Developing Algebraic Thinking in the Context of Arithmetic

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Abstract Using classroom episodes from grades 2–6, this chapter highlights four mathematical activities that underlie arithmetic and algebra and, therefore, provide a bridge between them. These are:

- understanding the behavior of the operations,
- generalizing and justifying,
- extending the number system, and
- using notation with meaning.

Analysis of each episode provides insight into how teachers recognize the opportunities to pursue this content in the context of arithmetic and how such study both strengthens students' understanding of arithmetic operations and enables them to develop ideas foundational to the study of algebra.

In recent years, the question, "What can be done in the elementary grades to prepare students for algebra?" has received a great deal of attention. The form of the question sometimes leads to a conception of preparation for algebra that focuses on doing formal algebra—or aspects of formal algebra—in lower grades. Rather, one might reframe the question as, "What are ways of thinking, modes of reasoning, and essential understandings that have their roots in arithmetic *and* are essential to algebra? What are the underlying connections between arithmetic and algebra?" These

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questions lead to a focus on finding instructional emphases that both serve the elementary grade goals of computational fluency and support students to develop the kind of reasoning that will lead to the need for, and meaningful use of, algebraic tools.

Several research teams have been pursuing these questions, as is represented by Kaput et al.'s (2008) anthology, *Algebra in the Early Grades*. Some of the groups represented in this collection of current work focus on introducing the concept of functions, providing tasks which invite students to create function rules to describe patterns of growth (e.g., Blanton 2008). Others organize their work around generalizations in the number system. For example, Carpenter et al. (2003) describe class discussion about true and false number sentences. Over the past decade, the authors of this chapter have been developing K-5 student curriculum and professional development materials for teachers in grades K-8 that address both of these strands of early algebra (Russell et al. 2008; Schifter et al. 2008a, 2008b). This paper draws from the part of our research that focuses on how students engage with generalizations about the behavior of the operations.

From our work with elementary and middle grade teachers, we have identified four mathematical activities that underlie both arithmetic and algebra and, therefore, provide a bridge between the two. These are:

- understanding the behavior of the operations,
- generalizing and justifying,
- extending the number system, and
- using notation with meaning.

These themes emerge from content at the heart of the elementary mathematics program, and can be highlighted and pursued by teachers who learn to recognize the opportunities that arise in their classrooms. Focusing on these aspects of arithmetic addresses two major goals: (1) It enables students to grow from arithmetic towards algebra, and (2) it strengthens their understanding of arithmetic operations and contributes to computational fluency.

In collaboration with teachers in grades K-8, we have been investigating how students articulate, represent, and justify general claims about the operations. We have also been examining how teachers can recognize the implicit generalizations that arise in the course of students' study of arithmetic and make them explicit objects of study in the classroom (Russell et al. 2006; Schifter et al. 2008c). An important component of this research is the close observation of classroom discourse by teachers, who carefully document and write about learning episodes in their own classrooms. Through discussion and analysis of these episodes at regular project meetings and via an electronic web-board, we consider evidence and develop ideas about students' early algebra experience.

In each of the next four sections of this chapter we focus on one of the four areas that links arithmetic and algebra. The examples come from videotaped lessons, lessons observed by project staff, and narratives written by teachers based on transcripts from their teaching.

Understanding the Behavior of the Operations

Computational fluency with the four basic arithmetic operations is a core of the elementary curriculum. In these years, students move from counting to computation. It is an expectation that students enter middle school with a firm grasp of addition, subtraction, multiplication, and division of, at least, whole numbers. Most students come into the secondary grades with procedures for solving basic arithmetic problems. Yet, even among students who carry out these procedures correctly, there are persistent problems as they make the transition from arithmetic to algebra. Many of these problems can be traced to lack of knowledge about the properties and behaviors of the operations. At best, these students may understand these properties in the context of arithmetic, but not access their knowledge in the new context of algebra. At worst, these students use memorized procedures correctly, but do not understand why they work or how they are based on properties of the operations.

What does it look like when students don't have sufficient experience with the behavior and properties of the operations when they reach algebra? What happens when only speed with computation and memorization of algorithms are foregrounded, while understanding falls into the background? Many teachers of algebra in the middle and high school note that students repeatedly make the same errors, for example:

$$-3 + -5 = 8$$
$$(a+b)^2 = a^2 + b^2$$
$$2(xy) = (2x)(2y)$$

Student Errors

Such errors can be persistent, even in the face of repeated correction. It is likely that students who make them see a resemblance in the patterns of the symbols to other, correct rules. For example, students who rely on memorization of calculation procedures may remember a rule informally expressed as "two negatives make a positive," but don't have other tools that help them determine that this rule applies to the *product* of two negative numbers, but not to the sum. Students who make the second error may incorrectly interpret the exponent as a number that behaves like a factor, so that $(a + b)^2$ is interpreted in the same way they would interpret $2(a + b)$. Or, if they do understand the meaning of the exponent, they are not able to access and apply the distributive property from their knowledge of multiplying whole numbers. In the third example above, students may be applying a rule to "multiply everything inside the parentheses by the number outside the parentheses," which would work for $2(x + y)$, but not for $2(xy)$. They incorrectly apply what they think is the distributive property and do not recognize an application of the associative property. In each case, properties of operations are over-generalized or misapplied.

In our work with elementary and middle grades teachers, we have been investigating how their students benefit from explicit study of the operations, for example, by examining calculation procedures as mathematical objects that can be described generally in terms of their properties and behaviors. By this study, we do not mean that students should learn the names of properties and state them as rules, as occurred in some curricula in the 1960s. Some of us who went to school at that time remember that we learned, for example, what the commutative, associative, and distributive properties were, but weren't quite sure why we were learning them or why they were so important. Rather, students use representations or story contexts to describe the behavior of the operations. For example, students might join two sets of cubes to illustrate addition, switching positions of the sets to show that changing the order of addends does not affect the sum. They might draw an image of some amount removed from a larger amount to demonstrate that as the amount removed (the subtrahend) increases, the result (the difference) decreases. Similarly, students might use arrays or equal groups of objects to illustrate the behavior of multiplication and division.

The following classroom episodes illustrate a grade 2 class investigating addition and subtraction and a grade 5 class investigating multiplication in this way.

Episode A: How Are Addition and Subtraction Different? (Grade 2)

In prior lessons in this second grade class (Schifter et al. 2008a, p. 114), the students had noticed that if you change the order of the numbers in an addition expression, the sum remains the same. Many students had been using this idea in their computation, but the teacher, Maureen Johnson, wanted them to consider this property of addition explicitly. During this class session, Ms. Johnson asked students to find pairs of numbers that add to 25. Then she brought students' attention to the question of whether the order of two addends can always be changed without affecting the sum.

