Formation of Pattern Generalization Involving Linear Figural Patterns Among Middle School Students: Results of a Three-Year Study

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Abstract This chapter provides an empirical account of the formation of pattern generalization among a group of middle school students who participated in a three-year longitudinal study. Using pre-and post-interviews and videos of intervening teaching experiments, we document shifts in students' ability to pattern generalize from figural to numeric and then back to figural, including how and why they occurred and consequences. The following six findings are discussed in some detail: development of constructive and deconstructive generalizations at the middle school level; operations needed in developing a pattern generalization; factors affecting students' ability to develop constructive generalizations; emergence of classroom mathematical practices on pattern generalization; middle school students' justification of constructive standard generalizations, and; their justification of constructive nonstandard generalizations and deconstructive generalizations. The longitudinal study also highlights the conceptual significance of multiplicative thinking in pattern generalization and the important role of sociocultural mediation in fostering growth in generalization practices.

Research on patterning and generalization at least in the last decade has empirically demonstrated the remarkable, albeit fundamental, view that individuals tend to see and process the same pattern P differently. Consequently, this means they are likely to produce different generalizations for P. For example, when we asked forty-two

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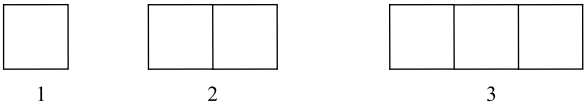
Square Toothpicks Pattern. Consider the sequence of toothpick squares below.



A. How many toothpicks will pattern 5 have? Draw and explain.
B. How many toothpicks will pattern 15 have? Explain.
C. Find a direct formula for the total number of toothpicks T in any pattern number n. Explain how you obtained your answer.
D. If you obtained your formula numerically, what might it mean if you think about it in terms of the above pattern?
E. If the pattern above is extended over several more cases, a certain pattern uses 76 toothpicks all in all. Which pattern number is this? Explain how you obtained your answer.
F. Diana's direct formula is as follows: $T = 4 \cdot n - (n - 1)$. Is her formula correct? Why or why not? If her formula is correct, how might she be thinking about it? Who has the more correct formula, Diana's formula or the formula you obtained in part C above? Explain.

Fig. 1 Adjacent squares pattern task

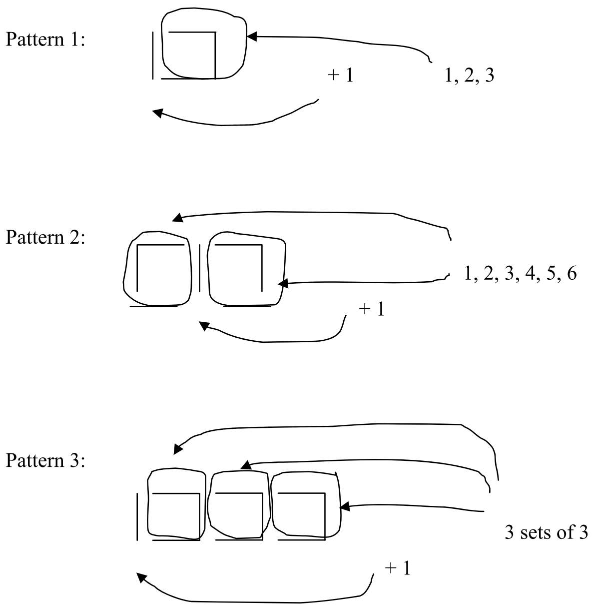
undergraduate elementary majors to establish a general formula for the total number of matchsticks at any stage in the *Adjacent Squares Pattern* shown in Fig. 1, Chuck obtained his generalization "$4 + (n - 1)3$" in the following manner:

> How many matchsticks are needed to form four squares? So ahm I'm looking for a pattern. For every square you add three more. So let's see. So that would be 4 plus 3 for two squares. Plus 3 more would be for three squares. So it's 10 matchsticks. So you have 4. So there would be 13. So 13 plus 3 more is 16. …So, for three squares, it would have to be two 3s. So there'd be two 3s. Three 3s is for four squares, and four 3s for five squares. For n squares, it would just be ahm n minus one 3s. (Rivera and Becker 2003, p. 69)

When we gave the same pattern in Fig. 1 to a group of middle school students three times over a two-year period, first when they were in sixth grade (after a teaching experiment) and then twice in seventh grade (before and after a teaching experiment), all of their generalizations consistently took the form $T = (n \times 3) + 1$. For example, in a clinical interview prior to the Year 2 teaching experiment, Dung, in seventh grade, initially set up a two-column table of values, listed down the pairs $(1, 4)$, $(2, 7)$, and $(3, 11)$ and noticed that "the pattern is plus 3 [referring to the dependent terms]." He then concluded by saying, "the formula, it's pattern number x 3 plus 1 equals matchsticks," with the coefficient referring to the common difference and the y-intercept as an adjustment value that he saw as necessary in order to match the dependent terms. When he was then asked to justify his formula, he provided the following faulty reasoning below in which he projected his formula onto the figures in a rather inconsistent manner (see Fig. 2 for an illustrative version).

> For 1 [square], you times it by 3, it's 1, 2, 3 [referring to three sides of the square] plus 1 [referring to the left vertical side of the square]. For pattern 2, you count the outside sticks

Fig. 2 Dung's justification of his direct formula for the Fig. 1 pattern



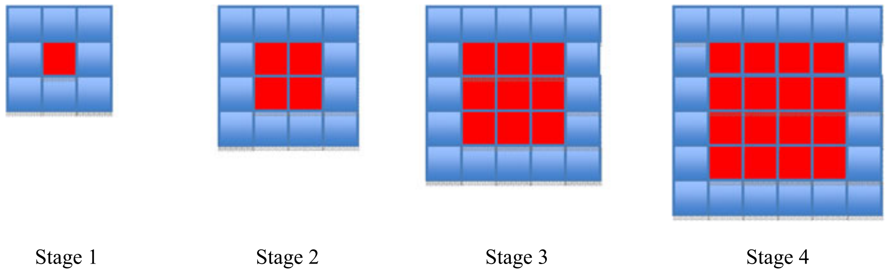


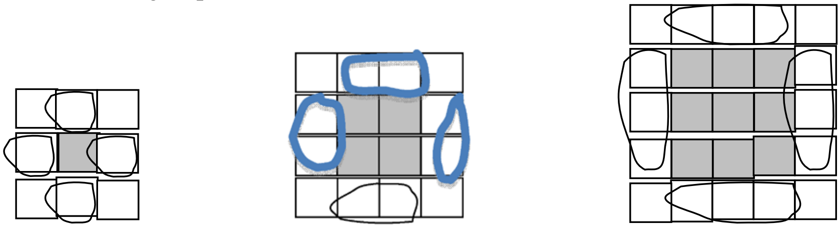
Fig. 3 Tile patio pattern

> and you plus 1 in the middle. For pattern 3, there's one set of 3 [referring to the last three sticks of the third adjacent square], two sets of 3 [referring to the next two adjacent squares] plus 1 [referring to the left vertical side of the first square].

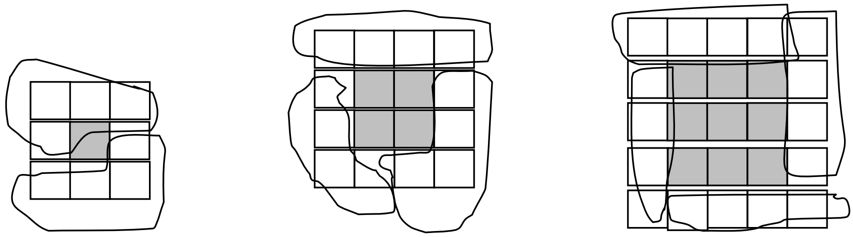
Also, by the end of the Year 2 study, none of Dung's classmates were able to come up with a general form similar to Chuck's. Further, when they were asked to explain an imaginary student's formula, $T = 4n - (n - 1)$, for the pattern in Fig. 1, they found this and other similar tasks difficult.

However, we found it interesting that when the students in Year 3 of the study were purposefully reoriented to a multiplicative thinking approach to patterning activity involving figural stages (i.e., pictorial patterns with known stages such as the one shown in Fig. 1), they finally settled on *figural-based generalizations*. For example, when they obtained a generalization for the *Tile Patio Pattern* in Fig. 3 during a teaching experiment, they developed at least three equivalent direct formulas that reflected the use of multiplicative reasoning (i.e., in relation to what they perceived

Che: "$W = 4n + 4$. So there's like 4 squares and then you add 4 to each one [corner]. ... And for patio #2, 2×4 is 8, so $1, 2, 3, \ldots, 8$, then you add 4. [There are] 4 groups of n."



Dina: "$4(n + 1)$ coz patio #1 you have two groups of 4, patio #2, 3 groups of 4, etc."



Dave: "$T = 2(n + 2) + 2n$. The top part, $2 + 1 = 3$. Then I multiplied by 2, the bottom, so that's 6. And the $2n$, so here's 1 [row 2 column 1 square] and 1 here [row 2 column 3 square], etc."

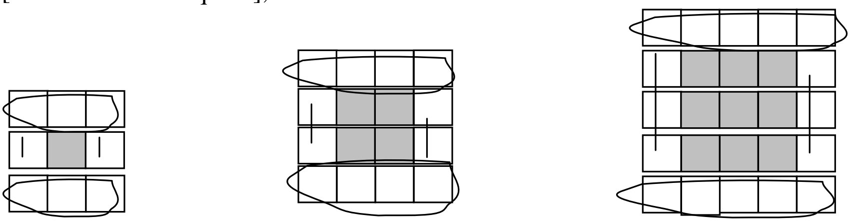


Fig. 4 Visuoalphanumeric generalizations of 8th graders on the Fig. 3 pattern

to be the repeated parts of the pattern). In Fig. 4, the students' constructed direct formulas are examples of figural-based generalizations in which the alphanumeric symbols in the formulas conveyed relationships that they have drawn figurally from the pattern. Such representations are, in fact, effects of particular (i.e., mathematical) ways of seeing and acquired knowledge and experience (Metzger 2006/1936).

In this chapter, we address issues relevant to the following two related questions: What is the content and structure of algebraic generalization that middle school students (i.e., Grades 6 to 8, ages 11 to 14) develop in the case of linear figural patterns? Further, to what extent are they capable of establishing and justifying their algebraic generalizations? By *algebraic generalization of a figural pattern*, we refer to, in Radford's (2008) words, the "[students'] capability of grasping a commonality noticed on some particulars (in a sequence), extending or generalizing this commonality to all subsequent terms, and being able to use the commonality to provide a direct expression of any term of the sequence" (p. 115). The two research questions address various aspects of what we label as *pattern generalization*, which involves constructing and justifying an algebraic generalization within the means available to a learner. Our notion of pattern generalization extends Radford's (2008) view to include justification. Also, we note that the above definition of algebraic generalization shares the basic conceptual intent surrounding all processes relevant to the task of *generalization*, which involves constructing an invariant and stable structure, property, attribute, or relation from particular known cases (or samples or domains) and extending, applying, and projecting it to the unknown cases or larger classes of cases (Dreyfus 1991). But we further refine Dreyfus's (1991) sense above by acknowledging the complex of factors (cognitive, cultural, extra-cultural such as linguistic and classroom practices, etc.) that influence the construction of a "generality" that, according to Dörfler (2008), is a way of practice of using and interpreting "signs, like graphs or letters, are not general by themselves" (p. 1).

In addressing the first research question, we initially survey relevant research in the area of middle school algebraic thinking. We then consider how findings in our three-year longitudinal research at the middle grades further confirm and/or extend the current knowledge base in this area. *Our response to the second research question* is grounded on how our students dealt with factors that influenced the manner in which they obtained their pattern generalizations. Our decision to investigate linear figural patterns has been drawn from our survey of various school mathematics curricula across states that show value and interest in this mathematical topic and its connections to other concepts as well.

Anticipating What Is to Come: Initial Reflections on Our Three-Year Data from the Clinical Interviews

Table 1 provides a summary of the results of the clinical interviews before and after every teaching experiment we conducted over the course of three years with our middle school students beginning at sixth grade. We briefly note the following observations:

- About 63% of the students in the Year 1 study employed figural-based strategies in obtaining a generalization for patterns that were mostly linear in content before a teaching experiment involving pattern generalization. But we also point out a dramatic shift to a numerical strategy (100%) after the teaching experiment in the same year.
- In the Year 2 clinical interviews with eight students, seven students maintained a numerical strategy in obtaining a generalization before and after a teaching experiment.
- In the Year 3 clinical interviews, a shift to figural-based strategies (about 69%) occurred.

