# An Approach to Geometric and Numeric Patterning that Fosters Second Grade Students' Reasoning and Generalizing about Functions and Co-variation

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**Abstract** In this chapter, we present illustrations of second grade students' reasoning about patterns and two-part function rules in the context of an early algebra research project that we have been conducting in elementary schools in Toronto and New York City. While the study of patterns is mandated in many countries as part of initiatives to include algebra from K-12, there is a plethora of evidence that suggests that the route from patterns to algebra can be challenging even for older students. Our teaching intervention was designed to foster in students an understanding of linear function and co-variation through the integration of geometric and numeric representations of growing patterns. Six classrooms from diverse urban settings participated in a 10–14-week intervention. Results revealed that the intervention supported students to engage in functional reasoning and to identify and express two-part rules for geometric and numeric patterns. Furthermore, the students, who had not had formal instruction in multiplication prior to the intervention, invented mathematically sound strategies to deconstruct multiplication operations to solve problems. Finally, the results revealed that the experimental curriculum supported students to transfer their understanding of two-part function rules to novel settings.

## Introduction

The study of patterns is now commonplace in elementary school curricula in many countries, arising out of initiatives to include algebra from Kindergarten through

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Grade 12 (e.g., Noss et al. 1997; Ontario Ministry of Training and Education 2005; Sasman et al. 1999; Warren 2000). The National Council of Teachers of Mathematics (NCTM) advocates that patterns should be taught from the first years of schooling with the expectation that students, as early as second grade, should be able to "analyze how both repeating and growing patterns are generated", and by the end of fifth grade should be able to "represent patterns and functions in words, tables and graphs" (NCTM 2000). It has been suggested that patterns can: (1) support students to understand the dependent relations among quantities that underlie mathematical functions (e.g. Carraher et al. 2008; Ferrini-Mundy et al. 1997; Mason 1996; Lee 1996); (2) serve as a concrete and transparent way for young students to begin to grapple with abstraction and generalization (Watson 2000; Noss and Hoyles 1996; Kieran 1992); and (3) support students to develop the language of conjecture and proof in communicating their reasoning about pattern rules (Kuchemann 2008; Moss and Beatty 2006b).

Developmentally, the inclusion of patterns seems to fit well with mathematics learning in the early years. We know that young children are fascinated with patterns (Ginsburg and Seo 1999) and are capable not only of noticing patterns but also of using this skill naturally to make sense of their world (Greenes et al. 2001).

However, the potential of pattern work to support algebra learning has not been substantially realized (e.g. Carraher et al. 2008; Dorfler 2008). An extensive literature review on patterning research—conducted primarily with older students—reveals that without specific targeted pedagogical supports even older students have significant difficulty finding algebraic rules for patterns, strongly suggesting that the route from perceiving patterns to finding useful rules and algebraic representations is difficult (English and Warren 1998; Kieran 1992; Lannin et al. 2006; Lee and Wheeler 1987; Orton and Orton 1999; Stacey and MacGregor 1999; Steele and Johanning 2004).

One challenge in moving from pattern study to algebra is the tendency of students to use additive strategies for identifying and describing patterns—that is, to focus on the variation within a single data set rather than on the relationship between two data sets (e.g., Orton et al. 1999; Rivera 2006; Rivera and Becker 2007; Warren 2006). While this recursive approach allows students to predict the "next" position of a pattern, it does not support co-variational thinking about a relationship across data sets to find the underlying function rule. As well, even when students begin to grasp two-part pattern rules, they often use incorrect proportional thinking or "whole object reasoning" to make predictions about the number of elements in a far position of a sequence (e.g., English and Warren 1998; Lee 1996; Orton 1997; Stacey 1989).

Over the last several years, however, there has been an increasing number of accounts of middle school students who, as part of dedicated instructional interventions, demonstrated the ability to work constructively with patterns. For example, in 2008, the journal *ZDM, Mathematics education* published a special issue that focused on patterns and generalizing problems (Becker and Rivera 2008). Research reported in this special issue by Amit and Neria, Radford, Rivera and Becker, and Steele analyzed strategies that middle school students employed in their pattern work. These studies had in common the use of an analytic framework (e.g. Stacey 1989; Lannin 2005; English and Warren 1998) that captured the progression of students' reasoning. They noted that students working on generalizing problems began by using an additive or recursive approach, then, if they were able, switched to explicit or functional reasoning to find far positions and general rules. While there were many promising results reported in these articles, there were also indications of limited strategy use. Amit and Neria (2008), who studied the patterning problem solving strategies of 139 gifted 11- to 13-year-olds, went so far as to conclude that it is only advanced mathematics students who are able to learn to generalize. In their words, "Because of the higher-order thinking involved in generalization, such as abstraction, holistic thinking, visualization, flexibility and reasoning, the ability to generalize is a feature that characterizes capable students and differentiates them from others."

However, we along with others believe that it is not patterns per se, but the ways that patterns are presented that may limit students' ability to engage in the higher order thinking that characterizes generalization. While there has been less research conducted with very young children, our present study with Grade 2 students in diverse urban settings joins the work of other researchers (e.g. Carraher et al. 2006, 2008; Carraher and Earnest 2003; Cooper and Warren 2008; Mulligan et al. 2004; Mulligan and Mitchelmore 2009) to examine the potential of pattern work to support algebraic thinking in the early elementary school years.