Teacher: These two numbers that we used, can we switch them around? Can we change the order and still get 25? I hear a lot of yeses. Who's not sure? So someone's not sure? Two people aren't so sure? If you feel sure, how would you explain that? Kwame?

Kwame: $18 + 7$. Change it around. That's $7 + 18$.

Teacher: So what do you want to say about that?

Kwame: It will still be 25.

Teacher: How come that's still 25?

Kwame: We didn't change the numbers.

Teacher: Does someone have another one they want to talk about? Kamika?

Kamika: $19 + 6$.

Teacher: OK. If I put the 6 first and then the 19, what will it be?

Kamika: 25, because you're just switching the numbers. You're not adding any more and you're not taking away any numbers. You're just changing them around.

Ms. Johnson then asked the class if they were sure this would work for all numbers. When they said yes, she asked if they could prove it: "Can anyone show me something that would prove it or explain it better?" She built two towers out of connecting cubes, one with 23 cubes and one with 2 cubes.

Latifa took the 2-cube tower and moved it rapidly back and forth from one side of the 23-cube tower to the other.

> Latifa: If you keep on switching it around, it will still make 25. Because you're not taking away or adding anything to it, so it will still be the same number.

Other students showed that they understood and agreed with Latifa's actions and words. Latifa used a representation of joining two sets of cubes to show that $23+2=2+23$, but she also used language to explain why this relationship would hold for any pair of numbers: If you change the order, nothing more is added and nothing is taken away, so the total stays the same.

Latifa's demonstration is an example of a phenomenon we see in many of our classroom examples: a representation showing specific quantities is talked about and thought about by students as representing a class of numbers. Although there are a specific number of cubes in each cube tower, the students can hold this model in their imagination to represent any pair of numbers—or any pair of numbers they can imagine (which, for second graders, may be the set of whole numbers or, at least, the whole numbers with which they are familiar and comfortable).

To find out whether students were, in fact, talking about any pair of numbers and not just those that sum to 25, Ms. Johnson asked them to consider numbers larger than they could easily add: "What about $175 + 266$?" Her students argued that $175 + 266$ and $266 + 175$ must both have the same sum, even though they had not attempted to carry out the addition. "It doesn't matter," they said. "You're not adding anything or taking anything away."

By now Ms. Johnson felt assured that the students in the class were, indeed, thinking in terms of a generalization, beyond the specific numbers of their examples, and they were able to describe the essential aspects of a representation to justify the claim. But she was also concerned that they should not overgeneralize. Were they thinking about a property that applies to addition, or were they thinking that this property would apply to any operation? She asked them whether they could apply their generalization to $7 - 3$: does $7 - 3$ equal $3 - 7$?

> Latifa: If you have 3 take away 7, but 3 doesn't have 7. So you can only do 7 and 3, because 3 is not a 7.
>
> Teacher: There is not enough in 3 to take away 7? Is that what you're saying? What if I had 3 and I want to take away 7, then how many could I take away?
>
> Latifa: You could only take away 3, to make 0.
>
> Kamika: After you use the 3, it's 3, 2, 1, 0, 0, 0. The 0 is going to keep on repeating itself until it gets to 7.

The question of what happens when one changes the order of the numbers in subtraction allows the possibility of introducing negative numbers. In fact, at a later point in the discussion, one student did raise this idea. Antoine stated, "That won't be 0, it would be negative 4… That means it's going lower. When you go lower than 0, that means negative 1, negative 2, negative 3, …" However, most students in the class, basing their ideas on their familiarity with positive numbers and a "take away" or removal model of subtraction, came to the conclusion that subtracting 7 from 3 is not possible. If you have 3 cookies and try to eat 7, you can only eat 3; then you have 0, and no more can be removed. As Latifa says, "you could only take away 3, to make 0." This reasoning was sufficient to convince students that $7-3 \neq 3-7$, and that the commutative property applies to addition, but not to subtraction, which was the teacher's purpose for this part of the lesson.

In this class, as in many primary classes, students noticed a regularity as they solved addition problems: $4+3$ and $3+4$ are both equal to 7; $5+8$ and $8+5$ are both equal to 13; and so forth. Students who notice such a regularity may be convinced it will always hold because they have encountered many examples and may apply the rule they have formulated in their computation. This teacher took the opportunity to make this regularity an explicit focus of investigation. She challenged her students to think about whether changing the order of the addends maintains the sum only for specific cases or whether it is true more generally and to explain how they knew. Keeping the symbols connected to a representation that demonstrates the action of addition allowed them to explain *why* their claim must be true. By presenting a contrasting case of subtraction, she checked to make sure they understood that their generalization applied specifically to the operation of addition. The students' explanation of the effect of changing the terms of a subtraction problem was, again, tied to their understanding of a model of the *action* of subtraction.

Episode B: Rounding Factors in a Multiplication Problem (Grade 5)

In order to focus on the behavior of the operations, teachers can pay attention to what regularities students are noticing, as the teacher did in the example above. Another site for determining which behaviors of the operations might be an important focus for a particular group of students is student errors, since errors are often related to the misapplication of basic properties of the operations. In the following example, students had been working on the problem, 17×36. After solving the problem and comparing results, students in the class knew that the correct product was 612. However, one student, Thomas, solved the problem this way:

> I round 17 to 20 and 36 to 40. I know that 20×40 is 800. Then I need to subtract the extra 3 (from rounding 17 to 20) and the extra 4 (from rounding 36 to 40). $800 - 3 - 4 = 793$. The answer is 793.

At this point, the teacher, Liz Sweeney, asked Thomas to put his method on the board and explain it to the class. Once Thomas—who also knew that the answer he had was incorrect—had finished his explanation, Ms. Sweeney asked the class to think through Thomas's method for homework, to consider how Thomas had been thinking about the problem, and why his reasoning didn't lead to the correct answer.

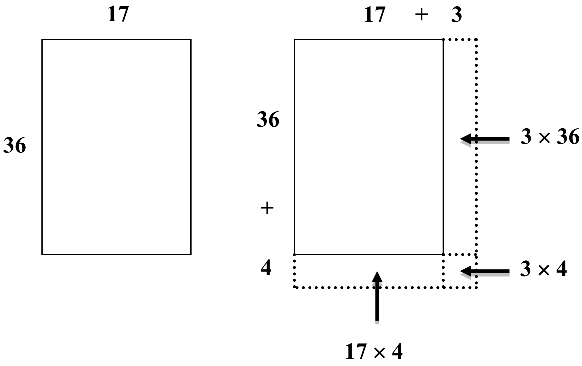
Why would Ms. Sweeney do this? This particular classroom episode is taken from a videotape (Schifter et al. 1999) that is used in a professional development seminar. Some teachers who watch this tape are horrified by the teacher's move—that she would focus on this incorrect solution and, even worse, ask students to work on it at home! While we might debate whether, strategically, we would or wouldn't send such an assignment home where it might be misinterpreted, the teacher's reasoning is clear, as she explained to the class. She saw that even students who easily computed the correct product were somewhat persuaded by Thomas's reasoning. This method looks like it should work—from the point of view of addition: students didn't automatically see why his method does not lead to the correct answer.

