Ultrafilter Extensions of Models*

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Dedication

To V.A. Uspensky on the 80th anniversary of his birth

Abstract. We show that any model $\mathfrak{A}$ can be extended, in a canonical way, to a model $\beta\mathfrak{A}$ consisting of ultrafilters over it. The extension procedure preserves homomorphisms: any homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ extends to a continuous homomorphism of $\beta\mathfrak{A}$ into $\beta\mathfrak{B}$. Moreover, if a model $\mathfrak{B}$ carries a compact Hausdorff topology which is (in a certain sense) compatible, then any homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ extends to a continuous homomorphism of $\beta\mathfrak{A}$ into $\mathfrak{B}$. This is also true for embeddings instead of homomorphisms.

We present a result in general model theory. We show that any model can be extended, in a canonical way, to the model (of the same language) consisting of ultrafilters over it such that the extended model inherits the universality property of the largest compactification.

Recall standard facts concerning topology of ultrafilters. The set βX of ultrafilters over a set X carries a natural topology generated by elementary (cl)open sets of form

$$\tilde{S} = \{u \in \beta X : S \in u\}$$

for all $S \subseteq X$. The space βX is compact Hausdorff, extremally disconnected (the closure of any open set is open), and in fact, the Stone–Čech (and also Wallman) compactification of the discrete space X, i.e. its *largest* compactification. This means that X is dense in βX (one lets $X \subseteq \beta X$ by identifying each $x \in X$ with the principal ultrafilter $\hat{x}$), and any continuous mapping h of X into any compact space Y can be uniquely extended to a continuous mapping $\tilde{h}$ of βX into Y. There is a one-to-one correspondence between filters over X and closed subsets of βX (a filter D corresponds to $\{u \in \beta X : D \subseteq u\}$ while a closed $C \subseteq \beta X$ corresponds to $\bigcap C$); in fact, the compactness of βX is equivalent to the claim that $\{u \in \beta X : D \subseteq u\}$ is nonempty for each D and thus unprovable in ZF alone (see [5]).

We show that if $F, \ldots, P, \ldots$ are operations and relations on X, there is a canonical way to extend them to operations and relations $\tilde{F}, \ldots, \tilde{P}, \ldots$ on βX, thus extending the model $\mathfrak{A} = (X, F, \ldots, P, \ldots)$ to the model $\beta\mathfrak{A} = (\beta X, \tilde{F}, \ldots, \tilde{P}, \ldots)$. We show that the extension procedure preserves homomorphisms: if h is a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$, then $\tilde{h}$ is a homomorphism of $\beta\mathfrak{A}$ into $\beta\mathfrak{B}$. Moreover, if $\mathfrak{B}$ carries a compact Hausdorff topology which is compatible in a certain sense,

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and h is a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$, then $\tilde{h}$ is a homomorphism of $\beta\mathfrak{A}$ into $\mathfrak{B}$; thus extended models inherit the universality property of the Stone–Čech (or Wallman) compactification. We note also that both facts remain true if one replaces homomorphisms by embeddings or some other relationships between models.

The construction, although it looks old and should be known, did not appear before, except one very particular case when models are semigroups [1].[1] The reader can also consult on semigroups of ultrafilters and their applications in various areas (number theory, algebra, dynamics, ergodic theory) in [2]; an analogous technique for non-associative groupoids and some infinitary generalizations are discussed in [3].

Definition of extensions. The first main theorem

Here we define the extensions of models by ultrafilters. Then we establish our first main result showing that the extension procedure preserves homomorphisms.

To extend a model $\mathfrak{A} = (X, F, \ldots, P, \ldots)$, we extend operations $F, \ldots$ on X, i.e. mappings of Cartesian products of X into X itself, and relations $P, \ldots$ on X, i.e. subsets of such products. Let us provide a slightly more general definition involving n-ary mappings of $X_1 \times \ldots \times X_n$ into Y, and n-ary relations that are subsets of $X_1 \times \ldots \times X_n$. We shall use it e.g. when we shall show that any mapping h of a certain type between models extends to $\tilde{h}$ of the same type.

Definition 1. Given an n-ary mapping $F : X_1 \times \ldots \times X_n \to Y$, let $\tilde{F} : \beta X_1 \times \ldots \times \beta X_n \to \beta Y$ be *defined* as follows:

$$\tilde{F}(u_1, \ldots, u_n) = \big\{S \subseteq Y : \{x_1 \in X_1 : \ldots \{x_n \in X_n : F(x_1, \ldots, x_n) \in S\} \in u_n \ldots\} \in u_1\big\}$$

for every $u_1 \in \beta X_1, \ldots, u_n \in \beta X_n$.

Lemma 2. *For all $z_1 \in X_1$ and $u_2 \in \beta X_2, \ldots, u_n \in \beta X_n$,*

$$\big\{x_1 : \big\{x_2 : \ldots \{x_n : F(x_1, x_2, \ldots, x_n) \in S\} \in u_n \ldots\big\} \in u_2\big\} \in \hat{z}_1$$
$$\textit{iff} \quad \big\{x_2 : \ldots \{x_n : F(z_1, x_2, \ldots, x_n) \in S\} \in u_n \ldots\big\} \in u_2 \,.$$

Proof. Clear.

Proposition 3. *If $F : X_1 \times \ldots \times X_n \to Y$, then $\tilde{F} : \beta X_1 \times \ldots \times \beta X_n \to \beta Y$. Moreover, the restriction of $\tilde{F}$ on* $\mathrm{dom}\,(F)$ *is F.*

Proof. By definition, $\mathrm{dom}\,(\tilde{F}) = \beta X_1 \times \ldots \times \beta X_n$, and a standard argument shows that values of $\tilde{F}$ are ultrafilters. It follows from Lemma 2 that for all $z_1 \in X_1, \ldots, z_n \in X_n$,

$$\big\{x_1 : \ldots \{x_n : F(x_1, \ldots, x_n) \in S\} \in \hat{z}_n \ldots\big\} \in \hat{z}_1 \;\leftrightarrow\; F(z_1, \ldots, z_n) \in S,$$

and therefore,

$$\tilde{F}(\hat{z}_1, \ldots, \hat{z}_n) = \hat{y} \quad \text{whenever} \quad F(z_1, \ldots, z_n) = y,$$

and thus $\tilde{F}$ extends F up to identification of x and $\hat{x}$.

[1] See also Remark at the end of the paper.

Let us discuss the construction.

First, in the unary case, an $F : X \to Y$ extends to $\tilde{F} : \beta X \to \beta Y$ by

$$\tilde{F}(u) = \big\{ S \subseteq Y : \{ x \in X : F(x) \in S \} \in u \big\}.$$

This gives the standard unique continuous extension of F. Indeed, it is easy to see that $\tilde{F}$ is continuous, and continuous extensions agreeing on a dense subset coincide.

