamma_+(t) = (\tilde{\gamma}_+(t), 0)$$

and

$$\gamma(t) = \begin{cases} \gamma_-(t), & \text{if } t \le -\bar{T}, \\ \gamma_0(t), & \text{if } -\bar{T} \le t \le \bar{T}, \\ \gamma_+(t), & \text{if } t \ge \bar{T} \end{cases}$$

is a sliding homoclinic orbit for the system (6.2.20), (6.2.21).

For concrete values of $\tau > 0$, we take $\beta = \frac{1}{8} + \frac{\tau^2}{2}$, compute Ω_τ and we solve (6.2.12) with initial values $z_s(\bar{T}) = \sqrt{\frac{1+\Omega_\tau}{\beta}}$, $y_s(\bar{T}) = \Omega_\tau$ to get $\widetilde{\gamma}_+(t)$ and $\gamma_+(t)$.

We now verify that system (6.2.20), (6.2.21) and $\gamma(t)$ satisfy conditions (H1)–(H4) of this section. We have already seen that (H1) is satisfied (see also Remark 6.2.2). Condition (H2) is also satisfied with $x_0 = (z^0, y_1^0, y_2^0) = (0,0,0)$. Note that in this example the discontinuity level is at $z = \Omega_\tau$ and not at $z = 0$ but we have observed that this fact does not make any difference. Now we verify (H3). It is trivial to verify that $h_+(\gamma(t)) < 0$ for $-\bar{T} \le t \le \bar{T}$, $h_-(\gamma(t)) > 0$ for $-\bar{T} \le t < \bar{T}$ and $h_-(\gamma(\bar{T})) = y_+(\bar{T})(1 - \beta y_+(\bar{T})^2 + \Omega_\tau) = 0$. So we check the last condition in (H3). We have:

$$\nabla_y h_-(\gamma_0(\bar{T})) = -2\begin{pmatrix} 1+\Omega_\tau \\ 0 \end{pmatrix} \quad \text{and} \quad k_-(\gamma_0(\bar{T})) = \begin{pmatrix} \Omega_\tau \\ 0 \end{pmatrix}$$

from which we obtain

$$\nabla_y h_-(\gamma_0(\bar{T}) k_-(\gamma_0(\bar{T}))^* = -2\Omega_\tau(1+\Omega_\tau) \neq 0\,.$$

Finally, we check (H4). By Remark 6.2.4 it is enough to prove that $(\dot{y}_1(-\bar{T}), 0)^* \notin \mathscr{S}'$ or, equivalently, that $(1,0)^* \notin \mathscr{S}'$. Now, the variational system of (6.2.20) along $\gamma_-(t)$ is given by:

$$\begin{aligned} \dot{z} &= y_-(t)z + (1 - 3\beta y_-(t)^2 + z_-(t))y_1, \\ \dot{y}_1 &= z, \\ \dot{y}_2 &= (1 + z_-(t))y_2\,. \end{aligned}$$

Since this system has the bounded solution at $-\infty$: $\left(0,0,e^{t+y_-(t)}\right)$, and $\dim \mathscr{S}' = 1$ it follows that $\mathscr{S}' = \text{span}\{(0,1)\}$ and hence $(1,0) \notin \mathscr{S}'$. Thus (H4) holds.

Finally we add a perturbation

$$\varepsilon g(t) = \varepsilon \begin{pmatrix} q(\omega t) \\ 0 \\ q_1(\omega_1 t) \end{pmatrix} \tag{6.2.23}$$

to (6.2.20), (6.2.21) and compute the Melnikov function. Here ω, ω_1 are positive constants and q_1, q_2 are almost periodic C^2-functions with bounded derivatives and their second order derivatives are uniformly continuous. To this end, we need to compute the solution $\psi(t)$ of the adjoint variational system:

$$\begin{aligned} \dot{z} &= -y_+(t)z - y_1 \\ \dot{y}_1 &= -(1 - 3\beta y_+(t)^2 + z_+(t))z \\ \dot{y}_2 &= -(1 + z_+(t))y_2 \end{aligned} \tag{6.2.24}$$

with $\psi(0) = (1,0,0)$ (see (6.2.5)). Since $y_2 = 0$ is invariant for system (6.2.24) we get $\psi(t) = (\psi_1(t), \psi_2(t), 0)$ where $(z,y) = (\psi_1(t), \psi_2(t))$ is a bounded (at $+\infty$) solution of

$$\begin{aligned} \dot{z} &= -y_+(t)z - y_1, \\ \dot{y}_1 &= -(1 - 3\beta y_+(t)^2 + z_+(t))z \end{aligned} \tag{6.2.25}$$

that is

$$\psi(t) = \begin{pmatrix} \dot{y}_+(t) \\ -\dot{z}_+(t) \\ 0 \end{pmatrix} e^{-\int_{\bar{T}}^t y_+(s)ds}$$

and the Melnikov function is

$$\mathscr{M}(\alpha) = \int_{\bar{T}}^{\infty} \dot{y}_+(t)\, e^{-\int_{\bar{T}}^t y_+(s)ds} q(\omega t + \alpha)dt\,.$$

Since

$$\begin{aligned} \lim_{\omega \to 0} \mathscr{M}(\alpha) &= q(\alpha) \int_{\bar{T}}^{\infty} \dot{y}_+(t)\, e^{-\int_{\bar{T}}^t y_+(s)ds} dt, \\ \lim_{\omega \to 0} \mathscr{M}'(\alpha) &= q'(\alpha) \int_{\bar{T}}^{\infty} \dot{y}_+(t)\, e^{-\int_{\bar{T}}^t y_+(s)ds} dt, \end{aligned}$$

we see that if $q(\alpha)$ has a simple zero at some $\alpha = \alpha_0$ and

$$\int_{\bar{T}}^{\infty} \dot{y}_+(t)\, e^{-\int_{\bar{T}}^t y_+(s)ds} dt \neq 0, \tag{6.2.26}$$

then $\mathscr{M}(\alpha)$ will have a simple zero at some α near to α_0 for $\omega > 0$ small. To check condition (6.2.26) we recall that

$$y_+(t) = \sqrt{\frac{1-u_\tau(\theta^-(t))}{\beta}} = \dot{\theta}^-(t)$$

so,

$$\int_{\bar{T}}^{t} y_+(s)ds = \theta^-(t) - \theta^-(\bar{T}) = \theta^-(t) - \theta_\tau^-. \tag{6.2.27}$$

Now, let $Y(\theta) = \sqrt{\frac{1-u_\tau(\theta)}{\beta}}$. Then:

$$y_+(t) = Y(\theta^-(t))$$

and

$$\dot{y}_+(t) = Y'(\theta^-(t))\dot{\theta}^-(t). \tag{6.2.28}$$

Plugging (6.2.27), (6.2.28) into (6.2.26) we obtain:

$$\begin{aligned}
&\int_{\bar{T}}^{\infty} \dot{y}_+(t)\,\mathrm{e}^{-\int_{\bar{T}}^{t} y_+(s)ds}\,dt = \mathrm{e}^{\theta_\tau^-}\int_{\bar{T}}^{\infty} \mathrm{e}^{-\theta^-(t)}\,Y'(\theta^-(t))\dot{\theta}^-(t)dt \\
&= \mathrm{e}^{\theta_\tau^-}\int_{\theta_\tau^-}^{0} \mathrm{e}^{-\theta}\,Y'(\theta)d\theta = \mathrm{e}^{\theta_\tau^-}\left[Y(\theta)\,\mathrm{e}^{-\theta}\Big|_{\theta_\tau^-}^{0} + \int_{\theta_\tau^-}^{0} \mathrm{e}^{-\theta}\,Y(\theta)d\theta\right] \\
&= \mathrm{e}^{\theta_\tau^-}\int_{\theta_\tau^-}^{0} \mathrm{e}^{-\theta}\,Y(\theta)d\theta - Y(\theta_\tau^-) \qquad (6.2.29)\\
&= \int_{\theta_\tau^-}^{0} \mathrm{e}^{\theta_\tau^- -\theta}\,Y(\theta)d\theta - \sqrt{\frac{1+\Omega_\tau}{\beta}} \\
&= \frac{1}{\sqrt{\beta}}\left(\int_{\theta_\tau^-}^{0} \mathrm{e}^{\theta_\tau^- -\theta}\,\sqrt{1-u_\tau(\theta)}d\theta - \sqrt{1+\Omega_\tau}\right).
\end{aligned}$$

We prove now that the expression (6.2.29) is negative for any $\tau > 0$. Using Cauchy-Schwarz-Bunyakovsky inequality we get

$$\int_{\theta_\tau^-}^{0} \mathrm{e}^{\theta_\tau^- -\theta}\,\sqrt{1-u_\tau(\theta)}d\theta \le \sqrt{\int_{\theta_\tau^-}^{0} \mathrm{e}^{\theta_\tau^- -\theta}\,d\theta}\sqrt{\int_{\theta_\tau^-}^{0} \mathrm{e}^{\theta_\tau^- -\theta}(1-u_\tau(\theta))d\theta}.$$

Next, we integrate

$$\int_{\theta_\tau^-}^{0} \mathrm{e}^{\theta_\tau^- -\theta}\,d\theta = 1 - \mathrm{e}^{\theta_\tau^-}$$

and

$$\begin{aligned}
\int_{\theta_\tau^-}^{0} \mathrm{e}^{\theta_\tau^- -\theta}(1-u_\tau(\theta))d\theta &= \int_{\theta_\tau^-}^{0} \mathrm{e}^{\theta_\tau^- -\theta}\left(1 - \mathrm{e}^{\theta/2}\cos(\tau\theta) + \frac{1}{2\tau}\,\mathrm{e}^{\theta/2}\sin(\tau\theta)\right)d\theta \\
&= 1 - \mathrm{e}^{\theta_\tau^-} + 2\,\mathrm{e}^{\theta_\tau^-/2}\sqrt{\frac{1}{1+4\tau^2}} = 1 - \mathrm{e}^{\theta_\tau^-} + \Omega_\tau.
\end{aligned}$$

Consequently:

$$\int_{\theta_\tau^-}^{0} \mathrm{e}^{\theta_\tau^- -\theta} \sqrt{1-u_\tau(\theta)}d\theta \le \sqrt{1-\mathrm{e}^{\theta_\tau^-}}\sqrt{1-\mathrm{e}^{\theta_\tau^-}+\Omega_\tau} < \sqrt{1+\Omega_\tau},$$

hence the expression (6.2.29) is negative for any value of $\tau > 0$. In summary, we obtain the following result.

Theorem 6.2.7. *Let $q(t)$ have a simple zero. Then there exist $\omega_0 > 0$ and $\varepsilon_0 > 0$ so that for $0 < |\omega| < \omega_0$ and $0 < |\varepsilon| < \varepsilon_0$, system*

$$\dot{x} = f_\pm(x) + \varepsilon g(t), \quad x \in \Omega_\pm \tag{6.2.30}$$

where $x = (z, y_1, y_2) \in \mathbb{R}^3$, $f_\pm(x)$ is as in (6.2.22) and $g(t)$ as in (6.2.23), is chaotic.

For example, if $q(t) = \cos t$ we get

$$\mathscr{M}(\alpha) = \int_{\bar{T}}^{\infty} \dot{y}_+(t)\,\mathrm{e}^{-\int_{\bar{T}}^{t} y_+(s)ds}\cos(\omega t+\alpha)dt$$

and then

$$\mathscr{M}(\alpha) - \imath\mathscr{M}'(\alpha) = \mathrm{e}^{\imath\alpha}\Psi_\tau(\omega), \quad \Psi_\tau(\omega) := \int_{\bar{T}}^{\infty} \dot{y}_+(t)\,\mathrm{e}^{-\int_{\bar{T}}^{t} y_+(s)ds}\,\mathrm{e}^{\imath\omega t}\,dt\,.$$

As a consequence if $\Psi_\tau(\omega) \neq 0$ then $\mathscr{M}(\alpha)$ has a simple zero. Since $\Psi_\tau(0) \neq 0$, $\Psi_\tau(\omega)$ is a nonzero analytical function. From Theorem 6.2.7 we know that (6.2.30) behaves chaotically for $|\omega| < \omega_0$ (and $|\varepsilon| < \varepsilon_0$) sufficiently small. However, for this particular example ($q(t) = \cos t$), (6.2.30) behaves chaotically also when ω is large. As a matter of fact, we have the following:

Theorem 6.2.8. *There exist continuous functions $F(\beta), D(\beta) : \left(\frac{1}{8}, \infty\right) \to (0, \infty)$ so that for any given constants $\beta > 1/8$, $\omega_1 > 0$, $\omega \in (0,\infty) \setminus [F(\beta), D(\beta)]$ and an almost periodic C^2–function $q_1(t)$ with bounded derivatives so that its second order derivative is uniformly continuous, there exists $\varepsilon_0 = \varepsilon_0(\beta, \omega, \omega_1, q_1(\cdot))$ so that for $0 < |\varepsilon| < \varepsilon_0$ and*

$$g(t) = \begin{pmatrix} \cos(\omega t) \\ 0 \\ q_1(\omega_1 t) \end{pmatrix}$$

system (6.2.20), (6.2.21) is chaotic. Moreover, it holds

$$\lim_{\tau \to 1/8_+} F(\beta) = 0, \quad \lim_{\beta\to\infty} F(\beta) = \frac{2\sqrt{2}}{\pi(2\sqrt{2}+1)} \doteq 0.235166\,,$$

$$\lim_{\beta \to 1/8_+} D(\beta) = \infty, \quad \lim_{\beta\to\infty} D(\beta) = \frac{3\sqrt{2}\pi}{2} + 4 - \sqrt{2} \doteq 9.25011\,.$$

Proof. We omit the proof of this theorem, since it is rather technical and refer the readers to [55] for more details. □

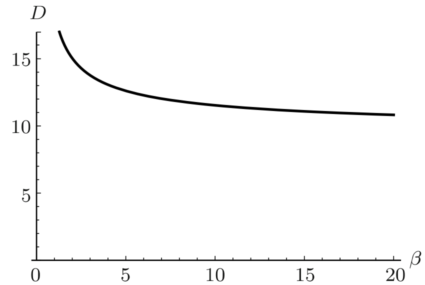
Here we only mention that

$$D(\beta) := B\left(\sqrt{8\beta - 1}/2\right),$$

where

$$B(\tau) := \sqrt{\frac{8(1+\tau^{-1}\mathrm{e}^{-\frac{\pi}{2\tau}})}{4\tau^2+1} + \frac{\sqrt{4+\tau^{-2}}}{2}\left(\frac{3\sqrt{2}\pi}{8}(\tau^{-2}+4\tau^{-1}+4)+(2+\tau^{-1})\left(2-\sqrt{1/2}\right)\right)}$$

and the graph of $D(\beta)$ in interval $(1/8, 20]$ looks like



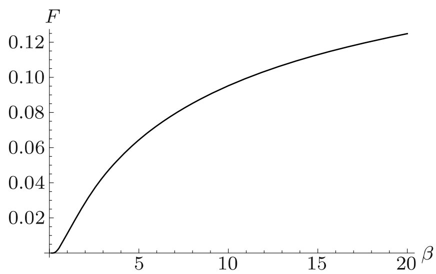
Furthermore, we have

$$F(\beta) := C\left(\sqrt{8\beta - 1}/2\right)$$

where

$$C(\tau) := \frac{2\sqrt{2}\tau}{\sqrt{4\tau^2+1}} \frac{\sqrt{1+\Omega_\tau} - \sqrt{1-\mathrm{e}^{\theta_\tau^-}}\sqrt{1-\mathrm{e}^{\theta_\tau^-}+\Omega_\tau}}{2\arctan 2\tau \sqrt{\dfrac{1+\mathrm{e}^{-\frac{\pi}{2\tau}}}{1+\Omega_\tau}} + \pi - \arctan 2\tau}$$

and the graph of $F(\beta)$ in interval $(1/8, 20]$ looks like



For instance, a numerical evaluation shows that for $\beta = 25$: $D(25) \doteq 0.1337$ and $F(25) \doteq 10.6489$, so for $\omega \in (0,\infty) \setminus [0.13, 10.65]$, system (6.2.20), (6.2.21) is chaotic for $\varepsilon \neq 0$.

Furthermore, since $\Psi_\tau(\omega)$ is analytical (cf Section 2.6.5), there is at most a finite number of $\omega_1, \dots, \omega_{n_\beta} \in [F(\beta), D(\beta)]$ so that for any $\omega > 0$ and $\omega \notin \{\omega_1, \dots, \omega_{n_\beta}\}$, there is a chaos like in Theorem 6.2.8. An open problem remains to estimate n_β. On the other hand, the statement of Theorem 6.2.8 can be extended as follows.

Theorem 6.2.9. *There exists a continuous function $G(\omega) : (0,\infty) \to \left[\frac{1}{8}, \infty\right)$ so that for any given constants $\omega \in (0,\infty)$, $\beta > G(\omega)$, $\omega_1 > 0$ and an almost periodic C^2–function $q_1(t)$ with bounded derivatives so that its second order derivative is uniformly continuous, there exists $\varepsilon_0 = \varepsilon_0(\beta, \omega, \omega_1, q_1(\cdot))$ so that for $0 < |\varepsilon| < \varepsilon_0$ and*

$$g(t) = \begin{pmatrix} \cos(\omega t) \\ 0 \\ q_1(\omega_1 t) \end{pmatrix},$$

system (6.2.20), (6.2.21) *is chaotic.*

We again refer the readers to [55] for more details. A lower bound $G(\omega)$ for β could be numerically estimated, but we do not carry out these awkward computations in this section. By Theorem 6.2.8, it would be enough to estimate $G(\omega)$ in the interval $[0.2, 9.3]$.

6.3 Outlook

The above results could be extended to other types of discontinuous homoclinics. First we could study impact systems like in [17, 18, 20, 21, 49]. Second we could develop Melnikov theory for grazing homoclinics which has not yet been done. Discontinuous systems with grazing orbits are investigated in [6, 22, 24, 57, 58].