We might ask, then, what is right or sensible about Thomas's method? In fact, in the operation of addition his idea works; one might add some amount to one or more addends, add the numbers, then subtract those amounts that had been added, for example:

$$17 + 36 = (17 + 3) + (36 + 4) - 3 - 4 = 20 + 40 - 3 - 4 = 60 - 7 = 53$$

Thomas's method is an example of taking a behavior of one operation and applying it to another operation where it doesn't work. By explicitly studying Thomas's method and why it doesn't work, students have to think through the properties of multiplication—in particular, the distributive property—in order to understand the role of the 3 and the 4 that Thomas added. In fact, using this problem with adults over many years, we have found that the exercise of starting with Thomas's steps of changing 17 to 20 and 36 to 40, and then figuring out how to complete the problem correctly (answering the question, what is it you have to subtract from 800?) is an excellent way for adults to revisit their understanding of multiplication and its properties.

By having teachers or students examine Thomas's strategy, we are not advocating that his procedure (completed in a way that it results in the correct product) is one that should be learned and used to solve multiplication problems. In Thomas's class, the teacher was not hoping that students would routinely alter multiplication problems in the way he had in order to solve them. His method does not necessarily make the problem easier to solve in the long run. However, figuring out what has been added to the product by changing the two factors gets at the heart of the meaning of multiplication and the distributive property, making this procedure worth studying. One way of representing the effect on the product of increasing the factors, as Thomas does, is illustrated below:

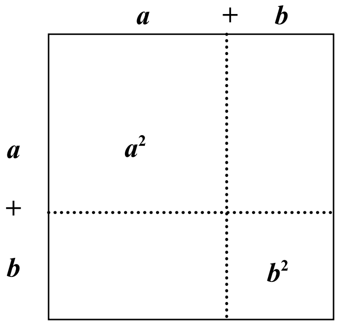


$(17 + 3) \times (36 + 4) =$

$(17 \times 36) + (17 \times 4) + (3 \times 36) + (3 \times 4) = 800$

therefore $17 \times 36 = 800 - (17 \times 4) - (3 \times 36) - (3 \times 4)$

This analysis requires representing the operation of multiplication in a way that manifests the distributive property (which may be hidden from students by some of the algorithms they use). Such visualization of the way factors are pulled apart and multiplied by parts of other factors applies to both arithmetic and algebraic contexts. The reasoning that students might engage in to decode Thomas's error is similar to the reasoning they might engage in to justify why $(a + b)^2$ is not equal to $a^2 + b^2$. Their understanding of the distributive property can be explicitly called upon, so that they can visualize that $(a + b)^2$ cannot possibly be equivalent to $a^2 + b^2$ unless a or b is equal to 0.



Ms. Sweeney reported that Thomas's error led to three days of deep thinking engaging the entire class. The students drew pictures of groups and presented arrays to explain what happens when the two factors are increased.

Generalizing and Justifying

A second mathematical activity that connects arithmetic to algebra is articulating, representing, and justifying generalizations about the operations. As seen in the episodes in the previous section, general ideas arise frequently in the course of students' study of arithmetic. For example, young students notice that when they change the order of addends, the sum does not change. Older students notice the same thing about multiplication expressions. Throughout the elementary grades, opportunities arise to investigate general claims about the operations that can be brought to the explicit attention of the students.

There are two aspects of engaging with general claims that we see teachers developing in the elementary grades:

- articulating particular general claims based on the regularities students notice in the behavior of numbers and operations
- developing a mathematical argument to justify a general claim for a class of numbers

The three classroom episodes in this section are examples of (1) a teacher helping her third graders focus on the articulation of a general claim; (2) a group of fifth graders who are developing both articulation and justification as they investigate equivalent addition expressions; and (3) fifth graders' representation-based proof of a generalization about multiplication.

Articulating General Claims

As students in the elementary grades are given opportunities to notice and discuss generalizations about number and operations, they encounter the need for language to describe the generalizations they are investigating. Young students often use words like "it" or "that," or use a gesture such as pointing, to indicate what they are describing. In math class, when a student says, for example, "I think it's true," it is important to clarify exactly what "it" means, both so that the student offering the idea can clarify his or her own thinking and so that other students do not make different assumptions about the nature of the assertion being considered. Putting reasoning into words can be challenging, for students or adults, but clarification of the language and clarification of the ideas appear to develop together for young students, as illustrated in the next example.

Episode C: Equivalent Expressions in Addition and Subtraction (Grade 3)

Alice Kaye's third graders had formulated a general claim about addition, which had been expressed by one of the students, Clarissa, as: "When you're adding two numbers together, you can take some amount from one number and give it to the other, and if you add those up, it will still equal the same thing." A few weeks later, Alice asked the class to consider subtraction: "By Clarissa's statement, we could say that we know this equation is true: $57 + 21 = 58 + 20$. Without even doing the addition, we would know that whatever $57 + 21$ equals, $58 + 20$ also equals that same total. Would it also be true to say that $57 - 21 = 58 - 20$?" Students quickly computed $57 - 21$ and $58 - 20$ and concluded that the differences are not equal, but students were puzzled about why this was true. As one student put it, "why wouldn't they be the same?"

After a couple of days of investigating this question and coming up with story contexts to illustrate their ideas, the class was considering two series of equations:

$25+0=25$	$25-0=25$
$24+1=25$	$26-1=25$
$23+2=25$	$27-2=25$
$22+3=25$	$28-3=25$
$21+4=25$	$29-4=25$
$20+5=25$	$30-5=25$
$19+6=25$	$31-6=25$

The set of subtraction equations had been generated using a story context that Todd had come up with:

> If Todd had 26 baseball cards, and his little brother stole 1, he'd have 25 cards left. What are other numbers of baseball cards Todd could start with, and how many would his little brother have to steal so that he would always have 25 cards left?

