# Chapter 3
# Fractional-Order Systems

## 3.1 Fractional LTI Systems

A general fractional-order system can be described by a fractional differential equation of the form

$$\begin{aligned} & a_n D^{\alpha_n} y(t) + a_{n-1} D^{\alpha_{n-1}} y(t) + \cdots + a_0 D^{\alpha_0} y(t) \\ & = b_m D^{\beta_m} u(t) + b_{m-1} D^{\beta_{m-1}} u(t) + \cdots + b_0 D^{\beta_0} u(t), \end{aligned} \tag{3.1}$$

where $D^\gamma \equiv {}_0D_t^\gamma$ denotes the Grünwald-Letnikov, the Riemann-Liouville or the Caputo's fractional derivative (Podlubny, 1999a). The corresponding transfer function of *incommensurate* real orders has the following form (Podlubny, 1999a):

$$G(s) = \frac{b_m s^{\beta_m} + \cdots + b_1 s^{\beta_1} + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + \cdots + a_1 s^{\alpha_1} + a_0 s^{\alpha_0}} = \frac{Q(s^{\beta_k})}{P(s^{\alpha_k})}, \tag{3.2}$$

or in the frequency domain it has form (Petráš et al., 2000):

$$G(j\omega) = \frac{b_m (j\omega)^{\beta_m} + \cdots + b_1 (j\omega)^{\beta_1} + b_0 (j\omega)^{\beta_0}}{a_n (j\omega)^{\alpha_n} + \cdots + a_1 (j\omega)^{\alpha_1} + a_0 (j\omega)^{\alpha_0}} = \frac{Q((j\omega)^{\beta_k})}{P((j\omega)^{\alpha_k})}, \tag{3.3}$$

where $a_k$ $(k = 0, \ldots, n)$, $b_k$ $(k = 0, \ldots, m)$ are constants, and $\alpha_k$ $(k = 0, \ldots, n)$, $\beta_k$ $(k = 0, \ldots, m)$ are arbitrary real or rational numbers and without loss of generality they can be arranged as $\alpha_n > \alpha_{n-1} > \cdots > \alpha_0$, and $\beta_m > \beta_{m-1} > \cdots > \beta_0$.

The incommensurate order system (3.2) can also be expressed in commensurate form by the multivalued transfer function (Bayat and Afshar, 2008)

$$H(s) = \frac{b_m s^{m/v} + \cdots + b_1 s^{1/v} + b_0}{a_n s^{n/v} + \cdots + a_1 s^{1/v} + a_0}, \quad v > 1. \tag{3.4}$$

Note that every fractional-order system can be expressed in the form (3.4) and the domain of the $H(s)$ definition is a Riemann surface with $v$ Riemann sheets (LePage, 1961).

In the particular case of *commensurate* order systems, it holds that $\alpha_k = \alpha k, \beta_k = \alpha k, 0 < \alpha < 1, \forall k \in Z$, and the transfer function has the following form:

$$G(s) = K_0 \frac{\sum_{k=0}^{M} b_k (s^\alpha)^k}{\sum_{k=0}^{N} a_k (s^\alpha)^k} = K_0 \frac{Q(s^\alpha)}{P(s^\alpha)}. \tag{3.5}$$

With $N > M$, the function $G(s)$ becomes a proper rational function in the complex variable $s^\alpha$ which can be expanded in partial fractions of the following form:

$$G(s) = K_0 \left[ \sum_{i=1}^{N} \frac{A_i}{s^\alpha + \lambda_i} \right], \tag{3.6}$$

where $\lambda_i \, (i = 1, 2, \ldots, N)$ are the roots of the pseudo-polynomial $P(s^\alpha)$ or the system poles which are assumed to be simple without loss of generality. The analytical solution of the system (3.6) can be expressed as

$$y(t) = L^{-1} \left\{ K_0 \left[ \sum_{i=1}^{N} \frac{A_i}{s^\alpha + \lambda_i} \right] \right\} = K_0 \sum_{i=1}^{N} A_i t^\alpha E_{\alpha,\alpha}(-\lambda_i t^\alpha), \tag{3.7}$$

where $E_{\mu,v}(z)$ is the Mittag-Leffler function defined as (2.3).