Table 1 Summary of pattern generalization

Year 1 Results	Before Teaching Experiment ($n = 29$)	After Teaching Experiment ($n = 11$)
Overall Visual	63%	0%
Overall Numeric	37%	100%
Constructive Standard Generalizations	0%	100%
Constructive Nonstandard Generalizations	0%	0%
Deconstructive Generalizations	0%	0%
Year 2 Results[a]	**Before Teaching Experiment ($n = 8$)**	**After Teaching Experiment ($n = 8$)**
Overall Visual	12%	25%
Overall Numerical	88%	75%
Constructive Standard Increasing Patterns	100%	100%
Constructive Standard Decreasing Patterns	38%	75%
Constructive Nonstandard Generalizations	0%	0%
Deconstructive Generalizations	50%	100%
Year 3[a]	**Before Teaching Experiment ($n = 18$; 5 new[b])**	**After Teaching Experiment ($n = 14$; 3 new[b])**
Overall Visual	67%[c]	71%
Overall Numeric	33%[c]	29%
Constructive Standard Generalizations	100%	100%
Constructive Nonstandard Generalizations	6%	36%
Deconstructive Generalizations	11%	86%

[a]Some tasks had multiple questions

[b]Did not participate in earlier two-year interviews

[c]More visual tasks than numerical

Considering three years of collected data, in this article we extrapolate factors that explain why such shifts in generalization strategies took place among the students over the course of three years. Also, we provide a description of the quality, content, and form of generalizations at each phase. Further, we point out the progress in students' ability to deal with various aspects and types of pattern generalization, which was especially evident in Year 3. Our overall intent in this chapter is to describe teaching-learning conditions that enable meaningful pattern generalization to occur at the middle school level.

Cognitive Issues Surrounding Pattern Generalization: What We Know from Various Theoretical Perspectives and Empirical Studies

Clarifying the Definition of Pattern Generalization

Several researchers have pointed out that the initial stage in generalization involves "focusing" or "drawing attention" on a candidate invariant property or relationship (Lobato et al. 2003), "grasping" a commonality or regularity (Radford 2006), and "noticing" or "becoming aware" of one's own actions in relation to the phenomenon undergoing generalization (Mason et al. 2005). Lee (1996) poignantly surfaces the central role of "perceptual agility" in patterning and generalization, which involves "see[ing] several patterns and [a] willing[ness] to abandon those that do not prove useful [i.e., those that do not lead to a formula]" (p. 95). Mason et al. (2005) points out as well how *specializing* on a particular case in a pattern on the route to a generalization necessitates acts of "paying close attention" to details, especially those aspects that change and/or stay the same, best summarized in Mason's (1996) well-cited felicitous phrase of "seeing the general through the particular." Results of our earlier work with 9th graders (Becker and Rivera 2005) and undergraduate majors (Rivera and Becker 2003) also confirm such a preparatory act whereby perception—as a "way of coming to know" an object or something property or fact about the object (Dretske 1990)—is necessary and fundamental in generalization. Of course, there are other researchers who emphasize the fundamental, genetic role of invariant acting in the construction of an intentional generalization (Dörfler 1991; Garcia-Cruz and Martinón 1997; Iwasaki and Yamaguchi 1997). In our longitudinal study, which focuses exclusively on the pattern generalization of figural objects, we affirm the above views about the nature of generalization.

Our contribution to the above characterizations deals with the mutually determining relationship between individual and sociocultural activity in the formation of pattern generalization. That is, while we acknowledge the constructivist nature of pattern generalization among individual students (every individual sees what s/he finds meaningful to see that influences how and what s/he constructs), collective action—that is, shared ways of seeing—makes the above characterizations even more meaningful than when performed in isolation.

As we have noted in the introduction, *pattern generalization* refers to both actions of *constructing* an algebraic generalization and *justifying* it on the basis of the students' repertoire of available explanatory mechanisms (Rivera 2010a, 2010b).

Constructing and justifying a generalization are two equally important tasks. In *constructing an algebraic generalization*, we expect closure in mathematical activity via the construction of a direct formula (i.e., a closed formula in function form). In the case of *justification*, in light of the cognitive level of middle school students who have just begun learning domain-specific knowledge and practices in algebra, we are more or less concerned with their capacity to reason, in the sense following Hershkovitz (1998), "to understand, to explain, and to convince" (p. 29). Knuth (2002), for instance, talks about the importance of having students perform a figural demonstration that explains, that is, using the relevant features in a figure in order to provide insights regarding a particular claim. Lannin's (2005) work with 25 US sixth graders had him pointing out how justification seemed to have been relegated to the "realm of geometric proofs" when, in fact, students' justifications in the context of pattern generalization could "provide a window for viewing the degree to which they see the broad nature of their generalizations and their view of what they deem as a socially accepted justification" (p. 232).

Types of Algebraic Generalization Involving Figural Patterns

There are two basic algebraic ways of developing a pattern generalization involving figural patterns. (For an extended list, see Rivera 2010a.) The first way involves what we classify as *constructive generalizations* (CG), which refer to those direct or closed polynomial formulas that learners construct from the known stages in a figural pattern as a result of cognitively perceiving figures that structurally consist of non-overlapping constituent gestalts or parts. For example, in the case of the Fig. 1 pattern, some students may perceive the stages as a sequence of growing squares that are produced by repeatedly adding three sides to form a new square. Dung's formula, "pattern number times 3 plus 1 equals matchsticks" in relation to Fig. 2 exemplifies a CG that exhibits the *standard* linear form $y = mx + b$, hence, it is a CSG. Chuck's direct expression for the Fig. 1 pattern, "$4 + (n - 1)3$," on the other hand, is an example of a constructive *nonstandard* generalization (CNG) since the terms in his expression still need further simplification.

The second way of developing a pattern generalization involves what we classify as *deconstructive generalizations* (DG), which refer to those direct or closed polynomial formulas that learners construct from the known stages as a result of cognitively perceiving figures that structurally consist of overlapping constituent gestalts or parts. Consequently, the corresponding general formulas involve a combined addition-subtraction process of separately counting each sub-configuration and taking away parts (sides or vertices) that overlap. For example, some students may initially infer the appropriate number of squares at each stage in the Fig. 1 pattern (i.e., stage 1 has one square, stage 2 has two squares that are adjacent to each other, stage 3 has three adjacent squares, ...) and then multiply that number by 4 (since there are four sides to a square) and subtract the appropriate number of overlapping sides (i.e., stage 2 has two groups of 4 sides with an overlapping "interior" side, stage 3 has three groups of 4 sides with two overlapping "interior" sides, ...). In a DG, further actions of deconstructing and decomposing are necessary in order to reveal the overlapping part(s).

In pattern generalization involving figural stages, we note that because there are many different ways of expressing a generalization for the same pattern, we foreground Duval's (2006) view about the cognitively complex requirements of semiotic representations—that is, a primary resolve is to assist learners to recognize the viability and equivalence of several generalizations that are drawn from several "semiotic representations that are produced within different representation systems" (p. 108). For example, Dung obtained his general formula for the Fig. 1 pattern by initially manipulating the corresponding numerical stages that he later justified figurally (Fig. 2), while Chuck established his formula for the same Fig. 1 pattern from the available figural stages. Both learners operated under two different representational systems and, thus, produced two different, but equivalent, direct expressions for the same pattern.

Methodology

This section is divided into five sections. The first section provides information about the middle school participants involved in the three-year study. The next two sections provide details of the teaching experiments on pattern generalization. The fourth section provides samples of the tasks used in the clinical interviews. The fifth section deals with matters involving data collection and analysis and relevant study protocols.

Classroom Contexts from Years 1 to 3 of the Study

In Fall 2005 and Fall 2006 (i.e., Years 1 and 2 of the study), the first author collaborated with two middle school mathematics teachers in developing and implementing two related design-driven teaching experiments on pattern generalization. From Fall 2007 to Spring 2008 (i.e., Year 3 of the study), the first author taught the class the whole academic year. The second author conducted the pre- and post-clinical interviews with the participating students in all three years of the study. Learnings from the pre-interviews were incorporated in the evolving teaching experiments with the participants, and the post-interviews were meant to assess students' abilities to establish and justify their generalizations, including the extent of influence of classroom practices in their developing capacity to generalize. In the Year 1 study, the sixth-grade class consisted of twenty-nine students (12 males, 17 females; mean age of 11; most of Southeast Asian origins). In the Year 2 study, three students moved to different schools and were replaced with six new students. In the Year 3 study, only fifteen students from the earlier two-year project were allowed to complete the project. They were then mixed with a new cohort of nineteen 7th and 8th grade students (22 females, 12 males) that together comprised an Algebra 1 class.

Nature and Content of Classroom Teaching Experiments in Years 1 and 2

A basic instructional objective of the Years 1 and 2 classroom teaching experiments on pattern generalization involves providing students with every opportunity to engage in problem-solving situations that would enable them to meaningfully acquire the formal mathematical requirements of algebraic generalization. The instructional theory that was initially used in Years 1 and 2 was Realistic Mathematics Education (RME). In RME, learners use models of their informal mathematical processes to assist them in developing models for more formal processes. Formalizing is, thus, seen as "growing out of their mathematical activity" and mathematizing, more generally, involves "expanding [their] common sense" with the same reality as "experiencing" in everyday life (Gravemeijer and Doorman 1999, p. 127).

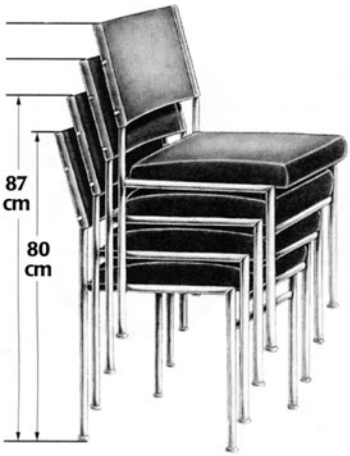
In the Year 1 teaching experiment on pattern generalization, two algebra units in the Mathematics-in-Context (MiC) curriculum were used. Also, taking note of the algebra requirements of the California state standards for sixth graders, sections were selected from the units *Expressions and Formulas* (MiC Team 2006b) and *Building Formulas* (MiC Team 2006a) that became the basis of a three-phase classroom teaching experiment on pattern generalization. In the first two phases, activities drawn from the two algebra units were used to foster the development of algebraic generalization through a series of horizontal and vertical mathematization tasks. Horizontal mathematization involves transforming real and experientially real problems into mathematical ones by using strategies such as schematizing, discovering relations and patterns, and symbolizing, while vertical mathematization involves reorganizing mathematical ideas using different analytic tools such as generalizing or refining of an existing model (Treffers 1987). In both units, the students initially explored horizontal activities that allowed them to build an informal mathematical model. They then engaged in vertical activities.

In the *Expressions and Formulas* unit, each section had the students starting out with a problem situation that involved using an arrow language notation to initially organize the situation and later to express relationships between two relevant quantities. An example is shown in Fig. 5. The arrow notation was meant to articulate the different numerical actions and operations that were needed to carry out a string of calculations in an activity. Also, the task situations were either stated in words or accompanied by tables, and they contained items that necessitated either a straightforward or a reverse calculation.

The *Patterns* section in the *Building Formulas* unit was the only one that was used in the teaching experiment because of constraints in the stipulated sixth-grade algebra requirements of the state's official mathematics framework. In this section, arrow language was employed less in favor of recursive formulas and direct formulas in closed, functional form. The students dealt with problem situations that consistently contained the following tasks relevant to generalizing: extending a near generalization problem physically (for example: drawing or demonstrating with the use of available manipulatives) and/or mentally (reasoning about

Stacking Chairs

The picture below shows a stack of chairs. Notice that the height of one chair is 80 centimeters, and a stack of two chairs is 87 centimeters high.



Damian suggests that the following formula can be used to find the height of a stack of these chairs:

number of chairs $\xrightarrow{-1}$ ___ $\xrightarrow{\times 7}$ ___ $\xrightarrow{+80}$ height

22. Explain what each of the numbers in the formula represents.

23. Alba thinks that a formula like this would do just as well:

number of chairs $\xrightarrow{\times}$ ___ $\xrightarrow{+}$ height

a. What numbers would Alba use in her formula? Explain how you determined these numbers.

b. Alba thought about making a formula like this:

number of chairs $\xrightarrow{+}$ ___ $\xrightarrow{\times}$ height

Will this work? Why or why not?