## Our Project

Over the last five years, we have been investigating new approaches to pattern teaching and learning that support students to forge connections amongst different representations of pattern. Our goal is to promote multiple ways of working with patterns (Mason 1996), and to foster what Lee (1996) has termed "perceptual agility—the ability to see multiple patterns coupled with a willingness to abandon those that do not prove useful for rule making" (p. 95). Our project to date has included intervention studies in 20 inner city elementary school classrooms (e.g., Beatty and Moss 2006a, 2006b; Beatty et al. 2006; London McNab and Moss 2004; Moss 2005; Moss and Beatty 2006a, 2006b; Moss et al. 2008). Further, it has served as a basis for a school district-wide professional development intervention (Beatty et al. 2008). This research began with a series of studies in second grade classrooms; it is the methods and data from these Grade 2 studies that we present in this chapter.

### *Our Approach: Theoretical*

The predominant theoretical inspiration for our research on patterning emanated from the theories about mathematical development of Case and colleagues (Case and Okamoto 1966). Case and his colleagues' previous work in mathematics development for number sense in the domains of whole number (e.g. Griffin and Case 1997) and rational number (e.g. Moss and Case 1999; Moss 2004) offered a model for the integration of numeric and spatial schemes that we paralleled in linking numeric and geometric representations of patterns. A central tenet of the instructional design of Case et al. is the focus on the development of students' visual/spatial schemes. The theoretical proposal is that the merging of the *numerical* and the *visual* provides the students with a new set of powerful insights that can underpin not only the early Learning of a new mathematical domain but subsequent Learning as well (Case and Okamoto 1966; Kalchman et al. 2001). As we elaborate below, our experimental patterning curriculum was designed to support students to forge connections between visual/spatial patterns in the form of geometric growth sequences and numeric patterns embedded in "Guess my rule" games. Our conjecture was that the merging of these two types of patterns would serve as a foundation to support students to gain an initial understanding of linear functions. To test this conjecture, we designed a lesson sequence that was pilot-tested, revised and refined over a two-year period, and implemented in 6 different Grade 2 classrooms.

## Context and Students

The 7- and 8-year old students in our study were from intact classrooms of between 20 and 22 students each, in urban settings in Toronto and New York City. These students represented diverse populations and a range of math competency. The classrooms were chosen because of the teachers' interest in learning more about this new approach and in involving their children in this study. Overall, the classrooms seemed to have in common an invitational sense of welcoming student contribution; the students were all accustomed to expressing their thoughts and reasoning in math, as in all subjects.

All of the students had previous experience with repeating patterns as part of the early years math curricula; however, none of the students had worked with growing patterns. Importantly, there had been no formal instruction in multiplication in any of the classrooms prior to the intervention. Although the activities in the intervention could be approached through multiplication, at no time was multiplication formally taught.

The length of the intervention varied from 10 to 14 lessons of about forty minutes each. In four of these classrooms, the interventions were taught by the classroom teachers with the help of research assistants; in the other classrooms, the interventions were taught by the first or second author with the assistance of the classroom teacher. It is important to note that there were research assistants in the classroom, who were able to work with Small groups or individual children, and that in some of the classrooms math was taught to only half the class at a time.

## Instructional Sequence

### *Visual Representation: Geometric Growing Patterns*

The lessons began by presenting students with the first three positions in a geometric growing pattern. These patterns were made of square tiles set out in arrays that grew by a given coefficient. To enable students to keep track of the ordinal position number of these tile patterns, position number cards were placed below the geometric arrays that represented that position of the pattern.

This clarified for students the functional relationship between the position number (independent variable) and the number of elements in each position (dependent variable). So, for example, for a pattern representing the relationship described by the equation $y = 3x$ (please see Fig. 1), students could easily connect the position number card "1" to the single row of three square tiles, the position number "2" to the 2 rows of three square tiles (6 tiles), the position number "3" to the 3 rows of three square tiles (9 tiles), and so on.

The initial challenges that the teacher posed were designed to focus students' attention on the relationship between the position number and the number of elements in each position, through the geometric configurations of the tile arrays. Referring once more to the pattern in Fig. 1, in the first lessons, the teacher's questions to the students followed a specific sequence: *If this pattern keeps growing in the same way, what would the next position look like? How many blocks would there be in the next position? What would the 10$^{th}$ position look like? How many blocks in the 10$^{th}$? What about the 100$^{th}$ position?* In subsequent lessons after the students had experience with the function machine activity "Guess my rule?" (see below), the teacher would go on to ask, *What if you had any position? What could the rule be?*

Next, two-part functions were introduced geometrically. To demonstrate the constant, a fixed number of tiles was placed at the top of the array; this configuration of tiles remained the same from position to position, while the array grew multiplicatively by one row for each new position. Because of the spatial representation of the constant as tiles that jutted out from the array (please see Fig. 2), the students began to refer to the constant as the "bump", and we made a deliberate decision to encourage their use of this natural language.

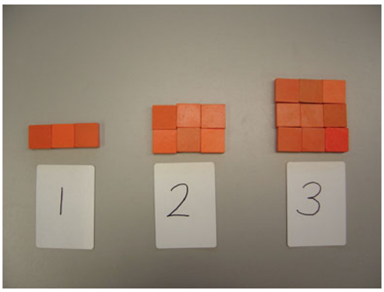



**Fig. 1** Position number cards

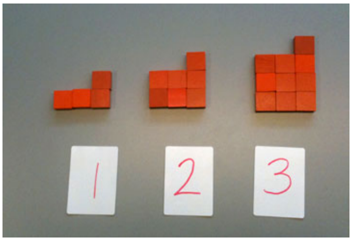



**Fig. 2** Composite functions: the "bump"

As the lessons progressed, students built their own patterns and challenged classmates to make conjectures for general rules.

