Overall Commentary on Early Algebraization: Perspectives for Research and Teaching

Carolyn Kieran

Arithmetic itself must be viewed with 'algebra eyes'
(Subramaniam & Banerjee, this volume)

The twenty-nine chapters of this volume on early algebraization, which include an introduction and commentary for each of the three main parts, reveal the rich diversity that characterizes the rapidly evolving field of early algebra. Cai and Knuth, in their introductory chapter, point out that the development of students' algebraic thinking in the earlier grades is not a new idea, but has been part of school practice in several countries around the world since the 1950s. Nevertheless, it was not until the mid-1990s that the idea took hold more broadly and that publications began to reflect the interest that researchers were investing in this area. Each new collection of writings since then has made advances on its predecessors as researchers continue in their efforts to unpack the central notions of school algebra and reflect on how they might be made accessible to the younger student at the elementary and middle school levels. This latest collection is no exception. With its three parts that articulate the ways in which researchers are currently conceptualizing early algebraization from curricular, cognitive, and instructional perspectives, this volume offers to researchers, teachers, curriculum developers, professional development educators, and policy makers alike some of the most recent thinking in the field.

The research that is presented within sheds light on how the term *algebraization* is being considered: *algebraization* concerns the nature of the thinking that is basic to algebra, along with the conceptual areas within early and middle school mathematics that can be exploited pedagogically in this early algebraic terrain, as well as the ways in which teachers can help students develop such thinking. The overall commentary that I have been invited to write attempts to synthesize the ways in which the researchers whose work is described in the chapters of this volume have

C. Kieran (✉)
Département de Mathématiques, Université du Québec à Montréal, Montreal, Canada
e-mail: kieran.carolyn@uqam.ca

J. Cai, E. Knuth (eds.), *Early Algebraization*, Advances in Mathematics Education,
DOI 10.1007/978-3-642-17735-4_29,

been shaping this triple aspect of *algebraization*. Their efforts will have an impact not only on the way in which children come to think about their mathematics at the elementary and middle school levels, but also on the way in which high school students come to engage with algebra.

Shaping the Notion of Algebraic Thinking within Early Algebra

The citation with which I chose to open this commentary chapter, one that is drawn from the Subramaniam and Banerjee chapter, states that arithmetic needs to be viewed with 'algebra eyes.' Elsewhere, Blanton and Kaput (2008) have referred to this phenomenon as *algebrafying* and have described it as transforming and extending the mathematics normally taught in elementary school toward algebraic thinking, with its intrinsic feature of generality, and including within this transformation "the establishing of classroom norms of participation so that argumentation, conjecture, and justification are routine acts of discourse" (p. 362). Taken together, these two references suggest that the developing of 'algebra eyes' involves seeing the general within arithmetic and that the more global mathematical reasoning processes of argumentation, conjecturing, and justification are routes toward this goal. However, as will be seen from the chapters within this volume, it involves much more than this.

More than a decade ago, Kieran (1996) offered the perspective that algebraic activity in school consists of three components: the generational; the transformational; and the global meta-level, which includes analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, justifying, proving, and predicting. Although these three types of activities were framed against the dual backdrop of both equation-based and function-based approaches, the ways in which they might be adapted for an early algebra context were left largely unarticulated.

Kaput (2008) has proposed a slightly different perspective on algebra. In his opening chapter of the anthology, *Algebra in the Early Grades*, he specified the two core aspects of algebraic reasoning to be (i) generalization and the expression of generalization in increasingly systematic, conventional symbol systems, and (ii) syntactically guided action on symbols within organized systems of symbols. Each of these core aspects is deemed, according to Kaput, to be found in varying degrees throughout the following three strands of algebra: algebra as the study of structures arising in arithmetic and in quantitative reasoning, algebra as the study of functions, and algebra as the application of modeling languages.

While Kieran (2004) has argued that algebraic thinking in the earlier grades could be construed in terms of the global, meta-level activity of algebra and be engaged in without the use of the letter-symbolic, Kaput's main thrust has been on the overarching role of generalization and its gradual symbolization. In any case, Radford, one of the chapter authors of this volume on *Early Algebraization*, emphasizes that "algebraic thinking is not about using or not using notations but about reasoning in certain ways."