In the course of the discussion about the two sets of equations, the teacher repeatedly asked her class to clarify what their general claim was as they were developing arguments to support it:

> Nora [looking at the chart]: So I guess it only works with adding, not subtracting.
>
> Teacher: What only works with adding? What's the "it?"
>
> Nora: The…um…the…the… [a long pause, but she cannot yet put into words what she was thinking was "working with addition"]
>
> Carl: There's both the same thing in the middle… 0, 1, 2, 3, 4, 5, 6 [pointing to the subtraction sequence] and 0, 1, 2, 3, 4, 5, 6 [pointing to the addition sequence]. But 26, 27, 28, 29, 30 is the other way from 24, 23, 22, 21, 20, 19.
>
> Clarissa: I noticed that's because it's going down, and this is going up… because in order to minus, you usually have to go up because if you did like $25 - 0 = 25$, $24 - 1$ would be 23. That would be if you did the same thing as this [pointing to the addition sequence].
>
> Teacher: And what's the "this" you're talking about?
>
> Clarissa: $25+0$, $24+1$, $23+2$. Because that would be adding one on, but you're subtracting one off.
>
> Todd: Since this one is going down [pointing to the first addend in the addition series of equations] this one [pointing to the second addend in the addition series of equations] has to go up, too. This column is going down, so this one has to go up.

Teacher: What's the idea about addition and subtraction that's being revealed here?

Jonah: I think the reason that both of these columns are going up in value is because if you want to get the same thing if you have a higher number to minus, you need a higher number to minus it to 25. But if you have a lower number to start with, you don't need as many numbers to get to 25.

Many kids: Ohhhhh! I get it!

Teacher: So can you use Todd's example to talk us through your idea? Todd's talking about always wanting to make sure he has 25 cards. Can you use that?

Jonah: If he starts with more, his brother has to take more to get to 25, because there's more cards to take.

Frannie: It sounds simple, but it really isn't.

Manuel: It's just like… he has a bigger number here. So he has to take away more in order to get to the number he wants to get.

Jamie: That's what I was going to say.

Sierra: Yeah… We knew that, and we thought everyone knew that, but now we just sort of figured it out.

Helen: Knew what?

Teacher: What is the idea, Sierra?

Sierra: The idea is that you need more to take away and get the same amount. If I had 26 and I minus 1, if you want the… That would be the same as if you wanted to have the same answer, … [you] could start with 27 and take away 2.

Addison: The reason why they're both going up is… Since it's higher, then you have to subtract more to get to that, but if it's less, you don't need as much to get to that number. It's less numbers to get to it.

In this episode, Ms. Kaye urges students to clarify what they mean by "it" and "this" as they are articulating their claim and explaining why they think the pattern for subtraction differs from the pattern for addition. As they build on each other's thinking to articulate *why* adding the same amount to both numbers in a subtraction expression results in an equivalent expression, they are simultaneously articulating more clearly *what* their generalization about subtraction is.

Developing a Mathematical Argument to Justify a General Claim

Articulation, representation, and justification of general claims do not occur for young students in a predictable sequence; rather, they develop together in the course of students' work. Representing particular instances of a regularity students have noticed leads to a clearer description of the claim as well as images of the mathematical relationships and structure that inform justification.

Episode D: Equivalent Addition Expressions (Grade 5)

In the following episode, students work with a representation at the same time that they are sorting out and stating a general claim. The teacher, Meg Lawson, has asked the students to write expressions equivalent to 32 using 2 addends. Not surprisingly, for fifth graders, they come up with many, for example:

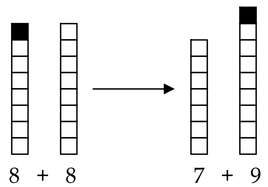
$16 + 16$
$30 + 2$
$28 + 4$
$10 + 22$
$15 + 17$

Ms. Lawson then writes on the overhead: $16 + 16 = 15 + 17$

> Teacher: I know you can calculate each side of this equation to find that each side equals 32. But if you didn't add each side, how would you know for sure that 16 + 16 equals 15 + 17? Think about explaining this to someone who couldn't add up these sums. Show with words and pictures how you know that 16 + 16 = 15 + 17.

In one small group, Fred, Carlson, and Laura work together.

> Laura [very excited]: The total just doesn't change. One number is just passing one over to the other number. See, this 16 gave 1 away and became 15 and the other one took it and is 17.
>
> Carlson: I have no idea what you just said.
>
> Teacher: Laura, instead of saying that again, can you show what you mean? Is there anything you can use or draw that would show what you understand?
>
> Laura puts together 2 sticks of 8 connecting cubes and demonstrates taking one cube from one stick and putting it onto the other.
>
> Laura: Look. This is 8 plus 8 which is 16. I can take one cube off of this stick and put it on the other stick and the total is the same.



Carlson and Fred are very quiet so the teacher asks, "Does this help us with 16 + 16 = 15 + 17?"

> Fred: Now it shows that 8 plus 8 is the same amount as 7 plus 9. But it works the same way. One number is smaller and the other number gets bigger.
>
> Laura: And by the same amount! Look—I could move 2 over to the other side and it would still work!

Teacher: Carlson, what do you think about this? Can you make any sense out of what Laura and Fred are saying?

Carlson: I think it's like the stuff is moving back and forth but the whole amount is staying the same. So you can take some away from an amount and the same plussed to another amount...

Carlson trails off, getting tangled in the words, and Ms. Lawson leaves them while they are working on the wording for their paper. They are very excited and almost laughing as they stumble over how complicated the words are. When Ms. Lawson returns a couple of minutes later, Laura has the cubes out again and is explaining to Carlson:

Laura: Look, it doesn't only work for 1 number of change. I can take any amount away as long as I add it to the other number so the total cubes don't change.

In her effort to convince Carlson of the generalization she has recognized, Laura has created a representation that proves that, for any two (whole number) addends, she can remove some amount from one addend and add it to the other without changing the total.

The task that the teacher gave her class has several characteristics that we have identified consistently, across grades, as helpful in engaging students in articulating and justifying general claims about the operations:

1. The task involves numbers and operations easily accessible to the students.
2. Students are asked to develop explanations about equivalence that do not rely on computing. (Even if students initially compute to convince themselves, they then move on to a different way of thinking about justification.)
3. Students are asked to use a representation of the operation as the basis for a general argument.

With these constraints and requirements, the students in this example began to shift from talking about specific numbers to talking in general terms. The first indication of this is how Laura used arbitrary numbers, 8 and 8 changed to 7 and 9, to represent 16 and 16 changed to 15 and 17. It is as if the particular numbers do not matter to her. At first her choice was confusing to Fred and Carlson, but when the teacher asked them, "does this help us with $16 + 16 = 15 + 17$?", Carlson talked in very general terms: "*the stuff* is moving back and forth but *the whole amount* is staying the same. So you can take *some* away from *an amount* and *the same* plussed to *another amount...*" By the end of small group time, Laura was able to articulate the claim clearly in general terms.

