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Early Algebraization

A Global Dialogue from Multiple Perspectives

Springer

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Foreword

Early Algebraization: A Global Dialogue from Multiple Perspectives is the second monograph in the Advances in Mathematics Education (AiME) series launched by Springer in 2009. The book follows in the tradition of *Theories of Mathematics Education* (Sriraman and English, monograph 1), stemming from a previous ZDM issues on early algebraic thinking (vol. 37, no. 1, 2005 and vol. 40, no. 1, 2008). That is, although it uses the previous issues as a basis for the current monograph, the monograph itself goes beyond simply revisiting the past. It conveys the present state of the art on existing research on early algebraization since 2005. The eight previous articles (five from vol. 37 and 3 from vol. 40) have been reworked and updated in addition to 18 new chapters from researchers involved in early algebraization research projects in different parts of the world, which include 4 commentaries on the scope of the research.

The book editors Jinfa Cai and Eric Knuth have compiled the book in three substantial parts between the bookends of a general introduction and an overall commentary addressing perspectives for research and teaching in this domain of inquiry. These three parts of the book examine curricular, cognitive and instructional components of early algebraization. Unlike the ZDM issue which was predominantly articles from researchers based in North America, this book contains ongoing research from different parts of the world, and initiates a global conversation on where the community stands in its research findings.

AiME is distinct from other mathematics education series because it attempts to draw the reader into a conversation, and be dialogic in its presentation. This is the purpose of soliciting commentaries from those that are able to synthesize ideas, expose them in a larger light of what is known, and directions in which they can be further pushed. This book continues in this tradition and attempts to draw us into the issues of understanding, implementing and assessing early algebraization in projects and curricula in different parts of the world. We hope this monograph is of value to the research community of mathematics educators interested in the role and significance of early algebraic thinking within the current research architecture. We appreciate the efforts of the authors that are a part of this book and thank the book editors (Cai & Knuth) for this superb book.

Gabriele Kaiser
Bharath Sriraman

Introduction

A Global Dialogue About Early Algebraization from Multiple Perspectives

Kilpatrick and Izsák (2008) quoted an anonymous editorial writer to start their chapter in the National Council of Teachers of Mathematics' 70th Yearbook: "If there is a heaven for school subjects, algebra will never go there. It is the one subject in the curriculum that has kept children from finishing high school, from developing their special interests and from enjoying much of their home study work. It has caused more family rows, more tears, more heartaches, and more sleepless nights than any other school subject." (p. 3) Even though there has been a dramatic change for the world 70 years ago when the editorial was written to nowadays, the status of algebra as a school subject has not changed much—algebra is important but many students experience difficulties (Kieran 2007; Loveless 2008; National Mathematics Advisory Panel [NMAP] 2008). In fact, algebra has been characterized as the most important "gatekeeper" in school mathematics.

Given its gatekeeper role as well as growing concern about students' inadequate understandings and preparation in algebra, algebra curricula and instruction have become focal points for policy makers and mathematics education researchers around the world (e.g., Bednarz et al. 1996; Lacampagne et al. 1995; RAND Mathematics Study Panel 2003; Stacey et al. 2004). An important emphasis, common around the globe, is the development of students' algebraic thinking in earlier grades. The development of students' algebraic thinking in earlier grades is not a new idea; in China and Russia, for example, algebraic concepts were introduced to elementary school students in the 50s and 60s. In other countries (e.g., Europe, North America), the discussion of integrating algebraic ideas into mathematics curricula in the earlier grades started in the 70s. In the past decade, however, there has been an increased emphasis on and wider acceptance for developing students' algebraic ideas and thinking in earlier grades, reflected in a number of influential policy documents. For example, in the United States, the NCTM proposed algebra as a content strand for all grade levels (NCTM 2000). In fact, it is widely accepted that to achieve the goal of "algebra for all", students in elementary and middle school must have experiences that better prepare them for more formal study of algebra in the later grades. Yet, only recently have researchers started to explore issues related to early algebraization.

Although a chapter on research on school algebra appeared in the *Handbook of the Research on Mathematics Teaching and Learning* (Grouws 1992), its focus was primarily on algebra at the secondary school level. In the *Second Handbook of the Research on Mathematics Teaching and Learning* (Lester 2007), there is again a chapter on algebra at the secondary school level, however, the volume now also includes a chapter on early algebra learning. In fact, this is the only chapter with such a focus in mathematics education research handbooks published in the past two decades. A similar trend can also be seen in publications directed toward teachers: In 1993, NCTM published a volume focused on research ideas for the elementary school classroom that did not include any chapters focused on early algebra learning; in contrast, NCTM recently published a similar volume (Lambdin and Lester 2010) that does include a chapter on early algebra learning. On one hand, such changes suggest that the field has known enough about early algebraization to synthesize research findings in the area. On the other hand, as Carraher and Schliemann (2007) recently pointed out: "Although there is some agreement that algebra has a place in the elementary school curriculum, the research basis needed for integrating algebra into the early mathematics curriculum is still emerging, little known, and far from consolidated." (p. 671) In fact, curriculum developers, educational researchers, teachers, and policy makers are just beginning to think about and explore the kinds of mathematical experiences and knowledge students in early grades need to be successfully prepared for the formal study of algebra in the later grades. This monograph is part of such an effort.

Early Algebraization

Traditionally, most school mathematics curricula separate the study of arithmetic and algebra—arithmetic being the primary focus of elementary school mathematics and algebra the primary focus of middle and high school mathematics. There is a growing consensus, however, that this separation makes it more difficult for students to learn algebra in the later grades (Kieran 2007). Moreover, based on recent research on learning, there are many obvious and widely accepted reasons for developing algebraic ideas in the earlier grades (Cai and Knuth 2005). The field has gradually reached consensus that students can learn and should be exposed to algebraic ideas as they develop the computational proficiency emphasized in arithmetic. In addition, it is agreed that the means for developing algebraic ideas in earlier grades is not to simply push the traditional secondary school algebra curriculum down into the elementary school mathematics curriculum. Rather, developing algebraic ideas in the earlier grades requires fundamentally reforming how arithmetic should be viewed and taught as well as a better understanding of the various factors that make the transition from arithmetic to algebra difficult for students.

The transition from arithmetic to algebra is difficult for many students, even for those students who are quite proficient in arithmetic, as it often requires them to think in very different ways (Kieran 2007; Kilpatrick et al. 2001). Kieran, for example, suggested the following shifts from thinking arithmetically to thinking algebraically: (1) A focus on relations and not merely on the calculation of a numerical answer; (2) A focus on operations as well as their inverses, and on the related idea of doing/undoing; (3) A focus on both representing and solving a problem rather than on merely solving it; (4) A focus on both numbers and letters, rather than on numbers alone; and (5) A refocusing of the meaning of the equal sign from a signifier to calculate to a symbol that denotes an equivalence relationship between quantities. These five shifts certainly fall within the domain of arithmetic, yet, they also represent a movement toward developing ideas fundamental to the study of algebra. Thus, in this view, the boundary between arithmetic and algebra is not as distinct as often is believed to be the case.

What is algebraic thinking in earlier grades then? Algebraic thinking in earlier grades should go beyond mastery of arithmetic and computational fluency to attend to the deeper underlying structure of mathematics. The development of algebraic thinking in the earlier grades requires the development of particular ways of thinking, including analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modeling, justifying, proving, and predicting. That is, early algebra learning develops not only new tools to understand mathematical relationships, but also new habits of mind. In this volume, we focus on the development of algebraic ideas in both elementary and middle schools.

Multiple Perspectives

In this volume, the authors address the issues of early algebraization from curricular, cognitive, and instructional perspectives. The inclusion of middle grades is desirable because of the critical transition from elementary to the middle grades, particularly related to algebra learning. The inclusion of issues related to curriculum, cognition, and instruction is based on the consideration that they are the three most fundamental perspectives for mathematics education. Curricula have a significant influence on what students learn (NCTM 2000) and have been found to contribute to mathematical performance differences in cross-national studies (Schmidt et al. 1996). Accordingly, the examination of curricula from various nations can provide a broader point of view regarding curricular approaches to integrating algebraic ideas into earlier grades as well as providing insights regarding the development of students' algebraic thinking.

Although curricula can provide elementary and middle school students with opportunities to develop their algebraic thinking, teachers are arguably the most important influence on what students actually learn. Thus, the success of efforts to develop students' algebraic thinking rests largely with the ability of teachers to foster such thinking.

The design of curricula and professional development programs as well as the enactment of instructional practices intended to support the development of students' algebraic thinking are all dependent, to a great extent, on what we know about students' algebraic thinking and its development. Thus it is critical to examine issues related to students' cognition in algebra learning.

As we look across this set of articles in this volume, with their variety of foci and perspectives, two cross-cutting themes surfaced. First is the importance of better integrating into current school mathematics practices opportunities for students to develop their algebraic thinking. These opportunities include both the design of curricula, at the elementary school level in particular, that pays explicit attention to making connections between arithmetic and algebra, and the recognition of opportunities to strengthen these connections as students progress through middle school. The second theme to emerge is the importance of supporting teachers' efforts to implement practices that foster the development of students' algebraic thinking. If future generations of students are to become better prepared for more formal study of algebra in the later grades, then likewise teachers must also be better prepared. The articles in this volume provide guidance and suggestions for continued work in the area of early algebra research regarding teachers' instructional practices and professional development.

One of the important features of this volume is its international in nature, which promotes a global dialogue on the topic. Research is presented from many parts of the world, including Australia, Canada, China, France, India, Italy, Japan, New Zealand, Russia, Singapore, South Korea, the United Kingdom, and the United States of America. Such a global dialogue will help us address issues related to early algebra learning and, ultimately, better prepare greater numbers of students for success in algebra.

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Jinfa Cai
Eric Knuth

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Part I
Curricular Perspective

Preface to Part I

Jinfa Cai and Eric Knuth

Although it is widely accepted that we should expect students in early grades to think algebraically, the real question is how can we prepare students in earlier grades to think algebraically? As we discussed earlier in this volume, developing algebraic ideas in earlier grades is not simply a matter of moving aspects of the traditional secondary school algebra curriculum down into the elementary school mathematics curriculum. Rather, developing algebraic ideas in the earlier grades requires fundamentally reforming how arithmetic should be taught as well as a identifying better ways to develop algebraic thinking from traditional arithmetic topics. In the chapters that follow, the authors focus on various aspects of curricula from different countries (including China, India, Japan, Russia, Singapore, and the United States) in order to examine how curriculum might be designed and delivered to help students develop algebraic habits of mind.

The chapters in this part provide illustrative examples of successful efforts to help students see algebra in the context of arithmetic. The chapters by Blanton and Russell et al. examine how instructional materials and school activities can be extended to support students' algebraic thinking. In Russell et al.'s chapter, they highlight four mathematical activities that underlie arithmetic and algebra, and discuss how these activities serve as an important bridge between the two domains. In Blanton's chapter, she proposes that elementary school mathematics should include curriculum and instruction that deliberately attends to how two or more quantities vary in relation to each other.

In the chapter by Cai et al., the authors analyzed Chinese and Singaporean curricula. They found that the Chinese and Singaporean curricula could be useful ref-

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erences for those wishing to help elementary students develop a stronger sense of the connections between arithmetic and algebra. Specifically, the Chinese and Singaporean curricula provide concrete examples of promising ways to integrate arithmetic and algebraic ideas in the earlier grades.

In the mathematics education research community, we know little about mathematics education in India. Thus, we are especially pleased to have a chapter from India by Subramaniam and Banerjee. The authors shared India's version of addressing the connection between arithmetic and algebra. In India, algebra is seen as foundational to arithmetic rather than as a generalization of arithmetic. Subramaniam and Banerjee present a framework that illuminates the arithmetic-algebra connection from an Indian perspective.

Schmittau's chapter presents a description of a curricular approach to elementary school mathematics based on the work of Russian psychologists Lev Vygotsky and V.V. Davydov. In contrast to curricular approaches that view number as foundational in children's early mathematical development, the curricular approach Schmittau describes views algebraic structure as foundational. In this approach, "traditional" arithmetic understanding is developed as students apply their algebraic understanding to concrete numerical instances. As Schmittau points out, this approach is very different from recent reform approaches that seek to introduce elements of algebra into the study of arithmetic.

In Watanabe's chapter, he presents an analysis of the Japanese elementary school (Grades 1 through 6) mathematics curriculum materials and notes that the study of functional relationships (patterns) is a major emphasis in Japanese elementary schools. However, the Japanese curriculum considers the ideas related to mathematical expressions, called "*shiki*" in Japanese, as a pillar of elementary school algebra.

In summary, the chapters that comprise this part highlight various curricular approaches, experiences, and practices that illustrate ways in which students in earlier grades can be better prepared to think algebraically.

Functional Thinking as a Route Into Algebra in the Elementary Grades

Maria L. Blanton and James J. Kaput

Abstract This chapter explores how elementary teachers can use functional thinking to build algebraic reasoning into curriculum and instruction. In particular, we examine how children think about functions and how instructional materials and school activities can be extended to support students' functional thinking. Data are taken from a five-year research and professional development project conducted in an urban school district and from a graduate course for elementary teachers taught by the first author. We propose that elementary grades mathematics should, from the start of formal schooling, extend beyond the fairly common focus on recursive patterning to include curriculum and instruction that deliberately attends to how two or more quantities vary in relation to each other. We discuss how teachers can transform and extend their current resources so that arithmetic content can provide opportunities for pattern building, conjecturing, generalizing, and justifying mathematical relationships between quantities, and we examine how teachers might embed this mathematics within the kinds of socio-mathematical norms that help children build mathematical generality.

The research reported here was supported in part by a grant from the U.S. Department of Education, Office of Educational Research and Improvement to the National Center for Improving Student Learning and Achievement in Mathematics and Science (R305A600007-98). The opinions expressed herein do not necessarily reflect the position, policy, or endorsement of supporting agencies. This chapter is a revised version of an article published in *ZDM—International Reviews on Mathematical Education*, *37*(1), 34–42. DOI 10.1007/BF02655895.

J.J. Kaput is deceased.

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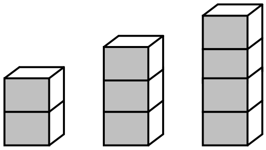
Introduction

Current research is redefining what we understand about the kind of mathematics that young children can and should learn (National Research Council [NRC] 2001). Consider the following Towers of Cubes problem (see Fig. 1) taken from the National Council of Teachers of Mathematics [NCTM] *Principles and Standards for School Mathematics* (2000, p. 160):



What is the surface area of each tower of cubes (include the bottom)? As the tower gets taller, how does the surface area change? What is the surface area of a tower with fifty cubes?

Fig. 1 Towers of cubes

In the not so distant past, such a problem was mostly absent from typical elementary school[1] curricula and instruction in the United States. While it might have appeared as an enrichment task, it was likely marginalized by the press towards computational skills (Thompson et al. 1994) and procedures that children were (and are) compelled to memorize as a signal of their readiness for higher mathematical thinking. Or, it might have appeared in an abbreviated, arithmetic form as "What is the surface area of a tower built of 3 cubes?" However, algebraic reasoning as an activity of generalizing mathematical ideas, using literal symbolic representations, and representing functional relationships, all implicit in this task, is no longer reserved for secondary grades and beyond, but is an increasingly common thread in the fabric of ideas that constitute mathematical thinking at the elementary grades.

The Challenge of Curriculum and Instruction

Simply put, young children today need to learn a different kind of mathematics than their parents learned. Some argue that they need to be "algebra ready" (e.g., National Mathematics Advisory Panel 2008). But what experiences make them ready for algebra, and for what kind of algebra are they being made ready? Romberg and Kaput (1999) maintain that understanding the increasingly complex mathematics of the 21st century will require children to have a type of elementary school experience that goes beyond arithmetic and computational fluency to attend to the deeper underlying structure of mathematics. It will require experiences that help children learn to recognize and articulate mathematical structure and relationships and to

[1]Elementary school refers here to grades PreK-5.

use these insights *of* mathematical reasoning as objects *for* mathematical reasoning. This type of elementary school experience has come to be embodied in what many refer to as *early algebra*,[2] and because its underlying purpose is to deepen children's understanding of the structural form and generality of mathematics and not just provide isolated experiences in computation, scholars increasingly agree that it is the avenue through which young children can become mathematically successful in later grades. Thus, our perspective on "algebra readiness" is that experiences in building, expressing, and justifying mathematical generalizations—for us, the heart of algebra and algebraic thinking—should be a seamless process that begins at the start of formal schooling, not content for later grades for which elementary school children are "made ready" through a singular, myopic focus on arithmetic.

But changing the mathematics elementary school children learn—their daily curriculum—is only part of the solution. As Blanton and Kaput note, "most elementary teachers have little experience with the kinds of algebraic thinking that need to become the norm in schools and, instead, are often products of the type of school mathematics instruction that we need to replace" (2005). However, these very teachers are central to reforms in children's school mathematical experiences. Moreover, the instructional materials in most elementary schools today are basal texts, and even newer, standards-based materials are just beginning to incorporate systematic approaches to the development of algebraic reasoning (Kaput and Blanton 2005). These constraints represent the challenge of building algebraic thinking into curriculum and instruction.

There are two issues implicit in the above discussion that this article aims to address: (1) how opportunities for algebraic thinking can be integrated into the elementary grades to prepare students for more powerful mathematics in later years, and (2) how elementary teachers can transform their own resources and instruction in ways that effect (1).

Functional Thinking as a Route to Algebraic Thinking

Early algebra can occur in several interrelated forms in the classroom.[3] We focus here on *functional thinking* as a strand by which teachers can build generality into their curriculum and instruction. We broadly conceptualize functional thinking to

[2]While there are multiple perspectives on early algebra, Lins and Kaput (2004) describe a general agreement among scholars that it involves "acts of deliberate generalization and expression of generality... [and] reasoning based on the forms of syntactically guided actions on those expressions".

[3]Kaput (2008) characterizes algebraic thinking as consisting of two core aspects: (1) making and expressing generalizations in increasingly formal and conventional symbol systems, and (2) reasoning with symbolic forms, including the syntactically guided manipulations of those symbolic forms. In turn, he argues that these core aspects cut across three longitudinal strands of school algebra: (1) Algebra as the study of structures and systems abstracted from computations and relations (e.g., algebra as generalized arithmetic); (2) Algebra as the study of functions, relations, and joint variation; and (3) Algebra as the application of a cluster of modeling languages to express and support reasoning about situations being modeled.

incorporate building and generalizing patterns and relationships using diverse linguistic and representational tools and treating generalized relationships, or functions, that result as mathematical objects useful in their own right. As the NCTM *Principles and Standards* (2000, p. 37) argues, children in the elementary grades should be able to

(1) Understand patterns, relations, and functions;
(2) Represent and analyze mathematical situations and structures using algebraic symbols;
(3) Use mathematical models to represent and understand quantitative relationships; and
(4) Analyze change in various contexts.

In addition, we use here three modes of analyzing patterns and relationships, outlined by Smith (2008), as a framework to discuss the kinds of functional thinking found in classroom data: (1) *recursive patterning* involves finding variation within a sequence of values; (2) *covariational thinking* is based on analyzing how two quantities vary simultaneously and keeping that change as an explicit, dynamic part of a function's description (e.g., "as x increases by one, y increases by three") (Confrey and Smith 1991); and (3) a *correspondence relationship* is based on identifying a correlation between variables (e.g., "y is 3 times x plus 2").

In what follows, we draw on data from a five-year research and professional development project in an urban school district (Kaput and Blanton 2005) and a subsequent graduate course for elementary teachers, taught by the first author, to examine how children think about functional relationships, its mathematical implications for later grades, and how instructional materials and school activities can be deepened and extended to support the development of functional thinking in the elementary grades.

Functional Thinking in the Elementary Grades

The idea of function has, for over a century, been regarded by mathematicians as a powerful, unifying idea in mathematics that merits a central place in the curriculum (Freudenthal 1982; Hamley 1934; Schwartz 1990). Indeed, the idea can be traced back to Leibniz (Boyer 1946). However, until very recently, the study of functions has been treated largely in the US as something to be learned in high school algebra, or even middle school mathematics. The perspective taken here is that the study of functions should be treated longitudinally and in its full richness beginning in early elementary school (NCTM 2000; Smith 2003).

But what capacity do young children have for functional thinking? Even though elementary school mathematics has more recently included recursive patterning, it has not attended pervasively to covariation or correspondence in functional thinking, especially in grades PreK-2. For instance, even NCTM (2000) suggests that, as late as fourth-grade, students might find a recursive pattern in the Towers of Cubes problem (see Fig. 1), and not until fifth-grade would they develop a correspondence relationship. Can elementary students, in fact, make the conceptual shift from simple recursive patterning to account for simultaneous changes in two or more variables? Moreover, at what grades can they do this? And can they, or in what ways can they, symbolize and operate on covariational or correspondence relationships in data?

Children's Capacity for Functional Thinking

Current research challenges the developmental constraints traditionally placed on young learners and their capacity for functional thinking. For example, researchers have found that elementary school children can develop and use a variety of representational tools to reason about functions, they can describe in words and symbols recursive, covarying, and correspondence relationships in data, and they can use symbolic language to model and solve equations with unknown quantities (e.g., Blanton 2008; Brizuela and Schliemann 2003; Brizuela et al. 2000; Carraher et al. 2008; Kaput and Blanton 2005; Moss et al. 2008; Schliemann and Carraher 2002; Schliemann et al. 2001).

While much of this research focuses on functional thinking in grades 3–5, we have found that students are not only capable of deeper functional analysis than previously thought, but that the genesis of these ideas appear at grades earlier than typically expected. In particular, we have found that the types of representations students use, the progression of mathematical language in their descriptions of functional relationships, the ways students track and organize data, the mathematical operations they use to interpret functional relationships, and how they express covariation and correspondence among quantities, can be scaffolded in instruction beginning with the very earliest grades, at the start of formal schooling (Blanton and Kaput 2004).

The following discussion draws on our research data to elaborate these capacities in children's functional thinking across elementary grades. We note that the data included here are intended to convey existence proofs of what is possible in children's thinking; our goal is not to examine the regularity with which functional thinking occurred in instruction.

The Development of Representational Infrastructure: Children's Use of Function Tables

Research, including early algebra research, suggests that students' flexibility with multiple representations both reflects and promotes deeper mathematical insights (Behr et al. 1983; Brizuela and Earnest 2008; Goldin and Shteingold 2001). Brizuela and Earnest note that "the connections between different representations help to resolve some of the ambiguity of isolated representations, [so] in order for concepts to be fully developed, children will need to represent them in various ways".

We found that teachers across the elementary grades were able to scaffold children's use of tables, graphs, pictures, words and symbols in gradually more sophisticated ways in order for them to make sense of data and interpret functional relationships (Blanton and Kaput 2004). For example, while students in grades PreK-1

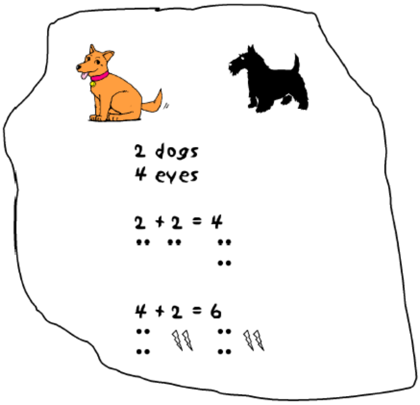


Fig. 2 Kindergarten students' representation for the numbers of eyes and eyes and tails for two dogs

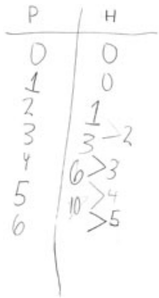


Fig. 3 A first-grader's t-chart for the Handshake Problem[5]

relied on counting visible objects or hand-written marks and registering their counts through inscriptions in t-charts[4] or through dots and hatch marks for eyes and tails (see Figs. 2 and 3), by second and third grade, students could routinely operate on data that had no iconic or tangible counterpart (e.g., tracking the number of eyes on ten dogs without pictures or physical objects). Moreover, while grades PreK-1 teachers typically led students in developing t-charts to organize their data, the responsibility for this began to shift to students during first grade. Figure 3 shows a t-chart, constructed by a first-grader, which records the total number of handshakes in a group of varying size (Blanton 2008).

We have found that the t-chart, or function table, becomes an important structure in children's mathematical reasoning. In the earlier grades (PreK-1), it provided a context to re-represent marks with numerals as children worked on the arithmetic of correspondence between quantity and numeral. But its introduction in these grades

[4]T-charts are teacher-termed function tables with a column of data for the independent variable followed by a column of data for the dependent variable.

[5]The Handshake Problem can be stated as follows: If 3 people are in a group, how many total handshakes would there be if every person shook hands with all people in the group once? How many handshakes would there be if there were 4 people in the group? Five people? Six people? Twenty people? Can you find a relationship between the number of people in the group and the total number of handshakes?

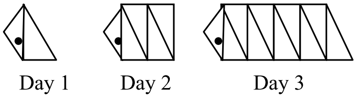


Fig. 4 The growing snake

as a tool for organizing covarying data also initiated its transformation from opaque to transparent object (Kaput 1995) in children's functional thinking as a representation that one could "look through" to "see" new relationships. The first grader's analysis of differences in the numbers of handshakes in Fig. 3 illustrates that as early as first grade, students can begin to transition beyond an understanding of t-charts as opaque objects—a place to record numbers—to a transparent object that can be used to determine relationships in data. We maintain that introducing such representational tools from the start of formal schooling can help spread the cognitive load across grades in a way that allows students in second and third grades (and beyond) to focus on more difficult tasks such as symbolizing correspondence and covariational relationships.

By second and third grades, we have found that students are able to use this tool transparently, as a mathematical object, in thinking about data. The following teacher narrative illustrates this algebraic reasoning with third-graders. The third-grade teacher who authored the narrative, Mrs. Gardiner, had designed a task in which students were to find the number of body parts a growing snake would have on day 10 and on day n, where each triangle equaled a body part. She drew the growing snake on the board for Days 1, 2, and 3 (see Fig. 4).

> The class worked on this problem for approximately 10 minutes. All organized their data with a t-chart. When I pulled the group together to discuss the problem, it was Karlie[6] who had her hand waving hard.... Karlie usually just sits and listens during math time, so her enthusiasm was very special. I called on her right away. 'I know that on day 10 the snake will have 101 body parts and I know that on day n the snake will have $n \times n + 1$. I know this because I used my t-chart and I looked for the relationship between n and body parts. This is the first time I saw the pattern, so please tell me I'm right!' she said excitedly.... The class had all come to pretty much the same answer.

This suggests to us that the t-chart helped structure Karlie's thinking about relationships between quantities. Unlike in the earlier elementary grades, where students were more likely to use t-charts opaquely as a storage system for numbers and were not yet able to attend to the meanings embedded in how data were positioned in the chart, the t-chart became the object, or tool, by which Karlie could compare data and find relationships. She was able to attend to how numbers were located in the chart and see through it to the relationships it made available to her. In this sense, we maintain that the t-chart had become transparent in how she used it to think about functions. Our point is that critical instruction in the earlier grades (PreK-1) can initiate the transition of representational tools from opaque to transparent objects in children's thinking so that children are able to shift their attention to more complex tasks in later elementary grades and beyond. This is exactly how

[6] All student names are pseudonyms.

mathematics has grown in power historically, as new representation systems were developed (including that of algebra itself) to increase the power of human thinking.

The Development of Students' Symbol Sense

One particularly vital aspect of early algebra is the transition from natural language to symbolic notational systems. If one's perspective is that development precedes learning, then the use of symbols as variables in elementary grades is, perhaps, not without controversy. However, we take the view here that learning promotes development and that it entails a pseudo-conceptual stage of concept formation in students' development of symbol sense. In describing the development of higher mental functioning in children, Vygotsky (1962) identified the notion of a *pseudo-concept* as an essential bridge in children's thinking to the final stage of concept formation. While the pseudo-concept a child possesses is phenotypically equivalent to that of an adult, it is psychologically different. As a result, the child is able to "operate with [the concept], to practice conceptual thinking, before he is clearly aware of the nature of these operations" (p. 69). This suggests that learning to think mathematically involves the acquisition of notational tools that are within the child's zone of proximal development, but not entirely owned by the child. In essence, it involves students' transition from an opaque to transparent use of symbols. Moreover, the dialectic between thought and language in learning (Vygotsky 1962) implies that symbolic notational systems are more fully conceptually formed in children's thinking as a result of children's interaction with them in meaningful contexts. In short, children can develop symbol sense as they have opportunity to use symbolic notation in meaningful ways (see also Brizuela et al. 2000).

We have found that, when curriculum and instruction provide opportunity for thinking about functional relationships, students can transition linguistically from iconic and natural language registers at grades PreK-1 to symbolic notational systems by grade 3 (Blanton and Kaput 2004). A first grade teacher described how one of her students made this transition while thinking about the Handshake Problem:

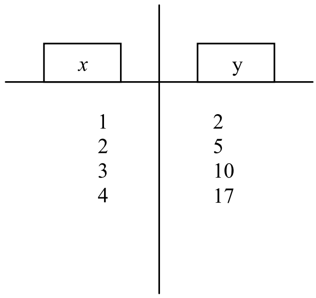
> I asked, 'Can I label one side [of the t-chart] 'people' and the other side 'handshake'?' One little boy said, 'Just write 'p' and 'h'.' I immediately stopped what I was doing. I asked, 'What did you say?' He continued to repeat what I heard him say. 'Awesome, how did you come up with that?' I probed. He continued, 'Well, 'people' begins with p and 'handshakes' begin with h.' (Blanton 2008, p. 43)

While this student's understanding of variable is certainly in its early stages (for example, care must be taken to ensure that the student does not confuse the variable as representing the object and not the quantity), the point here is that giving children opportunities to *begin* using symbolic representations can occur as early as first grade, and acquiring these more basic ideas in the early grades allows them greater cognitive room to explore more complex ideas in later elementary grades.

By third-grade, students can move beyond this more primitive act of symbolizing to describe and discuss functional relationships. We include the following teacher narrative to illustrate third-grade students using symbolic notation to think about the number of circle-shaped body parts on a growing caterpillar. In this vignette, Mrs. Gardiner has just described the Growing Caterpillar task to her students.[7]

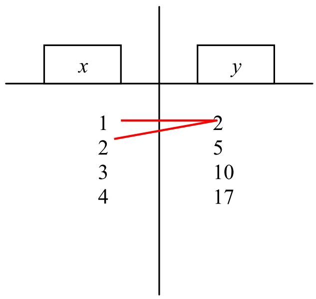
> I showed my students my caterpillar example and all I wanted them to see was how I developed the problem. I had no idea that they would begin to solve the problem. I couldn't stop them. There were hands going up all over the place. Everyone wanted to tell me the pattern they saw when they looked at the growth of the caterpillar. I said, 'Guys, I haven't even asked you the question yet'. 'But I see the pattern, Mrs. Gardiner', yelled Jak. 'Okay, what do you think the pattern is?' I asked. 'I think it is x times 2 plus one', he said. 'How many of you agree with Jak?' I questioned. 'I don't know. I have to do a t-chart', explained Meg. 'Well, then let's do that together on the board', I said. With the students' help, we drew the following t-chart on the board (see Fig. 5):

Fig. 5 T-chart for growing caterpillar



> 'Now that we have that on the board, I don't agree with Jak', said Meg. 'Why is that Meg?' I asked. 'Because if it was x times 2 plus 1, then x would be one and y would be three. And, it's not. It's $x = 1$ and $y = 2$', she explained.... [If x equaled 1, then by Jak's formula, y would be $2(1) + 1$, or 3, not 2, as the t-chart indicated.] The class struggled with the pattern for a long time. Then Shane saw a pattern that I had not seen. He came up to the t-chart on the board and with a red marker highlighted the pattern. It looked like this (see Fig. 6):

Fig. 6 Shawn's Pattern for Growing Caterpillar task



> So, what Shane was saying is that if you add $1 + 2 + 2$, it equals 5. If you then add $2 + 5 + 3$, it equals 10. This ... didn't help him find the formula, but it did help Joe! 'I see it, I know

[7] Growing Caterpillar was similar to Growing Snake except for the shape of the body parts. The growth rates for the snake and caterpillar were the same. Mrs. Gardiner had given Growing Caterpillar to students two weeks prior to Growing Snake.

> the formula!' Joe cried out. 'Well, what is it?' I prodded. 'It's $x \times x + 1 = y$', he said. At that moment, a loud group of 'Oh yeah's' could be heard in the room.... I asked everyone why this was algebra. I think Jak put it best. He said, 'Because we have people looking for patterns and relationships and we have them developing a formula'.

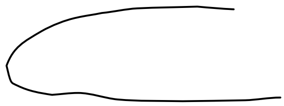
There are several points with respect to students' use of symbols that bear mentioning here. First, before any data were publicly recorded and without any prompting from the teacher, Jak proposed a symbolic relationship between an arbitrary day, x, and the number of caterpillar body parts. His spontaneous use of symbols conveys the generality with which he was beginning to think about functional relationships. Second, Meg was beginning to reason transparently with the t-chart and the symbolic relationship conjectured by Jak in order to refute his idea ("Now that we have that on the board, I don't agree with Jak"). That is, implicit in her refutation was her reasoning with both the meaning embedded in the structure of the t-chart, including the unique roles of dependent and independent variables, as well as the symbolic notation ("Because if it was x times 2 plus 1, then x would be one and y would be three. And, it's not. It's $x = 1$ and $y = 2$"). Meg's emerging transparent use of symbolic notation (as well as her evident understanding of equality, another critical issue in the development of children's algebraic thinking) is further indicated by her treatment of the expression 'x times 2 plus 1' and the dependent variable, y, as equivalent quantities.

Because the elementary grades often incorporate meaningful imagery and concrete experiences to support conceptual development, they, more so than secondary grades, can provide a rich, inquiry-based atmosphere for introducing symbolic notation. Thus, as with the development of representational infrastructure, we maintain that instruction should begin to scaffold students' thinking toward symbolic notation from the start of formal schooling so that students can transition from an opaque to transparent use of symbols as they progress through the elementary grades. Ultimately, elementary students who have learned to reason symbolically in meaningful ways will be much better prepared for the abstractions of more advanced mathematical thinking in later grades.

The Emergence of Thinking About Covariational and Correspondence Relationships

We have found it particularly compelling that, even as early as kindergarten, children can think about how quantities co-vary and, as early as first grade, can describe how quantities correspond (Blanton and Kaput 2004). For instance, in the task Cutting String (Blanton 2008; see also Cramer 2001), children are asked to look for a relationship between the number of cuts on a piece of string and the resulting number of pieces of string when the string is folded in a single loop (see Fig. 7). First graders were able to describe the relationship not only in recursive terms ("It gets two more each time"), but also in terms of a co-varying relationship "Every time you make one more snip it's two more" (Blanton 2008).

Fig. 7 Folded piece of string



In a task in which students were asked to describe the total number of eyes or the total number of eyes and tails for an increasing number of dogs, one kindergarten class described an additive covariational relationship between the numbers of eyes and dogs as "every time we add one more dog we get two more eyes". In first and second grade, students identified a multiplicative relationship of "doubles" and "triples" between the number of eyes and the number of eyes and tails, respectively, for an increasing number of dogs. The observation that the pattern "doubles" or "triples" suggests that students could attend to how quantities corresponded. For example, some quantity (in particular, the independent variable) needed to be doubled to get the total amount of eyes. Since data representing the total number of eyes (i.e., 2, 4, 6, 8...) were not doubled to get subsequent quantities of dog eyes (4 doubled does not yield the next value of 6; 6 doubled does not yield the next value of 8), this suggests that students were not looking for a recursive pattern such "add 2 every time" or "count by 2's", but a relationship between two quantities.

We recognize that some children might be responding to a known relationship without fully understanding its functional aspect. "Doubles", for example, is not uncommon in the vocabulary of early grades mathematics, and to say "it doubles" does not necessarily indicate a full conceptual understanding of correspondence or covariation, including an explicit understanding that the value of the independent variable is being doubled to obtain the value of the dependent variable. For some children, "doubles" could be code for a pattern recognized as adding by two's. However, these situations can prompt discussions that scaffold students' thinking about relationships between data, not just recursive patterning.

As the Growing Snake and Growing Caterpillar excerpts suggest, by third grade students can attend to how quantities co-vary and, moreover, symbolize relationships as a functional correspondence (e.g., "It's $x \times x + 1 = y$"). Thus, although the data on cutting string and dog eyes and tails illustrate a simple mathematical relationship for which some children used only natural language to describe covariational and correspondence relationships, we think this represents the critical kinds of experiences that children need in the earlier elementary grades in order to leverage deeper, more complex functional thinking in later elementary grades and beyond.

Implications of Children's Functional Thinking for Later Grades

The preceding discussion underscores how early algebra, and functional thinking in particular, can nurture the development of students' mathematical thinking in later grades. To begin with, it can help children build critical representational and linguistic tools for analyzing, describing and symbolizing patterns and relationships. Moreover, if teachers scaffold these ideas from the start of formal schooling, these experiences can provide a continuum of mathematical development whereby opaque symbols and tools can be transformed into transparent objects of functional thinking. T-charts and graphs become not just visual configurations, but structures embedded with meaning about relationships; symbols are no longer meaningless abstract inscriptions, but tools by which broader ideas can be mediated and communicated. Moreover, the elementary grades, because of its inclination towards concrete, tactile, and visual experiences in learning, can bridge the expression of mathematical ideas from natural, everyday language to symbolic notational systems in meaningful ways. For example, students in secondary grades are often given, *a priori*, a symbolic generalization about the commutative property of addition for real numbers a and b ($a + b = b + a$). In contrast, early algebra entails exploring this property through operations on particular numbers, then generalizing the property using everyday or symbolic language systems, where the symbolizing develops as a valid linguistic form of expression through children's interactions with number and operation (Carpenter et al. 2003).

All of these experiences—the development of representational and linguistic tools, the transformation of mathematical structures and symbols from opaque to transparent objects, and the integration of concrete, tactile, and visual experiences to support the development of mathematics with understanding—coalesce to build mathematical thinkers for whom abstract ideas are rooted in meaningful, concrete events. As a result, we argue that children for whom functional thinking is a routine part of mathematics in the elementary grades are better prepared than those who spend the first six or seven years of formal schooling fine-tuning arithmetic skills, procedures and facts.

Integrating Functional Thinking into Curriculum and Instruction

While much more could be said about children's capacity for functional thinking, our point thus far is that young children can identify and express functional relationships in progressively more symbolic ways and that instruction in the elementary grades that nurtures this kind of thinking can support students' mathematical thinking in later grades. Although this suggests a mandate for change in elementary school curricula, our reality is often working with teachers who have limited resources that, more often than not, focus on the development of children's arithmetic thinking. Moreover, curricular innovations alone, without the development of teachers' instructional and mathematical knowledge on how to build children's functional thinking, are not sufficient to produce real change in children's mathematical thinking. Smith (2003) notes that "elementary school teachers may create rich classroom experiences around patterns, yet not have a sense of how this topic ties into the ongoing mathematical development of their students, much less into the topic of functions" (p. 136). To address this, our early algebra work with teachers has involved three connected dimensions of change: (1) transforming teachers' instructional resource base, (2) using children's thinking to leverage teacher learning, and (3) creating classroom culture and practice to support algebraic thinking. In what follows, we address each of these and how they support the integration of functional thinking into curriculum and instruction.

Transforming Teachers' Resource Base to Support Students' Functional Thinking

In spite of limited resources or the lack of materials that integrate functional thinking in viable ways, elementary teachers can transform their existing instructional resource base to include the exploration of covariational and correspondence relationships. Our approach with teachers is to help them deliberately transform single-numerical-answer arithmetic problems to opportunities for pattern building, conjecturing, generalizing, and justifying mathematical relationships by varying the given parameters of a problem (Blanton and Kaput 2003). This is easily done with tasks such as the Telephone Problem, which might typically be posed as an arithmetic task with a single numerical answer:

How many telephone calls could be made among 5 friends if each person spoke with each friend exactly once on the telephone?

Stated this way, students simply need to compute a sum, although they might first draw a picture or diagram to keep track of the phone calls. Functional thinking can be introduced into the task by varying the number of friends in the group:

How many telephone calls would there be if there were 6 friends? Seven friends? Eight friends? Twenty friends? One hundred friends? Organize your data in a table. Describe any relationship you see between the number of phone calls and the number of friends in the group. How many phone calls would there be for n friends?

The tasks included here (e.g., Growing Snake, Growing Caterpillar, Towers of Cubes) are examples of this type of transformation; all are derived from single-numerical-answer tasks. For example, Towers of Cubes can be seen as an extension of the arithmetic problem "What is the surface area of a tower built of 3 one-inch cubes?" Similarly, Growing Snake can be seen as an extension of an arithmetic task in which students count the total number of body parts for a particular snake.

Varying Task Parameters Introduces Algebraic Thinking into the Curriculum

But how does this transformation lead to algebraic thinking or, specifically, functional thinking? First, varying a problem parameter enables students to generate a set of data that has a mathematical relationship, and using sufficiently large quantities for that parameter leads to the algebraic use of number. For example, in the Telephone Problem, finding the number of phone calls for a group whose size is large enough so that children cannot (or would not want to) model the problem and write down a corresponding sum to compute requires children to think about the structure in the numbers and how the numbers of phone calls for the various groups are related to the number of people in the group. From their analysis, children can identify a recursive pattern or conjecture a covariational or correspondence relationship between the total numbers of phone calls and the variations in the parameter that produces them. Moreover, depending on the grade and skill of the student, the teacher can scaffold students in describing their conjectures with symbolic notation. Children can then develop justifications for whether or not their conjectured relationships and patterns hold true. Finally, the mathematical generalizations that result, while important results in and of themselves, can become objects of mathematical reasoning as students become more sophisticated algebraic thinkers (Blanton 2008; Blanton and Kaput 2000). None of these processes occur if tasks remain purely arithmetic in scope.

As we describe elsewhere, our approach "recasts elementary mathematics in a profound way, not by ignoring its computational agenda, but by enlarging the agenda in ways that include the old in new forms that deliberately contextualize, deepen, and leverage the learning of basic skills and number sense by integrating them into the formulation of deeper mathematical understandings" (Kaput and Blanton 2005). In essence, a powerful result of transforming arithmetic tasks in this way is that children are doing many important things all at once, including building number sense, practicing number facts, building and recognizing patterns to model situations, and so forth (Blanton and Kaput 2003). In fact, this genre of tasks can provide large amounts of computational practice in a context that intrigues students and that avoids the mindlessness of numerical worksheets.

Transforming the Curriculum Empowers Teachers

Moreover, we have found that when teachers transform their own instructional resource base so that arithmetic tasks are extended to include opportunities for establishing and expressing mathematical generalizations, they are able to transcend constraints imposed by their existing school culture such as limited or inadequate resources, or even their own lack of experience with teaching algebraic thinking. Instead, they are able to see algebraic thinking as a fluid domain of thinking which permeates all of mathematics, not as a set of tasks or a prescribed curriculum. Thus, what we advocate, more so than an "early algebra curriculum" per se, involves the development of a habit of mind that transcends the particular resource being used and allows elementary teachers to see opportunities for algebraic thinking, and functional thinking in particular, in the mathematics they already teach, using the curriculum they have in place. After using only two functional thinking tasks with her students, one third-grade teacher wrote

> I had a new outlook on math. I knew I wanted to integrate algebraic thinking into every topic I did. The truth was that our curriculum was wonderful. It allowed plenty of ways to integrate this way of thinking. I just hadn't noticed up to this point (Blanton 2008, p. xii).

This sense of empowerment, as well as the development of an algebraic habit of mind, was later echoed by a first-grade teacher:

> [Functional thinking] activities at the beginning seemed like they were going to be hard to do, never mind creating my own. I've realized that they are a lot simpler to create and implement than I thought. I am really impressed with how these activities have shaped my way and my students' way of thinking algebraically. They have really opened my mind up about algebra and how, if we put it into a simple form, our students can do it! (Blanton 2008, p. 147)

Using Children's Functional Thinking to Leverage Teacher Learning

Integrating functional thinking into instruction does not rest solely on the particular materials the teacher chooses or develops. It requires an "algebra sense" by which teachers can identify occasions in children's thinking to extend conversations about arithmetic to those that explore mathematical generality. While the task one chooses can certainly support this, teachers also need the skills to interpret what children are writing about and talking about. In turn, a written or verbal record of student thinking can serve as a tool to engage teachers in thinking about content and practice. As teachers think collectively about how children make sense of data, whether and how they attend to how quantities relate, the kinds of meaning they derive from tables and graphs, and how they use symbols in describing and reasoning with mathematical ideas, they have the potential to build functional thinking into instruction in deeper and more compelling ways (Kaput and Blanton 2005).

The work of Cognitively-Guided Instruction (Carpenter and Fennema 1999) has been significant in bringing student thinking to the fore in how people conceptualize and engage in teacher professional development. More recently, researchers have extended this approach as a tool in the development of teachers' early algebraic thinking (Franke et al. 2001; Kaput and Blanton 2005). The assumption is that focusing on children's (algebraic) thinking in professional development builds teachers' capacity to identify classroom opportunities for generalization and to understand the representational, linguistic and symbolic tools that support this and the particular ways students use these to reason algebraically. Thus, if teachers are to build algebraic thinking into their instruction, they must become engaged in and by what students are saying, doing and writing as a catalyst for building their own classroom algebraic discourse. Moreover, they must be given occasions to use these classroom artifacts to negotiate mathematical and instructional knowledge within teacher communities of practice as a way to develop their *own* knowledge of algebra and teaching algebra. One fourth-grade teacher described her early experience in leading this work with her teacher peers:

> At our last professional development day, I told the other teachers that I have been doing algebra problems during my math workshop time. I told them the types of problems we did and how I have been implementing the problems in class. I told them it was a great way to get kids to look at numbers in different ways. I explained how it was more than algebra; it also helps kids practice basic arithmetic. I showed them samples of students' work. I even explained the importance of organizing data, finding a recursive pattern and finding a function. I talked so confidently about algebra that the teachers were intrigued. For the first time in my life, I was a math teacher! (Blanton 2008, p. 147)

Creating Classroom Culture and Practice to Support Functional Thinking

Building on children's functional thinking in instruction requires that a culture of practice that promotes this type of thinking exists. Classrooms in which children's functional thinking can thrive are those in which the teacher has established socio-mathematical norms of conjecturing, arguing, and generalizing in purposeful ways, where the arguments are taken seriously by students as ways of building reliable knowledge. Robust functional thinking requires children to interact with complex mathematical ideas, to negotiate new notational systems and to understand and use representational tools as objects for mathematical reasoning. It requires that the teacher respect and encourage these processes as standard practice on a daily basis, not as occasional enrichment treated as separate from the "regular" work of learning and practicing arithmetic.

The teacher narratives included here illustrate the kinds of classroom practice and culture that can support the development of children's functional thinking. For example, the Growing Caterpillar narrative depicts ways of doing mathematics in which the teacher (1) followed students' thinking in shaping a lesson's agenda ("I showed my students my caterpillar example and all I wanted them to see was how I developed the problem. I had no idea that they would begin to solve the problem. I couldn't stop them".), (2) placed the responsibility for conjecture, argumentation and justification with students ("Okay, what do you think the pattern is?", "How many of you agree with Jak?", "Why is that Meg?"), (3) cultivated children's use of representational structures as tools for reasoning (Meg: "I don't know. I have to do a t-chart"), (4) encouraged the use of symbolic notational systems as valid forms of mathematical expression (Jak: "I think it is x times 2 plus one"; Meg: "Because if it was x times 2 plus 1, then x would be one and y would be three. And, it's not. It's $x = 1$ and $y = 2$"; Joe: "I see it, I know the formula!.... It's $x \times x + 1 = y$"), and (5) used children's utterances to craft an idea-building, dialogic discourse that led to symbolizing a functional relationship. In short, these aspects of practice allowed children to construct a mathematical generalization about the caterpillar's growth.

Children's role in this process is critical; we are not advocating a form of practice in which children do not actively participate in the development of conjectures, the construction of arguments, the establishment of generalizations, or the use of notation, language, and tools for reasoning about functions. All of these experiences are critical components of the kind of classroom culture that makes functional thinking viable when it does occur.

Conclusion

This chapter elaborates the position that elementary school children are capable of functional thinking and that its study in the elementary grades can affect their success in mathematics in later grades. We propose that elementary grades mathematics extend beyond the fairly common, initial focus on recursive patterning to include curriculum and instruction that deliberately attends to how two or more quantities vary in relation to each other and that begins to scaffold these notions from the start of formal schooling. Because there is a fundamental conceptual shift that must occur in how teachers and students attend to data in recursive patterning as opposed to covariational or correspondence relationships, we speculate that the emphasis on recursive patterning that does occur in the early elementary grades curricula, could, if taught in isolation, impede the development of covariational and correspondence thinking about functions in later grades.