In keeping with Radford, the issue in coming to grips with *algebraic thinking* centers on what is meant by "reasoning in certain ways." As an aside, it is noted that scholars in the field of algebra education (be it at the high school level or earlier) have yet to distinguish *algebraic thinking* from *algebraic reasoning*. While the two terms are used interchangeably within this literature, classic approaches to the study of mathematical reasoning tend to focus, in general, on 'forms of reasoning,' be they deductive, inductive, abductive, or analogical (Jeannotte 2010). When viewed against the lens of classical-mathematical-reasoning terminology, the term *algebraic reasoning* risks being interpreted too narrowly to encompass adequately the various and diverse approaches to early algebra that are being considered within this volume. Thus, I have opted within this commentary to use whenever possible that which I consider to be the broader term, *algebraic thinking*. Taken as a whole, the chapters of this volume make significant strides in unpacking not only the nature and components of such thinking but also the manner in which it might be fostered by teachers of elementary and middle school students. Although my organizational structure and résumé of salient ideas from the chapters—the product of a diagonal cut through the volume—do not preserve the rich detail that constitutes the central contributions of the authors, I nevertheless attempt to point out within each of the sections below those chapter aspects that I consider inject something new and important into the development of the field of algebra education. The research that is presented in this volume, research that is shaping both our ways of thinking about the nature and components of algebraic thinking and the routes by which its growth might be encouraged, includes the following focal themes:

- Thinking about the general in the particular
- Thinking rule-wise about patterns
- Thinking relationally about quantity, number, and numerical operations
- Thinking representationally about the relations in problem situations
- Thinking conceptually about the procedural
- Anticipating, conjecturing, and justifying
- Gesturing, visualizing, and languaging.

Thinking about the General in the Particular

One of the pioneers of a generalization approach to the teaching and learning of algebra, John Mason, has described algebraic thinking as follows:

> Algebraic thinking is rooted in and emerges from learners' natural powers to make sense mathematically. At the very heart of algebra is the expression of generality. Exploiting algebraic thinking within arithmetic, through explicit expression of generality makes use of learners' powers to develop their algebraic thinking and hence to appreciate arithmetic more thoroughly. (Mason 2005, p. 310)

Nearly a dozen chapters in this volume express ideas that resonate with Mason's, that is, that the expression of generality is the core of algebraic thinking. Moreover, their focus is on both the process of generalizing that contributes to the production of such expressions of generality as well as the generalized product. Thus, generalizing is considered as both a route to, and a characteristic of, algebraic thinking.

For example, Rivera and Rossi Becker in their chapter draw our attention to their finding that "individuals tend to see and process the same pattern differently . . . and produce different generalizations for [that pattern]," while Britt and Irwin note that "successful application of operational strategies demands an awareness of the generality of the operational strategy." Russell, Schifter, and Bastable speak of "generalizing and justifying"; Koellner, Jacobs, Borko, Roberts, and Schneider, of "describing and generalizing patterns"; and Cai, Moyer, Wang, and Nie, of "the development of students' algebraic thinking related to . . . making generalizations." Both the process and product aspects of generalizing are explicitly found in Blanton and Kaput who, in their chapter within this volume, discuss "algebraic reasoning as an activity of generalizing mathematical ideas" and propose using these generalized ideas as "objects of mathematical reasoning." Similarly, Cooper and Warren argue for both grasping and expressing generalities. In addition, Radford discusses "dealing with generality through particular examples, in a manner that Balacheff (1987) calls 'generic example,' a way of seeing the general through the particular, as Mason (1996) puts it."

Radford, however, nuances the oft-found practice among many algebra-education researchers to identify nearly all generalization activity within this area as algebraic. His nuanced position is presented immediately below, within the focal theme of 'thinking rule-wise about patterns.'

Thinking Rule-Wise about Patterns

In his chapter that describes second graders' activity with pattern generalization, Radford argues that the process of grasping a commonality in a sequence and extending it to a few subsequent items does not mean that students are thinking algebraically. He points out that chimpanzees and birds can form commonalities too. Rather, what

> characterizes thinking as algebraic is that it deals with indeterminate quantities conceived of in analytic ways . . . indeterminacy and analyticity are in fact bound together in a schema or *rule* that allows the students to deal with any particular figure of the sequence, regardless of its size . . . the students' rule attests to a shift in focus: the student's focus is no longer specifically numeric . . . for the student's emerging understanding, what matters is not the [numeric] result; it is the rule, that is to say, the formula—the algebraic formula. (Radford, this volume)

Put succinctly, it is the shift from the purely numeric to the devising of a rule or calculation method involving indeterminates that constitutes a [pattern] generalization that is algebraic in nature. The precise articulation that Radford brings to the discussion of what is algebraic, and what is not, within the context of pattern generalization in early algebra is one that is important for the field. He identifies not only a distinction between students' using the visual and the numeric in action and their movement toward a more general kind of thinking that is neither visualized nor experienced directly, but also a distinction between this more general form of thought within patterning activity and algebraic thinking itself.