A fractional-order plant to be controlled can be described by a typical $n$-term linear homogeneous fractional-order differential equation (FODE) in time domain

$$a_n D_t^{\alpha_n} y(t) + \cdots + a_1 D_t^{\alpha_1} y(t) + a_0 D_t^{\alpha_0} y(t) = 0, \tag{3.8}$$

where $a_k \, (k = 0, 1, \ldots, n)$ are constant coefficients of the FODE; $\alpha_k \, (k = 0, 1, 2, \ldots, n)$ are real numbers. Without loss of generality, assume that $\alpha_n > \alpha_{n-1} > \cdots > \alpha_0 \geq 0$.

The analytical solution of the FODE (3.8) is given by general formula in the form (Podlubny, 1999a):

$$\begin{aligned} y(t) = {} & \frac{1}{a_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{\substack{k_0+k_1+\cdots+k_{n-2}=m \\ k_0 \geq 0; \ldots ; k_{n-2} \geq 0}} (m; k_0, k_1, \ldots, k_{n-2}) \\ & \times \prod_{i=0}^{n-2} \left( \frac{a_i}{a_n} \right)^{k_i} \mathscr{E}_m(t, -\frac{a_{n-1}}{a_n}; \alpha_n - \alpha_{n-1}, \alpha_n + \sum_{j=0}^{n-2} (\alpha_{n-1} - \alpha_j) k_j + 1), \end{aligned} \tag{3.9}$$

where $(m; k_0, k_1, \ldots, k_{n-2})$ are the multinomial coefficients and $\mathscr{E}_k(t, \lambda; \mu, v)$ is the function of Mittag-Leffler type introduced by Podlubny (Podlubny, 1999a). The function is defined by

$$\mathscr{E}_k(t, \lambda; \mu, v) = t^{\mu k + v - 1} E_{\mu,v}^{(k)}(\lambda t^\mu), \quad k = 0, 1, 2, \ldots \tag{3.10}$$

where $E_{\mu,\nu}^{(k)}(z)$ is $k$-th derivative of the Mittag-Leffler function of two parameters given by

$$E_{\mu,\nu}^{(k)}(z) = \sum_{i=0}^{\infty} \frac{(i+k)!\, z^i}{i!\, \Gamma(\mu i + \mu k + \nu)}, \qquad k = 0, 1, 2, \ldots. \tag{3.11}$$

The Laplace transform of the function $\mathscr{E}_k(t, \pm\lambda; \alpha, \beta)$ is (Podlubny, 1999a):

$$L\{\mathscr{E}_k(t, \pm\lambda; \alpha, \beta)\} = \frac{k!\, s^{\alpha-\beta}}{(s^\alpha \mp \lambda)^{k+1}}$$

for $s > |\lambda|^{1/\alpha}$.

The Laplace transforms for several other Mittag-Leffler type functions are summarized as follows (Gorenflo et al., 2004; Magin et al., 2009; Podlubny, 1999a):

$$\begin{aligned} L\{E_\alpha(-\lambda t^\alpha)\} &= \frac{s^{\alpha-1}}{s^\alpha + \lambda}, \\ L\{t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)\} &= \frac{1}{s^\alpha + \lambda}, \\ L\{t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha)\} &= \frac{s^{\alpha-\beta}}{s^\alpha + \lambda} \end{aligned} \tag{3.12}$$

for $s > |\lambda|^{1/\alpha}$.

A useful list of Laplace and inverse Laplace transforms of functions related to fractional calculus is presented in Appendix B and in (Chen et al., 2001).