Next, consider the binary case. $F : X_1 \times X_2 \to Y$ extends to $\tilde{F} : \beta X_1 \times \beta X_2 \to \beta Y$ by

$$\tilde{F}(u_1, u_2) = \big\{ S \subseteq Y : \{ x_1 \in X_1 : \{ x_2 \in X_2 : F(x_1, x_2) \in S \} \in u_2 \} \in u_1 \big\}.$$

This can be considered as the extension fulfilled in two steps: first one extends left translations, then right ones. In the extended F, all right translations are continuous; in other words, the groupoid $(\beta X, \tilde{F})$ is *right topological*. Moreover, all left translations by *principal* ultrafilters are continuous, and such an extension is unique.

The extensions of mappings of arbitrary arity have analogous topological properties: If $F : X_1 \times \ldots \times X_n \to Y$ and $1 \le i \le n$, then for every $x_1 \in X_1, \ldots, x_{i-1} \in X_{i-1}$ and $u_{i+1} \in \beta X_{i+1}, \ldots, u_n \in \beta X_n$, the mapping

$$u \mapsto \tilde{F}(\hat{x}_1, \ldots, \hat{x}_{i-1}, u, u_{i+1}, \ldots, u_n)$$

of βX_i into βY is continuous, moreover, $\tilde{F}$ is a unique such extension of F. A proof of this fact will be done in the next section (Lemma 13).

Definition 4. Given $P \subseteq X_1 \times \ldots \times X_n$, let $\tilde{P}$ be *defined* as follows:

$$\langle u_1, \ldots, u_n \rangle \in \tilde{P} \quad \text{iff}$$
$$\big\{ x_1 \in X_1 : \ldots \{ x_n \in X_n : \langle x_1, \ldots, x_n \rangle \in P \} \in u_n \ldots \big\} \in u_1$$

for every $u_1 \in \beta X_1, \ldots, u_n \in \beta X_n$.

Proposition 5. *If $P \subseteq X_1 \times \ldots \times X_n$, then $\tilde{P} \subseteq \beta X_1 \times \ldots \times \beta X_n$. Moreover, $\tilde{P} \cap (X_1 \times \ldots \times X_n)$ is P.*

Proof. By Lemma 2.

Let us discuss the construction.

If P is a unary relation on X, $P \subseteq X$, one has

$$u \in \tilde{P} \quad \text{iff} \quad P \in u.$$

(The definition involves n-tuples; a 1-tuple $\langle x \rangle$ is just x.) Thus $\tilde{P}$ is an elementary open set of βX; the extensions of all unary relations on X form the standard open basis of the topology of βX. As we noted, the $\tilde{P}$ are in fact clopen.

If P is a binary relation, $P \subseteq X_1 \times X_2$, one has

$$\langle u_1, u_2 \rangle \in \tilde{P} \quad \text{iff} \quad \big\{ x_1 \in X_1 : \{ x_2 \in X_2 : \langle x_1, x_2 \rangle \in P \} \in u_2 \big\} \in u_1.$$

There is an easier way to say the same. Let $\langle\,\rangle\tilde{}$ denote the extension of the pairing function $\langle\,\rangle$ (cf. Definition 11.1 in Hindman–Strauss' book, there $\langle\,\rangle\tilde{}$ is denoted by $\otimes$ and referred as a "tensor product"; another name that is used is a "Fubini product"). Then

$$\langle u_1, u_2\rangle \in \tilde{P} \quad \text{iff} \quad P \in \langle u_1, u_2\rangle\tilde{}.$$

This formula displays a similarity to the formula with unary P explicitly.

As for topological properties of extended binary relations, it is easy to see that for any $x_1 \in X_1$ and $u_2 \in \beta X_2$, the set $\{u_1 \in \beta X_1 : \langle u_1, u_2\rangle \in \tilde{P}\}$ is clopen in βX_1, and the set $\{u_2 \in \beta X_2 : \langle \hat{x}_1, u_2\rangle \in \tilde{P}\}$ is clopen in βX_2.

Likewise, if $\langle\,\rangle\tilde{}$ denotes the extension of taking n-tuples, one gets the following redefinition:

Proposition 6. *Let $P \subseteq X_1 \times \ldots \times X_n$. Then for all $u_1 \in \beta X_1, \ldots, u_n \in \beta X_n$,*

$$\langle u_1, \ldots, u_n\rangle \in \tilde{P} \quad \textit{iff} \quad P \in \langle u_1, \ldots, u_n\rangle\tilde{}\,.$$

Proof. Clear.

The extensions of relations of arbitrary arity have analogous topological properties: If $P \subseteq X_1 \times \ldots \times X_n$ and $1 \leq i \leq n$, then for every $x_1 \in X_1, \ldots, x_{i-1} \in X_{i-1}$ and $u_{i+1} \in \beta X_{i+1}, \ldots, u_n \in \beta X_n$, the subset

$$\{u \in \beta X_i : \langle \hat{x}_1, \ldots, \hat{x}_{i-1}, u, u_{i+1}, \ldots, u_n\rangle \in \tilde{P}\}$$

of βX_i is clopen. A proof of this fact is also postponed to the next section (Lemma 17).

Remark. It is worth to note that, strictly speaking, the symbol $\tilde{}$ carries an ambiguity (although the context usually leaves no doubts). First, given a relation P, one gets distinct extensions $\tilde{P}$ depends on its implicit arity. Say, let $P \subseteq X \times X$. If P is regarded as a binary relation on X, then $\tilde{P}$ is a binary relation on βX, while if P is considered as a unary relation on $X \times X$, then $\tilde{P}$ is a unary relation on $\beta(X \times X)$. Similarly for extensions of mappings. Second, the same object can have distinct extensions when regarded as a function or as a relation. Say, let P be a binary relation that is a function, and let F_P denote this unary function. If F_P is an injection, then $\tilde{P}$ and $\tilde{F}_P$ do not coincide: $\tilde{P} = P$, while $\tilde{F}_P \neq F_P$ whenever $\beta X \neq X$.

This case near characterizes relations coinciding with their extensions. Let us say that a relation P is *almost injective* iff for any i and all fixed $x_1, \ldots, x_{i-1}$, $x_{i+1}, \ldots, x_n$, the set

$$P_{x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n} = \{x_i : \langle x_1, \ldots, x_n\rangle \in P\}$$

is finite. Note that a unary relation is almost injective iff it is finite. Then it can be shown that $\tilde{P} = P$ iff P is almost injective. The 'only if' part assumes that any infinite set carries a non-principal ultrafilter, which is weaker than the compactness of βX but still unprovable in ZF. The result can be restated in ZF alone if we redefine almost injective relations by replacing "is finite" with "carries no ultrafilter" (in ZFC the definitions coincide, while in any model without ultrafilters all relations are almost injective).