Additional contributions from other chapters in this volume that bear on pattern generalization include the research by Rivera and Rossi Becker who describe middle schoolers' activity with more complex patterns, by Moss and McNab who discuss second graders' reasoning about linear function and co-variation through the integration of geometric and numeric representations of growing patterns, by Watanabe who provides details related to the functional underpinnings of patterning within the Japanese curriculum, and by Cai, Ng, and Moyer who do likewise with respect to the Singaporean curriculum.

Thinking Relationally about Quantity, Number, and Numerical Operations

Empson, Levi, and Carpenter point out that relational thinking is almost entirely neglected in typical U.S. elementary school classrooms. This reason alone would make all of the ten or so chapters dealing with this approach to the development of algebraic thinking required reading, for they offer a glimpse into what is possible within an early algebra context. However, these chapters offer even more, with their varying theoretical and cultural frameworks and rich descriptions of student and teacher work in this area.

According to Empson et al., *relational thinking* "involves children's use of fundamental properties of operations and equality to analyze a problem in the context of a goal structure and then to simplify progress towards this goal"; such thinking is also said to include *anticipating* those relations and actions that move one effectively toward the final goal of a given situation. These authors pit relational thinking against algorithmic thinking about operations where the goal structure can be summarized as 'do next'. An example of relational thinking that they provide involves a student who has to calculate $1/2 + 3/4$. This student unpacks $3/4$ as $1/2 + 1/4$ in anticipatory fashion and reasons that $1/2$ plus another $1/2$ is equal to 1, then plus another $1/4$ is $1\frac{1}{4}$. For Empson et al., to understand arithmetic is to think relationally about arithmetic, and thinking relationally about arithmetic involves the kind of property-based thinking that is used in algebra.

Several other chapters of this volume contribute equally important perspectives on relational thinking, especially with respect to the conceptual arena of 'unpacking number.' For example, Russell, Schifter, and Bastable describe how students benefit from "explicit study of the operations by examining *calculation procedures as mathematical objects* that can be described generally in terms of their properties and behaviors"; Subramaniam and Banerjee argue that "understanding and learning to 'see' the *operational composition* encoded by numerical expressions is important for algebraic insight"; and Cusi, Malara, and Navarra attend to both canonical and *non-canonical forms of numbers* in their work with teachers of early algebra. Similarly, Britt and Irwin promote algebraic thinking in the form of generalizing relationships for operations with emphasis on *relational and compensating operations*, by means of student tasks such as: "Jason uses a simple method to work out problems like $27 + 15 \ldots$ in his head. Jason's calculation is $30 + 12 = 42$. Show how to use Jason's method to work out $298 + 57$."

Other aspects of numerical unpacking are presented in the chapter by Cai, Ng, and Moyer who describe the Singaporean focus on 'doing and undoing' within the relationships between addition and subtraction, and between multiplication and division. They also draw our attention to the Singaporean curricular emphasis on 'abstract strategies,' which are clearly relational in nature. In a similar vein, but with a focus that is as much on quantity as it is on number, Watanabe synthesizes the Japanese course of study in mathematics at the elementary school level with its quantitative relations strand and attention to the 'writing and interpreting of mathematical expressions.'

The notion that algebra is about insight into quantities and their relationships is also reflected in the chapter by Subramaniam and Banerjee, who maintain that algebra is not so much a generalization of arithmetic as it is a foundation for arithmetic and who affirm that "arithmetic itself must be viewed with 'algebra eyes'." Britt and Irwin, as well, argue that the origins of algebraic thinking precede understanding of arithmetic and thus these researchers focus on developing such thinking in students from their earliest years in school. The ultimate embodiment of this position is found in the chapter by Schmittau. She first reminds us of Vygotsky's assertion that "the student who has mastered algebra attains 'a new higher plane of thought,' a level of abstraction and generalization that transforms the meaning of the lower (arithmetic) level." According to Schmittau, Davydov did not want students to wait until the secondary level of schooling and so sought to introduce theoretical or algebraic thinking earlier in the school experience. Schmittau describes the way in which students thereby begin the study of algebraic structure, even before they learn about number, by means of a focus on the theoretical (quantitative) characteristics of real objects.

While the stance of Schmittau is quite exceptional within this volume, much of the research within the theme of relational thinking could be said to have its roots in activity involving quantities. For example, Ellis states: "Quantities are attributes of objects or phenomena that are measurable; it is our capacity to measure them—whether we have carried out the measurements or not—that makes them quantities." Ellis, whose research is situated within a functional approach, argues further that a focus on functional relationships between quantities, rather than on numbers disconnected from meaningful referents, can ground the study of algebra, and functions in particular, in students' experiential worlds.