Consider a control function which acts on the FODE system (3.8) as follows:

$$a_n D_t^{\alpha_n} y(t) + \cdots + a_1 D_t^{\alpha_1} y(t) + a_0 D_t^{\alpha_0} y(t) = u(t). \tag{3.13}$$

By Laplace transform, we can get a fractional transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{a_n s^{\alpha_n} + \cdots + a_1 s^{\alpha_1} + a_0 s^{\alpha_0}}. \tag{3.14}$$

The fractional-order linear time-invariant (LTI) system can also be represented by the following state-space model (Matignon, 1998):

$$\begin{aligned} {}_0D_t^{\mathbf{q}} x(t) &= \mathbf{A} x(t) + \mathbf{B} u(t), \\ y(t) &= \mathbf{C} x(t), \end{aligned} \tag{3.15}$$

where $x \in \mathrm{R}^n$, $u \in \mathrm{R}^r$ and $y \in \mathrm{R}^p$ are the state, input and output vectors of the system and $\mathbf{A} \in \mathrm{R}^{n\times n}$, $\mathbf{B} \in \mathrm{R}^{n\times r}$, $\mathbf{C} \in \mathrm{R}^{p\times n}$, and $\mathbf{q} = [q_1, q_2, \ldots, q_n]^T$ are the fractional orders. If $q_1 = q_2 = \cdots = q_n \equiv \alpha$, system (3.15) is called a commensurate-order system, otherwise it is an incommensurate-order system.

The state transition matrix is

$$\begin{aligned}\mathbf{x}(t) &= \left[\mathbf{I}+\frac{\mathbf{A}\mathbf{x}(0)}{\Gamma(1+\alpha)}t^{\alpha}+\frac{\mathbf{A}^2\mathbf{x}(0)}{\Gamma(1+2\alpha)}t^{2\alpha}+\cdots+\frac{\mathbf{A}^k\mathbf{x}(0)}{\Gamma(1+k\alpha)}t^{k\alpha}+\cdots\right] \\ &= \left(\sum_{k=0}^{\infty}\frac{\mathbf{A}^k t^{k\alpha}}{\Gamma(1+k\alpha)}\right)\mathbf{x}(0)=\phi(\mathbf{t})\mathbf{x}(\mathbf{0}).\end{aligned} \tag{3.16}$$

Similar to conventional observability and controllability concept, the controllability is defined as follows (Matignon and D'Andrea-Novel, 1996): System (3.15) is *controllable* on $[t_0, t_{final}]$ if the controllability matrix

$$C_a = [B|AB|A^2B|\cdots|A^{n-1}B]$$

has rank $n$.

The observability is defined as follows (Matignon and D'Andrea-Novel, 1996): System (3.15) is *observable* on $[t_0, t_{final}]$ if the observability matrix

$$O_a = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has rank $n$.

A fractional-order system described by $n$-term fractional differential equation (3.13) can be rewritten into the state-space representation in the form (Dorčák et al., 2002; Yang and Liu, 2006):

$$\begin{bmatrix} {}_0D^{q_1}x_1(t) \\ {}_0D^{q_2}x_2(t) \\ \vdots \\ {}_0D^{q_n}x_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & & \vdots \\ -a_0/a_n & -a_1/a_n & \cdots & a_{n-1}/a_n \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1/a_n \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 \; 0 \cdots 0 \; 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \tag{3.17}$$

where $\alpha_0 = 0$, $q_1 = \alpha_1$, $q_2 = \alpha_{n-1} - \alpha_{n-2}, \ldots, q_n = \alpha_n - \alpha_{n-1}$, and with initial conditions:

$$x_1(0) = x_0^{(1)} = y_0,\; x_2(0) = x_0^{(2)} = 0, \ldots$$

$$x_i(0) = x_0^{(i)} = \begin{cases} y_0^{(k)}, & \text{if } i = 2k+1, \\ 0, & \text{if } i = 2k, \end{cases} \quad i \le n. \tag{3.18}$$

The $n$-term FODE (3.13) is equivalent to the system of Eqs. (3.17) with the initial conditions (3.18) if Caputo's derivative is considered.