24. These chairs are used in an auditorium and sometimes have to be stored underneath the stage. The storage space is 116 centimeters high.

a. How many chairs can be put in a stack that will fit in the storage space?

b. Describe your calculation with an arrow string.

25. For another style of chair, there is a different formula:

number of chairs $\xrightarrow{\times 11}$ ___ $\xrightarrow{+54}$ height

a. How are these chairs different from the first ones?

b. If the storage space were 150 centimeters high, would the following arrow string give the number of chairs that would fit? Why or why not?

150 $\xrightarrow{+11}$ ___ $\xrightarrow{-54}$ ___

Fig. 5 Arrow notation activity (MiC 2006b, p. 21)

it logically); calculating a far generalization task (i.e., finding a total number beyond stage 10) using either a figural or a numerical strategy; developing a general formula recursively and/or in closed, functional form, and; solving problems that involve inverse or reverse operations. In all problem situations, tables were presented and employed as an alternative representation for organizing the given data.

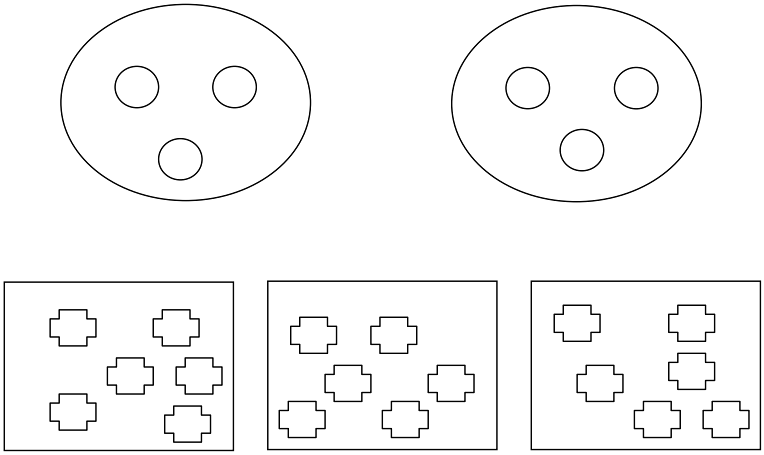


Fig. 6 Two multiplication tasks

Finally, the students dealt with tasks that asked them to reason and to make judgments about the equivalence of several different formulas for the same problem situation. In the third phase of the teaching experiment, they worked through several decontextualized patterning problems whose basic structure was similar to the ones that have been described in the paragraph above (see, for e.g., Figs. 1 and 3). Also, they explored problems that enabled them to develop both numerical and figural generalization.

In the Year 2 teaching experiment on pattern generalization, the same three-phase process occurred. The seventh-grade class used *Building Formulas* and portions of *Patterns and Figures* (MiC 2006a) in the first two phases with the third phase the same as in the description provided above.

Nature and Content of Classroom Teaching Experiments in Year 3

The Year 3 study on pattern generalization took place in three phases. The initial phase of learning pattern generalization focused on helping students develop a better understanding of multiplicative thinking, which actually was the unifying thread that connected all the algebra concepts and processes that were learned throughout the year. The students initially investigated counting activities that emphasized multiplicative thinking. For example, the activity in Fig. 6 asked the students to establish a mathematical expression involving multiplication. In the second phase, they explored pattern generalization activities that involve developing a structural analysis of a pattern (i.e., in terms of what stays the same and what changes in the pattern). In the third phase, they connected multiplicative thinking and structural analysis.

Nature and Content of Clinical Interview Tasks from Years 1 to 3

In the *Year 1 clinical interview prior to the teaching experiment* on pattern generalization, the students were given five tasks that addressed various aspects of pattern generalization. The task shown in Fig. 7a asked the students to determine a *near* and a *far* generalization item, with far items as arbitrarily referring to figural stages 10 and above in a given pattern and near items as pertaining to stages 9 and below. All five tasks required students to calculate a near and a far item. Three of the five tasks had figural patterns that show at least four consecutive initial stages. One task was presented numerically using a table of values. The fifth task began with an intermediate figural stage in some pattern and the students were asked to reconstruct a set of stages prior to the figural stage and then to use that knowledge to extend the pattern and deal with far items. In the *Year 1 clinical interview after the teaching experiment* on pattern generalization, analogous tasks were presented with some changes in the questions. For example, the task shown in Fig. 7b uses the same task structure in Fig. 7a but there is an increased emphasis in the following aspects of direct-formula construction: justification; the use of numerical or figural strategies; assessing for equivalence.

In the *Year 2 clinical interviews before and after the teaching experiment* on pattern generalization, tasks similar to Fig. 7b were presented to the students with the inclusion of two decreasing patterns.

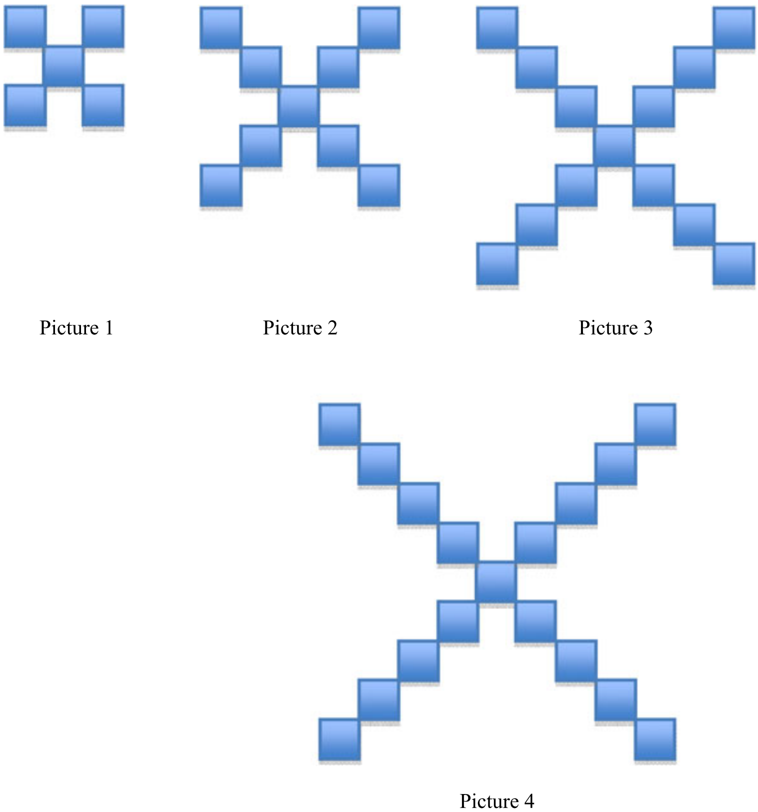
In the *Year 3 clinical interviews before and after the teaching experiment* on pattern generalization, the students were asked to justify a given direct formula of a given figural pattern (increasing and decreasing; see, for e.g., Fig. 8a) and to obtain several equivalent pattern generalizations for the same pattern (see, for e.g. Fig. 8b). A *semi-free construction* task was also added (see Fig. 9) in response to Dörfler's (2008) "plea for 'free' generalization tasks" (p. 153). Dörfler notes that patterns with well-defined stages impress on learners the view that "there is an expected direction of generalizing," which would then "intimate one and only one way [of continuing] a figural sequence" and, consequently, harbor "a strong regulating or even restrictive impact" on their thinking (p. 153). He recommends a different approach by asking students to think about (figural) patterns, as follows:

> How otherwise can one ask for, say, the number of matchsticks ... in an "arbitrary" item of the sequence? The situation would presumably be much more open if one asked simply "How can you continue?" or "What can you change and vary in the given figures?" ... I rather want to hint to possible further directions for research ... a plea for "free" generalization tasks not restricted by pre-given purposes. (Dörfler 2008, p. 153)

Data Collection and Analysis and Relevant Study Protocols

Each project year, we collected the following data: students' written work on various homework, classroom, and performance assessments involving pattern generalization; videos and transcripts of clinical interviews before and after every teaching

Tiles are arranged to form pictures like the ones below.



How many tiles does Picture 10 have? How do you know?
How many tiles does Picture 100 have? How do you know?
Find a formula to calculate the number of tiles in Picture "n." How did you obtain your formula?

Fig. 7a A sample Year 1 clinical interview task prior to a teaching experiment

experiment, and; videos and transcripts of relevant classroom episodes taken during a teaching experiment.

With respect to the analysis of data drawn from the clinical interviews, we engaged in several repeated processes of individual and shared reading within and across cases. We have carefully described this important step in several published research papers (Becker and Rivera 2005, 2006, 2007, 2008; Rivera and Becker 2008). Basically, individual cases were analyzed, developed, and later synthesized in order to construct individual cognitive maps with the aim of schematically capturing their generalizing schemes from problem to problem. Next, those individual cases were compared, analyzed, and categorized using grounded theory that enabled us to develop some empirical claims about aspects of their pattern generalizing ac-

Tiles are arranged to form pictures like the ones below.

See pattern in Fig. 7a

A. Find a direct formula that enables you to calculate the number of square tiles in Picture "n." How did you obtain your formula? If the solution has been obtained numerically, respond to the following question: Is there a way to explain your formula from the figures?

B. How many square tiles will there be in Picture 75? Explain.

C. Can you think of another way of finding a direct formula?

D. Two 6^{th} graders came up with the following two formulas:

Kevin's direct formula is: $T = (n \times 2) + (n \times 2) + 1$, where n means Picture number and T means total number of squares. Is his formula correct? Why or why not?

Melanie's direct formula is: $T = (n \times 2) + 1 + (n \times 2) + 1 - 1$, where n and T mean the same thing as in (D) above. Is her formula correct? Why or why not?

Which formula is correct: Kevin's formula, Melanie's formula, or your formula? Explain.

Fig. 7b Analogous Year 1 task given after a teaching experiment on pattern generalization

tivity. The results, findings, and observations we have developed were consequences of several iterated processes of reading and analyzing the within- and across-case studies in order to ensure greater validity.

The first author was also responsible for the analysis of data drawn from the classroom episodes. Relevant transcripts of key classroom episodes were obtained in order to provide additional support about the claim of shifts in pattern generalization practices of the participating students over the course of the longitudinal study.

Undergraduate student assistants videotaped all the classroom and collaborative group sessions involving pattern generalization. Each teaching experiment lasted three consecutive weeks on average. All clinical interviews were also videotaped. During a clinical interview, each student was requested to think aloud and to use the available and relevant manipulatives (pattern blocks, calculators, centimeter graphing paper, etc.) to help them deal with the tasks. In cases when a student incorrectly performed a calculation, the interviewer (second author) sought clarification to better assess the nature of the error. Also, in cases when a student had a difficult time articulating a verbal response, the interviewer sought clarification until both of them felt satisfied with the response.

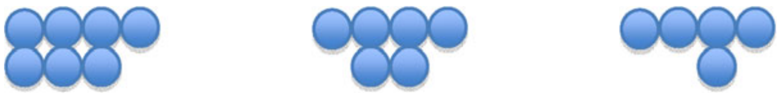
A. Consider the sequence of three figures shown below.

Stage 1 Stage 2 Stage 3

Three 8th grade students have been asked to find the total number of stars (S) at any given stage number (n). Explain how each student might be thinking of his or her formula.