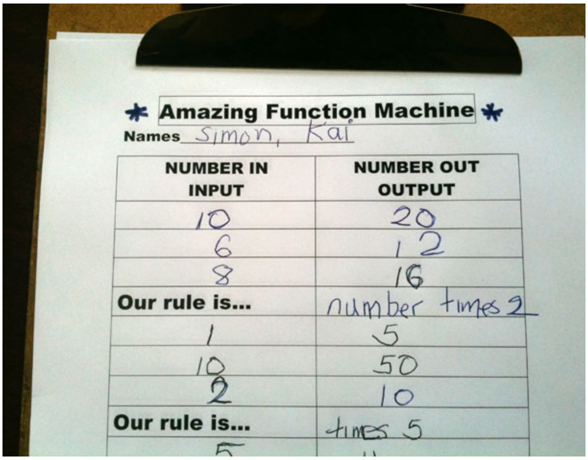
### *Numeric Representations: Function Machine*

We interspersed these visually based pattern lessons with numeric-based lessons that incorporated function machine activities (Carraher and Earnest 2003; Rubenstein 2002; Willoughby 1997). Please see Fig. 3a for an example of a function machine. As in the geometric lessons, in the first series of function machine activities, we focused on one-step multiplicative rules. To begin, the teacher led the activities; then the students took turns creating functional rules (e.g. "double the number and add 3 more") to challenge their classmates in the "Guess my rule" game. The teacher modeled the use of a T-table to record the non-sequential pairs of input and output numbers; please see Fig. 3b for an example of the Function machine T-table. Pairs of students generated between 3 and 5 examples of non-sequential pairs of input and output numbers, as clues to their rule. The children who were solving the challenges given to them by their classmates used T-tables to record the input and output numbers, and their iterative conjectures for what the rule might be. It was important that the numeric clues were non-sequential to allow students to focus on the "across" (on a T-table) or functional rule, rather than on the "down" pattern or "what comes next" strategy identified as interfering with functional generalizations. Further, the T-tables were used only to record the non-sequential clues in the "Guess my rule" game, but not to generate further pairs of values as is often done in many classrooms.

Because the children had not yet been taught multiplication, the coefficients we initially presented were confined to what we determined to be arithmetically manageable numbers—2, 3, 5 and 10—that they would have practiced as skip-counting in first grade. However, there were no such constraints on the numbers the students could choose when they were creating their own rules for the function machine, giving them the opportunity to experiment with even difficult or tricky arithmetic if they chose.

**Fig. 3** (**a**) Function machine. (**b**) Function machine T-table


(a)




(b)

## *Integration Activities: Pattern Sidewalk*

Both the geometric and the numeric activities offered students a chance to consider the idea of co-variation and function rules. The geometric activities specifically highlighted the direct connection between the position number and the structure of the corresponding arrays and number of tiles in each position. The function machine activities illuminated the idea of explicit rather than recursive rules. In order to integrate these two complementary approaches within a non-sequential presentation and exploration of geometric patterns, we designed what became known as a "pattern sidewalk". This is a large counting line placed on the classroom floor, with ordinal position numbers on each section of the sidewalk, from 1 to 10.

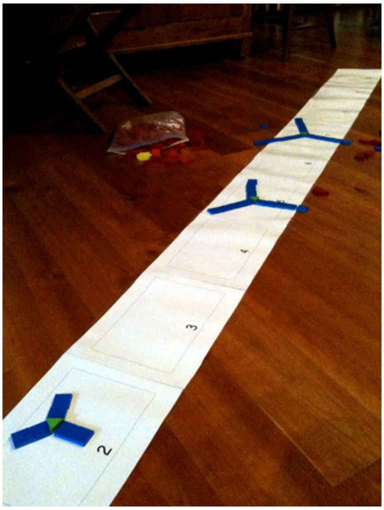


**Fig. 4** Pattern sidewalk

The pattern sidewalk activities were geometric, but paralleled those of the numeric function machine game. First, the teacher would build one position of a pattern that conformed to a secret rule on, for example, the third section of the sidewalk, and ask the children to consider a possible pattern rule that would fit with that configuration in that position. Then the teacher would build another position of the same rule on any other non-adjacent section of the sidewalk, for example, the seventh pattern position on the seventh section of the sidewalk. At this point the students were given the opportunity to revise their initial rule conjecture. Finally, another pattern position would be shown to the students, on its appropriate section of the sidewalk and students were allowed to guess the rule, and asked to show their guess by building another position of the pattern correctly on a new section of sidewalk. The students were then invited to take on the role of teacher and present their own challenges to their classmates.

These three components—the geometric pattern building, the function machine and the pattern sidewalk—comprised the main elements of the lesson sequence in all cases, and were introduced to the students in the specific order described above. However, as the students gained experience of increasingly complex patterns, the teacher/researcher would revisit these different components, adding new elements such as moving from one-step to two-step functions.

## *Role of the Teacher*

Another specific feature of this instruction that needs to be highlighted is the very particular ways in which the teachers engaged with the students, and the prompts and foci they adopted in their interactions with the children. Aligned with Radford's "theory of knowledge objectification" (Radford 2008), we shaped both the order of activities and the teachers' prompts to help students notice and make sense of the ways in which patterns stay the same and the ways in which they change, thus moving students towards generalization and algebraic reasoning. Specifically, the teachers focussed the students' attention and grounded their perceptions of *change* in the curriculum's sequenced arrays of tiles that grew by a multiplicative factor; the teacher also grounded the students' perception of *what stays the same* by drawing their attention to those configurations of tiles outside the arrays which remained constant for each position of a pattern. The teachers also carefully supported the children's learning by helping them draw connections between the idea of rules, as they are experienced in "Guess my rule" function machine activities, and the possibility that geometric pattern growth can also be predicted by a rule. Further, the teachers focused the children's attention on the relationship between the position number cards and the number of elements in, and structure of, the corresponding array, thus supporting these young students' emerging understandings of co-variation.