## 3.2 Fractional Nonlinear Systems

In this book, we will consider the general incommensurate fractional-order nonlinear system represented as follows:

$$\begin{aligned} {}_0D_t^{q_i} x_i(t) &= f_i(x_1(t), x_2(t), \ldots, x_n(t), t), \\ x_i(0) &= c_i, \; i = 1, 2, \ldots, n, \end{aligned} \tag{3.19}$$

where $c_i$ are initial conditions. The vector representation of (3.19) is:

$$D^{\mathbf{q}}\mathbf{x} = \mathbf{f}(\mathbf{x}), \tag{3.20}$$

where $\mathbf{q} = [q_1, q_2, \ldots, q_n]^T$ for $0 < q_i < 2$, $(i = 1, 2, \ldots, n)$ and $\mathbf{x} \in \mathrm{R}^n$.

The equilibrium points of system (3.20) are calculated via solving the following equation

$$\mathbf{f}(\mathbf{x}) = 0 \tag{3.21}$$

and we suppose that $E^* = (x_1^*, x_2^*, \ldots, x_n^*)$ is an equilibrium point of system (3.20).

## 3.3 Fractional-Order Controllers

### *3.3.1 Definition of Fractional-Order Controllers*

The fractional-order $PI^\lambda D^\delta$ (also $PI^\lambda D^\mu$ controller) controller (FOC) was proposed in (Podlubny, 1999a,b) as a generalization of the *PID* controller with integrator of real order $\lambda$ and differentiator of real order $\delta$. The transfer function of such controller in the Laplace domain has this form (Podlubny, 1999b; Podlubny et al., 2002):

$$C(s) = \frac{U(s)}{E(s)} = K_p + T_i s^{-\lambda} + T_d s^{\delta}, \qquad (\lambda, \delta > 0), \tag{3.22}$$

where $K_p$ is the proportional constant, $T_i$ is the integration constant and $T_d$ is the differentiation constant.

As we can see in Fig. 3.1, the internal structure of the fractional-order controller consists of the parallel connection, the proportional, integration, and derivative part (Dorf and Bishop, 1990). The transfer function (3.22) corresponds in time domain

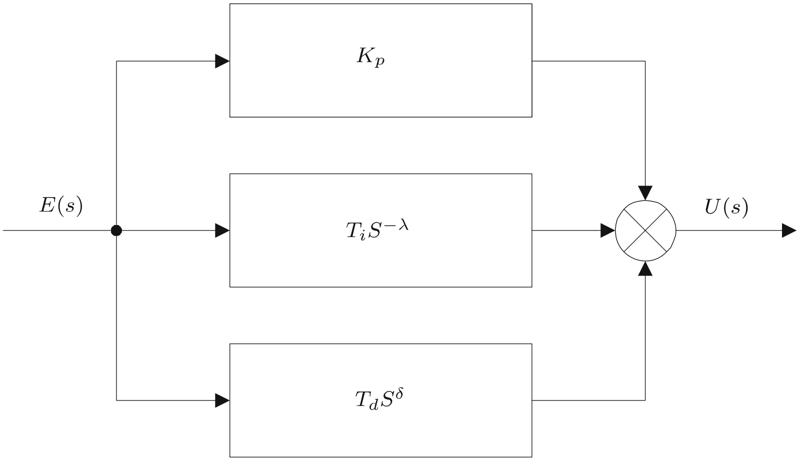


**Fig. 3.1** General structure of a $PI^{\lambda}D^{\delta}$ controller.

to fractional differential equation (3.23)

$$u(t) = K_p e(t) + T_i\, {}_0D_t^{-\lambda} e(t) + T_d\, {}_0D_t^{\delta} e(t), \tag{3.23}$$

or discrete transfer function given below:

$$C(z) = \frac{U(z)}{E(z)} = K_p + \frac{T_i}{(\omega(z^{-1}))^{\lambda}} + T_d(\omega(z^{-1}))^{\delta}, \tag{3.24}$$

where $\omega(z^{-1})$ denotes the discrete operator, expressed as a function of the complex variable $z$ or the shift operator $z^{-1}$.

Taking $\lambda = 1$ and $\delta = 1$, we obtain a classical *PID* controller. If $\lambda = 0$ and $T_i = 0$, we obtain a $PD^{\delta}$ controller, etc. All these types of controllers are particular cases of the fractional-order controller, which is more flexible and gives an opportunity to better adjust the dynamical properties of the fractional-order control system.