The multiple ways in which the above chapters open up the 'relational thinking' perspective on early algebra contribute substantially toward counteracting the traditional view of arithmetic as being simply about number facts and algorithms for number operations. Students who come to see number and its operations in terms of their inherent structural relations, that is, as objects that can be compared relationally in terms of their components, and who can use the fundamental properties of operations and equality within the kinds of activities that are described in this volume, could be said to be seeing their arithmetic with 'algebra eyes'. In high school algebra, students are often called upon to look for relationships in symbolic expressions in terms of underlying structure, such as for example, seeing $x^6 - 1$ both as $((x^3)^2 - 1)$ and as $((x^2)^3 - 1)$, and so being able to factor it in two ways (either as a difference of squares or as a difference of cubes). Even if literal symbols are not considered a constituent part of algebraic thinking within early algebra, it is clear that the unpacking of quantity, number, and numerical operations and seeing such unpacked objects in terms of their underlying structure has its parallels in the seeing of relationships in literal expressions at the high school level.

Thinking Representationally about the Relations in Problem Situations

A strongly held belief in algebra education is the notion that problem-solving contexts are foundational to algebraic activity. This stance is based to a certain extent on historical grounds whereby algebra grew in status to become the privileged tool for expressing general methods for solving whole classes of problems. However, the difficulties that students experience in generating equations to represent the relationships found in word problems is well known (Kieran 2007). Thus, research that leads to alternate forms of representation that both embody that which equations represent as well as prove to be more accessible to students, in particular younger students, is of great interest. Although much of the early algebraic activity related to the already described relational-thinking frame involves, at least implicitly, story problem contexts as opposed to purely numeric contexts, the Singaporean pictorial equation (or model method, as it is sometimes called), presented in the chapter by Cai, Ng, and Moyer, and also referred to by Watanabe, offers an analytic method for dealing with indeterminates in the representing of relationships in a problem situation—one that is well suited to the younger student.

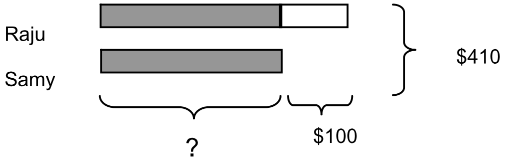
The Singaporean approach, as described by Cai, Ng, and Moyer, focuses on the use of pictorial equations so as "to analyze parts and wholes, generalize and specify, and do and undo." It is believed that, if children are provided with a means to visualize a problem, they will come to see the structural underpinnings of the problem. An example of the pictorial equation, which is drawn from the Cai, Ng, and Moyer chapter, is provided in Fig. 1.

The authors point out that, as students move to the higher grades of elementary school, the pictorial equations are used to solve algebra problems involving unknowns, emphasizing that the rectangles allow students to treat unknowns as if they were knowns. To solve for the unknown, students undo the operations that are implied by the pictorial equation. It is intended that pictorial equations provide a smooth transition to the more abstract forms of equations with their literal-symbolic notation that are encountered in the formal algebra of high school.

Another noteworthy approach to problem representation that is highlighted in the chapters by Cai, Ng, and Moyer and by Li, Peng, and Song involves the combining of various representations to encourage abstraction of central algebraic ideas. The

Fig. 1 Pictorial equation, drawn from the Cai, Ng, and Moyer chapter

Raju and Samy shared $410 between them. Raju received $100 more than Samy. How much money did Samy receive?
2 units = $410 - $100
= $310
1 unit = $155
Samy received $ 155



Chinese approach to developing algebraic thinking, which is described in both these chapters, provides students with opportunities to represent a quantitative relationship in a combination of different ways—an approach that the latter authors refer to as "teaching teaching with variations with variation." It is expected that students will use both arithmetic and algebraic approaches (from Grade 5 onward), and compare them. The authors suggest that the use of multiple approaches (which include the arithmetic, algebraic, pictorial, as well as other approaches for other types of situations) can foster a deeper understanding of the relationship between quantities, as well as their representation.

Thinking Conceptually about the Procedural

High school algebra has traditionally been viewed as a domain of school mathematics that is dominated by the procedural and where the notion of a conceptual component has been considered nothing short of an oxymoron. In his commentary on the instructional part of this volume, Mason argues that "a blinkered, procedurally oriented perspective on what school algebra is and could be inhibits and obstructs the take up of a richer and broader vision of what school algebra could be, and as far as I am concerned *must be* if mathematics education is going to develop."