Children's capacity for functional thinking raises the issue of how it might be nurtured by curriculum and instruction in the elementary grades. We advocate here a habit of mind, not just curricular materials, whereby teachers understand both how to transform and extend their current resources so that the mostly arithmetic content of the elementary grades can be extended to opportunities for pattern building, conjecturing, generalizing, and justifying mathematical relationships and how to embed this mathematics within the kinds of socio-mathematical norms that allow children to build mathematical generality. Generalizing is a human activity and an innate, natural capacity that young children bring to the classroom (Mason 2008). Curriculum and instruction should build on these natural abilities to provide a deeper, more compelling mathematical experience for young children.

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Developing Students' Algebraic Thinking in Earlier Grades: Lessons from China and Singapore

Jinfa Cai, Swee Fong Ng, and John C. Moyer

Abstract In this chapter, we discuss how algebraic concepts and representations are developed and introduced in the Chinese and Singaporean elementary curricula. We particularly focus on the lessons to be learned from the Chinese and Singaporean practice of fostering early algebra learning, such as the one- problem-multiple-solutions approach in China and pictorial equations approach in Singapore. Using the lessons learned from Chinese and Singaporean curricula, we discuss four issues related to the development of algebraic thinking in earlier grades: (1) To what extent should we expect students in early grades to think algebraically? (2) What level of formalism should we expect of students in the early grades? (3) How can

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we help students make a smooth transition from arithmetic to algebraic thinking? and (4) Are authentic applications necessary for students in early grades?

Introduction

Algebra has been characterized as the most important "gatekeeper" in mathematics. It is widely accepted that to achieve the goal of "algebra for all," students in elementary school should have experiences that prepare them for the more formal study of algebra in the later grades (National Council of Teachers of Mathematics [NCTM] 2000). However, curriculum developers, educational researchers, and policy makers are just beginning to explore the kinds of mathematical experiences elementary students need to prepare them for the formal study of algebra at the later grades (Britt and Irwin, this volume; Carpenter et al. 2003; Carraher and Schliemann 2007; Kaput 1999; Mathematical Sciences Education Board 1998; NCTM 2000; Schifter 1999; Stacey et al. 2004).

For example, there is evidence that U.S. students are ill-prepared for the study of algebra (Silver and Kenney 2001). One of the challenges teachers in the United States face is the lack of a coherent K-8 curriculum that can provide students with algebraic experiences that are both early and rich (Schmidt et al. 1996). Customarily, algebra has not been treated explicitly in the school curriculum until the traditional algebra course offered in middle school or high school (NCTM 2000). Moreover, according to a rigorous academic analysis by the American Association for the Advancement of Sciences (AAAS 2000), the majority of textbooks used for algebra in the United States have serious weaknesses. In addition, there is evidence that most elementary school teachers in the United States are not adequately prepared to integrate algebraic reasoning into their instructional practices (e.g., van Dooren et al. 2002). By comparison, it is plausible that the preparation of Chinese and Singaporean elementary school teachers benefits from their own elementary school education, in which the formal study of algebra begins much earlier than in the United States.

Most U.S. students do not start the formal study of algebra until eighth or ninth grade, and many of them experience difficulties making the transition from arithmetic to algebra because they have little or no prior experience with the subject (Silver and Kenney 2001). The unsatisfactory findings from national and international assessments (e.g., NAEP, TIMSS) indicate a need to develop U.S. students' algebraic thinking in the early grades. In several countries (e.g., China and Singapore), students begin the formal study of algebra much earlier. However, it is likely that, across these settings, curriculum developers, educational researchers, and policy makers faced challenges similar to those of their counterparts in the United States and other countries as they attempted to develop early and appropriate algebraic experiences for younger children (Cai and Knuth 2005; Carpenter et al. 2003; Kieran 2004; Stacey et al. 2004).

In this chapter, we discuss how algebraic concepts and representations are developed and introduced in the Chinese and Singaporean elementary curricula. We particularly focus on the lessons to be learned from the Chinese and Singaporean practice of fostering early algebra learning. Knowledge of the curricula and instructional practices of different nations can increase educators' and teachers' abilities to meet the challenges associated with the development of students' algebraic thinking.

We should state at the outset that it is not our intent in this paper to evaluate the curricula or the instructional practices used in China and Singapore. Instead, our focus is on studying and understanding how the curricula (and instruction) in the two countries contribute to the development of students' algebraic thinking. We believe that the understanding we gain by taking this international perspective will increase our ability to address the issues related to the development of students' algebraic thinking in elementary school. We need to support students' development of algebraic thinking in the early grades and help them appreciate the usefulness of algebraic approaches in solving various problems. Therefore, we are especially interested in the way the two curricula prepare students to make smooth transitions from informal to formal algebraic thinking.

In the sections that follow, we first present the unique features of the two curricula we studied. Then we discuss the lessons we can learn from China and Singapore regarding the development of students' algebraic thinking in earlier grades.

Features of the Chinese and Singaporean Curricula

We analyzed the national curricula of China and Singapore to understand how the authors intended to help students develop algebraic thinking in elementary school (Curriculum Planning & Development Division 2000; Division of Elementary Mathematics 1999). We were interested in the algebraic concepts included in the curriculum and in how these concepts are developed and represented. In particular, we analyzed each curriculum along three dimensions: (1) goal specification, (2) content coverage, and (3) process coverage (Cai 2004a). Results from the analysis can be found in Cai (2004b) and Ng (2004). The focus of this chapter is to identify the unique features of Chinese and Singaporean curricula and then to illustrate how mathematics teachers and researchers can use the international perspective to tackle difficult issues related to the development of algebraic thinking in early grades.

Algebra Emphases in the Chinese and Singaporean Curricula

The four goals of the algebra standard in the *Principles and Standards for School Mathematics* (NCTM 2000) are: Goal 1—Understand patterns, relations, and functions; Goal 2—Represent and analyze mathematical situations and structures using algebraic symbols; Goal 3—Use mathematical models to represent and understand quantitative relationships; and Goal 4—Analyze change in various contexts. We employed our previously developed case studies of the Chinese and Singaporean curricula (Cai 2004b; Ng 2004) to compare the emphases of each curriculum with these four goals. Our criteria for determining whether a curriculum emphasizes the development of a certain NCTM goal are that the curriculum must explicitly state a similar learning goal or must include extensive instructional activities clearly intended to help students achieve the goal.

There is a consistency between the curricular emphases of both the Chinese and Singaporean curricula. They both explicitly state that their main goal in teaching algebraic concepts is to deepen students' understanding of quantitative relationships. Furthermore, both curricula include extensive use of algebraic models that are intended to help achieve the overriding goal of deepening students' understanding of quantitative relationships. These two features indicate that both curricula emphasize the development of goal 3 of NCTM's algebra standard.

In fact, the algebraic emphasis in both Chinese and Singaporean elementary school mathematics is consonant with all but the fourth goal of the NCTM algebra standard. In China, the fourth goal is not addressed fully until the concept of function is formally introduced in junior high school, although qualitative analysis of change is done in elementary school. In Singapore, the fourth goal is not developed until students are in secondary school.

The Chinese Curriculum

The overarching algebra-related goal in the Chinese elementary curriculum is for students to better represent and understand quantitative relationships, numerically and symbolically. The main focus is on equations and equation solving. Variables, equations, equation solving, and function sense permeate the curriculum in grades 1 to 4. Equations and equation solving are formally introduced in the first half of grade 5. Once equation solving is introduced, it is applied to the learning of mathematical topics, such as fractions, percents, statistics, and proportional reasoning, in the second half of grade 5 and grade 6.

The term "variable" is not formally defined in Chinese elementary school mathematics. However, in the teacher's guide for the national curriculum, teachers are reminded that variables can represent many numbers simultaneously, that they have no place value, and that representations of variables can be selected arbitrarily. In Chinese elementary school mathematics, variable ideas are used in three different ways. First, they are used as place holders for unknowns in equation solving. In grades 1–2, for example, a question mark, a picture, a word, a blanket, or a box is used to represent the unknowns in equations. Second, variables are viewed as pattern generalizers or as representatives of a range of values. Specifically, words, rather than letters, are used to represent variables in order to help 3rd grade students understand the meanings of formulas. For example, after examining several specific examples, the formulas for the areas of rectangles and squares are represented as area of a rectangle $=$ its length $\times$ its width and area of a square $=$ its side $\times$ its side. However, teachers are counseled to emphasize the generalizable nature of the formulas. That is, for any rectangle, its area can be found by multiplying its length and width. In grades 5 and 6, letters are used to represent formulas for finding areas of squares, triangles, rectangles, trapezoids, and circles. The third use of variables is to represent relationships, such as direct proportionality ($y/x = k$) and inverse proportionality ($xy = k$).

The function concept is not formally introduced in the Chinese elementary curriculum. However, function ideas permeate the curriculum so that students' function sense can be informally developed. According to the curriculum guide, the pervasive use of function ideas in various content areas not only fosters students' learning of the content topics, but also provides a solid foundation for the future learning of advanced mathematical topics in middle and high schools. In the early grades, the Chinese curriculum provides students with many opportunities to develop function sense at a concrete, intuitive level. Function sense is first introduced in the context of comparing and operating with whole numbers in grade 1 using a one-to-one mapping. In grade 6, multiple representations (pictures, diagrams, tables, graphs, and equations) are used to represent functional relationships between two quantities. These functional relationships are embedded in the curricular treatments of circles, statistics, and proportional reasoning.

The Chinese elementary school curriculum is intended to develop at least three thinking habits in students. The first thinking habit is to examine quantitative relationships from different perspectives. Students are consistently encouraged, and provided with opportunities, to represent a quantitative relationship in different ways. Throughout the Chinese elementary school curriculum, there are numerous examples and problems that require students to identify quantitative relationships and represent them in multiple ways (Cai 2004b). For example, in the following problem from Grade 2, the quantitative relationship involves the amount of money paid to the cashier, the change, and the cost of two batteries: *Xiao Qing purchased two batteries. She gave the cashier 6 Yuans and got 4 Yuans in change back. How much does each battery cost*? The teacher's reference book recommends that teachers allow students to represent the quantitative relationship in different ways, such as the following:

The amount of money paid to the cashier − the cost of the two batteries = the change.

The cost of the two batteries + the change = the amount of money paid to the cashier.

The amount of money paid to the cashier − the change = the cost of the two batteries.

The second thinking habit is to solve a problem using both arithmetic and algebraic approaches. Expectations that students solve problems arithmetically in multiple ways and also algebraically in multiple ways can clearly be seen in the Chinese curriculum, and such expectations are common in Chinese classrooms by teachers (Cai 2004c). Furthermore, students are asked to make comparisons between arithmetic and algebraic ways of representing quantitative relationships. For example, in Grade 5, teachers and students can discuss and compare different ways to solve the following problem: *Liming elementary school has funds to buy 12 basketballs at 24 Yuans each. Before buying the basketballs, they decided to spend 144 Yuans of the funds for some soccer balls. How many basketballs can they buy*?

Arithmetic solutions:

Solution 1: Begin by computing the original funding and subtract the money spent on soccer balls: $(24 \times 12 - 144) \div 24 = 144 \div 24 = 6$ basketballs.

Solution 2: Begin by computing the number of basketballs that can no longer be bought: $12 - (144 \div 24) = 6$ basketballs.

Algebraic solutions:

Solution 3: Assume that the school can still buy x basketballs: $(24 \times 12 - 144) = 24x$.
Therefore, $x = 6$ basketballs.

Solution 4: Assume that the school can still buy x basketballs: $24 \times 12 = 24x + 144$.
Therefore, $x = 6$ basketballs.

Solution 5: Assume that the school can still buy x basketballs. $12 = (144 \div 24) + x$.
Therefore, $x = 6$ basketballs.

In the Chinese curriculum, after equation solving is introduced, students have opportunities to use an equation-solving approach to solve application problems as they learn statistics, percents, fractions, and ratios and proportions (Cai 2004b). This arrangement is built into the curriculum to deepen the students' understanding of quantitative relationships and to help students appreciate the equation-solving approach. For example, students in Grade 6 are encouraged to use four different methods to solve the following percent problem: *A factory modified its production procedures. After that, the cost of making one product was 37.40 Yuans which is 15% lower than the cost before the production procedures were modified. What was the cost of making the same product before the production procedures were modified?*

Solution 1: If the pre-modification cost is viewed as the unit 1, then the current cost is 15% less than the cost before modification.

Let $x =$ the cost before the modification.
$x - 15\%x = 37.4$
$(1 - 15\%)x = 37.4$
$85\%x = 37.4$
$x = 44$

Answer: The cost before the modification was 44 Yuans.

Solution 2: If the pre-modification cost is viewed as the unit, then 37.40 is 15% less, or $1 - 15\% = 85\%$ of the cost before the modification. Therefore, the cost before the modification is $37.4 \div 85\% = 44$.
Answer: The cost before the modification was 44 Yuans.

Solution 3: Similarly, the figure below can be used to represent the problem if the pre-modification cost is viewed as the unit. Since the current cost is 15% less than the cost before the modification, the current cost is 85% of the previous cost. Therefore, the cost before the modification was $37.4 \div 85\% = 44$.

|________________________________|

The cost before the modification

37.40 Yuans

|______________________|. . . . .⋮

The current cost 15%

Answer: The cost before the modification was 44 Yuans.

Solution 4: Because the 85% also represents a ratio, the ratio of costs before and after the modification is 85 to 100. It must be the same as 37.40 Yuans to x, where x is the cost before the modification.
Hence, $37.4/x = 85/100, x = 44$.
Answer: The cost before the modification was 44 Yuans.

Undoubtedly, the approach of asking students to use and compare these four different types of solutions (algebraic, arithmetic, pictorial, and ratio) is based on the principle that considering multiple perspectives can foster a deep understanding of the relationship between quantities.

According to the Chinese curriculum, solving a problem using both an arithmetic approach and an algebraic approach helps students build arithmetic and algebraic ways of thinking about problem solving. At the elementary school level, Chinese students solve problems like the examples given in this section, all of which can be solved arithmetically. As might be expected, it is common at the beginning of their transition to algebraic problem solving for students to wonder why they need to learn an equation-solving approach. However, after a period of time using both approaches, students come to see the advantages of using equations to solve these types of problems. In recent years, several researchers have discussed the notion of "algebra in arithmetic" (e.g., Britt and Irwin, this volume; Russell et al., this volume). While Chinese school mathematics does not explicitly claim this notion, the use of both arithmetic and algebraic approaches to solve problems can help show students the algebra in arithmetic. Although this practice contrasts the arithmetic and algebraic approaches, it also helps soften the boundaries between them.

There are three objectives in teaching students to solve problems both arithmetically and algebraically: (1) to help students attain an in-depth understanding of quantitative relationships by representing them both arithmetically and algebraically; (2) to guide students to discover the similarities and differences between arithmetic and algebraic approaches, so they can understand the power of a more general, algebraic approach; and (3) to develop students' thinking skills as well as flexibility in using appropriate approaches to solve problems. Post et al. (1988) indicated that "first-describing-and-then-calculating" is one of the key features that make algebra different from arithmetic. Comparisons between the arithmetic and algebraic approaches can highlight this unique feature.

The third thinking habit in Chinese school mathematics is to use inverse operations to solve equations. Starting in the first grade, subtraction is defined as the inverse of addition. Although the term "solve" is not used at grade 1, students learn to solve equations starting at grade 1 and they continue solving equations throughout the entire curriculum. For example, students in the first grade are guided to think about the following question: “If $1 + (\ ) = 3$, what is the number in ( )?” In order to find the number in ( ), the subtraction $3 - 1 = 2$ is introduced. Throughout the first grade, students are consistently asked to solve similar problems. In the second grade, multiplication and division with whole numbers are introduced in Chinese elementary school mathematics. Division is first introduced using equal sharing. Division is also presented as the inverse of multiplication: “What multiplied by $2 = 8$?” That is, “If $(\ ) \times 2 = 8$, what is the number in ( )?”

In addition, the Chinese curriculum emphasizes generalizing from specific examples. By examining specific examples, students are guided to create generalized expressions. Students develop this habit of mind at a variety of points in the curriculum, but especially when formulas for finding perimeters, areas, and volumes are introduced, when operational laws are presented, or when the averaging algorithm is discussed. In particular, generalizing is intertwined in the three habits of minds mentioned above.

The Singaporean Curriculum

In Singapore, some algebraic concepts are formally introduced in elementary grade six (age 12+). At this level children are taught how to construct, simplify, and evaluate algebraic expressions in one variable. The notion of letters as variables is introduced at this level. The concept of equations and other structural aspects of algebra are developed in lower secondary years (age 13+ onwards). However, the Singaporean elementary mathematics curriculum provides a wide variety of experiences to help younger children develop algebraic thinking, and this development is made possible by using “model methods” or “pictorial equations” to analyze parts and wholes, generalize and specify, and do and undo.

Solving arithmetic and algebra word problems is a key component at every level of the Singapore elementary mathematics curriculum (Curriculum Planning & Development Division [CPDD] 1999, 2000). In 1983, the Singapore Ministry of Education officially introduced into the elementary mathematics curriculum a heuristic involving diagram- or model-drawing. This heuristic was intended as a tool for solving arithmetic, as well as algebraic, word problems involving whole numbers, fractions, ratios and percents (Kho 1987). It was believed that if students were provided with the means to visualize a word problem—be it a simple arithmetic word problem or an algebra word problem—the structural underpinning of the problem would be made overt. Once children understood the structure of the problem, they were more likely to solve it (Kho 1987).

In the earlier grades, pictures of real objects are initially used to model problem situations, but then the pictures are replaced by the more abstract rectangles. For example, actual pictures of cars, and then rectangles, are used to solve the following problem in second grade: *Ali has 8 toy cars and David has 6 toy cars. How many*

Raju and Samy shared $410 between them. Raju received $100 more than Samy. How much money did Samy receive?
2 units = $410 - $100
= $310
1 unit = $155
Samy received <u>$ 155</u>

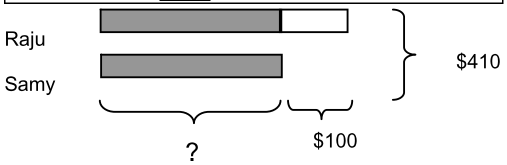


Fig. 1 Pictorial equation solving

toy cars do they have altogether? This pictorial approach, which becomes increasingly more complex as the grade level progresses is introduced in the first grade's teacher's guide. As students advance through the primary years, the model method is used to solve algebra problems involving unknowns, the part-whole concept and proportional reasoning. In each case, the rectangles allow students to treat unknowns as if they are knowns. This is because the unknowns are represented by unit rectangles that can be treated as if they are knowns, even though the unit represents an unknown number of objects.

Figure 1 is an example from Grade 5. Samy's rectangle or unit is the generator of all the relationships presented in the problem. Raju's rectangle is dependent upon Samy's, with Raju's share represented by a unit identical to Samy's plus another rectangle representing the relational portion of $100 more. Using the model drawing, a pictorial equation representing the problem is formed, and if the letter x replaces Samy's unit, then the algebraic equation $x + x + \$100 = \$410$ is produced. The use of the rectangle as a unit representing the unknown provides a pictorial link to the more abstract idea of letters representing unknowns. The entire structure of the model can be described as a pictorial equation.

In summary, children solve word problems using the "model method" to construct pictorial equations that represent all the information in word problems as a cohesive whole, rather than as distinct parts. To solve for the unknown, children undo the operations that are implied in the pictorial equation. This approach helps further enhance their knowledge of the properties of the four operations. The intent of using the "model method" described above is to provide a smooth transition from working with unknowns in less abstract form to the more abstract use of letters in formal algebra in secondary school.

Besides the use of pictorial equations to analyze part-whole relationships, the second big idea in the Singaporean curriculum is developed by exploring the structures in patterns. Students are provided with both numeric and geometric pattern recognition activities. Such activities require students to specify and then generalize the rule they construct to continue the pattern they see. The inclusion of many such activities in the curricular materials suggests that the Singapore primary mathematics syllabus places great importance on the thinking processes—generalizing and specifying.

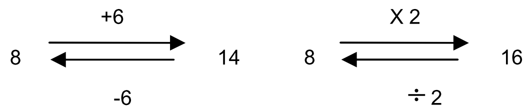


Fig. 2 Inverse operations in the second grade Singaporean curriculum

Furthermore children are provided a variety of activities that foster the development of other algebraic thinking habits—doing and undoing, building rules to represent functions, and also abstracting from computation (Driscoll 1999). For example, the process of "doing-undoing" is emphasized in the Singaporean teacher's guide, specifically when teachers first introduce the four operations. In particular, in the unit "Addition and Subtraction," notes in the second grade teacher's guide suggests that teachers highlight the relationship between addition and subtraction as well as multiplication and division, as evidenced by the example in Fig. 2 (TG2A, 1995, pp. 23–24).

The habit of mind, building rules to represent functions, as well as the notion of letters as variables, is developed using a functional approach. Through this approach, children engage in activities that help them develop the pointwise notion of function by looking for the relations exhibited in sets of ordered pairs representing the input and output of problem situations (e.g. $(1,2), (2,4), (3,6), \ldots, (x, 2x)$). Tasks move from simple numerical activities where children are engaged in doing and undoing to using letters to generalize the operation that produces the output from a given input. As children perform these tasks, they address the question, "What's the rule for this pattern?" This notion is first developed through supplementary activities in the second grade teacher's guide (Ng 2004) designed to help students memorize the multiplication facts and solve word problems (see Fig. 3).

The functional approach provides students in Grade 3 with experiences extending sequentially written number patterns, and with tasks that require them to look for number patterns presented in tables. Here students devise the rule linking numbers in one row/column with numbers in another row/column, thus continuing the informal introduction to pointwise functions begun in second grade. In the Singapore mathematics curriculum, these activities are generally presented to students within a context that is familiar to them.

The complexity of the doing and undoing process increases in Grade 3. For example, the teacher's guide (Ng 2004) challenges students to determine the input number from the output after two given operations. *Add 30 to a number. Then subtract 35 from the sum. The answer is 27. What is the original number?*

At Grade 6, a developmental approach to the informal introduction of functions (pictorial to symbolic) is used to introduce non-recursive rules and the concept of letter as variable.

Lessons from Chinese and Singaporean School Mathematics

What lessons can we learn from Chinese and Singaporean curricula about developing students' algebraic thinking in earlier grades? In the sections that follow we

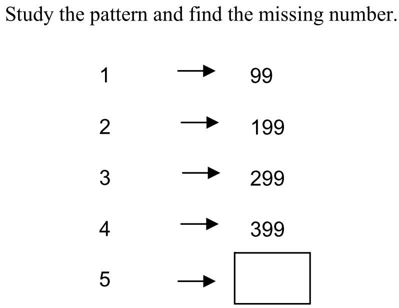


Fig. 3 Informal introduction to functions in the second grade Singaporean curriculum

present insights from our curricular analyses. In particular, we address the following issues: the reason curricula should expect students in early grades to think algebraically, the level of formalism and generalization expected of students, the nature of support for helping students make a smooth transition from arithmetic to algebraic thinking, and the role authentic applications play in fostering algebraic thinking.

Why Should Curricula Expect Students in Early Grades to Think Algebraically?

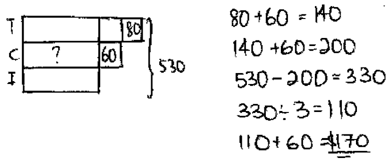
We realize that the question of whether we should expect students in early grades to think algebraically is not an issue these days. Based on recent research on learning, there are many obvious and widely accepted reasons for maintaining this expectation. However, we raise the question of why in order to offer a less obvious reason for developing algebraic ideas in the earlier grades, namely that resistance to algebra would be reduced if we could remove the misconception that arithmetic and algebra are disjointed subjects.

Although one can make an eloquent argument in favor of studying algebra at the secondary level (e.g., Usiskin 1995), in reality, the need to learn algebraic ideas currently is not as universally accepted as the need to learn arithmetic, history, or writing (Usiskin 1995). Even those who have taken an algebra course and have done well can live productive lives without ever using it. Therefore, many middle and high school students are not motivated to learn algebra. We believe that resistance to algebra can be more effectively addressed by helping students form algebraic habits of thinking starting at elementary school. If students and teachers routinely spent the first five or six years of elementary school simultaneously developing arithmetic and algebraic thinking (with differing emphases on both at different stages of learning), arithmetic and algebra would come to be viewed as being inextricably interconnected. We believe an important outcome would be that the study of algebra in secondary school would become a natural and non-threatening extension of the mathematics of the elementary school curriculum.

Although it is widely accepted that we should expect students in early grades to think algebraically, the real question is how can we prepare students in earlier grades

Fig. 4 Fifth grade Singaporean student solution

Furniture Problem: At a sale, Mrs. Tan spent $530 on a table, a chair and an iron. The chair cost $60 more than the iron. The table cost $80 more than the chair. How much did the chair cost?



to think algebraically? "Although there is some agreement that algebra has a place in the elementary school curriculum, the research basis needed for integrating algebra into the early mathematics curriculum is still emerging, little known, and far from consolidated" (Carraher and Schliemann 2007, p. 671). Our analyses indicate that the Chinese and Singaporean curricula could be useful references for those wishing to help elementary students develop a stronger sense of the connections between arithmetic and algebra. Specifically, the Chinese and Singaporean curricula provide concrete examples of promising ways to integrate arithmetic and algebraic ideas in the earlier grades.

Are Young Children Capable of Thinking Algebraically?

Our research clearly shows that elementary Chinese and Singaporean students are capable of using algebraic approaches to solve problems (Cai 2003, 2004b, 2004c; Ng and Lee 2005). For example, when 151 5th grade Singaporean students were asked to solve the "Furniture Problem," shown in Fig. 4, nearly a half of them correctly used the pictorial equation approach to solve the problem.

In comparative studies involving Chinese and U.S. 6th grade students, we found that U.S. 6th grade students tended to use concrete problem-solving strategies, while Chinese students tended to use generalized problem-solving strategies involving letter symbols as generalized representatives of ranges of values (Cai and Hwang 2002).

For example, in solving the Odd Number Pattern Problem shown in Fig. 5, the U.S. and Chinese students had almost identical success rates (70%) when they were asked to find the number of guests who entered on the 10th ring. However, the success rate for Chinese students (43%) was higher than that of the U.S. students (24%) when they were asked to find the ring number on which 99 guests would enter the party ($\chi 2(1, N = 253) = 10.23$, $p < .01$). This appears to be due to the fact that more Chinese than U.S. students used abstract strategies to answer the question. Indeed, fully 65% of Chinese students choosing an appropriate strategy for Question C used an abstract strategy, compared to only 11% for the U.S. sample. In contrast, the majority (75%) of U.S. students chose concrete strategies, compared to 29% of

Sally is having a party.
The first time the doorbell rings, 1 guest enters.
The second time the doorbell rings, 3 guests enter.
The third time the doorbell rings, 5 guests enter.
The fourth time the doorbell rings, 7 guests enter.
The guests keep arriving in the same way. On the next ring a group enters that has 2 more persons than the group that entered on the previous ring.

A. How many guests will enter on the 10th ring? Explain or show how you found your answer.
B. Write a rule or describe in words how to find the number of guests that entered on each ring.
C. 99 guests entered on one of the rings. What ring was it? Explain or show how you found your answer.

Fig. 5 Odd number pattern problem

the Chinese students. Abstract strategies (e.g. solve for n if $99 = 2n - 1$) are more efficient than concrete strategies (e.g., repeatedly adding 2 until 99 is reached or making an exhaustive table or list) to answer the third question, which involves "undoing" (i.e., finding the ring number when the number of entering guests is known). This Odd Number Pattern Problem was administered along with other tasks to some 4^{th}, 5^{th}, and 6^{th} Singaporean students (Cai 2003). It was found that 12% of the 4^{th} graders, 16% of the 5^{th} graders, and 37% of the 6^{th} graders used abstract strategies.

The findings from these studies about Chinese and Singaporean students' use of abstract strategies to solve problems like the Odd Number Pattern suggest that young children are capable of thinking algebraically. The findings also suggest that the Chinese and Singaporean approaches are beneficial for helping elementary students develop algebraic thinking.

How Can We Help Students to Think Arithmetically and Algebraically?

According to Kieran (2004), in the transition from arithmetic to algebra, students need to make many adjustments in the way they think, even those students who are quite proficient in arithmetic. Kieran particularly suggested the following five types of adjustments in developing an algebraic way of thinking: (1) Focus on relationships and not merely on the calculation of a numerical answer, (2) Focus on inverses of operations, not merely on the operations themselves, and on the related idea of doing/undoing, (3) Focus on both representing and solving a problem rather than on merely solving it, (4) Focus on both numbers and letters, rather than on numbers alone, (5) Refocus on the meaning of the equal sign. Helping students make a smooth transition from arithmetic to algebraic thinking is a common goal in the two Asian curricula. There are at least three ideas in the Chinese and Singaporean curricula that help students make the adjustments needed to develop algebraic ways of thinking.

The first is related to the use of inverse operations to solve equations. For example, in Chinese elementary schools, addition and subtraction are introduced simultaneously at the first grade, and subtraction is introduced as the inverse operation of addition (Cai 2004a, 2004b, 2004c). The idea of equation and equation solving permeates the introduction of both subtraction and division. Similar approaches are taken in the Singaporean curricula (see Fig. 2). There is no doubt that this inverse operation approach to subtraction and division can help students make two of the adjustments suggested by Kieran: (1) Focus on relationships and not merely on the calculation of a numerical answer, and (2) Focus on inverses of operations, not merely on the operations themselves, and on the related idea of doing/undoing.

The second idea is the use of pictorial equation solving in the Singaporean curriculum. Pictorial equation solving clearly can help students to *focus on both representing and solving a problem rather than on merely solving it*, as suggested by Kieran. Before the calculation of the numerical answer can begin, the model drawing has to make clear the relationships between the different objects. The pictorial equation solving approach also focuses on *both numbers and letters, rather than on numbers alone, as well as on both representing and solving a problem rather than on merely solving it.* Although rectangles, rather than letters, are used to represent variables in the pictorial equation solving approach, the model drawings show how children must learn to be flexible in their use of the rectangles, just as they do in later years when they use letters to represent variables. In a given question, some rectangles can be used to represent unknown values and others can be used to represent known parameters.

The third idea is that solving problems is done using both arithmetic and algebraic approaches in Chinese curriculum. This idea, incorporating the Chinese practice of using both arithmetic and algebraic approaches to solve problems, can help students to focus on both numbers and letters, rather than on numbers alone. Through the comparisons of the approaches without using letters (arithmetic approach) and with letters (algebraic approach), students are able to see the role of letters in the algebraic approaches. Incorporating the Chinese practice of using both arithmetic and algebraic approaches to solve problems can also *focus on relations and not merely on the calculation of numeric answers*, as well as *focus on both representing and solving a problem rather than on merely solving it*. In fact, the two different approaches can be viewed as different ways to represent quantitative relationships.

Are Authentic Applications Necessary for Students in Early Grades?

Some researchers and educators believe that the learning of algebraic ideas should always be anchored in real-world situations that the students are familiar with (e.g., Bell 1996). Some U.S. *Standards-based* curricula reflect this view (e.g., Senk and Thompson 2003). These curricula engage U.S. students in mathematical problems embedded in authentic contexts. The applied problem solving activities require U.S. students to explore contextualized problems in depth, construct strategies and approaches based on their understanding of mathematical relationships, utilize a variety of tools (e.g., manipulatives, computers, calculators), and communicate their mathematical reasoning through drawing, writing, and talking.

Others believe algebra does not have to be learned using real-world situations because the essence of algebra is not applied (Kieran 1992; Usiskin 1995). Rather, at its core, algebraic knowledge is an understanding of mathematical structures and relationships. So the work of algebra should be to abstract properties of operations and structures, and the goal should be to learn the abstract structures themselves, rather than to learn how the structures can be used to describe the real world.

While applications are important in both the Chinese and Singaporean curricula, the contexts of application problems are not truly authentic. In the case of the Chinese curriculum, for example, the main focus is on equations and the process of equation solving itself, rather than on the use of applications to provide insight into the equation solving process. As a result, the development of equation and equation solving ideas in the Chinese elementary mathematics curriculum is done in three interrelated stages: (1) the intuitive stage, (2) the introduction stage, and (3) the application stage. After Chinese students have been formally introduced to equations and equation solving, there are opportunities to use an equation-solving approach to solve application problems as they learn statistics, percents, fractions, and ratios and proportions (Cai 2004b). This arrangement is desirable in order to deepen the students' understanding of quantitative relationships and to help students appreciate the equation-solving approach.

Conclusion

This study analyzes and compares how algebraic concepts and representations are introduced and developed throughout the Chinese and Singaporean curricula. It provides an international perspective to the question of the kinds of algebraic experiences elementary school students should have. In particular, the study identifies unique features of each curriculum. Both curricula state that their main goal in teaching algebraic concepts is to deepen students' understanding of quantitative relationships, but the emphases and approaches to helping students deepen their understanding of quantitative relationships differ.

The Chinese elementary school curriculum emphasizes the examination of quantitative relationships from various perspectives. Students are consistently encouraged and provided with opportunities to represent quantitative relationships both arithmetically and algebraically. Furthermore, students are asked to make comparisons between arithmetic and algebraic ways of representing a quantitative relationship.

In Singapore, students are provided ample opportunities to make generalizations through number pattern activities. Equations are not introduced symbolically; instead, they are introduced through pictures. Such "pictorial equations" are used extensively to represent quantitative relationships. The "pictorial equations" not only provide a tool for students to solve mathematical problems, but they also provide a means for developing students' algebraic ideas.

In this chapter, we addressed four questions related to the development of algebraic thinking in earlier grades: (1) Why should curricula expect students in early grades to think algebraically? (2) Are young children capable of thinking algebraically? (3) How can we help students make a smooth transition from arithmetic to algebraic thinking? (4) Are authentic applications necessary for students in early grades? We found that both Chinese and Singaporean students in elementary school are expected to think algebraically. In fact, students in earlier grades are capable of thinking algebraically to solve problems. The earlier emphasis on algebraic ideas may indeed help students develop arithmetic and algebraic ways of thinking about problems.

Regarding the use of authentic applications in elementary school, both Chinese and Singaporean curricula include many application problems, but the contexts are not necessarily authentic. The Chinese elementary curriculum uses formal algebraic symbolism, but the Singaporean elementary curriculum does not.

It is important to indicate that any curriculum has a complex relationship to what actually occurs in classrooms. In this paper, the focus of our discussion has been on the intended treatment of algebraic ideas in the Chinese and Singaporean curricula. Nevertheless, the Chinese and Singaporean curricula may serve as concrete examples of what can be implemented when we try to address the issues related to the development of students' algebraic thinking in earlier grades.

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Developing Algebraic Thinking in the Context of Arithmetic

Susan Jo Russell, Deborah Schifter, and Virginia Bastable

Abstract Using classroom episodes from grades 2–6, this chapter highlights four mathematical activities that underlie arithmetic and algebra and, therefore, provide a bridge between them. These are:

- understanding the behavior of the operations,
- generalizing and justifying,
- extending the number system, and
- using notation with meaning.

Analysis of each episode provides insight into how teachers recognize the opportunities to pursue this content in the context of arithmetic and how such study both strengthens students' understanding of arithmetic operations and enables them to develop ideas foundational to the study of algebra.

In recent years, the question, "What can be done in the elementary grades to prepare students for algebra?" has received a great deal of attention. The form of the question sometimes leads to a conception of preparation for algebra that focuses on doing formal algebra—or aspects of formal algebra—in lower grades. Rather, one might reframe the question as, "What are ways of thinking, modes of reasoning, and essential understandings that have their roots in arithmetic *and* are essential to algebra? What are the underlying connections between arithmetic and algebra?" These

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questions lead to a focus on finding instructional emphases that both serve the elementary grade goals of computational fluency and support students to develop the kind of reasoning that will lead to the need for, and meaningful use of, algebraic tools.

Several research teams have been pursuing these questions, as is represented by Kaput et al.'s (2008) anthology, *Algebra in the Early Grades*. Some of the groups represented in this collection of current work focus on introducing the concept of functions, providing tasks which invite students to create function rules to describe patterns of growth (e.g., Blanton 2008). Others organize their work around generalizations in the number system. For example, Carpenter et al. (2003) describe class discussion about true and false number sentences. Over the past decade, the authors of this chapter have been developing K-5 student curriculum and professional development materials for teachers in grades K-8 that address both of these strands of early algebra (Russell et al. 2008; Schifter et al. 2008a, 2008b). This paper draws from the part of our research that focuses on how students engage with generalizations about the behavior of the operations.

From our work with elementary and middle grade teachers, we have identified four mathematical activities that underlie both arithmetic and algebra and, therefore, provide a bridge between the two. These are:

- understanding the behavior of the operations,
- generalizing and justifying,
- extending the number system, and
- using notation with meaning.

These themes emerge from content at the heart of the elementary mathematics program, and can be highlighted and pursued by teachers who learn to recognize the opportunities that arise in their classrooms. Focusing on these aspects of arithmetic addresses two major goals: (1) It enables students to grow from arithmetic towards algebra, and (2) it strengthens their understanding of arithmetic operations and contributes to computational fluency.

In collaboration with teachers in grades K-8, we have been investigating how students articulate, represent, and justify general claims about the operations. We have also been examining how teachers can recognize the implicit generalizations that arise in the course of students' study of arithmetic and make them explicit objects of study in the classroom (Russell et al. 2006; Schifter et al. 2008c). An important component of this research is the close observation of classroom discourse by teachers, who carefully document and write about learning episodes in their own classrooms. Through discussion and analysis of these episodes at regular project meetings and via an electronic web-board, we consider evidence and develop ideas about students' early algebra experience.

In each of the next four sections of this chapter we focus on one of the four areas that links arithmetic and algebra. The examples come from videotaped lessons, lessons observed by project staff, and narratives written by teachers based on transcripts from their teaching.

Understanding the Behavior of the Operations

Computational fluency with the four basic arithmetic operations is a core of the elementary curriculum. In these years, students move from counting to computation. It is an expectation that students enter middle school with a firm grasp of addition, subtraction, multiplication, and division of, at least, whole numbers. Most students come into the secondary grades with procedures for solving basic arithmetic problems. Yet, even among students who carry out these procedures correctly, there are persistent problems as they make the transition from arithmetic to algebra. Many of these problems can be traced to lack of knowledge about the properties and behaviors of the operations. At best, these students may understand these properties in the context of arithmetic, but not access their knowledge in the new context of algebra. At worst, these students use memorized procedures correctly, but do not understand why they work or how they are based on properties of the operations.

What does it look like when students don't have sufficient experience with the behavior and properties of the operations when they reach algebra? What happens when only speed with computation and memorization of algorithms are foregrounded, while understanding falls into the background? Many teachers of algebra in the middle and high school note that students repeatedly make the same errors, for example:

$$-3 + -5 = 8$$
$$(a+b)^2 = a^2 + b^2$$
$$2(xy) = (2x)(2y)$$

Student Errors

Such errors can be persistent, even in the face of repeated correction. It is likely that students who make them see a resemblance in the patterns of the symbols to other, correct rules. For example, students who rely on memorization of calculation procedures may remember a rule informally expressed as "two negatives make a positive," but don't have other tools that help them determine that this rule applies to the *product* of two negative numbers, but not to the sum. Students who make the second error may incorrectly interpret the exponent as a number that behaves like a factor, so that $(a + b)^2$ is interpreted in the same way they would interpret $2(a + b)$. Or, if they do understand the meaning of the exponent, they are not able to access and apply the distributive property from their knowledge of multiplying whole numbers. In the third example above, students may be applying a rule to "multiply everything inside the parentheses by the number outside the parentheses," which would work for $2(x + y)$, but not for $2(xy)$. They incorrectly apply what they think is the distributive property and do not recognize an application of the associative property. In each case, properties of operations are over-generalized or misapplied.

In our work with elementary and middle grades teachers, we have been investigating how their students benefit from explicit study of the operations, for example, by examining calculation procedures as mathematical objects that can be described generally in terms of their properties and behaviors. By this study, we do not mean that students should learn the names of properties and state them as rules, as occurred in some curricula in the 1960s. Some of us who went to school at that time remember that we learned, for example, what the commutative, associative, and distributive properties were, but weren't quite sure why we were learning them or why they were so important. Rather, students use representations or story contexts to describe the behavior of the operations. For example, students might join two sets of cubes to illustrate addition, switching positions of the sets to show that changing the order of addends does not affect the sum. They might draw an image of some amount removed from a larger amount to demonstrate that as the amount removed (the subtrahend) increases, the result (the difference) decreases. Similarly, students might use arrays or equal groups of objects to illustrate the behavior of multiplication and division.

The following classroom episodes illustrate a grade 2 class investigating addition and subtraction and a grade 5 class investigating multiplication in this way.

Episode A: How Are Addition and Subtraction Different? (Grade 2)

In prior lessons in this second grade class (Schifter et al. 2008a, p. 114), the students had noticed that if you change the order of the numbers in an addition expression, the sum remains the same. Many students had been using this idea in their computation, but the teacher, Maureen Johnson, wanted them to consider this property of addition explicitly. During this class session, Ms. Johnson asked students to find pairs of numbers that add to 25. Then she brought students' attention to the question of whether the order of two addends can always be changed without affecting the sum.

Teacher: These two numbers that we used, can we switch them around? Can we change the order and still get 25? I hear a lot of yeses. Who's not sure? So someone's not sure? Two people aren't so sure? If you feel sure, how would you explain that? Kwame?

Kwame: $18 + 7$. Change it around. That's $7 + 18$.

Teacher: So what do you want to say about that?

Kwame: It will still be 25.

Teacher: How come that's still 25?

Kwame: We didn't change the numbers.

Teacher: Does someone have another one they want to talk about? Kamika?

Kamika: $19 + 6$.

Teacher: OK. If I put the 6 first and then the 19, what will it be?

Kamika: 25, because you're just switching the numbers. You're not adding any more and you're not taking away any numbers. You're just changing them around.

Ms. Johnson then asked the class if they were sure this would work for all numbers. When they said yes, she asked if they could prove it: "Can anyone show me something that would prove it or explain it better?" She built two towers out of connecting cubes, one with 23 cubes and one with 2 cubes.

Latifa took the 2-cube tower and moved it rapidly back and forth from one side of the 23-cube tower to the other.

> Latifa: If you keep on switching it around, it will still make 25. Because you're not taking away or adding anything to it, so it will still be the same number.

Other students showed that they understood and agreed with Latifa's actions and words. Latifa used a representation of joining two sets of cubes to show that $23+2 = 2+23$, but she also used language to explain why this relationship would hold for any pair of numbers: If you change the order, nothing more is added and nothing is taken away, so the total stays the same.

Latifa's demonstration is an example of a phenomenon we see in many of our classroom examples: a representation showing specific quantities is talked about and thought about by students as representing a class of numbers. Although there are a specific number of cubes in each cube tower, the students can hold this model in their imagination to represent any pair of numbers—or any pair of numbers they can imagine (which, for second graders, may be the set of whole numbers or, at least, the whole numbers with which they are familiar and comfortable).

To find out whether students were, in fact, talking about any pair of numbers and not just those that sum to 25, Ms. Johnson asked them to consider numbers larger than they could easily add: "What about $175 + 266$?" Her students argued that $175 + 266$ and $266 + 175$ must both have the same sum, even though they had not attempted to carry out the addition. "It doesn't matter," they said. "You're not adding anything or taking anything away."

By now Ms. Johnson felt assured that the students in the class were, indeed, thinking in terms of a generalization, beyond the specific numbers of their examples, and they were able to describe the essential aspects of a representation to justify the claim. But she was also concerned that they should not overgeneralize. Were they thinking about a property that applies to addition, or were they thinking that this property would apply to any operation? She asked them whether they could apply their generalization to $7 - 3$: does $7 - 3$ equal $3 - 7$?

> Latifa: If you have 3 take away 7, but 3 doesn't have 7. So you can only do 7 and 3, because 3 is not a 7.

> Teacher: There is not enough in 3 to take away 7? Is that what you're saying? What if I had 3 and I want to take away 7, then how many could I take away?

> Latifa: You could only take away 3, to make 0.

> Kamika: After you use the 3, it's 3, 2, 1, 0, 0, 0. The 0 is going to keep on repeating itself until it gets to 7.

The question of what happens when one changes the order of the numbers in subtraction allows the possibility of introducing negative numbers. In fact, at a later point in the discussion, one student did raise this idea. Antoine stated, "That won't be 0, it would be negative 4… That means it's going lower. When you go lower than 0, that means negative 1, negative 2, negative 3, …" However, most students in the class, basing their ideas on their familiarity with positive numbers and a "take away" or removal model of subtraction, came to the conclusion that subtracting 7 from 3 is not possible. If you have 3 cookies and try to eat 7, you can only eat 3; then you have 0, and no more can be removed. As Latifa says, "you could only take away 3, to make 0." This reasoning was sufficient to convince students that $7-3 \neq 3-7$, and that the commutative property applies to addition, but not to subtraction, which was the teacher's purpose for this part of the lesson.

In this class, as in many primary classes, students noticed a regularity as they solved addition problems: $4+3$ and $3+4$ are both equal to 7; $5+8$ and $8+5$ are both equal to 13; and so forth. Students who notice such a regularity may be convinced it will always hold because they have encountered many examples and may apply the rule they have formulated in their computation. This teacher took the opportunity to make this regularity an explicit focus of investigation. She challenged her students to think about whether changing the order of the addends maintains the sum only for specific cases or whether it is true more generally and to explain how they knew. Keeping the symbols connected to a representation that demonstrates the action of addition allowed them to explain *why* their claim must be true. By presenting a contrasting case of subtraction, she checked to make sure they understood that their generalization applied specifically to the operation of addition. The students' explanation of the effect of changing the terms of a subtraction problem was, again, tied to their understanding of a model of the *action* of subtraction.

Episode B: Rounding Factors in a Multiplication Problem (Grade 5)

In order to focus on the behavior of the operations, teachers can pay attention to what regularities students are noticing, as the teacher did in the example above. Another site for determining which behaviors of the operations might be an important focus for a particular group of students is student errors, since errors are often related to the misapplication of basic properties of the operations. In the following example, students had been working on the problem, 17×36. After solving the problem and comparing results, students in the class knew that the correct product was 612. However, one student, Thomas, solved the problem this way:

> I round 17 to 20 and 36 to 40. I know that 20×40 is 800. Then I need to subtract the extra 3 (from rounding 17 to 20) and the extra 4 (from rounding 36 to 40). $800-3-4=793$. The answer is 793.

At this point, the teacher, Liz Sweeney, asked Thomas to put his method on the board and explain it to the class. Once Thomas—who also knew that the answer he had was incorrect—had finished his explanation, Ms. Sweeney asked the class to think through Thomas's method for homework, to consider how Thomas had been thinking about the problem, and why his reasoning didn't lead to the correct answer.

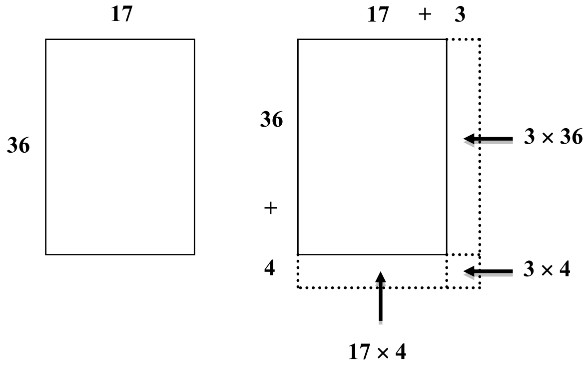
Why would Ms. Sweeney do this? This particular classroom episode is taken from a videotape (Schifter et al. 1999) that is used in a professional development seminar. Some teachers who watch this tape are horrified by the teacher's move—that she would focus on this incorrect solution and, even worse, ask students to work on it at home! While we might debate whether, strategically, we would or wouldn't send such an assignment home where it might be misinterpreted, the teacher's reasoning is clear, as she explained to the class. She saw that even students who easily computed the correct product were somewhat persuaded by Thomas's reasoning. This method looks like it should work—from the point of view of addition: students didn't automatically see why his method does not lead to the correct answer.

We might ask, then, what is right or sensible about Thomas's method? In fact, in the operation of addition his idea works; one might add some amount to one or more addends, add the numbers, then subtract those amounts that had been added, for example:

$$17 + 36 = (17 + 3) + (36 + 4) - 3 - 4 = 20 + 40 - 3 - 4 = 60 - 7 = 53$$

Thomas's method is an example of taking a behavior of one operation and applying it to another operation where it doesn't work. By explicitly studying Thomas's method and why it doesn't work, students have to think through the properties of multiplication—in particular, the distributive property—in order to understand the role of the 3 and the 4 that Thomas added. In fact, using this problem with adults over many years, we have found that the exercise of starting with Thomas's steps of changing 17 to 20 and 36 to 40, and then figuring out how to complete the problem correctly (answering the question, what is it you have to subtract from 800?) is an excellent way for adults to revisit their understanding of multiplication and its properties.