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Chapter 7
Concluding Related Topics

The final chapter of this book deals with three topics. In the first part, we study thoroughly the Melnikov function: its computation and structure. We also investigate an inverse problem: the construction of ODEs by prescribed homoclinic solutions. In the second part, as a by product of our investigation, is presented a result of the existence of a transversal homoclinic solution near a transversal homoclinic cycle. We end up with the third part devoted to blue sky catastrophes of periodic solutions.

7.1 Notes on Melnikov Function

7.1.1 Role of Melnikov Function

We know from Chapter 4 that the Melnikov method is an easy and effective method to detect chaotic dynamics in differential equations. In this section, we study the simplest case: The starting point is an autonomous system $\dot{x} = f(x)$, where x belongs to an open subset $\Omega \subset \mathbb{R}^n$, having a hyperbolic equilibrium x_0 and a *nondegenerate* homoclinic orbit $\phi(t)$, that is a non constant solution $\phi(t)$ so that $\lim_{t\to\pm\infty} \phi(t) = x_0$ and $\dot{\phi}(t)$ spans the space of bounded solutions of the variational system

$$\dot{x} = f'(\phi(t))x. \tag{7.1.1}$$

Then, associated with a given time periodic sufficiently smooth perturbation $\varepsilon h(t,x,\varepsilon)$, with ε sufficiently small, there is the Melnikov function (cf Section 4.1):

$$M(\alpha) := \int_{-\infty}^{+\infty} \psi^*(t)h(t+\alpha,\phi(t),0)\mathrm{d}t$$

with $\psi(t)$ being the unique (up to a multiplicative constant) bounded solution of the variational system

$$\dot{x} = -f'(\phi(t))^*x.$$

Note that $M(\alpha)$ is a periodic function having the same period as $h(t,x,\varepsilon)$. The basic result states that $M(\alpha)$ gives a kind of $O(\varepsilon)$-measure of the distance between the stable and unstable manifolds of the (unique) hyperbolic periodic solution $x_0(t,\varepsilon)$ of the perturbed system

$$\dot{x} = f(x) + \varepsilon h(t+\alpha, x, \varepsilon) \tag{7.1.2}$$

which is at an $O(\varepsilon)$-distance from x_0 [1]. Thus if $M(\alpha)$ has a simple zero at some points, then these two manifolds intersect transversally along a solution $\phi(t,\varepsilon)$ of (7.1.2) which is homoclinic to $x_0(t,\varepsilon)$. This transversality implies, by the classical Smale horseshoe construction, that a suitable iterate of the Poincarè map of the perturbed system exhibits chaotic behavior (cf Section 2.5).

In this section, we mainly consider the case where $h(t,x,0) = q(t)$ is a T-periodic perturbation independent of x, although in the next part some results are derived for the more general case. We also assume that $q(t)$ is C^1. Our first remark is that the Melnikov function is a bounded linear map from the space of T-periodic functions to itself, as it can be easily checked using the fact that $|\psi(t)| \leq C\mathrm{e}^{-\sigma|t|}$, for some positive real numbers C, σ. Moreover the average $\bar{M}$ of $M(\alpha)$ is:

$$\bar{M} := \frac{1}{2\pi}\int_0^{2\pi} M(\alpha)\, d\alpha = \int_{-\infty}^{+\infty} \psi^*(t)\mathrm{d}t \cdot \bar{q}.$$

Now, in many interesting cases, for example, when one deals with a second order conservative equation on $\mathbb{R}$, one has

$$\int_{-\infty}^{+\infty} \psi^*(t)\mathrm{d}t = 0$$

so that $\bar{M} = 0$. In this case Melnikov functions can be either zero or there are α_1 and α_2 so that $M(\alpha_1) < 0 < M(\alpha_2)$. This means that the Brouwer degree of $M(\alpha)$ in the interval $[\alpha_1, \alpha_2]$ is different from zero. This, in turns, implies a chaotic behaviour of some iterate of the Poincarè map (see Remark 3.1.9 and [2, 3]). This seems to be a good reason to study the kernel of the Melnikov map:

$$q(t) \mapsto \int_{-\infty}^{+\infty} \psi^*(t) q(t+\alpha)\mathrm{d}t.$$

This is the purpose of this section. Melnikov functions for two-dimensional mappings are investigated in [4, 5].

7.1.2 Melnikov Function and Calculus of Residues

We assume that

(a) $\phi(t) = \Phi(\mathrm{e}^t)$, where $\Phi(u)$ is a rational function on $\mathbb{C}$ so that $\Phi(u) \to 0$, and $\Phi(1/u) \to 0$ as $u \to 0$;

(b) $\psi(t) = \mathrm{e}^t\,\Psi(\mathrm{e}^t)$, where $\Psi(u)u \to 0$ as $|u| \to +\infty$, and $\Psi(u)$ is a rational function on $\mathbb{C}$.

From $h(t,x,\varepsilon) = h(t+T,x,\varepsilon)$ we deduce that $M(\alpha)$ is T-periodic. Let $\chi_{[-T/2,T/2]}$ be the characteristic function of the interval $[-T/2,T/2]$. Set $\omega = \frac{2\pi}{T}$ and $M_0(\alpha) = M(\alpha)\chi_{[-T/2,T/2]}(\alpha)$, $h_0(t,x) = h(t,x,0)\chi_{[-T/2,T/2]}(t)$, and for any $n \in \mathbb{Z}$, consider

$$\begin{aligned}
\hat{M}_0(n) &= \frac{1}{T}\int_{-T/2}^{T/2} M_0(\alpha)\,\mathrm{e}^{-\imath n\omega\alpha}\,\mathrm{d}\alpha \\
&= \frac{1}{T}\int_{-T/2}^{T/2}\int_{-\infty}^{+\infty} \psi^*(t)h(t+\alpha,\phi(t),0)\,\mathrm{e}^{-\imath n\omega\alpha}\,\mathrm{d}t\mathrm{d}\alpha \\
&= \int_{-\infty}^{+\infty} \psi^*(t)\frac{1}{T}\int_{-T/2}^{T/2} h(t+\alpha,\phi(t),0)\,\mathrm{e}^{-\imath n\omega\alpha}\,\mathrm{d}\alpha\mathrm{d}t \\
&= \int_{-\infty}^{+\infty} \mathrm{e}^t\,\Psi^*(\mathrm{e}^t)\hat{h}(n,\Phi(\mathrm{e}^t))\,\mathrm{e}^{\imath n\omega t}\,\mathrm{d}t\,,
\end{aligned}$$

where

$$\hat{h}(n,x) := \frac{1}{T}\int_{-\infty}^{+\infty} h_0(t,x)\,\mathrm{e}^{-\imath n\omega t}\,\mathrm{d}t = \frac{1}{T}\int_{-T/2}^{T/2} h(t,x,0)\,\mathrm{e}^{-\imath n\omega t}\,\mathrm{d}t \tag{7.1.3}$$

is the n-th Fourier coefficient of $h_0(t,x)$. We assume that

(c) For any $n \in \mathbb{Z}$ the function $\hat{h}(n,\Phi(x))$ extends to a meromorphic function $\hat{h}(n,\Phi(u))$ on $\mathbb{C}$ having the same poles as $\Phi(u)$.

Thus

$$F(n,u) := \Psi^*(u)\hat{h}(n,\Phi(u))$$

is meromorphic in $\mathbb{C}$, for any fixed $n \in \mathbb{Z}$, and its poles are either those of $\Psi(u)$ or those of $\Phi(u)$. Let us make some comments about the function $F(n,u)$. As $\Psi(u)$ and $\Phi(u)$ take real values when $u \in \mathbb{R}_+$, Schwarz reflection principle (cf Theorem 2.6.10) gives:

$$\overline{\Psi(\bar{u})} = \Psi(u) \quad \text{and} \quad \overline{\Phi(\bar{u})} = \Phi(u). \tag{7.1.4}$$

Second, we notice that being $\overline{\hat{h}(n,x)} = \hat{h}(-n,x)$ for any $x \in \Omega \subset \mathbb{R}^n$ we obtain

$$\overline{\hat{h}(n,\Phi(\bar{u}))} = \hat{h}(-n,\Phi(u))$$

because of the uniqueness of the analytical extension and hence

$$\overline{F(n,\bar{u})} = F(-n,u). \tag{7.1.5}$$

From (7.1.4) it follows that $\Psi(u)$ and $\Phi(u)$ have complex conjugate poles and hence the same holds for $F(n,u)$. Let $w_j = u_j \pm \imath v_j$, $j = 1,\dots,r$ be the poles of $F(n,u)$ (independent of $n \in \mathbb{Z}$). Note that the w_j do not belong to an angular sector around the positive real half–line, otherwise $\Psi(\mathrm{e}^t)$ will have singularities on the real line. Thus the poles of $F(n,\mathrm{e}^z)$ are given by

$$z = \operatorname{Log} w_j := \log |w_j| + \imath \operatorname{Arg} w_j$$

where $\operatorname{Arg} w \in (\beta, 2\pi - \beta)$ for some $\beta > 0$ and $\operatorname{Log} w_j$ is the *logarithm principal value*. We remark that $\operatorname{Arg} \bar{w}_j = 2\pi - \operatorname{Arg} w_j$. Finally, for any $u \in \mathbb{C} \setminus \{0\}$ so that $0 \leq \operatorname{Arg} u < 2\pi$, we set

$$u^{\imath \omega n} := \mathrm{e}^{\imath n \omega \operatorname{Log} u}.$$

Then we integrate the meromorphic function $\mathrm{e}^z F(n, \mathrm{e}^z)\, \mathrm{e}^{\imath n \omega z}$ on the boundary of the rectangle $\{-\rho \leq \Re z \leq \rho, 0 \leq \Im z \leq 2\pi\}$. For ρ sufficiently large Cauchy residue theorem implies:

$$\begin{aligned}
&2\pi\imath \sum_j \operatorname{Res}\left(\mathrm{e}^z F(n,\mathrm{e}^z)\, \mathrm{e}^{\imath n\omega z}, \operatorname{Log} w_j\right) \\
&= \int_{-\rho}^{\rho} \mathrm{e}^t F(n,\mathrm{e}^t)\, \mathrm{e}^{\imath n\omega t}\, \mathrm{d}t \\
&\quad - \int_{-\rho}^{\rho} \mathrm{e}^t F(n,\mathrm{e}^t)\, \mathrm{e}^{\imath n\omega t}\, \mathrm{e}^{-2\pi n\omega}\, \mathrm{d}t + \int_0^{2\pi} \mathrm{e}^{\rho}\, \mathrm{e}^{\imath y} F(n, \mathrm{e}^{\rho}\, \mathrm{e}^{\imath y})\, \mathrm{e}^{\imath n\omega\rho}\, \mathrm{e}^{-n\omega y}\, \imath \mathrm{d}y \\
&\quad - \int_0^{2\pi} \mathrm{e}^{-\rho}\, \mathrm{e}^{\imath y} F(n, \mathrm{e}^{-\rho}\, \mathrm{e}^{\imath y})\, \mathrm{e}^{-\imath n\omega\rho}\, \mathrm{e}^{-n\omega y}\, \imath \mathrm{d}y.
\end{aligned}$$

Now, the last two integrals on the right tend to zero as $\rho \to +\infty$ uniformly with respect to n. Hence, for any $n \neq 0$ we get (cf (2.6.1))

$$\begin{aligned}
\int_{-\infty}^{+\infty} \mathrm{e}^t F(n,\mathrm{e}^t)\, \mathrm{e}^{\imath n\omega t}\, \mathrm{d}t &= \frac{2\pi\imath}{1 - \mathrm{e}^{-2\pi n\omega}} \sum_j \operatorname{Res}\left(\mathrm{e}^z F(n,\mathrm{e}^z)\, \mathrm{e}^{\imath n\omega z}, \operatorname{Log} w_j\right) \\
&= 2\pi\imath \sum_j \operatorname{Res}\left(\frac{F(n,u) u^{\imath n\omega}}{1 - \mathrm{e}^{-2\pi n\omega}}, w_j\right).
\end{aligned}$$

Thus we have proved the following:

Theorem 7.1.1. *Under the conditions (a)–(c) the Fourier coefficients of the Melnikov function $M(\alpha)$ are given by:*

$$\hat{M}_0(n) = 2\pi\imath \sum_j \operatorname{Res}\left(\frac{F(n,u) u^{\imath n\omega}}{1 - \mathrm{e}^{-2\pi n\omega}}, w_j\right) \tag{7.1.6}$$

for $n \neq 0$, while

$$\hat{M}_0(0) = \frac{1}{T} \int_{-T/2}^{T/2} M_0(\alpha) \mathrm{d}\alpha = \int_{-\infty}^{+\infty} \psi^*(t) \hat{h}_0(0, \phi(t)) \mathrm{d}t$$

where $\hat{h}_0(n,x)$ has been defined in (7.1.3).

Using (7.1.5) we obtain:

$$\overline{\hat{M}_0(n)} = \overline{\int_{-\infty}^{+\infty} \mathrm{e}^t\, F(n, \mathrm{e}^t)\, \mathrm{e}^{\imath n\omega t}\, \mathrm{d}t} = \int_{-\infty}^{+\infty} \mathrm{e}^t\, \overline{F(n, \mathrm{e}^t)}\, \mathrm{e}^{-\imath n\omega t}\, \mathrm{d}t$$
$$= \int_{-\infty}^{+\infty} \mathrm{e}^t\, F(-n, \mathrm{e}^t)\, \mathrm{e}^{-\imath n\omega t}\, \mathrm{d}t = \hat{M}_0(-n).$$

Finally note that since $h(t,x,0)$ is T-periodic in t and C^1 we have (cf Section 2.1) $\hat{M}_0(n) = 0$ for any $n \in \mathbb{Z}$ if and only if $M_0(\alpha) \equiv 0$.

We conclude this section by giving a first example of application of the above result. Consider the Duffing-like equation:

$$\ddot{x} + x\left(\frac{x}{k} - 1\right) = \varepsilon\,[q_1(t)x + q_2(t)\dot{x}] \qquad (7.1.7)$$

where $k > 0$ and $q_1(t)$, $q_2(t)$ are 2π-periodic, C^1-functions. Setting $x_1 = x$ and $x_2 = \dot{x}$ we obtain the equivalent system

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = x_1\left(1 - \frac{x_1}{k}\right) + \varepsilon\,[q_1(t)x_1 + q_2(t)x_2]\,.$$

Let $\Phi_0(x) = \dfrac{6kx}{(x+1)^2}$. Then the homoclinic solution of the unperturbed system is given by $\phi(t) = \Phi(\mathrm{e}^t)$ where

$$\Phi(x) = \begin{pmatrix} \Phi_0(x) \\ x\Phi_0'(x) \end{pmatrix}.$$

Moreover:

$$\Psi(x) = \begin{pmatrix} -x\Phi_0''(x) - \Phi_0'(x) \\ \Phi_0'(x) \end{pmatrix}$$

and

$$h(t, x_1, x_2, \varepsilon) = \begin{pmatrix} 0 \\ q_1(t)x_1 + q_2(t)x_2 \end{pmatrix}.$$

Thus,

$$F(n,u) = q_n^1 \Phi_0(u)\Phi_0'(u) + q_n^2 u \Phi_0'(u)^2$$

where q_n^1 and q_n^2 are the Fourier coefficients of $q_1(t)$ and $q_2(t)$ respectively. From Theorem 7.1.1 we obtain then:

$$\hat{M}_0(n) = \delta_n^1 q_n^1 + \delta_n^2 q_n^2$$

where

$$\delta_n^1 = \frac{2\pi\imath}{1 - \mathrm{e}^{-2\pi n}}\,\mathrm{Res}\left(\Phi_0(u)\Phi_0'(u)u^{\imath n}, -1\right)$$

and

$$\delta_n^2 = \frac{2\pi\imath}{1-\mathrm{e}^{-2\pi n}} \operatorname{Res}\left(\Phi_0'(u)^2 u^{\imath n+1}, -1\right)$$

(note that -1 is the unique pole of $\Phi_0(u)$). Hence, using the fact that $\operatorname{Arg} u \in (0, 2\pi)$, we obtain the following expressions of δ_n^1, δ_n^2 for $n \neq 0$:

$$\delta_n^1 = -3\imath n^2 k^2 (n^2+1) \frac{\pi}{\sinh n\pi}, \quad \delta_n^2 = -\frac{6}{5} n k^2 \left(n^4 - 1\right) \frac{\pi}{\sinh n\pi}.$$

Thus $\hat{M}_0(0) = q_0^2 \int\limits_{-\infty}^{+\infty} \dot{\phi}(t)^2 dt = 6k^2 q_0^2/5$ and $\hat{M}_0(n) = 0$, for $n \neq 0$, are equivalent to

$$\frac{q_n^1}{q_n^2} = -\frac{\delta_n^2}{\delta_n^1} = \alpha_n := \frac{2\imath(n^2-1)}{5n}. \tag{7.1.8}$$

Note that, obviously, $\alpha_{-n} = \bar{\alpha}_n$ and then taking, for any integer $n \neq 0$, $q_n^1 = \alpha_n q_n^2$, $q_0^2 \in \mathbb{R}$, we get the following (cf Section 2.1)

Corollary 7.1.2. *Given any 2π-periodic function $q_2(t) \in H^3(\mathbb{R})$ with zero mean value, there exists a unique 2π-periodic function with zero mean value, $q_1(t) \in H^2(\mathbb{R}) \subset C^1(\mathbb{R})$ so that the Melnikov function of Eq. (7.1.7) vanishes identically on $\mathbb{R}$. Actually, for $n \neq 0$, the Fourier coefficients of $q_1(t)$ and $q_2(t)$ satisfy the relation (7.1.8). The map $q_2(t) \mapsto q_1(t)$ is linearly bounded and its kernel is the space* $\operatorname{span}\{\cos t, \sin t\}$. *That is, if $q_2(t) \in \operatorname{span}\{1, \cos t, \sin t\}$, the Melnikov map of Eq. (7.1.7) does not vanish identically for any nonconstant and nonzero 2π-periodic perturbation $\varepsilon q_1(t)x$.*

For example, if $q_2(t) = \cos 2t$ then $q_1(t) = -\frac{3}{5}\sin 2t$.

