The Algebraic Nature of Fractions: Developing Relational Thinking in Elementary School

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Abstract The authors present a new view of the relationship between learning fractions and learning algebra that (1) emphasizes the conceptual continuities between whole-number arithmetic and fractions; and (2) shows how the fundamental properties of operations and equality that form the foundations of algebra are used naturally by children in their strategies for problems involving operating on and with fractions. This view is grounded in empirical research on how algebraic structure emerges in young children's reasoning. Specifically, the authors argue that there is a broad class of children's strategies for fraction problems motivated by the same mathematical relationships that are essential to understanding high-school algebra and that these relationships cannot be presented to children as discrete skills or learned as isolated rules. The authors refer to the thinking that guides such strategies as relational thinking.

The research summarized in this chapter was supported in part by a grant from the National Science Foundation (ESI-0119732) and a Faculty Research Assignment from the University of Texas. The opinions expressed in this chapter do not necessarily reflect the position, policy, or endorsement of the National Science Foundation or UT. The authors would like to thank Jeremy Roschelle, Walter Stroup, Scott Eberle, and Brian Katz, along with two anonymous reviewers, for very helpful reviews of this chapter as it was being written and Annie Keith and her students for helping us to understand the potential for developing algebraic thinking.

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J. Cai, E. Knuth (eds.), *Early Algebraization*, Advances in Mathematics Education,
DOI 10.1007/978-3-642-17735-4_22,

Of what use ... is it to be able to see the end in the beginning? (Dewey 1974, p. 345).

Fractions and algebra are two topics in school mathematics that are considered critical to the curriculum and difficult to learn (National Council of Teachers of Mathematics 1998, 2000). Students' misconceptions and procedural errors for fractions and algebra, for example, have been well documented (Kerslake 1986; Matz 1982; Sleeman 1984; Stafylidou and Vosniadou 2004). Moreover, high-school students' poor performance in algebra has been blamed on their weak proficiency in fractions. According to a recent Math Panel report, for instance, the ability to perform fraction computations easily and quickly is one of the most critical prerequisites for algebra (U.S. Department of Education 2008).

We see the relationship between fractions and algebra differently. If there is an obstacle to learning algebra, it begins to form as children learn basic arithmetic. As a direct result of typical approaches to instruction in the U.S., American students tend to understand arithmetic as a collection of procedures, rather than in terms of conceptual relationships or general properties of number and operation. By the time the problem is exposed as children learn fractions, it is fairly entrenched, and it is only exacerbated by the fact that fractions are taught in isolation from whole numbers and that fraction operations are taught as a collection of procedures. Concrete materials and models may help children make critical connections (Lesh et al. 1987), but our take on the types of connections that are most fruitful for understanding fractions represents a departure from earlier lines of thinking.

In this chapter we present an alternative view on the relationship between fractions and algebra that (1) emphasizes the conceptual continuities between whole-number arithmetic and fractions; and (2) shows how the fundamental properties of operations and equality that form the foundations of algebra are used naturally by children in their strategies for problems involving operating on and with fractions. We ground this view in research on children's thinking to illustrate how algebraic structure emerges in young children's reasoning and can, with the help of the teacher, be made explicit. Specifically, we argue that there is a broad class of children's strategies for fraction problems motivated by the same mathematical relationships that are essential to understanding high-school algebra and that these relationships cannot be presented to children as discrete skills or learned as isolated rules. We refer to the thinking that guides such strategies as *relational thinking*.

These arguments are based on our research over the last 14 years, in which we have been studying how to provide opportunities for students to engage in relational thinking in elementary classrooms and how to use relational thinking to learn the arithmetic of whole numbers and fractions. We have focused on understanding children's conceptions and misconceptions related to relational thinking, how conceptions develop, how teachers might foster the development and the use of relational thinking to learn arithmetic, and how professional development can support the teaching of relational thinking. This research has included design experiments with classes and small groups of children (e.g. Falkner et al. 1999; Empson 2003; Koehler 2004; Valentine et al. 2004), case studies (Empson et al. 2006; Empson and

Turner 2006), and large-scale studies (Jacobs et al. 2007); and it has been synthesized in two books (Carpenter et al. 2003; Empson and Levi 2011).

In this chapter, we illustrate elementary school children's use of relations and properties of operations as a basis for learning fractions and argue that relational thinking is a critical foundation for learning algebra. We first define relational thinking and then we discuss how the use of relational thinking supports the development of children's understanding of arithmetic. At the same time we challenge the notion that an invigorated focus on fractions in the middle grades is the key to equipping students to learn algebra meaningfully (Hiebert and Behr 1988; U.S. Department of Education 2008). Instead, we argue that the key can be found in helping children to see the continuities among whole numbers, fractions, and algebra. Finally, we suggest that a model of the development of children's understanding of arithmetic that is based upon a concrete to abstract mapping is too simplistic. We propose instead that developing computational procedures based on relational thinking could effectively integrate children's learning of the whole-number and fraction arithmetic in elementary mathematics, in anticipation of the formalization of this thinking in algebra.

What Is Relational Thinking?

Relational thinking involves children's use of fundamental properties of operations and equality[1] to analyze a problem in the context of a goal structure and then to simplify progress towards this goal (Carpenter et al. 2003; see also Carpenter et al. 2005; Empson and Levi 2011). The use of fundamental properties to generate a goal structure and to transform expressions can be explicit or it can be implicit in the logic of children's reasoning much like Vergnaud's (1988) theorems in action.