By having teachers or students examine Thomas's strategy, we are not advocating that his procedure (completed in a way that it results in the correct product) is one that should be learned and used to solve multiplication problems. In Thomas's class, the teacher was not hoping that students would routinely alter multiplication problems in the way he had in order to solve them. His method does not necessarily make the problem easier to solve in the long run. However, figuring out what has been added to the product by changing the two factors gets at the heart of the meaning of multiplication and the distributive property, making this procedure worth studying. One way of representing the effect on the product of increasing the factors, as Thomas does, is illustrated below:

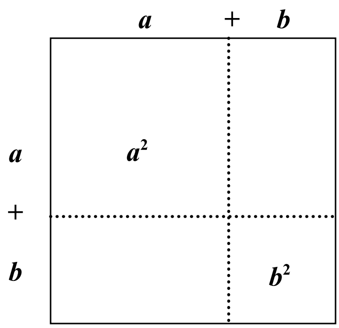


$(17 + 3) \times (36 + 4) =$

$(17 \times 36) + (17 \times 4) + (3 \times 36) + (3 \times 4) = 800$

therefore $17 \times 36 = 800 - (17 \times 4) - (3 \times 36) - (3 \times 4)$

This analysis requires representing the operation of multiplication in a way that manifests the distributive property (which may be hidden from students by some of the algorithms they use). Such visualization of the way factors are pulled apart and multiplied by parts of other factors applies to both arithmetic and algebraic contexts. The reasoning that students might engage in to decode Thomas's error is similar to the reasoning they might engage in to justify why $(a + b)^2$ is not equal to $a^2 + b^2$. Their understanding of the distributive property can be explicitly called upon, so that they can visualize that $(a + b)^2$ cannot possibly be equivalent to $a^2 + b^2$ unless a or b is equal to 0.



Ms. Sweeney reported that Thomas's error led to three days of deep thinking engaging the entire class. The students drew pictures of groups and presented arrays to explain what happens when the two factors are increased.

Generalizing and Justifying

A second mathematical activity that connects arithmetic to algebra is articulating, representing, and justifying generalizations about the operations. As seen in the episodes in the previous section, general ideas arise frequently in the course of students' study of arithmetic. For example, young students notice that when they change the order of addends, the sum does not change. Older students notice the same thing about multiplication expressions. Throughout the elementary grades, opportunities arise to investigate general claims about the operations that can be brought to the explicit attention of the students.

There are two aspects of engaging with general claims that we see teachers developing in the elementary grades:

- articulating particular general claims based on the regularities students notice in the behavior of numbers and operations
- developing a mathematical argument to justify a general claim for a class of numbers

The three classroom episodes in this section are examples of (1) a teacher helping her third graders focus on the articulation of a general claim; (2) a group of fifth graders who are developing both articulation and justification as they investigate equivalent addition expressions; and (3) fifth graders' representation-based proof of a generalization about multiplication.

Articulating General Claims

As students in the elementary grades are given opportunities to notice and discuss generalizations about number and operations, they encounter the need for language to describe the generalizations they are investigating. Young students often use words like "it" or "that," or use a gesture such as pointing, to indicate what they are describing. In math class, when a student says, for example, "I think it's true," it is important to clarify exactly what "it" means, both so that the student offering the idea can clarify his or her own thinking and so that other students do not make different assumptions about the nature of the assertion being considered. Putting reasoning into words can be challenging, for students or adults, but clarification of the language and clarification of the ideas appear to develop together for young students, as illustrated in the next example.

Episode C: Equivalent Expressions in Addition and Subtraction (Grade 3)

Alice Kaye's third graders had formulated a general claim about addition, which had been expressed by one of the students, Clarissa, as: "When you're adding two numbers together, you can take some amount from one number and give it to the other, and if you add those up, it will still equal the same thing." A few weeks later, Alice asked the class to consider subtraction: "By Clarissa's statement, we could say that we know this equation is true: $57 + 21 = 58 + 20$. Without even doing the addition, we would know that whatever $57 + 21$ equals, $58 + 20$ also equals that same total. Would it also be true to say that $57 - 21 = 58 - 20$?" Students quickly computed $57 - 21$ and $58 - 20$ and concluded that the differences are not equal, but students were puzzled about why this was true. As one student put it, "why wouldn't they be the same?"

After a couple of days of investigating this question and coming up with story contexts to illustrate their ideas, the class was considering two series of equations:

$25 + 0 = 25$	$25 - 0 = 25$
$24 + 1 = 25$	$26 - 1 = 25$
$23 + 2 = 25$	$27 - 2 = 25$
$22 + 3 = 25$	$28 - 3 = 25$
$21 + 4 = 25$	$29 - 4 = 25$
$20 + 5 = 25$	$30 - 5 = 25$
$19 + 6 = 25$	$31 - 6 = 25$

The set of subtraction equations had been generated using a story context that Todd had come up with:

> If Todd had 26 baseball cards, and his little brother stole 1, he'd have 25 cards left. What are other numbers of baseball cards Todd could start with, and how many would his little brother have to steal so that he would always have 25 cards left?

In the course of the discussion about the two sets of equations, the teacher repeatedly asked her class to clarify what their general claim was as they were developing arguments to support it:

> Nora [looking at the chart]: So I guess it only works with adding, not subtracting.
>
> Teacher: What only works with adding? What's the "it?"
>
> Nora: The... um... the... the... [a long pause, but she cannot yet put into words what she was thinking was "working with addition"]
>
> Carl: There's both the same thing in the middle... 0, 1, 2, 3, 4, 5, 6 [pointing to the subtraction sequence] and 0, 1, 2, 3, 4, 5, 6 [pointing to the addition sequence]. But 26, 27, 28, 29, 30 is the other way from 24, 23, 22, 21, 20, 19.
>
> Clarissa: I noticed that's because it's going down, and this is going up... because in order to minus, you usually have to go up because if you did like $25 - 0 = 25$, $24 - 1$ would be 23. That would be if you did the same thing as this [pointing to the addition sequence].
>
> Teacher: And what's the "this" you're talking about?
>
> Clarissa: $25 + 0$, $24 + 1$, $23 + 2$. Because that would be adding one on, but you're subtracting one off.
>
> Todd: Since this one is going down [pointing to the first addend in the addition series of equations] this one [pointing to the second addend in the addition series of equations] has to go up, too. This column is going down, so this one has to go up.

> Teacher: What's the idea about addition and subtraction that's being revealed here?
>
> Jonah: I think the reason that both of these columns are going up in value is because if you want to get the same thing if you have a higher number to minus, you need a higher number to minus it to 25. But if you have a lower number to start with, you don't need as many numbers to get to 25.
>
> Many kids: Ohhhhh! I get it!
>
> Teacher: So can you use Todd's example to talk us through your idea? Todd's talking about always wanting to make sure he has 25 cards. Can you use that?
>
> Jonah: If he starts with more, his brother has to take more to get to 25, because there's more cards to take.
>
> Frannie: It sounds simple, but it really isn't.
>
> Manuel: It's just like... he has a bigger number here. So he has to take away more in order to get to the number he wants to get.
>
> Jamie: That's what I was going to say.
>
> Sierra: Yeah... We knew that, and we thought everyone knew that, but now we just sort of figured it out.
>
> Helen: Knew what?
>
> Teacher: What is the idea, Sierra?
>
> Sierra: The idea is that you need more to take away and get the same amount. If I had 26 and I minus 1, if you want the... That would be the same as if you wanted to have the same answer, ... [you] could start with 27 and take away 2.
>
> Addison: The reason why they're both going up is... Since it's higher, then you have to subtract more to get to that, but if it's less, you don't need as much to get to that number. It's less numbers to get to it.

In this episode, Ms. Kaye urges students to clarify what they mean by "it" and "this" as they are articulating their claim and explaining why they think the pattern for subtraction differs from the pattern for addition. As they build on each other's thinking to articulate *why* adding the same amount to both numbers in a subtraction expression results in an equivalent expression, they are simultaneously articulating more clearly *what* their generalization about subtraction is.

Developing a Mathematical Argument to Justify a General Claim

Articulation, representation, and justification of general claims do not occur for young students in a predictable sequence; rather, they develop together in the course of students' work. Representing particular instances of a regularity students have noticed leads to a clearer description of the claim as well as images of the mathematical relationships and structure that inform justification.

Episode D: Equivalent Addition Expressions (Grade 5)

In the following episode, students work with a representation at the same time that they are sorting out and stating a general claim. The teacher, Meg Lawson, has asked the students to write expressions equivalent to 32 using 2 addends. Not surprisingly, for fifth graders, they come up with many, for example:

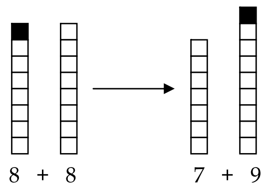
$16 + 16$
$30 + 2$
$28 + 4$
$10 + 22$
$15 + 17$

Ms. Lawson then writes on the overhead: $16 + 16 = 15 + 17$

> Teacher: I know you can calculate each side of this equation to find that each side equals 32. But if you didn't add each side, how would you know for sure that 16 + 16 equals 15 + 17? Think about explaining this to someone who couldn't add up these sums. Show with words and pictures how you know that 16 + 16 = 15 + 17.

In one small group, Fred, Carlson, and Laura work together.

> Laura [very excited]: The total just doesn't change. One number is just passing one over to the other number. See, this 16 gave 1 away and became 15 and the other one took it and is 17.
>
> Carlson: I have no idea what you just said.
>
> Teacher: Laura, instead of saying that again, can you show what you mean? Is there anything you can use or draw that would show what you understand?
>
> Laura puts together 2 sticks of 8 connecting cubes and demonstrates taking one cube from one stick and putting it onto the other.
>
> Laura: Look. This is 8 plus 8 which is 16. I can take one cube off of this stick and put it on the other stick and the total is the same.



Carlson and Fred are very quiet so the teacher asks, "Does this help us with $16 + 16 = 15 + 17$?"

> Fred: Now it shows that 8 plus 8 is the same amount as 7 plus 9. But it works the same way. One number is smaller and the other number gets bigger.
>
> Laura: And by the same amount! Look—I could move 2 over to the other side and it would still work!

Teacher: Carlson, what do you think about this? Can you make any sense out of what Laura and Fred are saying?

Carlson: I think it's like the stuff is moving back and forth but the whole amount is staying the same. So you can take some away from an amount and the same plussed to another amount…

Carlson trails off, getting tangled in the words, and Ms. Lawson leaves them while they are working on the wording for their paper. They are very excited and almost laughing as they stumble over how complicated the words are. When Ms. Lawson returns a couple of minutes later, Laura has the cubes out again and is explaining to Carlson:

Laura: Look, it doesn't only work for 1 number of change. I can take any amount away as long as I add it to the other number so the total cubes don't change.

In her effort to convince Carlson of the generalization she has recognized, Laura has created a representation that proves that, for any two (whole number) addends, she can remove some amount from one addend and add it to the other without changing the total.

The task that the teacher gave her class has several characteristics that we have identified consistently, across grades, as helpful in engaging students in articulating and justifying general claims about the operations:

1. The task involves numbers and operations easily accessible to the students.
2. Students are asked to develop explanations about equivalence that do not rely on computing. (Even if students initially compute to convince themselves, they then move on to a different way of thinking about justification.)
3. Students are asked to use a representation of the operation as the basis for a general argument.

With these constraints and requirements, the students in this example began to shift from talking about specific numbers to talking in general terms. The first indication of this is how Laura used arbitrary numbers, 8 and 8 changed to 7 and 9, to represent 16 and 16 changed to 15 and 17. It is as if the particular numbers do not matter to her. At first her choice was confusing to Fred and Carlson, but when the teacher asked them, "does this help us with $16 + 16 = 15 + 17$?", Carlson talked in very general terms: "*the stuff* is moving back and forth but *the whole amount* is staying the same. So you can take *some* away from *an amount* and *the same* plussed to *another amount…*" By the end of small group time, Laura was able to articulate the claim clearly in general terms.

Reflecting on this episode, Ms. Lawson wrote: "I wasn't sure if this question was going to be interesting to the 5th graders. I wondered if the idea would be so obvious that they wouldn't be able to engage. But most kids seemed very excited to show me, and each other, that they could see and understand what was happening. They looked like they felt very important as each group had a chance to share their findings." She noted that all of the small groups moved from explaining why $16 + 16 = 15 + 17$ to a more general argument for any addends, and two groups also realized that the amount being added/subtracted could be any amount.

Representation-Based Proof: Tools for Proving in the Elementary Grades

Despite some decades of emphasis on reasoning in national documents, many students expect mathematics to be about finding answers. They don't know what it means to state a general claim or, if they do, they don't know what it means to argue that the claim is true. It is not surprising that younger students might think that a few examples are sufficient to show that a general claim is true. For example, a second grader might argue that she has tried changing the order of addends lots of times, and the sum always stays the same, "so I think it's true for any numbers." Many students, as they progress through the grades, continue to believe that trying many examples is sufficient to prove a generalization. They never develop an understanding of what it might mean to state something general about how a class of numbers behaves under a particular operation or to justify such a claim in mathematics. The use of examples by both students and adults as sufficient proof is well-documented; even at the college level, many students are satisfied to accept a general claim on the evidence of a few examples (Harel and Sowder 1998, 2007; Kieran et al. 2002; Martin and Harel 1989; Recio and Godino 2001).

For example, in a 5^{th} grade class, students have noticed, through many examples, that if you double one factor in a multiplication expression and halve another factor, the product remains the same. They have come to accept this idea and routinely apply it as they solve multiplication problems. However, when their teacher asks them *why* doubling one factor and halving another results in an equivalent multiplication expression, their responses are largely assertions:

> Adele: It is the same product because they are equivalent. If you double one factor and halve the other it will result in the same product because it will stay the same product and not a wrong product.
>
> Therèse: When you double one number and you halve the other the result is the same product because they are equivalents, or the other way to say it is that they are in the same family.
>
> Gloria: When you double a factor and leave the other alone, the product becomes doubled. If you double one factor and halve the other factor the product stays the same but if you double one factor and not halve the other it will be wrong and if you halve both numbers it will be wrong.
>
> Kamala: There are no limitations to doubling and halving because you can halve any number to get a whole number or a mixed number and you can double any kind of number. For example I did $2 \times 12 = 4 \times 6$, and $7 \times 5 = 14 \times 2.5$. I think doubling and halving works with all numbers.

The students are convinced that halving and doubling will always work to maintain the same product, but, inexperienced with the kind of question the teacher is asking, their responses do not move in the direction of justification. Adele and Therèse assert, correctly, that if the expressions are equivalent, the products must be the same, but they do not show or describe why the new expression *must* be equivalent to the original expression. Gloria is correct that doubling only one factor results in doubling the product. If she could show why this is true, that could lead her towards an argument for doubling and halving. Kamala seems convinced that their general claim can work with both whole numbers and rational numbers, but she offers only examples to justify her assertions.

These students are typical of students just beginning to justify general claims; they have no experience with constructing mathematical arguments, but rely on examples or assertions. Within their statements there are some glimpses of ideas about multiplication that, if taken further, could lead to more complete mathematical arguments. How can they take the next step towards developing a justification for their claim?

Students in these grades do not have available to them the tools of formal proof. What they do have available to them are representations of the operations—drawings, models, or story contexts that can be used to represent specific numerical expressions, but can also be extended to model and justify general claims. In order to use representations to make mathematical arguments, students must develop strong images of the operations, images that embody their properties.

Elsewhere we have described and defined *representation-based proof* as the means for elementary and middle grade students to justify general claims (Schifter 2009) by reasoning from visual representations. As students gain experience in articulating, representing, and justifying generalizations in the context of number and operations, they learn to develop pictures, models, diagrams, or story contexts that represent the meaning of the operations, can accommodate a class of instances (for example, all whole numbers), and demonstrate, in the structure of the representation, how the conclusion of the claim follows from the premise.

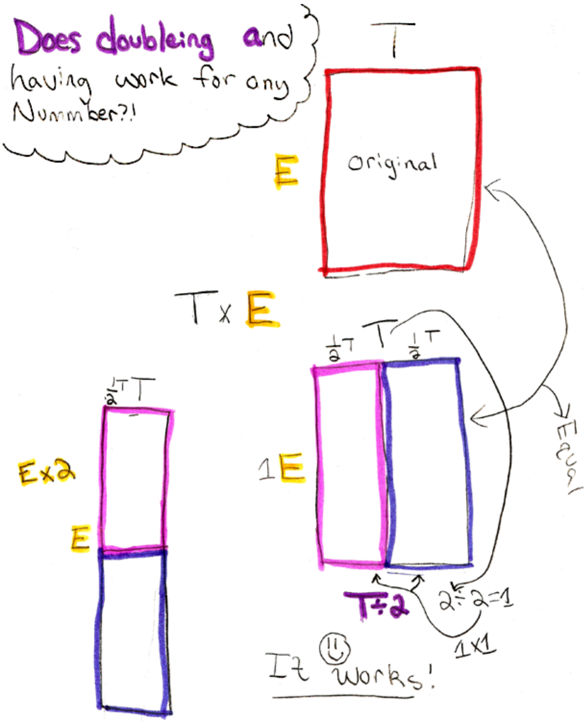
For example, in the first episode in this paper, second graders developed an argument for the commutative property of addition, based on a model of two cube towers. The teacher deliberately introduced that model into the discussion. The third graders investigating equivalent subtraction expressions used a story context about baseball cards to explain *why* one must increase both terms of a subtraction problem by the same amount to keep the same difference. Students typically begin by representing a particular instance of a general claim, then expand it to other instances, and, finally, modify the representation itself and the language they use to describe it so that it represents an infinite class of numbers. Fred, Carlson, and Laura use such general language in describing their cube towers. In the following episode, students who have been working on making and justifying general claims throughout the school year develop a representation-based proof.

Episode E: Equivalent Multiplication Expressions (Grade 5)

In the fifth grade described above, students had noticed the doubling/halving rule for multiplication, but were at the very beginning of work on justifying general claims. In another fifth grade class, students made the same claim—that if one factor of a multiplication expression is doubled and the other is halved, the product remains the same. After investigating and representing individual instances of this claim, the teacher presented the challenge to prove it:

> Teacher: Can you come up with a representation that shows this will always be true, no matter what numbers you start with? Make a picture, draw a model, but don't use any particular numbers.

In response, Trisha and Emily created the following poster.



In their rectangle marked "original," they represent the multiplication, $T \times E$, as the area of a rectangle with sides of lengths T and E. In the second picture, they have cut the rectangle in half and show ½T as a side equal in length to half of T. The same area ($T \times E$) is equal to two rectangles, each with area (½$T \times E$). By moving one of the smaller rectangles below the other, as shown in the third picture, they now have a rectangle with sides ½T and $2E$. Since its area (½$T \times 2E$) is equal to the area of the original rectangle, they have shown that ½$T \times 2E = T \times E$.

In later years, students might prove the same claim by invoking the commutative and associative laws of multiplication together with the multiplicative inverse and multiplicative identity. At this stage, they use as proof what they understand about multiplication as represented by the area of a rectangle and conservation of area.

The development of representations for the operations is critical to connecting arithmetic and algebra. Even students in upper elementary and middle grades who are fluent with computational procedures may not have developed images of the operations they can use when they encounter new contexts, for example, making the transition from using only numerical expressions to using symbolic notation in algebraic expressions. The use of pictures, diagrams, and story contexts to justify general claims appears to be accessible, powerful, and generative for elementary students.

Extending the Number System

In the second grade discussion of the commutative property of addition (Episode A), the focus was on how addition and subtraction behave differently; one is commutative, the other is not. But the discussion also allowed ideas about a different kind of number to be voiced. The second grade teacher made a decision not to pursue the idea of negative numbers at that time. But as students get older, discussions about generalizations provide openings for consideration of new kinds of numbers. Does a property they have justified for whole numbers, and perhaps now take for granted, still hold when expanding the number system to include fractions, decimals, or negative numbers?

As they consider new classes of numbers, students sort out which behaviors of the operations must remain consistent as the number system expands and which only appear to be general when considering certain classes of numbers. For example, consider these two statements:

- When you subtract any amount except 0, you end up with less than your original amount. (For any number $b \neq 0$, $a - b < a$.)
- If you add two numbers to get a third number, then you can subtract either addend from the sum to get the other addend. (If $a + b = c$, then $c - a = b$ and $c - b = a$.)

Students are likely to encounter both of these ideas when their view of numbers is limited to positive numbers. As their number system expands to include new classes of numbers, they need opportunities to examine which of the statements are still true. Students will find that the first statement is not true when b is a negative number, but the second statement is true for all numbers on the number line. The next two episodes illustrate students expanding a general claim in this way.

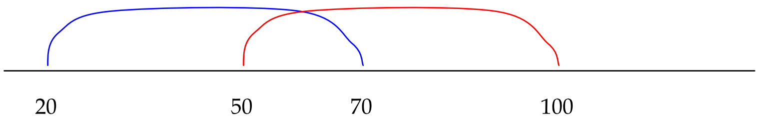
Episode F: Subtracting Negative Numbers (Grade 5)

The next episode illustrates how studies of the operations can support students' work on calculation and reasoning about new number domains.

These fifth graders were investigating equivalent subtraction expressions, like the third graders in Episode C. In this class, students began by generating expressions equal to 50 such as $70 - 20$ and $100 - 50$. As in many of these examples, the teacher, Marlena Diaz, chose numbers that were easy for the students because she wanted the focus of the discussion to be on the relationships of the numbers, not on computing results.

The students all knew that the difference in both expressions is 50. Implicit here is a generalization—add an amount to 70 and add the same amount to 20; the difference remains unchanged. But how do you know that the difference will always remain constant when the same amount is added to each number? This is the question Ms. Diaz posed to her class.

One fifth grader, Alex, came to the board and drew a number line on which he showed the distance between 100 and 50 and the distance between 70 and 20.



Then he explained, "You can see that the distance is the same. If you change one number, you change the other the same way. As long as both numbers change the same, you can make lots of new expressions." He was visualizing sliding the interval, which remains rigid, along the number line so that the beginning and end points change by the same amount, but the difference between those two points does not change.

Alex offered a representation of subtraction to justify his claim. Unlike the second graders who thought of subtraction as a process of removal, Alex relied on a different model of subtraction—finding the distance between two numbers on a number line. Alex's number line and explanation made sense to other students, and they realized they could generate many more expressions. Using Alex's image, his classmates were thinking of sliding the interval of 50 down the number line and proposed additional equivalent subtraction expressions, which Ms. Diaz listed on the board:

100 – 50
90 – 40
80 – 30
70 – 20
60 – 10
50 – 0

As this list was being generated, two students commented, as follows:

Patricia: We could keep adding to our list by changing both the numbers, but we are going to get to a point where we won't be able to change the numbers. That will happen when we get to 50.

Nicole: Yes, I agree with Trisha. Because if we look at Alex's number line we are going to get to zero and 50, and the jump will be 50, but then we are done.

As the discussion continued, additional ideas were offered.

Raul: But we could use the other numbers.

Teacher: What other numbers?

Raul: The negative numbers on the other side of zero.

Alex: I have one we can use. Let's use 40 and negative 10.

Teacher: How do you want me to write that on the chart?

Alex: Put 40, then the subtraction sign and then a negative 10.

Ms. Diaz wrote on the board, "$40 - -10 = 50$."

In her written account of this class session, Ms. Diaz reported, "At this point many of the students were talking at once... Several were pointing to the large classroom number line that extends to -40."

Josh: No way; you can't do that. How can you have a negative 10 and end up with 50?

Alex: It is like adding 10, because if you look on the number line you would have to jump 50 to get from negative 10 to 40. It is the same as we did with 100 and 50 and 70 and 20.

Teacher: So, Alex, how do you know that 40 minus negative 10 will give you 50?

Alex: Because you have to add 50 to negative 10 to get 40.

In this classroom, the students were discussing a generalization about subtraction of whole numbers and used a number line to clarify and justify it. Alex could see that their reasoning about whole numbers could extend to negative numbers. Furthermore, his reasoning brought him to articulate what subtraction of a negative number must mean. He applied what he understood about the relationship between addition and subtraction, as well as the image of "distance between" on the number line, to argue that $40 - (-10)$ must equal 50. For example, Alex reasoned that if $-10 + 50 = 40$, then $40 - (-10)$ must equal 50. As with whole numbers, if $a + b = c$, then $c - a = b$.

On the other hand, students must reconsider some of the generalizations they may have made about the behavior of subtraction in the context of whole numbers. For example, Josh says "No way; you can't do that. How can you have a negative 10 and end up with 50?" It is likely that Josh and other students hold an implicit belief, based on their experience with positive numbers, that the result of subtraction (the difference) is always less than the initial amount (the minuend). Josh may have been asking, "How can you subtract something from 40 and end up with a number larger than 40?" The students will need to reconcile these questions with the behavior of the operations in this new domain.

The ideas brought up in this discussion generated a great deal of interest and provided the class with the opportunity to think about subtraction of negative numbers and about the consistencies that should be maintained in the behavior of operations. Again, as for the students in Ms. Lawson's class (Episode D) who were considering $16 + 16 = 15 + 17$, a high level of enthusiasm for this kind of challenge was evidenced.

Episode G: Multiplication with Decimals (Grade 6)

A generalization can help students tie together ideas that at first seem unrelated and thereby strengthen their understanding of the foundations of arithmetic. In Jeanette McCorkle's sixth grade class, students had formulated the same doubling and halving rule that students were working on in Episode E. They had expanded that claim to include multiplying and dividing by any factor, not only 2, and had expressed their claim in symbolic notation as $A \times B = (A \times C) \times (B/C)$. Through the fall, they had encountered this idea a number of times, but always in the context of whole numbers. Like many of the teachers whose work is included in this chapter, the teacher of this class, Ms. McCorkle, frequently asked students to analyze expressions and equations without doing any computation. On one day in November, she posed a list of problems that focused on place value with decimals:

$25 \times 1 = 2.5 \times 10$

$25 \times 10 = 2.5 \times$ ____

$25 \times 100 = .25 \times$ ____

$25 \times .1 = 2.5 \times$ ____

$25 \times .01 = .25 \times$ ____

> Teacher: So look at this for a minute [the first equation above] and when you have decided if that is a true equation, without calculating, when you have a strategy for determining whether that is true, raise your hand and let me know.
>
> Fran: I'm not sure if this is right at all, but if 2.5 is timesed by 10, it means moving the decimal over one, and that is the same thing as 25 times 1.
>
> Britta: Well, 2.5 is ten times smaller than 25, and 10 is ten times bigger than 1.
>
> Charles: 2.5 times 10, if you multiply it by 10, you move it one to the right, so you're looking at 25 and 25.
>
> Mariah: I would think of 2.5 times 10 as two 10s and a half a 10, which is 25, so you have 25 and 25.
>
> Britta: This is like the problems we did before but A is divided by ten and B is multiplied by ten.

Britta's statement surprised Ms. McCorkle. She had not thought in advance about how this work would connect to the generalization they had articulated in earlier lessons. She had designed the lesson to address difficulties her students had exhibited with multiplication of decimals. As the class solved the rest of the problems, some students began by using mechanical methods, counting decimal places. For example, for the last problem in the group, one student explained:

> I'm sort of like, 25 times .01 equals .25 times, it has to be 1, because to get .25 you have to move the decimal over two, so then to get to 1 you have to move it two the other way.

As Ms. McCorkle interacted and questioned students, she urged them to consider what moving the decimal point means in terms of multiplication and division. After some time, she pulled the whole class together to talk about this issue with the goal of returning to Britta's earlier observation:

Teacher: You're looking at a number of decimal places relationship, and I want to expand that and talk about how one factor has been multiplied and another factor has been divided. The number of decimal places is just one way of talking about how factors have been altered.

Fran: It's not just about the decimal point, it's about multiplying and dividing the numbers.

Teacher: Exactly. I want to remind you about the pattern we were looking at last week and the week before, when Britta suggested that $A \times B = 2A \times ½B$, and George suggested that $A \times B = AC \times B/C$. Britta, how was 25 times 1 changed into 2.5 times 10?

Britta: The first factor was divided by 10 and the second factor was multiplied by 10.

Teacher: That's right. The decimal moved back means divided by 10, so to maintain this equality, what should happen to this factor?

Most students: Multiply by 10.

Britta: And it's still $A \times B = AC \times B/C$.

Ms. McCorkle wrote: "By following my students' thinking, I saw how some of them connected this page of problems directly to our previous work on the doubling and halving rule, which I did not expect."

Considering different kinds of numbers—fractions, decimals, negative numbers —is an opportunity for students to revisit the generalizations they have worked on with whole numbers. Through reconsidering these general claims, they identify the consistencies in the behavior of the operations as the number system to which they are applying those operations expands. Instead of operating with a new class of numbers as if they require a new set of rules (e.g., rules about counting or moving decimal points), they can extend and apply the foundational properties they have already encountered in operations with whole numbers.

Using Notation with Meaning

In the previous episode, as well as in Trisha and Emily's proof (episode E), students expressed general claims in symbolic notation. In the student curriculum we have developed, we introduce some use of algebraic notation in the elementary grades. However, we have been careful not to move too quickly. In order to support students' use of algebraic notation with meaning, they first need to spend a good deal of time articulating general claims clearly in words and then connecting those statements to arguments based on representations. The use of phrases that refer to a class of numbers, such as those used by Carlson in Episode D ("you can take some away from an amount and the same plussed to another amount") or second grader Kamika as she justifies the commutative property of addition in Episode A ("You're not adding any more and you're not taking away any numbers. You're just changing them around") are an important link to meaningful use of symbolic notation.

Using representations and story contexts to model general claims helps students develop meaning for the symbols of arithmetic. In particular, students' study of equivalent expressions, such as $16 + 16 = 15 + 17$ (Episode D) or $26 - 1 = 27 - 2$

(Episode C) provides the opportunity for meaningful use of the equal sign, signifying equivalence of expressions, rather than "now write down the answer" (Behr et al. 1980; Carpenter et al. 2003; Kieran 1981). Once students have considerable experience stating generalizations in words and justifying these general claims by using representations of the operations, they have images and explanations to which they can connect algebraic symbols. In our final episode, we see a group of students making this connection.

Episode H: Using Algebraic Notation to Represent Equivalent Addition Expressions (Grade 5)

This class of 5th graders has been investigating equivalent addition expressions as they looked at this sequence:

$$30 + 2$$
$$29 + 3$$
$$28 + 4$$
$$27 + 5$$
$$\vdots$$

Students considered a general claim based on this sequence—that if 1 is subtracted from one of the addends and added to the other addend, the sum is maintained—and developed some arguments to justify the claim. The teacher, Alina Martinez, introduced a cube representation (similar to what Laura uses in Episode D) to model addition as joining two quantities. Students talked about how they could move one cube from one quantity to the other quantity, maintaining the same sum because "you aren't adding any or taking any away … and since all the numbers are made up of ones, we can just move all those ones around." At this point in the discussion, Ms. Martinez judged that the students' ideas and images were quite clear and that symbolic notation would provide another representation with which they could continue to think about this idea.

> Teacher: You all are thinking about lots of numbers and trying to make sense of what is happening. It seems that you all are thinking that this is true about all numbers and you are trying to make convincing arguments. I wonder if we could write a sentence that wouldn't use numbers to show what is happening. Could we call these numbers up here on the chart just some numbers?
>
> Will: We could write letters for them. Like *n* for number, like *n one* and *n two*.
>
> Teacher: That's a great idea. One thing that mathematicians do sometimes is use different letters so they don't get confused. How about if we use *a* and *b*?
>
> Jonah: We could write *a* plus *b* equals a number.

Ms. Martinez then asked students to look at the cube representation of joining two quantities.

> Teacher: So let's use Jonah's idea and try to write down what we did to the two quantities. What are we doing to the a and the b in this pattern?

Ms. Martinez recorded a above the first addend and b above the second addend in the list of expressions:

$$\begin{array}{c} a \quad b \\ 30 + 2 \\ 29 + 3 \\ 28 + 4 \\ 27 + 5 \\ \cdot \\ \cdot \\ \cdot \end{array}$$

> Kathryn: We can write a plus b is the same as a take one away and b add one to it.
>
> Teacher [recording $a + b = (a - 1) + (b + 1)$]: How does this match what Kathryn said and what we did with the cubes?
>
> Reynold: We take one away like here in the pattern ... one goes up and one goes down.
>
> Amelia: Oh, look the minus one and the plus one is like a zero! That is why we don't change it. It is like staying on zero on the number line.

At this point, many of the children began talking in their groups excitedly. Ms. Martinez asked the small working groups to consider what would happen if more than one cube was moved from one quantity to the other. Several students then shared that they could move two cubes, three cubes, or lots of cubes and still maintain the sum. Ms. Martinez then asked the class if they thought the notation could be revised to accommodate this idea.

> Adena: We could write lots of them and change the numbers. [Adena is suggesting they write a series of number sentences, $a + b = (a - 1) + (b + 1)$; $a + b = (a - 2) + (b + 2)$, and so forth.]
>
> Will: Or we could write add any and take any away.
>
> Jonah: We could use another letter.
>
> Teacher: What do you all think?
>
> Adena: Put a c. Put a plus b equals c.
>
> Jonah: But put the c where the 1 is.
>
> Teacher [recording: $a + b = (a - c) + (b + c)$]: Do you mean like this, put the c where the 1 is? What does this mean now?
>
> Reynold: See, the c is the cubes you move around to the other side.

In this example, students move among their words, a representation, and the symbols, so that the words and representation are a referent for their thinking about notation: Kathryn's words, "a take one away and b add one to it," becomes $(a - 1) + (b + 1)$; similarly, "add any" becomes $b + c$ and "take any away" becomes $(a - c)$. The teacher asks students to consider how this notation matches their sequence of expressions and the cube model, and students are able to articulate these connections, for example, "the c is the cubes you move around to the other side."

Introducing this notation at a point at which students have already articulated their ideas in words and images allows them to maintain meaning for the symbols. But something else happens as well. Any representation can provide a different view, a new insight into the mathematical relationships that are represented. The symbolic representation in this case may make the $+1$ and -1 even more prominent. Even though students had noticed that "one goes up and one goes down" as they considered the sequence of expressions, Amelia now sees something new about this relationship: "the minus one and the plus one is like a zero! That is why we don't change it." In fact, she has come up with a new argument that involves the fact that the result of adding 1 and subtracting 1 is 0. Thus, the introduction of symbols in this case not only provides a concise expression of students' ideas but offers new ways of seeing the mathematical relationships.

We don't want to underestimate the complex issues students encounter as they begin to work with symbols. The error in which students simply substitute letters for words in an English sentence is well known, as in writing 6S = P to represent "there are 6 times as many students as professors" (Clement et al. 1981; Kaput and Sims-Knight 1983). This incorrect notation stems from using a letter as if it is an abbreviation for a word, standing for the thing itself rather than the quantity of that thing, and also perhaps from misinterpretation of the equal sign.

Students need time and experience to develop an understanding of the conventions for using algebraic notation and how the use of letters to represent variables differs from the use of multidigit numbers. Later in the lesson on multiplying decimals (Episode G) a student tries to rewrite the notation they have been using, $A \times B = AC \times B/C$, to accommodate decimals as: $A.B \times C = AB \times .C$. Grounded in the experience of multidigit numbers and the emphasis in these grades on decimal computation, it is not surprising that students might think there is a need to have a letter for each digit, and that the decimal point must be explicitly shown.

Furthermore, when negative numbers or fractions are introduced, students don't automatically realize they can use the same letters to represent them. In the equation, $a + b = c$, if a represents a negative number, many students think it now must be written as $-a$. Because there is no negative sign, a somehow *looks* positive. It takes experience to accept that a single symbol might represent a positive or negative value, a whole number or a fraction.

In all of these cases, students are making sensible choices, based on their experience with numbers. The transition to use of algebraic notation requires both connecting these new symbols to what they represent and also learning new conventions. For this reason, even though it may appear easy to make a transition to use of symbols in particular cases in the elementary grades, teachers' and students' experiences indicate that it makes sense to proceed cautiously with early introduction of algebraic notation.

Connecting Arithmetic and Algebra

The four aspects of early algebra discussed in this chapter have the potential to provide students with a strong foundation in whole number computation, which they can extend to their study of fractions, decimals, negative numbers, and algebraic symbols. Some might ask: How can work in algebra fit into an already crowded curriculum? We would argue that early algebra, defined in this way, not only provides crucial links between arithmetic and algebra, but also is an essential part of good arithmetic instruction.

As seen in the classroom episodes, investigation into these aspects of arithmetic—understanding the behavior of the operations, generalizing and justifying, extending the number system, and using notation with meaning—provides a means for students to re-examine and strengthen foundational understandings about the meaning of the operations and ways of thinking in mathematics.

Further, we are intrigued by the level of student engagement with investigation of general claims that teachers are seeing in their classrooms. Although it might be thought that this kind of reasoning is accessible only to "top" students, several of these examples come from schools in which there is a history of poor performance on standardized tests. We are accumulating documentation of how both students who have been relatively successful and relatively unsuccessful in grade-level computation as measured by school and district assessments are engaged by such investigations (Russell and Vaisenstein 2008; Schifter et al. 2009). Our hypothesis is that mathematical activities that connect arithmetic and algebra have the potential both to strengthen the foundations of computation for all students, perhaps especially for those who have relied on poorly understood procedures, and to intrigue many students, including those who excel in mathematics, with challenging questions about mathematical relationships.

Finally, many questions remain about what teachers need to know and understand in order to carry out this kind of instruction that links arithmetic to algebra. The isolation in which teachers often work, and the concomitant lack of communication between elementary and middle grades teachers, is one barrier to the kind of continuity that might be built in mathematics instruction from arithmetic to algebra. Elementary teachers need a better grasp of how their curriculum can embody ideas that are foundational to algebra and how these ideas might be made more explicit objects of study. Similarly, middle grades teachers need to know more about how to build on the work of the elementary grades and how to assess the ways students' conceptions of arithmetic may inform or undermine their understanding of algebra.

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The Role of Theoretical Analysis in Developing Algebraic Thinking: A Vygotskian Perspective

Jean Schmittau

Abstract Vygotsky asserted that the student who has mastered algebra attains "a new higher plane of thought" (Vygotsky 1986, p. 202), a level of abstraction and generalization that transforms the meaning of the lower (arithmetic) level. The development of this higher (algebraic) plane of thought not only precedes the development of arithmetic but becomes a major focus of the child's elementary education. It is characterized by orienting children to the most abstract and general level of mathematical understanding from the beginning of their formal schooling. This orientation to theoretical structure is mediated by the mastery of psychological tools which are not encountered as incidental to the solution of particular problem types but are instead the focus of explicit instruction. It is further characterized by the development of an adequate conceptual base, the incorporation of principles of dialectical logic, and the ascent from the abstract to the concrete in the development of conceptual content.

Introduction

Midway through the first grade curriculum of V.V. Davydov (Davydov et al. 1999) a child who is using the curriculum in a US school setting solves the following problem on a teacher made test: "Steven has a bag of jelly beans that weighs d kgs less than Jennifer's bag of candy bars. Jennifer has 10 kgs of candy bars. How much does Steven's bag weigh?" The child's answer: "10 kgs $-$ d kgs." Children studying Davydov's second grade curriculum enter their second day of debate on the problem: "$T - 4 - 4 =$?" Two opinions have been expressed and neither side

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has been able to convince the other. After an hour both opinions, T and $T - 8$, remain as options. Early in the class the next day the argument for $T - 8$ emerges as convincing. A seventh grade teacher comments that her students would have no idea how to solve this problem, and compares the first grade problem above to a multiple choice question on the state's eighth grade mathematics test, in which 1 inch of grass is cut from a lawn k inches tall, with students asked to identify which of four given expressions represent the remaining height of the grass. Finally, during the second semester of the Davydov third grade curriculum children solve rate-time-distance problems while a group of high school teachers observe the lesson, commenting that "our eleventh graders can't do these problems".

What is it in the Davydov curriculum that enables elementary school children to work with abstractions—so necessary to algebraic understanding—at a level that proves challenging for many US secondary students? The answer to this question begins with Vygotsky's assertion that the student who has mastered algebra attains "a new higher plane of thought" (Vygotsky 1986, p. 202), a level of abstraction and generalization that transforms the meaning of the lower (arithmetic) level. In the early decades of the twentieth century the Russian elementary curriculum focused on arithmetic, with algebra taught at the secondary level. Vygotsky saw this progression as a "rise from preconcepts (which the child's arithmetic concepts usually are) to true concepts, such as the algebraic concepts of adolescents" (1986, p. 202). Vygotsky held that only theoretical (also designated "scientific") concepts were real concepts; empirical concepts were not true concepts and hence, were designated as preconcepts above.

Empirical concepts are spontaneously derived from everyday experience, often by comparing and contrasting the empirical features of objects or phenomena. This occurs frequently in school settings as well, as for example, when polygons are compared and classified according to the number of their sides. A theoretical understanding of polygons, on the contrary, would be oriented to the central role of the triangle in their genesis, an observation made as far back in history as Aristotle. It is a matter of more than passing significance that mathematics concepts are quintessentially theoretical (or scientific) in nature (Schmittau 1993).

Scholars who had either influenced (as did the philosopher Hegel) or studied with Vygotsky, emphasized the role of theoretical thinking early in the child's development. Hegel held that a child should not be kept for too long in an empirical mode of thinking. And D. B. Elkonin found that the years best suited for theoretical learning were, in fact, the elementary school years, before interest shifted to peer relations and a focus on future careers in adolescence (Elkonin 1975). Consequently, in order to achieve the higher plane of thought envisioned by Vygotsky, Davydov saw clearly the disadvantages of a curriculum in which the order of the development of concepts was from arithmetic to algebra—an elementary curriculum comprised of empirical concepts (or preconcepts as Vygotsky called them in the above quote) followed by a secondary curriculum where students finally gained access to the realm of theoretical thought in mathematics.

Davydov and his colleagues sought therefore, to introduce theoretical or algebraic thinking earlier in the school experience. At a time when set theory was being adopted as a foundation for the "new mathematics" reform in the United States, Davydov (1975) observed that set theory was not as fully general as it appeared, and that this constituted a major drawback to building mathematics on sets as a foundation. He subscribed to the position held by Bourbaki, who also rejected the set theoretic foundation, asserting that it is not sets, but rather mathematical *structures* that constitute the essential content of mathematics (Bourbaki 1963; cited in Davydov 1975).

Thus although historically in mathematics and traditionally in education, algebra followed arithmetic, Vygotskian theory with its emphasis on scientific concepts and theoretical understanding, supports the reversal of this sequence in the service of orienting children to the most abstract and general level of understanding from the beginning of their formal schooling. However, given that elementary school children do not possess the sophisticated understandings of mathematicians or even the arithmetical knowledge of secondary students, it was by no means obvious how instruction might be designed to render algebraic structure preeminent at the elementary level, without imposing a conceptually sterile and largely unlearnable formalism.

The fact that the algebraic structure of positive scalar quantities is shared by the real numbers became the key to maintaining a theoretical focus while at the same time allowing for the accommodation of children's learning needs. Children could study scalar quantities such as the length, area, volume, and weight of real objects, which they can access visually and tactilely, discern their properties, and in this way equivalently access the algebraic structure of the real number system. This is the approach taken by Davydov's elementary curriculum, which stands as a major departure from conventional programs. By developing number from the measurement of quantities, Davydov's curriculum also breaks with the common practice of beginning formal mathematical study with number. Observing that culturally and in individual development, the concept of quantity is prior to that of number, he indicted the rush to number as a manifestation of ignorance of the real origins of concepts (Davydov 1990). In his first grade curriculum, so extensive is the foundation of investigation of (mostly continuous) quantities that number does not appear until the second semester.

The Davydov curriculum represents a departure from Piaget's proposed stages of development as well. Vygotsky, however, accepted neither Piaget's separation of instruction and development, nor the assumption that the latter had to precede the former. Rather, Vygotsky contended that *learning*, and in particular, the mastery of scientific concepts, *leads development*. He wrote, "The formal discipline of scientific concepts gradually transforms the structure of the child's spontaneous concepts and helps organize them into a system: this furthers the child's ascent to higher developmental levels" (Vygotsky 1986, p. 206).

An orientation to algebraic structure requires a focus on the most general and abstract characteristics of real phenomena, beginning with children's initial classroom encounters with such phenomena. This, in turn, requires the development of voluntary attention. And as is the case with scientific concepts and theoretical learning in general, pedagogical mediation is necessary.

This is accomplished in Davydov's curriculum by focusing children on the theoretical characteristics of real objects, objects with which they are familiar, asking them to compare such objects with respect to their length, area, volume, or weight, and to progressively refine such comparisons until they culminate in measurement itself, from which number is then defined. Children are confronted with tasks in which they determine that when adding a volume A to a volume B, for example, the result is the same as when the order of addition is reversed. Repeating this task in combining two different lengths of wood, or adding the weight of a pinecone and a pattern block in either order, children are developing the commutative property of addition of positive scalar quantities. Since they do not know the actual length of the wood or weight of the objects (that would require measurement, which will come later), the children label their result with letters, such as $T + C = C + T$, with the understanding that such a result is generalizable to any two quantities. It is noteworthy that initially the mere prizing out of the quantity from the empirical features, such as the shape and color of the objects in question, is the beginning of a theoretical orientation to the task.

Once children's comparisons have progressed to the actual measurement of quantities (through the laying off of a part of the quantity arbitrarily designated as a unit), a number is generated that is the measure of the quantity by the unit employed. The properties of quantities apply also to their numerical designations, which vary with the unit of measurement. Children have already established these generalized properties for *any* quantities, hence their extension to specific numerical designations of quantities is a concrete application of a general result previously obtained and expressed in the symbolism of algebra. Now algebra is no longer initially learned as a generalization of arithmetic, but rather as a generalization of the relationships between quantities and the properties of actions on quantities. With the introduction of measurement and the definition of number emanating there from, the application of the properties of quantities to their numerical designations represents the ascent from the abstract to the concrete, whereby the abstract essence of a concept is discerned in its concrete (here numerical) embodiments, which "fill in" and enrich, as it were, the conceptual content.

In the Davydov curriculum the development of the higher plane of thought alluded to by Vygotsky not only precedes the development of arithmetic but becomes a major focus of the child's elementary education. Further, throughout their three years of elementary schooling children are oriented to the highest levels of abstraction in their understanding of mathematical concepts. How this is accomplished and the role of psychological tools in this process is the subject of the next sections.

Orienting Children to Theoretical Concepts

The first grade curriculum introduces number after a semester of work on comparison of quantities and measurement. Considerable time during the second semester is spent on developing the concept of positional system through the development of many different number bases. This serves to establish a sufficiently generalized conceptual base for operations with whole numbers, and has foundational applicability to decimals and polynomials as well. It is difficult not to notice the theoretical level at which children are challenged to understand the material. Consider, for example, the following problems which reflect the level of abstraction and theoretical thinking expected of children studying the first grade curriculum.

(1) $\frac{A}{C} = 4 \qquad \frac{G}{C} = 6 \qquad A\ ?\ G$

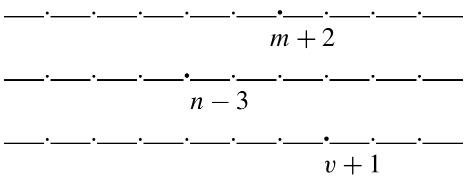
(2) $\frac{B}{T} = 7 \qquad \frac{K}{T} = ? \qquad B < K$

(3) Andrew had c stamps. After he was given some as a gift, he had g stamps. How many did he receive as a gift?

(4) There is a vase of flowers. r flowers died. t flowers are still fresh. How many flowers were in the vase originally?

(5) Write the number for each expanded form below:
Λ0 + Λ = Г00 + Д = Ω00 + Д0 + Λ =

(6) Locate the numbers m, n, and v on the number lines below.



(7) Compare the following numbers (by writing <, = or > in the blank):
ΛД___ΛД 0, Г0___Г00, Λ0 Г__Λ Г0

(8) Subtract the following numbers:
ΛΛΛ − Λ0 = ? Г0Λ − Λ = ? ДДД − Д00 = ?

(9) Daniel had some pencils. His friend gave him another 6 pencils. Now Daniel has 15 pencils. How many pencils did Daniel have originally? Which of the formulas below fits this problem?

a	$+b$	$=c$	a	$+b$	$=c$	a	$+b$	$=c$
15	6	x	x	6	15	6	x	15

(10) Make up two problems about pencils to fit the two formulas given in problem #9 other than the one that matches Daniel's situation.