Reflecting on this episode, Ms. Lawson wrote: "I wasn't sure if this question was going to be interesting to the 5th graders. I wondered if the idea would be so obvious that they wouldn't be able to engage. But most kids seemed very excited to show me, and each other, that they could see and understand what was happening. They looked like they felt very important as each group had a chance to share their findings." She noted that all of the small groups moved from explaining why $16 + 16 = 15 + 17$ to a more general argument for any addends, and two groups also realized that the amount being added/subtracted could be any amount.

Representation-Based Proof: Tools for Proving in the Elementary Grades

Despite some decades of emphasis on reasoning in national documents, many students expect mathematics to be about finding answers. They don't know what it means to state a general claim or, if they do, they don't know what it means to argue that the claim is true. It is not surprising that younger students might think that a few examples are sufficient to show that a general claim is true. For example, a second grader might argue that she has tried changing the order of addends lots of times, and the sum always stays the same, "so I think it's true for any numbers." Many students, as they progress through the grades, continue to believe that trying many examples is sufficient to prove a generalization. They never develop an understanding of what it might mean to state something general about how a class of numbers behaves under a particular operation or to justify such a claim in mathematics. The use of examples by both students and adults as sufficient proof is well-documented; even at the college level, many students are satisfied to accept a general claim on the evidence of a few examples (Harel and Sowder 1998, 2007; Kieran et al. 2002; Martin and Harel 1989; Recio and Godino 2001).

For example, in a 5^{th} grade class, students have noticed, through many examples, that if you double one factor in a multiplication expression and halve another factor, the product remains the same. They have come to accept this idea and routinely apply it as they solve multiplication problems. However, when their teacher asks them *why* doubling one factor and halving another results in an equivalent multiplication expression, their responses are largely assertions:

> Adele: It is the same product because they are equivalent. If you double one factor and halve the other it will result in the same product because it will stay the same product and not a wrong product.
>
> Therèse: When you double one number and you halve the other the result is the same product because they are equivalents, or the other way to say it is that they are in the same family.
>
> Gloria: When you double a factor and leave the other alone, the product becomes doubled. If you double one factor and halve the other factor the product stays the same but if you double one factor and not halve the other it will be wrong and if you halve both numbers it will be wrong.
>
> Kamala: There are no limitations to doubling and halving because you can halve any number to get a whole number or a mixed number and you can double any kind of number. For example I did $2 \times 12 = 4 \times 6$, and $7 \times 5 = 14 \times 2.5$. I think doubling and halving works with all numbers.

The students are convinced that halving and doubling will always work to maintain the same product, but, inexperienced with the kind of question the teacher is asking, their responses do not move in the direction of justification. Adele and Therèse assert, correctly, that if the expressions are equivalent, the products must be the same, but they do not show or describe why the new expression *must* be equivalent to the original expression. Gloria is correct that doubling only one factor results in doubling the product. If she could show why this is true, that could lead her towards an argument for doubling and halving. Kamala seems convinced that their general claim can work with both whole numbers and rational numbers, but she offers only examples to justify her assertions.

These students are typical of students just beginning to justify general claims; they have no experience with constructing mathematical arguments, but rely on examples or assertions. Within their statements there are some glimpses of ideas about multiplication that, if taken further, could lead to more complete mathematical arguments. How can they take the next step towards developing a justification for their claim?

Students in these grades do not have available to them the tools of formal proof. What they do have available to them are representations of the operations—drawings, models, or story contexts that can be used to represent specific numerical expressions, but can also be extended to model and justify general claims. In order to use representations to make mathematical arguments, students must develop strong images of the operations, images that embody their properties.

Elsewhere we have described and defined *representation-based proof* as the means for elementary and middle grade students to justify general claims (Schifter 2009) by reasoning from visual representations. As students gain experience in articulating, representing, and justifying generalizations in the context of number and operations, they learn to develop pictures, models, diagrams, or story contexts that represent the meaning of the operations, can accommodate a class of instances (for example, all whole numbers), and demonstrate, in the structure of the representation, how the conclusion of the claim follows from the premise.

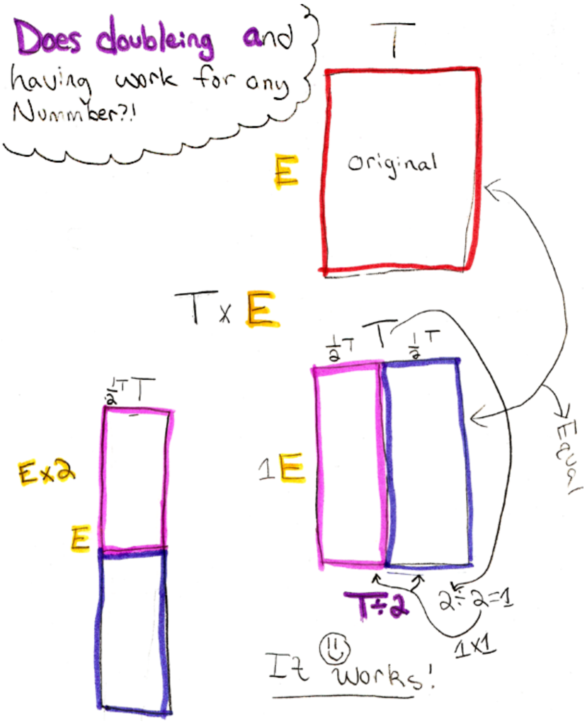
For example, in the first episode in this paper, second graders developed an argument for the commutative property of addition, based on a model of two cube towers. The teacher deliberately introduced that model into the discussion. The third graders investigating equivalent subtraction expressions used a story context about baseball cards to explain *why* one must increase both terms of a subtraction problem by the same amount to keep the same difference. Students typically begin by representing a particular instance of a general claim, then expand it to other instances, and, finally, modify the representation itself and the language they use to describe it so that it represents an infinite class of numbers. Fred, Carlson, and Laura use such general language in describing their cube towers. In the following episode, students who have been working on making and justifying general claims throughout the school year develop a representation-based proof.

Episode E: Equivalent Multiplication Expressions (Grade 5)

In the fifth grade described above, students had noticed the doubling/halving rule for multiplication, but were at the very beginning of work on justifying general claims. In another fifth grade class, students made the same claim—that if one factor of a multiplication expression is doubled and the other is halved, the product remains the same. After investigating and representing individual instances of this claim, the teacher presented the challenge to prove it:

> Teacher: Can you come up with a representation that shows this will always be true, no matter what numbers you start with? Make a picture, draw a model, but don't use any particular numbers.

In response, Trisha and Emily created the following poster.



In their rectangle marked "original," they represent the multiplication, $T \times E$, as the area of a rectangle with sides of lengths T and E. In the second picture, they have cut the rectangle in half and show ½T as a side equal in length to half of T. The same area ($T \times E$) is equal to two rectangles, each with area (½$T \times E$). By moving one of the smaller rectangles below the other, as shown in the third picture, they now have a rectangle with sides ½T and $2E$. Since its area (½$T \times 2E$) is equal to the area of the original rectangle, they have shown that ½$T \times 2E = T \times E$.