For example, to calculate $\frac{1}{2}+\frac{3}{4}$ a child may think of $\frac{3}{4}$ as equal to $\frac{1}{2}+\frac{1}{4}$ and reason that $\frac{1}{2}$ plus another $\frac{1}{2}$ is equal to 1, then plus another $\frac{1}{4}$ is $1\frac{1}{4}$. In a study by Empson (1999), several first graders reasoned this way when given a story problem involving these fractional quantities. This solution involves *anticipatory thinking*, a construct introduced by Piaget and colleagues (Piaget et al. 1960) to characterize the use of psychological structures to coordinate a goal with the subgoals used to accomplish it; thinking can involve several such coordinations. These students recognized that they could decompose $\frac{3}{4}$ into $\frac{1}{2}+\frac{1}{4}$, and that if they decomposed it this way, they could regroup to add $\frac{1}{2}+\frac{1}{2}$. In other words, they transformed $\frac{3}{4}$ to $\frac{1}{2}+\frac{1}{4}$ in anticipation of adding $\frac{1}{2}+\frac{1}{2}$. This solution involved thinking flexibly about both the quantity $\frac{3}{4}$ and about the operation, taken into account concurrently rather than separately as a series of isolated steps. Their thinking can be represented by the

[1]Essentially, we are referring here to the field properties and basic properties of equality (Herstein 1996; see also Carpenter et al. 2003; Empson and Levi 2011).

following equalities:

$$\frac{1}{2}+\frac{3}{4}=\frac{1}{2}+\left(\frac{1}{2}+\frac{1}{4}\right)=\left(\frac{1}{2}+\frac{1}{2}\right)+\frac{1}{4}=1+\frac{1}{4}=1\frac{1}{4}.$$

Although the first graders did not represent their reasoning symbolically in this way, their solution is justified in part by the implicit use of the associative property of addition, which we have represented explicitly here to highlight the logic of the their thinking.

Relational thinking is powerful because the applicability of fundamental properties such as the associative property of addition and the distributive property of multiplication over addition cuts across number domains and into the domain of algebra where one reasons about general quantities rather than specific numbers. Consider the expression $7a+4a$. A basic algebraic skill is to simplify this expression to $11a$, by application of the distributive property of multiplication over addition:

$$7a+4a=(7+4)a=11a.$$

The same property that justifies this transformation can also be used to justify that $70+40=110$ and $\frac{7}{5}+\frac{4}{5}=\frac{11}{5}$:

$$70+40=7\times 10+4\times 10=(7+4)\times 10=11\times 10=110,$$
$$\frac{7}{5}+\frac{4}{5}=7\times\frac{1}{5}+4\times\frac{1}{5}=(7+4)\times\frac{1}{5}=11\times\frac{1}{5}=\frac{11}{5}.$$

Yet addition of whole numbers and addition of fractions are taught in isolation from each other in the elementary curriculum, and they are often taught by rote, without reference either to the underlying properties or the process of deciding how and when to use a property. For example, to add fractions children are taught to first find a common denominator and then add the two numerators; many children remember this process as a series of steps to execute. They are not encouraged to draw on their understanding of the distributive property either to derive or to explain this procedure. Many children are therefore simply not prepared later to explicitly draw on the appropriate properties to justify why $7a+4a$ is $11a$, but $7a+4b$ is not $11ab$.

Children learn arithmetic with understanding when they are encouraged to use and develop their intuitive understanding of the properties of number and operation. Our research has led us to recast the meaning of *learning with understanding* in terms of thinking relationally: *to understand arithmetic is to think relationally about arithmetic*, because the coherence of operations on whole numbers and fractions is found at the level of the fundamental properties of operations and equality. Teaching arithmetic in general and fractions in particular primarily as a set of procedures fails to introduce children to the powerful reasoning structures that form the basis of our number system. On the other hand, if children enter algebra with a well developed ability to think relationally about operations, they are prepared to learn to reason meaningfully about and carry out transformations involving generalized expressions through the explicit application of algebraic properties. In the following section we show how these properties emerge and can be developed in the context of carrying out number operations involving fractions and we discuss their connections to learning algebra.

Use of Relational Thinking in Learning Fractions

Children's difficulties learning fractions have been well documented (Kerslake 1986; Stafylidou and Vosniadou 2004). The difficulty, however, may be in how fractions are taught rather than how intrinsically easy or hard they are to understand. Indeed, a conclusion we draw from our research is that fractions are not unduly difficult if instruction develops children's capacity for relational thinking.

A focus on relational thinking can transform fractions into a topic that children understand by drawing on and reinforcing the fundamental properties that govern reasoning about both whole-number and fraction quantities and operations. Children use relational thinking in their solutions to story problems (e.g., Baek 2008; Carpenter et al. 1998; Empson et al. 2006) and open number sentences (Carpenter et al. 2005). Teachers can cultivate children's use of relational thinking by using a combination of these types of problems. In this chapter we focus on children's relational thinking in the context of solving story problems.

Understanding Fractional Quantities Through Relational Thinking

Before children can learn to operate on or with fractions, they need to understand fractional as quantities. Because a fraction is defined by the multiplicative relationship between its two terms, a mature understanding of fractions as quantities is relational in nature. Young children can construct a relational understanding of fractions by solving and discussing Equal Sharing problems (Empson 1999; Empson and Levi 2011; Streefland 1993).