7.1.3 Second Order ODEs

In this section we consider the Melnikov function for the second order equation

$$\ddot{x} = f(x) + \varepsilon q(t)$$

where x belongs to an open interval $I \subset \mathbb{R}$, $f \in C^1(I, \mathbb{R})$, $q(t)$ is a C^1, T-periodic function, and $\ddot{x} = f(x)$ is assumed to have a hyperbolic equilibrium $x = 0 \in I$ and an associated homoclinic orbit $p(t) \in I$. In this case we have

$$\phi^*(t) = (p(t) \quad \dot{p}(t)), \quad \psi^*(t) = (-\ddot{p}(t) \quad \dot{p}(t)), \quad h^*(t,x) = (0 \quad q(t))$$

and hence $\psi^*(t)h(t,x) = \dot{p}(t)q(t)$. As in the previous section we assume that $p(t) = \Phi_0(\mathrm{e}^t)$. Then $\dot{p}(t) = \mathrm{e}^t\, \Phi_0'(\mathrm{e}^t)$, and

$$\Phi(u) = \begin{pmatrix} \Phi_0(u) \\ u\Phi_0'(u) \end{pmatrix}, \quad \Psi(u) = \begin{pmatrix} -u\Phi_0''(u) - \Phi_0'(u) \\ \Phi_0'(u) \end{pmatrix}.$$

Note that we have

$$F(n,u) = \Psi^*(u)\hat{h}(n,\Phi(u)) = \Phi_0'(u)\hat{q}_n$$

where $\hat{q}_n$ is the n-th Fourier coefficient of the periodic function $q(t)$. Thus in order that the analysis of the previous section is valid we see that we only need that $\Phi_0'(u)$ is rational and $\lim_{u\to\infty}\Phi_0'(u)u = 0$. Anyway, for simplicity, we also assume that $\Phi_0(u)$ is rational with the same poles as $\Phi_0'(u)$. Next:

$$\hat{M}_0(0) = \int_{-\infty}^{+\infty} \dot{p}(t)\mathrm{d}t \cdot \hat{q}_0 = 0$$

because $p(t)$ is homoclinic, and from (7.1.6) we obtain for $n \in \mathbb{Z}$, $n \neq 0$:

$$\hat{M}_0(n) = 2\pi\imath \sum_{w_j} \mathrm{Res}\left(\frac{\Phi_0'(u)\hat{q}_n u^{\imath n\omega}}{1-\mathrm{e}^{-2\pi n\omega}}, w_j\right) = \delta_n \hat{q}_n$$

where $\hat{q}_n$ is the n-th Fourier coefficient of the periodic function $q(t)$, w_j is the pole of $\Phi_0(u)$ and

$$\delta_n = \frac{2\pi\imath}{1-\mathrm{e}^{-2\pi n\omega}} \sum_{w_j} \mathrm{Res}\left(\Phi_0'(u)u^{\imath n\omega}, w_j\right). \tag{7.1.9}$$

Now, let γ_j be a circle around w_j so that no other pole w_i, $i \neq j$, is inside γ_j. We have, integrating by parts:

$$2\pi\imath\,\mathrm{Res}\left(\Phi_0'(u)u^{\imath n\omega}, w_j\right) = \int_{\gamma_j} \Phi_0'(u)u^{\imath n\omega}\mathrm{d}u = -\imath n\omega \int_{\gamma_j} \Phi_0(u)u^{\imath n\omega-1}\mathrm{d}u \tag{7.1.10}$$

and then

$$\delta_n = \frac{2\pi n\omega}{1-\mathrm{e}^{-2\pi n\omega}} \sum_{w_j} \mathrm{Res}\left(\Phi_0(u)u^{\imath n\omega-1}, w_j\right). \tag{7.1.11}$$

Next, assume that w_j is a pole of $\Phi_0(u)$ of multiplicity k. We have:

$$\begin{aligned}
&\mathrm{Res}\left(\Phi_0(u)u^{\imath n\omega-1}, w_j\right)\\
&= \frac{1}{(k-1)!}\frac{d^{k-1}}{du^{k-1}}\left[(u-w_j)^k\Phi_0(u)u^{\imath n\omega-1}\right]_{u=w_j}\\
&= \frac{1}{(k-1)!}\sum_{m=0}^{k-1}\binom{k-1}{m}\left\{\frac{d^{k-m-1}}{du^{k-m-1}}\left[(u-w_j)^k\Phi_0(u)\right]\cdot\right.\\
&\quad\left.(\imath n\omega-1)\cdots(\imath n\omega-m)u^{\imath n\omega-m-1}\right\}_{u=w_j}\\
&= \sum_{m=0}^{k-1}\frac{1}{m!}\mathrm{Res}\left((u-w_j)^m\Phi_0(u), w_j\right)\imath^m\frac{(n\omega+\imath)\cdots(n\omega+m\imath)}{w_j^{m+1}}\mathrm{e}^{\imath n\omega\,\mathrm{Log}\,w_j}.
\end{aligned} \tag{7.1.12}$$

Now, $\operatorname{Res}\left((u-w_j)^m\Phi_0(u),w_j\right)=0$ for $m\geq k$ because $(u-w_j)^m\Phi_0(u)$ is holomorphic in a neighborhood of w_j when $m\geq k$. Thus, denoting by r the maximum of the multiplicities of the poles w_j, we can extend the above sum up to $r-1$ obtaining:

$$\begin{aligned}\frac{\delta_n(1-\mathrm{e}^{-2\pi n\omega})}{2\pi n\omega} &= \sum_{j=1}^{N}\sum_{m=0}^{r-1}\frac{1}{m!}\operatorname{Res}\left((u-w_j)^m\Phi_0(u),w_j\right)\imath^m \\ &\quad (n\omega+\imath)(n\omega+2\imath)\cdots(n\omega+m\imath)\frac{\mathrm{e}^{-n\omega\operatorname{Arg}w_j}}{w_j^{m+1}}\,\mathrm{e}^{\imath n\omega\log|w_j|}\end{aligned} \tag{7.1.13}$$

with $w_1,\ldots,w_N$ being the poles of $\Phi_0(u)$. Let $\beta_0=\min\{\operatorname{Arg}w_j : j=1,\ldots,N\}\in(0,\pi]$ and r_0 be the greatest multiplicity of the poles of $\Phi_0(u)$ that belong to the half line $\operatorname{Arg}u=\beta_0$. Multiplying both sides of Eq. (7.1.13) by $\mathrm{e}^{n\omega\beta_0}$ we see that $\frac{1}{2\pi n\omega}\mathrm{e}^{n\omega\beta_0}\,\delta_n(1-\mathrm{e}^{-2\pi n\omega})$ is asymptotic, as $n\to+\infty$, to

$$\sum_{\operatorname{Arg}w_j=\beta_0}\operatorname{Res}\left(\Phi_0(u)u^{\imath n\omega-1},w_j\right)\mathrm{e}^{n\omega\beta_0}\,.$$

Now, again from Eq. (7.1.12) we see that for any pole w_j $\operatorname{Arg}w_j=\beta_0$, and multiplicity less than r_0 the quantity $n^{1-r_0}\operatorname{Res}\left(\Phi_0(u)u^{\imath n\omega-1},w_j\right)\mathrm{e}^{n\omega\beta_0}$ tends to zero as $n\to\infty$. Thus the leading term in $\sum_{w_j}\operatorname{Res}\left(\Phi_0(u)u^{\imath n\omega-1},w_j\right)$ is:

$$\sum_{\substack{\operatorname{Arg}w_j=\beta_0\\ \operatorname{mult}w_j=r_0}}\operatorname{Res}\left(\Phi_0(u)u^{\imath n\omega-1},w_j\right)$$

with $\operatorname{mult}w_j$ being the multiplicity of w_j. As a consequence, using also $\delta_{-n}=\overline{\delta_n}$, we obtain the following:

Theorem 7.1.3. *Let* $\beta_0=\min\{\operatorname{Arg}w_j : j=1,\ldots,N\}\in(0,\pi]$ *and* $r_0=\max\{\operatorname{mult}w_j : \operatorname{Arg}w_j=\beta_0\}$. *Then, if*

$$\liminf_{n\to\infty}\frac{\mathrm{e}^{n\omega\beta_0}}{n^{r_0-1}}\left|\sum_{\substack{\operatorname{Arg}w_j=\beta_0\\ \operatorname{mult}w_j=r_0}}\operatorname{Res}\left(\Phi_0(u)u^{\imath n\omega-1},w_j\right)\right|\neq 0, \tag{7.1.14}$$

there exists $\bar{n}$ *so that for any* $n\in\mathbb{Z}$, $|n|\geq\bar{n}$, *we have* $\delta_n\neq 0$. *As a consequence, the space of* T*-periodic functions* $q(t)$ *that the associated Melnikov function is identically zero is finite-dimensional.*

Condition (7.1.14) can be simplified a bit looking at Eq. (7.1.12). In fact setting r_0 in the place of k in that equation and multiplying by $\mathrm{e}^{\beta_0 n}n^{1-r_0}$ we see that only the term with $m=r_0-1$ survives. Thus we obtain the following:

Theorem 7.1.4. *Let* β_0, r_0 *be as in Theorem* 7.1.3. *Then, if*

$$\liminf_{n\to\infty}\left|\sum_{\substack{\operatorname{Arg} w_j=\beta_0\\ \operatorname{mult} w_j=r_0}}\frac{1}{w_j^{r_0}}\operatorname{Res}\left((u-w_j)^{r_0-1}\Phi_0(u),w_j\right)\mathrm{e}^{\imath n\omega\log|w_j|}\right|\neq 0, \qquad (7.1.15)$$

there exists $\bar{n}$ so that for any $n\in\mathbb{Z}$, $|n|\geq\bar{n}$, we have $\delta_n\neq 0$. As a consequence, the space of periodic T-functions $q(t)$ that the associated Melnikov function is identically zero is finite-dimensional.

Proof. As we have already observed, for any pole w_j, $\operatorname{Arg} w_j=\beta$ and $\operatorname{mult} w_j=r$ the quantity:

$$\frac{\mathrm{e}^{n\omega\beta}}{n^{r-1}}\operatorname{Res}\left(\Phi_0(u)u^{\imath n\omega-1},w_j\right)-\frac{\imath^{r-1}}{(r-1)!}\prod_{k=1}^{r-1}\left(\omega+\frac{k\imath}{n}\right)$$
$$\times\frac{1}{w_j^{r}}\operatorname{Res}\left((u-w_j)^{r-1}\Phi_0(u),w_j\right)\mathrm{e}^{\imath n\omega\log|w_j|}$$

tends to zero as n tends to infinity, and then the result follows from:

$$\omega^{r-1}\leq\left|\left(\omega+\frac{\imath}{n}\right)\cdot\,\cdots\,\cdot\left(\omega+\frac{(r-1)\imath}{n}\right)\right|\leq\left(\omega+\frac{r-1}{n}\right)^{r-1}.$$

The proof is finished. □

Remark 7.1.5. If $\Phi_0(u)$ has only one pole on the line $\operatorname{Arg} u=\beta_0$ with maximum multiplicity r_0, condition (7.1.15) of Theorem 7.1.4 is certainly satisfied. In fact in this case the left-hand side of (7.1.15) reads:

$$\frac{1}{w_j^{r_0}}\lim_{u\to w_j}(u-w_j)^{r_0}\Phi_0(u)$$

and cannot be zero because w_j is a pole of multiplicity r_0 of $\Phi_0(u)$.

Equation (7.1.13) has an interesting consequence when $\Phi(u)$ has only the simple poles w and $\bar{w}$ (we do not exclude that $w=\bar{w}$). In fact in this case we have the following result:

Theorem 7.1.6. *Assume that $\Phi_0(u)$ satisfies the assumption of the previous section and, moreover, that it has only the simple poles w and $\bar{w}$ (including the case that $\Phi_0(u)$ has only one simple pole $w=\bar{w}$). Then $\delta_n\neq 0$ for any $n\in\mathbb{Z}$, $n\neq 0$. Thus, for any T-periodic, nonconstant function, the associated Melnikov function is not identically zero.*

Proof. Let us consider, first, the case where $\Phi_0(u)$ has only the simple pole $w=\bar{w}<0$. We have

$$\frac{1-\mathrm{e}^{-2\pi n\omega}}{2\pi n\omega}\delta_n=\frac{1}{w}\operatorname{Res}\left(\Phi_0(u),w\right)\mathrm{e}^{-n\pi\omega}\,\mathrm{e}^{\imath n\omega\log|w|}$$
$$=\mathrm{e}^{-n\pi\omega}\,\mathrm{e}^{\imath n\omega\log|w|}\lim_{z\to 1}(z-1)\Phi_0(zw)\neq 0$$

because w is a simple pole of $\Phi_0(u)$. Now consider the case where $w \neq \bar{w}$ and assume, without loss of generality, that $0 < \operatorname{Arg} w < \pi$. We have:

$$\frac{1}{w}\operatorname{Res}\left(\Phi_0(u), w\right) = \frac{1}{w}\lim_{u\to w}\Phi_0(u)(u-w) = \lim_{z\to 1}\Phi_0(zw)(z-1),$$
$$\frac{1}{\bar{w}}\operatorname{Res}\left(\Phi_0(u), \bar{w}\right) = \frac{1}{\bar{w}}\lim_{u\to \bar{w}}\Phi_0(u)(u-\bar{w}) = \lim_{z\to 1}\Phi_0(z\bar{w})(z-1).$$

Since the above two limits exist we can evaluate them by changing z with $x \in \mathbb{R}$. We get:

$$\begin{aligned}\frac{1}{\bar{w}}\operatorname{Res}\left(\Phi_0(u), \bar{w}\right) &= \lim_{x\to 1}\Phi_0(\overline{xw})(x-1)\\ &= \lim_{x\to 1}\overline{\Phi_0(xw)(x-1)} = \overline{\frac{1}{w}\operatorname{Res}\left(\Phi_0(u), w\right)}.\end{aligned}$$

Setting $\lambda = w^{-1}\operatorname{Res}\left(\Phi_0(u), w\right)$ we obtain, from the above equation, and (7.1.13):

$$\frac{1-\mathrm{e}^{-2\pi n\omega}}{2\pi n\omega}\delta_n = \lambda w^{\imath n\omega} + \bar{\lambda}\bar{w}^{\imath n\omega}.$$

Thus, when $n \neq 0$, $\delta_n = 0$ if and only if

$$\frac{\lambda}{\bar{\lambda}} = -\left(\frac{\bar{w}}{w}\right)^{\imath n\omega}. \tag{7.1.16}$$

Now

$$\left(\frac{\bar{w}}{w}\right)^{\imath n\omega} = \mathrm{e}^{-n(\operatorname{Arg}\bar{w}-\operatorname{Arg}w)\omega} = \mathrm{e}^{-2n(\pi-\operatorname{Arg}w)\omega} = \alpha_n^2 > 0.$$

Thus $\lambda = -\alpha_n^2\bar{\lambda}$ and then $\bar{\lambda} = -\alpha_n^2\lambda$. So $\lambda = \alpha_n^4\lambda$ and hence $\alpha_n^2 = 1$, because $\lambda \neq 0$. But this means that w is real and negative and this contradicts $w \neq \bar{w}$. The proof is finished. □

We now give a closer look at the case where $\Phi_0(u)$ has two poles of multiplicity r_0 on the half-line $\operatorname{Arg} u = \beta_0$. Since

$$\frac{1}{w_j^{r_0}}\operatorname{Res}\left((u-w_j)^{r_0-1}\Phi_0(u), w_j\right) = \lim_{z\to 1}(z-1)^{r_0}\Phi_0(w_j z),$$

we see that we have to study the equation:

$$\lambda_1|w_1|^{\imath n\omega} + \lambda_2|w_2|^{\imath n\omega} = 0 \tag{7.1.17}$$

where $\lambda_j = \lim_{z\to 1}(z-1)^{r_0}\Phi_0(w_j z)$. Now, Equation (7.1.17) has a solution $n \in \mathbb{N}$ if and only if

$$\left|\frac{w_1}{w_2}\right|^{\imath n\omega} = -\frac{\lambda_2}{\lambda_1}.$$

Thus we have the following cases:

(i) $|\lambda_2| \neq |\lambda_1|$. In this case $\liminf_{n\to\infty} |\lambda_1 |w_1|^{\imath n\omega} + \lambda_2 |w_2|^{\imath n\omega}| \neq 0$;
(ii) $|\lambda_2| = |\lambda_1|$ and $\log(|\frac{w_1}{w_2}|)$ is a rational multiple of $T = \frac{2\pi}{\omega}$. In this case Eq. (7.1.17) has either a (least) solution $n_0 \in \mathbb{N}$, and hence it has infinite solutions of the type: $n = n_0 + kq$, $k \in \mathbb{Z}$, and $q \in \mathbb{Z}$ so that $\frac{q}{T}\log(|\frac{w_1}{w_2}|) \in \mathbb{Z}$, or $\liminf_{n\to\infty} |\lambda_1 |w_1|^{\imath n\omega} + \lambda_2 |w_2|^{\imath n\omega}| \neq 0$;
(iii) $|\lambda_2| = |\lambda_1|$ and $\log(|\frac{w_1}{w_2}|)$ is not a rational multiple of T. In this case we obtain $\liminf\limits_{n\to\infty} |\lambda_1 |w_1|^{\imath n\omega} + \lambda_2 |w_2|^{\imath n\omega}| = 0$.

As a consequence, Theorem 7.1.4 applies if either $|\lambda_2| \neq |\lambda_1|$, or $\log(|\frac{w_1}{w_2}|)$ is a rational multiple of T and $-\frac{\lambda_2}{\lambda_1}$ is not one of the (finite) values of $\left|\frac{w_1}{w_2}\right|^{\imath n\omega}$.