The reader is invited to actually solve the above problems in order to experience the level of abstraction and mathematical reasoning required and to notice one other very important element, viz. the manner in which they require the student to negotiate cognitive reversals and to consider a concept from virtually every angle. Note the different ways in which children are required to think and analyze in #1 and #2 (more than one solution is possible for #2, of course), and in #9 and #10. Note also the differing word orders in #3, #4, and #9. So-called "key word" approaches would be of little use with word problems such as these, nor can students rely on context cues gleaned from numbers. Problem #10 asks children to make up word problems to fit the equations, which requires them to reverse the perspective from which they analyze the problem situation. Students are not given collections of routine problems all of which can be worked in the same way; they are challenged to think and analyze, and to look at problem situations from different perspectives. In this sense, none are routine. And children must understand the structure of the number line in order to be successful with problem #6, since no beginning of the number line is designated, nor are there any numerical designations marking the points. Similarly, they must understand the structure of a positional system in order to correctly solve #5, #7, and #8, since there are only place holder (zero) and literal designations of positions. When the author confronted a group of forty middle school teachers with just a few of the above problems, they found especially those similar to #5, #7, and #8 so difficult that many of the teachers gave up trying to solve them. Yet these are representative of problems found in the first grade curriculum of Davydov.

Role of Psychological Tools

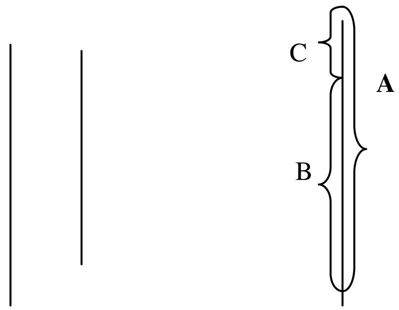
There is, however, another characteristic of the Vygotskian approach to the development of algebraic thinking that is also of critical importance for student mastery. It is the mediational role of psychological tools in the appropriation of theoretical understandings. This is exemplified by the representational “schematic”, the mastery of which is not merely incidental to but the explicit focus of instruction. According to Vygotsky, psychological tools, unlike their material counterparts that are directed toward action on the outer environment, are directed inward toward the control of one’s own behavior. What is the direction that the schematic gives to the problem solver? It orients the child’s attention to the internal relations among quantities, rather than to the empirical features of the problem or familiar aspects of numbers. By the time children reach the second grade, they generally have little need of actual objects (or “manipulatives”), preferring to work instead with the “schematics’ that focus them on, isolate, and express the mathematical actions in which they are engaged. These schematics are representations, but are not pictorial in nature as is often the case when US children model problem situations.

The Part-Whole Relation

An important schematic introduced in the first grade curriculum of Davydov is the “/\” representation for a part-whole relationship. Early in the first grade, for example, children determine that they can make two unequal volumes equal by adding to the smaller or subtracting from the larger the difference between the original quantities. If $A > B$, and the difference between volumes A and B is volume C, then the relationship among the three quantities can be modeled by the part-whole schematic in which A is the whole and B and C are the parts:

```
    A
   / \
  /   \
 B     C
```

The children then determine that if volume A is greater than volume B, $A = B + C$, where C is the difference between A and B. They may schematize their result with a "length" model, and symbolize it with three equations:



$A > B$

$A > B$ by C
$A = B + C$
$B = A - C$
$C = A - B$

In solving problem #3 that appears in the above list of problems, children would first analyze the problem structure, identify it as a part-whole structure, then determine that c, the number of stamps Andrew had originally, is a part and that g, the number of stamps he had after receiving more, is the whole. They would then model the quantitative relations expressed in the problem using the "/\" schematic, placing g at the top of the "/\" and c and ? at the bottom. Since the unknown part can be found by subtracting the known part from the whole, they set up the equation $x = g - c$. It is immediately obvious that this is a missing addend problem whose equation $g = c + x$, set up directly from the problem sequence, suggests addition. However, it is the schematic and its relationship to the equation that enables children using Davydov's curriculum to avoid the error suggested by the sequence of (additive) actions performed in the problem situation and to correctly subtract the part from the whole. Note that "counting on", a common strategy employed by children who learn arithmetic before algebra, is not an option where the numerical values of quantities are unknown, as is the case in an algebraic formulation wherein quantities have only literal designations.

In the Davydov curriculum, the situation that occurs in problem #3 above is explicitly addressed within the context of cutting a piece of wire from a coil. Here the action is always the same. But three separate problems can be formed from this action, each of which is simply delineated by an equation below:

$$\begin{array}{ccccccccccc} B & - & T = & R & \quad D & - & A = & N & \text{and} \quad G & - & C = M \\ x & & 9 & 6 & \quad 32 & & x & 17 & \quad 21 & & 9 \quad x \end{array}$$

In each case what is asked for is the number of meters of wire, which requires finding the value of x in each of the three problems. After completion of the third and final problem, it is noted that in all three of the exercises above the same action was performed, i.e., some wire was cut from a coil. Children must then determine why they sometimes added and sometimes subtracted in finding the amount of wire. The action on the wire in the coil was always to cut off a piece of the wire as each of the three literal equations indicates. However, the children must reason that in the case of the first problem ($B - T = R$), it is the amount of wire originally in the coil that is unknown, while the measures of the part removed and the part remaining are known. Consequently, they must add the parts to obtain the original length of wire in the coil prior to cutting off a piece. In the two subsequent cases, the part removed and the part remaining are to be found. Since the whole is known, subtraction is called for. And it is, of course, precisely this distinction that is relevant to problems #3, #4, #9, and #10 in the list of problems appearing above. The wire cutting problem renders explicit the difference between the action performed within a problem context and the action required to obtain the solution to a problem arising from within that context.

In addition, a fundamental dictum of dialectical logic is reflected throughout the curriculum, viz. that "the essence of a thing can be revealed only by considering the process of its *development*" (Davydov 1990, p. 288). Hence, the origin of number is traced back through its development from the progressive refinement of comparison of quantities to their measurement. Number does not appear "fully formed" as it were, as is typical in conventional textbook treatments. Nor do addition and subtraction appear as separate operations, but rather as dialectically interrelated actions that arise from the part-whole relation between quantities. The "/\" schematic serves to both focus children's attention on this relation and provide a means to represent it at an intermediate stage between its direct observation and its transformation into an inequality or equation.

Not infrequently, children are confronted with problems that are unsolvable. In our implementation of the Davydov curriculum (to our knowledge a first in the United States), children were asked to solve individually the following problem: "There were seven books on the bookshelf. Nine children entered the library and each child took a book from the shelf. How many books are now on the shelf?" None of the children answered "16" or "2". Instead they recognized that a part cannot be larger than the whole and all identified the problem as a "trap" (i.e., unsolvable). When asked to change the problem into one that had a solution, the following types of changes were made by the children.

(A) There were seven books on the shelf. Five children entered the library and each removed a book from the shelf. How many books are now on the shelf?
(B) There were 27 books on the shelf. Nine children entered the library and each took a book from the shelf. How many books are now on the shelf?
(C) There were seven books on the shelf. Nine children entered the library and each placed a book on the shelf. How many books are now on the shelf?

The children recognized that either the whole must be increased (B) or the part decreased (A) (or both), or the *action* on the objects must be changed. In (C), for example, both the books *initially* on the shelf and those *added* by the children, are now *parts*; whereas in the other formulations the first was the whole and the second was a part.

One may again discern in these problems an ascent from the abstract to the concrete. From the three equations generated from problem situations such as that arising from the comparison of two volumes (described above), children discern their application to the part-whole situation not only in problems such as #3, #4, #9, and #10 in the list provided above, but also to the specific numerical designations of quantities in problems such as the book problem, where they must reason in reverse, as it were, that the whole must be greater than the parts in order for the problem to have a solution. Because they have a theoretical orientation to the problem structure, they can analyze the relationships between quantities in problems without any numerical designations (such as problems #3 and #4 in the above list) to provide them with cues. And in the book problem they are not led by the numerical aspects of the problem to simply add the two numbers or subtract the smaller from the larger.

Another example of orientation to theoretical understanding is exemplified by a problem encountered early in the second grade curriculum, and mentioned in the opening paragraph of this chapter, viz.: "What is $T-4-4$?" After much discussion, the classroom debate centered on two possible answers: $T - 0$ ($= T$) or $T - 8$. Neither side in the discussion could convince the other of their point of view. Finally, as the debate entered its second day, the teacher asked a child who argued that the answer was $T-0$ to mark the parts and the whole in the statement: $14-4-4=14$ (since the child had argued that $T - 4 - 4 = T$). The child marked the 14 as the whole in both places where it appeared in the statement of equality, and marked each 4 as a part. The teacher then asked the child to mark the whole and the parts in the statement: $P - A - B = C$. Here the child correctly marked P as the whole, and A, B, and C as the parts, and then drew the following schematic:

$$\begin{array}{ccc} & P & \\ / & | & \backslash \\ A & B & C \end{array}$$

When the child realized that a similar schematic of the earlier answer would have to show 14 as the whole and 4, 4, and 14, as the parts, the child realized that in the original answer 14 had been treated simultaneously as the whole and also as one of the parts, and at the suggestion of classmates, changed the parts to 4, 4, and x (Lee 2002). What is striking here is that the child did not see the error until the relationship between the quantities was expressed algebraically in the abstract. The numbers in the problem had been the source of the error; the numbers had misled the child and the abstract algebraic representation of the mathematical structure corrected the error.

A negative illustration of the role of such psychological tools occurred in the case of a US child who was very quick with numerical computation, and refused to use schematics once numbers were introduced, protesting that they were unnecessary and cumbersome. This child was able to solve one-step word problems fairly well, but when two-step problems were introduced, the child "just picked numbers and calculated" (Lee 2002), ignoring the internal relationships among the quantities, and making many errors. And when no numbers were present, as in the following problem, the child's resulting reduction to "$A + B$" was predictable: Tanya picked

A mushrooms. Her grandmother picked *B* mushrooms more than Tanya picked. How many mushrooms did Tanya and her grandmother pick? (Davydov et al. 2000, p. 54). What is significant is that this child showed considerable ability to reason mathematically with simpler first grade tasks, but as a result of failure to master the important semiotic tools of analysis, was prone to error on tasks of greater complexity and consequently this capability did not develop along the lines of its initial promise.

By directing the focus to the underlying theoretical structure of a mathematical action or problem situation, the psychological tool serves to unite empirically disparate actions or phenomena. Addition and subtraction constitute a case in point, since the "/\" schematic reveals the part-whole structure common to both. This same psychological tool functions in the analysis of the underlying part-whole structure of so-called "fact families", the addends and sums that primary school children learn, and the part-whole structure of numbers represented in positional systems. Consequently, it figures prominently in children's ability to correctly answer questions #5, #7, and #8 in the list of problems above, without any need for recognizable numerals in the numbers they are asked to compare. The numbers can be compared simply by discernment of the theoretical (part-whole) structure of a number and its designation within a positional system. And later when children encounter equations such as $17 + 5x = 42$, they are by third grade generally able to work them without the use of the "/\", and simply write $5x = 42 - 17$, then $5x = 25$ and $x = 5$. And an equation such as $324 - x = 13$ is quickly transformed into $x + 13 = 324$ and $x = 311$. It is the child's orientation to theoretical structure, which has now largely become internalized through the mastery of the requisite psychological tools, that figures prominently in enabling the algebraic concepts to transform the meaning of the lower (numerical and empirical) concepts as Vygotsky indicated.

The Table as Psychological Tool

Rather than being directed to "make a table" the first time the need for one arises in the process of solving a particular type of problem, the construction of a table evolves through considerable stages beginning early in the third grade curriculum. Here again, as in the case of the "/\" schematic, a table is a psychological tool, and its construction is the subject of explicit exploration and instruction. And as is typical of the Davydov curriculum, every aspect of its construction is explored.

While doing research in Russia, the author watched beginning third graders solve problems such as the following:

Group the quantities below in several ways.

A = the height of the tree
B = 16 kg
C = the weight of the board
D = the age of the tree
E = 20 cm
M = the area of the board

H = 16 seconds
K = 24 meters
T = the length of the rope

Some children would group B, E, H, and K together under the designation "Known" and place all the other quantities under the designation "Unknown". Others would choose to group A and D under the designation "Tree", C and M under "Board", and T under "Rope", then group the other data under "Numbers". A third grouping might be comprised by A, E, K, and T grouped under "Length"; M under "Area"; B and C under "Weight"; and D and H under "Time". The children might then set up tables with the designated headings in a row across the top and the appropriate quantities listed under each heading.

As is typical of the Davydov curriculum, not only are children asked to solve problems such as the one above, but they are also given tables in which one or more numbers or letters appear under the designations "Time", "Weight", and "Length", and they are asked to generate a list containing a description of quantities that fit the categories in the table. A similar table might have categories such as "Elephant", "Recess", "Lesson", and "Log", and a third might contain only the two categories of "Known" and "Unknown". As with the first table, children are asked to supply a list of quantities that fit the categories in the last two tables. Thus they are required to think flexibly and to reverse their perspectives in solving these two types of problems.

From here children progress to more open ended problems that ask that they make up a list of quantities and group them according to property, object, and whether the quantities are known or unknown. It will be immediately noted that when working problems such as those above, these (i.e. property, object, and known/unknown) are the general categories. In the first problem, for example, "Length", "Area", "Weight", and "Time", are *properties*, whereas "Tree", "Board", and "Rope" are *objects*. Eventually it will be seen that in mathematics we are interested in the properties of quantities, and there will be tabular groupings such as are found in rate, time, and distance problems, wherein properties such as "Time" and "Distance" are listed horizontally in a row across the top of a table while the objects to which they pertain—Train #1 and Train #2, for example, or Bicycle and Runner—will appear in an adjacent vertical column. What is important to notice here is the thoroughness with which the children explore initially the element of arbitrariness in the designations and arrangements of a table, so that they can flexibly make use of this knowledge whenever they choose to construct a table to use as a psychological tool to organize the information given in a problem in a way that will assist them in its solution. At this point, however, they are asked only to generate the table, and this is illustrative of the manner in which the psychological tool itself becomes a matter of intentional and explicit instruction rather than being introduced as simply incidental to obtaining the solution to some problem.

The use of a table as a psychological tool to solve a problem will come later. In the interim children will be asked to make tables simply from narratives with no problem formulated and hence, no solution called for. For example, they may be asked to construct a table to organize information such as the following: "There are

20 oranges in a box, and there are 35 oranges in another box." Such a problem introduces the need to designate both rows and columns, and they may place "First Box" and "Second Box" in a column under what is obviously an object designation, and list "20 pieces" and "35 pieces" respectively in a second column under the heading "Quantity". This may be followed by a similar but slightly different narrative, viz. "There are 20 kg of oranges in one box and 35 kg of oranges in another one." Now they may use the same first column in their new table as in the previous one, but list "20 kg" and "35 kg" respectively in the second column under the category "Weight". Finally, the problem may be changed as follows: "There are 20 kg of oranges in one box and 35 oranges in another." Now while retaining the list of boxes in the first column, the children will have to discern the need to create two additional columns, one containing "20 kg" under the heading "Weight" and the other containing "35 pieces" under the heading "Quantity". These are three simple narratives, but they require children to pay close attention to the properties involved and how they need to be dealt with so that a table will accurately reflect the categories of its members.

Eventually a question will be attached to a simple narrative in a problem such as the following: "A small melon weighs m lbs. A watermelon is heavier than the melon and weighs n lbs. By how many pounds is the watermelon heavier than the melon?" The accompanying instruction directs the students to build a table for the problem. It will be immediately obvious to the reader that this is a problem that children could readily solve in the first grade using the "/\" schematic to discern that the weight of the melon was only a part of the weight of the watermelon, but the problem directs the children to construct and use a different psychological tool, namely a table. Requiring the use of a table for the solution to such a simple problem paves the way for its use in the solution of more complex problems that will be introduced later, such as those involving proportional reasoning. It allows the child to continue to focus mainly on the building and role of the newly introduced psychological tool (since s/he can by now easily solve the problem without it and is therefore not consumed with the need to focus on the solution), while at the same time introducing the possibility of applying the tool to the solution of a mathematical problem.

As they progress, children may be asked to build tables for two statements such as the following: "(A) A worker made 28 widgets before lunch and 42 widgets after lunch. (B) A worker made 28 widgets in 3 hours and 42 widgets during the next 3 hours." Now they must deal with variable quantities in the form of the number of widgets and in statement (B) also with time. They must identify not only the variable*(s)*, but the *process* and the separate *events* occurring during the process. Here, of course, the process is the making of widgets, and there are two events—before lunch and after lunch in the case of statement (A), and the first event (consisting of 2 hours) and the second event (consisting of 3 hours) in the case of statement (B). Children note that during the process of work, time and the amount of work changed. Consequently, these are the variables.

The children may build a table for statement (A) consisting of just two columns, viz. "Event" and "Quantity". Under "Event" they might list "I" and "II" referring to the first and second time periods, respectively, and "I + II" or "Total" for the third and final entry. And under "Quantity", entries 28, 42, and __, would appear opposite

Events I, II, and Total, respectively. In the case of statement (B) above, the students would need to add a third column, viz. “Time (hours)”, under which they would list 2, 3, and 5. Now the children might be asked to formulate a question for each of the above statements. Clearly, the question “How many widgets did the worker make altogether?” would be appropriate for both statements (A) and (B).

In a problem requiring a reversal in their perspective, children might be given a table with the Process of Buying, and Events listed as First Purchase, Second Purchase, and Total Purchase. The corresponding amounts (in dollars) might be listed as c, t, and $c + t$. Another column of amounts (in numbers) might list 3, 1, and 4, respectively. The children would then be asked to make up a text to fit the table.

Eventually, a table such as the following will be encountered. Two columns, one representing the variable “Weight (lbs)”, and another representing the variable “Number of Parts”, will be presented. The following entries appear under “Weight (lbs)”: 12, 24, 72, and 60. The corresponding entries under “Number of Parts” are 4, 8, 24, and 20. In a second table the entries under “Weight (lbs)” are the same as in the first table, but now the corresponding entries under “Number of Parts” are 4, 16, 32, and 24. Children are to name the process and both observe and discuss what is different concerning the variable listings in the two tables. Eventually they discern that in the first table as one of the variables increases several times, the other increases the same number of times. Such a process is called a *uniform* process and the variables are called *direct proportional* variables. The process described by the second table is not uniform.

The children have now been formally introduced to direct variation and later in the third grade they will use the knowledge they have acquired of building and using a table as a psychological tool to solve problems such as the following. “The speed of a car is 60 mph and the speed of a bicyclist is 4 times less. The bicyclist can travel from his house to the nearest city in 2 hours. How long will it take to go the same distance in the car?” The children also are able to construct a table to assist them in determining how many runs must be made by a truck and a trailer in order to transport 1080 tons of coal, if the truck can haul 30 tons per load and the trailer can carry twice as less coal per load (cf. Schmittau 2004). These are but two examples of the variety and types of problems for which children might choose to construct a table as an aid to finding a solution. Their extensive experience with situations in which tables may serve to organize information and to focus and direct attention to the problem structure and the relevant processes, events, and variables involved, render the table a flexible tool that can be of use in a wide variety of problem situations. This includes but is not limited to problems such as those involving rate, time, and distance, differential rates of work, and other sorts of problems involving proportional reasoning. Children may even construct a table to find missing dimensions of geometric figures for which only literal data are provided (Davydov et al. 2001).

In the manufacture of widgets problem it will be noted that the two events listed are parts of the whole event which is the total time worked. Consequently, the table might be thought of as embodying the part-whole structure of the “/\” schematic. This is the case as well in the melon problem preceding it, and some of the more difficult problems that follow. The table embeds this structure within itself, as a psychological tool capable of accommodating the greater conceptual complexity that occurs in problem situations involving processes with multiple variables.

Concluding Remarks

In our investigation of Davydov's elementary curriculum in what is, to the best of our knowledge, its first implementation in the US, we found that the US children evinced abstract and generalized understandings not unlike those the author observed among their Russian counterparts. (For information additional to that presented in this chapter cf. Schmittau 2004.) They were aided in developing their knowledge of algebraic structure by the schematic models, which functioned as "helpers" in the children's language and as psychological tools within the framework of Vygotskian theory (Vygotsky 1986).

By the end of the third grade curriculum, the children studying Davydov's program were solving applied problems involving proportional reasoning, differential rates of work, and rate, time and distance, that continued to challenge eleventh graders in their second year of formal algebra study in their regional high schools. Moreover, although Davydov's third grade curriculum (the final year of Russian elementary school) consisted of 969 problems (each year of the curriculum was composed entirely of very deliberately designed and sequenced problems), the class's teacher reported that students had no difficulties with the last 400 problems, despite the fact that objectively the level of difficulty of the problems themselves had increased considerably as indicated above. In fact, the more complex and difficult the problems became, the less difficulty the children had in solving them, and the more rapidly they progressed through them. This attests to the value of laying a broad in-depth conceptual foundation (extensive work with quantity as a concept antecedent to number) with a focus on theoretical (i.e., algebraic) structure, and to the mastery of psychological tools—two of which are discussed above—that function to orient students to that structure. Their appropriation of these psychological tools which are not encountered merely incidental to but are the explicit focus of instruction assist children in their engagement with "the formal discipline of scientific concepts" (Vygotsky 1986, p. 206).

It is worth noting also that the approach taken by the Davydov curriculum, following as it does Vygotskian theory and dialectical logic, is far removed from, for example, the simple introduction of equations into the elementary school mathematics curriculum. Teaching a method of solving equations by, for instance, "balancing" both sides of the equation, while it provides a means of obtaining a solution to the equation, does not ensure understanding of the theoretical essence of the relationship between the quantities represented, both known and unknown, and how that relationship arises from the progressive comparisons of quantities found in objects, through measurement and the origins of number, and subsequently finds expression in a variety of mathematical actions and situations. Similarly, the introduction of a table as a means to solve a particular type of problem fails to exploit its effectiveness as a psychological tool that can be flexibly applied wherever it is useful to organize information in the service of directing attention to the theoretical structure of a problem situation and its mathematical interrelationships. Such episodic introductions into the curriculum also fail to comply with the central tenet of dialectical logic concerning the necessity of tracing a concept through its entire developmental path as a necessary means to ensure its understanding.

In summary, the Vygotskian imperative concerning theoretical understanding as essential for the development of algebraic thinking is reflected throughout the elementary curriculum of Davydov. It is the *mastery of theoretical concepts* which "gradually transforms the structure of the child's spontaneous concepts and ... furthers the child's ascent to higher developmental levels" (Vygotsky 1986, p. 206).

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The Arithmetic-Algebra Connection: A Historical-Pedagogical Perspective

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Abstract The problem of designing a teaching learning approach to symbolic algebra in the middle school that uses students' knowledge of arithmetic as a starting point has not been adequately addressed in the recent revisions of the mathematics curriculum in India. India has a long historical tradition of mathematics with strong achievements in arithmetic and algebra. We review an explicit discussion of the relation between arithmetic and algebra in a historical text from the twelfth century, emphasizing that algebra is more a matter of insight and understanding than of using symbols. Algebra is seen as foundational to arithmetic rather than as a generalization of arithmetic. We draw implications from these remarks and present a framework that illuminates the arithmetic-algebra connection from a teaching-learning point of view. Finally, we offer brief sketches of an instructional approach developed through a design experiment with students of grade 6 that is informed by this framework, and discuss some student responses.

Introduction

Mathematics is widely believed in India to be the most difficult subject in the curriculum and is the major reason for failure to complete the school year in secondary school (National Centre for Educational Research and Training 2006). The education minister of a western Indian state recently complained that students spend vast amounts of time studying mathematics, with limited success and at the cost

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of neglecting other subjects and extracurricular activities. Similar complaints pressured the state government into removing the mandatory pass requirement in mathematics for the school exit examination in the year 2010. This is reflective of a trend among some school systems in India to make mathematics an optional subject in the school exit examination. Students' difficulties in mathematics may however have deeper causes located in the education system as a whole, which need to be addressed on multiple fronts. The nation wide annual ASER surveys, based on representative samples of rural schools, found very low levels of learning of mathematics in the primary grades (ASER report 2010). A survey of the most preferred, "top" schools in leading Indian metro cities found surprisingly low levels of conceptual understanding in science and mathematics (Educational Initiatives and Wipro 2006).

Efforts to address the issue of failure and low learning levels include an important reform of the school curriculum following the 2005 National Curriculum Framework (NCF 2005), which emphasized child-centered learning (National Centre for Educational Research and Training 2005). New textbooks for grades 1–12 following the NCF 2005 were brought out by the National Council of Educational Research and Training (NCERT) through a collaborative process involving educators and teachers. We shall refer to these as the "NCERT textbooks". The NCERT textbooks in mathematics have introduced significant changes in the instructional approach, especially in the primary grades. However, one of the issues that remain inadequately addressed in the new textbooks is the introduction to symbolic algebra in the middle grades, which follows a largely traditional approach focused on symbol manipulation. Since algebra is an important part of the secondary curriculum, bringing mathematics to wider sections of the student population, entails that more thought be given to how algebra can be introduced in a manner that uses students' prior knowledge. Our aim in this chapter is to articulate a framework that addresses the issue of transition from arithmetic to symbolic algebra, and to outline an instructional approach based on this framework that was developed by the research group at the Homi Bhabha Centre through a design experiment. In this section of the paper, we shall briefly sketch the background of the reform efforts, insofar as they are relevant to the teaching and learning of algebra.

In India, school education includes the following levels of schooling: primary: grades 1–5, middle or upper primary: grades 6–8, secondary: grades 9–10 and higher secondary: grades 11–12. The provision of school education is largely the domain of the state government, subject to broad regulations laid down by the central government. The vast majority of students learn from textbooks published and prescribed by the state or the central government. Following the reform process initiated by the central government through NCF 2005, many state governments have revised or are in the process of revising their own curricula and textbooks to align them with the new curriculum framework. In comparison to the earlier years, the mathematics curriculum and the NCERT textbooks at the primary level have changed significantly, while the middle school curriculum, where algebra is introduced continues largely unchanged (Tripathi 2007).

Algebra, as a separate topic, forms a large chunk of the middle and secondary school syllabus in mathematics and also underlies other topics such as geometry or trigonometry. Students' facility with algebra is hence an important determinant of success in school mathematics. Thus, as elsewhere, algebra is a gateway to higher learning for some pupils and a barrier for others. In the new NCERT textbooks, formal algebra begins in grade 6 (age 11+) with integer operations, the introduction of variables in the context of generalization, and the solution of simple equations in one unknown. Over the five years until they complete grade 10, pupils learn about integers, rational numbers and real numbers, algebraic expressions and identities, exponents, polynomials and their factorization, coordinate geometry, linear equations in one and two variables, quadratic equations, and arithmetic progressions.

The grade 6 NCERT mathematics textbook introduces algebra as a branch of mathematics whose main feature is the use of letters "to write formulas and rules in a general way" (Mathematics: Text book for class VI 2006, p. 221). It then provides a gentle introduction to the use of letters as variables, and shows how expressions containing variables can be used to represent formulas, rules for a growing pattern, relations between quantities, general properties of number operations, and equations. However, this easy-paced approach gives way to a traditional approach to the manipulation of algebraic expressions in grade 7, based on the addition and subtraction of like terms. The approach in the higher grades is largely formal, with real life applications appearing as word problems in the exercises. Thus, although an effort has been made in the new middle school textbooks to simplify the language, the approach is not significantly different from the earlier approach and does not take into account the large body of literature published internationally on the difficulty students face in making the transition from arithmetic to algebra and the preparation needed for it. (For details and examples, see Banerjee 2008b.)

The NCERT mathematics textbooks for the primary grades, have attempted to integrate strands of algebraic thinking. In a study of the primary mathematics curricula in five countries, Cai et al. (2005), have applied a framework that identifies the algebra strand in terms of algebra relevant goals, algebraic ideas and algebraic processes. Some of the elements identified by Cai et al. are also found in the NCERT primary mathematics textbooks. There is a consistent emphasis on identifying, extending, and describing patterns through all the primary grades from 1 to 5. "Patterns" have been identified as a separate strand in the primary mathematics syllabus, and separate chapters appear in the textbooks for all the primary grades with the title "Patterns." Children work on repeating as well as growing patterns in grade 2 and grade 3. Many other kinds of patterns involving numbers appear in these books: addition patterns in a 3×3 cell on a calendar, magic squares, etc. A variety of number puzzles are also presented at appropriate grade levels; some of the puzzles are drawn from traditional or folk sources (for an example, see Math-magic: Book 3 2006, pp. 92–94).

Simple equations with the unknown represented as an empty box or a blank appear in grade 2 and later. The inverse relation between addition and subtraction is highlighted by relating corresponding number sentences and is also used in checking column subtraction (Math-magic: Book 3 2006). Change also appears as an important theme in these textbooks. The quantitative relation between two varying quantities is discussed at several places: the weight of a growing child which, according to a traditional custom, determines the weight of sweets distributed on her birthday (Grade 3), the number of elders in each generation of a family tree, the annual growth of a rabbit population, the growth chart of a plant over a number of days (all in Grade 5, Math-magic: Book 5 2008). No letter symbols are used in these examples, and relationships are expressed in terms of numerical tables, diagrams or charts.

These strands in the NCERT primary school textbooks are not taken up and developed further in the NCERT middle school textbooks, which appear to begin afresh by introducing a symbolic approach to algebra. A part of the reason lies in the fact that the two sets of textbooks are produced by different teams, and the schedule of publication does not always allow for smooth co-ordination. (The grade 6 textbook, for example, was published two years before the grade 5 textbook.) Another reason, we hypothesize, is the pressure to build students' capabilities with symbolic algebra, which is needed for secondary school mathematics. Curriculum design involves striking a balance between different imperatives. The balance realized in the primary mathematics textbooks emphasizes immersion in realistic contexts, concrete activities, and communicating the view that mathematics is not a finished product (Mukherjee 2010, p. 14). The middle school curriculum is more responsive to the features of mathematics as a discipline and emphasizes the abstract nature of the subject. In the words of the coordinator for the middle school textbooks, "learners have to move away from these concrete scaffolds and be able to deal with mathematical entities as abstract ideas that do not lend themselves to concrete representations" (Dewan 2010, p. 19f).

Besides finding ways of building on the strands of algebraic thinking that are present in the primary curriculum and textbooks, a concern, perhaps even more pressing in the curriculum design context in India, is to find more effective ways for the majority of children to make the transition to the symbolic mathematics of secondary school. Algebra underlies a large part of secondary mathematics, and many students face difficulties of the kind that are identified in studies conducted elsewhere (Kieran 2006). A compilation of common student errors from discussions with teachers includes well-known errors in simplifying algebraic expressions and operating with negative numbers (Pradhan and Mavlankar 1994). Errors involving misinterpretation of algebraic notation and of the "=" sign are common and persistent (Rajagopalan 2010). Building on students' prior knowledge and intuition to introduce symbolic algebra remains one of the challenges facing mathematics curriculum designers, and it is yet to be adequately addressed.

In this chapter, we offer a perspective on the relationship between arithmetic and algebra and an example of a teaching approach developed by a research group at the Homi Bhabha Centre led by the authors to manage the transition from arithmetic to symbolic algebra. The key aspect of this approach is focusing on symbolic arithmetic as a preparation for algebra. Students work with numerical expressions, that is, expressions without letter variables, with the goal of building on the operational sense acquired through the experience of arithmetic. This, however, requires a shift in the way expressions are interpreted. The aim is not just to compute the value of an expression, but to understand the structure of the expressions. Numerical expressions offer a way of expressing the intuitions that children have about arithmetic and have the potential to strengthen this intuition and enhance computational efficiency. To enable this transition, numerical expressions must be viewed not merely as encoding instructions to carry out a sequence of binary operations, but as revealing a particular operational composition of a number, which is its "value." Thus facility with symbolic expressions is more than facility with syntactic transformations of expressions; it includes a grasp of how quantities or numbers combine to produce the resultant quantity. This view of expressions leads to flexibility in evaluating expressions and to developing a feel for how transforming an expression affects its value. We argue that understanding and learning to "see" the operational composition encoded by numerical expressions is important for algebraic insight. We elaborate on the notion of operational composition in a later section and discuss how this perspective informs a teaching approach developed through trials with several batches of students.

The idea that numerical expressions can capture students' operational sense or relational thinking has been explored in other studies (for example, Fujii and Stephens 2001). In appropriate contexts, students show a generalized interpretation of numbers in a numerical expression, treating them as quasi-variables. We will review these findings briefly in a later section. The idea that algebra can enhance arithmetical insight is a view that finds support in the Indian historical tradition of mathematics. Algebra is viewed not so much as a generalization of arithmetic, but rather as providing a foundation for arithmetic. An implication is that building on the arithmetic understanding of students is, at the same time, looking at symbols with new "algebra eyes." It is not widely known that Indian mathematicians achieved impressive results in algebra from the early centuries CE to almost modern times. The fact that Indian numerals and arithmetic were recognized as being superior and adopted first by the Islamic cultures and later by Europe is more widely known. The advances in arithmetic and algebra are possibly not unconnected, since arithmetic may be viewed as choosing a representation of the operational composition of a number in a way that makes calculation easy and convenient. In the next section, we shall briefly review some of the achievements in Indian algebra and discuss how the relation between arithmetic and algebra was viewed in the Indian historical tradition.

Arithmetic and Algebra in the Indian Mathematical Tradition

India had a long standing indigenous mathematical tradition that was active from at least the first millennium BCE till roughly the eighteenth century CE, when it was displaced by Western mathematics (Plofker 2009, p. viii). Some of the achievements of Indian mathematics worth highlighting are the appearance, in a text from 800 BCE, of geometrical constructions and statements found in Euclid's *Elements*, including the earliest explicit statement of the "Pythagoras theorem," discussion of the binomial coefficients and the Fibonacci series in a work dated to between 500 and 800 CE, the solution of linear and quadratic indeterminate equations in integers, a complete integer solution of indeterminate equations of the form $x^2 - Ny^2 = 1$ ("Pell's" equation) in a twelfth century text, the finite difference equation for the sine function in the fifth century CE and power series expansion for the inverse tangent function in the fourteenth century CE (Mumford 2010; Plofker 2009).

It is well attested that Indian numerals and arithmetic were adopted first by the Islamic civilization following exchanges between the two cultures around the eighth century CE, and later by Europe (Plofker 2009, p. 255). Indian algebra was also developed by this time as seen in the seventh century work of Brahmagupta, which we shall discuss below. However there are important differences between Arabic algebra (as found in al-Khwarizmi's work *al-jabr*, for example) and the algebra in Indian mathematical texts. Two of the main differences are that Arabic algebra avoided negative quantities, while Indian texts routinely used them, and Indian algebra used notational features such as tabular proto-equations and syllabic abbreviations for unknown quantities, while Arabic algebra was purely rhetorical (Plofker 2009, p. 258f).

We will first give an overview of how topics in arithmetic and algebra are organized in the central texts of Indian mathematics, and then turn to explicit statements about the relation between arithmetic and algebra. Indian texts containing mathematics from the first millennium CE are typically one or more chapters of a work dealing with astronomy. Purely mathematical texts appear only later, as for example, in the work of Bhaskara II in the twelfth century CE. The *Aryabhateeyam,* from the 5^{th} century CE, one of the oldest and most influential astronomical-mathematical texts, contains a single chapter on mathematics that includes arithmetic and the solution of equations.

The *Brahmasphuta Siddhanta* (c. 628 CE) by Brahmagupta, considered to be one of the greatest Indian mathematicians of the classical period, has two separate chapters dealing respectively with what we might classify as arithmetic and algebra. The word that Brahmagupta uses for the second of these chapters (algebra) is *kuttaka ganita* or "computation using *kuttaka*." *Kuttaka* (frequently translated as "pulverizer") is an algorithm for reducing the terms of an indeterminate equation, which is essentially a recasting of the Euclidean algorithm for obtaining the greatest common divisor of two natural numbers (Katz 1998). Interestingly, puzzles called *kuttaka* are found even now in folklore in India and require one to find positive integer solutions of indeterminate equations. (For an example, see Bose 2009.)

The "arithmetic" chapter in the *Brahmasphuta siddhanta* deals with topics such as the manipulation of fractions, the algorithm for cube roots, proportion problems of different kinds and the "rule of three" (a representation of four quantities in proportion with one of them unknown), the summation of arithmetic progressions and other kinds of series, miscellaneous computational tips, and problems dealing with geometry and geometrical measurement (Colebrooke 1817). The *kuttaka* or algebra chapter deals with techniques for solving a variety of equations. In the initial verses of this chapter, we find the oldest extant systematic description in the Indian tradition of rules of operating with various kinds of quantities: rules for operations with positive and negative quantities, zero, surds (irrational square roots of natural numbers), and unknown quantities. The approach of beginning the discussion of algebra by presenting the rules of operations with different kinds of numbers or quantities became a model for later texts. Laying out these rules at the beginning prepared the way for demonstrating results and justifying the procedures used to solve equations.

Mathematicians who came after Brahmagupta referred to algebra as *avyakta ganita* or arithmetic of unknown quantities, as opposed to *vyakta ganita* (arithmetic of known quantities). Others, starting from around the 9th century CE, have used the word *bijaganita* for algebra. *Bija* means "seed" or "element," and *bijaganita* has been translated as "computation with the seed or unknown quantity, which yields the fruit or *phala*, the known quantity (Plofker 2007, p. 467). The word *bija* has also been translated as "analysis" and *bijaganita* as "calculation on the basis of analysis" (Datta and Singh 1938/2001). "*Bijaganita*" is the word currently used in many Indian languages for school algebra.

Bhaskara II from the 12th century CE (the numeral "II" is used to distinguish him from Bhaskara I of an earlier period) devoted two separate works to arithmetic and algebra—the *Lilavati* and the *Bijaganita*, respectively, both of which became canonical mathematical texts in the Indian tradition. Through several remarks spread through the text, Bhaskara emphasizes that *bijaganita,* or analysis, consists of mathematical insight and not merely computation with symbols. Bhaskara appears to have thought of *bijaganita* as insightful analysis aided by symbols.

> Analysis (*bija*) is certainly the innate intellect assisted by the various symbols [*varna* or colors, which are the usual symbols for unknowns], which, for the instruction of duller intellects, has been expounded by the ancient sages… (Colebrooke 1817, verse 174)

At various points in his work, Bhaskara discourages his readers from using symbols for unknowns when the problem can be solved by arithmetic reasoning such as using proportionality. Thus after using such arithmetic reasoning to solve a problem involving a sum loaned in two parts at two different interest rates, he comments, "This is rightly solved by the understanding alone; what occasion was there for putting a sign of an unknown quantity? … Neither does analysis consist of symbols, nor are the several sorts of it analysis. Sagacity alone is the chief analysis … " (Colebrooke 1817, verse 110)

In response to a question that he himself raises, "if (unknown quantities) are to be discovered by intelligence alone what then is the need of analysis?", he says, "Because intelligence alone is the real analysis; symbols are its help" and goes on to repeat the idea that symbols are helpful to less agile intellects (*ibid.*).

Bhaskara is speaking here of intelligence or a kind of insight that underlies the procedures used to solve equations. Although he does not explicitly describe what the insight is about, we may assume that what are relevant in the context are the relationships among quantities that are represented verbally and through symbols. We shall later try to flesh out what one may mean by an understanding of quantitative relationships in the context of symbols.

The word "symbol" here is a translation for the sanskrit word *varna*, meaning color. This is a standard way of representing an unknown quantity in the Indian tradition—different unknowns are represented by different colors (Plofker 2009, p. 230). Bhaskara's and Brahmagupta's texts are in verse form with prose commentary interspersed and do not contain symbols as used in modern mathematics. This does not imply, however, that a symbolic form of writing mathematics was not present. Indeed, in the Bakshaali manuscript, which is dated to between the eighth and the twelfth centuries CE, one finds symbols for numerals, operation signs, fractions, negative quantities and equations laid out in tabular formats, and their form is closer to the symbolic language familiar to us. For examples of the fairly complex expressions that were represented in this way, see Datta and Singh (1938/2001, p. 13).

Bhaskara II also explicitly comments about the relation between algebra and arithmetic at different places in both the *Lilavati* and the *Bijaganita*. At the beginning of the *Bijaganita,* he says, "The science of calculation with unknowns is the source of the science of calculation with knowns." This may seem to be the opposite of what we commonly understand: that the rules of algebra are a generalization of the rules of arithmetic. However, Bhaskara clearly thought of algebra as providing the basis and the foundation for arithmetic, or calculation with "determinate" symbols. This may explain why algebra texts begin by laying down the rules for operations with various quantities, erecting a foundation for the ensuing analysis required for the solution of equations as well as for computation in arithmetic. Algebra possibly provides a foundation for arithmetic in an additional sense. The decimal positional value representation is only one of the many possible representations of numbers, chosen for computational efficiency. Algebra may be viewed as a tool to explore the potential of this form of representation and hence as a means to discover more efficient algorithms in arithmetic, as well as to explore other convenient representations for more complex problems.

At another point in the *Bijaganita,* Bhaskara says, "Mathematicians have declared algebra to be computation attended with demonstration: else there would be no distinction between arithmetic and algebra" (Colebrooke 1817, verse 214). This statement appears following a twofold demonstration, using first geometry and then symbols, of the rule to obtain integer solutions to the equation $axy = bx + cy + d$. Demonstration of mathematical results in Indian works often took geometric or algebraic form (Srinivas 2008), with both the forms sometimes presented one after the other. The role of algebra in demonstration also emerges when we compare the discussion of quadratic equations in the arithmetic text *Lilavati* and the algebra text *Bijaganita*. In the *Lilavati,* the rule is simply stated and applied to different types of problems, while in the *Bijaganita,* we find a rationale including a reference to the method of completing the square.

Algebra in earlier historical periods has often been characterized as dealing with "the solution of equations" (Katz 2001). While this view is undoubtedly correct in a broad sense, it is partial and misses out on important aspects of how Indian mathematicians in the past thought about algebra. Most importantly, they laid stress on understanding and insight into quantitative relationships. The symbols of algebra are an aid to such understanding. Algebra is the foundation for arithmetic and not just the generalization of arithmetic, implying that arithmetic itself must be viewed with "algebra eyes." Further, algebra involves taking a different attitude or stance with respect to computation and the solution of problems; it is not mere description of solution, but demonstration and justification. Mathematical insight into quantitative relationships, combined with an attitude of justification or demonstration, leads to the uncovering of powerful ways of solving complex problems and equations.

Making procedures of calculation more efficient and more accurate was often one of the goals of mathematics in the Indian tradition, and the discovery of efficient formulas for complex and difficult computations in astronomy was a praiseworthy achievement that enhanced the reputation of mathematicians. Thus not only do we find a great variety of procedures for simple arithmetic computations, but also for interpolation of data and approximations of series (Datta and Singh 1938/2001). The *karana* texts contain many examples of efficient algorithms (Plofker 2009, pp. 105ff). In the "Kerala school" of mathematics, which flourished in Southern India between the 14th and the 18th centuries CE, we find, amongst many remarkable advancements including elements of calculus, a rich variety of results in finding rational approximations to infinite series. Thus algebra was related as much to strengthening and enriching arithmetic and the simplification of complex computation as to the solution of equations. It was viewed both as a domain where the rationales for computations were grasped and as a furnace where new computational techniques were forged.

From a pedagogical point of view, understanding and explaining why an interesting computational procedure works is a potential entry point into algebra. Since arithmetic is a part of universal education, a perspective that views algebra as deepening the understanding of arithmetic has social validity. Thus, while algebra builds on students' understanding of arithmetic, in turn, it reinterprets and strengthens this understanding. In the remaining sections, we explore what this might mean for a teaching learning approach that emphasizes the arithmetic-algebra connection.

Building on Students' Understanding of Arithmetic

Modern school algebra relies on a more extensive and technical symbolic apparatus than the algebra of the *Bijaganita*. As students learn to manipulate variables, terms, and expressions as if they were objects, it is easy for them to lose sight of the fact that the symbols are about quantities. In the context of arithmetic, students have only learned to use symbols to notate numbers and to encode binary operations, usually carried out one at a time. Algebra not only introduces new symbols such as letters and expressions, but also new ways of dealing with symbols. Without guidance from intuition, students face great difficulty in adjusting to the new symbolism. So Bhaskara's precept that algebra is about insight into quantities and their relationships and not just the use of symbols is perhaps even more relevant to the learning of modern school algebra.

What do students carry over from their experience of arithmetic that can be useful in the learning of algebra? Do students obtain insight into quantitative relationships of the kind that Bhaskara is possibly referring to through their experience of arithmetic, which can be used as a starting point for an entry into symbolic algebra? Of course, one cannot expect such insight to be sophisticated. We should also expect that students may not be able to symbolize their insight about quantitative relationships because of their limited experience of symbols in the context of arithmetic.

Fujii and Stephens (2001) found evidence of what they call students' relational understanding of numbers and operations in the context of arithmetic tasks. In a missing number sentence like $746 + \_\_ - 262 = 747$, students could find the number in the blank without calculation. They were able to anticipate the results of operating with numbers by finding relations among the operands. Similar tasks have also been used by others in the primary grades (Van den Heuvel-Panhuizen 1996). Missing number sentences of this kind are different from those of the kind $13 + 5 = \_\_\_ + 8$, where the algebraic element is limited to the meaning of the "=" sign as a relation that "balances" both sides. Relational understanding as revealed in the responses to the former kind of sentence lies in anticipating the result of operations without actual calculation. Fujii and Stephens argue that in these tasks although students are working with specific numbers, they are attending to general aspects by treating the numbers as "quasi-variables."

Students' relational understanding, as described by Fujii and Stephens, is a form of operational sense (Slavit 1999), limited perhaps to specific combinations of numbers. The students' performance on these tasks needs to be contrasted with the findings of other studies. For example, Chaiklin and Lesgold (1984) found that without recourse to computation, students were unable to judge whether or not $685 - 492 + 947$ and $947 + 492 - 685$ are equivalent. Students are not consistent in the way they parse expressions containing multiple operation signs. It is possible that they are not even aware of the requirement that every numerical expression must have a unique value. It is likely, therefore, that students' relational understanding are elicited in certain contexts, while difficulties with the symbolism overpowers such understanding in other contexts. Can their incipient relational understanding develop into a more powerful and general understanding of quantitative relationships that can form the basis for algebraic understanding, as suggested by Bhaskara? For this to be possible, one needs to build an idea of how symbolization can be guided by such understanding, and can in turn develop it into a more powerful form of understanding. In a later study, Fujii and Stephens (2008) explored students' abilities to generalize and symbolize relational understanding. They used students' awareness of computational shortcuts (to take away six, take away ten and add four) and developed tasks that involved generalizing such procedures and using symbolic expressions to represent them.

Other efforts to build students' understanding of symbolism on the basis of their knowledge of arithmetic have taken what one may describe as an inductive approach, with the actual process of calculation supported by using a calculator (Liebenberg et al. 1999; Malara and Iaderosa 1999). In these studies, students worked with numerical expressions with the aim of developing an understanding of the structure by applying operation precedence rules and using the calculator to check their computation. These efforts were not successful in leading to an understanding of structure that could then be used to deal with algebraic expressions because of over-reliance on computation (Liebenberg et al.) or because of interpreting numerical and algebraic expressions in different ways (Malara and Ioderosa). The findings suggest that an approach where structure is focused more centrally and is used to support a range of tasks including evaluation of expressions, as well as comparison and transformation of expressions, may be more effective in building a more robust understanding of symbolic expressions.

We attempted to develop such an approach in a study conducted as a design experiment with grade 6 students during the two-year period 2003–2005. The teaching-learning approach evolved over five trials, with modifications made at the end of each trial based on students' understanding as revealed through a variety of tasks and our own understanding of the phenomena. The first year of the study consisted of two pilot trials. In the second year, we followed 31 students over three teaching trials. These students were from low and medium socio-economic backgrounds, one group studying in the vernacular language and one in the English language. Each trial consisted of $1\frac{1}{2}$ hours of instruction each day for 11–15 days. These three trials, which comprised the main study were held at the beginning (MST-I), middle (MST-II), and end of the year (MST-III) during vacation periods. The schools in which the students were studying followed the syllabus and textbooks prescribed by the State government, which prescribe the teaching of evaluation and simplification of arithmetic and algebraic expressions in school in grade 6 in a traditional fashion—using precedence rules for arithmetic expressions and the distributive property for algebraic expressions. Discussion with students and a review of their notebooks showed that only the vernacular language school actually taught simplification of algebraic expressions in class 6; the English school omitted the chapter.

These students joined the program at the end of their grade 5 examinations and were followed till they completed grade 6. They were randomly selected for the first main study trial from a list of volunteers who had responded to our invitation to participate in the program. The students were taught in two groups, in the vernacular and the English language respectively by members of the research team.

Data was collected through pre- and post-tests in each trial, interviews conducted eight weeks after MST-II (14 students) and 16 weeks after MST-III (17 students), video recording of the classes and interviews, teachers' log and coding of daily worksheets. The pre- and post-tests contained tasks requiring students to evaluate numerical expressions and simplify algebraic expressions, to compare expressions without recourse to calculation and to judge which transformed expressions were equivalent to a given expression. There were also tasks where they could use algebra to represent and draw inferences about a given problem context, such as a pattern or a puzzle. In the written tests, the students were requested to show their work for the tasks. The students chosen for the interview after MST-II had scores in the tests which were below the group average, at the average, and above the group average, and who had contributed actively to the classroom discussions. The same students also participated in the interviews after MST-III, and a few additional students were also interviewed. The interviews probed their understanding more deeply, using tasks similar to the post test. In particular, they probed whether student responses were mechanical and procedure-based or were based on understanding.