To solve a problem about equally sharing quantities, such as two pancakes shared among three children, children must partition the quantities equally and completely. Children's earliest, non-relational strategies often involve partitioning the pancakes into halves. In this example the two pancakes would yield four halves. A child using this strategy might then try to distribute the four halves into three groups. When the child discovers that there is a half left over, the child may then partition the extra half into half, and then partition each of those parts into half again, continuing until the parts get too small to partition. This solution is not relational in that it lacks anticipatory thinking. The child knows that it is necessary to partition the pieces to share them, but approaches the problem one step at a time, partitioning into halves without anticipating how the resulting parts are going to be shared.

Children begin to think relationally about fractional quantities when they begin to reason about the relationship between partitions into equal and exhaustive shares

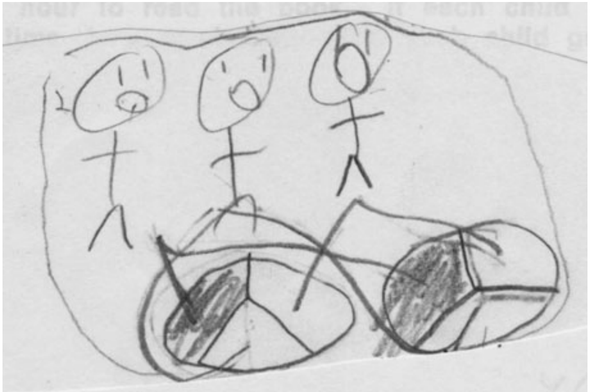


Fig. 1 Child's strategy for sharing 2 pancakes equally among 3 children, demonstrating emerging relational understanding of fractions

and the number of sharers. To solve two pancakes shared by three children, a child could decide to completely share the first pancake with all three children, and then to share the second pancake in the same way (Fig. 1). Alternatively, a child who began by distributing one half to each person might then decide to partition the left over half into three equal parts. In either case, a child who thinks about the number of people sharing *and* at the same time how to partition the things to be shared is in the process of developing a relational understanding of fractions.

These strategies implicitly use several important mathematical relationships. For ease of illustration, we concentrate on the strategy in which the child partitions each whole candy bar into thirds. Although young children are unlikely to use the following notation to represent their reasoning, it follows this logic:

$$2 \div 3 = (1 + 1) \div 3 = 1 \div 3 + 1 \div 3 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

This reasoning embodies the knowledge that three one-thirds make a whole pancake and that one pancake divided among three people yields one-third of a pancake to each. It also suggests an intuitive understanding of how a "distributive-like property" can be applied to division.[2]

A fully operationalized and explicit understanding of fractions as relational quantities develops gradually. Most basic to this understanding is that unit fractions are created by division or partitioning and that unit fractions are multiplicatively related to the whole:

$$1 \div n = \frac{1}{n} \quad \text{and} \quad \frac{1}{n} \times n = 1. \tag{1}$$

Multiple opportunities to combine unit-fraction quantities in solutions to Equal Sharing problems and to notate these solutions—such as "1 third and 1 third equals

[2]This property can be represented as $a \div c + b \div c = (a + b) \div c$, which is equivalent to $a \times \frac{1}{c} + b \times \frac{1}{c} = (a + b) \times \frac{1}{c}$. It is sometimes referred to as the right distributive property of division over addition. On the other hand, $a \div (b + c)$is not the same as $a \div b + a \div c$; that is, there is no left distributive property of division over addition.

2 thirds"— lead to the following more generalized relational understanding:

$$m \times \frac{1}{n} = \frac{m}{n}. \tag{2}$$

The conceptual connections between children's pictorial and symbolic representations of fractional quantities require prolonged attention to develop in a flexible, integrated way (Empson et al. 2006; Saxe et al. 1999). These relationships are initially grounded in children's informal knowledge of partitioning quantities, such as cupcakes and sandwiches. Children arrive at the generalized relational understanding represented by (1) and (2) above as the result of repeated opportunities to create, represent, and reason about these relationships in various interlinked forms over an extended period of time.

Understanding these basic relationships is absolutely critical to children's ability to reason with understanding about fraction operations and computations. Consider the case of Holly, a fifth grader who had been exposed to fraction instruction throughout her school career but did not understand fractions as relational quantities. She had learned that fractions involved partitioning wholes into parts, but she did not understand the relation between the parts and the whole. Fractional parts, to her, were entities unrelated to whole numbers. These limitations in her understanding were exposed in her solution to the following problem:

> Jeremy is making cupcakes. He wants to put $\frac{1}{2}$ cup of frosting on each cupcake. If he makes 4 cupcakes for his birthday party, how much frosting will he use to frost all of the cupcakes?

To solve the problem, Holly drew the picture in Fig. 2 and decided the answer was "four halves." Upon further questioning, it became clear that Holly did not see how these quantities could be combined; she insisted the answer was four halves and four halves only. It seemed instead that the entire circle partitioned in half represented the fraction $\frac{1}{2}$ for Holly, and it would have been nonsensical to combine them (akin to asking, "How much is 4 apples?"). For her, fractions existed separately from other numerical measures.

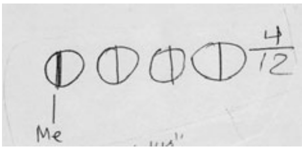


Fig. 2 Holly's written work for figuring four groups of half each, suggesting a non-relational understanding of fractions. (The 4 over what looks like 12 is Holly's way of writing 4 halves. She appears to be trying to remember syntactic features of the numeral and confounding "12" with "1/2")

Contrast Holly's solution to a third grader's solution to the following problem.