## Procedures and Measures: Grade 2 Interventions

In order to assess the potential of the intervention, we collected data from many different sources. Our major analyses were qualitative and descriptive, based on classroom artifacts, field notes, transcripts of videotaped classroom lessons and ad hoc interviews with students during the lessons. In addition, we gave each of the students in all of the research classrooms a short pre-test interview, that was administered again at the end of the intervention as a post-test, consisting of patterns in different representations. In keeping with the literature on patterning discussed above, this pre-/post-test was designed to assess changes in students' abilities to find "near" and "far" positions of patterns (e.g., Lannin 2003), to identify whether students relied on recursive strategies or functional reasoning and finally to assess students' abilities to find and express general pattern rules. At the end of the second year, we introduced an additional assessment in which we interviewed students to look specifically for transfer in their reasoning to a novel context.

## Results

The results presented here focus on four general areas: the way students developed their reasoning about pattern rules, the explicit aim of our research; students' constructed understandings of multiplication, an implicit research question; the use of zero as both co-efficient and position number, an unexpected result; and the transfer of understandings to a novel context.

## Finding Rules for Patterns and Generating Patterns Based on Given Rules

Our overall analyses for each year of the study revealed that students made significant gains in their ability to discern function rules for geometric growth patterns and reciprocally/conversely could also build patterns based on given rules. In contrast to findings from other studies that reveal the pervasiveness of recursive reasoning, the students in our research classrooms used a functional approach, which was evidenced in the way that they talked about the position number in relation to the number of elements in a position.

## Constructing a Pattern from a Rule: "A 'number times two, plus one' pattern?"

The following transcript of a conversation between Ricardo and the classroom teacher was initiated by the teacher as she walked around the class during a portion of a lesson where students were building patterns based on given rules. She asked Ricardo to build a "number times two, plus one" pattern. Ricardo, using the square tiles from the pattern block set, built the first four positions of a pattern (that could be described in the informal notation of this classroom as "$n \times 2 + 1$", placing position number cards below each position. Each position was comprised of a row of increasing numbers of square tiles, with one tile on top of the row. (Please see Fig. 5.)

Our transcript begins when the teacher asks Ricardo to explain his pattern:

Ricardo: See, this is the first position. *[Ricardo points to the first position of the pattern he built and then picks up the position card that he had placed*

**Fig. 5** Building a composite function

*under the first position].* So *[touching the row of 2 blocks],* so one times two is two, plus one *[points to the block on top]* is three. *[Keeping this same speech rhythm he picks up the second position card]* So, two times two is four *[points to the row of four blocks]* and ONE *[said emphatically]* makes five.

In his explanation of the next (third) position, Ricardo makes an error. As he is about to make the same mistake again for the fourth position, he catches himself and corrects his explanation:

*[He points to the position card on which is written a 3.*] So three times three [sic] is six and one *[points to the single tile]* is 7. [*He repeats the same set of actions for the fourth position—the final one he has built])* So, four times four... I mean four PLUS four... or, two TIMES four is eight and one makes nine.

Teacher: Well done. How many blocks would there be in the 10$^{\text{th}}$ position?

R: *[putting his hand over his eyes in thinking position and then rapidly dropping it and saying with a smile]* Twenty-one.

What was notable to us about Ricardo's explanation was his clear understanding of the co-variation of the position number and the number of tiles. What also was revealed in the interaction was the way that Ricardo constructed the pattern to clearly reflect his ability to distinguish the coefficient from the constant. Finally, this exchange also demonstrates this young student's fluency in being able to use his understanding of the function rule not only to build sequential pattern positions from 1 to 4 but also to predict quickly, easily and accurately the number of tiles that would be required for the 10th position—a far position.

## *Finding a Rule for a Given Pattern: "Position number times three, plus one"*

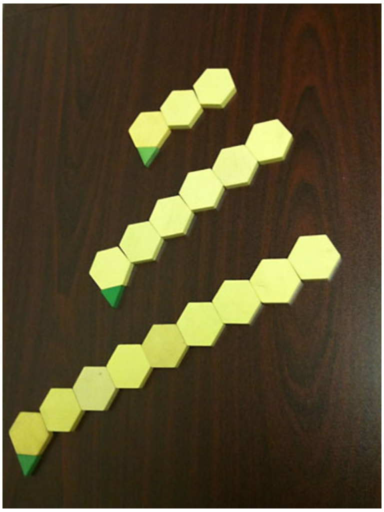
In this transcript, two students, Zoya and Marie, are sitting at a table examining the first three positions of a geometric pattern ($y = 3x + 1$) to determine the pattern rule. The figures each consisted of a planar column of yellow hexagonal pattern blocks with a single green triangle placed on the top, that grew by three yellow blocks each time (Fig. 6).

The students stare at the first three positions of the pattern:

Marie: It's a times 3 pattern, right?

Zoya: [*touches the blocks in the second pattern position]* Because this is a GROUP of 3 *[separates and points to one group of 3 in the second position, and then moves her finger in a circle around it],* and this is a GROUP of 3 *[points to the remaining group of hexagons in the second position; then she looks at the triangle and exclaims:]* Wait, oh but it can't be, because *[indicating the green triangle]* it's a whole block. So,

Fig. 6 Caterpillar pattern



never mind! *[pushes groups back together, and throws up her hands; the students seem stumped and continue to look at the pattern]*

Marie: I've got it. *[long pause as she stares at the pattern]* It is number times 3... No, position number times 3, plus 1.