Marcia's formula: $S = n \cdot 3 + 1$
Pete's formula: $S = (n + 1) \cdot 3 - 2$
Jayme's formula: $S = (n \cdot 4 + 1) - n$

B. Consider the sequence of three figures below.



Step 1 Step 2 Step 3

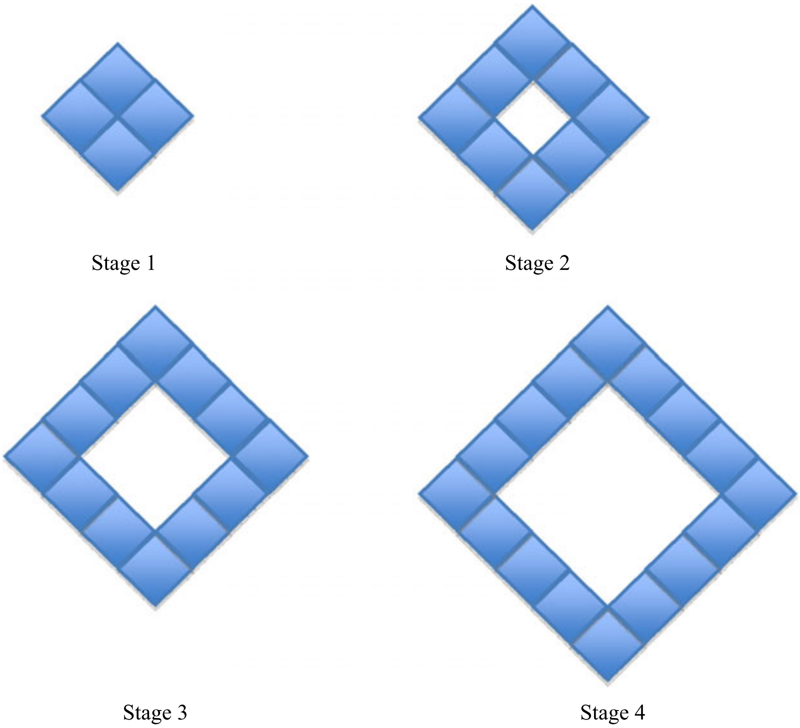
Two seventh grade students came up with two different direct formulas for the total number of circles (C) for any step number (n). How might each student be thinking about his formula?

Jake: $C = -n + 8$
Bharath: $C = 4 + 4 - n$

Fig. 8a Two sample Year 3 tasks given before and after a teaching experiment on pattern generalization (increasing and decreasing figural patterns)

Findings and Discussion Part 1: Accounting for Constructive and Deconstructive Generalizations

Findings in Our Study An analysis of Table 1 shows the predominant use of CSGs from Years 1 to 3. In fact, CNGs were not evident until Year 3. The process used to establish a CSG was predominantly numerical in Years 1 and 2 with a mean of 87.5% but a shift to figural took place in Year 3 at about the same percentage. Also, the generalizations that were developed in the case of all increasing and decreasing linear patterns were CSGs. Further, while the students were successful (100%) in establishing CSGs in the case of increasing linear patterns, they had considerable difficulty in the case of decreasing linear patterns (a success rate of 38% before a teaching experiment and 75% after). In the case of DGs, while none of the students could construct them by the end of the Year 2 study, they, however, Consider the sequence of four figures below.



Obtain two different ways (or formulas) that will enable you to find the total number of gray squares (S) at any stage number (n). Then explain why you think each way (or formula) makes sense to you.

1. Formula 1: ______________________
Explanation:

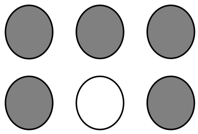
2. Formula 2: ______________________
Explanation:

Fig. 8b A sample Year 3 task given before and after a teaching experiment on pattern generalization

had considerable success in explaining them (from 50% to 100% before and after a teaching experiment, respectively).

Discussion Results drawn from our Years 1 and 2 study actually confirm findings from several research studies at the middle school level that also provide sufficient evidence indicating students' predisposition toward producing more CSGs than DGs. For example, when Taplin and Robertson (1995) asked 40 Australian 7th graders to establish a generalization for the pattern sequence in Fig. 1, while none

The figure below shows five gray circles that enclose a white circle. Call it stage 1.



Stage 1

1. First, find a way to continue the above figure so that you end up with several stages that altogether form a pattern of figures. Draw your figures below.
Your Stage 2:
Your Stage 3:
Your Stage 4:
2. Next, try to find a formula for the pattern of figures you constructed above. If a formula is not possible, describe your pattern in a general way.

Fig. 9 A semi-free construction task given before and after a teaching experiment on pattern generalization in the Year 3 study

of them could state an algebraic generalization, their incipient generalizations took the form of CS verbal statements. Seven students perceived four toothpicks that pertained to the original square in stage 1 and the repeated addition of 3 toothpicks each time from stage to stage. There were eight students who offered the CN verbal generalization, $3(n-1)+4$, although none offered an articulation that was as clear as Chuck's in the introduction above. Only one student began to think about the pattern in a deconstructive way; however, that student was not able to figure out the number of toothpicks that needed to be taken away despite seeing the pattern as consisting of overlapping squares. When the same problem was given to a cohort of four hundred thirty 12- to 15-year old Australian students, findings from English and Warren's (1998) study also showed that, among the less than 40% of students who successfully obtained a generalization, they expressed their generalities on this and other patterning tasks in constructive terms similar to what Taplin and Robertson (1995) found. For example, a student developed the general expression $2x+(x+1)$, where $2x$ refers to the top and bottom row sticks and $(x+1)$ to the column sticks in the Fig. 1 pattern, after seeing two invariant properties within and across stages.

While descriptions of CGs for figural patterns abound, the more important question involves the formation of CSG and CNG, in particular, how does *constructive objectification* come about? *First*, Radford (2003) notes that there are different semiotic means of objectification in relation to pattern stages, that is, possibly different ways in visibly surfacing attributes and properties of, or relationships among, stages with the use of signs and relevant processes or operations. *Second*, Radford (2003, 2006) advances the view that there are at least three layers of algebraic generalization—factual, contextual, and symbolic—based on his three-year longitudinal work with middle and junior high school students. *Third*, purposeful instruction through well-designed classroom teaching experiments could scaffold the development of closed forms of constructive generalizations in middle school children (Lannin et al. 2006; Martino and Maher 1999; Steele and Johanning 2004). In the following paragraphs, we dwell on cognitive-related issues at the entry stage of generality, that is, factual, since both contextual and symbolic layers are marked indications of further essentializing and increasing formality on the basis of the stated factual expressions.

At the pre-symbolic stage of factual generalizing involving increasing linear patterns, students oftentimes start with a recursive relation that is both additive and arithmetical in nature. As a matter of fact, studies done in different settings (for e.g., countries) and in different contexts (prior to formal instruction in algebra, during and/or after a teaching experiment, etc.) with middle school students have asserted the use of recursion as the entry (and, in some cases, the final) stage in factual generalizing (Becker and Rivera 2006; Bishop 2000; Lannin et al. 2006; Orton et al. 1999; Radford 2003; Sasman et al. 1999; Swafford and Langrall 2000). For example, in the case of increasing figural sequences, it is usually easy to first perceive the dependent terms as constantly being increased by a common difference. As soon as this takes place, students' thinking is then characterized in two ways, as follows:

- *First*, they see two consecutive stages as being different and, using the method of "differencing" (Orton and Orton 1999, p. 107), the same number of objects is constantly being added from one stage to the next, leading to a recursive, arithmetical generalization (of the type $u_n = u_{n-1} + c$, where c is the common difference). Then, some students further develop emergent factual generalizations from the arithmetical generalization. Two possible factual generalizations involving the Fig. 1 pattern are as follows: $4 + 3 + 3 + 3 + \cdots$; $1 + 3 + 3 + 3 + \cdots$.
- *Second*, a structural similarity is observed among and, thus, connects two or more stages in a relational way. Especially in the case of increasing linear patterns that figurally demonstrate growth, constructing a succeeding stage from a preceding one oftentimes involves a straightforward process of simply adding a constant number of objects on particular locations of the preceding stage. That is, the basic structure of the unit figure is perceived to stay the same despite the fact that equal amounts of objects are conjoined in various parts of the figure in a particular, predictable manner. Such method of construction does not necessitate making a figural change (in Duval's 1998 sense) on the part of the learner.

Radford (2003) further notes how in the factual stage of generalizing, invariant acting from one stage to the next operates at the concrete level that eventually leads to the abstraction of a numerical or operational scheme for the figural pattern. Hence, generalizations that have been mediated by such actions tend to be consequentially constructive and almost always standard (whether rhetorical, syncopated, or symbolic in form).

Even with pattern generalization tasks that require middle school students to first specialize (in Mason's 1996 sense) on the route to establishing a generality as a consequence of not being provided with the usual consecutive sequence of figural stages, many of them were predisposed to establishing constructive generalizations.

For example, Swafford and Langrall (2000) asked ten middle- to high-math achieving 6^{th} grade students to solve the Fig. 3 pattern prior to a formal course in algebra. In their case, the task began with a drawn 10 by 10 square grid in which the four borders of the grid are shaded. The students were asked to figure out the total number of squares on the border, and the task was repeated in a 5 by 5 grid. The students were then asked to describe how to determine the total number of squares in the border of an N by N grid. Results on this task show that, while none of the students offered a recursive rule, the general verbal descriptions ranged in form from the constructive to the deconstructive. When translated in symbolic form, two of the verbal generalizations were CNGs and obeyed the following forms: (1) $n + n + (n-2) + (n-2)$; (2) $n + (n-1) + (n-1) + (n-2)$. Only one student in their study offered a verbal DG that followed the form $4n - 4$. When the above task and other similar ones were given to eight 7^{th} grade students in Steele and Johanning's (2004) study in the context of a problem-solving enriched teaching experiment, only three students came up with DGs.

Findings and Discussion Part 2: Understanding the Operations Needed in Developing a Pattern Generalization

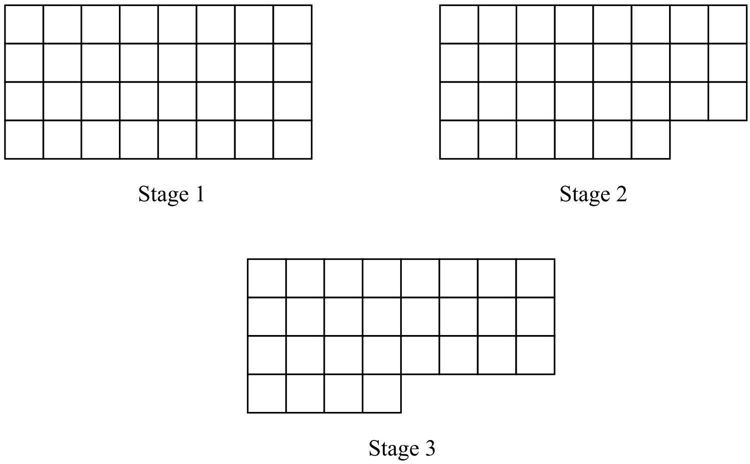
Findings in Our Study This section addresses issues our middle school participants had relative to developing DGs and CGs involving decreasing linear patterns. Due to space constraints, we illustrate in this section students' difficulties with decreasing linear patterns. Decreasing linear patterns could be expressed as CSGs in the form $y = mx + b$, where $m < 0$. In Year 2, the students' primary cognitive difficulty with decreasing patterns prior to a teaching experiment (with a success rate of 38%) was how to handle *negative differencing* and, especially, how to perform operations involving negative and positive integers. While we found that they were attempting to transfer the existing generalization process they developed in the case of increasing linear patterns, they could not, however, make sense of the negative integers and the relevant operations that were used with such types of numbers.

For example, in a clinical interview prior to a teaching experiment, Tamara easily obtained CSGs on two increasing linear patterns. Also, she was able to justify given several CNGs relative to another figural pattern. When she was then presented with the *Losing Squares Pattern* in Fig. 10 as a third task to analyze, she immediately saw that every stage after the first involves "minusing 2" squares. She used multiplication to count the total number of squares at each stage. In obtaining a direct formula, however, she was perturbed by the negative value of the common difference and said,

> I was trying to think of, just like the last time, I was trying to get a formula. ... I was thinking of trying to do with the stage number but I don't get it.

The presence of the negative difference, including the necessity of multiplying two differently signed numbers, partially and significantly hindered her from applying the method she learned in the case of increasing patterns. Tamara's situation

Take a look at the three different stages in the design below.



1. How many squares are there in stage 1? stage 2? stage 3?
2. How many squares are there in stage 10? How do you know for sure?
3. How many squares are there in stage 15? How do you know for sure?
4. Find a direct formula for the total number of squares in stage n, where n is a positive integer.
If you obtained your formula numerically, what might it mean in the context of the above pattern?
5. How many squares are there in stage 20? What might your answer mean in the context of the given pattern?
6. For what stage number will there be no more squares left? How do you know for sure?

Fig. 10 Losing squares pattern

exemplifies the thinking of those students interviewed who were also unsuccessful and, thus, unable to overcome such difficulties before (about 62%) and even after (25%) the teaching experiment in Year 2. Further, her thinking in relation to decreasing patterns after the teaching experiment captures the actions of those students who were also successful by the end of the Year 2 study (about 75%).