> Mr. W has 10 cups of frog food. His frogs eat $\frac{1}{2}$ a cup of frog food a day.
> How long can he feed his frogs before his food runs out?

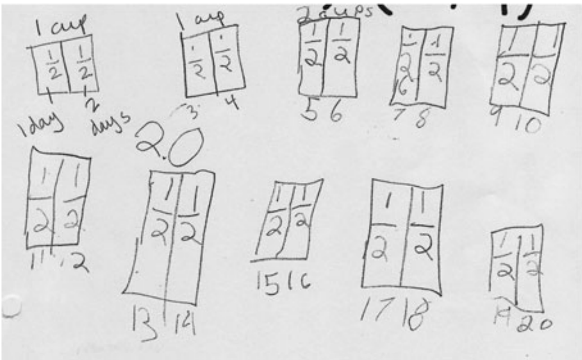


Fig. 3 A third grader's written work for figuring 10 groups of one half each, showing a relational understanding of fractions as quantities

The third grader, John, represented each cup of frog food with a rectangle, then divided each rectangle in half and notated "$\frac{1}{2}$" on each half to show how much food Mr. W's frogs could eat in a day (Fig. 3). He then counted these to arrive at an answer of 20 days. Unlike Holly, John used a relational understanding of the quantity $2 \times \frac{1}{2} = 1$ to construct a solution. John's solution represents a big step forward over Holly's. He might have gone further in his use of relational thinking by grouping the half cups in order to figure the total number of days more efficiently. For example, he could have reasoned that 2 half cups are one cup, 4 half cups are 2 cups and so on, until he reached the number of half cups in 10 whole cups. He also could have reasoned directly that 20 groups of $\frac{1}{2}$ are the same as 10 groups of 1. This type of reasoning, which takes into account both a relational understanding of fractional quantities and relations involving the operation of multiplication, is illustrated in the cases in the following section.

Use of Relational Thinking to Make Sense of Operations Involving Fractions

As children come to understand fractions as relational, they start to use this understanding to decompose and recompose quantities for the purpose of transforming expressions and simplifying computations. These manipulations are done purposefully and draw on (a) children's intuitive understanding of fractional quantities as relational described above and (b) children's relational understanding of operations cultivated in the context of whole-number reasoning and problem solving.

Children's strategies for multiplication and division word problems involving fractions can draw on and reinforce their growing understanding of the multiplicative nature of fractions. At the same time, the use of such problems supports the emergence of relational thinking about operations as children attempt to figure out how to make operations more efficient. Children's thinking becomes more anticipatory in that they begin to make choices about how to decompose and recompose fractions in the context of a goal structure that relates operations and quantities. This

Table 1 Combining groups using fundamental properties of multiplication

Equation representing child's thinking	Fundamental and other generalized properties of arithmetic on which child's thinking is based
$8 \times \frac{3}{8} = 8 \times (3 \times \frac{1}{8})$	Fractions represented as multiples of unit fractions
$= 8 \times (\frac{1}{8} \times 3)$	Commutative property of multiplication
$= (8 \times \frac{1}{8}) \times 3$	Associative property of multiplication
$= 1 \times 3 = 3$	Inverse and identity properties of multiplication

anticipatory thinking signals the purposeful use of fundamental properties of operations and equality and is in contrast with algorithmic thinking about operations in which the goal structure can be summarized as "do next."

A pivotal point in the growth of children's understanding is reached when children begin to use relational thinking to make repeated addition or subtraction of fractions more efficient by applying fundamental properties of operations and equality in their strategies for combining quantities. The emergence of relational thinking about operations in this context is facilitated by the need to combine several groups of equal size. For example, in one of the cases that follows, a fifth-grade student wanted to figure eight groups of three eighths each. The child reasoned that eight groups of one eighth each equals one, so three such groups would be three. This reasoning makes implicit use of the commutative and associative properties of multiplication (Table 1).

As children's understanding of fractions grows, basic relationships as illustrated in Table 1 serve as building blocks in more sophisticated relational thinking strategies. These strategies draw upon a variety of these properties in ways that are anticipatory rather than algorithmic and in ways that demonstrate a well connected understanding of number and operation. Most notably, these strategies are driven by each child's understanding and therefore cannot and should not be reduced to a generalized series of steps for all children to follow. In fact, a teacher would be hard pressed to explicitly teach these strategies, because each step is embedded in a goal structure that is specific to each child's relational understanding of the operations and quantities for a given problem. In the long run, this relational understanding of number and operations results in an efficiency in learning advanced mathematics, such as algebra.

To illustrate the types of relational thinking that elementary students are capable of using, we discuss two strategies generated by fifth and sixth graders in different classrooms. The teachers in these classrooms tended to place responsibility for generating and using conceptually sound strategies on each individual student.[3] This approach to instruction does not typify instruction in U.S. classrooms, and so the strategies we describe here are not representative of the current performance of U.S. children in the upper elementary grades (e.g., Hiebert et al. 2003). However, they are

[3]We have observed patterns both in the types of relational thinking used and how it develops, which are beyond the scope of this chapter (see Empson and Levi 2011).

representative of the types strategies that evolve in classrooms such as these—even if these classrooms are rare—and provide a study of the possibility of integrating fractions and algebra in the upper elementary grades.

Each problem involved division with a remainder to be taken into account in the quotient. For each case, we describe the strategy and then note how children used fundamental properties of operations and equality in their solutions.

Case 1: Measurement division.