Remark 7.1.7. (i) In this section we have assumed that $\Phi_0(u)$ and $\Phi_0'(u)$ are both rational functions on $\mathbb{C}$ with the same poles. However it may easily happen that the poles of $\Phi_0'(u)$ correspond to essential singularities of $\Phi_0(u)$. Nonetheless, the argument of this section hold even in this case, we simply do not have to integrate by parts as in (7.1.10) and use (7.1.9) instead of (7.1.11). For example Eq. (7.1.13) reads:

$$\frac{\delta_n(1-\mathrm{e}^{-2\pi n\omega})}{2\pi i} = \sum_{j=1}^{N}\sum_{m=0}^{r-1}\frac{1}{m!}\operatorname{Res}\left((u-w_j)^m\Phi_0'(u), w_j\right)\imath^m\,\mathrm{e}^{\imath n\omega\log|w_j|}$$
$$n\omega(n\omega+\imath)(n\omega+2\imath)\cdots(n\omega+(m-1)\imath)\frac{\mathrm{e}^{-n\omega\operatorname{Arg}w_j}}{w_j^m}$$

with r being the multiplicity of the pole w_j of $\Phi_0'(u)$. Thus Theorems 7.1.3 and 7.1.4 hold with the following changes:

r_0 is the maximum of the multiplicities of the poles of $\Phi_0'(u)$ and in Eqs. (7.1.14), (7.1.15), $\Phi_0(u)$ and $u^{\imath n\omega-1}$ have to be changed with $\Phi_0'(u)$, and $u^{\imath n\omega}$ respectively. Moreover, Theorem 7.1.6 holds as is (with $\Phi_0'(u)$ instead of $\Phi_0(u)$ of course). The proof goes almost in the same way, apart from that Eq. (7.1.16) has to be written as:

$$\frac{\lambda w}{\overline{\lambda w}} = -\left(\frac{\bar{w}}{w}\right)^{\imath n\omega} = -\alpha_n^2 \leq 0.$$

The rest of the proof is the same as λw instead of λ.

Note that the function

$$\Phi_0(u) = \arctan\left(\frac{3u}{2(u^2+1)}\right)$$

is an example of such a situation, since $\Phi_0'(u) = \frac{2}{1+4u^2} - \frac{2}{u^2+4}$. In the next section we give a method to construct a second order differential equation satisfied by $p(t) := \Phi_0(\mathrm{e}^t)$. Following this method we see that $p(t)$ satisfies:

$$\ddot{p} = \frac{1}{9}\frac{9-41\tan^2 p}{(\tan^2 p+1)^2}\tan p.$$

(ii) From Section 7.1.2 we know that δ_n is also given by:

$$\delta_n = \int_{-\infty}^{+\infty} \dot{p}(t)\, \mathrm{e}^{\imath n\omega t}\, dt.$$

Now, the function:

$$\delta(\xi) = \int_{-\infty}^{+\infty} \dot{p}(t)\, \mathrm{e}^{-\imath\omega\xi t}\, dt$$

tends to zero as $|\xi| \to \infty$ and the same holds for $\imath\xi\,\delta(\xi)$ and $(\imath\xi)^2\delta(\xi)$ because $p(t)$, $\dot{p}(t)$ tend to zero exponentially fast as $|t| \to \infty$ (and hence belong to $L^2(\mathbb{R})$), and $p(t)$ satisfies the equation $\ddot{p} = f(p)$. In fact we have, for example, integrating by parts:

$$\imath\omega\xi\,\delta(\xi) = \int_{-\infty}^{+\infty} \ddot{p}(t)\, \mathrm{e}^{-\imath\omega\xi t}\, dt = \int_{-\infty}^{+\infty} f(p(t))\, \mathrm{e}^{-\imath\omega\xi t}\, dt$$

and

$$(\imath\omega\xi)^2\delta(\xi) = \int_{-\infty}^{+\infty} f'(p(t))\dot{p}(t)\, \mathrm{e}^{-\imath\omega\xi t}\, dt.$$

As a consequence, $\xi\,\delta(\xi) \to 0$, $\xi^2\delta(\xi) \to 0$ as $|\xi| \to \infty$ and then $\delta_n \in \ell^1(\mathbb{Z}) \cap \ell^2(\mathbb{Z})$. Thus the series:

$$\sum_{n\in\mathbb{Z}} \delta_n\, \mathrm{e}^{\imath n\omega t}$$

is totally convergent to a continuous, T-periodic function $\Delta(t)$ whose n-th Fourier coefficient is precisely δ_n. Note that because $\delta_{-n} = \bar{\delta}_n$ we have:

$$\Delta(t) = \delta_0 + 2\sum_{n=0}^{+\infty} (\Re\delta_n)\cos nt - (\Im\delta_n)\sin nt.$$

Now, let $\phi_1(t)$ and $\phi_2(t)$ be two T-periodic functions on $\mathbb{R}$. For $t \in \mathbb{R}$, we set:

$$\phi_1 * \phi_2(t) = \frac{1}{T}\int_{-T/2}^{T/2} \phi_1(t-s)\phi_2(s)ds.$$

Then $\phi_1 * \phi_2(t)$ is T-periodic and its n-th Fourier coefficient is:

$$\begin{aligned}
&\frac{1}{T^2}\int_{-T/2}^{T/2}\int_{-T/2}^{T/2} \phi_1(t-s)\phi_2(s)ds\, \mathrm{e}^{-\imath n\omega t}\, dt \\
&= \frac{1}{T^2}\int_{-T/2}^{T/2}\left\{\int_{-T/2}^{T/2} \phi_1(\tau)\, \mathrm{e}^{-\imath n\omega\tau}\, d\tau\right\}\phi_2(s)\, \mathrm{e}^{-in\omega s}\, ds = \phi_1^{(n)}\phi_2^{(n)}
\end{aligned}$$

with $\phi_j^{(n)}$ being the n-th Fourier coefficient of $\phi_j(t)$. As a consequence, $\delta_n\hat{q}_n$ is the Fourier coefficient of both $M(\alpha)$ and $\Delta * q(\alpha)$, that is,

$$M(\alpha) = \Delta * q(\alpha) = \frac{1}{T}\int_{-T/2}^{T/2} \Delta(\alpha-s)q(s)ds = \frac{1}{T}\int_{-T/2}^{T/2} \Delta(s)q(\alpha-s)ds.$$

Finally, we note that the function $\Delta(\alpha)$ can be expressed by means of $\dot{p}(t)$ as follows:

$$M(\alpha) = \int_{-\infty}^{+\infty} \dot{p}(t-\alpha)q(t)dt = \sum_{k\in\mathbb{Z}} \int_{(2k-1)T/2}^{(2k+1)T/2} \dot{p}(s-\alpha)q(s)ds$$
$$= \int_{-T/2}^{T/2} \sum_{k\in\mathbb{Z}} \dot{p}(s+kT-\alpha)q(s)ds.$$

Now, the function $\sum_{k\in\mathbb{Z}} \dot{p}(kT-t)$ is T-periodic and continuous (actually analytic, since so is $\dot{p}(t)$). Thus:

$$\Delta(t) = T\sum_{k\in\mathbb{Z}} \dot{p}(kT-t).$$

7.1.4 Applications and Examples

In this section we apply the result of Section 7.1.3 to constructing a second order equation in $\mathbb{R}$ whose Melnikov function vanishes identically on an infinite number of independent 2π-periodic functions, or on any 2π-periodic function $q(t)$. To do this we will first prove a result allowing us to construct second order equations satisfied by prescribed homoclinic solutions. For completeness we will also give an example showing that this procedure can also produce non-rational differential equations. To start with, we make some remarks on the properties of the function $p(t)$ and the associated $\Phi_0(x)$. Since $p(t) \to 0$ as $|t| \to \infty$, there exists t_0 so that $\dot{p}(t_0) = 0$. Without loss of generality we can assume that $t_0 = 0$. Thus $p(t) = p(-t)$ because both satisfy the Cauchy problem:

$$\ddot{x} = f(x)$$
$$x(0) = p(0), \quad \dot{x}(0) = \dot{p}(0).$$

Possibly changing $f(x)$ with $-f(-x)$, we can also assume that $p_0 := p(0) > 0$. Thus $t\dot{p}(t) < 0$ for any $t \neq 0$. In fact if $\dot{p}(\tau) = 0$ for some $\tau > 0$ then $p(t)$ would be 2τ-periodic contradicting the fact that $p(t) \to 0$ as $|t| \to \infty$. Now, let $\Phi_0(x)$ be as in the previous section. Since we want that the equality $\Phi_0(x) = p(\log x)$ holds for any $x > 0$, we see that we have to assume that:

$$\Phi_0(x) = \Phi_0(1/x) \tag{7.1.18}$$

for any $x > 0$ and then $\Phi_0(u) = \Phi_0(1/u)$ because of uniqueness of the analytical extension. Thus, besides the pole w_j, $\Phi_0(u)$ has also the pole w_j^{-1} whose argument is $2\pi - \operatorname{Arg} w_j$ (here we assume that $0 < \operatorname{Arg} w_j \leq \pi$). Then, the function $\Phi_0(x)$ is increasing in $[0,1]$ and decreasing in $[1,\infty)$, moreover $\Phi_0(1) = p_0$. Thus there exist two functions $x_\pm(p)$ defined on $(0,p_0]$ so that

(i) $x_+(p)$ is decreasing on $(0,p_0]$ and $x_+(p_0) = 1$,

(ii) $x_-(p)$ is increasing on $[0, p_0]$ and $x_-(p_0) = 1, x_-(0) = 0$

satisfying:

$$\Phi_0(x_\pm(p)) = p. \tag{7.1.19}$$

Note that because of (7.1.18), we obtain

$$x_+(p) = \frac{1}{x_-(p)}$$

for any $p \in (0, p_0]$, moreover, being $\Phi_0'(1) = 0$ we get: $\lim_{p \to p_0} x_\pm'(p_0) = \mp\infty$. Now, $p(t)$ satisfies the equation:

$$\ddot{p}(t) = F(\mathrm{e}^t)$$

where $F(x) = x^2 \Phi_0''(x) + x\Phi_0'(x)$ is a rational function defined on a neighborhood of $x \geq 0$. Thus the point is to see whether $F(\mathrm{e}^t) = f(p(t))$ for some C^1-function $f(p)$. We note the following

$$F(1/x) = \frac{1}{x^2}\Phi_0''(1/x) + \frac{1}{x}\Phi_0'(1/x)$$

and using (7.1.18), we get

$$-\frac{1}{x^2}\Phi_0'(1/x) = \Phi_0'(x), \quad \frac{2}{x^3}\Phi_0'(1/x) + \frac{1}{x^4}\Phi_0''(1/x) = \Phi_0''(x)\,.$$

Thus it is easy to see that $F(x) = F(1/x)$ and then $F(x_-(p)) = F(x_+(p))$. Note that we also get $x^2 F'(x) = -F'(1/x)$. This last equation implies $F'(1) = 0$ (note that a similar conclusion holds for $\Phi_0'(1)$). We set

$$f(p) := F(x_-(p))\ (= F(x_+(p))), \quad p \in [0, p_0].$$

Note that we choose $x_-(p)$ so that $f(p)$ is continuous up to $p = 0$; moreover from (7.1.19) we see that either $x_-(p(t)) = \mathrm{e}^t$ or $x_+(p(t)) = \mathrm{e}^t$, but then, in any case $f(p(t)) = F(\mathrm{e}^t) = \ddot{p}(t)$. Thus we want to show that $f(p)$ can be extended in a C^1 way in a neighborhood of $[0, p_0]$. To this end we simply have to show that the limits: $\lim_{p \to p_0} \frac{d}{dp} F(x_-(p))$ and $\lim_{p \to 0} \frac{d}{dp} F(x_-(p))$ exist in $\mathbb{R}$. We have:

$$\lim_{p \to p_0} \frac{d}{dp} F(x_-(p)) = \lim_{p \to p_0} \frac{F'(x_-(p))}{\Phi_0'(x_-(p))} = \lim_{x \to 1} \frac{F'(x)}{\Phi_0'(x)} = \lim_{x \to 1} \frac{F''(x)}{\Phi_0''(x)} = \frac{F''(1)}{\Phi_0''(1)} \in \mathbb{R}.$$

Note that we have to assume that $\Phi_0''(1) \neq 0$ because otherwise $f(p_0) = 0$ and then $p(t) \equiv p_0$ will be another solution of the Cauchy problem $\ddot{p} = f(p)$, $p(0) = p_0$, $\dot{p}(0) = 0$. Hence $f(p)$ cannot even be Lipschitz continuous function in any neighborhood of p_0. Next we prove that the limit

$$\lim_{p \to 0} \frac{d}{dp} F(x_-(p)) = \lim_{x \to 0} \frac{F'(x)}{\Phi_0'(x)}$$

exists in $\mathbb{R}$. To this end we observe that $\Phi_0(x)$ being analytic, $x = 0$ has to be a zero of finite multiplicity, say, $k \geq 1$, of $\Phi_0(x)$, that is, $\Phi_0(x) = x^k G(x)$, with $G(0) \neq 0$. Then $F(x) = k^2 \Phi_0(x) + O(x^{k+1})$ and

$$\lim_{x\to 0} \frac{F'(x)}{\Phi_0'(x)} = \lim_{x\to 0} \frac{k^2 \Phi_0'(x) + O(x^k)}{\Phi_0'(x)} = k^2.$$

Furthermore we show that the limits $\lim\limits_{p\to p_0} \dfrac{d^2}{dp^2} F(x_-(p))$ and $\lim\limits_{p\to 0} \dfrac{d^2}{dp^2} F(x_-(p))$ exist in $\mathbb{R}$. As for the first we will see that the result holds without any further assumption on $\Phi_0(x)$, but the same does not hold in general for the second. We have:

$$\frac{d^2}{dp^2} F(x_-(p)) = \frac{F''(x_-(p))\Phi_0'(x_-(p)) - F'(x_-(p))\Phi_0''(x_-(p))}{\Phi_0'(x_-(p))^3}$$

and hence we are led to evaluate the two limits:

$$\lim_{x\to 1} \frac{F''(x)\Phi_0'(x) - F'(x)\Phi_0''(x)}{\Phi_0'(x)^3} \tag{7.1.20}$$

and

$$\lim_{x\to 0} \frac{F''(x)\Phi_0'(x) - F'(x)\Phi_0''(x)}{\Phi_0'(x)^3}. \tag{7.1.21}$$

Let us consider, first, the limit in (7.1.20). Since $F'(1) = \Phi_0'(1) = 0$ we apply L'Hopital rule and get:

$$\lim_{x\to 1} \frac{F''(x)\Phi_0'(x) - F'(x)\Phi_0''(x)}{\Phi_0'(x)^3} = \frac{1}{3\Phi_0''(1)} \lim_{x\to 1} \frac{F'''(x)\Phi_0'(x) - F'(x)\Phi_0'''(x)}{\Phi_0'(x)^2}$$

$$= \frac{1}{6\Phi_0''(1)^2} \lim_{x\to 1} \frac{F^{(iv)}(x)\Phi_0'(x) + F'''(x)\Phi_0''(x) - F''(x)\Phi_0'''(x) - F'(x)\Phi_0^{(iv)}(x)}{\Phi_0'(x)}$$

provided the last limit exists. Now, from $\Phi_0(1/x) = \Phi_0(x)$ we get:

$$-\Phi_0'''(1/x) = x^6 \Phi_0'''(x) + 6x^5 \Phi_0''(x) + 6x^4 \Phi_0'(x)$$

and then $\Phi_0'''(1) = -3\Phi_0''(1)$. Similarly $F'''(1) = -3F''(1)$. Thus we can apply again L'Hopital rule and obtain:

$$\lim_{x\to 1} \frac{F''(x)\Phi_0'(x) - F'(x)\Phi_0''(x)}{\Phi_0'(x)^3} = \frac{F^{(iv)}(1)\Phi_0''(1) - F''(1)\Phi_0^{(iv)}(1)}{3\Phi_0''(1)^3} \in \mathbb{R}.$$

Now we consider $\lim_{p\to 0} f''(p)$ that is the limit in (7.1.21). Recall that we set $\Phi_0(x) = x^k G(x)$, with $G(0) \neq 0$ and note that also $F(x)$ has $x = 0$ as a zero of multiplicity k. Thus the numerator of (7.1.21) has $x = 0$ as a zero of multiplicity (at least) $2k - 3$ while the denominator has $x = 0$ as a zero of multiplicity $3(k - 1)$.

Now a simple computation shows that $x = 0$ is actually a zero of the numerator of multiplicity (at least) $2(k-1)$, but in general this is the maximum we can expect. In fact one has:

$$F''(x)\Phi_0'(x) - F'(x)\Phi_0''(x) = k(k+1)(2k+1)x^{2(k-1)}G(x)G'(x) + O(x^{2k-1}). \tag{7.1.22}$$

Of course, this is not enough to prove that $f(p)$ is C^2 up to $p = 0$, unless $k = 1$. So we assume that $k \in \mathbb{N}$, $k > 1$, and the following holds:

$$\Phi_0(x) = x^k G_0(x^k)$$

where $G_0(0) \neq 0$. In this case, in fact the left-hand side of Eq. (7.1.22) vanishes at $x = 0$ (since $G'(0) = 0$) and we actually have:

$$F''(x)\Phi_0'(x) - F'(x)\Phi_0''(x) = 6k^5 G_0(x^k)G_0'(x^k)x^{3(k-1)} + O(x^{4k-3})$$

and then

$$\lim_{x\to 1} \frac{F''(x)\Phi_0'(x) - F'(x)\Phi_0''(x)}{\Phi_0'(x)^3} = \frac{6k^2 G_0'(0)}{G_0(0)^2} \in \mathbb{R}.$$

Let us rewrite what we have done as a theorem:

Theorem 7.1.8. *Let $\Phi_0(u) = u^k G(u)$, $k \geq 1$, be a rational function so that $G(0) \neq 0$ and the following hold:*

(i) $\Phi_0(u) = \Phi_0(1/u)$ *(that is $G(1/u) = u^{2k}G(u)$),*
(ii) $\Phi_0(x) > 0$ *when x is real and $x > 0$,*
(iii) $\Phi_0'(x) = 0$ *on $x > 0$ is equivalent to $x = 1$,*
(iv) $\Phi_0''(1) \neq 0$.