Lemma 7. *Let $h_1 : X_1 \to Y_1, \ldots, h_n : X_n \to Y_n$, and $G : Y_1 \times \ldots \times Y_n \to Z$. For all $S \subseteq Z$ and $u_1 \in \beta X_1, \ldots, u_n \in \beta X_n$, the following are equivalent:*

$$S \in \tilde{G}(\tilde{h}_1(u_1), \ldots, \tilde{h}_n(u_n)),$$

$$\big\{y_1 \in Y_1 : \ldots \{y_n \in Y_n : G(y_1, \ldots, y_n) \in S\} \in \tilde{h}_n(u_n) \ldots\big\} \in \tilde{h}_1(u_1),$$

$$\big\{x_1 \in X_1 : \ldots \{x_n \in X_n : G(h_1(x_1), \ldots, h_n(x_n)) \in S\} \in u_n \ldots\big\} \in u_1 .$$

Proof. The first and the second formulas are equivalent by definition of $\tilde{F}$.

That the second and the third formulas are equivalent can be proved by a straightforward induction on n. First one gets

$$\begin{aligned} & \big\{y_1 : \ldots \{y_n : G(y_1, \ldots, y_n) \in S\} \in \tilde{h}_1(u_n) \ldots\big\} \in \tilde{h}_1(u_1) \\ \text{iff} \quad & \big\{x_1 : h_1(x_1) \in \{y_1 : \ldots \{y_n : G(y_1, \ldots, y_n) \in S\} \in \tilde{h}_n(u_n) \ldots\}\big\} \in u_1 \\ \text{iff} \quad & \big\{x_1 : \{y_2 : \ldots \{y_n : G(h_1(x_1), y_2, \ldots, y_n) \in S\} \in \tilde{h}_n(u_n) \ldots\} \in \tilde{h}_2(u_2)\big\} \in u_1 . \end{aligned}$$

Then similarly

$$\begin{aligned} & \big\{y_2 : \ldots \{y_n : G(h_1(x_1), y_2, \ldots, y_n) \in S\} \in \tilde{h}_n(u_n) \ldots\big\} \in \tilde{h}_2(u_2) \\ \text{iff} \quad & \big\{x_2 : \ldots \{y_n : G(h_1(x_1), h_2(x_2), \ldots, y_n) \in S\} \in \tilde{h}_n(u_n) \ldots\big\} \in u_2 , \end{aligned}$$

etc. After n steps we obtain the required equivalence.

Corollary 8. *The following are equivalent:*

$$\begin{gathered} \langle \tilde{h}_1(u_1), \ldots, \tilde{h}_n(u_n) \rangle\tilde{} \in \tilde{P}, \\ P \in \langle \tilde{h}_1(u_1), \ldots, \tilde{h}_n(u_n) \rangle\tilde{}, \\ \{x_1 : \ldots \{x_n : \langle x_1, \ldots, x_n \rangle \in P\} \in \tilde{h}_n(u_n) \ldots\} \in \tilde{h}_1(u_1), \\ \{x_1 : \ldots \{x_n : \langle h_1(x_1), \ldots, h_n(x_n) \rangle \in P\} \in u_n \ldots\} \in u_1. \end{gathered}$$

Proof. The first and the second formulas are equivalent by Proposition 6, while the second and two last formulas are equivalent by Lemma 7 with $\langle\,\rangle$ as G.

Definition 9. Given a model $\mathfrak{A} = (X, F, \ldots, P, \ldots)$, let $\beta\mathfrak{A}$ denote the extended model $(\beta X, \tilde{F}, \ldots, \tilde{P}, \ldots)$.

As a corollary of Lemma 7, we get that continuous extensions of homomorphisms are homomorphisms.

Theorem 10 (The First Main Theorem). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two models. If h is a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$, then $\tilde{h}$ is a homomorphism of $\beta\mathfrak{A}$ into $\beta\mathfrak{B}$.*

Proof. Let $\mathfrak{A} = (X, F, \ldots, P, \ldots)$ and $\mathfrak{B} = (Y, G, \ldots, Q, \ldots)$.

Operations. As h is a homomorphism of (X, F) into (Y, G), we have for all $x_1, \ldots, x_n \in X$,

$$h(F(x_1, \ldots, x_n)) = G(h(x_1), \ldots, h(x_n)).$$

Then by Lemma 7, for all $u_1, \ldots, u_n \in \beta X$,

$$\begin{aligned} &\tilde{h}(\tilde{F}(u_1, \ldots, u_n)) \\ &= \big\{S : \{x_1 : \ldots \{x_n : h(F(x_1, \ldots, x_n)) \in S\} \in u_n \ldots\} \in u_1\big\} \\ &= \big\{S : \{x_1 : \ldots \{x_n : G(h(x_1), \ldots, h(x_n)) \in S\} \in u_n \ldots\} \in u_1\big\} \\ &= \tilde{G}(\tilde{h}(u_1), \ldots, \tilde{h}(u_n)), \end{aligned}$$

thus $\tilde{h}$ is a homomorphism of $(\beta X, \tilde{F})$ into $(\beta Y, \tilde{G})$.

Relations. As h is a homomorphism of (X, P) into (Y, Q), we have for all $x_1, \ldots, x_n \in X$,

$$\langle x_1, \ldots, x_n \rangle \in P \quad \text{implies} \quad \langle h(x_1), \ldots, h(x_n) \rangle \in Q.$$

We must verify that for all $u_1, \ldots, u_n \in \beta X$,

$$\langle u_1, \ldots, u_n \rangle \in \tilde{P} \quad \text{implies} \quad \langle \tilde{h}(u_1), \ldots, \tilde{h}(u_n) \rangle \in \tilde{Q},$$

thus $\big\{x_1 : \ldots \{x_n : \langle x_1, \ldots, x_n \rangle \in P\} \in u_n\} \ldots\big\} \in u_1$ implies

$$\big\{x_1 : \ldots \{x_n : \langle x_1, \ldots, x_n \rangle \in Q\} \in \tilde{h}(u_n) \ldots\big\} \in \tilde{h}(u_1).$$

By Corollary 8, the latter formula is equivalent to

$$\big\{x_1 : \ldots \{x_n : \langle h(x_1), \ldots, h(x_n) \rangle \in Q\} \in u_n \ldots\big\} \in u_1.$$

That h is a homomorphism means just $P \subseteq \{\langle x_1, \ldots, x_n \rangle : \langle h(x_1), \ldots, h(x_n) \rangle \in Q\}$. Therefore, the implication holds since $u_1, \ldots, u_n$ are filters, thus $\tilde{h}$ is a homomorphism of $(\beta X, \tilde{P})$ into $(\beta Y, \tilde{Q})$.