It can also be mentioned that there are many other considerations of the fractional-order controller (Xue and Chen, 2002). For example, we can mention several of them:

- *CRONE* controller (1st generation) (Oustaloup, 1995), characterized by the band-limited lead effect (Oustaloup, 1983):

$$C(s) = C_0 \left( \frac{1 + s/\omega_b}{1 + s/\omega_h} \right)^r \tag{3.25}$$

  where $0 < \omega_b < \omega_h$, $C_0 > 0$ and $r \in (0, 1)$.
  There are a number of real life applications of three generations of the CRONE controller such as the car suspension control (Oustaloup et al., 1996), flexible transmission (Oustaloup et al., 1995), and hydraulic actuator.

- Fractional lead-lag compensator (Raynaud and Zergaïnoh, 2000; Monje et al., 2008), given by

$$C(s) = k_c \left( \frac{s + 1/\lambda}{s + 1/x\lambda} \right)^r = k_c x^r \left( \frac{\lambda s + 1}{x\lambda s + 1} \right)^r, \tag{3.26}$$

$$r \in \mathrm{R},\ 0 < x < 1.$$

- Non-integer integral and its application to control (Manabe, 1961);
- *TID* compensator (Lurie, 1994), which has structure similar to a *PID* controller but the proportional component is replaced with a tilted component having a transfer function $s$ to the power of $(-1/n)$. The resulting transfer function of the *TID* controller has the form:

$$C(s) = \frac{T}{s^{1/n}} + \frac{I}{s} + Ds, \tag{3.27}$$

where $T$, $I$ and $D$ are the controller constants and $n$ is a non-zero real number, preferably between 2 and 3. The transfer function (3.27) more closely approximates an optimal transfer function and an overall response is achieved, which is closer to the theoretical optimal response determined by Bode (Bode, 1949).

### 3.3.2 Properties and Characteristics of Controller

It can be expected that $PI^\lambda D^\delta$ controller (3.22) may enhance the systems control performance due to more tuning knobs introduced, which is intuitively illustrated in Fig. 3.2.

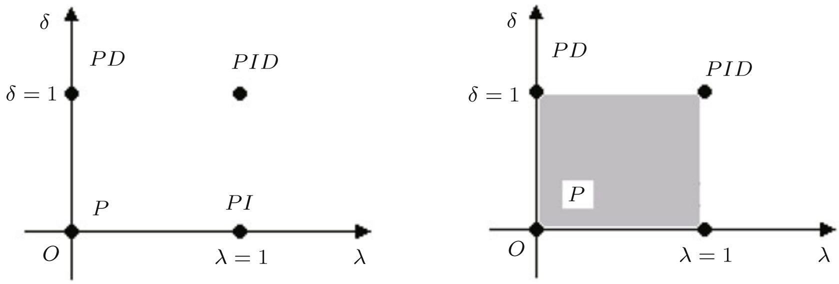


**Fig. 3.2** *PID* controller: from *points* to *plane*.

The $PI^\lambda D^\delta$ controller with complex zeros and poles located anywhere in the left-hand $s$-plane may be rewritten as

$$C(s) = K\frac{(s/\omega_n)^{\delta+\lambda} + (2\zeta s^{\lambda})/\omega_n + 1}{s^{\lambda}}, \tag{3.28}$$

where $K$ is the gain, $\zeta$ is the dimensionless damping ratio and $\omega_n$ is the natural frequency. Normally, we choose $\zeta < 1$. When $\zeta = 1$, the condition is called critical damping (Dorf and Bishop, 1990).

*Example 3.1.* : Let us consider the fractional-order controller (3.28) with the following parameters: $K = 6.5$, $\omega_n = 1$, $\zeta = 0.5$ and $\lambda = \delta = 0.5$.

In Fig. 3.3 are shown the Bode plots of the fractional $PI^{\lambda}D^{\delta}$ controller (3.28) with the above-mentioned parameters.