Then $\lim_{u\to\infty} u\Phi_0'(u) = 0$ and there exists a C^1–function $f(p)$ in a neighborhood of $[0, \Phi_0(1)]$ so that $p(t) = \Phi_0(\mathrm{e}^t)$ is the solution of the equation $\ddot{p} = f(p)$. Moreover, if $G(u) = G_0(u^k)$ for some rational function $G_0(u)$, $G_0(0) \neq 0$, the function $f(p)$ is C^2 in a neighborhood of $[0, \Phi_0(1)]$.

Proof. We only have to prove that $\lim_{u\to\infty} u\Phi_0'(u) = 0$. To this end we note that $G(1/u) = u^{2k}G(u)$ implies

$$G'(1/u) = -2ku^{2k+1}G(u) - u^{2k+2}G'(u)$$

and then

$$\lim_{u\to\infty} u\Phi_0'(u) = \lim_{u\to 0} \frac{\Phi_0'(1/u)}{u} = \lim_{u\to 0} \left[\frac{kG(1/u)}{u^k} + \frac{G'(1/u)}{u^{k+1}}\right]$$
$$= \lim_{u\to 0} -u^k\{uG'(u) + kG(u)\} = 0.$$

Finally note that condition $\Phi_0''(1) \neq 0$ can be also stated in terms of $G(u)$ since condition (i) implies $G'(1) = -kG(1)$ and then $\Phi_0''(1) = G''(1) - k(k+1)G(1)$. The proof is finished. □

One might wonder what kind of system one obtains starting with functions $\Phi_0(x)$ as in Theorem 7.1.8. Actually, since $\Phi_0(x)$ is a rational function one might expect that the function $F(x_-(p))$ is a rational function of p. However this is not generally true because $x_\pm(p)$ are in general far from being rational. To show this we start with the function

$$\Phi_0(x) := \frac{x(x^2+1)}{x^4+4x^2+1}.$$

There is no particular reason for the coefficient 4. It only has to be different from 2, otherwise the expression of $\Phi_0(x)$ can be simplified. It is easy to see that all the conditions of Theorem 7.1.6 are satisfied. Particularly we have:

$$\Phi_0(x) = \Phi_0\left(\frac{1}{x}\right); \quad \Phi_0(1) = \frac{1}{3}; \quad \Phi_0'(1) = 0; \quad \Phi_0'(0) = 1; \quad \Phi_0''(1) = -\frac{1}{9}.$$

Moreover we obtain the following expression for $F(x) = x^2\Phi_0''(x) + x\Phi_0'(x)$:

$$\begin{aligned} F(x) &= \frac{x(x^2+1)(x^8-16x^6+18x^4-16x^2+1)}{(x^4+4x^2+1)^3} \\ &= \Phi_0(x)\frac{(x^8-16x^6+18x^4-16x^2+1)}{(x^4+4x^2+1)^2}. \end{aligned}$$

In order to apply the above described procedure we have to solve the equation:

$$x(x^2+1) = (x^4+4x^2+1)p \tag{7.1.23}$$

with x being a function of p. Since $\Phi_0(x) = \Phi_0(1/x)$ we can solve (7.1.23) multiplying it by x^{-2} and setting $z = x + x^{-1}$. We obtain:

$$pz^2 - z + 2p = 0$$

which has the solution

$$z_\pm(p) = \frac{1 \pm \sqrt{1-8p^2}}{2p}. \tag{7.1.24}$$

Now, $x_-(p)$ and $x_+(p) = x_-(p)^{-1}$ are both solutions of the equation $x + x^{-1} = z_+(p)$, and not $x + x^{-1} = z_-(p)$, because, for $p = p_0 = \Phi_0(1) = 1/3$ we have $z_+(p_0) = 2$, $z_-(p_0) = 1$ and $x_+(p_0) = x_-(p_0) = 1$. Now, we want to construct $f(p) = F(x_-(p))$ where $x_-(p)$ is the unique solution of $\Phi_0(x) = p$ so that $0 \leq x_-(p) \leq 1$. We have $\Phi_0(x_-(p)) = p$ for any $0 \leq p \leq \frac{1}{3}$, and $x_-(\Phi_0(x)) = x$ for any $0 \leq x \leq 1$. So,

$$f(p) = pf_0(x_-(p))$$

where

$$f_0(x) = \frac{x^8-16x^6+18x^4-16x^2+1}{(x^4+4x^2+1)^2} = \frac{x^4-16x^2+18-16x^{-2}+x^{-4}}{(x^2+4+x^{-2})^2}.$$

Since $x_-(p)+x_-(p)^{-1}=z_+(p)$ we have

$$x_-^2(p)+x_-(p)^{-2}=z_+^2(p)-2,$$

and

$$x_-^4(p)+x_-(p)^{-4}=z_+^4(p)-4z_+^2(p)+2.$$

So

$$f_0(x_-(p))=\frac{z_+^4(p)-20z_+^2(p)+52}{(z_+^2(p)+2)^2}.$$

Plugging (7.1.24) in the above equation we obtain, after some algebra:

$$f_0(x_-(p))=7-6\sqrt{1-8p^2}-48p^2.$$

Thus we have seen that the second order equation

$$\ddot{x}=x(7-6\sqrt{1-8x^2}-48x^2) \tag{7.1.25}$$

has the homoclinic solution $p(t)=\dfrac{e^t(e^{2t}+1)}{e^{4t}+4e^{2t}+1}$. Note that Eq. (7.1.25) is defined in the interval $\left(-\frac{1}{2\sqrt{2}},\frac{1}{2\sqrt{2}}\right)$ that contains $\left[0,\frac{1}{3}\right]$.

We now give an example of equations whose associated Melnikov function vanishes on an infinite dimensional space of C^1-smooth and 2π-periodic functions. Take $a\in\mathbb{R}$, $a^2\neq 0,1$ and set:

$$\Phi_0(x)=\frac{|a^4-1|x^2}{(x^2+a^2)(a^2x^2+1)}.$$

Note that $\Phi_0(x)>0$, for $x\neq 0$, and changing a with a^{-1}, we obtain the same function, so we assume $a>1$. Moreover $\Phi_0(x)$ satisfies all the assumptions of Theorem 7.1.8 including $\Phi_0(u)=u^kG_0(u^k)$ with $k=2$. For example, one has:

$$\Phi_0''(1)=\frac{8a^2(1-a^2)}{(1+a^2)^3}$$

which is different from zero when $a^2\neq 0,1$. Now, the (simple) poles of $\Phi_0(u)$ are

$$w_1=\imath a,\quad \bar{w}_1=-\imath a,\quad w_2:=\imath a^{-1},\quad \bar{w}_2=-\imath a^{-1}$$

and thus Eq. (7.1.11) gives, after some algebra:

$$\delta_n=-\frac{\pi\imath n}{\sinh(n\frac{\pi}{2})}\sin(n\log a)$$

and we obtain the following:

(a) Taking $a = \mathrm{e}^{m\pi}$, $m \in \mathbb{N}$, we can construct a family of second order equation whose Melnikov function is identically zero, no matter what the 2π-periodic perturbation is.
(b) Taking $a = \mathrm{e}^{m\pi/2}$, $m \in \mathbb{N}$, we can construct a family of second order equation whose Melnikov function is identically zero on an infinite number of independent 2π-periodic perturbations but not for all.

To obtain an analytical expression of such systems, we proceed as in the previous example. The equation $\Phi_0(x) = p$ reads:

$$pa^2\left(x^2 + \frac{1}{x^2}\right) - a^4 + p + pa^4 + 1 = 0$$

and again can be solved by setting $z = x + x^{-1}$. We obtain:

$$z^2 = (a^2 - 1)\frac{a^2 + 1 - p(a^2 - 1)}{pa^2}$$

which has the solutions

$$z_{\pm}(p) = \pm\frac{\sqrt{p[a^4 - 1 - p(a^2 - 1)^2]}}{ap}.$$

It is not necessary to solve the equations $x + x^{-1} = z_{\pm}$. We only have to note that both $x_-(p)$ and $x_+(p) = x_-(p)^{-1}$ are solutions of the equation $x + x^{-1} = z_+(p)$, and not $x + x^{-1} = z_-(p)$, because $z_+(p_0) = 2$, $z_-(p_0) = -2$ and $x_-(p_0) = x_+(p_0) = 1$. Recall $p_0 = \Phi_0(1) = \frac{a^2-1}{a^2+1}$. Next we compute

$$F(x) = x^2\Phi_0''(x) + x\Phi_0'(x).$$

Since $F(x) = F(1/x)$ we expect that $F(x)$ can be expressed in terms of $z = x + x^{-1}$. An annoying computation shows that, in fact, $F(x) = G(x + x^{-1})$ where

$$G(z) = 4a^2(a^4 - 1)\frac{a^2z^4 - (a^4 + 4a^2 + 1)z^2 + 2(a^2 - 1)^2}{(a^2z^2 + (a^2 - 1)^2)^3}.$$

Thus $f(p,a) = F(x_-(p)) = G(z_+(p))$. After some algebra, we get:

$$f(p,a) = 4p\left(2p^2 - 3\frac{a^4 + 1}{a^4 - 1}p + 1\right) = 4p\left[2p^2 - 3p\coth(2\log a) + 1\right]. \quad (7.1.26)$$

Thus, in this case, $p(t) = \Phi_0(\mathrm{e}^t)$ is the solution of an analytic second order equation $\ddot{x} = f(x,a)$ so that when $a = \mathrm{e}^{m\pi}$ (or $a = \mathrm{e}^{m\pi/2}$), $m \in \mathbb{N}$, its Melnikov function vanishes identically on any 2π-periodic functions (or it is identically zero for infinitely many independent 2π-periodic functions but not for all). The geometrical meaning of this is that in spite of the fact that the perturbation of the equation is of the order $O(\varepsilon)$, the distance between the stable and unstable manifolds of the perturbed equation, along a transverse direction, is of the order (at least) $O(\varepsilon^2)$. This means that in order to study the intersection of the stable and the unstable manifolds, we have to look at the *second order Melnikov function*. By Section 4.1.4, for a C^2-equation $\ddot{x}+f(x)=\varepsilon q(t)$, this second order Melnikov function $M_2(\alpha)$ is given by (4.1.10). We now prove the following result.

Theorem 7.1.9. *For any $m \in \mathbb{N}$ and $c \neq 0$, the second order Melnikov function $M_2(\alpha)$ associated to the equation*

$$\ddot{x} = 4x(2x^2 - 3x\coth(2m\pi) + 1) + \varepsilon\left(\frac{c}{2} + q_{odd}(t)\right) \tag{7.1.27}$$

does not vanish identically on the complement of a codimension one closed linear subspace of the space $C^1_{odd,2\pi}$ of all C^1-smooth, $2\pi-$periodic and odd functions $q_{odd}(t)$. Moreover if a positive integer $k \in \mathbb{N}$ exists so that $q_{odd}(t+\frac{\pi}{k}) = -q_{odd}(t)$, $M_2(\alpha)$ changes sign in the interval $\left[0, \frac{\pi}{k}\right]$.

Proof. We emphasize the fact that many of the arguments of this proof can be used even for more general equations than (7.1.27) having a homoclinic orbit. For this reason we will write $f(x)$ instead of $4x(2x^2 - 3x\coth(2m\pi)+1)$, $q(t)$ instead of $\frac{c}{2}+q_{odd}(t)$ and $p(t)$ for the orbit homoclinic to the hyperbolic equilibrium $x=0$, in the first part of the proof. Note that the hyperbolicity of $x=0$ implies that $f'(0)>0$.

As a first step we simplify the expression of $M_2(\alpha)$ in the following way. Let $v_\alpha(t)$ be a bounded solution of the equation (cf Section 4.1.4)

$$\ddot{x} = f'(p(t))x + q(t+\alpha), \tag{7.1.28}$$

whose existence is guaranteed by the fact that $M(\alpha)=0$, and $u(t)$ be the unique 2π-periodic solution of the equation $\ddot{x} = f'(0)x + q(t)$. Then $r_\alpha(t) := v_\alpha(t) - u(t+\alpha)$ is a bounded solution of

$$\ddot{x} = f'(p(t))x + [f'(p(t)) - f'(0)]u(t+\alpha).$$

As a consequence, $r_\alpha(t) \to 0$ exponentially together with its first and second derivative (uniformly with respect to α) and $v_\alpha(t) = r_\alpha(t) + u(t+\alpha)$. Then

$$\begin{aligned} M_2(\alpha) =& \int_{-\infty}^{+\infty} \frac{d}{dt}[f'(p(t) - f'(0)]v_\alpha^2(t)dt \\ =& -2\int_{-\infty}^{+\infty} [f'(p(t)) - f'(0)]v_\alpha(t)\dot{v}_\alpha(t)dt \\ =& -2\int_{-\infty}^{+\infty} \Big([\ddot{v}_\alpha(t) - q(t+\alpha)]\dot{v}_\alpha(t) - f'(0)v_\alpha(t)\dot{v}_\alpha(t)\Big)dt. \end{aligned}$$

Now we observe that

$$2\lim_{n\to+\infty}\int_{-n\pi}^{n\pi}\ddot{v}_\alpha(t)\dot{v}_\alpha(t)dt = \lim_{n\to+\infty}\Big\{[\dot{r}_\alpha(n\pi)+\dot{u}(n\pi+\alpha)]^2 \\ -[\dot{r}_\alpha(-n\pi)+\dot{u}(-n\pi+\alpha)]^2\Big\} = 0$$

because $\dot{r}_\alpha(t)\to 0$ as $|t|\to+\infty$ and $u(t)$ is 2π-periodic. Similarly, using the fact that $r_\alpha(t)\to 0$ as $|t|\to+\infty$, we get:

$$2\lim_{n\to+\infty}\int_{-n\pi}^{n\pi}v_\alpha(t)\dot{v}_\alpha(t)dt \\ = \lim_{n\to+\infty}\Big\{[r_\alpha(n\pi)+u(n\pi+\alpha)]^2-[r_\alpha(-n\pi)+u(-n\pi+\alpha)]^2\Big\} = 0.$$

As a consequence,

$$M_2(\alpha) = 2\lim_{n\to+\infty}\int_{-n\pi}^{n\pi}\dot{v}_\alpha(t)q(t+\alpha)dt\,. \tag{7.1.29}$$

Note that $\lim_{n\to+\infty}\int_{-n\pi}^{n\pi}$ in Eq. (7.1.29) cannot be replaced by $\int_{-\infty}^{+\infty}$ because the convergence of this integral is not guaranteed. Now, in order to compute $v_\alpha(t)$, we first look for a fundamental matrix of the homogeneous equation $\ddot{x} = f'(p(t))x$. We already know that $\dot{p}(t)$ is a solution of the previous equation that satisfies also $\dot{p}(0) = 0$, and $\ddot{p}(0)\neq 0$. So we look for a solution $y(t)$ so that $y(0)\ddot{p}(0) = 1$ and $\dot{y}(0) = 0$. If $y(t)$ is such a solution, Liouville Theorem implies that (Section 2.5.1 and [6])

$$X(t) = \begin{pmatrix} y(t) & \dot{p}(t) \\ \dot{y}(t) & \ddot{p}(t)\end{pmatrix}$$

satisfies $\det X(t) = 1$, that is, $\dot{p}(t)\dot{y}(t)-\ddot{p}(t)y(t) = -1$. Integrating this equation we obtain:

$$y(t) = -\dot{p}(t)\int^t \frac{1}{\dot{p}(s)^2}ds.$$

Note that regardless of the constant we add to the integral, $c\dot{p}(t)$ vanishes for $t = 0$; however the constant is uniquely determined by the condition $\dot{y}(0) = 0$ (from which the equality $y(0)\ddot{p}(0) = 1$ follows). Let $\mu = \sqrt{f'(0)}$. From $p(t) = P(\mathrm{e}^{\mu t})$, we obtain:

$$y(t) = Y(\mathrm{e}^{\mu t}) \tag{7.1.30}$$

where

$$Y(x) = -\frac{1}{\mu^2}xP'(x)\int^x \frac{d\sigma}{\sigma^3[P'(\sigma)]^2}. \tag{7.1.31}$$

Specializing (7.1.31) to Eq. (7.1.27) where $P(x) = \frac{(a^4-1)x}{(x+a^2)(a^2x+1)}$, with $\mu = 2$, we obtain $Y(x) = Y_0(x)+Y_s(x)+Y_b(x)$ where

$$
\begin{aligned}
Y_0(x) &= \frac{3}{2}\frac{a^2(a^8+3a^4+1)}{a^4-1}\frac{x(x^2-1)\log x}{(x+a^2)^2(a^2x+1)^2},\\
Y_s(x) &= \frac{a^2}{8(a^4-1)}\left(x+x^{-1}\right),\\
Y_b(x) &= \frac{3}{4}\frac{a^4+1}{a^4-1} - \frac{a^{16}+52a^{12}+72a^8-4a^4-1}{16a^2(a^4-1)^2(x+a^2)} + \frac{a^{12}+29a^8+29a^4+1}{16(a^4-1)(x+a^2)^2}\\
&\quad - \frac{a^{16}+4a^{12}-72a^8-52a^4-1}{16a^4(a^4-1)^2(a^2x+1)} + \frac{a^{12}+29a^8+29a^4+1}{16a^4(a^4-1)(a^2x+1)^2}.
\end{aligned}
$$

Note that $Y'(1)=0$ and that $Y_0(x)+Y_b(x)$ is bounded on $[0,+\infty)$ while $Y_s(x)$ is unbounded near $x=0$ and infinity. Now, the variation of constants formula gives, for any solution of Eq. (7.1.28):

$$
\begin{aligned}
v_\alpha(t) =& c_1 y(t) + c_2\dot{p}(t) + \int_0^t \left[\dot{p}(t)y(s)-\dot{p}(s)y(t)\right]q(s+\alpha)ds\\
=& \left[c_1 - \int_0^t \dot{p}(s)q(s+\alpha)ds\right]y(t) + \dot{p}(t)\left[c_2+\int_0^t y(s)q(s+\alpha)ds\right].
\end{aligned}
$$

Then, from the boundedness of $q(t)$, the fact that $y(t)$ is of the order $\mathrm{e}^{\mu|t|}$ at $\pm\infty$ and $\dot{p}(t)$ is of the order $\mathrm{e}^{-\mu|t|}$ at $\pm\infty$, we see that the second term is bounded on $\mathbb{R}$. Hence $v_\alpha(t)$ will be bounded on $\mathbb{R}$ if and only if a constant c_1 exists so that

$$
\left[c_1 - \int_0^t \dot{p}(s)q(s+\alpha)ds\right]y(t)
$$

is bounded on $\mathbb{R}$, and this can happen (if and) only if

$$
c_1 = \int_0^{+\infty} \dot{p}(s)q(s+\alpha)ds = \int_{-\infty}^0 \dot{p}(s)q(s+\alpha)ds.
$$

This choice of c_1 is made possible by the fact that $M(\alpha)=0$ and gives:

$$
v_\alpha(t) = y(t)\int_t^\infty \dot{p}(s)q(s+\alpha)ds + \dot{p}(t)\left[c_2+\int_0^t y(s)q(s+\alpha)ds\right].
$$

Note that $v_\alpha(t)$ is bounded on $\mathbb{R}$ for any value of c_2. However we can make it unique by adding the condition $\dot{v}_\alpha(0)=0$. Since $\dot{y}(0)=0$ we see that this is equivalent to choosing $c_2=0$. That is,

$$
v_\alpha(t) = y(t)\int_t^\infty \dot{p}(s)q(s+\alpha)ds + \dot{p}(t)\int_0^t y(s)q(s+\alpha)ds. \tag{7.1.32}
$$

It is worth mentioning that Eq. (7.1.32) gives a bounded solution of Eq. (7.1.28) provided $p(t)$ is a homoclinic solution of $\ddot{x} = f(x)$, and $y(t)$ is defined as in (7.1.30) and (7.1.31).