Researcher: How do you know that?

Marie: Because, here is a group of 3, so that is times 3 and 1 makes 4.

Here we see the flexibility of students who were able to discard an initial conjecture of a pattern rule when the evidence (the built pattern) did not support their rule. We contrast the flexible reasoning of these very young students with findings of other researchers (e.g. Lee and Wheeler 1987; Stacey 1989; Stacey and MacGregor 1999) who all document older students' reluctance to change incorrect conjectures of rules in the face of contradictory evidence.

Marie offered an initial rule; Zoya immediately tried to support this conjecture by referring to the structure of the pattern. However, in trying to "prove" this rule, Zoya realized that the built pattern did not fit the rule, so they abandoned their initial conjecture. Eventually, Marie does figure out the correct rule, which she expresses in informal algebraic language as *position number times 3, plus 1*. Notable as well is the "groups of" language that illustrates one of the ways in which students constructed their multiplicative reasoning.

## *Students' Invention of Multiplication*

As mentioned previously, none of the classes had been taught multiplication prior to the patterning lessons. Perhaps amongst the most salient of our findings was the way that the pattern activities worked to support students to construct a robust understanding of multiplication, revealed in the diversity of approaches the students had "invented" and the deep conceptual orientation to multiplication they had constructed through continual experience with arrays of tiles. It appears that the arrays in the geometric growth patterns provided the students with a visual representation of multiplication as a set of relationships that they could construct and deconstruct. Schliemann et al. (2001) have suggested that operations such as multiplication may in fact be more effectively understood as functions.

As shown in the following transcripts, even arithmetically lower-achieving students who struggled to perform some of the required calculations were nonetheless able to use multiplication to explain their reasoning about pattern rules and the number of elements in pattern positions.

## *Deconstructing Multiplication: "Double the position, plus the position"*

One way in which students constructed their understanding of multiplication in the context of patterning was through a deconstruction of the operation. In this excerpt, taken in the context of a classroom lesson, Moni is presented with tile arrays representing the first four positions of a $y = 3x$ pattern in which the first position is a row of 3 tiles, the second position is two rows of 3 tiles, and so on. She explains her thinking about how the number of tiles in the fourth position (an array of 4 rows of 3 tiles each, or 3 columns of 4 tiles each) conforms to a general rule:

Moni: *[running her finger up and down one column of 4 tiles]* Here it would be 4 doubles; that would be 8... So when you put these two lines together it's 8. And here's [*indicates position number card]* 4. So you double the position, with the number.

Teacher: So, it's the number...?

Moni It's the position [number], plus the double of the position.

It is noteworthy that Moni chose to reason in a structural way rather than counting out the full number of tiles in her attempt to calculate the total. This type of reasoning typified the approach taken by many of the children. The geometric configuration (array) that Moni's explanation relied on clearly supported her deconstruction of multiplication; $3n$ is decomposed into both $n + 2n$ (*It's the position, plus the double of the position*) and $2n + n$ (*Double the position, with the number*). This visual/spatial reference that anchors her understanding further allows her to demonstrate correctly both the commutative property of addition and the distribution of multiplication over addition.

## *Using a Structural Understanding of Multiplication to Predict Far Positions: "It's 40 up, and 3 to the side"*

In this next transcript, Oscar was shown the first two positions of a pattern (representing the functional relationship $y = 3x + 5$) built with square and triangular tiles,

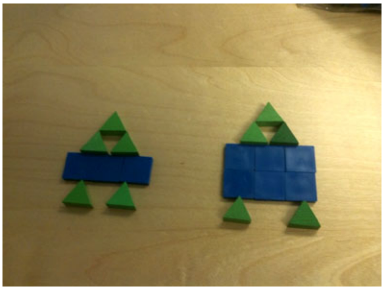


**Fig. 7** Photo of rocket ship pattern

designed to look like a "rocket ship" for astronauts. Each position included an array of a number of rows of 3 square tiles each, one row of 3 tiles for each astronaut. Below this array were two triangular tiles, positioned one on each side to look like rocket boosters; three triangular tiles above the array formed the nose of the rocket (Fig. 7).

After Oscar was shown the first two positions of this pattern, the researcher wondered if Oscar could use his understanding of the structure of the pattern to determine how many blocks there would be in the 40th (a far) position (there are not enough blocks on the table for Oscar to build it), so asks him what the pattern would look like for 40 astronauts:

Researcher: Do 40, first.

Oscar: Forty *[thinks]*. Forty would be 40 up, 40 up *[moving his finger along an imaginary column on the table]* and 3 to the side *[moves his finger sideways]* 'cause one astronaut is 3 blocks long.

Researcher: Oh, it's 40 up and 3 to the side. Can you figure out how many blocks that would be in all?

Oscar: That would mean 3 rows [sic] of 40. And 40 *[counts with one finger held up]* plus 40 *[counts on another finger]* is 80. And another 40 is... another 40 is.... *[turns to his partner]* What's 80 plus 40? *[the partner replies, "120."]* A hundred and twenty... So that's a hundred twenty. *[He now begins to put 5 triangle blocks down one at a time, very deliberately. He first places 2 at the base of a smaller array he has already built, then 3 triangles far above this configuration, apparently at the top of the imaginary much larger array that he is describing.]* Then a hundred and twenty one, a hundred and twenty two, a hundred and twenty three, a hundred and twenty four, a hundred and twenty five. That's the answer... Can I write that answer down before I lose it?