One of the central issues related to this widespread procedural orientation in algebra has been the lack in the past of any significant forward movement with respect to the question of that which might constitute the conceptual aspects of algebraic procedures. However, recent theoretical perspectives (e.g., Artigue 2002; Lagrange 2003) are offering a nuanced rethinking of the procedural in terms of the conceptual. Artigue and Lagrange argue that the learning of procedures has within itself a conceptual component. They point out that the technical activity of students, during the period of elaboration of techniques, contains an epistemic (i.e., conceptual) element that is so intertwined with the technical that one co-develops with the other. Examples (drawn from Kieran to appear) of conceptual understanding of algebraic procedures include: being able to see a certain form in algebraic expressions and equations (e.g., seeing that $x^2 + 5x + 6$ and $x^4 + 7x^2 + 10$ are both of the form

$ax^2 + bx + c$); being able to see relationships, such as the equivalence between factored and expanded expressions; and being able to see through algebraic transformations to the underlying change in form of the algebraic object and being able to explain and justify these changes.

Many of the research studies described in this volume reflect implicitly these new perspectives with their emphasis on the conceptual aspects of early algebra, as seen for example, in their attention to the structural face of arithmetic operations, viewed not just as procedures for calculation but also as relational objects. Such perspectives are beginning to break down the old dichotomy between the procedural and the conceptual by including a focus on the conceptual aspects of procedural operations. However, as is seen below, the breaking down of this old dichotomy between the procedural and conceptual brings with it some difficulties in naming and describing approaches to the teaching of algebra that are primarily procedurally oriented.

For example, functional and so-called structural approaches to curricula for middle schoolers are compared in the chapter by Cai, Moyer, Wang, and Nie. The functional approach is described as emphasizing the ideas of change and variation in situations and contexts, as well as the representation of relationships between variables, while the 'structural' approach is described as avoiding contextual problems so as to concentrate on working abstractly with symbols and following procedures in a systematic way. The authors assert that this latter approach uses "naked equations and [emphasizes] procedures for solving equations … all hallmarks of a structural focus." From their observations of classroom instruction, the authors report that the teaching of the functional approach involved a much higher level of conceptual emphasis while the so-called structural approach involved a much higher level of procedural emphasis. In particular, they found that a larger percentage of high cognitive demand tasks (procedures with connections) was implemented in the functional approach classrooms, while a larger percentage of low cognitive demand tasks (procedures without connections or involving memorization) was implemented in the structural approach classrooms.

However, Cai, Moyer, Wang, and Nie's use of the term *structural* is at variance with the way in which this term is used in other chapters of this volume. While Cai et al. associate *structural* with a low-cognitive-demand procedural (i.e., algorithmic) orientation, other authors tend to use the term within a more relational, conceptual focus, for example: "algebraic structure emerges in young children's reasoning and can, with the help of the teacher, be made explicit" (Empson et al.); "pupils focus … on relations, that is, on the structure of the sentence" (Cusi et al.); "the specific movement back and forth between these two representations, geometric and numeric, ultimately supported students to gain not only flexibility with, but also a structural sense of, two-part linear functions" (Moss and McNab); and "meaning is encoded in the structure or relationships between the components" (Cooper and Warren). The contrast between Cai et al.'s use of the term *structural* and the way in which it is used by other authors in the same volume is but one example that suggests a need for a more common terminology, but even more important is the urgency to grapple with the meanings of, and relation between, the *procedural* and the *conceptual* in early algebra.

The conceptual versus the procedural is also the theme of a study reported by Knuth, Alibali, Weinberg, Stephens, and McNeil, who compare relational thinking with that which they describe as 'operationally'-oriented thinking. They report that, despite having learned within a function-based curriculum, only a minority of the middle-school students that were tested demonstrated a relational understanding of the equal sign. Knuth et al. thus recommend that the concept of equivalence be given much more attention than it currently receives in the development of algebraic thinking at both the elementary and middle school levels.

Comparison between relational and 'procedural' emphases in instruction also constitutes the basis of the analysis in the chapter by Smith. She contrasts two 8th grade lessons on the topic of simultaneous equations. Smith describes one lesson as showing a procedural approach to the topic with students focusing on getting answers through a series of routine steps. The other lesson emphasized building generalized solution methods and understanding the relationships represented in systems of equations. Smith notes that the latter lesson "shows how problems that appear procedural can still be completed with conversations that provide rich mathematical connections, allowing students to begin to connect the relations and generalizations which characterize algebra." This observation by Smith is an important one that is consistent with the opening remarks of this section of my overall commentary: the procedural can be approached in a conceptual manner. Similarly, the chapter by Ellerton and Clements describes how procedures for solving decontextualized linear and quadratic equations and inequalities can be conceptualized in connected, relational ways. Both of these chapters, which offer a vision on how the learning of so-called formal algebraic procedures can be rendered conceptual, touch upon an area where further research is crucial, not only for high school algebra but also for early algebra.