Discussion A relevant issue we considered in relation to linear pattern generalization involves the operations that are employed in formulating CGs and DGs. Developing CGs in the case of increasing linear patterns requires students to have solid grounding in addition and multiplication of whole numbers. Developing DGs and CGs in the case of decreasing linear patterns necessitates knowledge relevant to manipulating addition, subtraction, and multiplication of integers (cf: English and Warren 1998; Stacey and MacGregor 2001).

Gelman and colleagues (Gelman 1993; Gelman and Williams 1998; Hartnett and Gelman 1998) have advanced and empirically justified a rational constructivist account of cognitive development among young children that presupposes the existence of innate or core skeletal mental structures (such as arithmetical structures) that enable them to easily develop and process new information as long as it is consistent with their core structures. Hartnett and Gelman (1998) write:

> As long as inputs are consistent with what is known, then novice's active participation in their Learning can facilitate knowledge acquisition. But when available mental structures are not consistent with the inputs meant to foster new Learning, such self-initiated interpretative tendencies can get in the way (p. 342).

Among middle school students who develop CSGs and CNGs in the case of increasing linear patterns, perhaps it is the case that their generalizations, which involve using the operations of addition and multiplication of whole numbers, map easily onto their current understanding of what numbers are and how such entities are used, represented, and manipulated. Thus, constructive generalizing will proceed naturally and smoothly. Moreover, middle school students are likely to associate increasing growth patterns with counting objects over several non-overlapping constituent gestalts and then use the addition and multiplication of counting numbers as useful operators in obtaining a final count. Hence, their core arithmetical structures assist in this developing capacity towards making constructive generalizations. This being the case, it is less likely that students will apprehend increasing patterns as being embedded in a figural process that involves the operation of subtraction via, say, the utilization of a figural change process of seeing sub-configurations and removing overlapping parts as in all cases of DGs.

In the case of decreasing linear patterns, students like Tamara have to first broaden their knowledge of multiplication to include two factors having opposite signs in order to establish, say, the formula $S = -2 \times n + 34$. However, we note that, as with the other students, while Tamara was able to explain the terms in her CSG consistently across increasing linear patterns, she was unable to justify the formulas she established for decreasing linear patterns.

Findings and Discussion Part 3: Factors Affecting Students' Ability to Develop CGs

Even when middle school students are found sufficiently capable of producing more CSGs than DGs and CNGs, we discuss three additional factors that influence their ability to establish the former.

Findings in Our Study In our Year 1 study, 69% of our sixth grade students' initial verbal generalizations (correct and incorrect) in relation to the *T Circle Pattern* in Fig. 11 could not be conveniently translated in closed form. Table 2 provides a list of these verbal responses. The remaining 31% provided verbal generalizations that are considered to be algebraically useful. The responses are listed in Table 3.

For example, when Dina was asked to obtain a generalization for the total number of dots in the Fig. 10 pattern, her circle chip-based stages in Fig. 12 revealed the

Fig. 11 *T* circle pattern

Table 2 Summary of non-algebraically useful verbal responses in relation to the Fig. 11 pattern (20 out of 29 students)

Frequency	Verbal Generalizations
14	Figure 1 has 1 circle, figure 2 has 3 circles, figure 3 has 5 circles, ..., figure 10 has 19 circles. So figure 100 has 190 circles since $10 \times 10 = 100$ and $10 \times 19 = 190$
1	Figure 1 has 1 circle, figure 2 has 3. So you're adding two each time. So figure 100 has $100 \times 2 = 200$ but the numbers are always 1 less than the actual multiple of 2. So figure 100 has 199 circles
1	Since the pattern is always adding 2, it's the same thing as multiplying by 2. So figure 100 should have $100 \times 2 = 200$ circles
1	Always keep adding 2
1	There is a pattern in the units digit. If figure 10 has 19 circles, then figure 20 has 29 circles, figure 30 has 39 circles, etc.
1	There is a pattern in the units digit. If figure 5 has 9 circles and figure 10 has 19 circles, then figure 15 should have 29 circles, figure 20 has 39 circles, etc.
1	Since figure 1 has 1 circle, then figure 5 has 5 circles. Since figure 2 has 2 row circles and 1 column circle, then figure 6 has 6 row circles and 1 column circle. Since figure 3 has 3 row circles and 2 column circles, then figure 7 has 7 row circles and 2 column circles. Since figure 4 has 4 row circles and 3 column circles, then figure 8 has 8 row circles and 3 column circles, etc.

Table 3 Summary of algebraically useful verbal responses in relation to the Fig. 11 pattern (9 out of 29 students)

Frequency	Verbal Generalizations
7	Since figure 2 has 1 circle in the column and 2 circles in the row, and figure 3 has 2 circles in the column and 3 circles in the row, then figure 10 has 9 circles in the column and 10 circles in the row
2	The pattern keeps adding two by adding a circle on the right side of the row and another circle at the top of the column

extent of her perception of the stages, that is, the stages just kept going up by twos and nothing else. Those who used a figural multiplicative strategy, on the other hand, initially employed analogical reasoning. Employing multiple instead of unit counting, their general statements reflect the invariant structure they thought was evident from stage to stage.

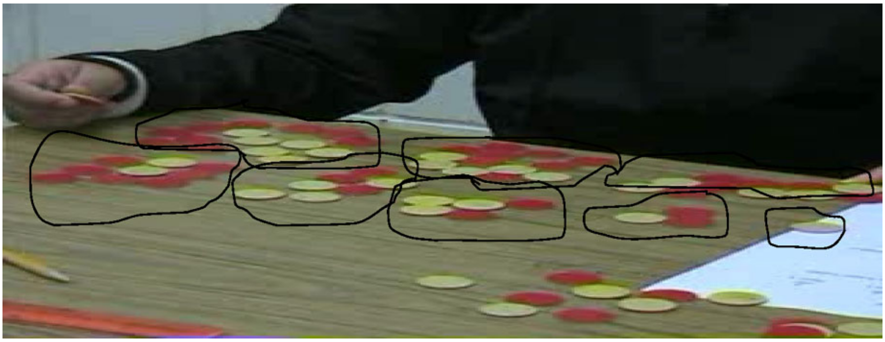
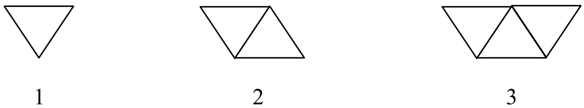


Fig. 12 Dina's interpretation of the Fig. 10 T circles pattern using colored chips

Fig. 13 The triangular toothpick pattern



Discussion Language and the use of variables and analogies are all important factors in direct-formula construction involving all CGs and DGs. Concerning language, Stacey and MacGregor (2001) point out the importance and necessity of the "verbal description phase" in the "process of recognizing a function and expressing it algebraically" (p. 150). Also, based on results drawn from Year 7 to Year 10 (ages 12 to 15) Australian students and their reflections on a national recommendation for a pattern-based approach to algebra, MacGregor and Stacey (MacGregor and Stacey 1992; Stacey and MacGregor, 2001) surface students' difficulties in "Transition[ing] from a verbal expression to an algebra rule" since "students with poor English skills" are oftentimes unable to "construct a coherent verbal description" and many of their "verbal description[s] cannot be [conveniently and logically] translated directly to algebra" (MacGregor and Stacey 1992, pp. 369–370).

Concerning variables, Radford (2006) points out the problematic status of variable use in students' expressions of generality. In his layers of algebraic generalization, the presence and use of variables in their proper form and meaning signal the accomplishment of the final stage of symbolic generalization. He notes that while some students may display knowledge of using algebraic language to express a CG, the variables used in such contexts have to reach their objective state of being desubjectified and disembodied placeholders. Radford's (2001) characterization of algebraic language at the layer of symbolic generalizing is best exemplified in the thinking of two small groups of 8th graders on the *Triangular Toothpick Pattern* in Fig. 13 who obtained the generalities $(n+n)+1$ and $(n+1)+n$ and perceived them as being different on the basis of having been derived from two different actions. Radford (2001) astutely points out that the use of variables to convey a generality has to evolve. In particular, when students employ a variable in relation to the independent term of the general expression, they need to eventually see that the variable has to shift meaning from being a "dynamic general descriptor of the figures in [a] pattern" to being "a generic number in a mathematical formula" (Radford 2000, p. 255). Thus, their general algebraic language in expressions of generality involves semantically transposing the independent variable from its ordinal character (indexical, positional, deictically-based) to the cardinal (as a "number capable of being arithmetically operated" (ibid.)).

Concerning analogies, since all linear patterns could take the CSG formula $y = mx + b$, perceiving and using analogies can quickly facilitate the generalizing process. While middle school students are likely to offer a constructive recursive expression, some have been documented to be capable of developing constructive analogical expressions in varying formats even prior to a formal study of algebra and algebraic notation (Becker and Rivera 2006; Bishop 2000; Lannin 2005; Stacey 1989; Swafford and Langrall 2000). Performing analogy involves "perceiv[ing] and operat[ing] on the basis of corresponding structural similarity in objects whose surface features are not necessarily similar" (Richland et al. 2004, pp. 37–38).

In our Year 1 study, we identified a possible source of difficulty among the sixth grade students in relation to constructing algebraically useful analogies for particular figural-based patterns. We distinguished between students who perceived and generalized additively from those who employed a multiplicative approach. Those students who used a figural additive strategy, on the one hand, were not thinking in analogical terms at all, and they frequently employed counting objects one by one from stage to stage. Further, when some of them were provided with manipulatives to copy figural stages that had been drawn on paper, their manipulative-constructed stages did not preserve the structure of individual stages like Dina in relation to Fig. 12 above; they, in fact, used the available manipulatives only as counters.

Findings and Discussion Part 4: A Three-Year Account of Classroom Mathematical Practices that Encouraged the Formation of Generalization Among Our Middle School Students

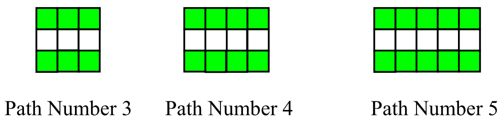
Findings in Our Study In this section, we describe how our middle school participants established six classroom mathematical practices on pattern generalization over the course of three teaching experiments that occurred over three consecutive years. We note that very few studies at the middle school level have focused on the manner in which students develop pattern generalizations over some extended timeframe. Thus, in this section, we aim to highlight how certain legitimate mathematical practices could be viewed not as conceptual, received objects that learners simply acquire rather unproblematically but as part of their individual and sociocultural developmental transformations drawn from and embodied in their activity with other learners.

Year 1 Classroom Practices: From Figurally- to Numerically-Driven CSGs

In the Year 1 study, four pattern generalization practices were constructed and became taken-as-shared in collaborative activity. Two of the practices had their origins in the first MiC unit they used in class (i.e., *Expressions and Formulas*). *First*, the students initially employed arrow strings as a method for organizing a sequence of arithmetical operations (see, for e.g., Fig. 5). They also explored the notion of equivalence through arrow strings that could either be shortened or lengthened depending on the nature of the numbers being manipulated. *Second,* the use of the arrow strings evolved as the students were asked to deal with more complicated problem situations that were still arithmetical in context. In several more sessions, they developed a connection between constructing an arrow string and a formula in such a way that they used arrow strings as a means of describing invariant operational schemes in the context of generalizing situations. In transitioning from the arrow strings to formulas, the students developed an understanding that a formula, like the arrow strings, consists of a starting number or input, a rule in the form of a sequence of operations, and an output value or expression.

Two additional practices emerged when the students began to generalize figural-based patterns that have been initially drawn from the *Patterns* section in the MiC unit *Building Formulas*. The *third* classroom practice that became taken-as-shared involves generalizing figurally and is exemplified in the classroom episode below in which the students were engaged in developing a formula for the total number of grey and white tiles for new path number n whose figural stages are shown in Figs. 14a and 14b. Initially, the students explored specific instances when $n = 3$ to 5, 9, 15, 30, and 100. In particular, they were not merely asked to obtain the output values but also to describe the patterns without actually drawing them explicitly. The class then generated a recursive rule for each tile type. In the episode below, the discussion that took place between the first author and the class shifted from the use of recursive rules to the construction of general expressions in relation to the new path patterns.