The first case comes from a combination fourth- and fifth-grade class, working on the following measurement division problem:

> It takes ____ of a cup of sugar to make a batch of cookies. I have $5\frac{1}{2}$ cups of sugar. How many batches of cookies can I make?

The students were given a variety of number choices for the divisor. In order of difficulty, these choices were $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, and $\frac{3}{8}$. Several students, including Jill, chose to work with $\frac{3}{8}$ of a cup of sugar.

Jill began her strategy by drawing upon the basic multiplicative relationship described above to generate familiar groupings of three eighths that would simplify the calculation (Table 1). She said she knew that 8 three-eighths would be 3, which meant that 4 three-eighths would be half that much, or $1\frac{1}{2}$, and 12 three-eighths would therefore be $4\frac{1}{2}$ (Fig. 4). At this point, she knew that she needed only 1 more cup to use up all $5\frac{1}{2}$ cups. Again Jill used the relationship between $\frac{3}{8}$ and 3 as a reference point. She said that because 8 three-eighths was 3, a third as many three-eighths would be a third as much, or 1. That is, $(\frac{1}{3} \times 8) \times \frac{3}{8}$ is 1, and $\frac{1}{3} \times 8$ is $\frac{8}{3}$ or $2\frac{2}{3}$. She concluded that she could make a total of $12 + \frac{8}{3}$ batches, which would be equal to $14\frac{2}{3}$ batches.

If we unpack Jill's description of her solution, we see that it involved setting subgoals that were readily solved using familiar relations. The solution of one sub-

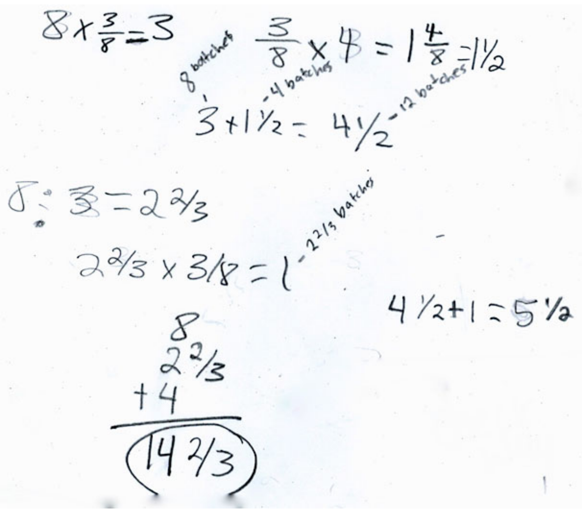


Fig. 4 Jill's written work for her strategy to solve $5\frac{1}{2}$ divided by $\frac{3}{8}$, suggesting implicit use of fundamental properties of operations and equality

goal provided a springboard for the next. Fundamental properties of operations and equality were implicit in the solution of each of the subgoals. Jill's ultimate goal was to find how many $\frac{3}{8}$ cups it would take to make $5\frac{1}{2}$ cups. She started with an overarching view of the problem that facilitated the formulation of a series of subgoals. The $5\frac{1}{2}$ cups could be partitioned into parts that would be easily divided by $\frac{3}{8}$. Then the parts could be combined.

Jill's first subgoal was to identify a multiple of $\frac{3}{8}$ that would give her a whole number that she might subsequently use as a building block to find how many $\frac{3}{8}$ cups it took to make $5\frac{1}{2}$ cups. Drawing implicitly on the kind of thinking described in Table 1, she started with the equation 8 groups of $\frac{3}{8}$ is 3.

Because she had only accounted for 3 of the $5\frac{1}{2}$ cups of flour in the problem, Jill now had to find how many $\frac{3}{8}$ cups it took to make $2\frac{1}{2}$ cups. She recognized that she could use the equation involving 8 groups of $\frac{3}{8}$ to make another $1\frac{1}{2}$ cups and that would leave exactly one cup to deal with. Essentially she used the multiplicative property of equality and the associative property of multiplication to transform the equation $8 \times \frac{3}{8} = 3$ as follows:

$$\frac{1}{2} \times \left(8 \times \frac{3}{8}\right) = \frac{1}{2} \times 3,$$
$$\left(\frac{1}{2} \times 8\right) \times \frac{3}{8} = 1\frac{1}{2},$$
$$4 \times \frac{3}{8} = 1\frac{1}{2}.$$

The next subgoal was to find how many $\frac{3}{8}$ cups it took to make the remaining one cup. Jill also used the equation $8 \times \frac{3}{8} = 3$ as the basis for addressing this subgoal. She again used the multiplicative property of equality and the associative property of multiplication to transform the core equation as shown below.

$$8 \times \frac{3}{8} = 3,$$
$$\frac{1}{3} \times \left(8 \times \frac{3}{8}\right) = \frac{1}{3} \times 3,$$
$$\left(\frac{1}{3} \times 8\right) \times \frac{3}{8} = 1,$$
$$\frac{8}{3} \times \frac{3}{8} = 1.$$

Note Jill might have simply used the reciprocal relation between $\frac{8}{3}$ and $\frac{3}{8}$ for this calculation, but she continued to build off of the equation $8 \times \frac{3}{8} = 3$. Although we believe it is likely that she did not intend to generate the reciprocal relationship between $\frac{8}{3}$ and $\frac{3}{8}$, we find its emergence here significant, because it illustrates how algebraic relationships can emerge fairly naturally in the context of children's relational reasoning. Problems such as this one provide experience with this relation.