The overall goal of the design experiment was to evolve an approach to learning beginning algebra that used students' arithmetic intuition as a starting point. The specific goal was to develop an understanding of symbolic expressions together with the understanding of quantitative relationships embodied in the expressions. Although this was done with both numerical and algebraic expressions, the approach entailed more elaborate work with numerical expressions by students compared to the approach in their textbooks. Students worked on tasks of evaluating expressions, of comparing expressions without calculation, and of transforming expressions in addition to a number of context-based problems where they had to generalize or explain a pattern. A framework was developed that allowed students to use a common set of concepts and procedures for both numerical and algebraic expressions. Details of the study are available in Banerjee (2008a). Here we shall briefly indicate how the teaching approach evolved, describe the framework informing the approach, and present some instances of students' responses to the tasks.

In the pilot study, students worked on tasks adapted from Van den Heuvel-Panhuizen (1996), and that were similar to the tasks used by Fujii and Stephens (2001). We found several instances of relational thinking similar to those reported by Fujii and Stephens. For example, students could judge whether expressions like $27 + 32$ and $29 + 30$ were equivalent and also give verbal explanations. One of the explanations used a compensation strategy: "the two expressions are equal because we have [in the first expression] taken 2 from 32 and given it to 27 [to obtain the second expression]." Students worked with a variety of such expressions, containing both addition and subtraction operations, with one number remaining the same or both numbers changed in a compensating or non-compensating manner (Subramaniam 2004). Some pairs were equivalent, and some pairs were not. For the pairs which were not equivalent, they had to judge which was greater and by how much. As seen in the explanation just cited, students used interesting strategies including some ad-hoc symbolism, but this did not always work. In general, when they attempted to compare the expressions by merely looking at their structure and not by computing, students made accurate judgements for expressions containing the addition operation but not for those containing the subtraction operation. Similarly they were not always successful in judging which expression was greater in a pair of expressions when the compensation strategy showed that they were unequal.

We noticed that students were separating out and comparing the additive units in the pairs of expressions but were comparing numbers and operation signs in inconsistent ways. This led to an approach that called attention more clearly to additive units in comparison tasks. However, an important moment in the evolution of the approach was the decision to use a structure-based approach for comparing as well as for evaluating numerical expressions. Other important aspects of the approach were dealing with arithmetic and algebraic expressions in a similar manner in the different tasks and relating these processes to algebraic contexts of generalizing and justification of patterns. We have described the evolution of the approach in greater detail elsewhere (Banerjee and Subramaniam submitted). Here we describe a framework that supports a structure based approach to working with numerical expressions on a range of tasks including evaluation, comparison and transformation.

The Arithmetic-Algebra Connection—A Framework

As we remarked earlier, learning algebra involves learning to read and use symbols in new ways. These new ways of interpreting symbols need to build on and amplify students' intuition about quantitative relationships. The view that algebra is the foundation of arithmetic, held by Indian mathematicians, entails that students need to interpret the familiar symbols of arithmetic also in new ways. The literature on the transition from arithmetic to algebra has identified some of the differences in the way symbols are used in arithmetic and algebra: the use of letter symbols, the changed interpretation of key symbols such as the "=" sign, and the acceptance of unclosed expressions as appropriate representations not only for operations but also for the result of operations (Kieran 2006). An aspect related to the last of the changes mentioned that we wish to emphasize is the interpretation of numerical and algebraic expressions as encoding the operational composition of a number.

The use of expressions to stand for quantities is related to the fact that, while in arithmetic one represents and thinks about one binary operation, in algebra we need to represent and think about more than one binary operation taken together. As students learn computation with numbers in arithmetic, they typically carry out a single binary operation at a time. Even if a problem requires multiple operations, these are carried out singly in a sequence. Consequently, the symbolic representations that students typically use in arithmetic problem-solving contexts are expressions encoding a single binary operation. In the case of formulas, the representation may involve more than one binary operation, but they are still interpreted as recipes for carrying out single binary operations one at a time. They do not involve attending to the structure of expressions or manipulating the expressions. Indeed, one of the key differences of the arithmetic approach to solving problems, as opposed to the algebraic, is that students compute intermediate quantities in closed numerical form rather than leaving them as symbols that can be operated upon. And these intermediate quantities need to be thought about explicitly and must be meaningful in themselves (Stacey and Macgregor 2000).

The representational capabilities of students need to be expanded beyond the ability to represent single binary operations before they move on to algebra. In the traditional curriculum, this is sought to be achieved by including a topic on arithmetic or numerical expressions, where students learn to evaluate expressions encoding multiple binary operations. However, students' work on this topic in the traditional curriculum is largely procedural, and students fail to develop a sense of the structure of expressions. As discussed earlier, students show relational understanding in certain contexts, but in general have difficulty in interpreting symbolic expressions.

One problem that arises when numerical expressions encode multiple binary operations is that such expressions are ambiguous with respect to operation precedence when brackets are not used. At the same time, one cannot fully disambiguate the expression using brackets since the excessive use of brackets distracts from the structure of the expression and is hence counter-productive. Students are, therefore, taught to disambiguate the expression by using the operation precedence rules. The rationale for this, namely, that numerical expressions have a unique value is often left implicit and not fully grasped by many students. Even if the requirement is made explicit, students are unlikely to appreciate why such a requirement is necessary. The transformation rules of algebra are possible only when algebraic expressions yield numerical expressions with a unique value when variables are appropriately substituted. Thus disambiguating numerical expressions is a pre-condition for the use of rules of transformations that preserve the unique value of the expression. Since students are yet to work with transformations of expressions, they cannot appreciate the requirement that numerical expressions must be unambiguous with regard to value.

In the traditional curriculum, students' work with numerical expressions is limited and is seen merely as preparatory to work with algebraic expressions. How does one motivate a context for work with numerical expressions encoding multiple binary operations? Student tasks with such expressions need to include three interrelated aspects—representational, procedural (evaluation of expressions), and transformational. To fully elaborate these aspects, we need to interpret expressions in a way different from the usual interpretation of an expression as encoding a sequence of such operations to be carried out one after another, a sequence determined by the visual layout in combination with the precedence rules. The alternative interpretation that students need to internalize is that such expressions express or represent the *operational composition* of a quantity or number. In other words, the expression reveals how the number or quantity that is represented is built up from other numbers and quantities using the familiar operations on numbers. This interpretation embodies a more explicit reification of operations and has a greater potential to make connections between symbols and their semantic referents. The idea of the operational composition of a number, we suggest, is one of the key ideas marking the transition from arithmetic to algebra.

Let us illustrate this idea with a few examples: (i) the expression $500 - 500 \times 20/100$ may indicate that the net price is equal to the marked price less the discount, which in turn is a fraction of the marked price, (ii) the expression $5 \times 100 + 3 \times 10 + 6$ shows the operational composition expressed by the canonical representation of a number (536) as composed of multiunits which are different powers of ten, (iii) the expression $300 + 0.6t$ may indicate cell phone charges as including a fixed rent and airtime charges at a fixed rate per unit of airtime. In examples (i) and (iii), the operational composition refers back to quantities identifiable in particular situations, while in example (ii) abstract quantities are put together or "operationally composed" to yield the number 536. It is important to preserve both these senses in unpacking the notion of operational composition.

By operational composition of a quantity, we mean information contained in the expression such as the following: what are the additive part quantities that a quantity is composed of? Are any of these parts scaled up or down? By how much? Are they obtained as a product or quotient of other quantities? The symbolic expression that denotes the quantity simultaneously reveals its operational composition, and in particular, the additive part quantities are indicated by the *terms* of the expression.

A refined understanding of operational composition includes accurate judgments about relational and transformational aspects. What is the relative contribution of each part quantity (each term) as indicated by the expression? Do they increase or decrease the target quantity? Which contributions are large, which small? How will these contributions change if the numbers involved change? How does the target quantity change when the additive terms are inverted, that is, replaced by the additive inverse of the given term? What changes invert the quantity as a whole? What are the transformations that keep the target quantity unchanged? If additive parts are themselves composed from other quantities, how do we represent and understand this?

The idea of the operational composition encoded by an expression is similar to the idea of a function but is more general and less precise. Looking at an expression as a function has a more narrow focus: how does the target quantity vary when one or more specific part quantities are varied in a systematic manner while retaining the form of the operational composition? When expressions are compared and judged to be equivalent, we judge that different operational compositions yield the same value. However, the idea of operational composition may play a role in developing the understanding of functions.

When we interpret expressions as encoding operational composition, we are not restricted to algebraic expressions. In fact, numerical expressions emerge as an important domain for reasoning about quantity, about relations and transformations, and for developing a structure based understanding of symbolic representation through the notion of operational composition. The pedagogical work possible in the domain of numerical expressions as a preparation for algebra expands beyond what is conceived in the traditional curriculum. Numerical expressions emerge as a domain for reasoning and for developing an understanding of the structure of symbolic representation.

When students' tasks focus on numerical expressions as encoding operational composition, attention is drawn to the relations encoded by the expression. Students are freed from the need to unpack the expression as a sequence of operations, fixed by a set of operation precedence rules. In the teaching approach that we developed, we emphasized ways of working with expressions that attend to the structure of expressions and are broadly aimed at developing an insight into quantitative relationships that must accompany working with symbols.

A simple numerical expression like $5 + 3$ is usually interpreted as encoding an instruction to carry out the addition operation on the numbers 5 and 3. In changing the focus to operational composition, the first transition that students make is to see the expression as "expressing" some information about the number 8. This information can be expressed verbally in various ways: 8 is the sum of 5 and 3, 8 is 3 more than 5, etc. Other expressions such as $6+2$ or 2×4 contain other information about the number 8, i.e., they encode different operational compositions of the number 8. Starting from this point, students move on to expressions with two or more operations of addition and subtraction. Each expression gives information about the number which is the "value" of the expression, and reveals a particular operational composition of the number.

What grounding concepts can scaffold students' attempts to study and understand the operational composition revealed in an expression? The basic level of information is contained in the *terms* or the additive units of the expression. Simple terms are just numbers together with the preceding "+" or "−" sign. Positive terms increase the value of the number denoted by the expression and negative terms decrease the value. Additive units are dimensionally "homogenous," and can be combined in any order.

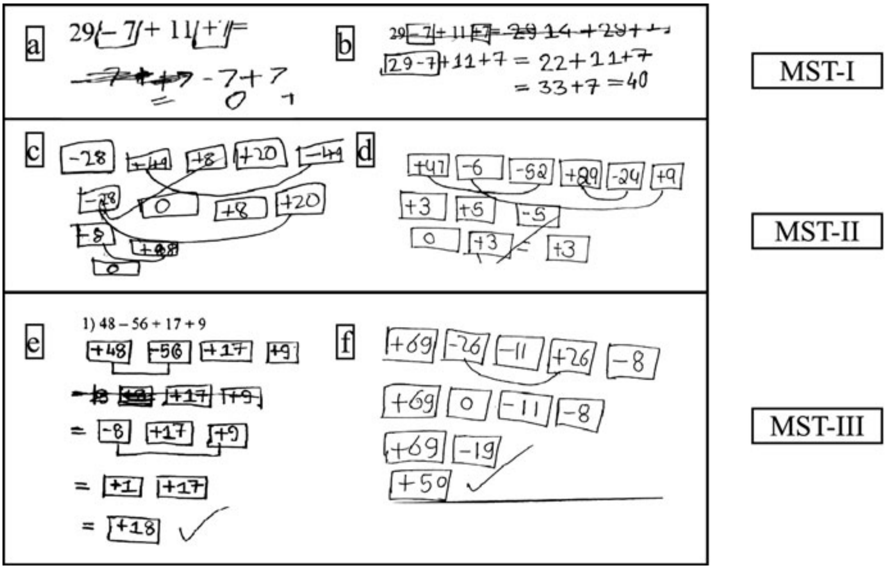

Fig. 1 Evaluation of expressions containing only simple terms by students using flexible ways in the three trials of the study (MST I, II and III)

This shift in perspective subtly turns attention away from procedure towards structure. In order to evaluate an expression, students do not need to work out and implement a sequence of binary operations in the correct order. Rather, to determine the value of the expression, they may combine simple terms in any order, keeping in view the compensating contributions of positive and negative terms. The concept of negative terms provides an entry point into signed numbers as encoding increase or decrease, which is one of the three interpetations of integers proposed by Vergnaud cited in Fuson (1992, p. 247). The approach of combining simple terms in any order, affords flexibility in evaluating an expression or in comparing expressions that is critical to uncovering structure. Thus students may cancel out terms that are additive inverses of one another; they may gather together some or all of the positive terms or the negative terms and find easy ways to compute the value of the expression by combining terms. Figure 1 shows students combining terms in flexible ways while evaluating expressions rather than proceeding according to operation precedence rules. Since the identification of additive units namely, terms, is the starting point of this approach, we have described this approach elsewhere as the "terms approach" (Subramaniam 2004; Banerjee and Subramaniam 2008).

Identifying the additive units correctly is one of the major hurdles that some students face. This is indicated by the frequency of such errors as "detachment of the minus sign" ($50 - 10 + 10 = 30$), and "jumping off with the posterior operation" ($115 - n + 9 = 106 - n$ or $106 + n$) (Linchevski and Livneh 1999). Although these errors are often not taken to be serious, they are widespread among students and impede progress in algebra. Not having a secure idea about the units in an expression and not knowing how they combine to produce the value may enhance the experience of algebra as consisting of arbitrary rules.

In working with transformations of expressions, some studies indicate that visual patterns are often more salient to students than the rules that the students may know for transforming expressions (Kirshner and Awtry 2004), suggesting that visual routines are easier to learn and implement than verbal rules. One advantage with the "terms approach" is the emphasis on visual routines rather than on verbal rules in parsing and evaluating an expression. Terms were identified in our teaching approach by enclosing them in boxes. In fact, the rule that multiplication precedes addition can be recast to be consistent with visual routines. This is done by moving beyond *simple terms*, which are pure numbers with the attached $+$ or $-$ sign, to *product terms*. In expressions containing "$+$," "$-$," and the "$\times$" operation signs, students learn to distinguish the product terms from the simple terms: the product terms contain the "$\times$" sign. Thus in the expression $5 + 3 \times 2$ the terms are $+5$ and $+3 \times 2$. In analyzing the operational composition encoded by the expression, or in combining terms to find the value of the expression, students first identify the simple and the product terms by enclosing them in boxes. The convention followed is that product terms must be converted to simple terms before they can be combined with other simple terms. Thus the conventional rule that in the absence of brackets multiplication precedes addition or subtraction is recast in terms of the visual layout and operational composition. Product terms are the first of the complex terms that students learn. Complex terms include product terms, bracket terms (e.g., $+(8-2\times 3)$) and variable terms (e.g., $-3 \times x$).

The approach included both procedurally oriented tasks such as evaluation of expressions and more structurally oriented tasks, such as identifying equivalent expressions and comparing expressions. As remarked earlier, one of the main features of the approach evolved only after the initial trials—the use of the idea of terms in the context of both procedurally and structurally oriented tasks. In the earlier trials, the use of the idea was restricted to structurally oriented tasks involving comparison of expressions, and the operation precedence rules were used for the more procedurally oriented tasks of evaluating expressions. By using the "terms idea" in both kinds of tasks, students began to attend to operational composition for both evaluating and comparing expressions, which allowed them to develop a more robust understanding of the structure of expressions. By supporting the use of structure for the range of tasks, this approach actually blurred the distinction between structural and procedural tasks. Students' written as well as interview responses revealed that they were relatively consistent in parsing an expression and that they appreciated the fact that evaluation of a numerical expression leads to a unique value (Banerjee 2008a; Banerjee and Subramaniam submitted).

In the students' written responses, we found a reduction as they moved from the first trial (MST-I) to the last (MST-III) in the common syntactic errors in evaluating numerical expressions or in simplifying algebraic expressions such as the conjoining error ($5 + x = 5x$), the detachment error described above, and the LR error (evaluating an expression from left to right and ignoring multiplication precedence). More importantly, students who were interviewed showed a reliance on identifying simple and complex terms to assess whether a particular way of combining terms was correct. Their understanding of procedural aspects was robust in the sense that they were able to identify and correct errors in a confident manner, when probed with alternative ways of computing expressions.

The interviews also revealed how some students were able to use their understanding of terms to judge whether two expressions were equal. One of the questions required students to identify which expression was numerically greater, when two expressions were judged to be unequal. Although this was not a question familiar to the students from classroom work, they were able to interpret the units or terms in the expression to make correct judgments. The following interview excerpt post-MST III from one of the better performing students illustrates how the idea of operational composition could be put to use in making comparisons:

> Interviewer: Ok. If I put $m = 2$ in this first expression [$13 \times m - 7 - 8 \times 4 + m$] and I put $m = 2$ in the original expression [$13 \times m - 7 - 8 \times m + 4$], would I get the same value?
>
> *BK: No.*
>
> Interviewer: It will not be. Why?
>
> *BK: Because it is* 8×4 *[in the first expression], if it [the value of m] is 4 here, then it would be the same value for both.*
>
> [The student is comparing the terms which are close but not equal: -8×4 and $-8 \times m$. She says that if m were equal to 4, the expressions will be equal, but not otherwise.]
>
> Interviewer: ... If I put $m = 2$ in (this) expression [$-7 + 4 + 13 \times m - m \times 8$] and $m = 2$ in the original expression [$13 \times m - 7 - 8 \times m + 4$], then would they be the same?
>
> *BK: Yes.*
>
> Interviewer: Why?
>
> *BK: Because, m is any number, if we put any number for that then they would be the same.*
>
> [Comparing the two expressions the student judges correctly that they are equal.]

Our study focused largely on expressions that encoded additive composition and, to a limited extent, combined it with multiplicative composition. Learning to parse the additive units in an expression is an initial tool in understanding the operational composition encoded by the expression. Multiplicative composition as encoded in a numerical expression is conceptually and notationally more difficult and requires that students understand the fraction notation for division and its use in representing multiplication and division together. In our study, multiplicative composition was not explored beyond the representation of the multiplication of two integers since students' understanding of the fraction notation was thought to be inadequate.

Even with this restriction, the study revealed much about students' ability to grasp operational composition and showed how this can lead to meaningful work with expressions as we have tried to indicate in our brief descriptions above. It is generally recognized that working with expressions containing brackets is harder for students. While this was not again explored in great detail in the study, we could find instances where students could use and interpret brackets in a meaningful way. In an open-ended classroom task where students had to find as many expressions as they could that were equivalent to a given expression, a common strategy was to replace one of the terms in the given expression, by an expression that revealed it as a sum or a difference. For example, for the expression, $8 \times x + 12 + 6 \times x$, students wrote the equivalent expression $(10 - 2) \times x + 12 + (7 - 1) \times x$, using brackets to show which numbers were substituted. This was a notation followed commonly by students for several such examples. Besides the use of brackets, this illustrates students using the idea that equals can be substituted one for the other, and that "unclosed" expressions could be substituted for "closed" ones. In the same task, students also used brackets to indicate use of the distributive property as for example, when they wrote for the given expression $11 \times 4 - 21 + 7 \times 4$ the equivalent expression $4 \times (7 + 11) - 21$.

The study also included work with variable terms and explored how students were able to carry over their understanding of numerical expressions to algebraic expressions. We found that students were capable of making judgments about equivalent expressions or of simplifying expressions containing letter symbols just as they were in working with numerical expressions. This did not, however, necessarily mean that they appreciated the use of algebraic symbols in contexts of generalization and justification (Banerjee 2008a). The culture of generalization that algebra signals probably develops over a long period as students use algebraic methods for increasingly complex problems.

We have attempted here to develop a framework to understand the arithmetic-algebra connection from a pedagogical point of view and to sketch briefly how a teaching approach informed by this framework might begin work with symbolic algebra by using students' arithmetic intuition as a starting point. Although the design experiment through which the teaching approach was developed was not directly inspired by the historical tradition of Indian mathematics, we have found there a source for clarifying the ideas and the framework that underlie the teaching approach. The view that understanding quantitative relationships is more important than just using symbols and the idea that algebra provides the foundation for arithmetic are powerful ideas whose implications we have tried to spell out. We have argued that symbolic expressions, in the first instance, numerical expressions, need to be seen as encoding operational composition of a number or quantity rather than as a set of instructions to carry out operations. We have also pointed to the importance, from a perspective that emphasizes structure, of working with numerical expressions as a preparation for beginning symbolic algebra.

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Shiki: A Critical Foundation for School Algebra in Japanese Elementary School Mathematics

Tad Watanabe

Abstract An analysis of the Japanese elementary school (Grades 1 through 6) mathematics curriculum materials reveals that the study of functional relationships (patterns) is a major emphasis in Japan, as is the case in curricula from other countries. However, the Japanese curriculum considers the ideas related to mathematical expressions, called "shiki" in Japanese, as the second pillar of elementary school algebra. This chapter elaborates how a Japanese textbook series attempts to realize this emphasis on writing and interpreting mathematical expressions.

The Final Report of the National Mathematics Advisory Panel (2008) identifies the major topics of school algebra. The Panel argues that mathematical experiences students encounter in elementary and middle schools must therefore address what the Panel labeled the Critical Foundations for school algebra (see Table 1). The items on this list come from three major components: fluency with whole numbers; fluency with fractions; and particular aspects of geometry and measurement. Although these topics identified by the Panel are certainly foundations for school algebra, there may be other elementary and middle school mathematics topics that are equally important for, and perhaps more directly related to, school algebra. The purpose of this chapter is to present the findings from an analysis of the treatment of algebra in the Japanese elementary school (Grades 1 through 6) mathematics curriculum materials and to examine what might be foundational for school algebra. The purpose of this study is not to make an evaluative judgment about the Japanese, nor the US, treatment of algebra in elementary schools but rather to help us more clearly understand our own practices.

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Table 1 Critical foundations for school algebra by the National Mathematics Advisory Panel (2008)

Fluency With Whole Numbers

1. By the end of Grade 3, students should be proficient with the addition and subtraction of whole numbers.
2. By the end of Grade 5, students should be proficient with multiplication and division of whole numbers.

Fluency With Fractions

1. By the end of Grade 4, students should be able to identify and represent fractions and decimals, and compare them on a number line or with other common representations of fractions and decimals.
2. By the end of Grade 5, students should be proficient with comparing fractions and decimals and common percents, and with the addition and subtraction of fractions and decimals.
3. By the end of Grade 6, students should be proficient with multiplication and division of fractions and decimals.
4. By the end of Grade 6, students should be proficient with all operations involving positive and negative integers.
5. By the end of Grade 7, students should be proficient with all operations involving positive and negative fractions.
6. By the end of Grade 7, students should be able to solve problems involving percent, ratio, and rate and extend this work to proportionality.

Geometry and Measurement

1. By the end of Grade 5, students should be able to solve problems involving perimeter and area of triangles and all quadrilaterals having at least one pair of parallel sides (i.e., trapezoids).
2. By the end of Grade 6, students should be able to analyze the properties of two-dimensional shapes and solve problems involving perimeter and area, and analyze the properties of three-dimensional shapes and solve problems involving surface area and volume.
3. By the end of Grade 7, students should be familiar with the relationship between similar triangles and the concept of the slope of a line.

School Algebra and Algebra in Early Grades

Kilpatrick and Izsak (2008) discusses how the role of “algebra” in the US school curricula has changed from its complete absence to a requirement for college bound students and then to a high school graduation requirement. In general, the movement in the United States has been to push algebra for more students, i.e., “algebra for all,” and earlier, i.e., “algebra in Grade 8.” *Principles and Standards for School Mathematics* by the National Council of Teachers of Mathematics (2000) includes “algebra” as one of the five common strands for pre-K through Grade 12 school mathematics.

As the target audience and the timing of school algebra was changing, the nature of school algebra itself was evolving as well. A traditional image of school algebra often centers on solving various types of equations and inequalities through symbol manipulation. Although solving equations and inequalities are still an important component of school algebra, in the last few decades, school algebra has become more broadly conceptualized. Although there are some differences, researchers and curriculum developers (e.g., Blanton and Kaput 2005; Kilpatrick and Izsak 2008; Usiskin 1988) seem to agree that school algebra includes the following aspects:

- Algebra as generalized arithmetic
- Algebra as a study of functions, patterns, and relationships
- Algebra as a tool for problem solving
- Algebra as a study of structures

Along with this broader conceptualization of school algebra, a consensus seems to have developed that students' experiences in elementary schools is a key factor in their success with school algebra. However, as Kieran (2004) noted, there does not appear to be a general consensus on what algebraic thinking in the early grades should look like. In fact, analyses of the elementary school curricula from China (Cai 2004), Korea (Lew 2004), Singapore (Ng 2004), the United States (Moyer et al. 2004), and Russia (Schmittau and Morris 2004) reveal that the approaches to algebra in the early grades vary significantly. However, these analyses serve as useful reference points to design, examine, and refine our own practices. The present study will expand the existing knowledge base by adding a case study of the treatment of algebra in the Japanese elementary school curriculum. In this chapter, I will report briefly the findings from the case study, and then provide a more in-depth analysis and discussion of one specific aspect in a textbook series.

Methodology

The present study tried to address the following research questions:

1. What are the big ideas of algebra in the Japanese elementary school mathematics curriculum?
2. What ideas related to algebra are included in the Japanese elementary school mathematics curriculum and when are they discussed?
3. How are those ideas developed in textbooks?

Materials In the present study, two documents published by the Japanese Ministry of Education, Culture, Sports, Science and Technology (hereafter the Ministry of Education) were examined. The first document is the national Course of Study (COS) for elementary school mathematics. This document specifies what topics are to be taught at what grade levels. The second document, often called *Teaching Guide*, elaborates and explains the COS in more details. In addition, two most widely used textbook series, including their teachers' manuals, were examined. Both series followed the 1989 COS. The content of the 1989 COS is very similar to the 2008 COS, which will be implemented beginning in the 2011–12 school year.

Analysis Framework The two documents published by the Ministry of Education were first analyzed to identify those topics that are related to algebra. This analysis initially used the four aspects of school algebra discussed earlier as a framework to specify the topics. These documents, but particularly *Teaching Guide*, were further analyzed to reveal any other topics related to the initially specified topics. Through these analyses, the big ideas of algebra in the Japanese elementary school mathematics curriculum were identified. Finally, the ways the identified topics were treated in the two textbook series were analyzed. The textbook analysis focused on how textbook authors intended to realize what was described in the Ministry documents. The analysis of the teachers' manuals played a significant role.

The materials analyzed for the study were all in Japanese. However, *Teaching Guide* for the 1989 COS was translated into English. In addition, a revised version of one of the series was also translated into English. The content of this version still followes the 1989 COS and is very similar, and often identical, to the version that was examined in the study. Therefore, when quoting from those documents, corresponding segments from the translated documents are used in this chapter.

Algebra in Japanese Curriculum

In the elementary school mathematics COS, the content is divided into four domains: Numbers and Calculations, Geometric Figures, Quantities and Measurements, and Quantitative Relations. In the Lower Secondary School (Grade 7 through 9) COS, there are three content domains: Number and Mathematical Expressions, Geometric Figures, and Quantitative Relations. Some may be surprised that there is no domain called algebra in these COS, even at the middle school level. However, further reading of the COS and *Teaching Guide* suggests that those topics that are often considered as "algebra" are found mostly in the Quantitative Relations domain. *Teaching Guide* for the 1989 COS states,

> The contents of this domain include... items which are useful in examining or manipulating contents in other domains... The objectives and contents of this domain cover a wide range, but can be divided into three categories: *idea of functions*, *writing and interpreting mathematical expressions*, and statistical manipulation. (Takahashi et al. 2004, p. 36, emphasis added)

Table 2 shows the grade level goal statements related to the Quantitative Relations domain from the 1989 COS and the focal points related to algebra from the National Council of Teachers of Mathematics' (NCTM) *Curriculum Focal Points* (2006). From this table, we can easily see that these focal points align well with the Math Panel's Critical Foundations.

However, both the Japanese curriculum and the NCTM emphasize other aspects of Elementary school mathematics that are related to algebra. Since the Quantitative Relations strand does not start until Grade 3, there was no explicit goal related to the domain in Grades 1 and 2. However, the Japanese COS certainly emphasizes the mastery of addition and subtraction operations in those grades. What the Japanese COS calls "idea of functions" is comparable to what *Focal Points* calls the examination of patterns in Grades pre-K through 5. However, in Japanese textbooks, there is almost no formal discussion of patterns such as repeating patterns

Table 2 Grade level goal statements in the quantitative relations domain of the Japanese Course of Study and algebra focal points in NCTM's *Focal Points*

	1989 Japanese COS	NCTM. *Focal Points*
Grades 1 & 2		Grade 1 ***Number and Operations*** **and** ***Algebra:*** Developing understandings of addition and subtraction and strategies for basic addition facts and related subtraction facts. Grade 2 ***Number and Operations and Algebra*****:** Developing quick recall of addition facts and related subtraction facts and fluency with multidigit addition and subtraction.
Grade 3	To help children become able to arrange data, and to use mathematical expressions and graphs, and to help children appreciate their meaning and become gradually able o represent or to investigate sizes and quantities and their mathematical relations.	***Number and Operations and Algebra*****:** Developing understandings of multiplication and division and strategies for basic multiplication facts and related division facts.
Grade 4	To help children become able to represent or consider quantities and their mathematical relations by using mathematical expressions or graphs, and further, to help them become able to investigate dependence relations between them.	***Number and Operations and Algebra*****:** Developing quick recall of multiplication facts and related division facts and fluency with whole number multiplication.
Grade 5	To help children become able to concisely represent mathematical expressions by using letters, and to investigate mathematical relations represented by them.	***Number and Operations and Algebra*****:** Developing an understanding of and fluency with division of whole numbers.
Grade 6	To help children deepen their idea of function through their understanding of proportion and become able to efficiently use it in considering quantitative relations.	***Algebra*****:** Writing, interpreting, and using mathematical expressions and equations.

and growing patterns. Rather, the emphasis in the Japanese textbooks is on two co-varying quantities. For example, in Grade 1, teachers are encouraged to organize the results of decomposing 10 into two numbers so that children might notice how the two numbers change, that is, as one number increases by 1, the other decreases by 1 (Watanabe 2008).

However, the Japanese curriculum also includes the study of mathematical expressions as an important component of the domain, which is not emphasized in *Focal Points*. The Japanese word translated to "mathematical expressions" is *shiki*, 式. This word cannot be translated into one English term as *shiki* includes expressions such as $3 + 5$, $x - 4$, and $\square \div 3$, as well as equations, $3 + 5 = 8$, $x - 4 = 7$, and $\square \div 3 = 7$. Moreover, even inequalities such as $x + 5 > 2$ are also considered as *shiki*. In the Japanese mathematics curriculum, writing and interpreting *shiki* is considered an important component of algebra. In fact, one of the content domains of Lower Secondary School Mathematics is titled "numbers and *shiki*." A former Ministry of Education official even translated it as "Numbers and Algebra" (Yoshikawa 2008, p. 14).

Mathematical Expressions in Japanese Curriculum

The emphasis on mathematical expressions in the Japanese curriculum is clearly evident in the following paragraph found in *Teaching Guide* for the 1989 COS:

> Tables, diagrams, graphs, and mathematical expressions are used to represent numbers and quantities and their relations in our daily life. Especially, mathematical expressions can be said to be a good way to express relations among numbers and quantities accurately, simply, and generally. By the way, mathematical expression is something in which certain symbols are arranged according to specific rules. If mathematical expression is taught to be something that only expresses how to do calculations, or as a mere convention, the understanding of the meaning that mathematical expression represents or how the expression works can be insufficient. So, when teaching mathematical expressions, it is important to interpret expressions, to manipulate expressions, and to be able to explain the process of transformation of expressions, as well as to become able to express phenomena and relations in concrete situations. Especially, it is important to focus on understanding the meaning that the mathematical expressions represent. (Takahashi et al. 2004, p. 38)

It is clear that mathematical expressions are not simply indicating what arithmetic operation is to be executed. Rather, mathematical expressions are used to represent phenomena and relationships, to promote and facilitate mathematical thinking, and to communicate reasoning processes.

In the following section, we will examine how Japanese textbooks deal with the ideas related to mathematical expressions, giving us more concrete examples. The textbook examples used in this section are from Hironaka and Sugiyama (2000), the English translation of a revised version of one of the series analyzed in the study.

Mathematical Expressions in Japanese Textbooks

In Japanese textbooks, writing mathematical expressions is emphasized as soon as children are introduced to the addition operation. Figure 1 shows the first 2 pages of the Grade 1 unit that introduces the addition operation.

Notice that they introduce the "math sentence"[1] as a way to represent what is happening in the problem situation. Furthermore, in Problems 2 and 3, students are gradually guided to write math sentences independently. Similar emphasis is observed each time a new operation is introduced and when the range of numbers with a specific arithmetic operation is expanded. Figure 2 shows two examples of such occasions from the textbook series.

[1]The original Japanese word here is *shiki*, but the translators decided to use the phrase "math sentence" instead of "mathematical expression."

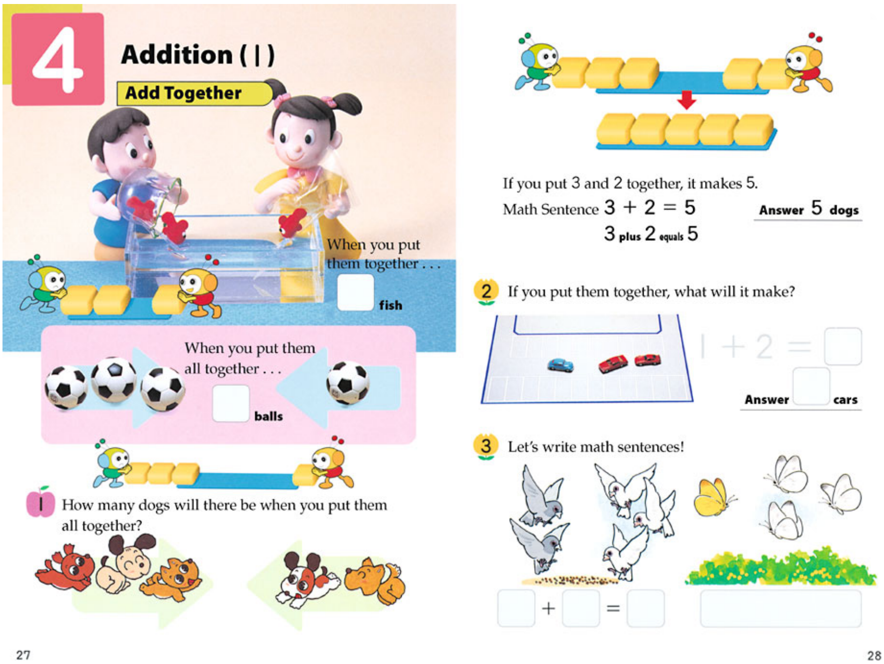


Fig. 1 The first 2 pages of the Grade 1 unit on addition. Students are gradually guided to write complete math sentences independently (Book 1, pp. 28 & 29)

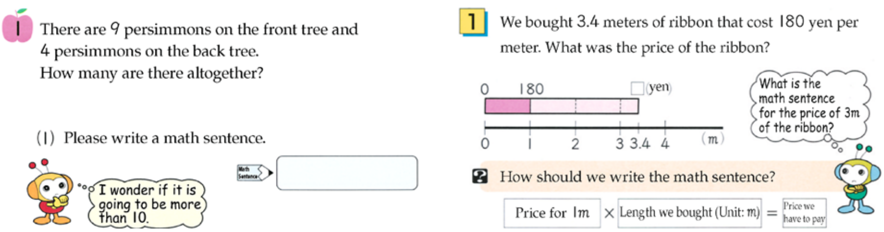
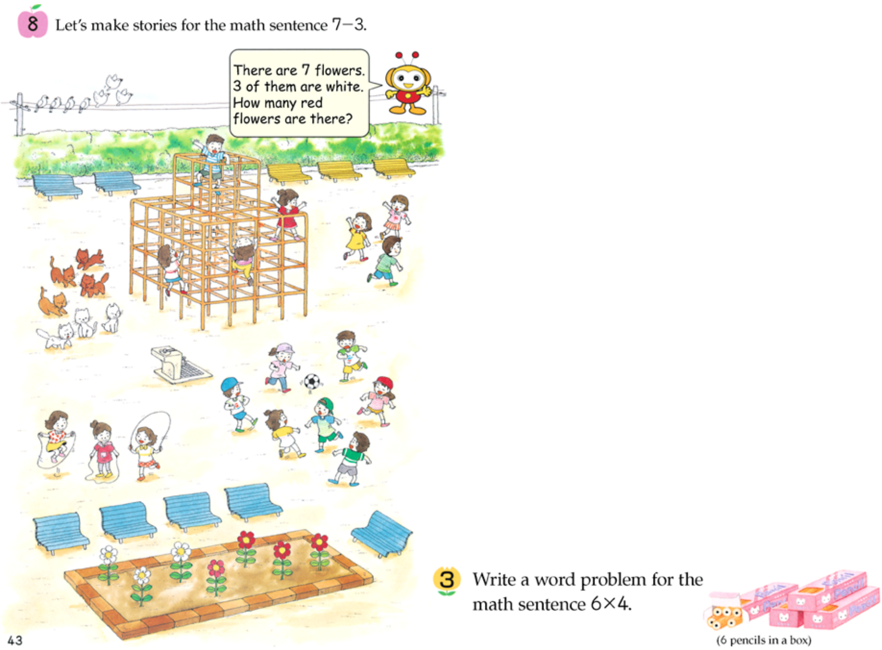


Fig. 2 Students are asked to write a math sentence first when the range of number is expanded—(**a**) Grade 1 sums greater than 10 (Book 1, p. 62), and (**b**) Grade 5 multiplication by decimal numbers (Book 5A, p. 26)

The first three examples showed how this textbook series emphasizes the writing of mathematical expressions starting from Grade 1. The emphasis on interpreting mathematical expressions also begins in Grade 1. When a new arithmetic operation is introduced, the textbook often asks students to write stories/problems that go with a specific mathematical expression. Figure 3(a) is found at the end of the first unit on subtraction in Grade 1. Students have studied two meanings of subtraction, take-away and comparison, in this unit. The illustration includes several instances of both situations. Figure 3(b) comes from the multiplication unit in Grade 2. It is worth noting here that this problem is given to students in the first sub-unit where



(a) (b)

Fig. 3 Students are often asked to write stories/problems for the given math sentences (**a**, Book 1, p. 43; **b**, Book 2B, p. 17)

the meaning and representation of multiplication are introduced. In other words, students have yet to study the actual multiplication facts, 6×4. Therefore, the emphasis is to interpret the mathematical expression.

The idea of using mathematical sentences to represent students' own thinking process is often developed through a problem like the one shown in Fig. 4.

This problem is found at the end of the two multiplication units in Grade 2. The problem not only asks students to find different ways to find the total number of chocolates but also to describe their thinking processes. Although the pupil's page does not explicitly state it, the intent is that students will use mathematical expressions to represent their (or the ones presented in the book) thinking processes. Thus, Mami's thinking process may be represented by the following set of math sentences:

$6 \times 3 = 18$
$3 \times 2 = 6$
$18 + 6 = 24$

In Grade 2, students write a set of math sentences, but in Grade 4, students learn to write compound math sentences by using parentheses. Figure 5 shows the illustration found on the opening pages of the unit in which students learn to write, interpret and calculate mathematical expressions involving parentheses.

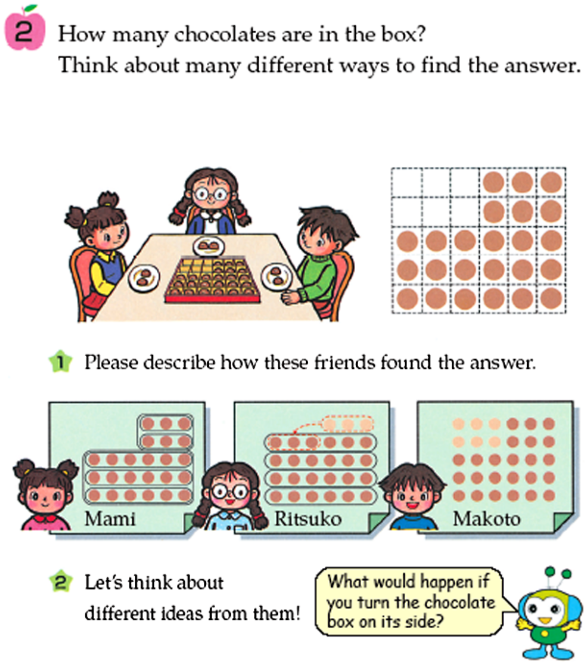


Fig. 4 Students are asked to represent thinking processes using mathematical expressions (Book 2B, p. 44)

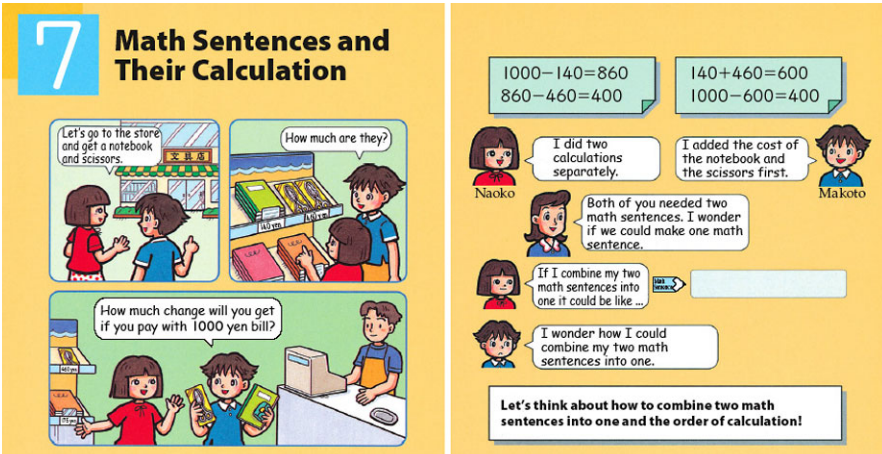


Fig. 5 Grade 4 unit on compound math sentences—mathematical expressions with parentheses (Book 4A, pp. 70 & 71)

It is interesting to note that students learn about the order of operations in the context of writing compound math sentences. Naoko's math sentences can be simply combined into 1000 − 140 − 460. However, Makoto's thinking cannot be written as 1000 − 140 + 460. From such situations, students realize the need for an agreement on the order of operations and new notations for grouping of quantities.

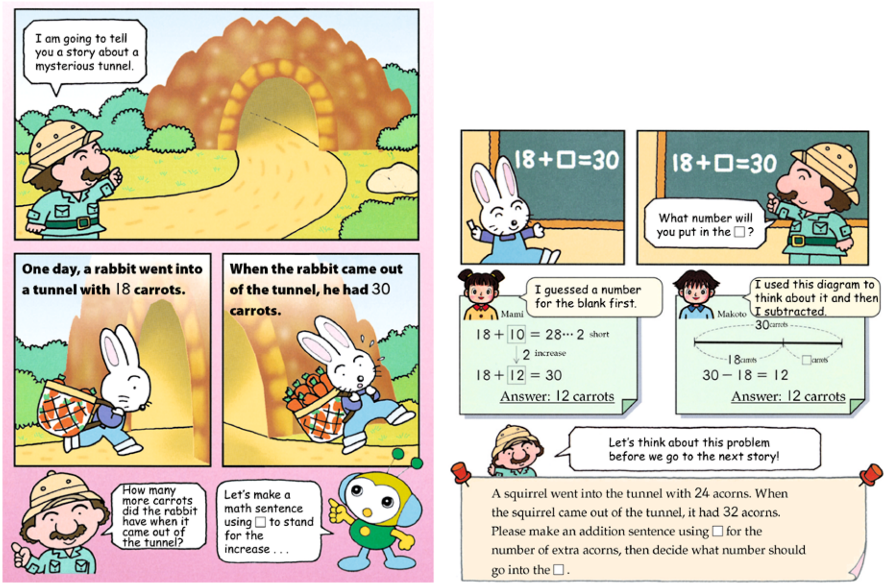


Fig. 6 Grade 3 unit on writing mathematical expressions with □ (Book 3B, pp. 65 & 66)

In Grade 3, students also learn to write math sentences with missing quantities. Figure 6 shows the first 2 pages of the unit in which students will learn to write math sentences using the symbol, □.

Prior to this unit, students have learned missing number type problems. For example, students previously learned a missing addend subtraction problem similar to the one shown in Fig. 6 in Grade 2. Therefore, solving these problems is not necessarily the main focus of the unit. Rather, the emphasis is to help students write mathematical expressions using □ for an unknown quantity. The textbooks use □ to indicate unknown quantities in diagrams and even in some blank math sentences, like $5 + \square = 8$, but this unit is the first time where students are actually asked to write math sentences with □.

Symbols such as □, △, and ◯ are initially used as unknowns, and students are often asked to find what number can replace the symbol. However, starting in Grade 4, students are exposed to these symbols used as variables. For example, in the unit on writing mathematical expressions with parentheses, students study the distributive property formally. In that unit, the property is expressed using symbols as shown in Fig. 7.

As you can see, students are encouraged to substitute different numbers to verify the property. Through such experiences, students are expected to develop an understanding that multiple values can be substituted in a single symbol.

In Grade 4, students also learn to write mathematical expressions using symbols such as, □, △, and ◯, as variables. This is done in the context of exploring relationships between two quantities. Figure 8 shows an example of such problems.

Fig. 7 Symbols are used as variables in Grade 4 (Book 4A, p. 76)

These are two properties of operations that are related to math sentences with parentheses.

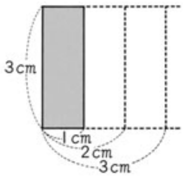
(■+●)×▲=■×▲+●×▲

(■−●)×▲=■×▲−●×▲

3 Use different numbers in ■, ●, and ▲ above to verify the sentences above.

Fig. 8 Students express the relationship between two quantities using mathematical expressions with symbols (Book 4B, p. 58)

① There is a rectangle that is 3cm long and 1cm wide.
Let's investigate how the area changes when the width increases to 2cm, 3cm, . . .



(1) Complete the table below by filling in the areas.

Width (cm)	1	2	3	4	5	6	7
Area (cm²)							

(2) Regarding the widths as □ and the areas as ○ cm², please express the relationship between □ and ○ in a math sentence.

Based on these experiences with mathematical expressions, students are introduced to the use of letters as variables in Grade 5 in the 1989 COS and in Grade 6 in the 2008 COS (see Fig. 9). However, as you can see from Fig. 9, letters are introduced as simply replacing familiar symbols such as □, △, and ◯. Students have already begun to learn about the concept of variables, and they have also studied how symbols can be used in mathematical expressions to represent relationships among quantities. Furthermore, they are familiar with problems requiring finding the missing number represented by those symbols (see Fig. 6). Thus, their focus in this unit is really much more notational than conceptual.

In Fig. 6, you notice that the textbook includes two strategies to determine the missing number, guess-then-adjust and using a diagram. Students learned how to represent problems using the diagram like the one shown in Fig. 6 at the end of Grade 2. The Japanese curriculum materials emphasize linear models in general, and in Grade 4, the textbook introduces another type of linear model. Consider the following problem:

Teams A and B are making 40 posters for the sports festival. Team B will make 8 more than Team A. How many posters will they each make?

Although it may be possible to represent this problem in a single tape/segment, it may be difficult to use it if there are more than 2 quantities involved. Thus, the textbook suggests a diagram like the one shown in Fig. 10. This diagram can easily

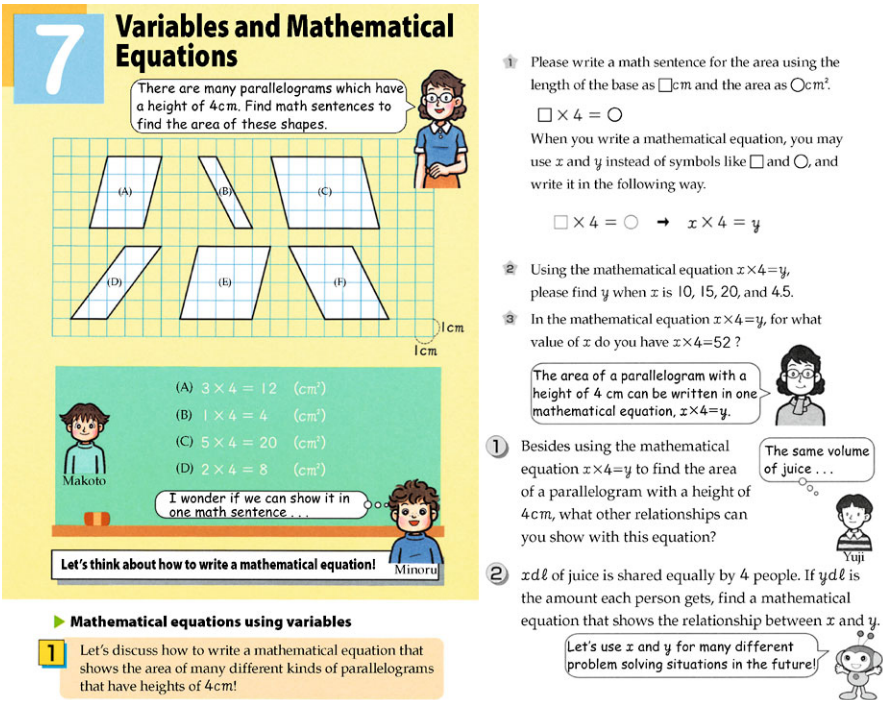
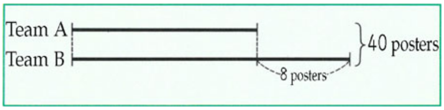


Fig. 9 Letters as variables are introduced after students have learned to work with mathematical expressions with symbols like □, △, and ◯ (Book 5A, pp. 85 & 86)

Fig. 10 A new type of linear model is introduced in Grade 4 (Book 4A, p. 40)



be adjusted even if there are more than 2 quantities. Furthermore, from the diagram, students may be able to come up with different ways to calculate the answer. For example, if Team A were to make 8 more posters, then the two teams would make 48, i.e., 40 + 8, posters altogether, each making the same number. So, dividing it by 2 will give you the number of posters Team B will make. On the other hand, if Team B were to make 8 fewer posters, then the two teams would make 32, i.e., 40 − 8, posters. Since each team would make the same number of posters, you can divide 32 by 2 to determine the number of posters Team A will make.