Topological properties of extensions. The second main theorem

Here we describe specific topological structure of our extensions. Then we establish our second main result showing that the extensions are universal in the class of models carrying a topology with similar properties.

We start from an explicit description of extensions of (unary) mappings to arbitrary compact Hausdorff spaces.

Definition 11. If $F : X \to Y$ where Y is a compact Hausdorff topological space, let $\tilde{F} : \beta X \to Y$ be *defined* as follows:

$$\tilde{F}(u) = v \quad \text{iff} \quad \{v\} = \bigcap_{A \in u} \mathrm{cl}_Y(F``A).$$

It is routine to check that the intersection consists of a single point, so the definition is correct, and that $\tilde{F}$ is a continuous extension of F, unique since Y is Hausdorff.

If the compact space is βY, the ultrafilter $\tilde{F}(u)$ can be rewritten in a form closer to that we known already.

Lemma 12. *If $F : X \to \beta Y$, then*

$$\tilde{F}(u) = \big\{S \subseteq Y : \{x \in X : F(x) \in \tilde{S}\} \in u\big\}.$$

Proof. It easily follows from the definition that

$$\tilde{F}(u) = \{S \subseteq Y : (\forall A \in u)\ (\exists x \in A)\ \ F(x) \in \tilde{S}\}.$$

It remains to verify

$$(\forall A \in u)\ (\exists x \in A)\ \ F(x) \in \tilde{S} \quad \text{iff} \quad \{x \in X : F(x) \in \tilde{S}\} \in u.$$

'If' uses the fact that u is a filter, while 'only if' uses that u is ultra.

In particular, if $F : X \to Y \subseteq \beta Y$ with Y discrete, then $\tilde{F}$ in the sense of the first definition coincide with $\tilde{F}$ in the sense of this new definition, thus witnessing we do not abuse notation.

Lemma 13. *Let $F : X_1 \times \ldots \times X_n \to Y$. For each i, $1 \le i \le n$, and for every $x_1 \in X_1, \ldots, x_{i-1} \in X_{i-1}$ and $u_{i+1} \in \beta X_{i+1}, \ldots, u_n \in \beta X_n$, the mapping $\tilde{F}_{x_1,\ldots,x_{i-1},u_{i+1},\ldots,u_n}$ of βX_i into βY defined by*

$$u \mapsto \tilde{F}(x_1, \ldots, x_{i-1}, u, u_{i+1}, \ldots, u_n)$$

is continuous. Moreover, $\tilde{F}$ is the only such extension of F.

Proof. We shall show that $\tilde{F}$ can be constructed by fixing successively all but one arguments and extending resulting unary functions. First we describe the construction and verify that the constructed extension has the required continuity properties. Then we verify that it coincides with $\tilde{F}$.

Step 1. Fix all but the last arguments: $x_1 \in X_1, \ldots, x_{n-1} \in X_{n-1}$, and put

$$f_{x_1,\ldots,x_{n-1}}(x) = F(x_1, \ldots, x_{n-1}, x).$$

Thus $f_{x_1,\ldots,x_{n-1}} : X_n \to Y$. We extend it to $\tilde{f}_{x_1,\ldots,x_{n-1}} : \beta X_n \to \beta Y$ and put

$$F_1(x_1, \ldots, x_{n-1}, u) = \tilde{f}_{x_1,\ldots,x_{n-1}}(u).$$

Thus $F_1 : X_1 \times \ldots \times X_{n-1} \times \beta X_n \to \beta Y$. It is obvious from the construction that F_1 is continuous in its last argument (since then it coincides with $\tilde{f}_{x_1,\ldots,x_{n-1}}$). And it is continuous in any other of its arguments (since then its domain is discrete).

Step 2. Fix all but the $(n-1)$th arguments: $x_1 \in X_1, \ldots, x_{n-2} \in X_{n-2}$, $u_n \in \beta X_n$, and put

$$f_{x_1,\ldots,x_{n-2},u_n}(x) = F_1(x_1, \ldots, x_{n-2}, x, u_n).$$

Thus $f_{x_1,\ldots,x_{n-2},u_n} : X_{n-1} \to \beta Y$. We extend it to $\tilde{f}_{x_1,\ldots,x_{n-2},u_n} : \beta X_{n-1} \to \beta Y$ and put

$$F_2(x_1, \ldots, x_{n-2}, u, u_n) = \tilde{f}_{x_1,\ldots,x_{n-2},u_n}(u).$$

Thus $F_2 : X_1 \times \ldots \times \beta X_{n-2} \times \beta X_n \to \beta Y$. The mapping F_2 is continuous in its $(n-1)$th argument (since then it coincides with $\tilde{f}_{x_1,\ldots,x_{n-2},u_n}$). Moreover, it is continuous in its nth argument whenever the fixed $(n-1)$th argument is in X_{n-1} (since then it coincides with F_1).

Arguing so, after $n-1$ steps we get $F_{n-1} : X_1 \times \beta X_2 \times \ldots \times \beta X_n \to \beta Y$, which is continuous in its ith argument whenever any jth fixed argument is in X_j, for all i, $1 \le i \le n$, and all $j < i$.

Step n. Fix all but the first arguments: $u_2 \in \beta X_2, \ldots, u_n \in \beta X_n$, and put

$$f_{u_2,\ldots,u_n}(x) = F_{n-1}(x, u_2, \ldots, u_n).$$

Thus $f_{u_2,\ldots,u_n} : X_1 \to \beta Y$. We extend it to $\tilde{f}_{u_2,\ldots,u_n} : \beta X_1 \to \beta Y$ and put

$$F_n(u, u_2, \ldots, u_n) = \tilde{f}_{u_2,\ldots,u_n}(u).$$

Thus $F_n : \beta X_1 \times \ldots \times \beta X_n \to \beta Y$. The mapping F_n is continuous in its first argument (since then it coincides with $\tilde{f}_{u_2,\ldots,u_n}$). Moreover, it is continuous in its ith argument whenever any jth fixed argument is in X_j, for all i, $1 \le i \le n$, and all $j < i$.

The uniqueness of such an extension follows from the uniquiness of continuous extensions of unary mappings by induction.