In later years, students might prove the same claim by invoking the commutative and associative laws of multiplication together with the multiplicative inverse and multiplicative identity. At this stage, they use as proof what they understand about multiplication as represented by the area of a rectangle and conservation of area.

The development of representations for the operations is critical to connecting arithmetic and algebra. Even students in upper elementary and middle grades who are fluent with computational procedures may not have developed images of the operations they can use when they encounter new contexts, for example, making the transition from using only numerical expressions to using symbolic notation in algebraic expressions. The use of pictures, diagrams, and story contexts to justify general claims appears to be accessible, powerful, and generative for elementary students.

Extending the Number System

In the second grade discussion of the commutative property of addition (Episode A), the focus was on how addition and subtraction behave differently; one is commutative, the other is not. But the discussion also allowed ideas about a different kind of number to be voiced. The second grade teacher made a decision not to pursue the idea of negative numbers at that time. But as students get older, discussions about generalizations provide openings for consideration of new kinds of numbers. Does a property they have justified for whole numbers, and perhaps now take for granted, still hold when expanding the number system to include fractions, decimals, or negative numbers?

As they consider new classes of numbers, students sort out which behaviors of the operations must remain consistent as the number system expands and which only appear to be general when considering certain classes of numbers. For example, consider these two statements:

- When you subtract any amount except 0, you end up with less than your original amount. (For any number $b \neq 0$, $a - b < a$.)
- If you add two numbers to get a third number, then you can subtract either addend from the sum to get the other addend. (If $a + b = c$, then $c - a = b$ and $c - b = a$.)

Students are likely to encounter both of these ideas when their view of numbers is limited to positive numbers. As their number system expands to include new classes of numbers, they need opportunities to examine which of the statements are still true. Students will find that the first statement is not true when b is a negative number, but the second statement is true for all numbers on the number line. The next two episodes illustrate students expanding a general claim in this way.

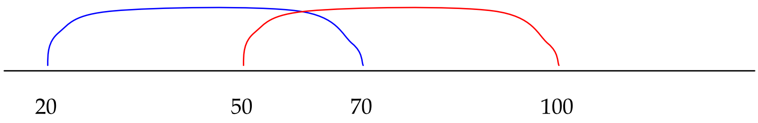
Episode F: Subtracting Negative Numbers (Grade 5)

The next episode illustrates how studies of the operations can support students' work on calculation and reasoning about new number domains.

These fifth graders were investigating equivalent subtraction expressions, like the third graders in Episode C. In this class, students began by generating expressions equal to 50 such as $70 - 20$ and $100 - 50$. As in many of these examples, the teacher, Marlena Diaz, chose numbers that were easy for the students because she wanted the focus of the discussion to be on the relationships of the numbers, not on computing results.

The students all knew that the difference in both expressions is 50. Implicit here is a generalization—add an amount to 70 and add the same amount to 20; the difference remains unchanged. But how do you know that the difference will always remain constant when the same amount is added to each number? This is the question Ms. Diaz posed to her class.

One fifth grader, Alex, came to the board and drew a number line on which he showed the distance between 100 and 50 and the distance between 70 and 20.



Then he explained, “You can see that the distance is the same. If you change one number, you change the other the same way. As long as both numbers change the same, you can make lots of new expressions.” He was visualizing sliding the interval, which remains rigid, along the number line so that the beginning and end points change by the same amount, but the difference between those two points does not change.

Alex offered a representation of subtraction to justify his claim. Unlike the second graders who thought of subtraction as a process of removal, Alex relied on a different model of subtraction—finding the distance between two numbers on a number line. Alex’s number line and explanation made sense to other students, and they realized they could generate many more expressions. Using Alex’s image, his classmates were thinking of sliding the interval of 50 down the number line and proposed additional equivalent subtraction expressions, which Ms. Diaz listed on the board:

100 – 50
90 – 40
80 – 30
70 – 20
60 – 10
50 – 0

As this list was being generated, two students commented, as follows:

Patricia: We could keep adding to our list by changing both the numbers, but we are going to get to a point where we won’t be able to change the numbers. That will happen when we get to 50.

Nicole: Yes, I agree with Trisha. Because if we look at Alex’s number line we are going to get to zero and 50, and the jump will be 50, but then we are done.

As the discussion continued, additional ideas were offered.

Raul: But we could use the other numbers.

Teacher: What other numbers?

Raul: The negative numbers on the other side of zero.

Alex: I have one we can use. Let’s use 40 and negative 10.

Teacher: How do you want me to write that on the chart?

Alex: Put 40, then the subtraction sign and then a negative 10.

Ms. Diaz wrote on the board, “$40 - -10 = 50$.”

In her written account of this class session, Ms. Diaz reported, “At this point many of the students were talking at once… Several were pointing to the large classroom number line that extends to -40.”

Josh: No way; you can’t do that. How can you have a negative 10 and end up with 50?

Alex: It is like adding 10, because if you look on the number line you would have to jump 50 to get from negative 10 to 40. It is the same as we did with 100 and 50 and 70 and 20.

Teacher: So, Alex, how do you know that 40 minus negative 10 will give you 50?

Alex: Because you have to add 50 to negative 10 to get 40.

In this classroom, the students were discussing a generalization about subtraction of whole numbers and used a number line to clarify and justify it. Alex could see that their reasoning about whole numbers could extend to negative numbers. Furthermore, his reasoning brought him to articulate what subtraction of a negative number must mean. He applied what he understood about the relationship between addition and subtraction, as well as the image of “distance between” on the number line, to argue that $40 - (-10)$ must equal 50. For example, Alex reasoned that if $-10 + 50 = 40$, then $40 - (-10)$ must equal 50. As with whole numbers, if $a + b = c$, then $c - a = b$.

On the other hand, students must reconsider some of the generalizations they may have made about the behavior of subtraction in the context of whole numbers. For example, Josh says “No way; you can’t do that. How can you have a negative 10 and end up with 50?” It is likely that Josh and other students hold an implicit belief, based on their experience with positive numbers, that the result of subtraction (the difference) is always less than the initial amount (the minuend). Josh may have been asking, “How can you subtract something from 40 and end up with a number larger than 40?” The students will need to reconcile these questions with the behavior of the operations in this new domain.

The ideas brought up in this discussion generated a great deal of interest and provided the class with the opportunity to think about subtraction of negative numbers and about the consistencies that should be maintained in the behavior of operations. Again, as for the students in Ms. Lawson’s class (Episode D) who were considering $16 + 16 = 15 + 17$, a high level of enthusiasm for this kind of challenge was evidenced.