The textbook continues to use a linear model to represent various problems in Grades 5 and 6. In Grade 5, students learn to use a linear model to solve problems shown in Fig. 11. These problems can be solved using linear equations, or systems of linear equations, but the focus here is for students to learn to represent problem situations using linear diagrams. From the diagrams, students can identify different solution methods, using only elementary school arithmetic. Clearly, each step of the arithmetic solution processes correspond to the solution processes with linear

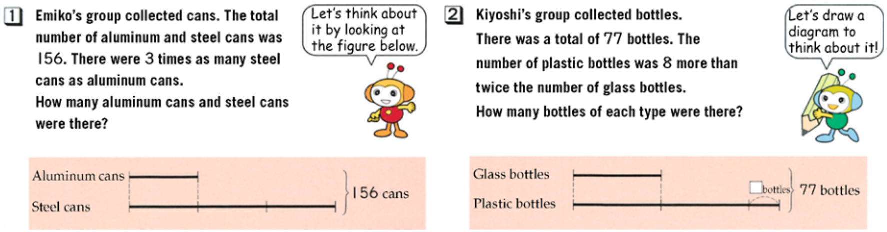


Fig. 11 Grade 5 textbook uses linear models to solve more complex problems (Book 5A, pp. 48 & 49)

equations. (See Watanabe et al. 2010 for a more detailed discussion of the way Japanese textbooks introduce and develop diagrams to support students' thinking.)

Discussion

As it was noted earlier, pre-K through Grade 5 focal points (NCTM 2006) are well aligned with the content of the Math Panel's Critical Foundations for School Mathematics.[2] In addition, NCTM recommends the importance of the study of patterns in various "connections" in pre-K through Grade 5. Starting in Grade 6, *Focal Points* begin emphasizing the understanding of mathematical (symbolic) Representations.

It is rather surprising that both the Math Panel and NCTM completely omit the reference to written/symbolic representations in elementary grades. The Math Panel seems to focus primarily on computational fluency. Even though they use the term "proficiency," they state that by the term, "the Panel means that students should understand key concepts, achieve automaticity as appropriate (e.g., with addition and related subtraction facts), develop flexible, accurate, and automatic execution of the standard algorithms, and use these competencies to solve problems" (p. xvii). This emphasis seems to be consistent with the Panel's major topics of algebra, which seems to take on the perspective of algebra as generalized arithmetic.

NCTM's *Focal Points*, in alignment with their *Standards* (NCTM 2000), take a much broader view of algebra, which incorporates all four conceptualizations of algebra discussed earlier. Thus, they emphasize the importance of the study of patterns and functions in pre-K through Grade 5. NCTM (2000) also discusses how learning about various properties of operations is the beginning of algebraic thinking. *Focal Points* also makes references to representations several times, but they seem to focus on representations using concrete materials or visual representations such as graphs and diagrams.

In contrast, the Japanese curriculum documents consider the study of functional relationships (patterns) and the ideas related to mathematical expressions (*shiki*) as

[2]Note that neither *Focal Points* nor the Math Panel report is a mathematics curriculum. They are mentioned in this discussion because both documents offer frameworks for a school mathematics curriculum.

the two pillars of elementary school algebra. Moreover, the Japanese curriculum materials emphasize the writing and interpreting of mathematical expressions as a major focus in the domain of quantitative relations. In addition, the Japanese curriculum considers that a goal of this domain "is to understand the contents of other domains using the ideas and methods discussed in this domain" (Takahashi et al. 2004, p. 36).

This emphasis on mathematical expressions makes sense mathematically. For example, a Grade 4 focal point discusses the importance of students' understanding of the distributive property. However, it seems like a true appreciation and understanding of this property requires students to write composite mathematical expressions. Consider the following problem:

A fruit basket contains 5 apples, 8 oranges, and 4 bananas. If you buy 3 baskets, how many fruits are there all together?

Students can write their solution steps using sets of math sentences:

$$\begin{array}{l} 5+8+4=17 \\ 17\times 3=51 \end{array}, \quad \text{or} \quad \begin{array}{l} 3\times 5=15 \\ 3\times 8=24 \\ 3\times 4=12 \\ 15+24+12=51 \end{array}$$

Clearly, it is important for students to understand that both of these thinking processes are mathematically valid. However, to understand the distributive property they must understand that each set of math sentences can be written as a composite math sentence and that those two math sentences are equal; that is, $3 \times (5 + 8 + 4) = 3 \times 5 + 3 \times 8 + 3 \times 4$. Similarly, the need for understanding of the order of operations seems to arise when students write compound mathematical expressions.

Perhaps mathematical/symbolic representations are not receiving much attention because there is a perception that formal algebra instruction in the past overemphasized the symbolic manipulation of mathematical expressions (Kilpatrick and Izsak 2008). Whether or not that perception is correct, the fact remains that symbolic manipulation is a component of school algebra. With the development of computing technologies, actual manipulation of mathematical expressions themselves may not be as critical as it used to be. However, the availability of computing technologies increases the importance of students' ability to write and interpret mathematical expressions. Moreover, it is widely known that many children consider the equal sign as simply a do-something symbol, and such an understanding can hinder their understanding of algebra (for example, Knuth et al. 2006). Kieran (2004) lists "a refocusing on the meaning of equal sign" (p. 141) as one of the adjustments from arithmetical thinking to algebraic thinking. The Japanese curriculum materials, however, seem to take the position that mathematical expressions, not just the equal sign, are a central feature of mathematics.

Overall, the treatment of algebra in the Japanese elementary school mathematics curriculum has many similarities to those of other countries. Kieran (2004) noted that the emphasis on quantitative relationships is one commonality among the elementary school curricula from China, Korea, Russia, Singapore, and the United States. The Japanese curriculum also shares the same emphasis. The Japanese curriculum introduces literal symbols in upper elementary grades like the other Asian curricula. The explicit emphasis on helping students develop diagrams to support students problem solving is similar to the Singaporean approach. However, the emphasis on writing and interpreting mathematical expressions seems to be a unique feature of the Japanese approach. Fujii (2003) noted "the importance of recognizing the potentially algebraic nature of arithmetic, as distinct from trying to move children from arithmetic to algebra" (p. 62). The Japanese elementary mathematics curriculum seems to embody this perspective.

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Commentary on Part I

Jeremy Kilpatrick

During the nineteenth century, the study of algebra moved into the secondary school curriculum as colleges and universities began to require it for admission (Kilpatrick and Izsák 2008). Coming after an extensive treatment of arithmetic in the elementary grades, school algebra was commonly introduced formally as a generalization of that arithmetic, with an emphasis on symbol manipulation and equation solving. Given the well-established status of algebra in the secondary curriculum, mathematics educators today confront the question of, in the words of Subramaniam and Banerjee, how "to manage the transition from arithmetic to symbolic algebra."

The U.S. National Mathematics Advisory Panel (2008, pp. 17–18) addressed the transition question by identifying what the panel called Critical Foundations for School Algebra, which comprise three clusters of concepts and skills: (1) fluency with whole numbers, (2) fluency with fractions, and (3) particular aspects of geometry and measurement. Watanabe observes that although those items are certainly foundational, they are neither exhaustive nor even the components of the elementary school grades mathematics curriculum that are, perhaps, or ought to be, most directly related to school algebra. As Blanton and Kaput note, although the National Mathematics Advisory Panel focused on getting learners ready for the study of algebra,

> experiences in building, expressing, and justifying mathematical generalizations . . . should be a seamless process that begins at the start of formal schooling, not content for later grades for which elementary school children are "made ready" through a singular, myopic focus on arithmetic.

Russell, Schifter, and Bastable pose the curriculum question differently: "How can work in [early] algebra fit into an already crowded curriculum?" Their response is echoed in the other chapters in this part: early algebra "not only provides crucial links between arithmetic and algebra, but also is an essential part of good arithmetic instruction."

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Algebra First

The dominant theme of the six chapters offering a curricular perspective on early algebra learning is that algebra is not something to be postponed until arithmetic has been mastered but instead ought to be present in the curriculum from the beginning. This approach has been taken in those school mathematics curricula in which arithmetic is handled in general terms and functional thinking is encouraged. As Izaak Wirszup noted, such an approach characterizes school mathematics in some countries but not others:

> In Europe, in other countries [including Russia], they start arithmetic combined with geometry, and arithmetic is not calculations alone. They introduce algebraic concepts and algebraic thinking from the first grade, from kindergarten. Variables, place holders, empty squares, question marks, these are the way variables are smuggled in without explaining, only through examples. And in the elementary grades you learn about equations, inequalities, and systems of equations. So all this experience results in the fact that the transition in Europe from arithmetic to algebra is almost invisible, while in the United States after some eight years of endless, meaningless calculations, you are given a one-year algebra course, the first-year algebra course, with variables and polynomials and exponents and equations. (quoted in Roberts 2010, p. 56)

The most radical argument for incorporating algebra into the curriculum of the early grades, as explained by Schmittau, was given by Vygotsky, who argued not only that thinking moves from the abstract to the concrete rather than vice versa but also that consequently the algebraic "plane of thought" needs to be developed before arithmetic is developed. Vygotsky's follower Davydov said that rather than viewing the process of generalization as an induction from examples,

> the specific examples should be seen as carrying the generalization within them; the generalization process ought to be one of enrichment rather than impoverishment. Instead of thinking of generalization as moving from the concrete to the abstract, we should think of it as beginning with the abstract and moving to the "intellectually concrete" and then on to an enriched abstraction. (Kilpatrick 1990, pp. xv–xvi)

Implementing the elementary curriculum developed by Davydov, Schmittau demonstrates the value of a systematic, theory-based approach to algebraic structure in helping children learn to solve challenging mathematics problems.

Less radical but certainly equally challenging approaches to early algebra are outlined by the other authors in the section. Blanton and Kaput emphasize functional thinking and the need to consider both covariation and correspondence between variables; Cai, Ng, and Moyer discuss some curriculum practices in China and Singapore that promote algebraic thinking; Russell, Schifter, and Bastable elaborate mathematical activities that can provide a bridge between arithmetic and algebra; Subramaniam and Banerjee show how children's arithmetic intuition can be used to help them interpret and evaluate numerical and algebraic expressions; and Watanabe demonstrates how Japanese textbook materials help children write and interpret symbolic mathematical expressions. In every case, arithmetic is treated as not simply a venue for learning and doing calculations but rather as an arena for developing children's ideas about quantities and their interrelationships, representations, and use.

A Curriculum Topic

Several of the authors in the section point out that introducing algebraic thinking into the elementary school grades entails not the introduction of new topics but rather new approaches to existing topics. The question of how the curriculum is affected, therefore, depends on one's concept of the curriculum. If curriculum is a topic list, nothing changes. But if curriculum is the set of experiences that learners have, then the change can be profound. To illustrate, consider a common topic from the elementary school curriculum: *the area of a rectangle as the product of its length and width*.

This topic commonly appears about the time that children are learning the operation of multiplication of whole numbers. In the new Portuguese program of mathematics for basic education (Ponte et al. 2007), for example, students in the third and fourth grades are expected to understand and use formulas to calculate the area of the square and the rectangle (p. 25). In the United States, several of the Grade 3 measurement and data standards in the Common Core State Standards Initiative (see http://www.corestandards.org/the-standards/mathematics) address the area of a rectangle. For example, one standard reads as follows: "Multiply side lengths to find areas of rectangles with whole-number side lengths in the context of solving real world and mathematical problems, and represent whole-number products as rectangular areas in mathematical reasoning."

A third-grade teacher who wanted to follow the approach advocated by Blanton and Kaput would hardly stop at having the children find the areas of some rectangles given their sides. The children would also look at patterns formed as sides and areas vary together. They might, for example, explore how, for a fixed width, changes in the length are reflected in changes in the area or how, for a length 2 units longer than a varying width, the width and area covary. Similarly, a teacher who wanted to pursue the Chinese "one problem multiple solutions" approach outlined by Cai, Ng, and Moyer might offer the children a problem like the following: *The area of a rectangle is 120 square centimeters, and one side is 16 centimeters. How long is the other side?* The children would be asked to represent the quantitative relationship in several different ways both arithmetic and algebraic. A teacher wanting to develop algebraic thinking by engaging in the approach promoted by Subramaniam and Banerjee might ask students to judge whether rectangles with dimensions 33×14 and 11×42 have the same area and to explain why or why not. In every case, the children would have opportunities to investigate and use the aspects of arithmetic identified by Russell, Schifter, and Bastable: "understanding the behavior of the operations [in this case multiplication and division], generalizing and justifying, extending the number system [in this case to fractions or decimals], and using notation with meaning." Few teachers would want to deny children opportunities to develop such understanding and ways of thinking simply because the elementary school curriculum is "already crowded."

Numerical Patterns

The National Mathematics Advisory Panel (NMAP 2008) made a strong recommendation "that 'algebra' problems involving patterns should be greatly reduced in the NAEP [National Assessment of Educational Progress]" (p. 59), arguing that "at Grade 4, most of the NAEP algebra items relate to patterns or sequences (Daro et al. 2007)" (p. 59) and that neither mathematical considerations nor comparative analyses of curricula support "the prominence given to patterns in PreK-8" (p. 59). Regarding comparative analyses, the panel cites a paper by Schmidt and Houang (2007) in making its claim that "patterns are not emphasized in high-achieving countries" (NMAP 2008, p. 59). According to the Schmidt and Huang analysis, a majority of the countries whose eighth graders were high achieving in the 1995 Trends in International Mathematics and Science Study (TIMSS) did not address the topic of "patterns, relations, and functions" until Grade 8.

The NMAP claim, however, is not supported by the chapters at hand. Cai, Ng, and Moyer point out that in Singapore, already in second grade, children are looking at "sets of ordered pairs representing the input and output of problem situations" and are being asked to find rules for the patterns they are observing in those situations. Watanabe argues that the "idea of functions" in the Japanese course of study is essentially the same thing as the American "examination of patterns in Grades Pre-K through 5." And one might note that even though China did not participate in the 1995 TIMSS, in the words of Cai, Ng, and Moyer, "function ideas permeate the [elementary school] curriculum."

In the Daro et al. (2007) study of the validity of the NAEP mathematics assessment, mathematicians who reviewed the items complained that there were not only too many items dealing with patterns but also too many that were flawed because they asked that a sequence be extended without specifying the rule for pattern generation. Those complaints are apparently the source of the NMAP's argument that "mathematical considerations" do not support prominent attention being given to numerical patterns in the first eight grades. Which is worse: asking children, given a sequence of numbers, to conjecture what the rule might be that would give the next one, or denying them an opportunity to address such problems because, of course, neither the rule nor the next term is unique mathematically? Fortunately, the chapters at hand support efforts to give children chances to explore numerical patterns.

Word Problems

Consider the following word problem:

> Some pupils gathered 1800 kilograms of maple and acacia seeds. There were five times as many acacia seeds gathered as maple seeds. How many kilograms of seeds of each type were gathered? (Yaroshchuk 1969, p. 79)

A problem of this sort might easily appear in a first-year algebra textbook in the United States, with the expectation that students would solve it using two equations and two unknowns—the number of kilograms of maple seeds and the number of kilograms of acacia seeds. In the Soviet Union a half century ago, however, it was given to fourth-grade students of average ability, and they were expected to solve it by recognizing that it is a problem "in parts" and that they needed to solve it by finding the total number of parts (six) and then finding the size of one part.

One need not have children recognize the problem as being of a certain "type" to appreciate that they ought to be solving such problems well before they have learned to solve systems of linear equations. By reasoning through the arithmetic operations in the problem along the lines promoted in the various chapters in this part, children can learn how such problems are structured, learn how to represent them in various ways, and begin to develop their algebraic thinking skills.

Multiple Perspectives

Although the chapters in this monograph are divided among three perspectives, one can see in the six chapters on the curricular perspective that other perspectives are present as well. For example, Blanton and Kaput claim that their work shows that children are "capable of deeper functional analysis than previously thought" and that those ideas "appear at grades earlier than typically expected." Further, they argue that curriculum change alone, without attention to developing teachers' knowledge of both instruction and mathematics, is "not sufficient to produce real change in children's mathematical thinking." Thus, cognitive and instructional perspectives play a prominent role in their chapter. Similarly, the other chapters offer ample evidence that young children are capable of impressive feats of algebraic reasoning and that proposed changes in the arithmetic curriculum ought to be accompanied by instructional changes as well. As Subramaniam and Banerjee conclude, children need to learn arithmetic as more than "a set of instructions to carry out operations"; the expressions of arithmetic and algebra encode quantitative relationships that children can and should learn to reason with.

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Part II
Cognitive Perspective

Preface to Part II

Eric Knuth and Jinfa Cai

Enhancing the nature of algebra in the earlier grades requires substantial effort on the part of teachers insofar as they are responsible for ensuring students have the means as well as the opportunities to engage in learning to reason algebraically. Such efforts, however, are predicated in part on understanding the nature of students' algebraic thinking and ways to foster its development. The chapters in this part present research that focuses on the development of students' algebraic thinking across a range of grade levels (from primary grades to intermediate/middle grades). Many of the chapters also underscore different schools of thought regarding early algebra; in particular, these chapters illustrate the nature of students' algebraic thinking as it develops within primarily *arithmetic contexts* (e.g., generalizing about arithmetic computations) as well as within primarily *algebraic contexts* (e.g., functions and functional thinking). As a collection, the chapters in this section illustrate that regardless of grade level, arithmetic or algebraic context, or even country, young children are very capable of developing the ability to think algebraically.

The chapter by Britt and Irwin provides compelling evidence that emphasizing algebraic thinking within arithmetic has a positive influence on students' algebraic thinking in later grades. They present results from two studies that suggest that students in the intermediate grades who were provided with early algebra experiences in the primary grades outperform on algebraic tasks their peers who received a more traditional curriculum and, moreover, such early algebra experience continues to pay dividends when students are in secondary school. The chapter by Cooper and Warren also considers the influence of an algebraic emphasis during the primary grades on the development of students' algebraic thinking. In their work, they focus primarily on students' abilities to generalize from a variety of both arithmetic-based

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situations (e.g., compensation principles generalized from computational problems) and representational forms. Their results highlight the importance that understanding and communicating the features of representational forms play in students' abilities to generalize. Moss and London McNab, in their chapter, also provide evidence that children in the early grades can learn to reason algebraically; in particular, their work illustrates the development of students' thinking about linear functions and co-variation in the context of growing patterns. Interestingly, their work suggests that the instructional emphasis on linear functions and co-variation also had a positive effect on aspects of students' understanding of multiplication—a staple of instruction in a traditional arithmetic curriculum. Finally, in Radford's chapter, he discusses from an epistemological perspective the relationship between arithmetic and algebraic thinking, addressing the question of what counts as arithmetic and what counts as algebra. He then situates this discussion in the context of young children's first encounter with algebraic concepts, and also uses this context to discuss the limits and possibilities of introducing algebra in the early grades.

The remaining chapters in this part continue the focus on students' algebraic thinking, however, the focus now shifts to the intermediate/middle grades. Knuth and colleagues present results from a cross-sectional study that focused on students' understanding of two fundamental algebraic ideas (equivalence and variable)—ideas that can be developed in the primary grades—and how their understanding affected their performance on algebraic tasks that required use of these ideas. In the chapter by Cai and his colleagues, they compare the effects of curricula on the longitudinal development of students' algebraic thinking. In particular, they compare the effects on students' algebraic thinking of a reform-based curriculum that takes a functional approach to teaching algebra to a more traditional curriculum that takes a structural approach to teaching algebra. A functions-based approach also underscores the chapters by Ellis, Izsák, and Rivera and Becker. In her chapter, Ellis illustrates how building on students' capabilities to reason with quantities can serve as a powerful means of fostering the development of student's understanding of linear and quadratic functions. In Izsák's chapter, he reviews three decades worth of research on students' understanding of algebraic and graphical representations of functions. He then discusses two significant advances in this area of research: insight into students' criteria for evaluating representations as well as into students' coordination of shifts within and between representations and problem situations. Finally, in the chapter by Rivera and Becker, they detail the results of a longitudinal study that focused on students' abilities to generalize the underlying functional relationship for various linear patterns. They provide a detailed account of the development of students' abilities to generalize and the factors that influenced their development.

Although the chapters in this part focused on students' algebraic thinking and its development from a variety of perspectives, collectively, the chapters present a common message: students are capable of learning to reason algebraically in the early grades (prior to formal algebra), and curriculum and instruction should build on such capabilities.

Algebraic Thinking with and without Algebraic Representation: A Pathway for Learning

Murray S. Britt and Kathryn C. Irwin

Abstract The origins of algebraic thinking precede understanding of arithmetic, as shown in a study of children aged 4–7. A mathematics curriculum introduced in some New Zealand schools in 1999, The New Zealand Numeracy Project, now encourages this algebraic thinking within arithmetic. The underlying framework for this curriculum is described, with examples of the type of thinking encouraged. The effect of this emphasis on the algebra underpinning arithmetic operations was examined in two further studies. One of these involved students in their final year of elementary and intermediate school, at age 12. This study showed that on a test that focused on students' awareness of the underlying algebraic structure of arithmetic, those students who had been included in the new curriculum in its early stages outperformed those who had received a traditional curriculum. A later study followed a cohort of students who received the new curriculum through their two intermediate school years (aged 11–12) and into their first year of high school at age 13, when traditional algebra is introduced. The results of this study showed that students who had developed their understanding of the interrelationship of mathematical relationships for additive, multiplicative and proportional operations could display this understanding algebraically. The ramifications of these findings for further teaching algebraic thinking with or without algebraic representation led to a proposal for a 'pathway for algebraic thinking' accessible to all students.

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Introduction

In a plenary address to the North American Chapter of the international Group for the Psychology of Mathematics Education, Radford (2006) presented a compelling argument in support of a generalization approach to algebra in which he noted that "the algebraic generalization of a pattern rests on the noticing of a local commonality that is then generalized to all terms of the sequence and serves as a warrant to build expressions of elements of the sequence that remains beyond the perceptual field" (p. 5). As well, Radford added a third element, using the commonality to provide a direct expression or rule to specify any term of the sequence. But while it is customary to require that learners use the symbols of algebra to express such rules, Radford acknowledged that rule-making proceeds through various layers of awareness articulated through different semiotic systems; words, gestures, pictures, graphs and symbols. We illustrate some of these layers of awareness of generality in a study that contributed to an evaluation of The New Zealand Numeracy Project (New Zealand Ministry of Education 2007a, http://www.nzmaths.co.nz/teaching-numeracy), a national project in which students throughout New Zealand are encouraged to devise and experiment with a range of mental operational strategies in arithmetic (Irwin and Britt 2005a). In that study we argued that students who could apply a range of mental operational strategies to solve different numerical problems were disclosing an awareness of the relationships of the numbers involved as well as the underlying structure of the strategy. We claimed that successful application of such operational strategies demanded an awareness of the generality of the operational strategy, thereby illustrating algebraic thinking. Students' explanation of their thinking revealed that they were treating the numbers as if they were variables. Fujii and Stephens (2001) refer to numbers used in this way as quasi-variables. The results of our study led to a view of algebra, particularly as it pertains to the pedagogy of introductory algebra, in which we do not see algebra as following arithmetic so that arithmetic has an ending that coincides with the beginning of algebra. Instead our view is consonant with those of Hewitt (1998, p. 20) who argued that algebra enables arithmetic to be carried out, and of Steffe (2001, p. 563), who argued that children's knowledge of number together with numerical operational knowledge that is effective and reliable is essentially algebraic in nature. We further argue that the roots of this algebraic, or generalized, thinking precede the introduction of numbers as unschooled children demonstrate an understanding of such operations with unnumbered quantities.

While Carraher et al. (2006) and several of the studies included in Kaput et al. (2008) argue for the early inclusion of algebraic symbols as a valuable tool for early algebraic thinking (e.g. Brizuela and Earnest 2008; Carraher et al. 2008; and Dougherty 2008), we have primarily followed the reasoning of Fujii and Stephens (2001) and Hewitt (1998) that emphasizes algebraic thinking in order to understand arithmetic. Mason (2008) also pointed to children's innate power to reason and generalize in a manner that leads them to understanding the underlying structure of arithmetic that is central to mathematical development. Our case, which focuses on this crucial role for generalization, is consistent with those expressed by this second group of writers. Prior to their introduction to the semiotics of algebra, young children need to work successfully with several layers of awareness of generality that involve expression of generality in words, in pictures and graphs as well as with numerical symbols acting as quasi-variables.

Much has been written about the difficulties encountered during the transition from arithmetic to algebra (see for example, Herscovics and Linchevski 1994; Filloy and Rojano 1989). But as Carraher et al. (2006, p. 89) argue, acceptance of such a transition arises from an impoverished view of elementary school mathematics in which mathematical generalization is postponed until the onset of algebra instruction. We argue similarly. The notion of algebra in arithmetic, in which generalization provides the basis for successful numerical operational thinking, dismisses the claim for transition and offers algebra for all through algebraic thinking with and without the symbols of algebra.

In the discussions that follow we refer to students by year group, which is the New Zealand nomenclature and because years of schooling do not necessarily match school grades in the US and elsewhere. All children in New Zealand begin school on their 5th birthday and that year of entry to compulsory schooling is called Year 0 or Year 1 depending on when the birthday falls. While they may be moved around between classes in the first two to three years at school, they usually remain in the same year group from Year 3 onwards. The primary school system goes from Year 0 through Year 8 (roughly age 5 to 12). While some children remain in the same primary school up to the end of Year 8, most children attend a separate intermediate school for Years 7 and 8 (ages 11 to 12). Secondary school begins at Year 9 (typically aged 13) and goes on for five years to Year 13 (about age 17). Algebra is traditionally introduced formally in Year 9. The curriculum areas algebra, geometry, trigonometry, and statistics are not separate subjects as they often are in the US. They are taught with different emphases within the curriculum heading, Mathematics and Statistics.

In this chapter, we begin our discussion of algebraic generality by illustrating how the development of algebraic thinking can evolve, without recourse to the symbols of algebra, from students developing an ongoing awareness of the underlying structure of operational strategies in arithmetic.

First we describe a study that demonstrated preschool children's understanding of generalization in operations. Next we draw on the New Zealand Numeracy Project to illustrate the development of operational strategies in arithmetic for students at different ages, and then we focus on two recent studies in which we attempted to ascertain what level of strategy development was likely to be necessary for students to extend their expressions of generality from using numbers themselves as quasi-variables to a semiotic layer of awareness that embraces the literal symbols of algebra.

Children's Understanding of Generalities for Operations Before Schooling

While the main focus of this chapter is the effect of the New Zealand Numeracy Project, it is useful to discuss an earlier study (Irwin 1996). This study explored young children's understanding of operations before they are introduced to arithmetic. Children aged 4 through 7 were asked what happened to a total quantity under three conditions. The total quantity was made up of two parts, in this case, two small boxes of sweets, called "lollies" in New Zealand. They were asked what happened to the total quantity if a doll, Ernie©, took a sweet from one of the parts, if he added or removed a sweet to or from one of the parts, if he moved a sweet from one of the parts to the other, or if he took away a sweet from one part but the interviewer added a different sweet to the other part. In effect, they were asked for their understanding of compensation and covariation of the whole with changes to one of the parts. Children aged 4 were certain that the total quantity would stay the same if an item was moved or replaced and would increase or decrease if one of the parts was altered. At age 5 and 6 they could explain these relationships, sometimes with a principle in their own language that showed that they understood this as a generality. For example, one child said the total number in a compensatory move would be the same, "The same, except Ernie put one of the lollies from here to here". Expressed algebraically, they understood that if $P_1 + P_2 = W$, then $(P_1 - k) + (P_2 + k) = W$. Here, W stands for the total (the whole) uncounted number of sweets, P_1 and P_2 are the generalized unknown (uncounted) number of sweets in the separate containers (the parts comprising the whole), and k stands for the number of lollies transferred from one container to the other.

However, if they were given a similar task with numbers only, using doubles facts that they knew such as $5 + 5$, and asked whether or not it would be the same as $4 + 6$, most were unsuccessful until age 7. As one child phrased it using a visual image for the equality of $10 + 10$ and $11 + 9$, "…because if you put one of the group of 11 over to the 9 group they would both be 10 and that means 20." Since young children understand this concept when no numbers are attached, it may be that the complexity of learning to understand numbers distracts students from the knowledge that they had in a proto-quantitative form (Resnick 1992) before going to school.

The New Zealand Numeracy Project built on this understanding of young children's knowledge as well as the studies of Steffe et al. (1988) and Wright (1994). A major intention of this Numeracy Project was that learners should develop rather than lose their initial ability to understand the way in which numerical quantities can be manipulated.

Algebraic Thinking and the New Zealand Numeracy Project

In 1999, the New Zealand Ministry of Education introduced a professional development program in mathematics known as the Numeracy Development Project, motivated by the need to improve students' number sense and understanding of operations by introducing a flexible approach to solving problems in numerical situations (see for example, McIntosh et al. 1992; Slavit 1999; Wright 1994). At first the project was intended for Years 1–3 students (aged 5–7). In 2001, the project was extended to Years 4–6 students (aged 8–10). In 2002, following a pilot study, the project was expanded to include some 13,600 Year 7–10 students (aged 11–

14). By the end of 2008, approximately 95% of New Zealand elementary and intermediate schools, 40% of secondary schools, and 85% of schools that teach in Māori, had been involved in two years of numeracy professional development (New Zealand Ministry of Education 2009b, http://www.nzmaths.co.nz/annual-evaluation-reports-and-compendium-papers). Nearly all elementary schools were using at least some aspects of the Numeracy Project and an increasing number of secondary schools were becoming involved. The major aspects of this project are now incorporated into the New Zealand Mathematics (New Zealand Ministry of Education 2007b, http://nzcurriculum.tki.org.nz/the_new_zealand_curriculum/learning_areas/mathematics_and_statistics).

A basis of the Numeracy Project is the Number Framework (New Zealand Ministry of Education 2008, http://www.nzmaths.co.nz/sites/default/files/Numeracy/2008numPDFs/NumBk1.pdf). In this, a distinction is made between strategy and knowledge. The strategy section describes the thinking students use to mentally calculate answers to numerical problems. The knowledge section describes the key items of number knowledge that students need to learn and without which they will be unable to broaden and advance their repertoire of strategies. The two are linked in that operational strategies create new knowledge through consistent use and knowledge provides a foundation for the development of new operational strategies. Tables 1 and 2 (see New Zealand Ministry of Education 2008, http://www.nzmaths.co.nz/sites/default/files/Numeracy/2008numPDFs/NumBk1.pdf, pp. 15–17) show the progression of operational strategies that we argue forms a strong basis for ongoing opportunities for students to develop an awareness of generality and hence of algebraic thinking.

The operational strategies illustrated in Table 2 involve part-whole thinking in which students recognize that numbers are abstract units that can be partitioned and then recombined in different ways to facilitate numerical calculation.

An essential part of the Numeracy Project is a diagnostic test administered in whole or part as an interview with individual students. The intention of the interview, which is used in different forms from ages 5 through 14, and includes questions of increasing complexity so that students are not asked to make responses that are clearly beyond them, is to provide information about the knowledge and mental strategies of the student (see, New Zealand Ministry of Education 2009a, http://www.nzmaths.co.nz/gloss-forms). Teachers use the interview data to help devise teaching programs to match the learning needs of students assessed as being at similar stages across the three operational domains: addition and subtraction, multiplication and division, and proportions and ratio described in the Number Framework (see Tables 1 and 2). Students are encouraged to work mentally as this provides information that can be analysed more fully than answers from a paper and pencil test.

A sample of additive, multiplicative and proportional tasks from Form H of the interview is given below:

Task (9): On a hot day the tomato plants absorbed 1.5 litres of water. On a cold day they absorbed 0.885 litres (885 mL). How much more water did the plants absorb on the hot day than the cold day?

Table 1 The number framework for stages 1 to 4 operational strategies that involve counting

Global stage	Addition and subtraction	Multiplication and division	Proportions and ratio
	Operational domains		
Stage Emergent	Emergent • Unable to count or form a given set of up to ten objects		
Stage 1 One to one counting	*One-to-one counting* • Able to count a set of objects • Unable to form a set of objects to solve simple addition and subtraction problems	*One-to-one counting* • Able to count a set of objects • Unable to form a set of objects to solve simple multiplication and division problems	*Unequal sharing* • Unable to divide a region or set into two or four equal parts
Stage 2 Counting from one on materials	*Counting from one* • Counts objects to solve simple addition and subtraction problems • Needs to use materials such as counters or fingers	*Counting from one* • Solves simple multiplication and division problems by counting one-to-one with the aid of materials	*Equal sharing* • Able to divide a region or set into a given number of equal parts using materials
Stage 3 Counting from one by imaging	*Counting from one* • Counts objects by visualizing or imaging • Unaware of 10 as a counting unit • Solves multi-digit problems by counting all the objects	*Counting from one* • Counts all the objects in simple multiplication and division problems by imaging the objects • Uses materials to solve multiplication and division problems with larger numbers	*Equal sharing* • Able to share a region or set into a given number of equal parts using materials or by imaging
Stage 4 Advanced counting	*Counting on* • Counts on or back to solve simple addition and subtraction problems	*Skip counting* • Skip counts to solve simple multiplication and division problems using materials or imaging	

Task (10): Bas needs to buy 114 cans of soft drink for the volley ball club party? How many 6-packs should he get?

Task (11): The dog ate three-eighths of an 800 gram can of jollymeat. The cat ate three-quarters of a 400 gram can. Which ate more, the dog or the cat?

Since 1999, when it began as a modest experiment, the Numeracy Project has been subjected to a number of evaluations (see New Zealand Ministry of Education 2009b, http://www.nzmaths.co.nz/annual-evaluation-reports-and-compendium-papers). These evaluations have provided ongoing national data for the project over an eight-year period. Evidence including measures of effect sizes related to

Table 2 The number framework for stages 5–8 operational strategies that involve part-whole thinking

Operational domains			
Global stage	Addition and subtraction	Multiplication and division	Proportions and ratio
Stage 5	*Early addition and subtraction*	*Multiplication by repeated addition*	*Fraction of a number by addition*
Early additive	• Uses a limited range of mental partitioning & compensation strategies, to solve addition and subtraction problems. E.g., $8 + 7$ is $8 + 8 - 1$ and $39 + 26 = 40 + 25 = 65$	• Uses a combination of known multiplication facts and repeated addition. E.g., 4×6 is $(6 + 6) + (6 + 6) = 12 + 12 = 24$ • Uses known multiplication facts with repeated addition, to anticipate the result of division. E.g., $20 \div 4 = 5$ since $5 + 5 = 10$ and $10 + 10 = 20$	• Uses addition facts to find the fraction of a number. E.g., $\frac{1}{3}$ of 12 is 4 since $4 + 4 + 4 = 12$ • Solves division problems mentally using halving or deriving from known addition facts. E.g., when 7 pies are shared among 4 children each gets 1 pie plus $\frac{1}{2}$ of a pie plus $\frac{1}{4}$ of a pie
Stage 6	*Advanced addition and subtraction of whole numbers*	*Derived multiplication*	*Fraction of a number by addition and multiplication*
Advanced additive - early multiplicative	• Can estimate answers and solve mentally addition and subtraction problems that involve whole numbers by choosing appropriately from a broad range of advanced mental strategies. E.g., $324 - 86 = 324 - 100 + 14$ and $1242 - 986 = 1242 + 14 - (986 + 14)$	• Uses a combination of known multiplication facts and mental strategies to derive answers to multiplication and division problems. E.g., $4 \times 8 = 2 \times 16 = 32$ (doubling and halving) and 9×6 is $(10 \times 6) - 6 = 54$	• Uses repeated halving or known multiplication and division facts to solve problems that involve finding fractions of a set or region, and division with remainders. E.g., $\frac{1}{3}$ of $36 = 12$ since $3 \times 10 = 30, 6 \div 3 = 2$, and $10 + 2 = 12$
Stage 7	*Addition and subtraction of decimals and integers*	*Advanced multiplication and division*	*Fractions, ratios, and proportions by multiplication*
Advanced multiplicative - early proportional	• Can estimate answers and solve mentally addition and subtraction problems that involve decimals, integers and related fractions by choosing appropriately from a broad range of advanced mental strategies. E.g., $3.2 + 1.95 = 3.2 + 2 - 0.05 = 5.2 - 0.05 = 5.15$	• Chooses appropriately from a broad range of mental strategies to estimate answers and solve multiplication and division problems. E.g., 24×6 is $(20 \times 6) + (4 \times 6)$ or $25 \times 6 - 6$; $81 \div 9 = 9$ so $81 \div 3 = 3 \times 9$; and $4 \times 25 = 100$, so $92 \div 4 = 25 - 2 = 23$	• Uses a range of multiplication and division strategies to estimate answers and solve problems with fractions, proportions, and ratios. E.g., $13 \div 5 = (10 \div 5) + (3 \div 5) = 2\frac{3}{5}$; 3 : 5 is equivalent to 24:40 since $8 \times 3 = 24$ and $8 \times 5 = 40$

Table 2 (Continued)

Operational domains			
Global stage	Addition and subtraction	Multiplication and division	Proportions and ratio
Stage 8	*Addition and subtraction of fractions*	*Multiplication and division of decimals, multiplication of fractions*	*Fractions, ratios and proportions by re-unitising*
Advanced proportional	• Uses a broad range of mental partitioning strategies to estimate answers and solve problems that involve adding and subtracting fractions including decimals • Combines ratios and proportions. E.g., 20 counters in ratio of 2:3 combined with 60 counters in ratio of 8:7 gives a combined ratio of 1:1	• Chooses appropriately from a broad range of mental strategies to estimate answers and solve problems that involve the multiplication and division of decimals and the multiplication of fractions. E.g., $4.2 \div 0.25 = (4.2 \times 4) \div (0.25 \times 4) = 16.8 \div 1$	• Chooses appropriately from a broad range of mental strategies to estimate answers and solve problems that involve fractions proportions and ratios. E.g., 6:9 is equivalent to 16:24 since $6 \times 1\frac{1}{2} = 9$ and $16 \times 1\frac{1}{2} = 24$ or $9 \times 2\frac{2}{3} = 24$ and $6 \times 2\frac{2}{3} = 16$

the average attainment in number of different ethnic groups and of issues related to the validity of the framework and reliability related to the test administration by classroom teachers have supported the continuing development of the project (see Thomas et al. 2006; Young-Loveridge 2006), and have led to the establishment of standards for students' learning in number for each of the first four levels of a new national curriculum for years 1 through 8 (New Zealand Ministry of Education 2007b).

An example of activities in a New Zealand classroom will help clarify the type of tasks that students undertake. In this example we describe what a 7-year old student, who we called Mary, might have done as she engaged with a series of tasks drawn from the Numeracy Project material for students advancing from Stage 4 (Advanced Counting) to Stage 5 (Early Additive) strategy activity.

Mary's initial task was to use the tens-frames (see Fig. 1) to help devise a sensible non-counting strategy to work out $9 + 4$.

Mary's teacher would have begun by asking her to put counters on a tens-frame to show 9 and then a further 4 counters on another tens-frame to show 4 (Fig. 1a). Figure 1b shows the outcome of Mary's actions that transform $9 + 4$ into $10 + 3$ so leading to her recognition of 13 as the answer.

Mary might then have gone on to use the tens-frames to figure out the solution to several similar tasks, $8 + 5$, $7 + 6$, and $9 + 7$ before being challenged by her teacher to see if she could work out $19 + 4$, $27 + 6$, and $38 + 7$ without recourse to the tens-frames. She could revert to using the tens-frames if she was unsure what to do. She might be asked by her teacher to explain her thinking after each response. For $19 + 4$, she might have said that she took one from the 4 and put it on the 19. So in her mind she could see 20 and 3 making a total of 23. She might similarly transform

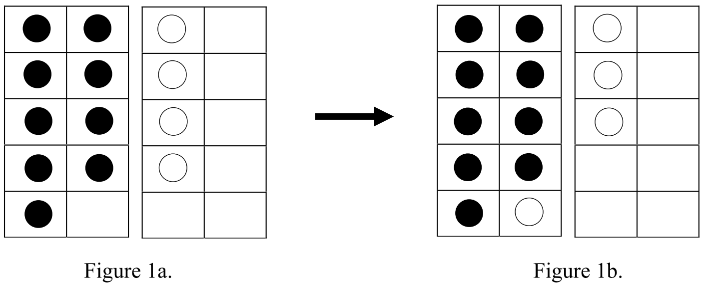

Fig. 1 Using tens-frames to develop early understanding of additive compensation

$27+6$ into $30+3=33$ and $38+7$ into $40+5=45$ each time describing an additive compensation strategy for which the underlying structure or generalization may be represented algebraically as $a+b=(a+c)+(b-c)$.

There are several aspects associated with Mary's thinking that warrant further analysis. Firstly, we argue that the role assigned to Mary's use of the tens-frames is one of 'Image-Making', an early level of understanding drawn from Pirie and Kieren's Model for the Growth of Mathematical Understanding (Pirie and Kieren 1989, 1994). In summarizing their model, Pirie and Kieren (1989, p. 8) claim that, "Mathematical understanding can be characterized as leveled but non-linear. It is a recursive phenomenon and recursion is seen to occur when thinking moves between levels of sophistication. Indeed each level of understanding is contained within succeeding levels. Any particular level is dependent on the forms and processes within and, further, is constrained by those without." Mary's Image-making here is largely constrained by her existing number knowledge. She could count to 99, she knew the pairs of whole numbers that add to ten, and she could interpret 2-digit whole numbers according to the place-value system for tens and ones. Without this number knowledge, which in Pirie and Kieren's model is referred to as the 'Primitive Knowing' level of understanding, it is unlikely that Mary would make much progress with building the images to establish a basis for fully understanding addition of whole numbers.

In the Numeracy Project, the recursive theory of mathematical understanding, proposed by Pirie and Kieren (1989) provides the basis of a teaching model, to be used alongside Project teaching materials (New Zealand Ministry of Education 2007a). In this Numeracy Project Teaching Model, students begin with a 'Using Materials' phase designed to build concrete images that reflect their thinking and in so doing reflect the Image-Making level proposed by Pirie and Kieren. As Mary manipulates the counters on the tens-frames, she notices the space/s to be filled on one frame to make it complete and also where she could get the counter/s from to do the filling of the space/s. She is likely to have begun to develop an awareness of the consequence of this compensation action (Pirie and Kieren's Image Having Level) and so visualize the structure of the transformation that she uses subsequently to solve the more challenging tasks, $19+4$, $27+6$, and $38+7$. In the Teaching Model, these tasks are included within a 'Using Imaging' phase and are designed to challenge Mary's thinking towards an awareness of the transformations involved in the tasks but in the absence of the tens-frames. Further, she is also likely to have begun to isolate the features common to each task. This level of Pirie and Kieren is called 'Property Noticing' and corresponds to the 'Using Number Properties' phase in the Project's Teaching Model. Here at this level Mary demonstrates her understanding by explaining the roles of the various elements in the transformation. Such explanation, where she uses both numbers and words to express generality of the transformation, indicates she had engaged in algebraic thinking. It is helpful to note that, as with the Pirie and Kieren's model, the Numeracy Project's Teaching Model incorporates opportunities for folding-back. For example, if Mary had not succeeded with the tasks at the Using Imaging phase she would fold back to work with more tasks in the Using Materials phase. She would then advance once again to the Using Imaging phase when she had demonstrated success with these tasks.

Students' Algebraic Thinking in the Last Year of Intermediate School (Age 11–12)

In 2002, we carried out an evaluation of one aspect of the Numeracy Project that is reported more fully in Irwin and Britt (2005a). The goal of the evaluation was to gauge if the students who had participated in the Numeracy Project used operational strategies deemed algebraic in nature more successfully than students from the same year cohort who had not participated in the project. We wanted to test our conjecture that project students would show a greater awareness of the algebraic structure of problems in arithmetic. We devised a 21-item test comprising six sections: compensation in addition, $x + y = (x + a) + (y - a)$; compensation in subtraction, $x - y = (x + a) - (y + a)$; compensation in the distributive law of multiplication over addition, $xy = x(y + a) - xa$; equivalence with sums and differences in which one of four structures is, 'If $x + a = b$, then $x = b - a$'; compensation in multiplication, $xy = (ax)(\frac{y}{a})$; and equivalence with fractional values, again in which one of four structures is, 'If $\frac{a}{b} = \frac{an}{x}$, then $x = b \times n$ or bn'. The following shows the test items for Section A: Compensation in addition.

Example 1. $47 + 25 = 50 + 22 = 72$
Example 2. $67 + 19 = 66 + 20 = 86$

Use a strategy like those shown above to work out each of the following.

(1) $97 + 56$
(2) $268 + 96$
(3) $4613 + 987$

The study involved 899 Year 8 students from four schools of which 431 participated in the Numeracy Project in 2002 and 468 had not participated in this Project. Year 8 students, the last year of intermediate school, were chosen for the study because they had not been taught formal algebra prior to the study. The test was subjected to a Rasch analysis with reliability 0.88 as estimated by Kuder-Richardson's formula 20. A subsequent analysis of the students' results showed that for every test item, the proportion of students who participated in the project and were deemed to have completed the item successfully was greater than the proportion that was attained by those who had not participated in the project. The results suggested strongly that students' involvement in the Numeracy Project was likely to lead, not only to improved outcomes for the arithmetic involved, but to improved algebraic thinking skills for current and future algebraic activity. These results are in accord with the views of a number of researchers such as MacGregor and Stacey (1999) and Orton and Orton (1994) who have argued that the understanding of and skill in using arithmetical relations are a necessary pre-requisite for learning algebra. While we do not disagree with this we feel that we are able to explain why this is likely to be the case. In our view, it is not merely that students involved in such arithmetic activity develop greater skill in this area but rather it is because they have already begun their algebraic development as a consequence of their participation in the Numeracy Project where a considerable amount of their time had been given to working with the strategies as described in the Project Number Framework (see Tables 1 and 2). That work, as we have previously argued, demands the development of awareness of generality representing algebraic thinking.

The Growth of Algebraic Thinking from Numbers to Symbols: A Longitudinal Study

Following the study above, in which we assessed the difference between students engaged in the Numeracy Project and those not in the Numeracy Project, we carried out a Longitudinal Study, Britt and Irwin (2008), in which we examined the growth in algebraic thinking across the important move for students from intermediate school, which is part of the elementary school system in New Zealand, to secondary school. Our hope was that the algebraic thinking skills gained in intermediate school would not be lost when students proceeded to secondary school, where traditionally algebra has meant 'doing arithmetic with letters', which Mason et al. (2005, p. 309) contend, "has proved fruitless for countless generations".

We reasoned that students, who had developed an awareness of the generality in a range of numerical operational strategies through their participation in the Numeracy Project, would be able to extend their algebraic thinking to include the standard alphanumeric symbols of algebra. Students who could generalize in this way are at the 'Representational Level of Generalizing' (Rivera 2006). We argued that it was likely they would be able to capitalize on these generalizing skills as they progressed through secondary school.

A new test of algebraic thinking, devised specifically for this study, was trialled and revised before it was given in each of three consecutive years to students aged 12, 13 and 14 at years 8, 9 and 10 respectively. The intention was to gather data in each of the three years on students as they progressed from year 8 through 10. The test explored similar properties of operations to those explored by Schifter et

Section A

Jason uses a simple method to work out problems like $27 + 15$ and $34 + 19$ in his head.

