Chapter 7
Conclusion

In this book the fractional-order nonlinear systems and methods for their numerical simulation and stability analysis are presented. By illustrative examples we have shown chaotic behaviour of such systems and studied their dynamics. We presented the examples of electrical, mechanical, hydrodynamical, chemical, biological, economical, and the other chaotic systems. We studied only the state trajectories (attractors) and we avoided bifurcation analysis and Poincaré maps.

Some authors consider the attractors of chaotic systems a numerical error (Yao, 2010). In fact, deterministic chaos exists if the Lyapunov exponent of the system is positive (Parker and Chua, 1989). We also presented the so-called instability measure as a condition to determine chaos in fractional-order systems. Computation of strange attractors in fractional-order nonlinear systems is very important and therefore we have to find appropriate approximation methods. Utilization of methods in the form of rational polynomial leads to high-order systems. In this case we must consider different initial conditions and large numerical errors which are amplified by the systems constants and approximation polynomial constants. We recommend using a method in the form of FIR filter with a large number of coefficients because it works more accurately and numerical errors are much smaller than those of the methods in the form of IIR filter (Vinagre et al., 2003). However, the time of computation is longer because of the number of coefficients.

In this book we also mention a total order of fractional-order systems. The system order in such case is equal to the sum of particular fractional orders of differential equations. The conclusion of this work confirms the conclusions of the works (Arena et al., 2000; Hartley et al., 1995; Podlubny, 1999) that there is a need to refine the notion of the order of a system which cannot be considered only by the total order of differentiation. For fractional-order differential equations the number of terms in equations and the number of equations are more important than the order of differentiation.

We have considered examples of chaotic fractional-order systems which exhibit chaotic behavior, with total order less than three except Duffing's, Van der Pol's oscillators and Lotka-Volterra system with total order less than two, and memristor based Chua's oscillator with hyperchaos and total order less than four. We

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have shown chaotic systems with several types of nonlinearities as for example piecewise-linear nonlinearity, cross product, square and cubic power and so on. For these fractional-order chaotic systems we have made:

- mathematical description,
- stability investigation,
- numerical solution,
- Matlab routines for simulation,
- Matlab/Simulink models (for two systems).

The Matlab functions have been created for all described chaotic systems and they are listed in Appendix A. The Simulink models have been created only for two systems, namely, fractional-order Chua's and Volta's systems as a general guide for such kind of system simulation.

There are a large number of fractional-order chaotic systems that are not described in this book. This number rapidly grows (e.g., Caponetto et al., 2010; Hilfer, 2000; West et al., 2002; Zaslavsky, 2005, etc.). To have a closer picture we refer to several additional references but we have to note that this list is not complete. For illustration we can mention additional well-known fractional-order chaotic systems, e.g., delayed fractional-order chaotic systems (Deng et al., 2007; Guo, 2006), hyperchaotic systems (Ahmad, 2005b; Deng et al., 2009; Matouk, 2009), fractional-order HIV model (Ye and Ding, 2009), fractional-order multi-scroll attractors system (Ahmad, 2005a; Deng and Lu, 2007), fractional-order 3-D quadratic autonomous system with 4-wing attractor (Wang et al., 2010), fractional-order Sprott's electronic oscillator and mechanical "jerk" model (Ahmad and Sprott, 2003), fractional neuron network system (Zhou et al., 2008), etc. In addition, we note that there are various modifications of Chen's system as, for example, hybrid Lorenz-Chen system (Lian at al., 2007) or Chen-Lee system (Tam and Tou, 2008). There are also many works which report possible electronic implemenation of such type chaotic systems (e.g., Li et al., 2009; Tavazoei et al., 2008, etc.) and its utilization, for instance, in a chaotic secure communication scheme (Kiani-B et al., 2009).

Various numerical methods may also be used in chaotic attractor computations. In addition to the proposed algorithm based on Grünwald-Letnikov definition of the fractional derivative, a modified matrix approach (Podlubny, 2000; Podlubny et al., 2009) or the Adams-Bashforth-Moulton type predictor-corrector scheme (Deng, 2007a,b; Diethelm et al., 2005; Ford and Simpson, 2001) can be successfully used. The frequency-based methods are not sufficient because as shown in (Tavazoei and Haeri, 2007a,b, 2008), false chaos can be observed in systems, which are not chaotic. It is influenced by approximation error.