The fact that Oscar could predict the 40th position (from only two examples) revealed his growing understanding of multiplication as an array and the structure of the linear functions we had been working with Further, Oscar's reasoning illustrates the understanding students developed of the co-variation of the position number and of the number of elements in a given position. Finally, this short excerpt is indicative of the kind of excitement this work generated in the students, the "big numbers" they were willing to engage with and the kind of effort they were willing to make.

### *The Discovery of Zero*

Our sequence of lessons required the children to move back and forth between numeric and geometric expressions of function rules; the lessons also moved back and forth between teacher and student generation of patterns and rules. In the course of creating their own rules within both the numeric and the visual/spatial investigations of pattern, students in different research classrooms independently made the discovery of zero as a powerful mathematical idea and arithmetic tool. Below we present examples taken from different classrooms revealing how students used zero in their pattern designs, first as a coefficient and then as a position number.

### *Zero as a Coefficient: "Zero groups of 4 million is zero"*

The first example comes from a classroom lesson at a time when pairs of students were working independently to create function machine challenges for their fellow classmates. The researcher approached two students, Clarice and Emma who had already invented a rule and had created written pairs of input and output numbers.

Researcher: Okay, I'm going to give you an input number you don't already have, and can you give me an output? Ready, my input is 2.
Clarice: It's going to be 5.
Researcher: Okay, input number is 17.
Clarice: 5, 5, 5!
Researcher: Input number is 672!
Clarice & Emma: 5, 5, 5, 5, 5! *[laughing]:*
Researcher: Wow. Can I ask you a question—what does that mean? How many groups of the input number are there?
Clarice: If it's times zero, it would always be zero. Zero groups of 4 million is zero!
Researcher: Zero—there are just no groups of them.
Clarice: And then plus 5, so it's always 5.

We were surprised, as there had been no prior discussion of zero in any context. As this lesson progressed, Emma and Clarice had the opportunity to sit at the function machine and present their challenge to their classmates. While many were stumped, one student asserted, "It's the number, minus itself, plus 5". We were interested to note that Emma and Clarice were flexible in being able to accept this different rule as an expression of the same relationship, something older students have difficulty doing (e.g., Lannin 2003; Lee 1996; Mason 1996; Stacey 1989.)

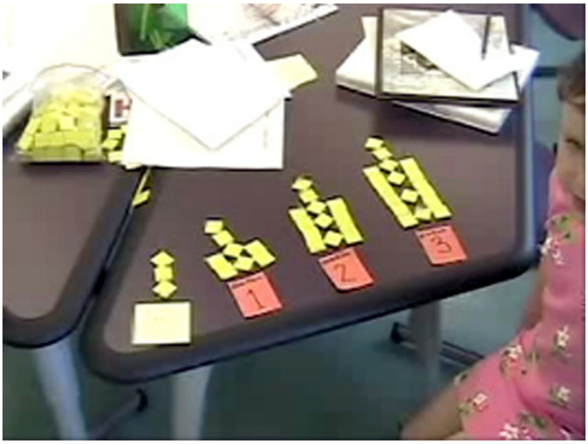


**Fig. 8** The zero-th position

## *Zero as a Position Number: "the zero-th position"*

Whereas the two girls in the example above were excited to use zero as a trick, in the next excerpt we see Anna, pictured in Fig. 8 with her geometric pattern that offered no clear visual organizational clues for her classmates to identify the constant. Anna suggested that the "zero-th" position was a "help" to them in discovering her pattern rule:

Researcher: Can you show the class the pattern that you have built?

Anna: *[holding a Small piece of paper in her hand, with a rule eon it written by a researcher]* So what our rule was, was—this is our second one—it's times 5 plus 3. *[gestures to the pattern in general and then points to a position card she has made with a zero and indicates the three tiles above it]* And the zero-th position helps you a lot, it gives you a big clue. *[she splays her 3 fingers as she tries to indicate the 3 tiles that are over the "zero-th position" card].* It's 3 *[touching all 3 at once]*—this is the bump, cause the bump stays the same, but there's no GROUPS of 3 *[according to the pattern rule].* So it's the bump. It helps you a lot because it identifies what the bump would look like. So it's like 5 *[pointing to the first position configuration]*, and then plus the bump. *[pointing to the second position]* Five and then 5, plus the bump. *[pointing to the third position]* Five, and then 5, and then 5 *[holding her hands over each group of 5]*, plus the bump.

The inclusion of zero was Anna's and her partner's idea. There were no position cards with zero written on them. As she prepared the challenge for her classmates, she had requested a blank card to make a zero-th position card. Anna was aware that the position number always indicated how many groups there were, regardless of how many tiles were in each group; so she determined that if she used zero as a position number, then there would be no groups at all, isolating the constant and making it easy to identify. Thus, she was giving her classmates what she determined to be a big hint to finding the rule for her visually complicated pattern.

### Transfer of Structure

As we observed the students over the course of the research lessons, we could see that they were gaining fluency with geometric patterns and becoming increasingly successful with function machine games, and integrating these features within their work in the pattern sidewalk. However, what we could not tell through observation was the robustness of the students' acquired understandings of the two-part function structure (represented formally as $y = mx + b$) and whether they could transfer their new understandings to other mathematical contexts. Accordingly, at the end of the second year of our Grade 2 interventions, we interviewed pairs of students using a word problem—a novel context—that was a narrative representation of a two-part function. The problem was presented only orally; there were no visual representations, and the students were not given the opportunity to write or draw to find the solution. There was nothing in the word problem that resembled what they had done on patterning in the research lessons, and no verbal cues that linked the word problem to what we had done in the classroom. The word problem is as follows:

> Charlotte really wants to buy a scooter. But she doesn't have enough money. The scooter she wants costs $100! From the tooth fairy, Charlotte already has saved $10. But she decides to earn more money by walking her neighbour's dog, Sparky. For each day that Charlotte walks Sparky, her neighbour will pay her $5.