Now, we write

$$q(t) = q_{even}(t) + q_{odd}(t)$$

where $q_{even}(-t) = q_{even}(t)$ and $q_{odd}(-t) = -q_{odd}(t)$. Then the solution $v_0(t)$ of the equation $\ddot{x} = f'(p(t))x + q(t)$ satisfies $v_0(t) = v_{even}(t) + v_{odd}(t)$ where $v_{even}(t)$ is the (unique) bounded solution of

$$\ddot{x} = f'(p(t))x + q_{even}(t), \quad \dot{x}(0) = 0$$

while $v_{odd}(t)$ is the (unique) bounded solution of

$$\ddot{x} = f'(p(t))x + q_{odd}(t), \quad \dot{x}(0) = 0.$$

From $p(t) = p(-t)$, and the uniqueness of the solutions we get $v_{even}(t) = v_{even}(-t)$ and $v_{odd}(t) = -v_{odd}(-t)$ and then

$$M_2(0) = 2 \lim_{n\to+\infty} \int_{-n\pi}^{n\pi} \dot{v}_{even}(t)q_{odd}(t) + \dot{v}_{odd}(t)q_{even}(t)dt.$$

Now, we consider the situation where $q_{even}(t) = \frac{c}{2} \neq 0$ is constant and different from zero. We obtain immediately:

$$\int_{-n\pi}^{n\pi} \dot{v}_{odd}(t)q_{even}(t)dt = c\, v_{odd}(n\pi).$$

Next, let $u_{odd}(t)$ be the unique bounded solution of $\ddot{x} = f'(0)x + q_{odd}(t)$. From the uniqueness we see that $u_{odd}(t)$ is 2π-periodic and odd, moreover $v_{odd}(t) - u_{odd}(t)$ is a bounded solution of $\ddot{x} = f'(0)x + [f'(p(t)) - f'(0)]v_{odd}(t)$ and hence tends to zero exponentially as $|t| \to +\infty$. As a consequence,

$$\lim_{n\to+\infty} v_{odd}(n\pi) = \lim_{n\to+\infty} u_{odd}(n\pi).$$

On the other hand, $-u_{odd}(-n\pi) = u_{odd}(n\pi) = u_{odd}(-n\pi)$ because of oddness and periodicity. As a consequence, $u_{odd}(n\pi) = 0$ and then

$$M_2(0) = 2 \lim_{n\to+\infty} \int_{-n\pi}^{n\pi} \dot{v}_{even}(t)q_{odd}(t)dt = 2\int_{-\infty}^{\infty} \dot{v}_{even}(t)q_{odd}(t)dt$$

with the last equality being justified by the fact that $v_{even}(t) + \frac{c}{2f'(0)}$ tends to zero, as $|t| \to +\infty$, together with its first derivative, being a bounded solution of

$$\ddot{x} = f'(p(t))x - \frac{c}{2f'(0)}[f'(p(t)) - f'(0)].$$

At this point we note that when $q_{odd}(t+\frac{\pi}{k}) = -q_{odd}(t)$ we have $v_{\pi/k}(t) = v_{even}(t) - v_{odd}(t)$ and hence it is easy to see that

$$M_2(\pi/k) = 2\lim_{n\to+\infty}\int_{n\pi}^{n\pi} \dot{v}_{\pi/k}(t)\left[\frac{c}{2} - q_{odd}(t)\right]dt$$
$$= -2\int_{-\infty}^{\infty} \dot{v}_{even}(t)q_{odd}(t) = -M_2(0)$$

and the theorem follows provided we prove that $M_2(0) \neq 0$.

Now, from Eq. (7.1.32) we obtain:

$$v_{even}(t) = \frac{c}{2}\left(\dot{p}(t)\int_0^t y(s)ds - p(t)y(t)\right) = \frac{c}{2}v(t)$$

where $v(t)$ is defined by the equality. We note that $v(t)$ is the bounded solution of $\ddot{x} = f'(p(t))x + 1$, with $\dot{x}(0) = 0$ and that $v(t) + \frac{1}{f'(0)}$ tends to zero exponentially, as $|t| \to +\infty$, together with its derivative. Moreover,

$$M_2(0) = c\int_{-\infty}^{\infty} \dot{v}(t)q_{odd}(t)dt = c\int_0^{2\pi} r(t)q_{odd}(t)dt$$

where

$$r(t) = \sum_{k\in\mathbb{Z}} \dot{v}(t+2k\pi) \tag{7.1.33}$$

is $2\pi-$periodic and odd. From $p(t) = P(\mathrm{e}^{\mu t})$ and $y(t) = Y(\mathrm{e}^{\mu t})$ we see that $v(t) = V(\mathrm{e}^{\mu t})$ where

$$V(x) = xP'(x)\int_1^x \frac{Y(\sigma)}{\sigma}d\sigma - P(x)Y(x).$$

Note that $V(x)$ is linear in $Y(x)$. Applying the above considerations to Eq. (7.1.27) (hence with $\mu = 2$) we obtain after some integrations:

$$V(x) + \frac{1}{4} = \frac{3a^2x(a^4+1)(1-x^2)\log x}{4(a^2x+1)^2(x+a^2)^2}$$
$$+\frac{x[(a^{12}+23a^8+23a^4+1)(x^2+1)+16a^2(a^8+4a^4+1)x]}{16a^2(a^2x+1)^2(x+a^2)^2}.$$

We set

$$\widetilde{M}_2(\alpha) := \int_{-\infty}^{+\infty} \dot{v}(t)q_{odd}(t+\alpha)dt.$$

Then $\widetilde{M}_2(\alpha)$ is $2\pi-$periodic and $M_2(0) = c\widetilde{M}_2(0)$. Expanding $\widetilde{M}_2(\alpha)$ into its Fourier series we get:

$$\widetilde{M}_2(\alpha) = -\sum_{n\in\mathbb{Z}} \imath n\gamma_n q_n \mathrm{e}^{\imath n\alpha}$$

with q_n being the n-th Fourier coefficient of $q_{odd}(t)$ and

$$\gamma_n = \int_{-\infty}^{+\infty} \left[V(\mathrm{e}^{2t}) + \frac{1}{4} \right] \mathrm{e}^{\imath nt}\, dt.$$

Note that $\imath n\gamma_{-n}/(2\pi)$ are also the Fourier coefficients of the function $r(t)$ defined in (7.1.33). Since $q_{odd}(t)$ is an odd real function we easily get $q_n = \imath c_n$ where c_n are real numbers so that $c_n = -c_{-n}$. Thus

$$\widetilde{M}_2(\alpha) = \sum_{n\in\mathbb{Z}} n\gamma_n c_n \mathrm{e}^{\imath n\alpha}\,. \tag{7.1.34}$$

With $\widetilde{M}_2(\alpha)$ being a real valued function, we also get: $\bar{\gamma}_n = \gamma_{-n}$. Moreover, arguing as in Section 7.1.2 we can evaluate the Fourier coefficients of $\widetilde{M}_2(\alpha)$ by means of residues. For $a = \mathrm{e}^{m\pi}$ and $n \neq 0$, an annoying computation shows that:

$$\gamma_n = \frac{(-1)^{nm}\pi n}{8\sinh\left(n\frac{\pi}{2}\right)} \left[\cosh^2(2m\pi) - 6m\pi\coth(2m\pi) + 2\right],$$

that is,

$$M_2(0) = c\cdot C_m \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{(-1)^{nm} n^2 c_n}{\sinh\left(n\frac{\pi}{2}\right)} = 2c\cdot C_m \sum_{n>0} \frac{(-1)^{nm} n^2 c_n}{\sinh\left(n\frac{\pi}{2}\right)}$$

with C_m being a positive constant. Since $M_2(0) = 0$ gives a codimension one closed linear subspace of $C^1_{odd,2\pi}$, the proof is finished. □

We conclude this section with a remark. Letting $m \to +\infty$ in equation

$$\ddot{x} = 4x(2x^2 - 3x\coth(2m\pi) + 1) \tag{7.1.35}$$

we obtain the equation

$$\ddot{x} = 4x(2x^2 - 3x + 1) \tag{7.1.36}$$

which has two *heteroclinic* connections to the equilibria $x = 0$ and $x = 1$ (Figure 7.1 A). Since the Melnikov function of Eq. (7.1.35) is identically zero for any 2π-periodic perturbation of the equation, one might wonder whether this fact holds for the Melnikov functions associated with the heteroclinic orbits of Eq. (7.1.36). The answer to this question is negative as it can be easily seen by direct evaluation of the Fourier coefficients of the Melnikov function. In fact, let us consider, for example, the heteroclinic solution of (7.1.36) going from $x = 0$ to $x = 1$:

$$p_\infty(t) = \frac{\mathrm{e}^{2t}}{\mathrm{e}^{2t}+1} = R(\mathrm{e}^t)$$

where $R(x) = \frac{x^2}{x^2+1}$. Applying the procedure described in this section we see that the Fourier coefficients of the Melnikov function are given by $\delta_n q_n$ where $\delta_0 = 1$ and for $n \neq 0$:

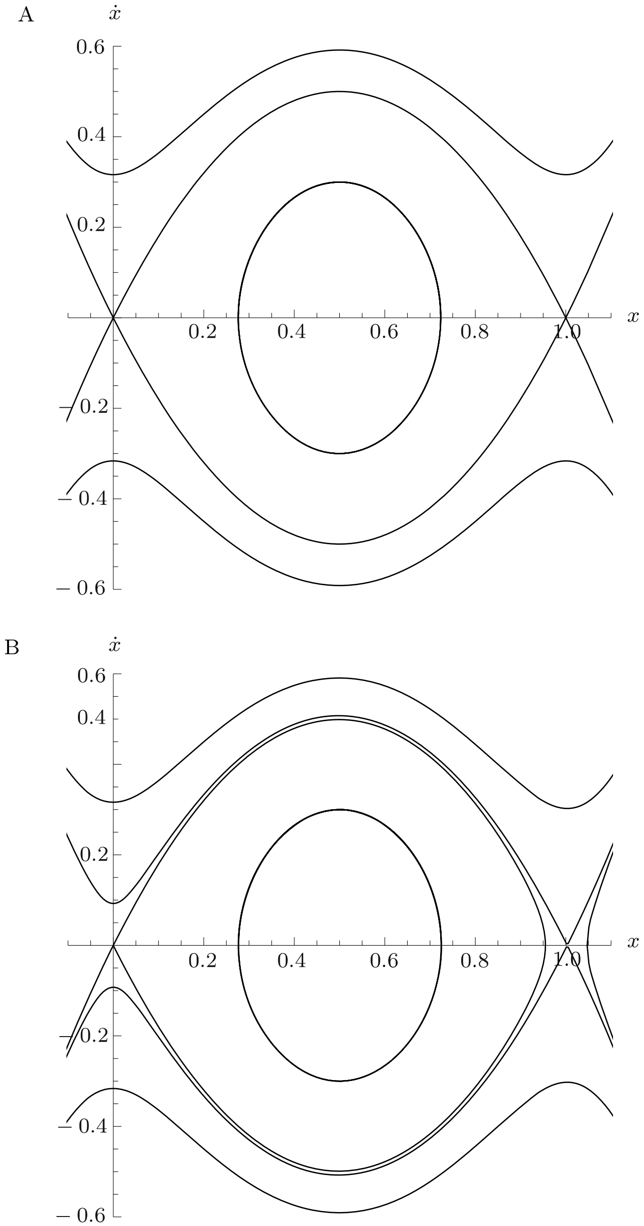

Fig. 7.1 A: The phase portrait of (7.1.36) nearby the heteroclinic cycle. B: The phase portrait of (7.1.35) with $m = 0.6$ nearby the homoclinic orbit.

$$\delta_n = \frac{2n\pi}{1-\mathrm{e}^{-2n\pi}}\left[\mathrm{Res}\left(\frac{u^{\imath n+1}}{u^2+1},\imath\right)+\mathrm{Res}\left(\frac{u^{\imath n+1}}{u^2+1},-\imath\right)\right] = \frac{n\pi}{2\sinh\left(n\frac{\pi}{2}\right)}.$$

Geometrically, this strange behaviour depends on the fact that the homoclinic solution of (7.1.35) gets orbitally closer and closer (as $m \to \infty$) to the *heteroclinic cycle* but not to any of the heteroclinic orbits (Figure 7.1 B). As a matter of fact, setting

$$p_m(t) = \frac{e^{2t}(e^{4m\pi} - 1)}{(e^{2t} + e^{2m\pi})(e^{2t+2m\pi} + 1)},$$

the Melnikov function associated with a heteroclinic solution of (7.1.36) is the limit, for $m \to \infty$ of either:

$$\int_0^{+\infty} \dot{p}_{2m}(t)q(t+\alpha)dt$$

or

$$\int_{-\infty}^{0} \dot{p}_{2m}(t)q(t+\alpha)dt$$

and these are not zero in general. To see this, consider, for example, the heteroclinic solution of (7.1.36), $p_\infty(t)$. We have, for $t \leq 0$:

$$0 \leq \dot{p}_\infty(t+m\pi) - \dot{p}_m(t) = \frac{1}{2\cosh^2(m\pi - t)} \leq 2\,e^{2t}\,.$$

From Lebesgue's theorem we get then:

$$\lim_{m\to+\infty} \int_{-\infty}^{0} [\dot{p}_\infty(t+m\pi) - \dot{p}_m(t)]b(t)dt = 0$$

for any L^∞-function $b(t)$, and hence:

$$\begin{aligned}\int_{-\infty}^{\infty} \dot{p}_\infty(t)q(t+\alpha)dt &= \lim_{m\to+\infty} \int_{-\infty}^{2m\pi} \dot{p}_\infty(t)q(t+\alpha)dt \\ &= \lim_{m\to+\infty} \int_{-\infty}^{0} \dot{p}_\infty(t+2m\pi)q(t+\alpha)dt = \lim_{m\to+\infty} \int_{-\infty}^{0} \dot{p}_{2m}(t)q(t+\alpha)dt.\end{aligned}$$

A similar argument shows that

$$\int_{-\infty}^{\infty} \dot{p}_\infty(t+\pi)q(t+\alpha)dt = \lim_{m\to+\infty} \int_{-\infty}^{0} \dot{p}_{2m+1}(t)q(t+\alpha)dt.$$

Finally we note that $\coth(2\pi) \doteq 1.00000698$, while $\coth(1.2\pi) \doteq 1.001064$, so for $m = 1$ the phases portraits of (7.1.35) and (7.1.36) graphically coincide. For this reason we consider $m = 0.6$ in Figure 7.1 B.

7.2 Transverse Heteroclinic Cycles

The purpose of this section is to show the existence of a transversal homoclinic orbit near a transversal heteroclinic cycle. This gives a chaos near a transversal heteroclinic cycle. To start with, we extend and study the above relationship between (7.1.35) and (7.1.36) for more general systems. More precisely, we study the relationship between

$$\ddot{x} = 4x(2x^2 - 3x\coth(2m\pi) + 1) + \varepsilon q(t)\,, \qquad (7.2.1)$$

and

$$\ddot{x} = 4x(2x^2 - 3x + 1) + \varepsilon q(t)\,. \qquad (7.2.2)$$

To this end, and to help the readers in understanding the assumptions we make, we observe that the difference between the r.h.s. of Eqs. (7.2.1) and (7.2.2), given by:

$$12x^2(1 - \coth(2m\pi)) = -\frac{24x^2}{\mathrm{e}^{4m\pi} - 1}$$

tends to zero as $m \to \infty$ uniformly on compact sets and the same holds for its derivative with respect to x. Hence we consider a family of T-periodic differential equations

$$\dot{x} = f_m(t,x) = f_m(t+T,x) \qquad (7.2.3)$$

where either $m \in \mathbb{N}$ or $m = \infty$, $t \in \mathbb{R}$, and $x \in \Omega$, an open and bounded subset of $\mathbb{R}^n$. We assume that $f_m(t,x)$ are C^2 functions in $(t,x) \in \mathbb{R} \times \Omega$, and that the following conditions hold:

(a) $\dot{x} = f_\infty(t,x)$ has a *transversal heteroclinic cycle* in Ω made of two hyperbolic T-periodic solutions $p_i(t)$, $i = 0,1$, and two heteroclinic orbits $p_\infty^{(01)}(t)$ and $p_\infty^{(10)}(t)$ connecting them, that is,

$$\begin{aligned} &\lim_{t\to-\infty}\left[p_\infty^{(01)}(t) - p_0(t)\right] = \lim_{t\to\infty}\left[p_\infty^{(10)}(t) - p_0(t)\right] = 0\,,\\ &\lim_{t\to\infty}\left[p_\infty^{(01)}(t) - p_1(t)\right] = \lim_{t\to-\infty}\left[p_\infty^{(10)}(t) - p_1(t)\right] = 0\,. \end{aligned} \qquad (7.2.4)$$

(b) $f_\infty(t,x)$ is a regular perturbation of $f_m(t,x)$, that is,

$$\begin{aligned} &\sup_{(t,x)\in\mathbb{R}\times\Omega} |f_m(t,x) - f_\infty(t,x)| \to 0 \quad \text{as} \quad m \to \infty\,,\\ &\sup_{(t,x)\in\mathbb{R}\times\Omega} |D_2 f_m(t,x) - D_2 f_\infty(t,x)| \to 0 \quad \text{as} \quad m \to \infty\,. \end{aligned} \qquad (7.2.5)$$

Note that by transversality of the heteroclinic cycle we mean that the stable and unstable manifolds of the periodic orbits $p_i(t)$ intersect transversally along both $p_\infty^{(01)}(t)$ and $p_\infty^{(10)}(t)$.