Fig. 14a Urvashi's tile patterns (MiC Team 2006a, p. 2)



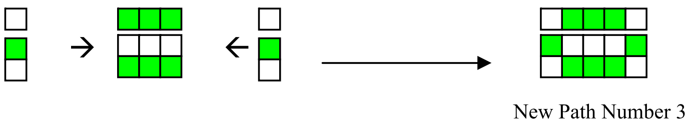
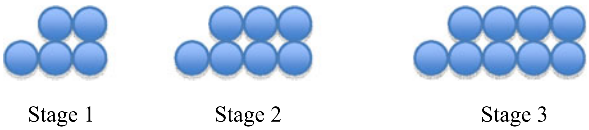


Fig. 14b Urvashi's design for new path 3 (MiC Team, 2006a, p. 3)

FDR: *Suppose I want you to describe new path 1,025. That's a big number. I want you to figure out the total number of white and grey tiles for new path 1,025. Emily, how do we do this?*
Emily: *The whites will be 2,054?*
Ford: *That's the grey.*
Emily: *It is?*
Ford: *Yeah, the white's the middle.*
Emily: *1,029.*
FDR: *Why 1,029?*
Emily: *Because it's in the middle and in the corners it has four.*
FDR: *Alright. What about the grey ones? Mark.*
Mark: *The grey ones are 2,052.*
FDR: *Why 2,052?*
Mark: *Because you added the top and the bottom and then you add the two middle.*
FDR: *Okay, this will be a challenge for some of you. Can you find a formula for me? Suppose, I say, I'm going to use a variable, new path number n. n could mean 1, 2, 3, 4, all the way to 1025. All the way to a billion.*
Dung: *n plus 4 equals white.*
FDR: *Why* $n + 4$ *equals white?*
Dung: *Coz n is the number of whites in the middle plus 4 whites on the sides.*
FDR: *Does that make sense? [Students nodded in agreement.] What about the grey ones? The grey ones are a bit more difficult. What's a formula for the number of grey ones?*
Che: *n times 2 and then you plus 2.*
FDR: *It's* $n \times 2 + 2$. *What about if I express it as n plus?*
Deb: *n plus n plus 2.*
FDR: $n + n + 2$. *Are they the same?*
Jack: *Yes.*
FDR: *Why?*
Nora: *You have two grey ones.*
FDR: *Yes, you have the two gray ones plus the two on both sides. So now if I know these formulas here, can I figure out new path number 50,000?*
Students: *Yeah.*
FDR: *So how do we do this, using the formula here. Number of whites. n plus 4 for whites. What do we do?*
Tamara: *It's 50,004.*
FDR: *What about the grey ones?*
Mark: *100,002.*

One indication of the students' individual appropriation of learnings from the above social event involves their work on succeeding figural patterns. The formulas they established were all CSGs that they oftentimes justified in figural terms. From the above discussion, the students acquired an understanding of using figural generalizing in explicitly articulating structural similarities among the available pattern

Fig. 15 Two layer circles pattern



stages and, hence, figurally identifying properties or relationships that remained stable and invariant over a sequence of known stages. Further, they learned to express those properties or relationships in algebraic form, including the need to justify the reasonableness and validity of the direct formulas. We classify such formulas as *figural-based representations*.

The *fourth* classroom practice came about when the students tackled the *Two Layer Circles Pattern* (Fig. 15). All the students initially perceived a recursive relation with the constant addition of one circle per layer. Two groups of students offered the figural-based formula $C = (n + 1) + (n + 2)$, where n represents figure number and C stands for the total number of circles, which they established analogically. That is, since Fig. 1 had two and three circle rows, Fig. 2 had three and four circle rows, and so on, then figure n had to have $(n + 1)$ and $(n + 2)$ circle rows. The first author then suggested organizing the two sets of numerical values in the form of a table without making any recommendation that might have encouraged a numerical strategy. The basic purpose in introducing the table in several classroom instances was primarily to foster students' growth in their representational skills, that is, patterns could also be expressed in tabular form. In the classroom episode below, Anna shared her group's thinking with the class which eventually was taken as shared and became the fourth classroom practice, that of generalizing numerically using differencing, which was reflective of an appropriation of a standard institutional numerical strategy.

Anna: *We made up a formula. Like we got the figures until figure 5, and we tried it with other ones. We got* $n \times 2 + 3$*, where n is the figure number and timesed it by 2. So* 5×2 *equals 10, plus 3, that's 13. So for figure 25, it's 53.*

FDR: *I like that formula. So tell me more. So your formula is?*

Anna: $n \times 2 + 3$.

FDR: *So how did you figure this out?*

Anna: *First we were like making the numbers to 25. We kept adding 2 and for figure 25, it was 53.*

FDR: *Wait. So you kept adding all the way to 25?*

Anna: *Yeah.... Then we used our chart. Then finally we figured out that if we timesed by 2 the figures and plus 3, that would give us the answer.*

FDR: *Does that make sense? [Students nodded in agreement.] So what Anna was suggesting was that if you look at the chart here, Anna was suggesting that you multiply the figure number by 2, say, what's* 1×2*?*

Tamara: *2.*

FDR: *2. And then how did you [referring to Anna's group] figure out the 3 here?*

Anna: *Because we also timesed it with figure number 13.*

FDR: *What did you have for figure 13?*
Anna: *That was 29. And then* 13×2 *equals 26 plus 3.*
FDR: *Alright, does that work? So what they were actually doing is this. They noticed that if you look at the table, it's always adding by 2. You see this? [Students nodded.] They were suggesting that if you multiply this number here [referring to the common difference 2 by figure number, say figure number 1, what's* 1×2*?*
Students: *2.*
FDR: *Now what do you need to get to 5? What more do you need to get to 5? [Some students said "3" while others said "4."] Is it 4 or 3?*
Students: *3.*
FDR: *It's 3 more. So what is* 1×2*?*
Students: *2.*
FDR: *Plus 3?*
Students: *5. [The class tested the formula when* $n = 2$, *3, and 25.]*

Year 2 Practice: Continued Use of Numerically-Driven CSGs and a Refinement in the Case of Decreasing Linear Patterns

In the Year 2 study, the students were once again involved in a teaching experiment that focused on linear patterning. While the first author observed that the students, in seventh grade, seemed to have remembered how to generalize patterns figurally (weak) and numerically (strong), results of our clinical interviews with a subgroup of ten students prior to the teaching experiment confirmed this observation.

In the classroom episode below, the students were asked to obtain an algebraic generalization for increasing and decreasing linear patterns in both figural and numerical forms. Emma and her group (with Dave below as a member) have been consistently applying the shared practice of generalizing numerically. However, Emma introduced her process of "zeroing out" in the case of decreasing linear patterns that resulted in a further refinement of the numerical generalizing process.

FDR: *Alright. So I have my* x *and my* y. *[FDR sets up a table of values consisting of the following pairs:* $(1, 17)$, $(2, 14)$, $(3, 11)$, $(4, 8)$, $(5, 5)$, $(6, 2)$.*] So what's the answer to this one?*
Dave: $y = -3x + 20$. *[FDR writes the formula on the board.]*
FDR: *This is always the problem, here [pointing to the constant 20]. Before we figure that out, how did you figure out the* -3*?*
Dave: *The difference between the ys, between the numbers.*
FDR: *So what's happening here [referring to the dependent terms]. Is this increasing by 3 or decreasing by 3?*
Students: *Decreasing by 3.*
FDR: *So if it's decreasing by 3, what's our notation?*
Students: *Negative.*

FDR: *Alright, so negative 3. So this one is clear [referring to the slope]. Look at this. This one I get [the slope]. If you keep doing that [i.e., differencing], it's always true. That's why you have this. The difficult part is this [referring to the constant 20].*

Emma raised her hand and argued as follows:

Emma: *If you did a negative times a positive, it's gonna be a negative. So what I'd do is zero it out.*
FDR: *So what do you mean by zero out?*
Emma: *So like if it's* -3 *times 1, that's* -3 *[referring to the product of the common difference* (-3) *and the first independent term (1)]. ... So I'd zero out by adding 3.*
FDR: *So you try to zero out by adding 3. So, what does that mean?*
Emma: *Coz a* -3 *plus 3 equals 0.*
FDR: *So what's the purpose of zeroing out?*
Emma: *So it's easier to add to 17. Coz if it's 0, all you have to do is add 17.*
FDR: *So you're suggesting if you're adding 3 here, if this is* -3 *plus 3, that goes 0. So what do you do with the plus 3 here?*
Emma: *Just remember it and write it down.*
FDR: *Suppose I remember it, adding 3. So how does that help me?*
Emma: *Then ahm it's easier to add to 17. So just add 17 [to 3 to get 20].*

The class then tried Emma's method in a different example. The first author asked the class to first generate a table of values, and they came up with the following (x, y) pairs: $(1, 10)$, $(2, 8)$, $(3, 6)$, $(4, 4)$, $(5, 2)$. Using Emma's method, one student offered the general formula $y = -2x + 12$, where the constant 12 was obtained after initially adding the common difference and its opposite to get 0 (i.e., $-2 + 2 = 0$) and then adding 2 to the first dependent term to yield the constant value of 12 (i.e., $2 + 10 = 12$). The class then verified that the formula worked in any instance of the sequence. Finally, when the first author asked if there was a limitation to Emma's strategy, Emma quickly pointed out that "it only works for 1" (i.e., when the case of $n = 1$ is known) and that her method would fail when the initial independent term was any other number besides 1. Hence, the *fifth* mathematical practice that became taken-as-shared was generalizing numerically using Emma's zeroing out strategy, which was a further refinement of an institutional practice involving decreasing linear patterns.

Year 3 Practices: A Third Shift Back to Figural-based Generalization and the Consequent Occurrence of CSGs, CNGs, and DGs

Prior to the Year 3 teaching experiment on pattern generalization, the students explored activities involving multiplicative thinking. In the following episode, they obtained a mathematical expression for the two sets of figures shown in Fig. 6.

FDR: *So what mathematical expression corresponds to what you see here [referring to the top set]?*
Francis: *6 circles.*
FDR: *Yes, there are 6 circles but I want a mathematical expression that shows how you got 6.*
David: *Add them one by one.*
FDR: *Yes, you can certainly add them one by one. But are there other ways of getting 6?*
Eric: *2 times 3. [FDR writes the answer on the board.]*
FDR: *So what do you mean by* 2×3*, Eric?*
Eric: *It means 2 threes.*
FDR: *Uhum, 2 threes or we say 2 groups of threes. Does that make sense? [Students nod in agreement.] Okay, so now what expression works with the second item here [referring to the bottom set]?*
Salina: *Three groups of 6.*
FDR: *Uhum, and how do we write that using multiplication?*
Students: *3 times 6.*
FDR: *Times meaning what?*
Salina: *Groups of.*

For homework, the students were given similar figural tasks that required them to construct different mathematical expressions with a focus on articulating multiplicative relationships.

The following day, they worked in pairs to obtain a pattern generalization for the Fig. 3 pattern. Considering the fact that there were old and new participants in the Year 3 study, the first author and the classroom teacher saw to it that every table that had two pairs of students had at least one experienced student who could guide the remaining table members in setting up a direct formula. During the classroom discussion, the students offered three different constructive generalizations (Fig. 4) that the class then assessed for equivalence. In obtaining a pattern generalization, the students first addressed structural issues of what stayed the same and what changed from stage to stage. Then they found a multiplicative expression for those aspects or parts that changed from stage to stage and then added a number corresponding to the remaining parts that stayed the same. This process became the *sixth* mathematical practice that was taken as shared in class.

A further refinement in this figural-based strategy occurred when the students began to inspect a particular stage number in a pattern in terms of multiples of the stage number and then either added or subtracted a number corresponding to the remaining parts. For example, Che in Fig. 3 circled four groups of stage 1, four groups of stage 2, four groups of stage 3, ..., four groups of stage n, which justifies her use of the expression $4n$. Then she saw added 4 corresponding to the four corners, which led her to conclude that her formula made sense.

Findings and Discussion Part 5: Middle School Students' Capability in Justifying CSGs

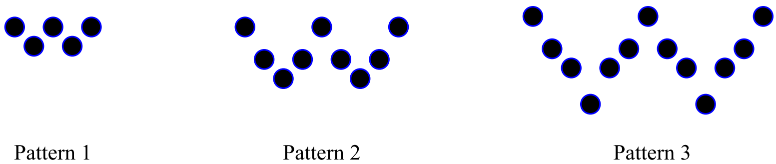
Findings in Our Study Results of the Year 1 teaching experiment indicate differing levels of competence in providing justifications. In particular, based on a follow-up clinical interview with nine students after the teaching experiment on pattern generalization, they justified in several different ways on five linear patterns, as follows:

1. They employed *extension generation* (7%), which involves using more examples to verify the correctness of a formula.
2. Some used a *generic case* (7%), which involves describing a perceived structural similarity in an imagined general instance.
3. Some employed *formula projection* (22%), a figural-based explanation that involves demonstrating the validity of a direct formula as it is seen on the given figural stages.
4. Some used *formula appearance match* (71%), a numerical-based explanation that involves merely fitting the formula onto the corresponding generated table of values that have been initially drawn from the figural stages (Rivera and Becker 2009a, 2009b).