Finally, Jill put the parts together using the additive property of equality and the distributive property.

$$8 \times \frac{3}{8} + 4 \times \frac{3}{8} + \frac{8}{3} \times \frac{3}{8} = 3 + 1\frac{1}{2} + 1 = 5\frac{1}{2}$$

and

$$\begin{aligned} 8 \times \frac{3}{8} + 4 \times \frac{3}{8} + \frac{8}{3} \times \frac{3}{8} &= \left(8 + 4 + \frac{8}{3}\right) \times \frac{3}{8} \\ &= 14\frac{2}{3} \times \frac{3}{8}. \end{aligned}$$

The kinds of thinking implicit in Jill's strategy are directly related to the kinds of thinking that are involved in solving algebra problems with meaning. She started with a primary goal—to find how many $\frac{3}{8}$-cups were in $5\frac{1}{2}$ cups—which subsequently guided the formulation of her subgoals. To make progress on solving the problem, she transformed the primary goal into a series of subgoals for which she had a ready solution. This practice is fundamental to high-school algebra in which a series of properties of operations and equality are often used to simplify a complex equation. For example, to solve a linear equation in one unknown, students set subgoals that entail finding successively simpler equations that are closer to the goal of finding an equation of the form $x =$ a number. As in the above example, the subgoals are met by repeated application of fundamental properties of operations and equality. Similarly, the goal of solving a quadratic equation is transformed into subgoals of solving simple linear equations by applying a corollary of the zero property of multiplication ($a \times b = 0$, if and only if $a = 0$ or $b = 0$).

To address each of the subgoals, Jill essentially constructed and transformed relationships of equality using fundamental properties of operations and equality in ways that were strikingly similar to the thinking used in constructing and solving equations in formal treatments of algebra. She drew on anticipatory thinking in transforming the equations into equations that could be put together to solve the problem. In other words she consistently constructed and transformed equations in ways that brought her closer to the solution of the basic problem. Again, that is essentially what solving algebra equations is all about.

Case 2: Partitive division.

Our second case involves a sixth-grade boy, Keenan, who solved the following problem:

> Two thirds of a bag of coffee weighs 2.7 pounds. How much would a whole bag of coffee weigh?

This problem involves partitive division and differs from the previous division problem in that the goal is to find out how much per group rather than to find out how many groups. Keenan's strategy included the transformation of quantities for the

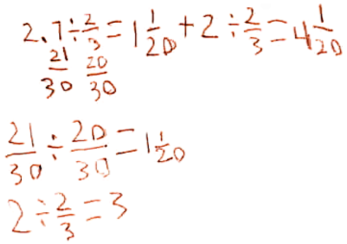


Fig. 5 Keenan's strategy to solve 2.7 divided by $\frac{2}{3}$, suggesting implicit use of fundamental properties of operations and equality

purpose of simplifying calculations as well as the flexible use of several fundamental properties of operations and equality (Fig. 5).

To start, Keenan recognized that the problem was a division problem and wrote $2.7 \div \frac{2}{3}$. He remarked, "Two divided by $\frac{2}{3}$ is going to be really easy, all I really need to worry about is the seven tenths. Seven tenths divided by $\frac{2}{3}$ isn't easy to think about so [long pause] if I make them both thirtieths, it would be easier." He notated his thinking so that it read:

$$\frac{21}{30} \div \frac{20}{30}$$

and then said, "21 thirtieths divided by 20 thirtieths is just the same as 21 divided by 20 which is one and one twentieth." He notated his answer:

$$\frac{21}{30} \div \frac{20}{30} = 1\frac{1}{20}.$$

Keenan then said, "Now all I have to do is 2 divided by $\frac{2}{3}$, which is 3." When asked how he knew that so quickly he said, "2 divided by $\frac{1}{3}$ would be 6 since you have 3 groups of $\frac{1}{3}$ in each 1, so 2 divided by $\frac{2}{3}$ would be 3 since $\frac{2}{3}$ is twice as big as $\frac{1}{3}$." He then extended his notation as follows:

$$2.7 \div \frac{2}{3} = 1\frac{1}{20} + 2 \div \frac{2}{3} = 1\frac{1}{20} + 3 = 4\frac{1}{20}.$$

This strategy incorporates several instances of relational thinking. Keenan began by decomposing 2.7 into .7 + 2. This choice involved anticipatory thinking in that he analyzed the problem to see what relationships he might draw upon to simplify his calculations, rather than simply begin to execute a series of steps to solve the problem. He used the commutative property of addition and a "distributive-like" property to simplify the division. Although he did not notate this step, his thinking could be represented as:

$$2.7 \div \frac{2}{3} = (2 + .7) \div \frac{2}{3} = (.7 + 2) \div \frac{2}{3} = .7 \div \frac{2}{3} + 2 \div \frac{2}{3}.$$

Of note is his correct use of this distributive-like property for division. This division relationship is generalizable and can be justified with the distributive property of multiplication over addition in conjunction with the inverse relationship between multiplication and division (see footnote 2).

Next Keenan facilitated the computation of $.7 \div \frac{2}{3}$ by transforming .7 into $\frac{21}{30}$ and $\frac{2}{3}$ into $\frac{20}{30}$ and then using these transformed quantities as follows:

$$.7 \div \frac{2}{3} = \frac{21}{30} \div \frac{20}{30} = 21 \div 20 = 1\frac{1}{20}.$$

Again, Keenan used anticipatory thinking to produce equivalent fractions for the purpose of simplifying the division.