Episode G: Multiplication with Decimals (Grade 6)

A generalization can help students tie together ideas that at first seem unrelated and thereby strengthen their understanding of the foundations of arithmetic. In Jeanette McCorkle’s sixth grade class, students had formulated the same doubling and halving rule that students were working on in Episode E. They had expanded that claim to include multiplying and dividing by any factor, not only 2, and had expressed their claim in symbolic notation as $A \times B = (A \times C) \times (B/C)$. Through the fall, they had encountered this idea a number of times, but always in the context of whole numbers. Like many of the teachers whose work is included in this chapter, the teacher of this class, Ms. McCorkle, frequently asked students to analyze expressions and equations without doing any computation. On one day in November, she posed a list of problems that focused on place value with decimals:

$$25 \times 1 = 2.5 \times 10$$

$$25 \times 10 = 2.5 \times \_\_\_\_$$

$$25 \times 100 = .25 \times \_\_\_\_$$

$$25 \times .1 = 2.5 \times \_\_\_\_$$

$$25 \times .01 = .25 \times \_\_\_\_$$

> Teacher: So look at this for a minute [the first equation above] and when you have decided if that is a true equation, without calculating, when you have a strategy for determining whether that is true, raise your hand and let me know.
>
> Fran: I'm not sure if this is right at all, but if 2.5 is timesed by 10, it means moving the decimal over one, and that is the same thing as 25 times 1.
>
> Britta: Well, 2.5 is ten times smaller than 25, and 10 is ten times bigger than 1.
>
> Charles: 2.5 times 10, if you multiply it by 10, you move it one to the right, so you're looking at 25 and 25.
>
> Mariah: I would think of 2.5 times 10 as two 10s and a half a 10, which is 25, so you have 25 and 25.
>
> Britta: This is like the problems we did before but A is divided by ten and B is multiplied by ten.

Britta's statement surprised Ms. McCorkle. She had not thought in advance about how this work would connect to the generalization they had articulated in earlier lessons. She had designed the lesson to address difficulties her students had exhibited with multiplication of decimals. As the class solved the rest of the problems, some students began by using mechanical methods, counting decimal places. For example, for the last problem in the group, one student explained:

> I'm sort of like, 25 times .01 equals .25 times, it has to be 1, because to get .25 you have to move the decimal over two, so then to get to 1 you have to move it two the other way.

As Ms. McCorkle interacted and questioned students, she urged them to consider what moving the decimal point means in terms of multiplication and division. After some time, she pulled the whole class together to talk about this issue with the goal of returning to Britta's earlier observation:

Teacher: You're looking at a number of decimal places relationship, and I want to expand that and talk about how one factor has been multiplied and another factor has been divided. The number of decimal places is just one way of talking about how factors have been altered.

Fran: It's not just about the decimal point, it's about multiplying and dividing the numbers.

Teacher: Exactly. I want to remind you about the pattern we were looking at last week and the week before, when Britta suggested that $A \times B = 2A \times ½B$, and George suggested that $A \times B = AC \times B/C$. Britta, how was 25 times 1 changed into 2.5 times 10?

Britta: The first factor was divided by 10 and the second factor was multiplied by 10.

Teacher: That's right. The decimal moved back means divided by 10, so to maintain this equality, what should happen to this factor?

Most students: Multiply by 10.

Britta: And it's still $A \times B = AC \times B/C$.

Ms. McCorkle wrote: "By following my students' thinking, I saw how some of them connected this page of problems directly to our previous work on the doubling and halving rule, which I did not expect."

Considering different kinds of numbers—fractions, decimals, negative numbers—is an opportunity for students to revisit the generalizations they have worked on with whole numbers. Through reconsidering these general claims, they identify the consistencies in the behavior of the operations as the number system to which they are applying those operations expands. Instead of operating with a new class of numbers as if they require a new set of rules (e.g., rules about counting or moving decimal points), they can extend and apply the foundational properties they have already encountered in operations with whole numbers.

Using Notation with Meaning

In the previous episode, as well as in Trisha and Emily's proof (episode E), students expressed general claims in symbolic notation. In the student curriculum we have developed, we introduce some use of algebraic notation in the elementary grades. However, we have been careful not to move too quickly. In order to support students' use of algebraic notation with meaning, they first need to spend a good deal of time articulating general claims clearly in words and then connecting those statements to arguments based on representations. The use of phrases that refer to a class of numbers, such as those used by Carlson in Episode D ("you can take some away from an amount and the same plussed to another amount") or second grader Kamika as she justifies the commutative property of addition in Episode A ("You're not adding any more and you're not taking away any numbers. You're just changing them around") are an important link to meaningful use of symbolic notation.

Using representations and story contexts to model general claims helps students develop meaning for the symbols of arithmetic. In particular, students' study of equivalent expressions, such as $16 + 16 = 15 + 17$ (Episode D) or $26 - 1 = 27 - 2$

(Episode C) provides the opportunity for meaningful use of the equal sign, signifying equivalence of expressions, rather than "now write down the answer" (Behr et al. 1980; Carpenter et al. 2003; Kieran 1981). Once students have considerable experience stating generalizations in words and justifying these general claims by using representations of the operations, they have images and explanations to which they can connect algebraic symbols. In our final episode, we see a group of students making this connection.

Episode H: Using Algebraic Notation to Represent Equivalent Addition Expressions (Grade 5)

This class of 5th graders has been investigating equivalent addition expressions as they looked at this sequence:

$$30 + 2$$
$$29 + 3$$
$$28 + 4$$
$$27 + 5$$
$$\vdots$$

Students considered a general claim based on this sequence—that if 1 is subtracted from one of the addends and added to the other addend, the sum is maintained—and developed some arguments to justify the claim. The teacher, Alina Martinez, introduced a cube representation (similar to what Laura uses in Episode D) to model addition as joining two quantities. Students talked about how they could move one cube from one quantity to the other quantity, maintaining the same sum because "you aren't adding any or taking any away … and since all the numbers are made up of ones, we can just move all those ones around." At this point in the discussion, Ms. Martinez judged that the students' ideas and images were quite clear and that symbolic notation would provide another representation with which they could continue to think about this idea.

> Teacher: You all are thinking about lots of numbers and trying to make sense of what is happening. It seems that you all are thinking that this is true about all numbers and you are trying to make convincing arguments. I wonder if we could write a sentence that wouldn't use numbers to show what is happening. Could we call these numbers up here on the chart just some numbers?
>
> Will: We could write letters for them. Like *n* for number, like *n one* and *n two*.
>
> Teacher: That's a great idea. One thing that mathematicians do sometimes is use different letters so they don't get confused. How about if we use *a* and *b*?
>
> Jonah: We could write *a* plus *b* equals a number.