Some remarks on chaos control have been noted as well. We mentioned several control strategies and synchronization techniques and by illustrative examples presented control of chaos via feedback methods. Two such methods are described: (i) digital state-space proportional feedback controller, and (ii) sliding mode controller.

Finally, for detecting chaotic behavior in the system an instability measure can be used. Computation of the Lyapunov exponents is sometimes impossible and instead of these exponents, the instability measure is a sufficient condition for detecting chaos in the fractional-order chaotic systems.

As has been demonstrated, the idea of fractional calculus requires one to reconsider dynamic system concepts that are often taken for granted. So by changing the order of a system from integer to real, we also move from a three-dimensional system to infinite dimension. A lot of tasks have been opened, namely, stability analysis of uncertain nonlinear fractional-order systems, conditions to determine chaos, control strategies and so on. They should be considered in further work.

Besides mentioned one we also have to note a problem related to an identification of the fractional-order chaotic system parameters (Al-Assaf et al., 2004). It is a difficult task because any change in the system fractional derivative orders or system coefficients generates completely different time response. It is necessary to find an effective identification technique in order to obtain dynamical models that represent the given measured chaotic data in finite time. It could bring a lot of possible applications such as, for example, modelling of the macroeconomic performance of the countries (Petráš and Podlubny, 2007), or many other interesting phenomena with chaotic nature.

References

Ahmad W. M. and Sprott J. C., 2003, Chaos in fractional-order autonomous nonlinear systems, *Chaos, Solitons and Fractals*, **16**, 339–351.

Ahmad W. M., 2005a, Generation and control of multi-scroll chaotic attractors in fractional order systems, *Chaos, Solitons and Fractals*, **25**, 727–735.

Ahmad W. M., 2005b, Hyperchaos in fractional order nonlinear systems, *Chaos, Solitons and Fractals*, **26**, 1459–1465.

Arena P., Caponetto R., Fortuna L. and Porto D., 2000, *Nonlinear Noninteger Order Circuits and Systems – An Introduction*, World Scientific, Singapore.

Al-Assaf Y., El-Khazali R. and Ahmad W., 2004, Identification of fractional chaotic system parameters, *Chaos, Solitons and Fractals*, **22**, 897–905.

Caponetto R., Dongola G., Fortuna L. and Petráš I., 2010, *Fractional Order Systems: Modeling and Control Applications*, World Scientific, Singapore.

Deng H., Li T., Wang Q. and Li H., 2009, A fractional-order hyperchaotic system and its synchronization, *Chaos, Solitons and Fractals*, **41**, 962–969.

Deng W., Li Ch. and Lu J., 2007, Stability analysis of linear fractional differential system with multiple time delays, *Nonlinear Dynamics*, **48**, 409–416.

Deng W., 2007a, Short memory principle and a predictor-corrector approach for fractional differentional equations, *Journal of Computational and Applied Mathematics*, **206**, 174–188.

Deng W., 2007b, Numerical algorithm for the time fractional Fokker-Planck equation, *Journal of Computational Physics*, **227**, 1510–1522.

Deng W. and Lu J., 2007, Generating multi-directional multi-scroll chaotic attractors via a fractional differential hysteresis system, *Physics Letters A*, **369**, 438–443.

Diethelm K., Ford N. J., Freed A. D. and Luchko Yu., 2005, Algorithms for the fractional calculus: A selection of numerical methods, *Comput. Methods Appl. Mech. Engrg.*, **194**, 743–773.

Ford N. and Simpson A., 2001, The numerical solution of fractional differential equations: speed versus accuracy, *Num. Anal. Report 385*, Manchester Centre for Computational Mathematics.

Guo L. J., 2006, Chaotic dynamics of the fractional-order Ikeda delay system and its synchronization, *Chinese Physics*, **15**, 301–305.