Anticipating, Conjecturing, and Justifying

Up to now in this commentary, the characterization of the nature and components of algebraic thinking as reflected in the chapters of this volume has been the main thrust. These characteristics have included thinking about the general in the particular, thinking rule-wise about patterns, thinking relationally about quantity, number, and numerical operations, thinking representationally about the relations in problem situations, and thinking conceptually about the procedural. Clearly, one of the main routes to the development of such algebraic thinking is generalizing, a process that was touched upon in an earlier section. In addition to generalizing, other routes to the development of algebraic thinking that are emphasized within this volume include anticipating, conjecturing, explaining, and justifying. Still other chapters add questioning, wondering, and discussing to this list.

With respect to the role of anticipating within algebraic activity, Boero (2001) has elsewhere argued that:

> A common ingredient of all the processes of transformation (without, before and/or after formalisation) is *anticipation*. In order to direct the transformation in an efficient way, the

> subject needs to foresee some aspects of the final shape of the object to be transformed related to the goal to be reached, and some possibilities of transformation. This 'anticipation' allows planning and continuous feed-back. (p. 99)

Reflecting this point of view, Empson, Levi, and Carpenter in their chapter emphasize the role played by anticipation within relational thinking. As was described briefly above, the example they provide of a student's relational approach to adding $1/2$ and $3/4$ included anticipatory thinking. They argue further that the solution "involved thinking flexibly about both the quantity $3/4$ and about the operation, taken into account concurrently rather than separately as a series of isolated steps." Another perspective on anticipation and the role it can play is highlighted in the chapter by Moss and McNab. In the report of their study of 2nd graders, they discuss how the process of designing and presenting their own growing patterns to classmates provided the students with the opportunity to anticipate how their classmates might respond. According to Moss and McNab, this kind of anticipation and planning adds an extra metacognitive dimension to students' algebraic thinking, thereby enriching the learning potential of the activity.

Conjecturing, generalizing, and justifying are central to the developing of algebraic thinking, according to Blanton and Kaput. In their chapter within this volume, these authors suggest further that tasks ought not only to involve these processes but also build upon systematic variation in the values of problem parameters: "Deliberately transform single-numerical-answer arithmetic problems to opportunities for pattern building, conjecturing, generalizing, and justifying mathematical relationships by varying the given parameters of a problem." But how, Blanton and Kaput ask rhetorically, does this transformation lead to algebraic thinking or, specifically, functional thinking? They respond: "Varying a problem parameter enables students to generate a set of data that has a mathematical relationship, and using sufficiently large quantities for that parameter leads to the algebraic use of number."

Other chapters that signal the importance of justifying within the development of algebraic thinking include that of Russell, Schifter, and Bastable, who describe students' constructing of mathematical arguments to justify general claims for classes of numbers. The authors point out that, although younger students lack the tools of formal proof, they do have available to them ways of representing the operations—drawings, models, or story contexts that they can use to represent specific numerical expressions, but which can also be extended to model and justify general claims. They argue specifically that the development of representations for the operations is critical to connecting arithmetic and algebra. This is clearly an area that invites further research—research on the ways in which operations might effectively be represented by drawings and models, and used as tools for justifying general claims, within the context of early algebra.

Subramaniam and Banerjee, in a historical passage within their chapter, offer a quote attributed to Bhaskara: "Mathematicians have declared algebra to be computation attended with demonstration: else there would be no distinction between arithmetic and algebra." The way in which Indian mathematicians in the past thought about algebra provides, according to Subramaniam and Banerjee, the foundation for the way in which the two of them conceptualize algebraic thinking in terms of justification:

> Algebra involves taking a different attitude or stance with respect to computation and the solution of problems, it is not mere description of solution, but demonstration and justification. Mathematical insight into quantitative relationships combined with an attitude of justification or demonstration, leads to the uncovering of powerful ways of solving complex problems and equations.

Both anticipation and justification are inherent to the theoretical frame presented by Morselli and Boero in their chapter. These authors use Habermas' theory of rationality as a tool for analyzing students' use of algebraic language in mathematical modeling and proving. In their adaptation of Habermas' construct of rational behavior, the authors propose the following three dimensions of rational behavior: epistemic rationality, which concerns both "coherency between the algebraic model and the modeled situation" and the "manipulation rules of the system of signs"; teleological rationality, which consists of the "transformations and interpretations that are useful to the aims of the activity"; and communicative rationality, which includes "not only communication with others (explanation of the solving processes, justification of the performed choices, etc.) but also communication with oneself." As the authors point out, students may carry out certain operations correctly and thereby satisfy the requirements of epistemic rationality; however, they may not have adequately anticipated the aims of the activity and thereby do not satisfy the requirements of teleological rationality. In addition, this model with its communicative-rationality dimension allows for a focus on explanation and justification.