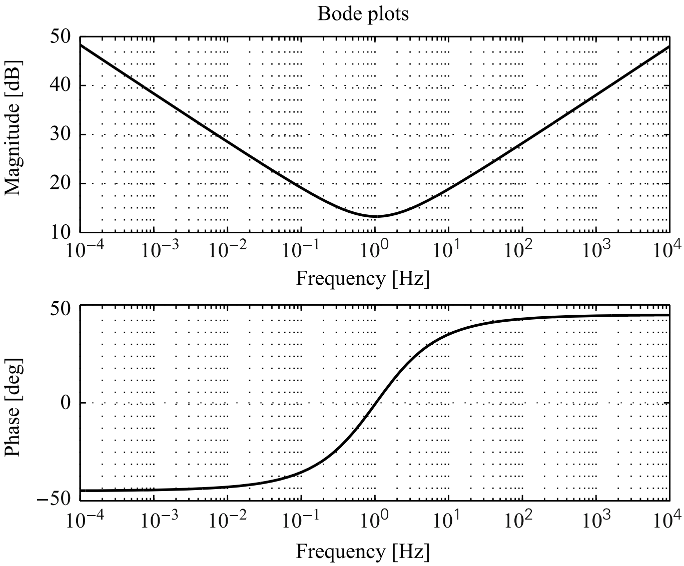


**Fig. 3.3** Bode plots of $PI^{\lambda}D^{\delta}$ controller (3.28) with $K = 6.5$, $\omega_n = 1$, $\zeta = 0.5$ and $\lambda = \delta = 0.5$.

The slopes of the magnitude and the value of the phase for low and high frequencies can be selected, which are related with the relative stability and the high frequencies gained by Vinagre (2000). Asymptotically, at low frequency the slope will be $-\lambda dB/dec$ and the phase will be $-\lambda\frac{\pi}{2}$, and at high frequency the slope will be $\delta dB/dec$ and the phase will be $\delta\frac{\pi}{2}$.

For a wide class of controlled objects we recommend the fractional $PI^{n}D^{\delta}$ controller, which is a particular case of $PI^{\lambda}D^{\delta}$ controller, where $\lambda = n$, $n \in N$ and $\delta \in R$. Integer-order integrator is important for steady-state error cancellation but on the other hand, the fractional integral is also important for obtaining a *Bode's ideal loop transfer function* response with constant phase margin for desired frequency range (Aström, 2000; Bode, 1949; Manabe, 1961; Tustin et al., 1958).

### 3.3.3 Design of Controller Parameters and Implementation

The tuning of $PI^{\lambda}D^{\delta}$ controller parameters is determined according to the given requirements. These requirements are, for example, the damping ratio, the steady-state error ($e_{ss}$), dynamical properties, etc. One of the methods being developed is the method of dominant roots (Petráš, 1999; Petráš and Dorčák, 2003), based on the given stability measure and the damping ratio of the closed control loop. Assume that the desired dominant roots are a pair of complex conjugate root as follows:

$$s_{1,2} = -\sigma \pm j\omega_d,$$

designed for the damping ratio $\zeta$ and natural frequency $\omega_n$. The damping constant (stability measure) is $\sigma = \zeta\omega_n$ and the damped natural frequency of oscillation $\omega_d = \omega_n\sqrt{1-\zeta^2}$. The design of parameters: $K_p$, $T_i$, $\lambda$, $T_d$ and $\delta$ can be computed numerically from characteristic equation. More specifically, for simple plant model $P(s)$, this can be done by solving

$$\min_{K_p,T_i,\lambda,T_d,\delta} ||C(s)P(s)+1||_{s=-\sigma\pm j\omega_d}.$$

Another possible way to obtain the controller parameters is using the tuning formula, based on gain $A_m$ and phase $\Phi_m$ margins specifications for crossover frequency $\omega_{cg}$. Gain and phase margins have always served as important measures of robustness. The equations that define the phase margin and the gain crossover frequency are expressed as (Monje et al., 2008; Vinagre, 2000):

$$|C(j\omega_{cg})P(\omega_{cg})|_{\mathrm{dB}} = 0\,\mathrm{dB}$$

$$\arg(C(j\omega_{cg})P(\omega_{cg})) = -\pi + \Phi_m$$

The above equations are often used also for the so-called auto-tuning techniques. For instance, relay auto-tuning process has been widely used in industrial application and it was already modified for the fractional-order controllers (Monje et al., 2008).