We also note that, in our study, because the students in Year 1 initially developed the emergent practice of figural-based generalizing, they were in fact constructing and validating their direct formulas at the same time. For example, Dung established and justified his direct expression $n + 4$ for the total number of white tiles in Fig. 14b as soon as he saw "the number of white [square tiles] in the middle plus [the] 4 white [tiles] on the sides." Also, Che, Deb, and Nora established and justified their direct expressions, $n \times 2 + 2$, when they perceived "two grey [squares] plus the two squares on both sides [in a given figural stage]." All four students came up with their justifications above after empirically verifying them on several extensions and then either employing formula projection or imagining a generic case that highlights the invariant properties common to all stages. The formula appearance match was used only later after the class developed the emergent practice of generalizing numerically.

When the students in our Year 1 study fully appropriated the above numerical strategies in establishing CSGs, as exemplified in the thinking of Anna and Emma above in relation to the Fig. 14b pattern, we observed a shift that took place from a figural to a numerical mode of generalizing among them. In fact, in both the pre- and post-clinical interviews in the Year 2 study, very few (about 19%) initiated a figural-based approach with most of them developing numerical-based CSGs (about 82%). Consequently, the shift affected their capacity to justify algebraic generalizations correctly on the basis of faulty responses that used either formula projection or formula appearance match. For example, Dung, in two clinical interviews when he was in sixth grade, primarily established and justified his generalizations figurally and oftentimes with the use of a generic example. However, in two clinical interviews when he was in seventh grade, Dung primarily established his generalizations

W-Dot Sequence Problem. Consider the following sequence of W-patterns below.



A. How many dots are there in pattern 6? Explain.
B. How many dots are there in pattern 37? Explain.
C. Find a direct formula for the total number of dots D in pattern n. Explain how you obtained your answer. If you obtained your formula numerically, explain it in terms of the pattern of dots above.
D. Zaccheus's direct formula is as follows: $D = 4(n + 1) - 3$. Is his formula correct? Why or why not? If his formula is correct, how might he be thinking about it? Which formula is correct: your formula or his formula? Explain.
E. A certain W-pattern has 73 dots altogether. Which pattern number is it? Explain.

Fig. 16a W-dot pattern task

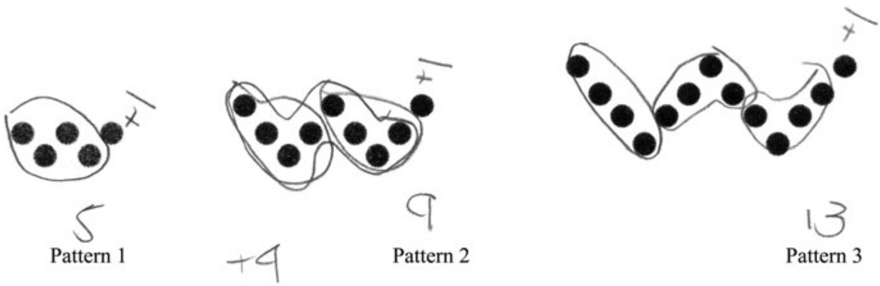


Fig. 16b Anna's figural justification of the W-dot pattern in Fig. 16a

numerically and justified inconsistently using formula projection. An example of a faulty argument that uses formula appearance match is exemplified in the thinking of Anna who first developed the generalization $D = n \times 4 + 1$ numerically for the figural pattern in Fig. 16a. When she was then asked to justify her formula, she circled 1 group of 4 circles, 2 groups of 4 circles, and three groups of 4 circles in patterns 1, 2, and 3, respectively, beginning on the left and then referred to the last circle as the y-intercept (Fig. 16b). As a matter of fact, in the post-interview in Year 2, only three of the eight students saw the sequence in Fig. 16a in the same way Dung perceived it (Fig. 16c).

Discussion The phenomenological shift from the figural to numerical modes in establishing generalizations involving figural linear patterns among our middle school

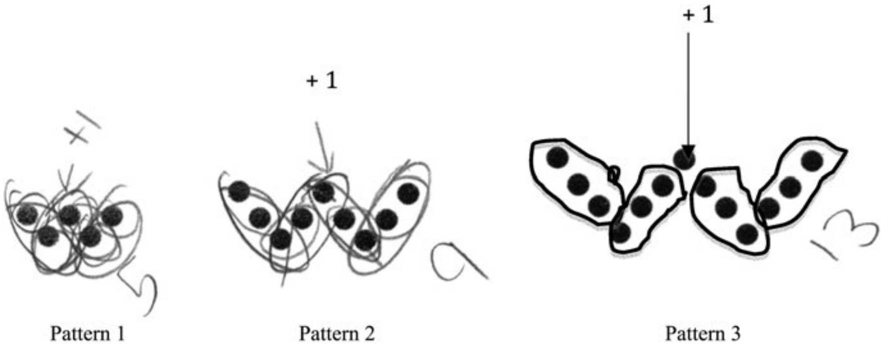


Fig. 16c Dung's figural justification of the pattern in Fig. 16a

students in the first two years of the study is not uncommon in empirical accounts of cognitive development. Induction studies in developmental psychology have demonstrated shifts in students' abilities to categorize (from perceptual to conceptual; from object- or attribute-oriented to relation-oriented, etc.). Also, Davydov (1990) has noted similar occurrences of change on the basis of his work on generalization with Soviet students, including his critique of mathematics instruction that seems to favor one process over the other.

Based on the empirical data we collected in Years 1 and 2, the shift from the figural to the numerical could be explained initially in terms of the predictive and methodical nature of the established numerical strategy (as exemplified by Anna's group thinking relative to the Fig. 15 pattern). That is, the students found them to be compact and easy to use particularly in far generalization tasks that asked them to determine an output value for a large input value. What was difficult with figural strategies, which could be dispensed with the established numerical strategy, was the *cognitive perceptual distancing* that was necessary in order to: (1) figurally apprehend and capture invariance in an algebraically useful manner; (2) selectively attend to aspects of sameness and differences among figural stages and; (3) create a figural schema or a mental image of a consistent generic case and then transform the schema or image into symbolic form. In terms of Radford's (2006) definition of algebraic generalization of a pattern—grasping of a commonality, applying the commonality to all the terms in the pattern, and providing a direct expression for the pattern—the almost, albeit not fully, automatic process of numerical generalizing requires only a surface grasp of a commonality (i.e., a common difference in the case of a linear pattern) that would then be used to set up a direct expression. In particular, when the students surfaced a commonality among stages in a numerical generalizing process involving linear patterns, most of them did not even establish it figurally since the corresponding numerical representation was sufficient for their purpose.

In articulating our argument of a figural-to-numerical shift in mode of generalizing in the first two years of our study, we have already noted how most of them could correctly establish CSGs numerically but had difficulty justifying them. We also discussed how some of them employed formula projection in an inconsistent (faulty) manner. Another significant source of difficulty in justifying CSGs was the students' misconstrual of the multiplicative term in the general form $y = mx + b$ for linear patterns. Toward the end of the Year 1 teaching experiment, they would oftentimes express their algebraic generalization in the form $O = n \times d + a$, where the variable O refers to the total number of objects being dealt with (like matchsticks, circles, squares, etc.), n the pattern number, d the common difference, and a the adjusted value. For example, the general form for the pattern sequence in Fig. 1 is $T = n \times 3 + 1$. The students would then justify their formula by locating n groups of 3 matchsticks respecting invariance along the way. In the Year 2 study, they learned more about the commutative property, which then encouraged them to write all their generalizations in the equivalent form $O = dn + a$. This became a source of confusion among some of them because they interpreted the expressions $n \times d$ and $d \times n$ as referring to the same grouping of objects. For example, in the clinical interviews that we conducted immediately after the Year 2 teaching experiment, some of those who wrote the form $D = 4n + 1$ for the sequence in Fig. 16a justified its validity by looking for 4 groups of, say, 2 circles in pattern 2 when, in fact, they should have been looking for 2 groups of 4 circles. Thus, the algebraic representation proved to be especially confusing among those who established their generalizations numerically because of their misinterpretations involving some of the mathematical concepts and properties relevant to integers (such as the commutative law for multiplication).

The final shift in Year 3, from numerical to figural mode of generalizing, as a matter of fact, settled the above issues. Because the students understood the relationship between multiplicative thinking and grouping relations, they reinterpreted their pattern structural analysis in terms of how grouping could be accomplished so that it is stable and consistent across stages. For example, in Fig. 3, Che noted the four corner squares that stayed the same. She also saw stability in grouping the middle parts on all four sides from stage to stage. Hence, her direct formula, $W = 4n + 4$, captured her figural interpretation of the structure that she saw in Fig. 3.

Findings and Discussion Part 6: Middle School Students' Capability in Constructing and Justifying CNGs and DGs

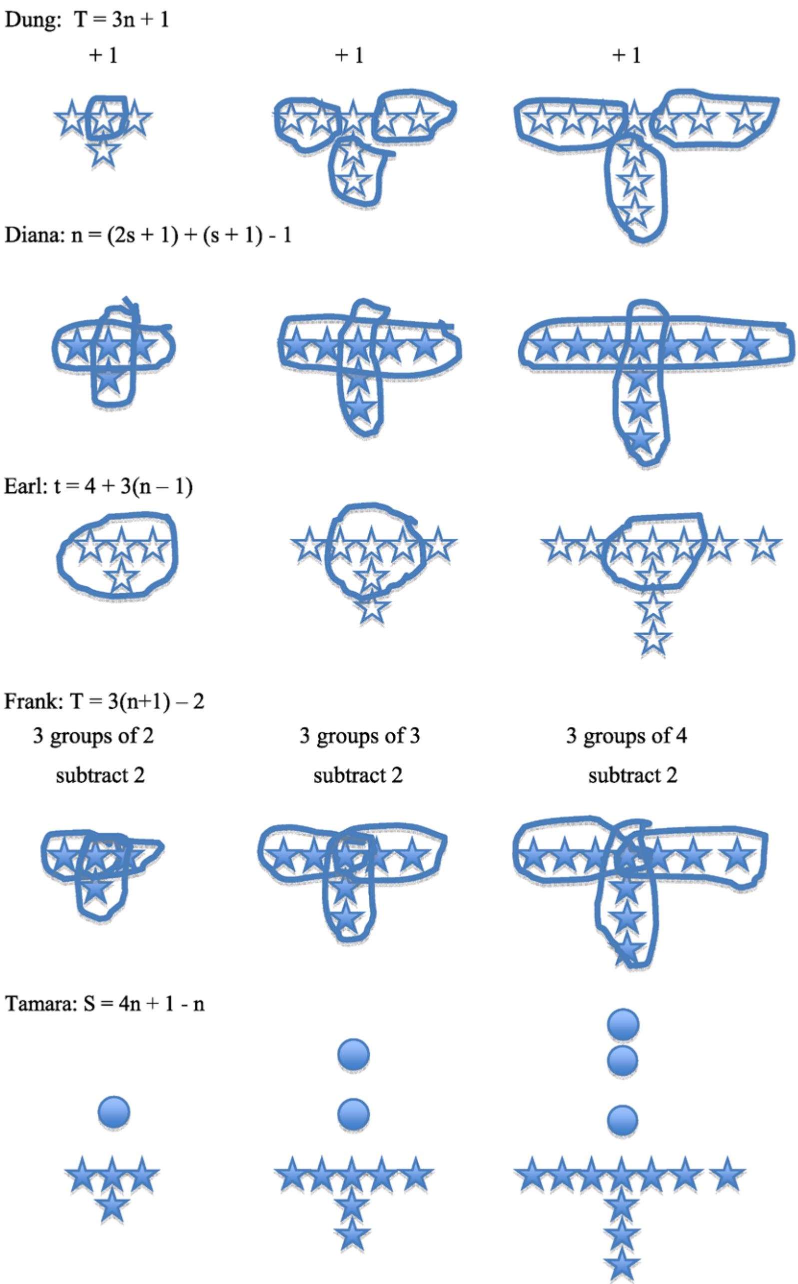
Findings in Our Study Considering the results drawn from our longitudinal work (and, in fact, relevant patterning studies discussed in this paper), we can conclude with sufficient sample that the task of establishing and justifying CNGs and DGs could be both easy and difficult for middle school students. For most students, competence in pattern generalization that leads to a CNG and DG could be considered as an effect of acquired knowledge and experience. We have found that individual and classroom-generated practices on pattern generalization with minimal scaffolding from the teacher, while helpful in many simple cases of linear patterns, appear limited in many respects. In our three-year study, the first two years in which the students were numerically driven to producing CSGs constrained them from obtaining more complex and equivalent generalizations for the same pattern. The numerical method of table differencing assisted in simplifying the process of constructing direct formulas, however, it had a negative effect on the students' ability to justify. We note as well the limited form in which such formulas took shape at least in the case of linear figural patterns, that is, they were oftentimes CSGs.