Problem	Jason's calculation
$27 + 15$	$30 + 12 = 42$
$34 + 19$	$33 + 20 = 53$

1) Show how to use Jason's method to work out $298 + 57$
2) Show how to use Jason's method to work out $35.7 + 9.8$
3) Use Jason's method to work out what goes in the space: $58 + n = 60 +$ ______
4) Use Jason's method to work out what goes in the space: $9.9 + k = 10 +$ ______
5) Use Jason's method to work out what goes in the space: $a + b = (a + c) +$ ______

Fig. 2 Section A: Additive compensation in the algebraic thinking test

al. (2008) with somewhat younger students. It comprised four sections, one for each arithmetical operation. First, a solution was presented showing how a hypothetical student had solved two problems using a compensation strategy for the operation. The first item for each operation involved generalizing the demonstrated method with 3-digit whole numbers, chosen to discourage students' use of computational approaches while at the same time encouraging the use of the illustrated compensation approach. The second item required them to demonstrate the operation with decimal fractions. The inclusion of decimal fraction items arose partly out of a previous study (Irwin and Britt 2004) in which we noted that decimal fraction questions increased the level of complexity of the task. We also wanted to provide further opportunity to gauge the effect of extending numerical generalizing beyond whole numbers as is required when letters are used as generalized numbers. The third item asked students to show how the operation would work when one of the numbers in the particular item was represented by a letter as a generalized unknown, the fourth item asked them the same question when the item involved letters and a decimal fraction, and the final item asked them to identify missing letters in an algebraic identity representing the compensation operation. Figure 2 shows the additive compensation tasks of the algebraic thinking test.

Four intermediate schools and four secondary schools agreed to participate in this study. The intermediate schools were all participants in the Numeracy Project. The secondary schools were then chosen by finding out what secondary school the majority of students from the intermediate school moved to. Intermediate and secondary schools were paired in this way. Of the four pairs of schools two pairs of schools were on the outskirts of each of two large cities in New Zealand. Students who came, on average, from middle-income families attended three of the pairs of schools and students from higher income families attended one pair of schools. We asked schools to give the same test to all students in year 8 at Intermediate school in 2004, year 9 at secondary school in 2005, and year 10 in 2006. In addition, each secondary school was asked to give the test to all of their year 9 and 10 students in each of the three years. Our main focus was on the students who took the test for each of the three years, thus giving us three data points for each of the students. Irwin and Britt (2005b, 2006) reported on progress during the first two years of the study. There were 116 students who participated in all three years of the Longitudinal Study.

Teachers in all schools gave the algebraic thinking test to their students during the final term of the year in question. The tests were then returned to the authors for marking. The criterion for success was that an item was deemed correct only if the compensation method was used correctly for that operation, disregarding answers that were correct if students used a vertical algorithm but with no evidence of generalization of compensation.

Three analyses were carried out to evaluate the data gathered from the tests of algebraic thinking over three years. The first analysis explored the correlation of individual student's scores on the algebraic thinking test with the results of individual interviews (see the earlier section on algebraic thinking and the New Zealand Numeracy Project and its evaluation) on the three strategy scores of the Numeracy Project Diagnostic Interview at the end of year 9 (first year of secondary school). The second analysis looked at the algebraic thinking test scores for students who took the test in years 8, 9 and 10 in 2004, 2005 and 2006 respectively. The third analysis was both quantitative and qualitative, and took a closer look at what was happening in the one pair of schools where the mean scores on the algebraic thinking test were significantly superior to the other schools. This third analysis is not discussed here but is discussed in Britt and Irwin (2008).

In 2005, the scores that year 9 students gained on the algebraic thinking test and on the Numeracy Project Diagnostic Interview were correlated for the three secondary schools in the Secondary Numeracy Project. These students had been given the Numeracy Project diagnostic interview described earlier in the section on 'Algebraic thinking and the New Zealand Numeracy Project'. While scores for strategy stages in additive, multiplicative, and proportional thinking are reported individually in the diagnostic interview, we added the scores for the stages together, reasoning that students needed to have good ability in all skills to be a competent mathematics student in this age group.

The correlations between students' scores for these strategy stages on the diagnostic interview, and on the algebraic thinking test was 0.47 with significance < 0.01. Thus we were assured that individual interviews demonstrated that students who were flexible in their thinking about numerical problems as fostered by the Numeracy Project were more likely to be the same ones who could transfer from using numbers flexibly to using letters to express the generalizations than those who had not benefited from the Numeracy Project. The flexible thinkers we argued were capable of algebraic thinking.

A further analysis of the diagnostic strategy scales shows that students who were above the median on the Numeracy Project diagnostic test were those who demonstrated algebraic thinking on our test. This means that students who could demonstrate multiplicative and proportional reasoning as well as additive reasoning were likely to transfer this flexible numerical thinking to algebraic items that involve students expressing their generalizations algebraically.

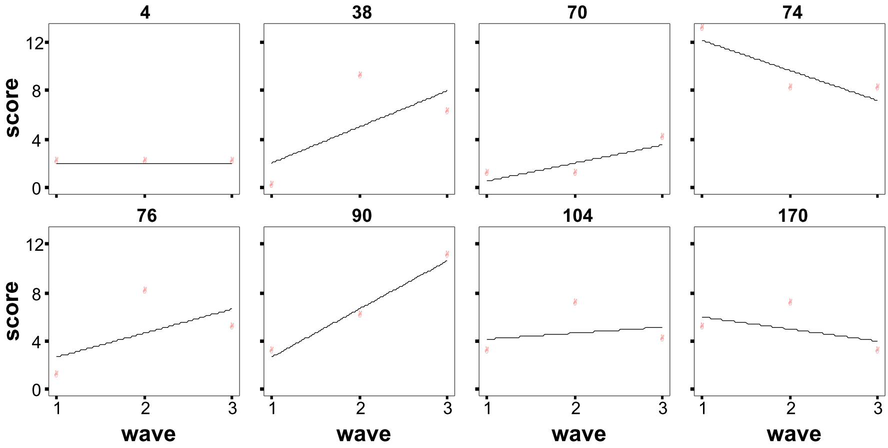

Fig. 3 Each panel represents a randomly selected student (ID at top of panel). Waves 1, 2, and 3 refer to Years 8, 9, and 10 respectively. The *data points* show a student's score at each wave, and the *straight lines* are least-squares fits

Table 3 Correlation coefficients among the students' scores over three years

		Year 8	Year 9	Year 10
Year 8	Pearson Correlation	1	.640*	.639*
Year 9	Pearson Correlation	.640*	1	.714*
Year 10	Pearson Correlation	.639*	.714*	1
	N	116	116	116

*significant at the 0.01 level (2-tailed)

We were particularly interested in the growth of algebraic thinking across the three years. This analysis was of the scores of the 116 students who took the algebraic thinking test on three occasions.

The students varied both in their initial attainment at the end of Year 8 and in their rate of development over the three-year period. An indication of this variability is shown in Fig. 3, which presents the results of a random selection of eight students. The figure shows that some students improved over the three years, some did not change, and some declined in performance. In order to take into account this variability among students over each year, a random coefficient analysis was undertaken. In this analysis, the coefficients for the intercept and slope of the regression are allowed to differ among the students. (Figure 3 illustrates examples of the different intercepts and slopes of the fitted line for each student.) Accommodating for this variation is especially desirable in any Longitudinal Study because of the interdependence among responses when the same person is measured on several occasions (Twisk 2003). Table 3 shows the correlation coefficients of the scores over the three years.

Figure 4 shows the average performance of all the students who attended a pair of schools over three years. With the exception of School Pair 4, the mean scores

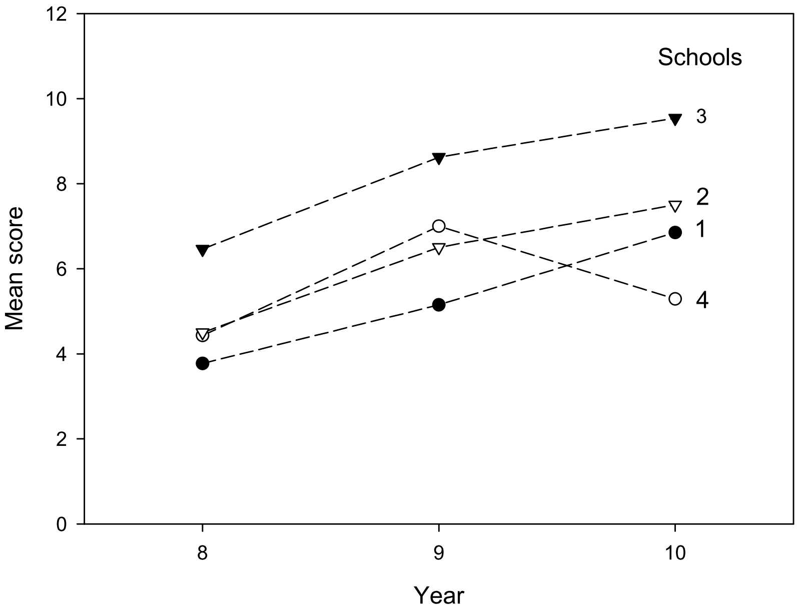

Fig. 4 Mean algebraic thinking test score of students attending each of the four pairs of schools over three years

of students in each of the schools improved over the three years. In examining this figure it should be borne in mind that the number of students from each pair of schools differed widely: 13 students in school-pair labelled "1", 14 students in the school-pair "2", 61 students in school-pair "3", and 28 students in school-pair "4". The low figure for students who were tested on three separate occasions can be accounted for in several ways. Some students at the intermediate schools chose to move from their intermediate school to a secondary school other than the nominated secondary school, some moved away from the school district, and others, particularly those from schools from lower socio-economic homes, had poor school attendance records.

Scores from intermediate school 3 were significantly different from intermediate schools 2 and 4 ($p < 0.05$) but not from intermediate school 1 ($p = 0.081$) because of the large variance in the small number of student scores at that school (t-test). The difference between schools when all three years were taken into consideration approached significance $F(3, 112) = 2.620$, ($p = 0.054$).

The random coefficients analysis of the data was undertaken by means of SPSS's mixed linear model. Two models were fitted. The simpler model estimated, without differentiating among the schools, the average score at Year 8 and the average rate of improvement over each subsequent year. The model's estimated score at Year 8 was 5.7 and the estimated improvement per year was 1.3 points. The heavy line labelled "all" in Fig. 5 shows this model.

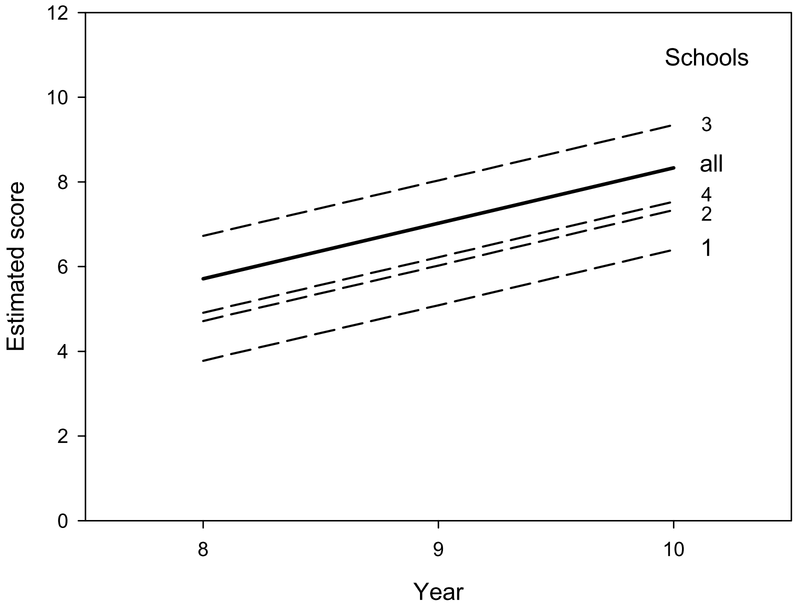

Fig. 5 Two best-fitting models: (**a**) the *heavy line*, labelled "*all*", shows the estimated score at Year 8 and the estimated improvement over three years of the average student, without differentiating among the schools; and (**b**) a model of the estimated score at Year 8 and the estimated improvement of the average student in each of the pairs of schools

The other model took into consideration the effect of attending different pairs of schools. This model provided a marginally better fit to the data than the simpler one. If this is taken to show a worthwhile improvement in the model's fit, then the resulting estimated initial score and improvement rate for students from each school are shown in Fig. 5 by the lines labelled "1", "2", "3", and "4". For both models, the rate of change (1.3 points per year) represented a significant improvement.

Discussion

In the study of students at the end of intermediate school (Irwin and Britt 2005a), we showed that students who had participated in the New Zealand Numeracy Project were more able than students who had not participated in the Numeracy Project to demonstrate an awareness of the underlying algebraic structure of the operational strategies they used to solve problems in arithmetic. We noted that this awareness of structure amounted to an awareness of generality and argued that such students were therefore engaging in algebraic thinking. We also claimed that, rather than using the literal symbols of algebra, these students who had no prior experience with such symbols, were thinking of the numbers themselves as variables. We referred to these as quasi-variables (Fujii and Stephens 2001) and subsequently argued that the arithmetic that students used to solve problems was rooted in algebra. That is, without algebra there could be no arithmetic (Hewitt 1998; Mason et al. 2005). In the Longitudinal Study (Britt and Irwin 2008), we extended this relationship between algebra and arithmetic by showing that students, who had developed algebraic thinking in multiplicative and proportional situations at intermediate school were capable of algebraic thinking that involved representing numerical generalities with the special symbols of algebra. We also showed that these algebraic thinking skills for such students continually improved as they progressed through the first year of secondary school. These analyses taken together suggest strong support for a positive relationship between success in the Numeracy Project and subsequent algebraic thinking in which students use the literal symbols of algebra to express algebraic generalizations.

The support for a link between the effect of the Numeracy Project and the growth of algebraic thinking skills as students progress from Year 8 through Year 10 is yet further strengthened by the analysis of the data shown in Fig. 4. While the numbers of students contributing from different schools to the sample for the Longitudinal Study differed widely, the two linear models (Fig. 5) that were fitted to the data provided complementary evidence for an overall improvement of performance over the three years. As a result of this analysis and the previous analyses of student performance, we contend that we have demonstrated the existence of a strong ongoing positive link between the effects of the Numeracy Project and the development of algebraic thinking that involves expressing generality with the standard algebraic symbols.

A Pathway for Algebraic Thinking

In New Zealand, it has been customary for students to begin formal algebra at secondary school. A typical algebra curriculum drawn from locally produced textbooks for Year 9 began by focusing on functional thinking activities such as patterns in growing match-stick designs, skill development with the syntax of operational algebra, and solving simple linear equations. The teaching was typically instructional rather than explorative with little or no attention given to generality. Students mostly found algebra taught in this manner confusing and therefore difficult.

The New Zealand Numeracy Project has led to a new perspective of algebra among many New Zealand teachers, particularly primary and intermediate-school teachers. The developing awareness that the seeking of generality in any mathematical task is essentially algebraic in nature has offered primary teachers, many of whom have a jaundiced view of algebra, a new emphasis in their teaching of mathematics. Careful consideration of the three studies detailed in this chapter together suggests an alternative pathway for teaching and learning beginning algebra or algebraic thinking. We argue that this pathway should provide opportunities for all students to work with several layers of awareness of generality in all areas of their mathematics curriculum prior to any formal introduction to algebra.

Such a pathway might firstly involve very young children engaging in activities of a proto-quantitative nature where counting is not needed and in which they describe in their own words generalizations such as for example, those involved in compensation thinking, "If $P_1 + P_2 = W$, then $(P_1 - k) + (P_2 + k) = W$". This is the same emphasis offered by Dougherty (2008). Attention might then turn to students developing an array of counting strategies before using those strategies and associated counting knowledge to help devise and broaden a range of operational strategies in arithmetic where generality may be expressed through the use of quasi-variables. Finally, students who have worked successfully with quasi-variables might be further challenged to use the symbols of algebra to represent and work with mathematical generalizations drawn from a range of numerical and measurement situations as well as figural representations (see Rivera 2006; Rivera and Becker 2008).

In New Zealand, first steps towards establishing a pathway for algebraic thinking have been taken during a recent redevelopment of national curricula. The changes in the New Zealand mathematics curriculum for students aged 5 to 13, where the formerly separate Number and Algebra strands have been combined to form a single Number and Algebra strand in the newly designated Mathematics and Statistics curriculum (New Zealand Ministry of Education 2007b) have arisen as a result of the overall success of the Numeracy Project initiative (see New Zealand Ministry of Education 2009b, http://www.nzmaths.co.nz/annual-evaluation-reports-and-compendium-papers) and of the research findings related to algebraic thinking that have been discussed earlier in this chapter.

As a consequence of this reconsideration of the role of algebra in the New Zealand mathematics curriculum, a number of classroom activities that encourage algebraic thinking have been devised and trialled in classroom settings. Figure 6 shows an example of a sequence of such tasks in which students, assessed at Stage 6 (See Table 2—Advanced Additive), are asked to look for and express generality in words, with quasi-variables where numbers act as if they are variables, and with the literal symbols of algebra.

Students working on these tasks are asked to use three semiotic systems to express layers of generality: firstly with numbers as quasi-variables, then with words and finally with the literal symbols of algebra. For example, the design in Fig. *A2* might be expressed numerically as 2 lots of 47 black circles plus 2 lots of 45 white circles for the coaster with 47 circles on each edge and in words as, 'The total number of circles is two lots of the number of circles on an edge (black circles) side plus two lots of two fewer than the number of circles on a side (white circles). The algebraic expression is then $2 \times x + 2 \times (x - 2)$ which is more usually written as $2x + 2(x - 2)$. The algebraic expressions for the designs in Fig. *A3* and Fig. *A4* are $4 + 4(x - 2)$ and $4(x - 1)$ respectively.

As a follow-up to the tasks in Coaster Design, students themselves figure out how to reduce the different algebraic expressions to the same form, likely here to be $4x - 4$, interpreted as, 'four times the number of circles on an edge less one circle that has already been counted twice for each corner'. Students might also be encouraged to devise algebraic expressions to represent a range of challenging figural representations. For example, they might look for generality in coaster designs similar to the square-shaped designs in Fig. 6 but with circles along the edges of regular

Task A: Coaster Designs

1. The drink's coaster in Figure *A1* has 5 circles painted on each edge. The coaster in Figure *A2* shows a different design

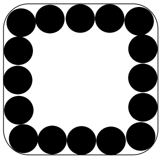

Figure *A1*

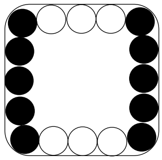

Figure *A2*

Using the design in Figure A2, work out the total number of circles for a coaster with 100, 47, 139 circles on each edge. Explain your working in words as well as with numbers.

2. Work out the total number of circles for a coaster with 100, 47, 139 circles on each edge. Explain your working in words as well as with numbers.

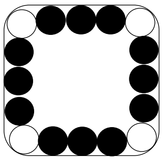

Figure *A3*

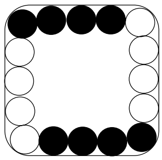

Figure *A4*

3. Suppose the coaster had *x* circles on each edge (*x* stands for any number). For each design shown in the figures above, write an algebraic expression for the total number of circles in the design. Explain your algebraic expression in words.

Fig. 6 Expressing generality with and without the literal symbols of algebra

triangles, pentagons, hexagons, and even n-gons. Rivera (2006) notes that students who identify generality in figural representation and are able to express that generality algebraically may be classified as predominantly Figural Generalizers working at the Representational Level of Generalizing.

While figural representations such as those shown in Fig. 6 can offer a rich source of challenging tasks requiring students to seek and express generality, the tasks in Fig. 7 also provide similar opportunities. These tasks do not need students to be figural generalizers as proposed by Rivera (2006), but they nevertheless provide opportunity to learn the structure and syntax of algebraic generality where letters stand for generalized unknown numbers. The tasks in Fig. 7 comprise separate learning progressions that deepen awareness of generality while at the same time assist students to hone their mental skills in working with compensation strategies in addition and subtraction.

Each item in the separate progression illustrates the structure of a group of similar items. The groups of items so formed are sequenced in a manner to draw attention to the generality inherent in the particular strategy. Students express such generality with quasi-variables and with algebraic symbols. Task *B1* focuses on additive compensation operations. In the first item, $7 + \square = 9 + \square$, any number can go in the first box but a number two less than the first chosen number must go in the second box

Task B1: Compensation in addition

$7 + \square = 9 + \square$

$52 + \square = 54 + \square$

$1.9 + \square = 2 + \square$

$5 + n = 2 + \square$

$\square + 28 = n + 30$

$50 + (n + 2) = \square + n$

$x + 9 = (x - 3) + \square$

Task B2: Compensation in subtraction

$55 - 34 = 56 - \square$

$40 - \square = 38 - 21$

$\square - 5 = 43 - 3$

$n - 67 = (\quad) - 70$

$40 - p = 50 - (\quad)$

$14 - n = 15 - (\quad)$

$20 - (\quad) = 15 - m$

Fig. 7 Generalizing additive and subtractive compensation strategies using quasi-variables and algebraic symbols

to preserve the equivalence. Task *B2* involves compensation in subtraction. Thus in $55 - 34 = 56 - \square$, 55 has been increased by 1 so the number in the box must also increased by one in order to preserve the difference represented by $55 - 34$. Alternatively, in $20 - (\ ) = 15 - m$, 20 has been decreased by 5. So, in order to maintain the difference between the pairs of values on either side of the equation the number missing between the parentheses must be 5 more than m, that is $m + 5$.

However, in order for all students to achieve greatest benefit from working with activities that encourage algebraic thinking at all levels teachers themselves will need to also develop their own algebraic thinking skills. This could be achieved through the adoption in teacher development programs of an "algebrafication" strategy, similar to that proposed by Kaput and Blanton (2001). In such programs, teachers might continually seek out generality in all aspects of mathematics and find helpful ways, including using the symbols of algebra, to express generality. They might also be encouraged to devise and apply wherever possible, a range of sensible flexible mental operational strategies to carry out everyday tasks in arithmetic that embody algebraic thinking.

New Zealand teachers and their students have had a growing involvement in the Numeracy Project since 1999. The initial experimental nature of the project together with its strong evaluative component and accompanying teacher professional development, has led to improvements in skill level and confidence in teaching numeracy. Increasingly, teachers are developing awareness that their work entails considerably more than helping students get answers to problems involving numerical situations. They are supported by, the Numeracy Project resource material and a curriculum that encourages the concept of algebra within arithmetic. This national "experiment" has led to professional development for most elementary school teachers and for an increasing number of secondary school teachers. The challenge now is to enable all teachers to see the benefit of an approach that includes seeing algebra within arithmetic and the development of school teaching programs in mathematics that reflect the notion of seeking and expressing generality that forms the basis of the Pathway for algebraic thinking described in this chapter.

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Examining Students' Algebraic Thinking in a Curricular Context: A Longitudinal Study

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Abstract This chapter highlights findings from the LieCal Project, a longitudinal project in which we investigated the effects of a *Standards*-based middle school mathematics curriculum (CMP) on students' algebraic development and compared them to the effects of other middle school mathematics curricula (non-CMP). We found that the CMP curriculum takes a functional approach to the teaching of algebra while non-CMP curricula take a structural approach. The teachers who used the CMP curriculum emphasized conceptual understanding more than did those who used the non-CMP curricula. On the other hand, the teachers who used non-CMP curricula emphasized procedural knowledge more than did those who used the CMP curriculum. When we examined the development of students' algebraic thinking related to representing situations, equation solving, and making generalizations, we found that CMP students had a significantly higher growth rate on representing-situations tasks than did non-CMP students, but both CMP and non-CMP students

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had an almost identical growth in their ability to solve equations. We also found that CMP students demonstrated greater generalization abilities than did non-CMP students over the three middle school years.

Algebra readiness has been characterized as the most important "gatekeeper" in school mathematics (Cai and Knuth 2005). Given its gatekeeper role as well as growing concerns about students' inadequate preparation in algebra, algebra curricula and instruction have become focal points of mathematics education research (e.g., Bednarz et al. 1996; Carpenter et al. 2003; Katz 2007). In particular, researchers have tried to understand the nature of students' algebraic thinking as well as the role that different types of curricula play in its development (Katz 2007; Kieran 2007; National Research Council 2004). The purpose of this chapter is to compare how two different types of curricula affect the development of students' algebraic thinking over the three middle school years (grades 6–8).

Reviews of research on the teaching and learning of algebra clearly show the need to study the development of students' algebraic thinking over time (Jones et al. 2002; Kieran 2007). NCTM (2000) suggests that algebraic thinking should be developed across all grade levels, but we know very little about how we should characterize the growth of students' algebraic thinking. For example, in their review, Jones et al. (2002) explicitly called for future research to build a general model that characterizes the growth of students' algebraic thinking over time.

Although research in cognition has shown that students' experiences out of school have a substantial effect on their learning and problem solving (e.g., Lave 1988; Resnick 1987), classroom instruction is still considered a central component in the development and the organization of students' thinking and learning (Cai 2004; Detterman 1993; Rogoff and Chavajay 1995; Wozniak and Fischer 1993). Bruner (1990/1998) proposed that it is culture and education, not biology, that shape human life and the human mind. Similarly, Gardner (1991) argued that once a child reaches age six or seven, the influence of culture and schooling "...has become so pervasive that one has difficulty envisioning what development could be like in the absence of such cultural supports and constraints" (p. 195). Because classroom instruction plays such a central role in the development of students' thinking, we situated our investigation of the development of middle school students' algebraic thinking within a context of comparative instruction using two different types of curricula—*Standards*-based and traditional. By doing so, we have been able to examine how the use of different types of curricula affect the development of students' algebraic thinking over the middle school years.

Standards-Based and Traditional Curricula in the United States

In the late 1980s and early 1990s, the National Council of Teachers of Mathematics (NCTM) published its first round of *Standards* documents, which provided recommendations for reforming and improving K-12 school mathematics. In the

Standards and related documents, the discussions of goals for mathematics education emphasize the importance of thinking, understanding, reasoning, and problem solving, with an emphasis on connections, applications, and communication (e.g., NCTM 1989, 1991; National Research Council 1989). This view stands in contrast to the more conventional view of mathematics education, which involves the memorization and recitation of decontextualized facts, rules, and procedures, with an emphasis on the application of well-rehearsed procedures to solve routine problems.

With extensive support from the U.S. National Science Foundation, a number of school mathematics curricula aimed to align with the recommendations in the *Standards* were developed and field-tested in the United States (see Senk and Thompson 2003 for details of these NSF-funded curricula). The Connected Mathematics Program (CMP) is one of the so-called *Standards*-based school mathematics curricula developed with the support of the National Science Foundation. The CMP curriculum is a complete middle-school mathematics program that was identified as an exemplar by the U.S. Department of Education. The stated intent of CMP is to build students' understanding of major ideas in number, algebra, geometry, measurement, data analysis, and probability through explorations of real-world situations and problems (Lappan et al. 2002). Because NSF-funded curricula like CMP not only look very different from commercially developed mathematics curricula, but also claim to have different learning goals, they provide an interesting context for examining the impact of curriculum on the development of students' algebraic thinking. By situating our examination of the development of algebraic thinking in a curricular context, we can investigate the role that curriculum plays in students' learning of mathematics in general and in their acquisition of algebraic concepts in particular (NCTM 1989; National Research Council 2004; RAND Mathematics Study Panel 2003; Senk and Thompson 2003; Usiskin 1999).

LieCal Project

This chapter draws on findings from our NSF-funded LieCal Project (Longitudinal Investigation of the Effect of Curriculum on Algebra Learning). The LieCal Project was designed to longitudinally compare the effects of the Connected Mathematics Program (CMP) to the effects of more traditional middle school curricula (hereafter called non-CMP curricula) on students' learning of algebra. The project was conducted in 14 middle schools in an urban school district serving a diverse student population. When the project began, 27 of the 51 middle schools in the district had adopted the CMP curriculum, and the remaining 24 middle schools had adopted more traditional curricula. Seven CMP schools were randomly selected from the 27 schools that had adopted the CMP curriculum. After the seven CMP schools were selected, seven non-CMP schools were chosen based on comparable demographics. According to district ratings, the CMP and non-CMP groups of schools each comprised two schools that were above average in overall educational performance, three schools that were average, and two schools that were below average. About

700 CMP students in 25 classes and 600 non-CMP students in 22 classes participated in the study when they were 6th graders. We followed these 1,300 students as they progressed from grade 6 to 8. Approximately 85% of the participants were minority students: 64% African American, 16% Hispanic, 4% Asian, and 1% Native American. Male and female students were about evenly distributed.

By comparing longitudinally the effects of the CMP curriculum to the effects of other middle-school mathematics curricula on students' learning of algebra, the LieCal Project has been able to provide: (1) A profile of the intended treatment of algebra in the CMP curriculum with a contrasting profile of the intended treatment of algebra in the non-CMP curricula; (2) a profile of classroom experiences that CMP students and teachers have, with a contrasting profile of experiences in non-CMP classrooms; and (3) a profile of student performance resulting from the use of the CMP curriculum, with a contrasting profile of student performance resulting from the use of non-CMP curricula. Accordingly, the project was designed to answer three sets of research questions: (1) What are the similarities and differences between the intended treatment of algebra in the CMP curriculum and in the non-CMP curricula? (2) What are key features of the CMP and non-CMP experience for students and teachers, and how might these features explain performance differences between CMP and non-CMP students? and (3) What are the similarities and differences in performance between CMP students and a comparable group of non-CMP students on tasks measuring a broad spectrum of mathematical thinking and reasoning skills, with a focus on algebra?

In the next two sections, we first highlight the differences between the CMP and non-CMP curricula and then the differences between classroom instruction using CMP and non-CMP curricula. The goal of these two sections is to provide a curricular context in which to examine and understand the development of students' algebraic thinking. In the following two sections, we describe the methods we used to investigate the development of students' algebraic thinking and then present our findings related to the development of students' algebraic thinking.

Highlights of the Differences between CMP and Non-CMP Curricula

We conducted detailed analyses of CMP and one of the non-CMP curricula,[1] Glencoe *Mathematics: Concepts and Applications* (Bailey et al. 2006a, 2006b, 2006c), and found significant differences between them (Cai et al. 2010c; Nie et al. 2009). Overall, our research revealed that the CMP curriculum takes a functional approach to the teaching of algebra, and the non-CMP curriculum takes a structural approach. The functional approach emphasizes the important ideas of change and variation

[1] In this section, we only discuss the differences between CMP and one of the non-CMP curricula (Glencoe Mathematics). Although there are differences among the non-CMP curricula, the differences between the CMP and each of the non-CMP curricula are similar.

in situations and contexts. It also emphasizes the representation of relationships between variables. The structural approach, on the other hand, avoids contextual problems in order to concentrate on developing the abilities to generalize, work abstractly with symbols, and follow procedures in a systematic way (Cai et al. 2010c). In this section, we highlight specific differences in the ways that the CMP curriculum and the non-CMP curriculum (1) define variables, (2) define equation solving, (3) introduce equation solving, and (4) use mathematical problems to develop algebraic thinking. We focused on these four aspects in this chapter because they are fundamental to algebra learning.

Defining Variables

Because of the importance of variables in algebra, and in order to appreciate the differences between the CMP and non-CMP curricula, it is necessary to understand how the CMP and non-CMP curricula introduce variable ideas. The learning goals of the CMP curriculum characterize variables as quantities used to represent relationships. In contrast, learning goals in the non-CMP curriculum characterize variables as placeholders or unknowns. The CMP curriculum does not formally define variable until 7^{th} grade. However, the CMP's definition of variable as a quantity rather than a symbol makes it convenient to use variables informally to describe relationships long before formally introducing the concept of variable in 7^{th} grade. Once CMP defines variables as quantities that change or vary, it uses them to represent relationships. The non-CMP curriculum formally defines a variable in 6^{th} grade as a symbol (or letter) used to represent a number. It treats variables predominantly as placeholders by using them to represent unknowns in expressions and equations.

Defining Equations

With its emphasis on relationships, CMP clearly approaches the concept of *variable* functionally. On the other hand, the non-CMP curriculum's focus on variable as a symbol points toward their structural approach. It is not surprising, therefore, that the concept of *equation* is defined functionally in CMP, but structurally in the non-CMP curriculum.

In CMP, the functional approach to equation is a natural extension of its development of the concept of variable as a changeable quantity used to represent relationships. At first, CMP expresses relationships between variables with graphs and tables of real-world quantities rather than with algebraic equations. Later, when CMP introduces equations, the emphasis is on using them to describe real-world situations. Rather than seeing equations simply as objects to manipulate, students learn that equations often describe relationships between varying quantities that arise from meaningful, contextualized situations (Bednarz et al. 1996).

In the non-CMP curriculum, the definition of variable as a symbol develops naturally into to the use of "naked" equations and puts an emphasis on procedures for solving equations. These are all hallmarks of a structural focus. For example, non-CMP curriculum defines an equation as "...a sentence that contains an equals sign, =" through examples like $2 + x = 9$, $4 = k - 6$, and $5 - m = 4$. Students are told that the way to solve an equation is to replace the variable with a value that results in a true sentence.

Introducing Equation Solving

CMP and the non-CMP curriculum use functional and structural approaches, respectively, to introduce equation solving. These are consistent with the approaches they use to define equations. In the CMP curriculum, equation solving is introduced within the context of discussing linear relationships. The initial treatment of equation solving does not involve symbolic manipulation as found in most conventional curricula. Instead, the CMP curriculum introduces students to linear equation solving by making visual sense of what it means to find a solution using a graph. Its premise is that a linear equation in one variable is, in essence, a specific instance of a corresponding linear relationship (equation) in two variables. It relies heavily on the context in which the equation itself is situated and on the use of a graphing calculator.

After CMP introduces equation solving graphically, the symbolic method of solving linear equations is finally broached. It is introduced within a single contextualized example, where each of the steps in the equation-solving process is accompanied by a narrative that demonstrates the connection between what is happening in the procedure and in the real-life situation. In this way, CMP justifies the equation-solving manipulations through contextual sense-making of the symbolic method. That is, CMP uses real-life contexts to help students understand the meaning of each step of the symbolic method of equation solving, including why inverse operations are used, as shown in Table 1.

In the non-CMP curriculum, contextual sense-making is not used to justify the equation-solving steps as it is in the CMP curriculum. Rather, the non-CMP curriculum first introduces equation solving as the process of finding a number to make an equation a true statement. Specifically, *solving* an equation is described as replacing a variable with a value (called the *solution*) that makes the sentence true. Equation solving is introduced in the non-CMP curriculum symbolically by using the additive property of equality (equality is maintained if the same quantity is added to or subtracted from both sides of an equation) and the multiplicative property of equality (equality is maintained if the same non-zero quantity is multiplied by or divided into both sides of an equation).

In 6^{th} grade, the non-CMP curriculum (Bailey et al. 2006a) formally introduces equation solving with inverse operations by way of an activity that uses a cup to stand for an unknown. The appropriate number of cups and counters used as manipulatives in the activity are initially positioned to exactly represent the equation's

Table 1 An example of equation solving in CMP (Lappan et al. 2002b, p. 55)

The Unlimited Store allows any customer who buys merchandise costing over $30 to pay on the installment plan. The customer pays $30 down and then pays $15 a month until the item is paid for. Suppose you buy a $195 CD-ROM drive from the Unlimited Store on an installment plan, How many months will it take you to pay for the drive? Describe how you found your answer.

Thinking	Manipulating the Symbol
"I want to buy a CD-ROM drive that costs $195. To pay for the drive on the installment plan, I must pay $30 down and $15 a month."	$195 = 30 + 15N$
"After I pay the $30 down payment, I can subtract this from the cost. To keep the sides of the equation equal, I must subtract 30 from both sides"	$195 - 30 = 30 - 30 + 15N$
"I now owe $165, which I will pay in monthly installments of $15."	$165 = 15N$
"I need to separate $165 into payments of $15. This means I need to divide it by 15. To keep the sides of the equation equal, I must divide both sides by 15."	$\frac{165}{15} = \frac{15N}{15}$
"There are 11 groups of $15 in $165, so it will take 11 months."	$11 = N$

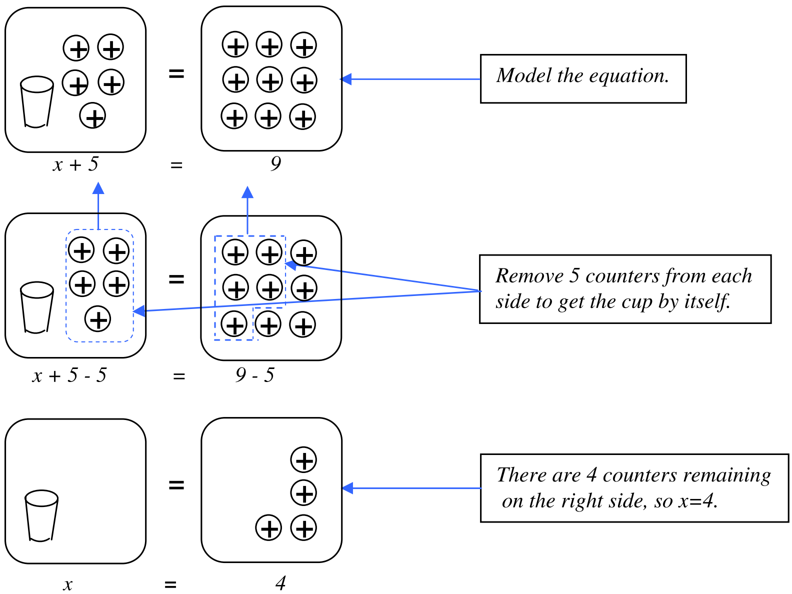

Fig. 1 Introduction of equation solving in the non-CMP curriculum

symbols. They are then used to illustrate each step of the symbolic manipulations (see Fig. 1).

Using manipulatives as described above is referred to as "Method 1" and is typically shown adjacent to an example illustrating the corresponding solution using the strictly symbolic "Method 2." In this way, the non-CMP curriculum illustrates

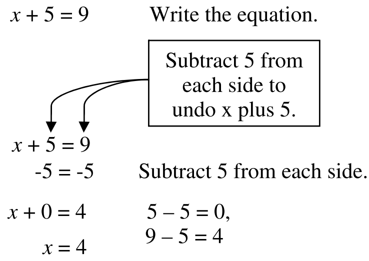

Fig. 2 Symbolic representation of solving an equation in the non-CMP curriculum

Table 2 Percentages of various tasks in CMP and non-CMP curricula

	Memorization	Procedures without Connections	Procedures with Connections	Doing Mathematics
CMP ($n = 920$)	0.43	27.93	61.52	10.11
Non-CMP ($n = 2391$)	4.6	74.57	18.24	2.59

how each manipulative step is comparable to a symbolic step in a solution based on the algebraic properties of equality, which is shown through vertical work. The following is an example of Method 2 showing how to solve a one-step equation (see Fig. 2).

Using Mathematical Problems

The extent of the differences between the CMP and non-CMP curricula can also be highlighted through an analysis of mathematical problems. Using a scheme developed by Stein et al. (1996), we classified the mathematical tasks in the CMP curriculum and non-CMP curriculum (Bailey et al. 2006a, 2006b, 2006c) into four increasingly demanding categories of cognition: memorization, procedures without connections, procedures with connections, and doing mathematics. As Table 2 shows, significantly more tasks in the CMP curriculum than in the non-CMP curriculum are higher-level tasks (procedures with connections and doing mathematics) ($\chi^2(3, N = 3311) = 759.52$, $p < .0001$).

We further analyzed the problems in the CMP and non-CMP curricula that involve linear equations by classifying them into three categories:

(1) One equation with one variable (1equ1va)—e.g., $2x + 3 = 5$;
(2) One equation with two variables (1equ2va)—e.g., $y = 6x + 7$;
(3) Two equations with two variables (2equ2va)—e.g., the system of equations $y = 2x + 1$ and $y = 8x + 9$.

Figure 3 shows the percentage distribution of the problems involving linear equations in the two curricula. These two distributions are significantly different

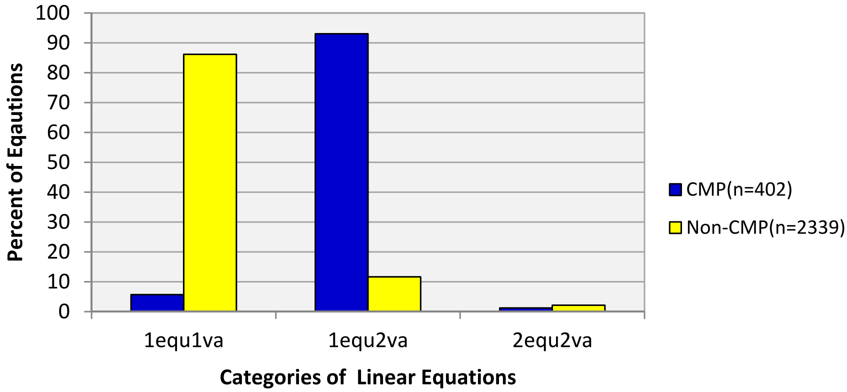

Fig. 3 Percentage distribution of problems involving linear equations in CMP curriculum and the non-CMP curriculum

($\chi^2(2, N = 2741) = 1262.0$, $p < .0001$). The CMP curriculum includes a significantly greater percentage of "one equation with two variables" problems than does the non-CMP curriculum ($z = 35.49$, $p < .0001$). On the other hand, the non-CMP curriculum includes a significantly greater percentage of "one equation with one variable" problems than does the CMP curriculum ($z = 34.145$, $p < .0001$). These results resonate with the findings that we reported above. Namely, that the CMP curriculum emphasizes an understanding of the relationships between the variables of equations, rather than an acquisition of the skills needed to solve them. In fact, of the 402 equation-related problems in the CMP curriculum, only 33 of them (about 8% of the linear equation solving problems) involve decontexualized symbolic manipulation of equation solving. However, the non-CMP curriculum includes 1,550 problems involving decontexualized symbolic manipulation of equations (nearly 70% of the linear equation solving problems in the curriculum).

Highlights of the Differences between CMP and Non-CMP Classroom Instruction

We conducted 620 detailed lesson observations of CMP and non-CMP lessons over a three-year period. Approximately half of the observations were of teachers using the CMP curriculum. The other half were observations of teachers using non-CMP curricula. Two retired mathematics teachers conducted and coded all the observations. The observers received extensive training that included frequent checks for reliability and validity throughout the three years. Over the course of the 6^{th}-grade year, for example, we checked the reliability of the observers' coding three times. The reliability achieved during the three sessions averaged 79% perfect agreement, using the criterion that the observers' coded responses were considered equivalent only if they were identical (i.e., perfect match). The reliability averaged 95% using the following criteria: (a) If an item or sub-item was "scored" using an ordinal scale, then the observers' coded responses were considered equivalent if they differed by at most one unit; (b) If an item or sub-item (e.g. representation) was "scored" by choosing from a list of alternatives all the words/phrases that characterize it, then the observers' coded responses were considered equivalent if they had at least one choice in common (e.g. symbolic and pictorial vs. pictorial). We reached similar high reliabilities for the 7^{th} and 8^{th} grade observations.

Each class of LieCal students was observed four times, during two consecutive lessons in the fall and two in the spring. The observers recorded extensive information about each lesson in a 28-page project-developed observation instrument. During each observation, the observer made a minute-by-minute record of the lesson on specially designed form. This record was used later to code the lesson. The coding section of the LieCal observation instrument has three main components: (1) the structure of the lesson and use of materials, (2) the nature of the instruction, and (3) the analysis of the mathematical tasks used in the lesson (see Moyer et al. 2010 for details about the lessons observed and observation instrument used). The analyses revealed striking differences between classroom instruction using the CMP and non-CMP curricula (Cai et al. 2009; Moyer et al. 2009). In this section, we discuss the differences related to two important instructional variables: (1) the level of conceptual and procedural emphases in the lessons, and (2) the cognitive demand of the instructional tasks implemented.

Conceptual and Procedural Emphases

The second component of the coding section includes twenty-one 5-point Likert scale questions that the observers used to rate the nature of instruction in a lesson. Of the 21 questions, four of them are designed to assess the extent to which a teacher's lesson has a conceptual emphasis. Another four of the questions are designed to determine the extent to which a teacher's lesson has a procedural emphasis. Factor analysis of the LieCal observation data confirmed that the four procedural-emphasis questions loaded on a single factor, as did the four conceptual-emphasis questions.

There was a significant difference across grade levels among the levels of conceptual emphasis in the CMP and non-CMP instruction ($F = 53.43$, $p < 0.001$). The overall (grades 6–8) mean of the summated ratings of conceptual emphasis in CMP classrooms was 13.41, while the overall mean of the summated ratings of conceptual emphasis in non-CMP classrooms was 10.06. The summated ratings of conceptual emphasis were obtained by adding the ratings on the four items of the conceptual-emphasis factor in the classroom observation instrument. That implies that the mean rating on the conceptual-emphasis items was 3.35 ($\frac{13.41}{4}$) for CMP instruction and 2.52 ($\frac{10.06}{4}$) for non-CMP instruction. That is, CMP instruction was rated 0.40 points above the midpoint, while non-CMP instruction was rated 0.5 points below the midpoint. The bottom line is that CMP instruction was rated an average of about 4/5 of a point higher (out of 5) on each conceptual emphasis

Table 3 Emphasis on conceptual understanding

	Grade 6* (CMP: $n = 96$; non-CMP: $n = 87$)	**Grade 7*** (CMP: $n = 101$; non-CMP: $n = 95$)	**Grade 8*** (CMP: $n = 108$; non-CMP: $n = 92$)	**Overall*** (CMP: $n = 305$; non-CMP: $n = 274$)
CMP	**14.51** (3.70)[a]	**12.52** (3.70)	**13.27** (3.65)	**13.41** (3.76)
Non-CMP	**9.44** (2.50)	**10.11** (2.31)	**10.61** (2.73)	**10.06** (2.55)

*$p < .001$

[a]numbers in parenthesis are standard deviations

Table 4 Emphasis on procedural knowledge

	Grade 6* (CMP: $n = 96$; non-CMP: $n = 87$)	**Grade 7*** (CMP: $n = 101$; non-CMP: $n = 95$)	**Grade 8*** (CMP: $n = 108$; non-CMP: $n = 92$)	**Overall*** (CMP: $n = 305$; non-CMP: $n = 274$)
CMP	**11.67** (3.03)[a]	**11.70** (3.05)	**11.48** (3.44)	**11.61** (3.18)
Non-CMP	**13.77** (3.58)	**14.24** (3.32)	**15.41** (3.27)	**14.49** (3.44)

*$p < .001$

[a]numbers in parenthesis are standard deviations

item than was non-CMP instruction, which is a significant difference ($t = 11.44$; $p < 0.001$).

On the other hand, non-CMP lessons had significantly more emphasis on the procedural aspects of learning than did the CMP lessons. The procedural-emphasis ratings for the non-CMP lessons were significantly higher than were the procedural-emphasis ratings for the CMP lessons ($F = 37.77$, $p < 0.001$). Also, the overall (grades 6–8) mean of summated ratings of procedural emphasis in non-CMP classrooms (14.49) was significantly greater than was the overall mean of the summated ratings of procedural emphasis in CMP classrooms, which was 1.61 ($t = -9.43$, $p < 0.001$). The summated ratings for the procedural emphasis were obtained by adding the ratings on the four items of the procedural-emphasis factor. That implies that the mean rating on the procedural emphasis items was 3.62 ($\frac{14.49}{4}$) for non-CMP instruction and 2.91 ($\frac{11.61}{4}$) for non-CMP instruction. On average, non-CMP instruction was rated about 7/10 of a point higher (out of 5) on each procedural emphasis item than was CMP instruction, which is a significant difference.

Instructional Tasks

Using a scheme developed by Stein et al. (1996), we also classified the instructional tasks implemented in CMP and non-CMP classrooms into four increasingly demanding categories of cognition: memorization, procedures without connections,

Table 5 Percentages of tasks with different levels of cognitive demand in CMP and non-CMP classrooms

	Memorization	Procedure Without Connections	Procedure With Connections	Doing Mathematics
CMP ($n = 623$)	2	53	34	11
Non-CMP ($n = 695$)	11	79	9	1

Table 6 Percentages of tasks with high cognitive demand in each grade level

	6^{th} Grade	7^{th} Grade	8^{th} Grade	Overall
CMP	62	28	42	45
Non-CMP	12	6	13	10

Note: High cognitive demand tasks refer to tasks involving procedures with connection or doing mathematics

procedures with connections, and doing mathematics. Table 5 illustrates the percentage distributions of the cognitive demand of the instructional tasks implemented (or enacted) in CMP and non-CMP classrooms. A chi-square test shows that the percentage distributions in CMP and non-CMP classrooms are significantly different ($\chi^2(3, N = 1318) = 219.45$, $p < .0001$). The difference confirms that a larger percentage of high cognitive demand tasks (procedures with connection or doing mathematics) was implemented in CMP classrooms than was implemented in non-CMP classrooms ($z = 14.12$, $p < .001$). On the other hand, a larger percentage of low cognitive demand tasks (procedures without connection or memorization) were implemented in non-CMP classrooms than was implemented in CMP classrooms (Cai et al. 2010a, 2010b).