Last but not least, we should mention the optimization algorithm based on the integral absolute error (IAE) minimization (Podlubny, 1999b):

$$\mathrm{IAE}\,(t) = \int_0^t |e(t)|dt = \int_0^t |w(t)-y(t)|dt,$$

where $w(t)$ is the desired value of closed control loop and $y(t)$ is the real value of closed control loop. This method does not ensure the desired stability measure of the closed control loop. Measure of stability has to be checked out additionally by some known method as, for example, frequency method described in literature (Petráš and Dorčák, 1999).

Implementation techniques for the FOC have been described in several works. Some proposal can be found in the work by Vinagre (2000). An analogue implementation was proposed in the book by Petráš et al. (2002) and a digital implementation was suggested in the book by Caponetto et al. (2010).

It can be expected that $PI^{\lambda}D^{\delta}$ controller (3.22) may enhance the systems control performance due to more tuning knobs introduced. Actually, in theory, $PI^{\lambda}D^{\delta}$ itself is an infinite-dimensional linear filter due to the fractional order in differentiator or integrator.

We comment that since *PID* control is ubiquitous in industry process control, fractional-order *PID* control will also be ubiquitous when tuning and implementation techniques are well developed (Caponetto et al., 2010; Chen, 2006; Chen et al., 2008, 2009; Monje et al., 2008; Petráš, 1999).

## References


Aström K. J., 2000, *Model Uncertainty and Robust Control*, COSY project.

Bayat F. M. and Afshar M., 2008, Extending the root-locus method to fractional-order systems, *Journal of Applied Mathematics*, Article ID **528934**.

Bode H. W., 1949, *Network Analysis and Feedback Amplifier Design*, Tung Hwa Book Company, Shanghai.

Caponetto R., Dongola G., Fortuna L. and Petráš I., 2010, *Fractional Order Systems: Modeling and Control Applications*, World Scientific, Singapore.

Chen Y. Q., 2006, Ubiquitous Fractional Order Controls? *Proc. of the Second IFAC Symposium on Fractional Derivatives and Applications (IFAC FDA06)*, July 19–21, Porto, Portugal.

Chen Y. Q., Petráš I. and Vinagre B. M., 2001, A List of Laplace and Inverse Laplace Transforms Related to Fractional Order Calculus, http://people.tuke.sk/ivo.petras/foc_laplace.pdf.

Chen Y. Q., Bhaskaran T. and Xue D., 2008, Practical tuning rule development for fractional order proportional and integral controllers, *ASME Journal of Computational and Nonlinear Dynamics*, **3**, 021403-1 – 021403-8.

Chen Y. Q., Petráš I. and Xue D., 2009, Fractional order control – A tutorial, *Proc. of the American Control Conference, ACC 2009.*, June 10–12, 2009, St. Louis, USA, 1397–1411.

Dorčák Ľ, Petráš I., Koštial I. and Terpák J., 2002, Fractional-order state space models, *Proc. of the International Carpathian Control Conference*, Malenovice, Czech republic, May 27-30, 193–198.

Dorf R. C. and Bishop R. H., 1990, *Modern Control Systems*, Addison-Wesley, New York.

Gorenflo R., Luchko Yu. and Rogosin S., 2004, Mittag-Leffler type functions: notes on growth properties and distribution of zeros, Preprint No. **A-97-04**, Fachbereich Mathematik und Informatik, Freie Universitat Berlin, Germany.

LePage W. R., 1961, *Complex Variables and the Laplace Transform for Engineers*, McGraw-Hill, New York.

Lurie B. J., 1994, Three-Parameter Tunable Tilt-Integral-Derivative (TID) Controller, *United States Patent, 5 371 670*, USA.