Ms. Martinez then asked students to look at the cube representation of joining two quantities.

> Teacher: So let's use Jonah's idea and try to write down what we did to the two quantities. What are we doing to the *a* and the *b* in this pattern?

Ms. Martinez recorded *a* above the first addend and *b* above the second addend in the list of expressions:

$$\begin{array}{c} a \quad b \\ 30 + 2 \\ 29 + 3 \\ 28 + 4 \\ 27 + 5 \\ \cdot \\ \cdot \\ \cdot \end{array}$$

> Kathryn: We can write *a* plus *b* is the same as *a* take one away and *b* add one to it.
>
> Teacher [recording $a + b = (a - 1) + (b + 1)$]: How does this match what Kathryn said and what we did with the cubes?
>
> Reynold: We take one away like here in the pattern ... one goes up and one goes down.
>
> Amelia: Oh, look the minus one and the plus one is like a zero! That is why we don't change it. It is like staying on zero on the number line.

At this point, many of the children began talking in their groups excitedly. Ms. Martinez asked the small working groups to consider what would happen if more than one cube was moved from one quantity to the other. Several students then shared that they could move two cubes, three cubes, or lots of cubes and still maintain the sum. Ms. Martinez then asked the class if they thought the notation could be revised to accommodate this idea.

> Adena: We could write lots of them and change the numbers. [Adena is suggesting they write a series of number sentences, $a + b = (a - 1) + (b + 1)$; $a + b = (a - 2) + (b + 2)$, and so forth.]
>
> Will: Or we could write add any and take any away.
>
> Jonah: We could use another letter.
>
> Teacher: What do you all think?
>
> Adena: Put a *c*. Put *a* plus *b* equals *c*.
>
> Jonah: But put the *c* where the 1 is.
>
> Teacher [recording: $a + b = (a - c) + (b + c)$]: Do you mean like this, put the *c* where the 1 is? What does this mean now?
>
> Reynold: See, the *c* is the cubes you move around to the other side.

In this example, students move among their words, a representation, and the symbols, so that the words and representation are a referent for their thinking about notation: Kathryn's words, "*a* take one away and *b* add one to it," becomes $(a - 1) + (b + 1)$; similarly, "add any" becomes $b + c$ and "take any away" becomes $(a - c)$. The teacher asks students to consider how this notation matches their sequence of expressions and the cube model, and students are able to articulate these connections, for example, "the *c* is the cubes you move around to the other side."

Introducing this notation at a point at which students have already articulated their ideas in words and images allows them to maintain meaning for the symbols. But something else happens as well. Any representation can provide a different view, a new insight into the mathematical relationships that are represented. The symbolic representation in this case may make the $+1$ and -1 even more prominent. Even though students had noticed that "one goes up and one goes down" as they considered the sequence of expressions, Amelia now sees something new about this relationship: "the minus one and the plus one is like a zero! That is why we don't change it." In fact, she has come up with a new argument that involves the fact that the result of adding 1 and subtracting 1 is 0. Thus, the introduction of symbols in this case not only provides a concise expression of students' ideas but offers new ways of seeing the mathematical relationships.

We don't want to underestimate the complex issues students encounter as they begin to work with symbols. The error in which students simply substitute letters for words in an English sentence is well known, as in writing 6S = P to represent "there are 6 times as many students as professors" (Clement et al. 1981; Kaput and Sims-Knight 1983). This incorrect notation stems from using a letter as if it is an abbreviation for a word, standing for the thing itself rather than the quantity of that thing, and also perhaps from misinterpretation of the equal sign.

Students need time and experience to develop an understanding of the conventions for using algebraic notation and how the use of letters to represent variables differs from the use of multidigit numbers. Later in the lesson on multiplying decimals (Episode G) a student tries to rewrite the notation they have been using, $A \times B = AC \times B/C$, to accommodate decimals as: $A.B \times C = AB \times .C$. Grounded in the experience of multidigit numbers and the emphasis in these grades on decimal computation, it is not surprising that students might think there is a need to have a letter for each digit, and that the decimal point must be explicitly shown.

Furthermore, when negative numbers or fractions are introduced, students don't automatically realize they can use the same letters to represent them. In the equation, $a + b = c$, if a represents a negative number, many students think it now must be written as $-a$. Because there is no negative sign, a somehow *looks* positive. It takes experience to accept that a single symbol might represent a positive or negative value, a whole number or a fraction.

In all of these cases, students are making sensible choices, based on their experience with numbers. The transition to use of algebraic notation requires both connecting these new symbols to what they represent and also learning new conventions. For this reason, even though it may appear easy to make a transition to use of symbols in particular cases in the elementary grades, teachers' and students' experiences indicate that it makes sense to proceed cautiously with early introduction of algebraic notation.

Connecting Arithmetic and Algebra

The four aspects of early algebra discussed in this chapter have the potential to provide students with a strong foundation in whole number computation, which they can extend to their study of fractions, decimals, negative numbers, and algebraic symbols. Some might ask: How can work in algebra fit into an already crowded curriculum? We would argue that early algebra, defined in this way, not only provides crucial links between arithmetic and algebra, but also is an essential part of good arithmetic instruction.

As seen in the classroom episodes, investigation into these aspects of arithmetic—understanding the behavior of the operations, generalizing and justifying, extending the number system, and using notation with meaning—provides a means for students to re-examine and strengthen foundational understandings about the meaning of the operations and ways of thinking in mathematics.

Further, we are intrigued by the level of student engagement with investigation of general claims that teachers are seeing in their classrooms. Although it might be thought that this kind of reasoning is accessible only to "top" students, several of these examples come from schools in which there is a history of poor performance on standardized tests. We are accumulating documentation of how both students who have been relatively successful and relatively unsuccessful in grade-level computation as measured by school and district assessments are engaged by such investigations (Russell and Vaisenstein 2008; Schifter et al. 2009). Our hypothesis is that mathematical activities that connect arithmetic and algebra have the potential both to strengthen the foundations of computation for all students, perhaps especially for those who have relied on poorly understood procedures, and to intrigue many students, including those who excel in mathematics, with challenging questions about mathematical relationships.

Finally, many questions remain about what teachers need to know and understand in order to carry out this kind of instruction that links arithmetic to algebra. The isolation in which teachers often work, and the concomitant lack of communication between elementary and middle grades teachers, is one barrier to the kind of continuity that might be built in mathematics instruction from arithmetic to algebra. Elementary teachers need a better grasp of how their curriculum can embody ideas that are foundational to algebra and how these ideas might be made more explicit objects of study. Similarly, middle grades teachers need to know more about how to build on the work of the elementary grades and how to assess the ways students' conceptions of arithmetic may inform or undermine their understanding of algebra.

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