It remains to verify that F_n coincides with $\tilde{F}$. We have:

$$\begin{aligned} &F_1(x_1, \ldots, x_{n-1}, u_n) = \tilde{f}_{x_1,\ldots,x_{n-1}}(u_n) \\ &= \big\{S : \{x : f_{x_1,\ldots,x_{n-1}}(x) \in S\} \in u_n\big\} \\ &= \tilde{F}(\hat{x}_1, \ldots, \hat{x}_{n-1}, u_n). \end{aligned}$$

Then

$$\begin{aligned} &F_2(x_1, \ldots, x_{n-2}, u_{n-1}, u_n) = \tilde{f}_{x_1,\ldots,x_{n-2},u_n}(u_{n-1}) \\ &= \big\{S : \{x_{n-1} : f_{x_1,\ldots,x_{n-2},u_n}(x_{n-1}) \in \tilde{S}\} \in u_{n-1}\big\} \\ &= \big\{S : \{x_{n-1} : F_1(x_1, \ldots, x_{n-2}, x_{n-1}, u_n) \in \tilde{S}\} \in u_{n-1}\big\} \\ &= \big\{S : \{x_{n-1} : \tilde{f}_{x_1,\ldots,x_{n-1}}(u_n) \in \tilde{S}\} \in u_{n-1}\big\} \\ &= \big\{S : \{x_{n-1} : \{T : \{x : f_{x_1,\ldots,x_{n-1}}(x) \in T\} \in u_n\} \in \tilde{S}\} \in u_{n-1}\big\} \\ &= \big\{S : \{x_{n-1} : S \in \{T : \{x : f_{x_1,\ldots,x_{n-1}}(x) \in T\} \in u_n\}\} \in u_{n-1}\big\} \\ &= \big\{S : \{x_{n-1} : \{x_n : f_{x_1,\ldots,x_{n-1}}(x_n) \in S\} \in u_n\} \in u_{n-1}\big\} \\ &= \tilde{F}(\hat{x}_1, \ldots, \hat{x}_{n-2}, u_{n-1}, u_n). \end{aligned}$$

Likewise we get $F_n(u_1, \ldots, u_n) = \tilde{F}(u_1, \ldots, u_n)$, as required.

Remark. This description of continuity of extended mappings cannot be improved. If some of $u_1, \ldots, u_{i-1}$ is non-principal, then the mapping $\tilde{F}_{u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_n}$ of βX_i into βY defined by

$$u \mapsto \tilde{F}(u_1, \ldots, u_{i-1}, u, u_{i+1}, \ldots, u_n)$$

is not necessarily continuous. E.g. let F be a usual (binary) addition of natural numbers; then the mapping $u \mapsto u_1 \tilde{+} u$ is discontinuous. Also for fixed only $x_1 \in X_1, \ldots, x_{i-1} \in X_{i-1}$, the $(n-i+1)$-ary mapping $\tilde{F}_{x_1,\ldots,x_{i-1}}$ of $\beta X_i \times \ldots \times \beta X_n$ into βY defined by

$$\langle u_i, \ldots, u_n \rangle \mapsto \tilde{F}(x_1, \ldots, x_{i-1}, u_i, \ldots, u_n)$$

is not necessarily continuous. E.g. let $F(x_1, x_2, x_3) = x_2 + x_3$ and use the previous observation.

To name shortly the established topological property of $\tilde{F}$, let us introduce a terminology.

Definition 14. Let $X_1, \ldots, X_n, Y$ be topological spaces, and let $C_1 \subseteq X_1, \ldots, C_n \subseteq X_n$. We shall say that an n-ary function $F : X_1 \times \ldots \times X_n \to Y$ is *right continuous* w.r.t. $C_1, \ldots, C_n$ iff for each i, $1 \leq i \leq n$, and every $c_1 \in C_1, \ldots, c_{i-1} \in C_{i-1}$ and $x_{i+1} \in X_{i+1}, \ldots, x_n \in X_n$, the mapping

$$x \mapsto F(c_1, \ldots, c_{i-1}, x, x_{i+1}, \ldots, x_n)$$

of X_i into Y is continuous. If all the C_i coincide with, say C, we shall say that F is right continuous w.r.t. C.

In particular, F is right continuous w.r.t. the empty set iff for any $x_2 \in X_2, \ldots, x_n \in X_n$, the mapping

$$x \mapsto F(x, x_2, \ldots, x_n)$$

of X_1 into Y is continuous. Clearly, a unary F is right continuous iff it is continuous. If the operation is binary, the right continuity w.r.t. the empty set means that all right translations are continuous, and usually referred as “right continuity”, see e.g. [2]. If F is right continuous w.r.t. the whole $X_1, \ldots, X_n$, it is called *separately continuous*.

The following proposition notes obvious properties of compositions of right continuous functions.

Proposition 15. (i) *Let $F : X_1 \times \ldots \times X_n \to Y$ be right continuous w.r.t. $C_1, \ldots, C_n$, and let $g : Y \to Z$ be continuous. Then $H : X_1 \times \ldots \times X_n \to Z$ defined by*

$$H(x_1, \ldots, x_n) = g(F(x_1, \ldots, x_n))$$

is right continuous w.r.t. $C_1, \ldots, C_n$.
(ii) *Let all $f_1 : X_1 \to Y_1, \ldots, f_n : X_n \to Y_n$ be continuous, and let $G : Y_1 \times \ldots \times Y_n \to Z$ be right continuous w.r.t. $D_1, \ldots, D_n$. Then $H : X_1 \times \ldots \times X_n \to Z$ defined by*

$$H(x_1, \ldots, x_n) = F(h_1(x_1), \ldots, h_n(x_n))$$

is right continuous w.r.t. $f_1^{-1} D_1, \ldots, f_n^{-1} D_n$. □

Proof. Clear.

Definition 16. We shall say that an algebra is *right topological* with C a *topological center* iff all its operations are strongly right continuous w.r.t. C.

In this terms, Lemma 13 states that for any algebra $\mathfrak{A} = (X, F, \ldots)$, its extension $\beta\mathfrak{A} = (\beta X, \tilde{F}, \ldots)$ is right topological with X a topological center.

Lemma 17. *Let $P \subseteq X_1 \times \ldots \times X_n$. For every i, $1 \leq i \leq n$, and for any $x_1 \in X_1, \ldots, x_{i-1} \in X_{i-1}$ and $u_{i+1} \in \beta X_{i+1}, \ldots, u_n \in \beta X_n$, the subset*

$$\tilde{P}_{x_1,\ldots,x_{i-1},u_{i+1},\ldots,u_n} = \{u \in \beta X_i : \langle \hat{x}_1, \ldots, \hat{x}_{i-1}, u, u_{i+1}, \ldots, u_n \rangle \in \tilde{P}\}$$

of βX_i is clopen.