Keenan then computed $2 \div \frac{2}{3}$. This computation appeared to be routine for him; however, he justified it as follows:

$$2 \div \frac{2}{3} = 2 \div \left(2 \times \frac{1}{3}\right) = 2 \div \left(\frac{1}{3} \times 2\right) = \left(2 \div \frac{1}{3}\right) \div 2$$

with an associative-like property of division. Again he used a generalizable principle, similar to the distributive-like principle he used above, that could be justified with formal properties but is rooted in a relational understanding of fractional quantities and division.

Like Jill, Keenan had a unified view of the entire problem and its parts. This view allowed him to set subgoals to address the parts individually with the understanding that the answers to the problems addressed by these subgoals could be reassembled into the whole. As was the case with Jill, Keenan drew on anticipatory thinking and a fluid understanding of how expressions and equations could be transformed. Once again the parallels with the kind of thinking used in symbolic treatments of algebra are striking.

Discussion of Cases

The strategies used by these two elementary aged children to divide fractions illustrate the power of relational thinking and its algebraic character. Children's thinking in these examples was anticipatory in that their strategies were driven by a goal structure premised on relational thinking. These strategies contrast with the goal structure in the execution of standard algorithms as they are typically learned which can be summarized as "do next."

Further, the thinking displayed by these children resembles the "competent reasoning" that proficient mathematical thinkers use to compare rational numbers, as reported by Smith (1995). Based on an analysis of 30 students' solutions to order and equivalence problems, Smith argued that competent reasoning is characterized by the use of strategies that exploit the specific numerical features of a problem and often apply only to a restricted class of fractions. These strategies were reliable and efficient. He contrasted this reasoning with the use of generalized, all purpose strategies—such as conversion to a common denominator to compare fractions—

which the students in his study tended to use as a last resort.[4] These findings led Smith to conclude "the analysis of skilled reasoning with rational numbers should … move beyond a focus on particular strategies to examine the character of the broader knowledge *system* that has those strategies as components" (p. 38). We are proposing that this system consists of children's informal algebra of fractional quantities and that it is expressed in children's relational thinking.

From this perspective, Jill and Keenan represent students who are in the process of developing a knowledge base for reasoning about fractions in ways that can be characterized as proficient and competent. Moreover, this knowledge base is integrated with properties of whole-number operations and relations and anticipates the algebra of generalized quantities, typically taught in the eighth or ninth grade. We are not suggesting that standard algorithms for fraction operations have no place in developing fluency with number operations. Proficient thinkers use them when they see no way to exploit the number relationships in a problem. However, we are arguing that when children are supported to develop relational thinking in elementary school, their knowledge of generalized properties of number and operation becomes explicit and can serve as a foundation for learning high-school algebra in ways that mitigate the development of mistakes and misconceptions.

A Conjecture Concerning Relational Thinking as a Tool in Learning New Number Content

Jill and Keenan used fundamental properties of operations and equality and other notable relationships, such as $\frac{3}{8} \times 8 = 3$, as tools in their strategies to divide fractions. The use of these relationships was coordinated within a goal structure and is a hallmark of relational thinking. In this section we discuss a critical and perhaps surprising implication of a focus on relational thinking in the elementary curriculum with respect to the role of other types of tools such as concrete materials and models in facilitating the development of children's understanding of number operations involving fractions (as well as decimals and integers—which are beyond the scope of this chapter).

Some approaches to teaching for understanding emphasize the use of concrete materials such as base-ten blocks or fraction strips to model abstract relationships (e.g., Van de Walle 2007). The use of such materials has at times been seen as a universal remedy to children's difficulties in understanding mathematics. Several studies have shown, however, that concrete materials alone are insufficient at best and at worst, ineffective (Brinker 1997; Resnick and Omanson 1987;

[4] This approach to numerical reasoning is not unique to children. Dowker (1992) reported that professional mathematicians prefer to approach computational estimation in the same ways, that is, by exploiting specific numerical features of a problem rather than using a generalized algorithm that works in all cases.

Uttal et al. 1997). In a review of this research, Sophian (2007) noted that manipulatives are symbols themselves and how they map to mathematical notation and processes is best appreciated by those who already understand the mapping (p. 157). Teachers can show students how to manipulate these materials to perform calculations involving fractions just as they show students how to manipulate symbols to perform calculations. Some students may remember steps involving materials more easily than they remember symbolic algorithms, but in neither case are they necessarily reasoning about the relationships involved in each step or more globally in the problem.

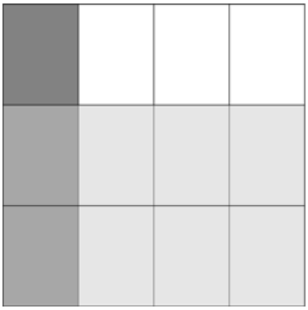
In contrast, when manipulatives and other types of models are used as tools to think with, rather than to simply generate an answer, they can play a critical role in the development of children's understanding (Carpenter and Lehrer 1999; Koehler 2004; Martin and Schwartz 2005). For example, the images that children create and reason about as they partition quantities such as cupcakes and sandwiches in their solutions to story problems can help children conceptualize fractions in terms of basic relationships such as $\frac{2}{3} = 2 \times \frac{1}{3}$ (Empson and Levi 2011).

Keeping in mind this valuable use of visualizing tools, we propose a shift in relative emphasis as the curriculum turns to advanced number operations: *Children's use of relational thinking can and should drive the development of new content and concrete materials and models should be used to support the emergence of relational thinking*. Jill's and Keenan's strategies for division of fractions described above cannot be mapped in any straightforward way onto the manipulation of concrete materials and so do not appear to represent an abstraction of their operations on concrete materials. Instead, these strategies (1) incorporated a relational understanding of fractions and (2) were planned and executed (sometimes in an emergent sense) on the basis of each child's understanding of fundamental properties of operations and equality.