By using the implicit function theorem, it is not difficult to show (cf [1] and Remark 4.1.7) that the conditions (a) and (b) imply that for any m sufficiently large, the T-periodic nonlinear system

$$\dot{x} = f_m(t,x)$$

has unique T-periodic solutions $q_m(t)$ and $r_m(t)$ so that $\sup_{t\in\mathbb{R}} |q_m(t) - p_0(t)| \to 0$ and $\sup_{t\in\mathbb{R}} |r_m(t) - p_1(t)| \to 0$ as $m \to \infty$. Moreover, both $q_m(t)$ and $r_m(t)$ are hyperbolic.

The purpose of this section is to prove the following:

Theorem 7.2.1. *There exists $m_0 \in \mathbb{N}$ so that for any $m \in \mathbb{N}$, $m > m_0$, system* (7.2.3) *has an orbit $p_m(t)$ homoclinic to $q_m(t)$ so that*

$$\begin{aligned}
&\sup_{t\le 0} |p_m(t) - p_\infty^{(01)}(t+mT)| \to 0 \quad as \quad m \to \infty\,, \\
&\sup_{t\ge 0} |p_m(t) - p_\infty^{(10)}(t-mT)| \to 0 \quad as \quad m \to \infty\,.
\end{aligned} \tag{7.2.6}$$

Moreover, the stable and unstable manifolds of the periodic solution $q_m(t)$ of system (7.2.3) *intersect transversely along the homoclinic solution $p_m(t)$.*

Proof. To simplify the proof, we first replace x with $y = x + p_0(t) - q_m(t)$. We obtain the family of equations

$$\dot{y} = \hat{f}_m(t,y)$$

where

$$\hat{f}_m(t,y) = f_m(t, y - p_0(t) + q_m(t)) + f_\infty(t, p_0(t)) - f_m(t, q_m(t))$$

and

$$\hat{f}_\infty(t,y) = f_\infty(t,y)\,.$$

Note that the family $\hat{f}_m(t,y)$ satisfies assumptions (a) and (b). Hence, without loss of generality, we suppose in this proof that for any $m \in \mathbb{N}$ system (7.2.3) has the periodic solution $p_0(t)$.

From the transversality assumption of the heteroclinic cycle (cf Lemma 2.5.2) it follows that the linear system

$$\dot{x} = D_2 f_\infty(t, p_\infty^{(01)}(t))x \tag{7.2.7}$$

has an exponential dichotomy on $\mathbb{R}$ with projection, say, Q_∞, that is, the fundamental matrix $X_\infty(t)$ of (7.2.7) satisfies

$$\begin{aligned}
&\|X_\infty(t) Q_\infty X_\infty^{-1}(s)\| \le K \mathrm{e}^{-\delta(t-s)}, \quad s \le t\,, \\
&\|X_\infty(t) [\mathbb{I} - Q_\infty] X_\infty^{-1}(s)\| \le K \mathrm{e}^{\delta(t-s)}, \quad t \le s
\end{aligned}$$

for a constant $\delta > 0$. Similarly there exists a projection P_∞ so that the fundamental matrix $Y_\infty(t)$ of the linear system

$$\dot{x} = D_2 f_\infty(t, p_\infty^{(10)}(t))x$$

satisfies

$$\|Y_\infty(t)P_\infty Y_\infty^{-1}(s)\| \le K\,\mathrm{e}^{-\delta(t-s)}, \quad s \le t\,,$$
$$\|Y_\infty(t)[\mathbb{I}-P_\infty]Y_\infty^{-1}(s)\| \le K\,\mathrm{e}^{\delta(t-s)}, \quad t \le s\,.$$

Now, it is clear that the fundamental matrix $X_\infty(t+mT)X_\infty^{-1}(mT)$ of the linear system $\dot{x} = D_2 f_\infty(t, p_\infty^{(01)}(t+mT))x$ has also an exponential dichotomy on $\mathbb{R}$ with projection matrix

$$Q_\infty(mT) = X_\infty(mT)Q_\infty X_\infty^{-1}(mT),$$

and similarly, the fundamental matrix $Y_\infty(t-mT)Y_\infty^{-1}(-mT)$ of the linear system $\dot{x} = D_2 f_\infty(t, p_\infty^{(10)}(t-mT))x$ has an exponential dichotomy on $\mathbb{R}$ with projection matrix

$$P_\infty(-mT) = Y_\infty(-mT)P_\infty Y_\infty^{-1}(-mT).$$

We seek for a solution $p_m(t)$ of the nonlinear system (7.2.3), with $m \in \mathbb{N}$, sufficiently large, so that (7.2.6) holds. Hence, setting

$$x_1(t) = p_m(t) - p_\infty^{(01)}(t+mT), \quad x_2(t) = p_m(t) - p_\infty^{(10)}(t-mT)\,,$$

we look for a pair of functions $(x_1(t), x_2(t))$ so that

$$\sup_{t\le 0} |x_1(t)| \quad \text{and} \quad \sup_{t\ge 0} |x_2(t)|$$

are small, and actually tend to zero as $m \to +\infty$, satisfying:

$$\begin{aligned}
&\dot{x}_1 - D_2 f_\infty(t, p_\infty^{(01)}(t+mT))x_1 = h_m^-(t,x_1) \quad \text{for} \quad t \le 0\,,\\
&\dot{x}_2 - D_2 f_\infty(t, p_\infty^{(10)}(t-mT))x_2 = h_m^+(t,x_2) \quad \text{for} \quad t \ge 0\,,\\
&x_2(0) - x_1(0) = b_m := p_\infty^{(01)}(mT) - p_\infty^{(10)}(-mT)\,,
\end{aligned} \tag{7.2.8}$$

where

$$\begin{aligned}
h_m^-(t,x) = \;& f_\infty(t, x + p_\infty^{(01)}(t+mT)) - f_\infty(t, p_\infty^{(01)}(t+mT))\\
&- D_2 f_\infty(t, p_\infty^{(01)}(t+mT))x\\
&+ f_m(t, x + p_\infty^{(01)}(t+mT)) - f_\infty(t, x + p_\infty^{(01)}(t+mT))\,,\\
h_m^+(t,x) = \;& f_\infty(t, x + p_\infty^{(10)}(t-mT)) - f_\infty(t, p_\infty^{(10)}(t-mT))\\
&- D_2 f_\infty(t, p_\infty^{(10)}(t-mT))x\\
&+ f_m(t, x + p_\infty^{(10)}(t-mT)) - f_\infty(t, x + p_\infty^{(10)}(t-mT))\,.
\end{aligned}$$

Note that $b_m = o(1)$ as $m \to \infty$. Let $\rho > 0$ be a fixed positive number so that the closure of the sets:

$$\left\{x + p_\infty^{(01)}(t) \mid t \in \mathbb{R},\, |x| \le \rho\right\}, \quad \left\{x + p_\infty^{(10)}(t) \mid t \in \mathbb{R},\, |x| \le \rho\right\}$$

is contained in Ω. Then note that

$$\begin{aligned} \sup_{t\in\mathbb{R}_\pm,|x|\le\rho} |h_m^\pm(t,x)| &= \Delta^\pm(|x|)|x| + o_1^\pm(1)\,, \\ \sup_{t\in\mathbb{R}_\pm,|x|\le\rho} |D_2 h_m^\pm(t,x)| &= \Delta^\pm(|x|) + o_2^\pm(1) \end{aligned} \tag{7.2.9}$$

where

$$\Delta^-(r) = \sup_{t\in\mathbb{R},|x|\le r} |D_2 f_\infty(t, x + p_\infty^{(01)}(t+mT)) - D_2 f_\infty(t, p_\infty^{(01)}(t+mT))|,$$
$$\Delta^+(r) = \sup_{t\in\mathbb{R},|x|\le r} |D_2 f_\infty(t, x + p_\infty^{(10)}(t+mT)) - D_2 f_\infty(t, p_\infty^{(10)}(t+mT))|$$

are positive increasing functions so that $\Delta^\pm(r) \to 0$ as $r \to 0$ uniformly with respect to $m \in \mathbb{N}$ (see (7.2.5)) and, for example

$$o_1^-(1) = \sup_{t\in\mathbb{R},|x|\le r} |f_m(t, x + p_\infty^{(01)}(t+mT)) - f_\infty(t, x + p_\infty^{(01)}(t+mT))| \to 0$$

as $m \to \infty$, uniformly with respect to $t \in \mathbb{R}$ and $|x| \le \rho$, because of assumption (b). Of course, a similar conclusion holds as far as $o_1^+(1)$ and $o_2^\pm(1)$ are concerned.

Owing to the exponential dichotomy, any solution of the first two equations in (7.2.8) whose sup-norm in $(-\infty, 0]$ is less than a given $r > 0$ satisfies

$$\begin{aligned} x_1(t) = {} & X_\infty(t+mT)[\mathbb{I} - Q_\infty]X_\infty^{-1}(mT)\xi \\ & + \int_{-\infty}^t X_\infty(t+mT)Q_\infty X_\infty^{-1}(s+mT)h_m^-(s, x_1(s))ds \\ & - \int_t^0 X_\infty(t+mT)[\mathbb{I} - Q_\infty]X_\infty^{-1}(s+mT)h_m^-(s, x_1(s))ds \end{aligned} \tag{7.2.10}$$

and similarly

$$\begin{aligned} x_2(t) = {} & Y_\infty(t-mT)P_\infty Y_\infty^{-1}(-mT)\eta \\ & + \int_0^t Y_\infty(t-mT)P_\infty Y_\infty^{-1}(s-mT)h_m^+(s, x_2(s))ds \\ & - \int_t^\infty Y_\infty(t-mT)[\mathbb{I} - P_\infty]Y_\infty^{-1}(s-mT)h_m^+(s, x_2(s))ds\,. \end{aligned} \tag{7.2.11}$$

A classical argument shows (cf Section 4.1) that the maps defined by the right-hand sides define contractions on the appropriate spaces $C_b^0(\mathbb{R}_-, \mathbb{R}^n)$ and $C_b^0(\mathbb{R}_+, \mathbb{R}^n)$ of bounded continuous functions on $\mathbb{R}_-$ and $\mathbb{R}_+$ respectively, provided $m \ge m_0$ is sufficiently large, $6K|\xi| < r$, $6K|\eta| < r$, and $\|x_i\| < r$ where $r > 0$ is such that

$$3K\delta^{-1}\Delta^\pm(r) < 1.$$

Let $x_1(t,\xi,m)$, $x_2(t,\eta,m)$ be the solutions of the above fixed point equations. From Eq. (7.2.10) and the properties of the functions $h_m^{\pm}(s,x)$ we easily obtain

$$\sup_{t\le 0}|x_1(t,\xi,m)| \le K|\xi| + 2K\delta^{-1}\left[\Delta^-(r)\sup_{t\le 0}|x_1(t,\xi,m)| + o_1^-(1)\right]$$

and then

$$\sup_{t\le 0}|x_1(t,\xi,m)| \le 3K|\xi| + o_1(1) < r \tag{7.2.12}$$

where $o_1(1)\to 0$ as $m\to+\infty$ uniformly with respect to ξ, η. Similarly:

$$\sup_{t\ge 0}|x_2(t,\eta,m)| \le 3K|\eta| + o_2(1) < r \tag{7.2.13}$$

where $o_2(1)\to 0$ as $m\to+\infty$ uniformly with respect to ξ, η. In order to find $p_m(t)$ we have to solve the equation:

$$\begin{aligned} &P_\infty(-mT)\eta - [\mathbb{I}-Q_\infty(mT)]\xi \\ &= \int_0^\infty Y_\infty(-mT)[\mathbb{I}-P_\infty]Y_\infty^{-1}(s-mT)h_m^+(s,x_2(s,\eta,m))ds \\ &\quad + \int_{-\infty}^0 X_\infty(mT)Q_\infty X_\infty^{-1}(s+mT)h_m^-(s,x_1(s,\xi,m))ds + b_m\,. \end{aligned} \tag{7.2.14}$$

Now, according to (7.2.4) and (7.2.5), $D_2f_\infty(t,p_\infty^{(01)}(t+mT))$ and $D_2f_\infty(t,p_\infty^{(10)}(t-mT))$ tend to $D_2f_\infty(t,p_1(t))$, uniformly in compact intervals in $\mathbb{R}$ as $m\to\infty$. Hence from Lemma 2.5.1 the projections $P_\infty(-mT)$ and $Q_\infty(mT)$ tend, as $m\to\infty$, to the projection $\mathscr{P}$ of the dichotomy of $\mathbb{R}$ of the linear system along $p_1(t)$:

$$\dot{x} = D_2f_\infty(t,p_1(t))x\,.$$

Thus for any $m\in\mathbb{N}$ sufficiently large $\|P_\infty(-mT)\|$ and $\|\mathbb{I}-Q_\infty(mT)\|$ are bounded below by a positive constant. Then, using (7.2.9), (7.2.12) and (7.2.13), we see that the right-hand side of (7.2.14) is bounded by a term like

$$3K^2\delta^{-1}[\Delta^-(3K|\xi|+o_1(1))|\xi| + \Delta^+(3K|\eta|+o_2(1))|\eta|] + o(1) \tag{7.2.15}$$

where $o(1)\to 0$ as $m\to\infty$ uniformly with respect to $|\xi|$, $|\eta|$. Thus by using the implicit function theorem, we see that (7.2.14) can be uniquely solved for $\xi=\xi_m$ and $\eta=\eta_m$. Moreover, since the expression in (7.2.15) tends to zero as $\xi\to 0$, $\eta\to 0$ and $m\to+\infty$, we easily see, from the uniqueness, that ξ_m and η_m tend to zero as $m\to\infty$. We set

$$p_m(t) = \begin{cases} x_1(t,\xi_m,m) + p_\infty^{(01)}(t+mT), & \text{if} \quad t\le 0\,, \\ x_2(t,\eta_m,m) + p_\infty^{(10)}(t-mT), & \text{if} \quad t\ge 0\,. \end{cases}$$

Then $p_m(t)$ satisfies (7.2.6) because of (7.2.12), (7.2.13) and the fact that $|\xi_m|, |\eta_m| \to 0$ as $m \to +\infty$ and hence $|t|$ sufficiently large remains in a small neighborhood of the periodic orbit $p_0(t)$. Thus, because of the saddle node property of hyperbolic periodic solutions, $p_m(t)$ is homoclinic to $p_0(t)$.