Proof. Let

$$f_{x_1,\ldots,x_{i-1},u_{i+1},\ldots,u_n}(u) = \langle \hat{x}_1, \ldots, \hat{x}_{i-1}, u, u_{i+1}, \ldots, u_n \rangle\tilde{}.$$

The mapping $f_{x_1,\ldots,x_{i-1},u_{i+1},\ldots,u_n}$ of βX_i into $\beta(X_1 \times \ldots \times X_n)$ is continuous by the previous lemma. Hence

$$\begin{aligned} \tilde{P}_{x_1,\ldots,x_{i-1},u_{i+1},\ldots,u_n} &= \{u \in \beta X_i : \langle \hat{x}_1, \ldots, \hat{x}_{i-1}, u, u_{i+1}, \ldots, u_n \rangle \in \tilde{P}\} \\ &= \{u \in \beta X_i : P \in \langle \hat{x}_1, \ldots, \hat{x}_{i-1}, u, u_{i+1}, \ldots, u_n \rangle\tilde{}\} \\ &= \{u \in \beta X_i : P \in f_{x_1,\ldots,x_{i-1},u_{i+1},\ldots,u_n}(u)\} \\ &= \{u \in \beta X_i : f_{x_1,\ldots,x_{i-1},u_{i+1},\ldots,u_n}(u) \in \tilde{Q}\} \end{aligned}$$

where Q is P considered as a unary relation on $X_1 \times \ldots \times X_n$, thus $\tilde{Q}$ is a unary relation on $\beta(X_1 \times \ldots \times X_n)$. Since Q is clopen, so is its preimage $\tilde{P}_{x_1,\ldots,x_{i-1},u_{i+1},\ldots,u_n}$ under the continuous mapping $f_{x_1,\ldots,x_{i-1},u_{i+1},\ldots,u_n}$.

To name shortly the established topological property of $\tilde{P}$, let us introduce a terminology.

Definition 18. Let $X_1, \ldots, X_n$ be topological spaces, and let $C_1 \subseteq X_1, \ldots, C_n \subseteq X_n$. We shall say that an n-ary relation $P \subseteq X_1 \times \ldots \times X_n$ is *right open* w.r.t. $C_1, \ldots, C_n$ iff for each i, $1 \leq i \leq n$, and every $c_1 \in C_1, \ldots, c_{i-1} \in C_{i-1}$ and $x_{i+1} \in X_{i+1}, \ldots, x_n \in X_n$, the subset

$$P_{c_1,\ldots,c_{i-1},x_{i+1},\ldots,x_n} = \{x \in X_i : \langle c_1, \ldots, c_{i-1}, x, x_{i+1}, \ldots, x_n \rangle \in P\}$$

of X_i is open. That a relation is *right closed* (or *right clopen*, etc.) is *defined* likewise.

In particular, P is right open w.r.t. the empty set iff for every $x_2 \in X_2, \ldots, x_n \in X_n$, the subset

$$P_{x_2,\ldots,x_n} = \{x \in X_1 : \langle x, x_2, \ldots, x_n \rangle \in P\}$$

of X_1 is open. Clearly, a unary P is right open iff it is open. Likewise for right closed (right clopen, etc.) relations.

The following proposition notes an obvious interplay of right open (right closed, right clopen, etc.) relations and right continuous functions.

Proposition 19. (i) *Let* $F : X_1 \times \ldots \times X_n \to Y$ *be right continuous w.r.t.* $C_1, \ldots, C_n$*, and let* $Q \subseteq Y$ *be open. Then*

$$P = \{\langle x_1, \ldots, x_n \rangle \in X_1 \times \ldots \times X_n : F(x_1, \ldots, x_n) \in Q\}$$

is right open w.r.t. $C_1, \ldots, C_n$*.*
(ii) *Let all* $F_1 : X_1 \to Y_1, \ldots, F_n : X_n \to Y_n$ *be continuous, and let* $Q \subseteq Y_1 \times \ldots \times Y_n$ *be right open w.r.t.* $D_1, \ldots, D_n$*. Then*

$$P = \{\langle x_1, \ldots, x_n \rangle \in X_1 \times \ldots \times X_n : \langle F_1(x_1), \ldots, F_n(x_n) \rangle \in Q\}$$

is right open w.r.t. $F_1^{-1} D_1, \ldots, F_n^{-1} D_n$*.*
Both clauses also hold for right closed (right clopen, etc.) relations.

Proof. Clear.

Definition 20. Let $\mathfrak{A} = (X, F, \ldots, P, \ldots)$ be a model equipped with a topology, and $C \subseteq X$. We shall say that $\mathfrak{A}$ is *right open*, and C is its *topological center* iff all its operations are right continuous w.r.t. C and all its relations are right open w.r.t. C. Likewise for *right closed* (*right clopen*, etc.) models.

Note that if the model is an algebra (i.e. does not have relations), each of these properties means that the algebra is right topological with C a topological center.
In this terms, two last lemmas state the following.

Corollary 21. *For any model* $\mathfrak{A}$*, its extension* $\beta\mathfrak{A}$ *is right clopen with* $\mathfrak{A}$ *a topological center.*

Proof. Lemmas 13 and 17.

The following theorem concerns rather arbitrary right open and right closed models with dense topological centers than ultrafilter extensions.

Theorem 22. *Let* $\mathfrak{A}$ *be a right open model,* $\mathfrak{B}$ *a Hausdorff right closed model, and* $\mathfrak{C} \subseteq \mathfrak{A}$ *a dense submodel and a topological center of* $\mathfrak{A}$*. Let* h *be a continuous mapping of* $\mathfrak{A}$ *into* $\mathfrak{B}$ *such that*

(i) $h \restriction \mathfrak{C}$ *is a homomorphism, and*
(ii) $h``\,\mathfrak{C}$ *is a topological center of* $\mathfrak{B}$*.*

Then h *is a homomorphism of* $\mathfrak{A}$ *into* $\mathfrak{B}$*.*

Proof. Let $\mathfrak{A} = (X, F, \ldots, P, \ldots)$ and $\mathfrak{B} = (Y, G, \ldots, Q, \ldots)$.
Operations. We argue by induction on arity of F (and G).
Step 1. Fix $c_1, \ldots, c_{n-1} \in C$ and put for all $x \in X$ and $y \in Y$,

$$f_{c_1,\ldots,c_{n-1}}(x) = F(c_1, \ldots, c_{n-1}, x),$$
$$g_{h(c_1),\ldots,h(c_{n-1})}(y) = G(h(c_1), \ldots, h(c_{n-1}), y).$$