Hartley T. T., Lorenzo C. F. and Qammer H. K., 1995, Chaos on a fractional Chua's system, *IEEE Trans. Circ. Syst. Fund. Theor. Appl.*, **42**, 485–490.

Hilfer R., 2000, *Application of fractional calculus in physics*, World Scientific, Singapore.

Kiani-B A., Fallahi K., Pariz N. and Leung H., 2009, A chaotic secure communication scheme using fractional chaotic systems based on an extended fractional Kalman filter, *Communications in Nonlinear Science and Numerical Simulation*, **14**, 863–879.

Li X. F., Chlouverakis K. E. and Xu D. L., 2009, Nonlinear dynamics and circuit realization of a new chaotic flow: A variant of Lorenz, Chen and Lü, *Nonlinear Analysis: Real World Applications*, **10**, 2357–2368.

Lian Q. D., Qiao W. and Hong G., 2007, Chaotic attractor transforming control of hybrid Lorenz-Chen system, *Chinese Phys. B*, **17**.

Matouk A. E., 2009, Stability conditions, hyperchaos and control in a novel fractional order hyperchaotic system, *Physics Letters A*, **373**, 2166–2173.

Parker T. S. and Chua L. O., 1989, *Practical Numerical Algorithm for Chaotic Systems*, Springer, New York.

Petráš I. and Podlubny I., 2007, State space description of national economies: the V4 countries, *Computational Statistics & Data Analysis*, **52**, 1223–1233.

Podlubny I., 1999, *Fractional Differential Equations*, Academic Press, San Diego.

Podlubny I., 2000, Matrix approach to discrete fractional calculus, *Fractional Calculus and Applied Analysis*, **3**, 359–386.

Podlubny I., Chechkin A., Škovránek T., Chen Y. Q. and Vinagre B. M. J., 2009, Matrix approach to discrete fractional calculus II: Partial fractional differential equations, *Journal of Computational Physics*, **228**, 3137–3153.

Tam L. M. and Tou W. M. S., 2008, Parametric study of the fractional-order Chen-Lee system, *Chaos, Solitons and Fractals*, **37**, 817–826.

Tavazoei M. S. and Haeri M., 2007a, Unreliability of frequency-domain approximation in recognising chaos in fractional-order systems, *IET Signal Proc.*, **1**, 171–181.

Tavazoei M. S. and Haeri, M., 2007b, A necessary condition for double scroll attractor existence in fractional-order systems, *Physics Letters A*, **367**, 102–113.

Tavazoei M. S., Haeri M., Jafari S., Bolouki S. and Siami M., 2008, Some Applications of Fractional Calculus in Suppression of Chaotic Oscillations, *IEEE Trans. Ind. Electron.*, **55**, 4094–4101.

Tavazoei M. S. and Haeri M., 2008, Limitations of frequency domain approximation for detecting chaos in fractional order systems, *Nonlinear Analysis*, **69**, 1299–1320.

Vinagre B. M., Chen Y. Q. and Petráš I., 2003, Two direct Tustin discretization methods for fractional-order differentiator/integrator, *J. Franklin Inst.*, **340**, 349–362.

Wang Z., Sun Y., Qi G. and van Wyk B. J., 2010, The effects of fractional order on a 3-D quadratic autonomous system with four-wing attractor, *Nonlinear Dyn.*, DOI: 10.1007/s11071-010-9705-7.

West B. J., Bologna M. and Grigolini P., 2002, *Physics of Fractal Operators*, Springer, New York.

Yao L. S., 2010, Computed chaos or numerical errors, *Nonlinear Analysis: Modelling and Control*, **15**, 109–126.

Ye H. and Ding Y., 2009, Nonlinear Dynamics and Chaos in a Fractional-Order HIV Model, *Mathematical Problems in Engineering*, Article ID **378614**.

Zaslavsky G. M., 2005, *Hamiltonian Chaos and Fractional Dynamics*, Oxford University Press, Oxford.

Zhou S., Li H. and Zhu Z., 2008, Chaos control and synchronization in a fractional neuron network system, *Chaos, Solitons and Fractals*, **36**, 973–984.