To complete the proof of the theorem we have to show that the stable and unstable manifolds $W_m^s(p_0)$ and $W_m^s(p_0)$ of the solution $p_0(t)$ of (7.2.3) with $m \geq m_0$ intersect transversely along $p_m(t)$. From the hyperbolicity of the periodic solution $p_0(t)$ and the roughness of exponential dichotomies, for any $m \geq m_0$, the linear systems

$$\dot{x} = D_2 f_m(t, p_m(t))x \tag{7.2.16}$$

have an exponential dichotomy on $\mathbb{R}_-$ with projections Q_m, that is, the fundamental matrix $X_m(t)$ of (7.2.16) satisfies:

$$\begin{aligned} \|X_m(t)Q_m X_m^{-1}(s)\| &\leq k\mathrm{e}^{-\delta(t-s)}, \quad s \leq t \leq 0, \\ \|X_m(t)[\mathbb{I} - Q_m]X_m^{-1}(s)\| &\leq k\mathrm{e}^{\delta(t-s)}, \quad t \leq s \leq 0. \end{aligned}$$

Moreover, from (7.2.4), (7.2.5), (7.2.6) it follows [7] that the projections Q_m can be chosen so that

$$\lim_{m\to\infty} |Q_m - Q_\infty(mT)| = 0. \tag{7.2.17}$$

On the other hand, Eq. (7.2.16) has also an exponential dichotomy on $\mathbb{R}_+$ with projection, say, P_m and we can similarly assume that

$$\lim_{m\to\infty} |P_m - P_\infty(-mT)| = 0. \tag{7.2.18}$$

We now describe the unstable manifold $W_m^u(p_0)$ of $p_0(t)$. Let $x_m(t,\xi)$ be the solution of (7.2.3) so that $x(0) = \xi$. We have

$$W_m^u(p_0) = \left\{ \widetilde{\xi} \in \mathbb{R} : \lim_{t\to-\infty} |x_m(t,\widetilde{\xi}) - p_0(t)| = 0 \right\}.$$

Because of the exponential dichotomy, the solutions occurring in the definition of $W_m^u(p_0)$ can be written as $x_m(t,\widetilde{\xi}) = z_m(t) + p_m(t)$, where $z_m(t) = z_m(t,\xi)$ is the unique solution of the implicit equation:

$$\begin{aligned} z_m(t) = X_m(t)[\mathbb{I} - Q_m]\xi &+ \int_{-\infty}^{t} X_m(t)Q_m X_m^{-1}(s) h_m(s, z_m(s))ds \\ &- \int_t^0 X_m(t)[\mathbb{I} - Q_m]X_m^{-1}(s) h_m(s, z_m(s))ds, \end{aligned} \tag{7.2.19}$$

where $\xi = \widetilde{\xi} - p_m(0)$ and

$$h_m(t,z) = f_m(t, z + p_m(t)) - f_m(t, p_m(t)) - D_2 f_m(t, p_m(t))z. \tag{7.2.20}$$

Note that (7.2.19) defines $z_m(t)$ for $t \leq 0$, however $z_m(t)$ can be extended up to any finite time mT and satisfies the same formula. Moreover, because of the uniqueness, we have $z_m(t,0) = 0$. Thus the tangent space of $W^u_\infty(p_0)$ at the point $p^{01}_\infty(mT)$ is spanned by the vectors

$$X_\infty(mT)\,[\mathbb{I} - Q_\infty]\,\xi = [\mathbb{I} - Q_\infty(mT)]X_\infty(mT)\xi$$

while the tangent space of $W^u_m(p_0)$ at the point $p_m(0)$ is spanned by the vectors like $[\mathbb{I} - Q_m]\xi$ where we used the identity $X_m(0) = \mathbb{I}$. Thus

$$T_{p_m(0)}W^u_m(p_0) = \mathscr{N}Q_m\,, \quad T_{p^{(01)}_\infty(mT)}W^u_\infty(p_0) = \mathscr{N}Q_\infty(mT)\,.$$

Similarly

$$T_{p_m(0)}W^s_m(p_0) = \mathscr{R}P_m\,, \quad T_{p^{(10)}_\infty(-mT)}W^s_\infty(p_0) = \mathscr{R}P_\infty(-mT)\,.$$

Thus in order to show the transversality of the intersection of $W^s_m(p_0)$ and $W^u_m(p_0)$ along $p_m(t)$ we have to show that $\mathbb{R}^n = \mathscr{R}P_m \oplus \mathscr{N}Q_m$. But, we have already seen that $Q_\infty(mT)$ and $P_\infty(-mT)$ tend, as $m \to \infty$, to the projection $\mathscr{P}$ of the dichotomy on $\mathbb{R}$ of the linear system along $p_1(t)$:

$$\dot{x} = D_2 f_\infty(t, p_1(t))x\,.$$

So, using also (7.2.17), (7.2.18), we have

$$\lim_{m\to\infty} \|Q_m - \mathscr{P}\| = 0$$

and similarly

$$\lim_{m\to\infty} \|P_m - \mathscr{P}\| = 0\,.$$

Thus we can assume $m_0 \in \mathbb{N}$ is so large that

$$\mathbb{R}^n = \mathscr{R}P_m \oplus \mathscr{N}Q_m$$

for any $m \geq m_0$ and then the stable and unstable manifolds $W^s_m(p_0)$ and $W^u_m(p_0)$ intersect transversally along $p_m(t)$. The proof is finished. □

Remark 7.2.2. Theorem 7.2.1 holds also for the periodic solutions $r_m(t)$.

As an application of this result we can consider the family of second order equations (7.2.1) whose limiting equation, for $m \to \infty$, is (7.2.2). We know that the unperturbed limit equation (7.1.36) has the heteroclinic cycle made of the two heteroclinic connections $p_\infty(t)$ and $p_\infty(-t)$. Moreover the Melnikov functions associated with both heteroclinic orbits have a transverse zero at least for infinitely many 2π-periodic C^1-functions $q(t)$. Hence, for any $\varepsilon \neq 0$ sufficiently small, $\alpha_\pm(\varepsilon)$ exist so that Eq. (7.2.2) has hyperbolic periodic solutions $p_0(t,\varepsilon)$, $p_1(t,\varepsilon)$ and bounded solutions $p_\pm(t,\varepsilon)$ so that

$$\sup_{t\in\mathbb{R}} |p_0(t,\varepsilon)| \to 0\,, \quad \sup_{t\in\mathbb{R}} |p_1(t,\varepsilon)-1| \to 0,$$
$$\sup_{t\in\mathbb{R}} |p_+(t,\varepsilon)-p_\infty(t-\alpha_+(\varepsilon))| \to 0,$$
$$\sup_{t\in\mathbb{R}} |p_-(t,\varepsilon)-p_\infty(-t-\alpha_-(\varepsilon))| \to 0$$

as $\varepsilon \to 0$ and the same holds for the $t-$derivative. Moreover the variational equations of (7.2.2) along $p_\pm(t,\varepsilon)$ have an exponential dichotomy on $\mathbb{R}$. Since $p_\infty(t)$ tends to $x=0$ when $t \to -\infty$ and to $x=1$ when $t \to +\infty$, and the periodic solutions $p_0(t,\varepsilon)$, $p_1(t,\varepsilon)$ have the saddle point property, we easily obtain that the solutions $\{p_0(t,\varepsilon), p_+(t,\varepsilon), p_1(t,\varepsilon), p_-(t,\varepsilon)\}$ form a heteroclinic cycle, which is transverse thanks to the exponential dichotomy of the linear systems. Thus, the result of this section applies and we obtain the following:

Theorem 7.2.3. *If the function $q(t)$ in system* (7.2.1) *is such that the Melnikov functions associated with $p_\infty(\pm t)$ of the limiting equation* (7.2.2) *have transverse zeroes, then for any given $0<|\varepsilon|<\varepsilon_0$, sufficiently small, there exists $m(\varepsilon)$ so that for any $m>m(\varepsilon)$ system* (7.2.1) *has a transversal homoclinic orbit $p_{m,\varepsilon}(t)$ with such a $q(t)$, therefore,*

$$\sup_{t\le 0} |p_{m,\varepsilon}(t)-p_+(t+mT,\varepsilon)| \to 0 \quad \text{as} \quad m\to\infty\,,$$
$$\sup_{t\ge 0} |p_{m,\varepsilon}(t)-p_-(t-mT,\varepsilon)| \to 0 \quad \text{as} \quad m\to\infty\,.$$

Theorem 7.2.1 can be related to the following result [8]:

Lemma 7.2.4. *Let $f:\mathbb{R}^n \to \mathbb{R}^n$ be a C^1-diffeomorphism possessing hyperbolic fixed points p_1 and p_0. If $W^u_{p_1}$ transversally intersects $W^s_{p_2}$ and $W^s_{p_1}$ transversally intersects $W^u_{p_2}$, then $W^u_{p_i}$ transversally intersects $W^s_{p_i}$ for $i=1,2$.*

Indeed, assuming that (a) holds. we take $f_m(t,x)=f_\infty(t,x)$. Then we get a transversal homoclinic solution for $\dot{x}=f_\infty(t,x)$, so the existence of a transverse heteroclinic cycle gives a chaos. Lemma 7.2.4 is proved in [8] by using the λ-lemma. Our proof, instead, is based on such notions as exponential dichotomies and roughness. We emphasize the fact that our proof is more constructive than the one given in [8] and also leads to Eq. (7.2.6) which allows of locating, within some small error, the homoclinic orbit. Finally we note that the above results can be directly extended to transverse heteroclinic cycles consisting of a finite number of transverse heteroclinic orbits connecting hyperbolic periodic solutions. Such transverse heteroclinic cycles occur often for symmetric systems of ODEs [9].

7.3 Blue Sky Catastrophes

Typically a family of periodic orbits of scalar second order equations terminates in either to equilibria or to heteroclinic/homoclinic orbits. In the last case the minimal periods of periodic orbits increase to infinity as orbits approach heteroclinic/homoclinic cycles, i.e. we have a *period blow-up* or *blue sky catastrophe* [10–12]. Clearly this phenomenon occurs in the Smale-Birkhoff homoclinic theorem 2.5.4. The purpose of this section is to survey certain results in this direction. Other types of blue sky catastrophes are investigated in [3].

7.3.1 Symmetric Systems with First Integrals

In this section, we consider a smooth system

$$\dot{u} = f(u) \tag{7.3.1}$$

with some symmetry conditions, that is, either (7.3.1) is *equivariant* (i.e. $f(Tu) = Tf(u)$ for a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ with $T^p = \mathbb{I}$ for some $p \in \mathbb{N}$) or (7.3.1) is periodic (i.e. $f(u+\tau) = f(u)$ for some $\tau \in \mathbb{R}^n \setminus \{0\}$). Furthermore, we suppose that (7.3.1) has a heteroclinic orbit $\gamma(t)$ to hyperbolic equilibrium points p_0, p_1 satisfying $Tp_0 = p_1$, if (7.3.1) is equivariant, or $p_0 + \tau = p_1$, if (7.3.1) is periodic.

If (7.3.1) is equivariant then we look for solutions satisfying

$$Tu(t) = u(t+\omega) \quad \forall t \in \mathbb{R} \tag{7.3.2}$$

for a $\omega > 0$, and if (7.3.1) is periodic then we look for solutions satisfying

$$u(t) + \tau = u(t+\omega) \quad \forall t \in \mathbb{R} \tag{7.3.3}$$

for a $\omega > 0$. Now we can state the following result [9].

Theorem 7.3.1. *Let* (7.3.1) *satisfy the above assumptions. Furthermore, assume that the variational system* $\dot{v} = Df(\gamma(t))v$ *has the unique bounded solution* $\dot{\gamma}(t)$ *up to a multiplicative constant. If* (7.3.1) *has a symmetric smooth first integral* $H : \mathbb{R}^n \to \mathbb{R}$ *(i.e. either* $H(Tu) = H(u)$ *if* (7.3.1) *is equivariant, or* $H(u+\tau) = H(u)$ *if* (7.3.1) *is periodic) and* $DH\big(\gamma(0)\big) \neq 0$, *then there is a* $\omega_0 > 0$ *so that*

(a) *If* (7.3.1) *is equivariant then it has a* ωp*–periodic solution* u_ω *for any* $\omega \geq \omega_0$ *satisfying* (7.3.2) *and this solution accumulates on the set* $\overline{\cup_{k=1}^{p}\{T^k\gamma(t) \mid t \in \mathbb{R}\}}$ *as* $\omega \to \infty$.
(b) *If* (7.3.1) *is periodic then it has a solution* u_ω *for any* $\omega \geq \omega_0$ *satisfying* (7.3.3) *and this solution accumulates on the set* $\overline{\cup_{k\in\mathbb{Z}}\{\gamma(t) + k\tau \mid t \in \mathbb{R}\}}$ *as* $\omega \to \infty$.

Theorem 7.3.1 is a generalization of results of [10–12] to symmetric conservative systems.

7.3.2 D'Alembert and Penalized Equations

Let $\mathscr{M}$ be a smooth orientable submanifold of $\mathbb{R}^n$. Let D_t be the *covariant derivative* along the tangent bundle $T\mathscr{M}$ and $P_z : \mathbb{R}^n \to T_z\mathscr{M}$ be the orthogonal projection on $T_z\mathscr{M}$ along the normal space $T_z\mathscr{M}^{\perp}$. We recall that given a smooth curve $z(t)$ on $\mathscr{M}$ and a vector field $Y(t) \in T_{z(t)}\mathscr{M}$, the covariant derivative $D_tY(t)$ is defined as

$$D_tY(t) = P_{z(t)}\dot{Y}(t)$$

(see [13, p. 305–306]). It is well known [14] that the constrained motion on $\mathscr{M}$ of a second order smooth ODE given by

$$\ddot{z} + F(z) = 0 \tag{7.3.4}$$

is determined by the *D'Alembert equation* of the form

$$D_t\dot{z} + P_zF(z) = 0\,. \tag{7.3.5}$$

By *penalized equation* of (7.3.4), instead, we mean the equation [14, 15]

$$\ddot{z} + F(z) + \varepsilon^{-2}G(z) = 0 \tag{7.3.6}$$

where $\varepsilon > 0$ is small and $G(z)$ is a smooth function vanishing on $\mathscr{M}$ so that for any $x \in \mathscr{M}$, $G'(x)$ is positively definite on the normal space $T_x\mathscr{M}^{\perp}$. Often, in the applications, $G(x)$ is the gradient $\nabla U(x)$ of a smooth function vanishing on $\mathscr{M}$. Thus the previous condition means that $U(x) \geq 0$ in a neighborhood of $\mathscr{M}$ and any point of $\mathscr{M}$ is a strict local minimum of $U(x)$ in the direction normal to $\mathscr{M}$.

Setting $z_1 = z$, $z_2 = \varepsilon\dot{z}$, Eq. (7.3.6) reads:

$$\begin{aligned} \varepsilon\dot{z}_1 &= z_2, \\ \varepsilon\dot{z}_2 &= -G(z_1) - \varepsilon^2F(z_1) \end{aligned}$$

which has the form of a singularly perturbed system (cf Section 4.4.1). Passing to the time $\tau = t/\varepsilon$ and setting $\varepsilon = 0$, we obtain the system

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -G(z_1) \end{aligned}$$

whose local set of equilibria is $(z_1, z_2) = (x, 0)$, $x \in \mathscr{M}$, and the Jacobian matrix at these points is

$$\hat{J}(x) = \begin{pmatrix} 0 & \mathbb{I} \\ -G'(x) & 0 \end{pmatrix}.$$

Now, since $G'(x)v = 0$ for any $v \in T_x\mathscr{M}$, we see that zero is an eigenvalue of $\hat{J}(x)$, for any $x \in \mathscr{M}$ and that the eigenvectors of the zero eigenvalue are those $v \in \mathbb{R}^{2n}$ so that $v = \begin{pmatrix} u \\ 0 \end{pmatrix}$ with $u \in T_x\mathscr{M}$. Next, if $\lambda \neq 0$ is another non-zero eigenvalue of $\hat{J}(x)$ there should exist $v, w \in \mathbb{C}^n$ so that

$$w = \lambda v, \quad -G'(x)v = \lambda w.$$

Thus $-\lambda^2$ is one of the positive eigenvalues of $G'(x)$ and v is an eigenvector of $-\lambda^2$. Hence we are in a situation where geometric singular perturbation theory [16] cannot be applied, because the manifold of equilibria is far from being *normally hyperbolic*. On the contrary, the presence of positive eigenvalues of $G'(x)$ makes the manifold of equilibria not normally hyperbolic and the problem is resonant. We speak of *elliptic singularly perturbed problem*. Now we can state the following result [17, 18].

Theorem 7.3.2. *Suppose that $F(z) \in C^3$, $G(z) \in C^5$ and $\mathscr{M}$ is an orientable C^5-smooth submanifold with codimension m of $\mathbb{R}^n$. Moreover assume that the following conditions hold:*

(1) $G(z) = 0$ *for any* $z \in \mathscr{M}$ *and* $P_{z_0}F(z_0) = 0$ *for some* $z_0 \in \mathscr{M}$;
(2) $T_z\mathscr{M}^\perp$ *has an orthonormal basis* $\{n_j(z) \mid j = 1, \dots, m\}$ *so that* $G'(z)n_j(z) = \lambda_j^2(z)n_j(z)$, *with* $\lambda_j^2(z) \geq \lambda^2 > 0$, *for any* $z \in \mathscr{M}$;
(3) *the D'Alembert equation* (7.3.5) *has a (nontrivial) symmetric solution* $\gamma_0(t) = \gamma_0(-t) \in \mathscr{M}$ *homoclinic to the equilibrium* z_0 *which is hyperbolic for the dynamics of* (7.3.5) *restricted on* $\mathscr{M}$;
(4) $z(t) = \dot{\gamma}_0(t)$ *is the unique bounded solution, up to a multiplicative constant, of the variational equation of* (7.3.5) *along* $\gamma_0(t)$ *so that* $z(t) \in T_{\gamma_0(t)}\mathscr{M}$ *and*

$$[n_j'(\gamma_0(t))z(t)]^*\dot{\gamma}_0(t) + n_j^*(\gamma_0(t))\dot{z}(t) = 0$$

for any $j = 1, \dots, m$.

Then, there exist $M > 0$, $\delta > 0$, $\mu > 0$, $\varepsilon_0 > 0$ so that for any $\varepsilon \in (0, \varepsilon_0)$ there exists a subset $S_\varepsilon \subset (-3\delta^{-1}\ln\varepsilon, \mu\varepsilon^{-1/2})$ with the Lebesgue measure satisfying

$$m(S_\varepsilon) \geq \frac{1}{2}[\mu\varepsilon^{-1/2} + 3\delta^{-1}\ln\varepsilon]$$

and for any $T \in S_\varepsilon$ penalized equation (7.3.6) *has a $2T-$periodic solution $z(t)$ so that $z(t) = z(-t)$ and the following holds.*

$$\sup_{-T \leq t \leq T} |z(t) - \gamma_0(t)| \leq M\varepsilon.$$

Moreover, there is a $T_\varepsilon > 0$ so that for any $T \geq T_\varepsilon$ the D'Alembert equation (7.3.5) *has a $2T-$periodic solution $\tilde{z}(t) \in \mathscr{M}$ so that $\tilde{z}(t) = \tilde{z}(-t)$ and the following holds*

$$\sup_{-T \leq t \leq T} |\tilde{z}(t) - \gamma_0(t)| \leq M\varepsilon.$$

Hence Theorem 7.3.2 deals with the problem of existence of periodic solutions of penalized equation (7.3.6) for $\varepsilon > 0$ sufficiently small, when the D'Alembert equation (7.3.5) restricted to $\mathscr{M}$ has a homoclinic orbit $\gamma_0(t) \in \mathscr{M}$. A problem similar to this is considered in [19] where it is assumed that $\gamma_0(t)$ is a T-periodic solution of the D'Alembert equation. It is proved, there, that if T satisfies a non-resonance condition then penalized equation (7.3.6) has a T-periodic solution near $\gamma_0(t)$. On the other hand, we have a continuum layer of periodic solutions of penalized equation (7.3.6) near $\gamma_0(t)$ with large periods. This is a kind of partial blue sky catastrophe for penalized equation (7.3.6). But the D'Alembert equation (7.3.5) has a blue sky catastrophe on $\mathscr{M}$ near $\gamma_0(t)$.

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