The functions $f_{c_1,\ldots,c_{n-1}}$ and $g_{h(c_1),\ldots,h(c_{n-1})}$ are continuous (since $c_1,\ldots,c_{n-1}$ are in C, C is a topological center of $\mathfrak{A}$, and $h``C$ is a topological center of $\mathfrak{B}$). Therefore the functions $h \circ f_{c_1,\ldots,c_{n-1}}$ and $g_{h(c_1),\ldots,h(c_{n-1})} \circ h$ (both of X to Y) are continuous too (as compositions of continuous functions). Moreover, they agree on the dense subset C of X (since $\mathfrak{C}$ is a subalgebra and $h \restriction C$ is a homomorphism), i.e. for all $c \in C$,

$$h(f_{c_1,\ldots,c_{n-1}}(c)) = g_{h(c_1),\ldots,h(c_{n-1})}(h(c)).$$

Hence (as Y is Hausdorff) they coincide, i.e. for all $x \in X$,

$$h(f_{c_1,\ldots,c_{n-1}}(x)) = g_{h(c_1),\ldots,h(c_{n-1})}(h(x)).$$

Thus we proved that for all $c_1,\ldots,c_{n-1} \in C$ and $x_n \in X$,

$$h(F(c_1,\ldots,c_{n-1},x_n)) = G(h(c_1),\ldots,h(c_{n-1}),h(x_n)).$$

Step 2. Fix $c_1,\ldots,c_{n-2} \in C$ and $x_n \in X$, and put for all $x \in X$ and $y \in Y$,

$$f_{c_1,\ldots,c_{n-2},x_n}(x) = F(c_1,\ldots,c_{n-2},x,x_n),$$
$$g_{h(c_1),\ldots,h(c_{n-2}),h(x_n)}(y) = G(h(c_1),\ldots,h(c_{n-2}),y,h(x_n)).$$

Again, the functions $f_{c_1,\ldots,c_{n-2},x_n}$ and $g_{h(c_1),\ldots,h(c_{n-2}),h(x_n)}$ are continuous (since $c_1,\ldots,c_{n-2}$ are in C, C is a topological center of $\mathfrak{A}$, and $h``C$ is a topological center of $\mathfrak{B}$). Therefore the compositions $h \circ f_{c_1,\ldots,c_{n-2},x_n}$ and $g_{h(c_1),\ldots,h(c_{n-2}),h(x_n)} \circ h$ (both of X to Y) are continuous too. Moreover, they agree on the dense subset C of X (by Step 1), i.e. for all $c \in C$,

$$h(f_{c_1,\ldots,c_{n-2},x_n}(c)) = g_{h(c_1),\ldots,h(c_{n-2}),h(x_n)}(h(c)).$$

Hence they coincide, i.e. for all $x \in X$,

$$h(f_{c_1,\ldots,c_{n-2},x_n}(x)) = g_{h(c_1),\ldots,h(c_{n-2}),h(x_n)}(h(x)).$$

Thus we proved that for all $c_1,\ldots,c_{n-2} \in C$ and $x_{n-1},x_n \in X$,

$$h(F(c_1,\ldots,c_{n-2},x_{n-1},x_n)) = G(h(c_1),\ldots,h(c_{n-2}),h(x_{n-1}),h(x_n)).$$

After n steps, we get $h(F(x_1,\ldots,x_n)) = G(h(x_1),\ldots,h(x_n))$ for all $x_1,\ldots,x_n \in X$, thus showing that h is a homomorphism of (X,F) into (Y,G), as required.

Relations. Assuming $\langle x_1,\ldots,x_n\rangle \in P$, we shall show $\langle h(x_1),\ldots,h(x_n)\rangle \in Q$ by induction on n.

Step 1. First we suppose $c_1,\ldots,c_{n-1} \in C$. Pick arbitrary neighborhood V of $h(x_n)$. Since h is continuous, there exists a neighborhood U of x_n such that $h``U \subseteq V$. The set $U \cap P_{c_1,\ldots,c_{n-1}}$ is open ($P_{c_1,\ldots,c_{n-1}}$ is open as $c_1,\ldots,c_{n-1}$ are in the topological center C) and nonempty (x_n belongs to it), and so there is $c \in C \cap U \cap P_{c_1,\ldots,c_{n-1}}$ (since C is dense). Therefore, we have $\langle c_1,\ldots,c_{n-1},c\rangle \in P$, and so $\langle h(c_1),\ldots,h(c_{n-1}),h(c)\rangle \in Q$ (since $h \restriction C$ is a homomorphism).

So we see that any neighborhood of $h(x_n)$ has a point y with $\langle h(c_1), \ldots, h(c_{n-1}), y\rangle \in Q$. Since the set

$$Q_{h(c_1),\ldots,h(c_{n-1})} = \{y : \langle h(c_1), \ldots, h(c_{n-1}), y\rangle \in Q\}$$

is closed (as $h(c_1), \ldots, h(c_{n-1})$ are in the topological center $h``C$), it has the point $h(x_n)$. Thus we proved that whenever $c_1, \ldots, c_{n-1} \in C$ and $\langle c_1, \ldots, c_{n-1}, x_n\rangle \in P$, then

$$\langle h(c_1), \ldots, h(c_{n-1}), h(x_n)\rangle \in Q.$$

Step 2. Now we suppose $c_1, \ldots, c_{n-2} \in C$ and $x_n \in X$. Pick arbitrary neighborhood V of $h(x_{n-1})$. Since h is continuous, there exists a neighborhood U of x_{n-1} such that $h``U \subseteq V$. Again, the set $U \cap P_{c_1,\ldots,c_{n-2},x_n}$ is open and nonempty, so there is $c \in C \cap U \cap P_{c_1,\ldots,c_{n-2},x_n}$. Hence, $\langle c_1, \ldots, c_{n-2}, c, x_n\rangle \in P$, and so $\langle h(c_1), \ldots, h(c_{n-2}), h(c), h(x_n)\rangle \in Q$ (by Step 1).

So any neighborhood of $h(x_{n-1})$ has a point y with $\langle h(c_1), \ldots, h(c_{n-2}), y, h(x_n)\rangle \in Q$. Since the set

$$Q_{h(c_1),\ldots,h(c_{n-2}),h(x_n)} = \{y : \langle h(c_1), \ldots, h(c_{n-2}), y, h(x_n)\rangle \in Q\}$$

is closed, it has the point $h(x_{n-1})$. Thus we proved that whenever $c_1, \ldots, c_{n-2} \in C$ and $\langle c_1, \ldots, c_{n-2}, x_{n-1}, x_n\rangle \in P$, then

$$\langle h(c_1), \ldots, h(c_{n-2}), h(x_{n-1}), h(x_n)\rangle \in Q.$$

After n steps, we conclude that whenever $\langle x_1, \ldots, x_n\rangle \in P$, then $\langle h(x_1), \ldots, h(x_n)\rangle \in Q$, thus h is a homomorphism of (X, P) into (Y, Q), as required.