Other routes considered important in the fostering of algebraic thinking include questioning and discussing. For example, Izsák, in his chapter on the complexity of students' thinking in the act of generating and interpreting problem representations, recommends that teachers elicit this student thinking and engage in classroom conversations that include explicit comparisons of different approaches, thereby encouraging the emergence of more powerful algebraic representations. Other related pedagogical interventions considered important by the authors for developing algebraic thinking within problem-solving and problem-representation situations are proposed in the chapter by Koellner, Jacobs, Borko, Roberts, and Schneider: posing questions to move the students forward in their thinking, having students explain and justify their own thinking, and probing more deeply into relevant and challenging ideas.

However, just as little is known about the way that students generalize (Radford, this volume), even less is known about the ways that students come to anticipate, conjecture, and justify. The manner is which students' engagement with these processes leads to algebraic thinking is an area of research that could prove fruitful for years to come.

Gesturing, Visualizing, and Languaging

Although generalizing has already been discussed in terms of being both a characteristic of and a route to algebraic thinking, we return to it once more, even if briefly—this time using the lens of gesturing, visualizing, and languaging, as suggested in the chapters by Radford, by Moss and McNab, and by Cooper and Warren.

From his study on patterning with 2nd graders, Radford notes that the progression in the grasping of the regularity within a pattern linked two kinds of components, both a spatial and a numerical one. This link was mediated by a complex interaction of various senses, such as the visual, the motor, and the aural, as well as by language and rhythm. As students began to think about larger figures, gestures and words helped them to visualize these non-present figures. They generalized and could produce both spatial descriptions of the unspecified figures and the sought-for numerical totals by means of their calculators, even if the majority of the students were not yet stating explicitly the operations being used in terms of unknown numbers. Throughout, both the teacher and the students made extensive use of gestures, acting out, rhythm, and words—most of the senses, in fact. In a related way, the study by Moss and McNab, which also involved 2nd graders in a patterning sequence, highlights the centrality of the visual in interaction with the numeric in evoking students' initial algebraic thinking. Similarly, the roles played by the kinaesthetic, the visual, and the verbal are underlined by Cooper and Warren in their studies of generalization among 3rd to 5th graders.

Elsewhere, Radford (2010) has contrasted his view of the 'sensuous' nature of thinking with that of a purely mental conception of thinking: "Thinking is considered a sensuous and sign-mediated reflective activity embodied in the corporeality of actions, gestures, and artifacts … the adjective *sensuous* refers to a conception of thinking that is inextricably related to the role that the human senses play in it. Thinking is a versatile and sophisticated form of sensuous action where the various senses *collaborate* in the course of a multi-sensorial experience of the world" (p. 4).

The cultural-semiotic lens that Radford brings to his analysis of the role played by the senses in arriving at a pattern generalization in the context of early algebraic thinking provides a valuable viewpoint on the process of generalization. This view broadens considerably existing perspectives on the mental nature of the generalizing process, opening up the construct to the consideration of factors that up to now have largely been ignored, and so suggests an area for further research in the study of algebraic thinking with younger students.

The View of Algebraic Thinking that Emerges from this Volume

The authors of the chapters in this volume provide support for their point of view that algebra in elementary and early middle school is not all about literal symbols but rather is about ways of thinking—thinking about the general in the particular, thinking rule-wise about patterns, thinking relationally about quantity, number, and numerical operations, thinking representationally about the relations in problem situations, and thinking conceptually about the procedural. The processes that constitute these ways of thinking include generalizing, anticipating, conjecturing, justifying, gesturing, visualizing, and languaging. The conceptual areas within early and middle school mathematics that serve as the terrain for such thinking involve not the traditional content of high school algebra but rather the content of arithmetic, including elements of function and change. However, the arithmetic being engaged in is far removed from the usual fare of number facts, algorithms for number operations, and single-numerical-answer problems. The emphasis is rather on seeing within arithmetic not only its inherent regularities, equivalences, multiple ways of conceptualizing numerical relations and analyzing and representing quantitative relationships, but also its functional face involving patterning, analyzing how quantities vary, and identifying correlations between problem variables. As Kilpatrick points out in his commentary on the curricular part of the volume, "if curriculum is a topic list, nothing changes; but if curriculum is the set of experiences that learners have, then the change can be profound."