### Circumventing Whole Object Reasoning

The first transcript comes from a post-intervention interview in which two students, although not asked to state a function rule, clearly demonstrate their ability to discern and use a rule:

Researcher: Okay, kids. Now you really have to listen hard. I have a question for you. I don't have any paper or anything. *[Reads the problem out loud.]* How much money will Charlotte have altogether at the end of the first day of walking Sparky?

Mai: She. . . .

Juanita: 15.

Researcher: How do you know?

Juanita: She already has 10, and then she gets 5.

Researcher: How much money will she have altogether at the end of Day 2? How about on the second day?

Mai: 20.
Juanita: Hey!
Researcher: Why, is that what you were going to say, Juanita?! How much money will she have altogether at the end of Day 5?
Juanita: 20…25…. Just a second. *[counting by fives on her fingers]*
Mai: 35.
Researcher: How did you get 35?
Mai: Because 5 × 5 is 25, plus 10 is 35.
Researcher. Is that how you did it, Juanita?
Juanita: Yeah.
Researcher: How much money will she have altogether at the end of Day 10?
Mai: 60.
Researcher How did you get 60?
Juanita: Well, on the 5$^{th}$ day is 25, and 25 and 25 is 50, plus 10 is 60.
Juanita. Same with me.
Researcher: What day would it be if Charlotte has $70 altogether?
Juanita: I think the… twelfth.

The inappropriate use of proportional reasoning or "whole object reasoning" in the context of patterning problems is well documented. For this reason, the sequence of questions in our interview progressed from asking the students how much money Charlotte would have in 5 days, to how much she would have in 10 days. A whole object strategy, which could be anticipated, would produce an incorrect answer of 70. That is, if 5 days equals $35, then 10 days would be double that, or $70. However, Juanita, like the majority of students in the Grade 2 research classrooms, gave the correct answer of 60.

While Juanita and Mai had not been asked to express a rule explicitly, they appeared to understand what the rule was and how to use it to predict positions of the pattern, as evidenced in their responses.

Further, the students' ability to correctly answer the final question in the narrative problem (*how many days would it be if Charlotte has $70*) is a further indication of their agility and robust understanding. They could use their explicit understanding of the coefficient and constant to reason backwards, i.e. to begin with the number of elements (money) in an unknown position and find the position (day).

## *Informal Algebraic Expressions of Rules in the Sparky Problem*

In the excerpts below, from another classroom, the researcher gave students the opportunity to explain their thinking and to propose a general rule for the Sparky problem. The responses below are impressive in that these students were able to extract the functional relationships inherent in the Sparky problem, and express them in syncopated language (Sfard 1995). The three examples below of rules offered by Luca, Tomas and Stella show increasing levels of abstraction of rules:

Luca: It's counting by 5s with a 10 bump.

Tomas: Oh, I get it—it's a groups of 5 pattern with a 10 bump because she [Charlotte] already had 10 dollars from the tooth fairy.

Another student, Stella went on to notice that the constant was larger than the coefficient, which had not been the Case in the geometric problems that the students engaged with as part of the intervention. Her references to the geometric, as well as those of the previous students, in talking about "groups of" and the "bump", illustrate the crucial role of the visual/spatial representation in their ability to transfer.

Stella: It's always the day [ordinal position number] times 5, plus 10. So there's 10 bumps and 5 normal things, more bumps than normal things—that's weird!

Taken together, in the order that they are presented, these three excerpts reveal the increasing degree of formalization of the students' explanations of rules. Our conjecture is that the role of spatial-inspired terms like "bumps" and "normal things" was fundamental in ensuring the abstraction required to tackle the purely numeric Sparky pattern. We see this generalization as related to what Radford (2003) has called algebraic contextual generalizations. Further, we concur with Radford that adherence to conventions is not necessarily an indicator of algebraic thinking: "It is not notations which make thinking algebraic; it is rather the way the general is thought about" (2008, p. 84).

## Discussion

The Grade 2 students in our research classrooms did not rely on recursive reasoning in their solutions to patterning problems, nor did they use inappropriate proportional ("whole object") strategies, both of which have been indicated in the literature as common problems even among older students. Rather, they appeared to develop a fairly robust understanding of two-part function rules through their engagement with the curriculum: they could predict how a pattern would grow, find general rules for geometric and numeric patterns, and construct patterns based on given rules. As well, our results revealed that the students were able to transfer their understanding of rules to a new (narrative) context, both finding and applying a rule.

The invention of multiplication has been noted in other studies of young students engaging with patterns; however, the diversity and quality of the approaches these students invented seemed noteworthy. Not only did the students find mathematically sound ways to deconstruct multiplication operations to solve problems, but some students also, at their own initiative, experimented with the effect of using zero as either the position number or the coefficient.

Finally, the students appeared to enjoy the lessons and seemed intrigued by the geometric presentation of patterns. They were interested in making, justifying and testing conjectures, were flexible in their general approach, and were excited to explore different ways of creating difficult challenges for their classmates. This contrasts with the concerns expressed by scholars, such as Mason (1996) and Hewitt (1992), who noted that when geometric sequences are introduced, often students produce a table of values from which they extract a closed form formula which they check with only one or two figural examples. The question arises: what are the characteristics of our program that may have supported these very young students in their productive and flexible approach to patterning activities?