The following theorem states the universal property of $\mathfrak{A}$ completely analogous to that of the Stone–Čech (or Wallman) compactification.

Theorem 23 (The Second Main Theorem). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two models, and let $\mathfrak{B}$ be compact Hausdorff right closed. Let h be a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ such that $h``\mathfrak{A}$ is a topological center of $\mathfrak{B}$. Then $\tilde{h}$ is a homomorphism of $\beta\mathfrak{A}$ into $\mathfrak{B}$.*

Proof. By Corollary 21 and Theorem 22.

Note that the First Main Theorem (Theorem 10) follows from this one.

Generalizations

Here we note that the results establishing universality of ultrafilter extensions of models w.r.t. homomorphisms remain true if one replaces homomorphisms by more general relationships between models.

The concepts of homotopy and isotopy are customarily used for groupoids, especially, in quasigroup theory. Let us give a general definition of homotopy and isotopy between arbitrary models.

Definition 24. Let F and G be n-ary operations on X and Y resp. Mappings $h, h_1, \ldots, h_n$ of X into Y form a *homotopy* of (X, F) into (Y, G) iff

$$h(F(x_1, \ldots, x_n)) = G(h_1(x_1), \ldots, h_n(x_n))$$

for all $x_1, \ldots, x_n \in X$. The homotopy is an *isotopy* iff all the $h, h_1, \ldots, h_n$ are bijective.

Definition 25. Let P and Q be n-ary relations on X and Y resp. Mappings $h_1, \ldots, h_n$ of X into Y are a *homotopy* of (X, P) into (Y, Q) iff

$$P(x_1, \ldots, x_n) \quad \text{implies} \quad Q(h_1(x_1), \ldots, h_n(x_n))$$

for all $x_1, \ldots, x_n \in X$. The homotopy is an *isotopy* iff all the $h_1, \ldots, h_n$ are bijective and

$$P(x_1, \ldots, x_n) \quad \text{iff} \quad Q(h_1(x_1), \ldots, h_n(x_n)).$$

Note that when all the $h, h_1, \ldots, h_n$ coincide, then the homotopy is an homomorphism (and the isotopy is an isomorphism). In particular, homotopies of unary relations are homomorphisms.

If $\mathfrak{A}$ and $\mathfrak{B}$ have more than one operation or relation, there are various ways to define homotopies (and isotopies) between them, the weakest of which is as follows.

Definition 26. A family H of mappings of X into Y form a *homotopy* of $\mathfrak{A}$ into $\mathfrak{B}$ iff for any m-ary operation F in $\mathfrak{A}$ there are mappings $h, h_1, \ldots, h_m$ in H forming a homotopy of (X, F) into (Y, G) with the corresponding operation G in $\mathfrak{B}$, and for any n-ary relation P in $\mathfrak{A}$ there are mappings $h_1, \ldots, h_n$ in H forming a homotopy of (X, P) into (Y, Q) with the corresponding relation Q in $\mathfrak{B}$. The homotopy H is an *isotopy* iff all mappings in H are bijective.

Obviously, a homotopy H is a homomorphism iff $|H| = 1$. In general, the size of H can be regarded as a degree of its dissimilarity to a homomorphism.

Proposition 27. *Let $F : X \to Y$.*

(i) *If F is surjective, then so is $\tilde{F}$.*
(ii) *If F is injective, then so is $\tilde{F}$. Moreover, $(\tilde{F})^{-1} = (F^{-1})^{\sim}$.*
(iii) *If F is bijective, then $\tilde{F}$ is a homeomorphism of βX onto βY.*

Proof. (i) We must show that for any $v \in \beta Y$ there is $u \in \beta X$ such that $\tilde{F}(u) = v$, i.e.

$$S \in v \quad \text{iff} \quad \{x : F(x) \in S\} \in u$$

for all $S \subseteq Y$. Given v, let

$$D = \big\{\{x : F(x) \in S\} : S \in v\big\}.$$

D has the finite intersection property: Given $S', S'' \in v$, we have $\{x : F(x) \in S'\} \cap \{x : F(x) \in S''\} = \{x : F(x) \in S' \cap S''\}$, so this set is in D (since $S' \cap S''$ is in v).

Let u be any ultrafilter that extends D. Then u is as required: The 'only if' part holds by definition of u. To verify the 'if' part, notice that if $S \notin v$ then $Y \setminus S \in v$, and so $\{x : F(x) \in Y \setminus S\} \in u$, whence it follows $\{x : F(x) \in S\} \notin u$ (as preimages of disjoint sets are disjoint).

(ii) We must show that if $u', u'' \in \beta X$ are distinct, then so are $\tilde{F}(u'), \tilde{F}(u'') \in \beta Y$, i.e. there is $T \in \tilde{F}(u') \setminus \tilde{F}(u'')$, and thus $\{x : F(x) \in T\} \in u' \setminus u''$. As $u' \neq u''$, there is $S \in u' \setminus u''$. Since F is injective, we have $\{x : F(x) \in F``S\} = S$, so we can put $T = F``S$.

The equality $(\tilde{F})^{-1} = (F^{-1})\tilde{}$ follows immediately.

(iii) This follows from (i) and (ii).

Remark. Clause (i) uses the assumption that any filter extends to an ultrafilter, which is, as we mentioned above, equivalent to the compactness of βX.

By using Lemma 7 and Proposition 27 and modifying arguments of the proofs of our main results, one gets the following generalization (we leave details for the reader).

Theorem 28. *Both Main Theorems (Theorems 10 and 23), as well as Theorem 22, remain true by replacing homomorphisms with homotopies, isotopies, and embeddings.* □

Question. Characterize relationships between models such that both theorems remain true by replacing homomorphisms with these relationships.

Another interesting question is about theories of extended models.

Question. Characterize formulas that are preserved under β.

In [4] we answer the question for the case when the formulas are identities and the models are groupoids.

Finally, let us mention that certain types of ultrafilters form submodels of extended models. In particular, so are κ-complete ultrafilters, for any given κ. A proof generalizes the proof in [5] given for groupoids.

Remark. When I prepared the paper for publishing, I recognized V. Goranko's unpublished manuscript *Filter and Ultrafilter Extensions of Structures: Universal-algebraic Aspects*, 2007 (the author said me the first version was written ten years before). Goranko extends models by arbitrary filters. However his filter extension of operations does not work for ultrafilters, so he defines this case separately, in the same way that in Definition 1 here. His extension of relations differs from that given in Definition 4. Goranko proved a theorem analogous to the First Main Theorem, both theorems coincide for ultrafilter extension of operations. He asks whether there is another ultrafilter extension of operations that would be "better". Perhaps the Second Main Theorem can be considered as the negative answer.

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