An additional, but non-negligible, thread running through almost all the chapters is that algebraic thinking does not develop unaided in students. The role of the teacher is crucial. For example, Blanton and Kaput emphasize that "it requires an 'algebra sense' by which teachers can identify occasions in children's thinking to extend conversations about arithmetic to those that explore mathematical generality"; Radford, within a patterning context, points to the importance of the teacher asking students to come up with an *idea* of how to find the total before using actual numbers, thereby encouraging the emergence of the generic aspects of the spatial configuration; and Russell, Schifter, and Bastable recount a teacher's pivotal requesting of her students to "make a picture, draw a model, but not use any particular numbers." Sriraman and Lee remark in their commentary on the cognitive part of the volume that "algebraic thinking can be cultivated from the early grades on if teachers are cognizant of non-symbolic modes of reasoning." The examples that are provided throughout the volume of the ways in which teachers are instrumental in assisting their students to come to think algebraically about their arithmetic point to the complexity of being "cognizant of non-symbolic modes of reasoning." It involves being cognizant of not only the characteristics and components of algebraic thinking, as well as the centrality of certain process-related routes to the development of such thinking, but also novel approaches to tasks, forms of questioning, key examples to focus on, appropriate ways of reacting to students' responses, and a manner of capitalizing on students' contributions so as to help make them accessible to the class at large. As has been emphasized several times throughout this volume, students learn to see algebraically because appropriate learning environments have been designed and put into place according to specific mathematical and pedagogical ideas. Despite the considerable advances that have been made in this field of early algebra, as reflected in the chapters of this volume, much still remains to be done.

References

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning*, *7*, 245–274. doi:10.1023/A:1022103903080.

Balacheff, N. (1987). Processus de preuve et situations de validation [Proving processes and validation situations]. *Educational Studies in Mathematics*, *18*, 147–176.

Blanton, M. L., & Kaput, J. J. (2008). Building district capacity for teacher development in algebraic reasoning. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the Early Grades* (pp. 361–388). New York: Routledge.

Boero, P. (2001). Transformation and anticipation as key processes in algebraic problem solving. In R. Sutherland, T. Rojano, A. Bell, & R. Lins (Eds.), *Perspectives on School Algebra* (pp. 99–119). Dordrecht, The Netherlands: Kluwer Academic.

Jeannotte, D. (2010). L'apport pour la formation et la pratique enseignante: analyse et synthèse de différents modèles de raisonnement mathématique dans la littérature scientifique [Contributing to teacher training and teaching practice: Analysis and synthesis of different models of mathematical reasoning in the scientific literature]. In V. Freiman (Ed.), *Actes du congrès Groupe des Didacticiens des Mathématiques 2010*. Moncton, New Brunswick, Canada: Comité scientifique GDM.

Kaput, J. J. (2008). What is algebra? What is algebraic reasoning. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the Early Grades* (pp. 5–17). New York: Routledge.

Kieran, C. (1996). The changing face of school algebra. In C. Alsina, J. Alvarez, B. Hodgson, C. Laborde, & A. Pérez (Eds.), *Eighth International Congress on Mathematical Education: Selected Lectures* (pp. 271–290). Seville: S.A.E.M. Thales.

Kieran, C. (2004). Algebraic thinking in the early grades: What is it? *The Mathematics Educator*, *8*(1), 139–151.

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels: Building meaning for symbols and their manipulation. In F. K. Lester Jr. (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 707–762). Charlotte, NC: Information Age Publishing.

Kieran, C. (to appear). The false dichotomy in mathematics education between conceptual understanding and procedural skills: An example from algebra. In K. Leatham (Ed.), *Vital Directions in Mathematics Education Research*. New York: Springer.

Lagrange, J.-B. (2003). Learning techniques and concepts using CAS: A practical and theoretical reflection. In J. T. Fey (Ed.), *Computer Algebra Systems in Secondary School Mathematics Education* (pp. 269–283). Reston, VA: National Council of Teachers of Mathematics.

Mason, J. (1996). Expressions of generality and roots of algebra. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to Algebra* (pp. 65–86). Dordrecht, The Netherlands: Kluwer Academic.

Mason, J., with Graham, A., & Johnston-Wilder, S. (2005). *Developing Thinking in Algebra*. London: Sage Publications.

Radford, L. (2010). Signs, gestures, meanings: Algebraic thinking from a cultural semiotic perspective. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the 6th Congress of the European Society for Research in Mathematics Education (2009)*. Lyon, Fr: Institut National de Recherche Pédagogique. http://www.inrp.fr/editions/editions-electroniques/cerme6/plenary-1.