One fairly popular way to introduce fraction multiplication, for example, is by using an area model (Izsák 2008). Its advantages are that it is generalizable—it can be used to multiply any two fractions—and it is "concrete" so children can "see" the multiplication. To multiply $\frac{1}{4} \times \frac{2}{3}$ using this model, for example, a rectangular unit is divided into thirds and two of the thirds are shaded. Then the rectangle is divided into fourths orthogonally to the original partition into thirds. Based on this partitioning, one fourth of the rectangle is shaded. The intersection of the shaded parts (Fig. 6) represents the product of one fourth and two thirds. The model might be used to develop understanding of multiplication of fractions, but the use of this model is easily proceduralized, especially if it is introduced before children have had opportunities to make and integrate relational connections between quantities and operations. (See teachers' own difficulties with the proceduralization of this model, reported in Izsák 2008.)

Using relational thinking, children might approach a problem such as this one in any number of ways employing strategies that involve the application of generalized properties of arithmetic. For example, a child could reduce the calculation to operating on unit fractions, by applying the distributive property of multiplication over addition. A child might say, "a quarter of $\frac{1}{3}$ is $\frac{1}{12}$ so a quarter of $\frac{2}{3}$ has to be $\frac{1}{12}$ plus

Fig. 6 Area representation of the product of $\frac{1}{4} \times \frac{2}{3}$



$\frac{1}{12}$ which is $\frac{2}{12}$." This thinking could be formally represented:

$$\frac{1}{4} \times \frac{2}{3} = \frac{1}{4}\left(\frac{1}{3} + \frac{1}{3}\right) = \left(\frac{1}{4} \times \frac{1}{3}\right) + \left(\frac{1}{4} \times \frac{1}{3}\right) = \frac{1}{12} + \frac{1}{12} = \frac{2}{12}.$$

A child could also transform the calculation into an easier one through an implicit use of the associative property of multiplication. The reasoning might be "a quarter of $\frac{1}{3}$ is $\frac{1}{12}$ so a quarter of $\frac{2}{3}$ has to be $\frac{1}{12}$ times 2 which is $\frac{2}{12}$:"

$$\frac{1}{4} \times \frac{2}{3} = \frac{1}{4} \times \left(\frac{1}{3} \times 2\right) = \left(\frac{1}{4} \times \frac{1}{3}\right) \times 2 = \frac{1}{12} \times 2 = \frac{2}{12}.$$

These strategies represent the same types of relational thinking that we saw in children's strategies for division and mirror the types of relational thinking that children use in whole-number multiplication (Baek 2008). They are two examples of possible strategies that are driven by relational thinking instead of the potentially rote use of a concrete model. With some experimentation, the reader should be able to generate strategies for the multiplication of any two fractions that incorporate the same fundamental properties of operations and equality and are robust and generalizable.

In summary, our conjecture is that if instruction is focused on developing relational thinking with whole numbers throughout the early grades, the role of concrete materials in introducing and developing understanding of operations on fractions and decimals will likely change. Concrete materials would be used to support the development of relational thinking rather than simply as tools to calculate answers or justify algorithms. Further, encouraging children to construct and use procedures based on relational thinking would help them to integrate learning number operations across different number domains.

Conclusion

The kinds of activity and thinking illustrated in this chapter are not isolated examples, and they do not represent mathematics that should be reserved for only a limited number of students or as supplementary enrichment (Carpenter et al. 2003). The results of a recent study by Koehler (2004) document that young children of a wide range of abilities are able to learn to think about relations involving the distributive property and that instruction that focuses on relational thinking as illustrated in these examples supports the learning of basic arithmetic concepts and skills.

In this chapter we have argued that a focus on relational thinking can address some of the most critical perennial issues in learning fractions with understanding. One of the defining characteristics of learning with understanding is that knowledge is connected (Bransford et al. 1999; Carpenter and Lehrer 1999; Greeno et al. 1996; Hiebert and Carpenter 1992; Kilpatrick et al. 2001). Not all connections, however, are of equal value, and we propose that our conception of relational thinking can sharpen mathematics educators' conceptions of what learning with understanding looks like. Students who engage in relational thinking are using a relatively small set of fundamental principles of mathematics to establish relations. Thus, relational thinking can be seen as one way of specifying the kinds of connections that are productive in learning with understanding. We have presented several such connections made by children in elementary grades in the context of generating strategies for problems involving multiplication and division of fractions.

We have further argued that relational thinking is a critical precursor—perhaps the most critical—to learning algebra with understanding, because if children understand the arithmetic that they learn, then they are better prepared to solve problems and generate new ideas in the domain of algebra. However, relational thinking is almost entirely neglected in typical U.S. classrooms with the unfortunate result that children experience all types of learning difficulties as they move beyond arithmetic into learning algebra. Some proposed solutions focus on a renewed emphasis on prerequisite skills (e.g., U.S. Department of Education), while others emphasize the use of concrete materials and models (e.g., Lesh et al. 1987). We have presented an alternative view of how to address these difficulties that centers on cultivating children's implicit use of fundamental properties of the real-number system to solve arithmetic problems, to better align the concepts and skills learned in arithmetic and algebra. At the heart of this view is the reciprocal relationship between arithmetic and algebra as it is revealed in children's reasoning about quantity.

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