In Year 3 of the study, when the students acquired knowledge of multiplicative thinking and found ways to link such thinking in patterning activity, the resulting generalizations they produced and justified included CNGs and DGs. Results of the Year 3 clinical interviews with fourteen students after the teaching experiment on patterning and generalization show continued use of CSGs (100%), then DGs (86%), and finally CNGs (36%). Dina and Dave in Fig. 4 constructed and justified two equivalent CNGs for the Fig. 3 pattern. Figure 17 shows the generalizations of five students on the *T Stars Pattern* that ranged in complexity from CSGs to CNGs to DGs.

We discuss briefly the nonlinear pattern generalization of Diana, 7$^{\text{th}}$ grader from Cohort 2, whose pattern of growing segment-triangles is shown in Fig. 19 on a free construction task in Fig. 18 that was given after a teaching experiment on pattern generalization. We should point out that Diana ignored the differences in lengths of the diagonal and horizontal line segments, a fact that applies to a significant number of students who saw segment length as unimportant on this task. Despite that fact, Diana clearly identified an underlying structure in her growing pattern. Hence, assuming all segments are equal, her stage 1 triangle consists of two diagonal segment-sides, a horizontal base that has two segments, and with no interior segments. She then doubled each segment in stage 1 so that in stage 2, each diagonal side has two segments, the horizontal base has four segments, and two interior horizontal segments. She then circled in two colors to distinguish the groupings she was counting, one the interior horizontal segments, and the other, the outer segments on the perimeter of the triangle. In her written description of what to her stayed the same and what changed, she wrote:

> Number 1 [the original triangle] will stay in all of them. The $x(x-1)$ is for the lines in the middle of the triangle. The $+4x$ is for the triangle borders. It's really short for $2x+2(x)$. But it was pretty much the same.

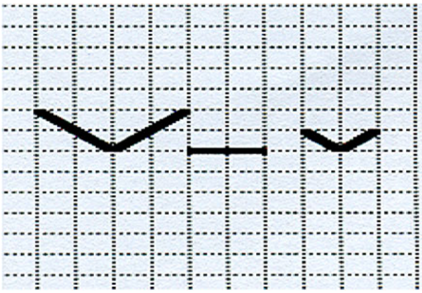
To illustrate, Diana counted the interior horizontal segments of her growing triangle pattern as: 1 group of 0 segment in stage 1; 2 groups of 1 segment in stage 2; 3 groups of 2 segments in stage 3; 4 groups of 3 segments in stage 4; 5 groups of 4 segments in stage 5 *leading to the expression* $n(n-1)$. Then she counted the segments on the perimeter of the growing triangle in two parts. Part A pertains to the two diagonal sides of the growing triangle: 2 groups of 1 segment in stage 1; 2 groups of 2 segments in stage 2; 2 groups of 3 segments in stage 3 *leading to* $2n$. Part B pertains to the base of the growing triangle: 2 groups of 1 segment in stage 1; 2 groups of 2 segments in stage 2; 2 groups of 3 segments in stage 3; 2 groups of 4 segments in stage 4 *leading to* $2n$. Clearly, central to her pattern generalization was her understanding of multiplicative thinking that enabled her to count in ways that corresponded to how she was circling the parts of her figures. Finally, she simplified her pattern of $L=n(n-1)+4n$ to $L=n^2+3n$.



n	1	2	3	4	5
S	4	7	10	13	16

Fig. 17 Year 3 students' work on the *T* stars pattern in Fig. 8a

From the following three figures below, pick at least two figures to create a pattern sequence of five stages. Use the attached grid paper to draw your four additional stages.



1. What stays the same and what changes in your pattern?
2. Obtain a generalization for your pattern either by describing it in words or by constructing a formula. How do you know that your generalization works?

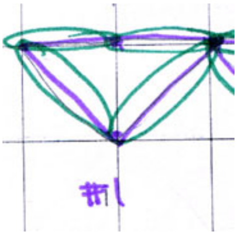
Fig. 18 Semi-free construction task

Discussion Central to the students' success in the Year 3 study was the sociocultural mediation that took place in the context of activities that encouraged them to explicitly engage in multiplicative thinking. When they began to see the significance of multiplicative thinking on matters that involve grouping and invariance in relation to patterning activity, their pattern generalization further progressed in ways that could not be simply done in the case of the numerically driven method of table differencing. We should note, however, that the students interviewed by the end of the Year 2 study were all successful in justifying given DGs. But their success was task-sensitive with some of them providing correct justification in one task and then an incorrect justification in some other task.

For example, results of the two clinical interviews in our Year 2 study separated by a teaching experiment show that almost all the students had more difficulty dealing with the Fig. 16a pattern than the Fig. 1 pattern. Results of the clinical interviews prior to the teaching experiment show only one student correctly justifying a DG in the case of the Fig. 16a pattern and six students in the case of the Fig. 1 pattern. Further, all students interviewed after the teaching experiment were able to justify the DG for the Fig. 1 pattern, but only six students in the case of the Fig. 16a pattern. Thus, it seems that some overlaps in a deconstructive generalization task are easier to see than others. For example, the students above found it easier to see overlaps among the shared adjacent sides of the squares (Fig. 1) than the shared interior vertices in a W-dot formation (Fig. 16a).

In a reported study by Steele and Johanning (2004), their middle school participants found DGs difficult at least in the context of their teaching experiment. The authors asked eight U.S. 7th graders to generalize five linear and three quadratic problem situations that pertained to growth, change, size, and shape. Their results show that, in the case of tasks that contained figural stages, only three students were

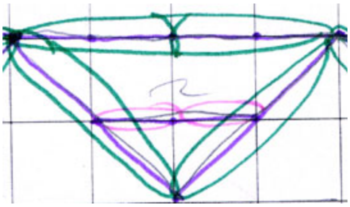
Stage 1



Interior horizontal segements:
1 group of 0

Perimeter segments:
Diagonals: 2 groups of 1
Horizontal (Top Base): 1 group of 2

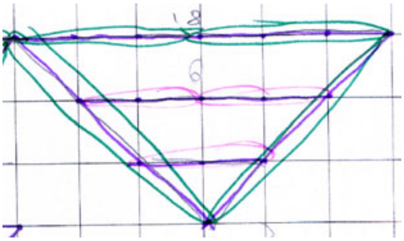
Stage 2



Interior horizontal segements:
2 groups of 1

Perimeter segments:
Diagonals: 2 groups of 2
Horizontal (Top Base): 2 groups of 2

Stage 3



Interior horizontal segements:
3 group of 2

Perimeter segments:
Diagonals: 2 groups of 3
Horizontal (Top Base): 2 groups of 3

Stage n

Interior horizontal segements:
n group of $(n - 1)$

Perimeter segments:
Diagonals: 2 groups of n
Horizontal (Top Base): 2 groups of n

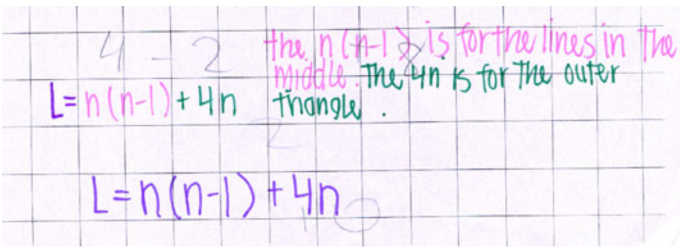


Fig. 19 Diana's pattern generalization in relation to Fig. 18 task

able to establish and justify DGs (or "well-connected subtracting-out schemas"). The notion of multiplicative thinking was not used in their teaching experiment.

In the Year 1 clinical interviews after the teaching experiment on pattern generalization, none of them were found to be capable in establishing and justifying a DG. Further, in clinical interviews in Year 2 after the teaching experiment on pattern generalization, none of them were capable of constructing DGs. However, there was a marked gain in their ability to interpret and justify a stated DG (with a success rate of 50% to 100% in pre- and post-clinical interviews, respectively). All the students interviewed saw the overlapping sides in the adjacent squares pattern in Fig. 1 and six could see the overlapping interior vertices in the case of the W-dot pattern in Fig. 16a. We further note that despite their success in justifying, seeing an overlap was not immediate for most of the students; it became evident only after they had initially employed formula appearance match followed by formula projection. Of course, some students employed formula projection incorrectly. For example, Jana justified the subtractive term 3 in Zaccheus's DG (item D in Fig. 16a) in the following manner:

FDR: So if you look at this [referring to the formula (item D, Fig. 16a) in which Jana substituted the value of 2 for n], this one's four times two plus one, right? And then minus 3. So how might he be looking at 4 times 2 plus 1 and then minus 3?

Jana: Uhum, the 2 is for the pattern number.

FDR: Uhum. Because when Zaccheus was thinking about it, he said multiply 4 by $n + 1$ *and then take away 3. So how might he be thinking about it?*

Jana: Like it's gonna be 3 [referring to $2 + 1$*] and then it's gonna be 12 [referring to* 4×3*]. But I counted there's only 9, so he has to subtract 3.*

FDR: So how might he be doing that? Suppose I do this? [FDR builds pattern 2 with circle chips in which the three overlapping "interior" vertices are colored differently.]

Jana: Hmm, like he has this group of 4 [Jana sees only two sides in W in pattern 2 with the top middle interior dot connecting the two sides. Hence, one side has 4 dots.].

FDR: Is there a way to see these 4 groups of 3 here [referring to pattern 2]?

Jana: Like he imagines there's 3 and he has to subtract 3.

FDR: So can you try it for other patterns? [Jana builds pattern 4.]

Jana: He has 1 group of 4. So there's 3 groups of 4 and he imagines 3 more [to form 4 groups of 4] and then he subtracts them [the three circles added].

FDR: So he imagines there's three more. But why do you think he would add and then take away?

Jana: Because there's supposed to be 4 groups of 4 and then you don't have enough of these ones [circles] so he adds 3. You add these ones.

Conclusion

This paper began with two broad questions that have guided the longitudinal study summarized in this work: What is the nature of the content and structure of generalization involving figural patterns among middle school students? To what extent are they capable of establishing and/or justifying more complicated generalizations? Various patterning studies that have been conducted at the middle grades level provide strong evidence that students' generalizations shift from the recursive to the closed, constructive form. In this article, we discussed in some detail at least three epistemological forms of generalization involving figural linear patterns, namely: CSG, CNG, and DG. The general forms are further classified according to perceptual complexity. CSGs are the easiest for most middle school students to establish and, thus, most prevalent. CNGs and DGs are relatively difficult and less prevalent. This classification scheme of generalizations emerged from detailed analyses of students' pattern generalization over three years. Also, it elucidates the content and structure of such generalizations.

We have also discussed how students' approaches to establishing generalizations are intertwined with their justification schemes. Further, results drawn from our longitudinal work show shifts in pattern generalization schemes among middle school students at least in the case of figural patterns. We note two consequences. *First*, we highlight changes in their representational skills and fluency, that is, from being verbal (situated) to symbolic (formal) and to figural (formal). *Second*, the phenomenological shifts affect the manner in which they justify their generalizations. We have documented at least four types of justifications, namely: extension generation; generic example use; formula projection, and; formula appearance match. The entry level of justification oftentimes involves generating extensions (i.e., calculating and/or producing more stages after the initial ones). Students who then generalize numerically without having a strong figural foundation are most likely to employ formula appearance match and use formula projection inconsistently. Students who understand multiplicative thinking in relation to figural patterning activity oftentimes employ formula projection but the success and validity of such formulas are relative to the associated structural analyses.

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