Magin R. L., Feng X. and Baleanu D., 2009, Solving the fractional order Bloch equation, *Concepts in Magnetic Resonance Part A*, **34A**, 16–23.

Manabe S., 1961, The non-integer integral and its application to control systems, *ETJ of Japan*, **6**, 83–87.

Matignon D., 1998, Generalized fractional differential and difference equations: stability properties and modelling issues, *Proc. of the Math. Theory of Networks and Systems Symposium*, Padova, Italy.

Matignon D. and D'Andrea-Novel B., 1996, Some results on controllability and observability of finite-dimensional fractional differential systems, *Computational Engineering in Systems Applications*, Lille, France, IMACS, IEEE-SMC, **2**, 952–956.

Monje C. A., Vinagre B. M., Feliu V. and Chen Y. Q., 2008, Tuning and auto-tuning of fractional order controllers for industry application, *Contr. Eng. Pract.*, **16**, 798–812.

Oustaloup A., 1983, *Systèmes Asservis Linéaires d'Ordre Fractionnaire: Théorie et Pratique*, Editions Masson, Paris.

Oustaloup A., 1995, *La Derivation Non Entiere: Theorie, Synthese et Applications*, Hermes, Paris.

Oustaloup A., Mathieu B. and Lanusse P., 1995, The CRONE control of resonant plants: application to a flexible transmission, *European Journal of Control*, **1**, 113–121.

Oustaloup A., Moreau X. and Nouillant M., 1996, The CRONE suspension, *Control Engineering Practice*, **4**, 1101–1108.

Petráš I., 1999, The fractional-order controllers: methods for their synthesis and application, *Journal of Electrical Engineering*, **50**, 284–288.

Petráš I. and Dorčák Ľ., 1999, The frequency method for stability investigation of fractional control systems, *Journal of SACTA*, **2**, 75–85.

Petráš I., Dorčák Ľ., O'Leary P., Vinagre B. M. and Podlubny I., 2000, The modelling and analysis of fractional-order control systems in frequency domain, *Proc. of the ICCC'2000 Conference*, May 23-26, High Tatras, 261–264.

Petráš I., Podlubny I., O'Leary P., Dorčák Ľ. and Vinagre B. M., 2002, *Analogue Realization of Fractional Order Controllers*, Faculty of BERG, Technical University of Kosice.

Petráš I. and Dorčák Ľ., 2003, Fractional-order control systems: Modelling and simulation, *Fractional Calculus and Applied Analysis*, **6**, 205–232.

Podlubny I., 1999a, *Fractional Differential Equations*, Academic Press, San Diego.

Podlubny I., 1999b, Fractional-order systems and $PI^{\lambda}D^{\mu}$-controllers, *IEEE Transactions on Automatic Control*, **44**, 208–213.

Podlubny I., Petráš, I., Vinagre, B.M., O'Leary P. and Dorčák Ľ., 2002, Analogue realizations of fractional-order controllers, *Nonlinear Dynamics*, **29**, 281–296.

Raynaud H. F. and Zergaïnoh A., 2000, State-space representation for fractional order controllers, *Automatica*, **36**, 1017–1021.

Tustin A., Allanson J. T., Layton J. M. and Jakeways R. J., 1958, The design of systems for automatic control of the position of massive objects, *The Proceedings of the Institution of Electrical Engineers*, **105C**.

Vinagre B. M., Podlubny I., Hernandez A., and Feliu V., 2000, On realization of fractional-order controllers, *Proc. of the Conference Internationale Francophone d'Automatique*, Lille, Jule 5-8, 945–950.

Xue D. and Chen Y. Q., 2002, A comparative introduction of four fractional order controllers, *Proc. of the* $4^{th}$ *World Congress on Intelligent Control and Automation*, June 10 - 14, Shanghai, China.

Yang C. and Liu F., 2006, A computationally effective predictor-corrector method for simulating fractional order dynamical control system, *Australian and New Zealand Industrial and Applied Mathematics Journal*, **47**, C168–C184.