Analyses of our findings over the many iterations of our studies suggest a number of distinct but overlapping factors that may have contributed to our students' ability to work with patterns. We draw your attention to three factors in particular. First, is the design of the curriculum with its deliberate movement back and forth between, and then bridging of, geometric and numeric representations of growing patterns through the idea of "rules". Second, and related, is the primacy given to the visual and to the way that the curriculum design deliberately focuses students' attention on the spatial/geometric pattern formations. And third, inherent in the design of the instructional sequence is the emphasis on student invention. In the sections that follow we briefly elaborate on these factors.

## *The Curriculum with Its Focus on Integration*

The theoretical framework that underpinned this research, and served as a heuristic for the design of the curriculum, came from previous work of Case and colleagues on children's mathematical development. Specifically, we were guided by the proposal of Case et al. that children's development in a domain of mathematics (e.g., whole number, rational number) is underpinned by the integration of the children's visual schemas on the one hand, and their numeric understandings on the other, for the mathematics domain in question (Please see Kalchman et al. 2001 for details of this theory). Further, as we mentioned earlier, this theoretical framework helped to establish a developmentally grounded sequence for our intervention. Students first worked with geometric (tile array) representations and then moved on to numeric (function machine) patterns. This sequence served to help the students to consolidate and extend their separate understandings in both the geometric and numeric domains. These separate understandings were bridged for the students by the concept of (function) *rule* which enabled the students to begin to move between the visual and the numeric with increasing flexibility. The subsequent introduction of the pattern sidewalk, in its use of a non-sequential geometric representation of pattern, also fostered this integration and flexibility. Our preliminary conjecture is that it was the specific movement back and forth between these two representations, geometric and numeric, that ultimately supported the students to gain not only flexibility with, but also a structural sense of, two-part linear functions, thus supporting/enabling the students to discern and understand pattern rules in contexts that were new to them.

## *Prioritizing Visual Representations of Pattern*

While we believe that the back and forth movement was critical to the flexible reasoning that students ultimately were able to demonstrate, we also suggest that the students' initial grounding in the visual geometric context was also significant in the effectiveness of the curriculum. All through the lessons and interviews with students we were made aware of how their reasoning was underpinned by their interpretations and analyses that were based on geometric figures. When probed for explanations of rules, the students focused on how a pattern grew in relation to the position number; how the addition of the constant or "bump" was related to the coefficient, or multiplicative; how parts of the pattern changed and parts stayed the same based on their visual configurations. Finally, even in their post-intervention explanations of the "Sparky" narrative word problem, many students referred to the two elements of the two-part function in geometric terms: "Oh, I get it—it's a groups of 5 pattern with a 10 bump because she already had 10 dollars from the tooth fairy."

Indeed, a number of researchers have reported on the support provided by figural representations for students working with generalizing problems (e.g., Carraher et al. 2008; Healy and Hoyles 2000; Lannin 2005; Noss et al. 1997; Rivera and Becker 2008; Sasman et al. 1999; Stacey 1989). When visual representations are prioritized, and students are supported to focus on the figural patterns as a way of discerning general rules, they are better able to find, express and justify functional rules.

However, research has also shown that, overwhelmingly, students and adults have a strong tendency to ignore the geometric properties of figural patterns, and to focus instead on the number of elements in the given pattern. The focus on the visual in our program appears to have had a double advantage for students: providing a rich context in which to analyze growth and change, and also supporting students to be aware of covariation.

## *Pedagogy and Student Inventions*

In the opening sections of this article we discussed the particular ways that in which the teachers interacted with the students and how they focussed the children's attention on salient features of the instructional sequence to support the children's learning. It is also our proposal that another significant contribution to the success of the intervention was the ongoing insertion into the learning sequence of the children's own inventions: specifically the geometric patterns they designed and also the challenges they created for their classmates with the function machine.

As we mention in earlier sections of this chapter, inasmuch as there was a continuous movement back and forth between geometric and numeric representations of patterns, so too was there movement back and forth from the standpoint of the pedagogy: for example, in iterative fashion teachers modeled geometric one-step patterns, and then students designed and presented their own patterns; teachers presented challenges with the function machine, and then students in pairs followed their lead and designed their own challenges for their classmates. The evidence, based on the transcripts of classroom lessons, is clear in revealing that these student-invented challenges created excitement, interest and motivation among the children. They also may have served other important purposes. First was the opportunity to practice. Students in Grade 2 had little or no experience with growing patterns prior to the intervention, and many held firmly to the belief that patterns could only repeat. Creating their own patterns gave the grade 2 students the opportunity to discover and experience how linear growing patterns worked or did not work. In addition, by creating their own geometric and numeric patterns, students had the time and space to invent and then practice multiplication. Also, in the course of developing challenges for their classmates, the students had the opportunity to take on an additional perspective in anticipating how their classmates might respond. In our view, this kind of anticipation and planning added an extra dimension (metacognitive) to students' thinking, thus enriching the learning potential of the lessons.

## Concluding Thoughts

Typically, patterns are taught in the early years as repeating, with children asked to find "what comes next". As Blanton and Kaput point out, this limited view does not capitalize on the potential of patterns to support later mathematics learning (Blanton and Kaput 2004). A number of researchers have included a focus on young children and patterns (e.g. Carraher et al.; Mitchelmore and Mulligan; Mulligan, Prescott & Mitchelmore; Warren & Cooper), investigating ways of promoting algebraic thinking, generalizing and awareness of structure through the use of patterning. We join with these researchers in trying to illuminate the potential of pattern work for young children. Our findings suggest that, with appropriate instruction, the study of patterns can support students of *all levels* of mathematics abilities to foster the kinds of mathematical thinking that Kieran suggests is fundamental to algebraic reasoning: "analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modeling, justifying, proving, and predicting" (Cai and Knuth 2005, pg. 1)

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