Table 6 shows the percentage of high cognitive demand tasks implemented in the observed lessons at each grade level. Not only did CMP teachers implement a significantly higher percentage of cognitively demanding tasks than did non-CMP teachers across the three grades, but also within each grade (z values range from 6.06–11.28 across the three grade levels, $p < .001$).

Over 50% of the CMP lessons implemented at least one high level task (involving either procedures with connections or doing mathematics), but only 19% of the non-CMP lessons did so ($z = 8.91$, $p < .001$). Nearly 80% of the non-CMP lessons implemented low-level tasks involving procedures without connections, which is significantly greater than that of the CMP lessons (45%) ($z = 8.13$, $p < .001$).

Students' Development of Algebraic Thinking: Methodological Considerations

Given the significant differences between the CMP and the non-CMP curricula themselves, as well as the resulting instruction, how does the development of algebraic thinking differ between students in CMP and non-CMP classrooms? Before we present results that address this question, we first discuss the methodological consideration for examining students' development of algebraic thinking.

The Focus of Algebraic Thinking

Middle school algebra lays the foundation for the acquisition of tools for analyzing quantitative relationships, for solving problems, and for stating and proving generalizations (Bednarz et al. 1996; Carpenter et al. 2003; Kaput 1999; Mathematical Sciences Education Board 1998; RAND Mathematics Study Panel 2003). There are many important features of algebraic thinking (e.g., Cuoco et al. 1996; Kieran and Chalouh 1993; Mason 1996; NCTM 1989; Zazkis and Liljedahl 2002). In this chapter, we focus on the following three components of algebraic knowledge: representing situations, solving equations, and making generalizations. Mastering these three components requires the acquisition of the fundamental habits of minds (Cuoco et al. 1996) involved in thinking algebraically (Sfard 1995). Not only are these three components commonly accepted as important, they are also listed in the National Assessment of Educational Progress's Framework for 8th graders in the United States (NAEP 2006).

Representing Situations One of the important aspects of algebraic thinking is to interpret and represent quantitative situations (Mayer 1987; Kieran 1996). To solve word problems algebraically, students first need to generate equations to represent the quantitative relationships involved. Quantitative situations usually involve additive propositions, relational propositions, or both. For example, "Jake and Tom have 30 marbles altogether" is an additive proposition. "One pound of shrimp costs $3.50 more than one pound of fish" is a relational proposition. Researchers (e.g., Cocking and Mestre 1988) have found that students have difficulty interpreting and representing quantitative relationships, especially those that involve relational propositions.

Equation Solving Being able to solve equations is a basic and important aspect of algebraic thinking. In fact, when the topic of algebra in school mathematics is brought up, the first thing that usually comes to people's minds is equation solving. Historically, equation solving has played a central role in the development of other aspects of mathematics, and in solving real-life problems. Even though there has been a major shift in the landscape of school mathematics in recent years (Chazan 2008), learning to solve equations is still an essential element in the study of algebra (Mathematical Sciences Education Board 1998).

Making Generalizations Making generalizations from patterns is at the heart of mathematical thinking in general and of algebraic thinking in particular (Lee 1996; Steen 1988). School mathematics has increasingly emphasized the use of pattern to develop students' ability to make and express generalizations (NCTM 1989; Orton and Orton 1999).

Tasks and Data Analysis

In this section, we present findings about the development of students' algebraic thinking. These findings are based on data that we collected in the LieCal Project. In the LieCal Project, we used multiple measures to examine the effect of curriculum on students' learning. Besides using the state test, we developed 32 multiple-choice items and 13 open-ended tasks to assess students' Learning (see Cai et al. 2010a, 2010b for details). Of the 32 multiple-choice items, six of them were representing-situations tasks and another six were equation-solving tasks. The open-ended tasks required the students to provide answers and explain how they got them. These items and tasks were administered to 1,300 middle school students four times (fall 2005, spring 2006, spring 2007, and spring 2008).

Below, we report the results of the six representing-situations items and the six equation-solving items. From one testing administration to another, two of the six representing-situations tasks were identical, and the other four were parallel. The design was similar for the six equation-solving tasks. We also report the results on one of the open-ended tasks. The open-ended task was a making-generalizations task that was common across all forms and administrations. Appendix shows sample items.

For representing-situations tasks and equation-solving tasks, we mainly used scaled scores in our analysis. A scaled score is a generic term for a mathematically transformed student raw score on an assessment. Even though we used parallel items, it is still possible that students responded to the parallel items differently than they did to the original items. Parallel items were created through piloting and expert judgment. Using scaled scores, assessment results can be placed on the same scale even though students responded to different (but parallel) tasks at different times. In particular, we used the two identical representing-situations items to scale students' performance on the representing situations, and used the two identical equation-solving items to scale students' performance on equation solving. Since the scaled scores estimated the students' ability to represent situations or solve equations as a whole, the scaled scores could not be used in an item analysis. When an item analysis was needed, we used raw scores.

For the making-generalizations task, we conducted a qualitative analysis that captured the correctness of students' answers and the kinds of strategies they employed. When we analyzed the students' solution strategies, we paid particular attention to their use of concrete or abstract strategies.

Findings about the Development of Students' Algebraic Thinking

We summarize the data on the development of students' algebraic thinking in three sub-sections, each focusing on one of the three types of tasks: representing situations, solving equations, and making generalizations.

Representing Situations

Repeated Measures ANOVA Table 7 shows the mean scaled scores on representing-situations tasks for both CMP and non-CMP students across the four testing administrations. Repeated measures analysis of variance showed that both CMP and non-CMP students showed significant growth from the fall of 2005 (6^{th} grade) to the spring of 2008 (8^{th} grade) ($F = 275.73$, $p < .001$). CMP students started a bit lower than non-CMP students in the fall of 2005, but by the spring of 2008, the CMP students performed better than did the non-CMP students. The CMP and non-CMP students not only had different *patterns* of growth, but also different *rates* of growth, as shown in Fig. 4. A repeated measure ANOVA with mixed design analysis indicates that CMP students had a significantly higher growth rate than did non-CMP students on representing-situations items ($F = 2.61$, $p < .05$).

HLM Analysis Because the data is hierarchical in nature, we also used multilevel statistical models to capture student achievement changes over time and to analyze

Table 7 Mean scaled scores and standard deviations for CMP and non-CMP students on tasks involving representing situations

	Fall 2005	Spring 06	Spring 07	Spring 08
CMP Students	449	501	536	563
($n = 312$)[a]	(92)	(94)	(96)	(91)
Non-CMP Students	461	502	544	554
($n = 309$)	(90)	(92)	(94)	(89)

[a]The number of students reported in this table only includes those students who took all four assessments from fall 2005 to spring 2008

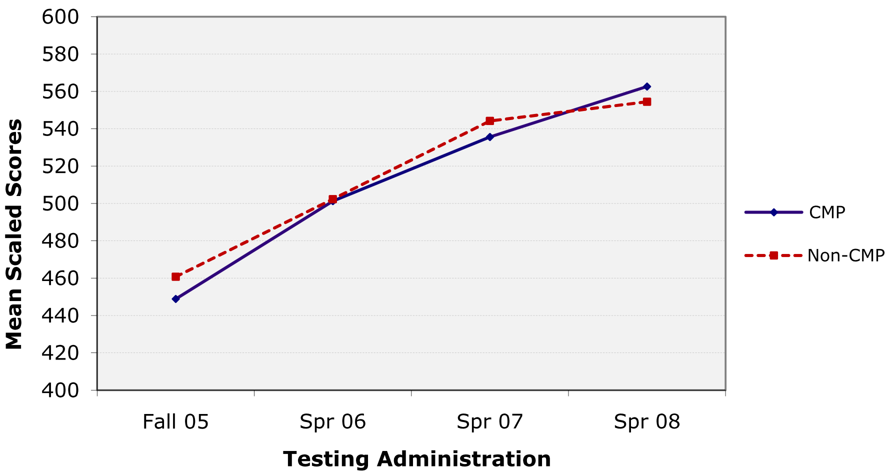

Fig. 4 Mean scaled scores for CMP and non-CMP students on representing situations

the development of algebraic thinking in a curricular context (Raudenbush and Bryk 2002). In particular, we used growth curve modeling to examine the longitudinal effect of curriculum, taking into account both student level variables (e.g., gender and ethnicity) and classroom variables (conceptual and procedural emphases) (see Cai et al. 2009 for details). Similar to what was found using the repeated measures ANOVA, the growth curve modeling showed that CMP students had a significantly higher growth rate on the representing-situations tasks than did non-CMP students ($t = 2.24$, $p < .05$). The level of conceptual emphasis in instruction had a positive impact on the growth rate of students' performance on these tasks ($t = 2.79$, $p < .05$). In fact, we found that a unit increase in the level of conceptual emphasis resulted in an increase of 4.26 scaled-score points per year in the students' growth rate on the representing-situation tasks. The level of procedural emphasis in instruction, however, did not have a statistically significant impact on the growth rate of students' performance on the representing situations tasks ($t = -0.64$, $p = .53$).

Using growth curve modeling techniques, we were able to control for classroom variables related to conceptual and procedural analysis. When we did so, we found that the difference between CMP and non-CMP students' growth rate on representing-situations tasks was no longer significant ($t = 1.38$, $p = .17$). This is to say that when the teacher's conceptual and procedural emphases were at the same level, there was no difference between CMP and non-CMP students with respect to their growth on representing situations tasks. Similarly, the impact of the level of conceptual understanding in instruction also became insignificant when the students' CMP status was controlled ($t = 1.35$, $p = .18$). This is to say that when the students were either all CMP students or all non-CMP students, there was no difference between conceptual and procedural emphases with respect to students' growth on representing situation tasks.

Item Analysis To further examine how CMP and non-CMP students performed across the four assessments, we conducted item analyses using raw scores. For both CMP and non-CMP students, Fig. 5 shows the means of the percent scores on items involving additive and relational propositions. As was expected, both CMP and non-CMP students performed better on the items involving additive propositions than on

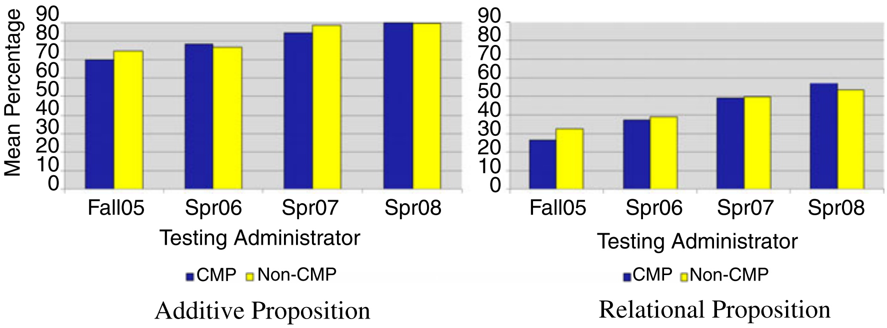

Fig. 5 Mean percentages of CMP and non-CMP students on items involving additive and relational propositions

the items involving relational propositions. By the spring of 2008, the vast majority of the students were able to answer questions related to additive propositions correctly, but only a bit over a half of them were able to answer questions involving relational propositions correctly.

From the fall of 2005 to the spring of 2008, both CMP and non-CMP students had greater increases on items involving relational propositions than on additive propositions, even though they had higher mean percentages on items involving additive propositions than on relational propositions. CMP students showed greater increases than non-CMP students on both types of items, although the CMP students' additive proposition increase was not significantly greater than the non-CMP students'. In fact, on items involving additive propositions, the increase for CMP students was about 20% from the fall of 2005 to the spring of 2008, but the increase was about 15% for the non-CMP students ($z = 1.64$, $p = .10$). On items involving relational proposition, the increase for CMP students was over 30% from the fall of 2005 to the spring of 2008, but the increase was about 25% for the non-CMP students ($z = 2.88$, $p < .01$).

Even though the increase over the three years was higher for CMP students than non-CMP students, an item analysis revealed that both CMP and non-CMP students showed very similar error patterns. For example, when solving items involving relational propositions, the most common error for both CMP and non-CMP students was to make an incorrect "literal translation" (Cocking and Mestre 1988). For instance, for the relational proposition "One pound of shrimp costs $3.50 more than one pound of fish," the most common error for both groups was to respond that "shrimp's cost per pound +$3.50 = fish's cost per pound," rather than "shrimp's cost per pound = $3.50+ fish's cost per pound."

Solving Equations

Repeated Measures ANOVA Table 8 shows the mean scaled scores on tasks involving equation solving for both CMP and non-CMP students across the four testing administrations. Both CMP and non-CMP students showed significant growth

Table 8 Mean scaled scores and standard deviations for CMP and non-CMP students on tasks involving equation solving

	Fall 2005	Spring 06	Spring 07	Spring 08
CMP Students	459	466	505	505
($n = 312$)[a]	(80)	(76)	(93)	(91)
Non-CMP Students	498	504	543	543
($n = 309$)	(90)	(92)	(94)	(89)

[a]The number of students reported in this table only includes those students who took all four assessments from fall 2005 to spring 2008

from the fall of 2005 (6^{th} grade) to the spring of 2008 (8^{th} grade) ($F = 177.72$, $p < .001$). CMP students started lower than non-CMP students in the fall of 2005, and the advantage of the non-CMP students continued for all three years. Overall, the growth rates for CMP and non-CMP students were almost identical, about 45 scaled-score points from the fall of 2005 to the spring of 2008. A repeated measures ANOVA with mixed design analysis indicated that there was no significant difference between the growth rates of CMP and non-CMP students over the three years ($F = .45$, $p = .717$). This finding suggests that students had similar gains in equation solving ability, regardless of the curriculum used.

HLM Analysis Similar to what we found using the repeated measures ANOVA, growth curve modeling showed that CMP and non-CMP students had comparable growth rates over the three years on equation-solving tasks ($t = -0.39$, $p = .70$). Instructional differences in the level of conceptual emphasis did not have a significant impact on the growth rate of equation solving skills ($t = 1.12$, $p = .26$), nor did instructional differences in the level of procedural emphasis ($t = 0.50$, $p = .62$).

Making Generalizations

In this section, we report the results from one of the 13 open-ended tasks, namely the Doorbell task (see Appendix). The Doorbell task includes four questions. The answer to each succeeding question requires more generalization than the previous one. To effectively convey how CMP and non-CMP students' ability to generalize grew over the three years, we will report the differences between the fall of 2005 and the spring of 2008 data.

Success Rates on Questions A, B, and C Table 9 shows the success rates for CMP and non-CMP students on questions A, B, and C. To answer question A, students need to find the number of guests entering on the 10th ring. The CMP students' success rate on this question increased significantly from 55% in the fall of 2005 (6^{th} grade) to 76% in the spring of 2008. The success rate of the non-CMP students also increased significantly (62% to 73%) from the fall of 2005 to the spring of 2008. However, the success rate of CMP students increased significantly more than did the success rate of the non-CMP students ($z = 3.52$, $p < .001$).

Table 9 Percentages of students having correct answers for questions A, B, and C of the Doorbell problem

	Fall 2005			Spring 08		
	QA	QB	QC	QA	QB	QC
CMP Students ($n = 296$)	55	4	3	76	19	15
Non-CMP Students ($n = 299$)	62	5	3	73	21	10

The success rates were lower on the later questions for both CMP and non-CMP students. For example, in spring 2008 only about 20% of the CMP and non-CMP students correctly answered question B (How many guests will enter on the 100th ring?). Furthermore, question C (299 guests entered on one of the rings. What ring was it?) was correctly answered by only 3% of the CMP and 3% of the non-CMP students in the fall of 2005. By the spring of 2008, the percentages had increased to 15% for CMP students and 10% for non-CMP students. Even though still only small proportions of the CMP and non-CMP students were able to answer question C correctly in the spring of 2008, the increase for CMP students was significantly greater than that for the non-CMP students ($z = 1.99$, $p < .05$). This finding suggests that both CMP and non-CMP students increased their generalization abilities over the middle school years. It also suggests that, on average, CMP students developed greater generalization abilities than non-CMP students over the middle school years.

Solution Strategies An examination of the students' solution strategies confirmed this finding. We coded the solution strategies for each of these questions into two categories: abstract and concrete. Students who chose an abstract strategy generally followed one of two paths. Students who followed the first path noticed that the number of guests who enter on a particular ring of the doorbell equals two times that ring number minus one (i.e., $y = 2n - 1$), where y represents the number of guests and n represents the ring number. Students on the second path recognized that the number of guests who enter on a particular ring equal the ring number plus the ring number minus one (i.e., $y = n + (n - 1)$). Using their generalized rule, these students were able to determine the ring number at which 299 guests entered. Those who used a concrete strategy also generally took one of two paths. Those on the first path made a table or a list, while those on the second path noticed that each time the doorbell rang two more guests enter than did on the previous ring and so added 2's sequentially to find an answer.

In the fall of 2005, one CMP student and none of the non-CMP students used an abstract strategy to correctly answer question A, but in the spring of 2008, nearly 9% of the CMP students and 9% of the non-CMP students used abstract strategies to correctly answer question A.

Table 10 shows the percentages of CMP and non-CMP students who used concrete or abstract strategies to correctly answer questions B and C. In the spring of 2008, nearly 20% of the CMP students and 19% of non-CMP students used an abstract strategy to correctly answer question B. Only a small proportion of the CMP and non-CMP students used abstract strategies to correctly answer question C in the spring of 2008. However, the rate of increase for the CMP students who used abstract strategies from the fall of 2005 to the spring of 2008 was significantly greater than that for non-CMP students ($z = 2.58$, $p < .01$). Thus, these results confirmed that both CMP and non-CMP students increased their generalization ability over the middle school years. However, on average, the CMP students developed their generalization ability more fully than did non-CMP students.

Table 10 Percentages of students having correct answers for questions B and C of the Doorbell problem

	Fall 2005			Spring 08		
	Concrete Strategy	Abstract Strategy	No Strategy	Concrete Strategy	Abstract Strategy	No Strategy
QUESTION B						
CMP Students ($n = 296$)	27	1	72	14	20	66
Non-CMP Students ($n = 299$)	20	3	77	14	19	67
QUESTION C						
CMP students ($n = 296$)	9	0	91	11	9	80
Non-CMP Students ($n = 299$)	11	2	87	14	5	81

Success Rate for Question D The Question D asked students to write a rule or describe in words how to find the number of guests that entered on each ring. The percentages of getting correct rule for Question D of Doorbell for CMP students are .3% in the fall of 2005, 6.4% in the spring of 2006, 3.7% in the spring of 2007, and 15.9% in the spring of 2008. The percentages of getting correct rule for Question D of Doorbell for Non-CMP students are 1.3% in the fall of 2005, 5.4% in the spring of 2006, 6.0% in the spring of 2007, and 15.7% in the spring of 2008. There is no significant difference between CMP and Non-CMP students in each grade level in terms of the percentages of the students who got the correct rule to find the number of guests that entered on each ring. The rates of increase from the fall of 2005 to the spring of 2008 between CMP and non-CMP are similar ($z = .40$, $p = .35$).

Conclusions and Instructional Implications

Middle school algebra lays the foundation for the acquisition of tools for representing and analyzing quantitative relationships, for solving problems, and for stating and proving generalizations. Given recent efforts at curriculum reform, there is an urgent need to understand the role that curriculum plays in students' learning of mathematics in general and in the acquisition of algebraic concepts in particular (NCTM 1989; National Research Council 2004; RAND Mathematics Study Panel 2003; Senk and Thompson 2003; Usiskin 1999). A study like the one presented in this chapter is significant not only because it investigates the development of students' algebraic thinking in middle grades, but also because it examines explicit connections between the acquisition of algebraic concepts and the manner in which algebra is taught and learned using two different types of curricula.

The findings presented here showed that across the middle school years, no matter which type of mathematics curriculum the students in the LieCal Project used, they all showed significant growth on representing problem situations, solving equations, and making generalizations. On the one hand, students became much more capable of representing problem situations, solving equations, and making generalizations in the spring of 2008 than they were in the fall of 2005. For example, it is encouraging that the vast majority of the students were able to represent additive propositions correctly by 8th grade, and also that some 8th graders were able to make and represent generalizations in the Doorbell problem situation. On the other hand, in an absolute sense, the performance level attained by 8th grade is not very encouraging. For example, only slightly over a half of the 8th graders were able to successfully represent the relational propositions, and only about 70% of the 8th graders were able to solve simple equations. In particular, only a very small proportion of the 8th graders were able to use abstract strategies and generalize the pattern in the Doorbell problem.

What should we expect from middle school students in terms of algebra? The National Assessment of Educational Progress (2006) has listed a set of expectations for 8th graders. For example, by grade 8, students are expected to write algebraic expressions, equations, or inequalities to represent a situation, to solve linear equations or inequalities (e.g., $ax + b = c$ or $ax + b = cx + d$ or $ax + b > c$) and to generalize a pattern appearing in a numerical sequence or table or graph using words or symbols. We believe that it is reasonable to expect greater proficiency representing situations, solving equations and making generalizations than the students in this study have shown.

How can we expect and foster greater proficiency representing situations, solving equations, and making generalizations than what the students in this study have shown? In this chapter, we examined the impact of instruction using two different types of curricula on the development of students' algebraic thinking over three years (grades 6–8). Students who used the CMP curriculum showed significantly higher growth rates than did non-CMP students on both the representing-situations tasks and the making-generalization task across the three middle school years. On the equation-solving tasks, the growth rates for CMP and non-CMP students were similar. These findings suggest that the use of the CMP curriculum has a positive impact on students' development of algebraic thinking, as measured by the representing-situations tasks and the making-generalization task.

In *Standards*-based curricula like CMP, one important focus is on developing students' conceptual understanding and higher-order thinking skills. Will *Standards*-based curricula's attention to the development of students' higher-order thinking skills come at the expense of the development of basic mathematical skills? The findings presented in this paper showed that the development of students' higher-order thinking skills does not necessarily come at the expense of the development of basic mathematical skills when using *Standards*-based curricula like CMP. In fact, students using the CMP curriculum had growth rates in basic mathematical skills that were similar to those of the non-CMP students.

Why would the students using the CMP curriculum show significantly greater growth than the students using non-CMP curricula on the representing-situations tasks and the making-generalization task? The answer to this question might be related to the nature of curriculum and instruction using CMP and non-CMP curricula. CMP explicitly uses a functional approach to define equations and introduce equation solving. In particular, in the introduction of equation solving, CMP emphasizes the connections between a situation and an equation used to represent it. Moreover, not only did the CMP curriculum include more cognitively demanding mathematical problems than the non-CMP curricula, but also the teachers in classrooms using CMP curricula implemented more cognitively demanding instructional tasks. Research has shown that tasks with higher cognitive demand provide better learning opportunities for students (Doyle 1988; Hiebert and Wearne 1993; Stein et al. 1996). Therefore the CMP students' more frequent engagement in cognitively demanding tasks is likely to have contributed to the CMP students' superior performance on the representing-situations tasks and the making-generalizations task.

Appendix: Sample Tasks

Representing Situations

Which number sentence is correct?

One pound of shrimp costs \$3.50 more than one pound of fish.

a. shrimp's cost per pound = fish's cost per pound + \$3.50
b. shrimp's cost per pound + \$3.50 = fish's cost per pound
c. shrimp's cost per pound + fish's cost per pound = \$3.50
d. shrimp's cost per pound = fish's cost per pound − \$3.50

Solving Equations

Find the value of x so that $x - 5 = 5$

(a). 0 (b). 1 (c). 10 (d). 25

Making Generalizations

Sally is having a party.
The first time the doorbell rings, 1 guest enters.
The second time the doorbell rings, 3 guests enter.
The third time the doorbell rings, 5 guests enter.
The fourth time the doorbell rings, 7 guests enter.

Keep going in the same way. On the next ring a group enters that has 2 more persons than the group that entered on the previous ring.

A. How many guests will enter on the 10th ring? Explain or show how you found your answer.
B. How many guests will enter on the 100th ring? Explain or show how you found your answer.
C. 299 guests entered on one of the rings. What ring was it? Explain or show how you found your answer.
D. Write a rule or describe in words how to find the number of guests that entered on each ring.

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Years 2 to 6 Students' Ability to Generalise: Models, Representations and Theory for Teaching and Learning

Tom J. Cooper and Elizabeth Warren

Abstract Over the last three years, in our Early Algebra Thinking Project, we have been studying Years 3 to 5 students' ability to generalise in a variety of situations, namely, compensation principles in computation, the balance principle in equivalence and equations, change and inverse change rules with function machines, and pattern rules with growing patterns. In these studies, we have attempted to involve a variety of models and representations and to build students' abilities to switch between them (in line with the theories of Dreyfus 1991, and Duval 1999). The results have shown the negative effect of closure on generalisation in symbolic representations, the predominance of single variance generalisation over covariant generalisation in tabular representations, and the reduced ability to readily identify commonalities and relationships in enactive and iconic representations. This chapter uses the results to explore the interrelation between generalisation, and verbal and visual comprehension of context. The studies evidence the importance of understanding and communicating aspects of representational forms which allowed commonalities to be seen across or between representations. Finally the chapter explores the implications of the results for a theory that describes a growth in integration of models and representations that leads to generalisation.

From 1999–2001, we were asked by the writers of the new Queensland state mathematics syllabus for our advice with regard to algebra in the elementary years. We

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developed a framework for early algebra based on our knowledge and beliefs at that time. These were driven by a structural view of mathematics and algebra (Kieran 1990; Sfard 1991), a cognitive perspective on learning (Hiebert and Carpenter 1992; English and Halford 1995), and an appreciation of students' difficulties with variables and the cognitive gap between arithmetic and algebra (Linchevski and Herscovics 1996; Usiskin 1988). This framework (Warren and Cooper 2001) was used as a basis of the new Queensland syllabus which was finalised in 2004 (Queensland Studies Authority 2003).

In 2002, we undertook a pilot study of Year 2 students' learning of early algebra. With the support of the Queensland state education department, we successfully applied for a national grant for a longitudinal project, entitled *Early Algebra Thinking Project* (EATP), to study Years 2 to 6 students' learning of early algebra. This study was based on our syllabus framework (Warren and Cooper 2001), our ideas of effective teaching through connecting multiple representations (Dreyfus 1991; Duval 2002; Halford 1993; Hiebert and Carpenter 1992), and Krutetskii's (1976) generic pedagogies of generalising, flexibility and reversing (see Cooper et al. 2006). The early algebra framework and the representations we used gave rise to many different examples of generalisation.

This chapter is a sweep across the five years of the pilot and main study for a cohort of Years 2 to 6 students, a sweep that describes what we found and where we now wish to go. It initially describes a theoretical framework with regard to our development of early algebraic thinking and generalisation and representation. It also describes the focus and design of the project, and discusses generalisation results with respect to patterning, functional thinking, equations and equivalence, and arithmetic principles. Finally, it draws conclusions and implications, and relates findings to an EATP theory with regard to a teaching/learning trajectory designed to support the development of the ability to generalise (Warren and Cooper 2008).

Special attention is given, in the results section, to functional thinking and arithmetic principles as these have not been analysed to the same extent as patterning and equations and equivalence in other EATP publications. The chapter particularly focuses on generalisation abilities of young children, factors that affect these generalisation abilities, instructional strategies that are effective in developing generalisations, and sequences of models and representations that facilitate generalisation.

Perspectives on the Mathematics of Early Algebra

In its most powerful form, we viewed algebra as an abstract system, a system with interactions that reflected the structure of Arithmetic (Usiskin 1988). Like Dienes (1961), Skemp (1978) and Wilson (1976), we saw the importance of algebra in terms of how it represented the principles (e.g., commutative principle and balance principle), structures of mathematics (e.g., field, group and equivalence class) and not in terms of the "behaviours" of algebra (such as simplification and factorisation). In line with Radford (2006), we did not see algebra as the manipulation of letters but rather as a system characterised by indeterminacy of objects, an analytic nature of thinking and symbolic ways of designating objects.

Structure and Approach In terms of structures, we determined the importance in laying the groundwork for both the equivalence class and the field structures, particularly as the equals sign and the operational principles have been identified as sources of difficulty in the development of algebra for many years (Behr et al. 1980; Herscovics and Linchevski 1994; Kieran 1990). We also appreciated the need to develop the two approaches to mathematics, relation and change, from the start.

Mathematics has been categorised by Scandura (1971) as having only three foci: *things*; *relations* between things; and *transformations* (changes) between these things, but with every relation capable of being seen as a transformation and every transformation capable of being seen as a relationship. This is particularly true for number and operations; for example, the addition of 2 and 3 can be seen in relational or static terms as "*balance*", $2 + 3 = 5$, and in transformational or dynamic terms as "*change*", $2 \to +3 \to 5$. From the relation perspective, equals is equivalence or "same value as" whilst from the transformation perspective, equals is seen as a two-way mapping in which the change is reversed (a move from a uni-directional mode of reading an equation to a multi-directional processing of information—Linchevski 1995).

Abstract Schema and Reification We were impressed by Ohlsson's (1993) discussion of abstract schema (which we determined in mathematics as being the same as the mathematical idea of principle), that states the most powerful ideas in mathematics are schema where meaning is encoded in the structure or relationships between the components and not in the form of the components (e.g., $a + b = b + a$ has meaning independent of what the a and b represent—the meaning is in how the a and b are related within the rule). Ohlsson (1993) argued that abstract schemas (and, therefore, for our needs, principles) allow for cross-domain transfer, are highly portable, and provide the sets of rules by which formal abstract thinking operates. Ohlsson (1993) saw Piaget's (1985) process of reflective abstraction as a process whereby abstract schemas are created from reflection on regularities in cognitive operations.

Along with Piaget's (1985) idea of abstraction, Sfard's (1991) idea of reification from operational to structural knowledge, where a concept is recognised as "a fully fledged mathematical object" (p. 14), appeared to provide a perspective on algebraic development that illuminated the difficulties in learning algebra (see Warren and Cooper 2001, for a discussion of these difficulties). We appreciated that number could be reified from counting everyday items to the object of thought that allows, for example, numbers to be added without reference to sets of items. We also saw that, at a later time, variable could be reified, in turn, to the object of thought when students were experienced with mathematical ideas that are true for any number. Thus, we came to regard algebra as a second level reification, a reification of a previous reification, indicating that algebra difficulties might be reduced if number was fully reified, that is, no longer a set of items, and generalisation was an outcome for as many number activities as possible.

Framework and EATP In our discussions with the Queensland mathematics syllabus writing committee, a framework (Warren and Cooper 2001) was developed that placed early algebra activities within a special Patterns and Algebra strand and as part of the Number strand of the Queensland Years 1–10 mathematics syllabus (Queensland Studies Authority 2003). This framework encompassed: (i) *pattern and functions*, the study of repeating and growing patterns and of early functional thinking (focusing on change); (ii) *equivalence and equations*, the study of equivalence, equations and expressions; and (iii) *arithmetic generalisation*, the study of number that involves generalisation to principles. The framework also saw early algebra as the development of mental models based on relationships between real world instances, symbols, language, drawings and graphs, particularly those that enabled the modelling of real situations that contained unknowns and variables.

It was evident that the basis of these activities is the ability of students to generalise: (i) from position to term in numerical and visual growing patterns (e.g., Warren 2005b); (ii) from tables of values to relationships between numbers down and across columns, and to graphs in patterning and functional situations (e.g., Warren 2005a; Warren and Cooper 2007); (iii) from particular examples to general principles (e.g., Dougherty and Zilliox 2003; Warren 2005b); and (iv) from real-world situations to abstract representations (Carraher et al. 2006; Schliemann et al. 2001; Warren 2005b). Thus, improving one's ability to generalise lies at the foundation of efforts to enhance participation in and learning of algebra. As Dienes (1961) argued: "If this generalisation [from an initial class of small familiar numbers to 'any number'] does not take place, algebra cannot possibly be understood" (pp. 289–290). Generalisation also lies at the heart of mathematics; as Lannin (2005), supported by Kaput (1999) and Mason (1996), argued: "Statements of generality and discovering generality are at the very core of mathematical activity."

In EATP, we have been studying the act of generalisation, in particular, pattern rules with growing patterns, change and inverse change rules with function machines and tables of values, balance principle in equivalence and equations, compensation principles in computation, and abstract representations of change (e.g., tables, arrow diagrams, graphs) and relationship (equations). We have studied these acts of generalisation across a variety of situations, in diverse contexts, and with a range of representations, looking particularly at the relationships between representations and growth of algebraic thinking. This has reinforced our position that generalisation appears to be a major determiner of growth in algebraic thinking and preparation for later learning of algebra.

Representation and Generalisation

Our research in EATP was based on sequences of *teaching experiments* using the conjecture driven approach of Confrey and Lachance (2000) with the aim to produce both theoretical analyses and instructional innovations (Cobb et al. 2000). We postulated hypothetical learning trajectories within each class and formulated conjectures (in terms of both mathematical content and pedagogy) about envisaged learning processes and specific means that might support these processes. Our basis for these conjectures was multi-models (e.g., balance and number line together) and multi-representations (i.e., connecting language, diagrams and figures, symbol systems and graphs).

Models and Representations

There has been general consensus for some time that mathematical ideas are represented externally (i.e., concrete materials, pictures/diagrams, spoken words, written symbols) and internally (i.e., mental models and cognitive representations of the mathematical ideas underlying the external representations) (Putnam et al. 1990). Models and representation are related; models are ways of thinking about abstract concepts (e.g., balance for equivalence) and representations are various forms of the models (e.g., physical balances, balance diagrams, balance language, equations as balance). Mathematical understanding is the number and strength of the *connections* in a student's internal network of *mental models* and *representations* (Hiebert and Carpenter 1992). It has long been argued that generalising mathematics structures involves determining what is preserved and what is lost between the specific structures which have some isomorphism (Gentner and Markman 1994; Halford 1993). An example of such a structure is what is common between the subtraction algorithm for whole numbers, decimal numbers, common fractions and measures. This is referred to in the literature as the Mapping Instruction approach (English and Halford 1995; Peled and Segalis 2005).

Dreyfus (1991) argued that learning proceeds through four states, namely, using one representation, using more than one representation in parallel, making links between parallel representations, and integrating representations and flexibly moving between them. Duval (2002) extends Dreyfus' argument; he argues that mathematics comprehension results from coordination of at least two representation forms or *registers* and that there are four registers; the multifunctional registers of *natural language* and *figures/diagrams*, and the mono-functional registers of *notation systems* (symbols) and *graphs*. Again extending Dreyfus, Duval contends that learning involves moving from *treatments* where students stay within one register (e.g., carrying out calculations while remaining strictly in the one notation system) to *conversions* where students change register without changing the objects being denoted (e.g., passing from natural language of a relationship to using letters to represent it) and finally to *coordination* of registers. He argues that learning also requires building understanding of the mathematical processing performed in each register (Duval 1999). A further distinction between Dreyfus and Duval is that Duval also suggests that representations play an epistemological and pedagogical role in teaching and learning. It is this distinction that guided us in our classroom interactions.

Generalisation

Literature with respect to patterns, principles and abstract representations, and mathematics induction provides a framework for generalisation. For patterns and tables of values, Lannin (2005) distinguishes between two types of generalisation: *recursive* (single variant or sequential between terms, e.g., add 2 to get the next term) and *explicit* (covariant or between term and position, e.g., the 25th term is $2 \times 25 - 1$). For each of these types, Radford (2003) argues that generalisation develops through three levels: (i) *factual*, where the generalisation focus remains at the level of the material to be generalised, such as "the four counters" and is gesture and rhythm driven to show the generalisation; (ii) *contextual*, where the focus is on more abstract and descriptive terms such as "the next figure" and is language driven to explain the generalisation; and (iii) *symbolic*, where algebraic notation (including letters) is used to describe the generalisation.

For principles and abstract representations, research in *Measure Up* (Dougherty and Zilliox 2003) and EATP (Warren 2005b; Warren and Cooper 2007) has shown that very young students can generalise the Equivalence Class principles from activity in numberless situations (work pioneered by Davydov 1975). Other research (e.g., Carraher et al. 2006) has shown that young students can generalise to abstract representations, and that such activity results in better understandings of mathematical structures in later years (Morris 1999). Goodson-Espy (1998) argues for Sfard's (1991) sequence of interiorisation, condensation and reification and Cifarelli's (1988) reflective abstraction levels of: recognition, re-presentation, structural abstraction and structural awareness for generalisation to principles. Peled and Segalis (2005) argue for the Mapping Instruction approach to teaching (English and Halford 1995) which focuses on identifying similarities between isomorphic procedures (e.g., what is the same in the processes for 34–16 and 3 weeks 4 days subtract 1 week 6 days).

Overall, Harel's (2001) two different forms of mathematics-induction generalisation appear applicable to all types of generalisation. He argued that there were two forms: (i) *results* generalisation, developing a generality from a few examples; and (ii) *process* generalisation, developing a generality from a few examples and then justifying it in terms that show its applicability to all examples or any number. His distinction appears similar to Radford's (2003, 2006) induction-generalisation and Lannin's (2005) empirical-generic distinctions. In particular, Radford (2006) suggested that true generalisation involves noticing a local commonality and then generalising this across all terms (perceiving the particular and then using this to conceive the general). With regard to these forms of generalisation, Lannin (2005) distinguished between generalisation from *iconic/visual* and *numerical* representations, arguing that iconic is a better representation to lead to process generalisation.

Finally, Radford (2003, 2006) purported that generalisation involves two components: (i) *grasping* a generality (phenomenological) through noticing how a local commonality holds across all terms; and (ii) *expressing* a generality (semiotic) through gestures, language and algebraic symbols. The act of grasping a generality through extending a local commonality (process generalisation) has led to a study of strategies of which some (e.g., trial and error) are not strongly supported and others (e.g., restructuring visual presentations) have some support. In particular, research into using tables has had advocates for use and non use (e.g., Herbert and Brown 1997; Orton and Orton 1999; Warren 1996, 2006). Expression of generalisation has also attracted support with Redden (1996) and Stacey and MacGregor

(1995) suggesting that natural language is a prerequisite for algebraic notation. By contrast, Bloedy-Vinner (1995), Kaput (1992), Ursini (1991) and Warren (2005b) have shared misgivings with regard to this conjecture. Fujii and Stephens (2001) identified the notion of a quasi-variable as a bridge between arithmetic and algebraic notation. From their perspective this involves the recognition that a number sentence or group of number sentences can indicate an underlying mathematical relationship which remains true whatever the numbers used are. Our research suggests that this idea is extendable to generalisation, to give a notion of *quasi-generalisation* as a step towards full generalisation. In this extension, quasi-generalisation is where students are able to express the generalisation in terms of specific numbers. It appears often to be the case that students can apply a generalisation to many numbers, and even to an example of "any number", before they can provide a generalisation in language or symbols (Warren 2005b; Warren and Cooper 2007). We have found that a *quasi-generalisation* in the elementary context appears to be a necessary precursor to expressing the generalisation in natural language and algebraic notation.

Focus of EATP

In EATP, we designed instruction to: (i) have grasping and expressing generalisation as its major outcome, distinguishing between process, results, recursive and explicit types of generalisation in patterns and tables, and ensuring that time is spent on generalising principles and abstract representations; (ii) use a variety of models (e.g., balance, line, function machine) and representations (e.g., natural language, figures/diagrams, symbols and, in later years, graphs) to achieve this outcome; and, (iii) build students' abilities to switch between representations and models (Dreyfus 1991; Duval 1999). Given the paucity of literature concerning the development of algebraic thinking at the elementary level, Bruner's theory (1966) was utilised to assist us in selecting representations for the interventions. Our selection broadly followed the enactive to iconic to symbolic sequence unless another imperative intervened. We considered mathematical development as cumulative rather than replacement and thus integrated various models and representations from different levels (e.g., an iconic picture of balance with enacting the number line). Based on our belief that no one model or system of representations provides all of the answers, we used comparison of and transition between models and representations to support the emergence of algebraic thinking (e.g., using the balance and number line models in unison). Overall, the instruction focused on connecting Duval's four registers to real world situations and the acting out of these situations by the students themselves with physical materials. It also focused on connecting particular representations emerging from different perspectives (relationship and change—Scandura 1971), particularly the arrow based symbol system for change with the equation based system for relationship.

Our use of models and their representations, particularly in their physical or concrete representation form, were endowed with two fundamental components (Filloy and Sutherland 1996), namely, the ability to translate and generalise to abstraction. Translation encompasses moving from the state of things at a concrete level to the state of things at a more abstract level with the model acting as an analogue for the more abstract. Abstraction is believed to begin with exploration and use of processes or operations performed on lower level mathematical constructs (English and Sharry 1996; Sfard 1991).

In line with Filloy and Sutherland (1996), we accepted that models often hide what is meant to be taught and present problems when abstraction from the model is left to the pupil. Thus, we saw teacher intervention as a necessity for detachment from the model to construction of the new abstract notion. As we implemented the models and representations, we engaged in classroom conversations with the young students and continually explored new signs that would assist students to extract the essence of the mathematics embedded in the exploration (Radford's, 2003, semiotic nodes). We saw expression and language as essential to this journey as they gave subtle shades of meaning that arise from the students' thinking (Tall 2004). Thus, EATP was based in the socio-constructivist theory of learning, inquiry based discourse and the simultaneous use of multi-representations to build new knowledge (Warren 2006).

The goals of the EATP were to: (i) investigate Years 2 to 6 (6 to 12 year old) children's abilities to reason algebraically and, in particular, to generalise arithmetic and algebraic situations; (ii) identify key transitions in the children's development of algebraic reasoning and generalisation; (iii) construct a model of this development; (iv) develop instructional strategies effective in facilitating this development; and (v) develop professional development processes that facilitate teacher learning of these approaches.

Focus of Chapter

This chapter will report generally on the design of EATP and its overall findings with regard to generalisation, and particularly on Year 5 (9–10 years old) functional-thinking and Years 3–4 (7–9 years old) mathematics-principles lessons as cases of how generalisation was studied and how multi-representations were used. The particular research questions to be answered by the chapter are: (i) at what age can young children generalise patterns, tables of values, principles and abstract representations; (ii) what factors enable and inhibit the development of these abilities; and (iii) what instructional strategies are effective in this development. For the final question, the chapter will look across the sequence of models, representations used and their relative efficacy to draw conclusions with regard to a theory of model and representation use for effective generalisation (and abstraction).

Design of EATP

The methodology adopted for EATP was *longitudinal* and *mixed method* using a *design research* approach, namely, a series of *teaching experiments* that followed a cohort of students across elementary Year 2 to Year 6 based on the conjecture driven approach of Confrey and Lachance (2000). It was predominantly *qualitative* and *interpretive* (Burns 2000) but with some quantitative analysis of pre-post tests. In each year, the teaching experiments investigated the students' learning in lessons on patterning and functional thinking (using the change perspective), equivalence and equations (using the relationship perspective), and principles of arithmetic (related, in particular, to the Equivalence Class and Field structures).

EATP was based on a re-conceptualisation of content and pedagogy for algebra in the elementary school and as such the teaching experiments were exploratory in nature; seeking to identify the fundamental cognitive building blocks on which student understanding of function could be constructed. The representations chosen were intended to be inclusive of all children; however, the necessity to respond to individual student needs was a position we acknowledged from the outset. Multiple sources of data were collected, and only those findings for which there was triangulation were considered in the analysis. Adequate time was spent in the field observing the lessons to substantiate the reliability of the collected data (Davis and Maher 1997).

Participants and Procedure The *participants* across the years were a cohort of students, and their teachers, from 5 inner city Queensland state schools with socio-economic status varying from upper to working class. These students moved through Year 2 to Year 6 during the time of the study (although in the Pilot Year, Year 2 (6–7 years old), there was only one middle class state school involved). Thus the sample comprised 3 classes (70–85 students) and 3 teachers in Year 2 and 10 classes (220–270 students) and 10 teachers in Years 3 to 6. All schools were following the new Queensland Years 1–10 Mathematics Syllabus (Queensland Studies Authority 2003) which has a new "Patterns and Algebra" strand from Year 1.

For each teaching experiment, the *procedure* was that we developed and taught four one-hour lessons to two classes and prepared detailed lesson plans for the other teachers to follow. In this way, all classes could receive instruction from all the teaching experiments, and the teachers could have professional learning with respect to a new area of content that they had to teach in the future. Detailed lesson plans and professional learning sessions to introduce them were required because, although all teachers were well credentialed (all had four years training, in line with Queensland policy), the mathematics component of their training was small and, like most elementary teachers in Queensland, they were not confident in teaching mathematics (Nisbet and Warren 2000). As algebra was a new mathematics content area for them, requiring thinking that had not previously been explored, the teachers had no teaching experience and were unsure of how to conduct these lessons.

In common with normal practice in Queensland elementary schools, the mathematics taught in the classes had a preponderance of activities related to the development of pen and paper algorithms and the acquisition of computational skills. The teachers used some materials in their teaching, but the use of these materials was highly prescribed, accompanied by stipulated language and recording processes, and imitative and teacher directed (Baturo et al. 2003).

The teaching in this study was based on models and representations (see Figs. 1 to 6 for models and representations) that have been used before, particularly in the sixties and seventies. However, the use of these materials was linked to: (i) a social semiotic approach to teaching; (ii) an inquiry approach to classroom management in which students shared their thinking; and (iii) lesson plans and worksheets that connected representations in terms of drawings, real life stories and symbols.

Instruments and Analysis The *instruments* used in the teaching experiments were: (i) *observations*, all lessons in the teaching experiments taught by the researchers and some lessons taught by teachers were recorded with field notes and videotaped, with one camera fixed on the teacher and class as a whole and another camera moving around the class focusing on student activity of interest; (ii) *interviews*, planned yearly interviews with teachers and a sample of students and ad hoc unplanned interviews with students who showed interesting activity during lessons were conducted and audiotaped; (iii) *reflections*, feedback from teachers and other researchers/research assistants on their perceptions of the lessons and student learning were recorded in written and/or oral form (audiotaped); (iv) *tests*, yearly written tests were conducted to ascertain students' development of algebraic thinking across the years, and pre-post written tests covering the content in teaching experiments were administered before and after the experiments with all pen and paper tests collected and marked; and (v) *artefacts,* all worksheet materials completed by students (students directed to show working and not to rub out any errors) were collected during all lessons, marked and collated.

For *analysis*, the videotapes of the classroom observations and the audiotapes of the interviews were transcribed, and the pre-post tests and students' worksheets were graded and results placed in Excel spreadsheets. An interpretive, descriptive protocol analysis was used in exploring the data. This entailed a situated discourse analysis which provides salient segments omitting where necessary students' repetitions.

These data were combined with field notes and written reflections to provide rich descriptions of each lesson and each teaching experiment, descriptions that contain information on relationships between teachers' teaching actions and students' learning responses in relation to records of performance and performance change. These descriptions were then analysed for evidence of student learning and generalisation processes that followed from that learning. Finally, data across lessons and teaching experiments were compared for similarities and differences in order to construct theory with regard to integration of models and representations that leads to Years 2–6 students' development of algebraic reasoning.

Findings and Discussion

EATP teaching experiments have focused on generalisation in patterns and functions (the change perspective) and principles and abstract representations (the relationship perspective). We will discuss each of these in turn.

Patterns

Results in patterns have shown that young students can determine pattern rules (up to symbols with letters) for growing patterns from tables of numbers and visual structures of counters (Warren 2005b; Warren and Cooper 2007). Results also indicated that generalising patterns with tables is easier to teach while generalising with visuals provides more equivalent solutions (Warren 2005a). Further, results show that young students develop these abilities from recursive to explicit (using Lannin's 2005 terms) and via quasi-generalisation (building on Fujii and Stephens' 2001 term) (Warren 2005b; Warren and Cooper 2007, 2008). With further analysis, it also appears likely that results will show pattern-generalisation development moves from factual to contextual to algebraic (as in Radford 2003) and that pattern visuals result in better process generalisation (similar to Lannin 2005).

Results have also shown that students can generalise relationships between different materials within repeating patterns across many repeats. In these activities, students break the pattern into repeats, record in a table the number of each material against the number of repeats, and generalise across the columns. The students have also shown an ability to use these tables to generalise equivalent fractions and equivalent ratios (Warren 2005a).

A lesson that illustrates some of these findings is a Year 5 (9–10 years old) lesson that focused on teaching growing pattern generalisation without tables. Students were required to find the general pattern rule for growing patterns built from blocks and sticks (as in Fig. 1) directly from the visual geometry of the objects.

This proved very difficult and most students found it hard to notice a commonality in the examples given, which they could extend to a general rule. Students were asked to find a step sufficiently large enough (e.g., the 22nd step) that directed attention towards commonality but not so large that it prevented any commonalities being noticed (Radford 2006). They were then asked to state the rule for large numbers (e.g., the 237th step), in language, and for a variable n. Only a few students could find the rule for a large number; even *quasi-generalisation* appeared to be difficult for this cohort of students.

The method that was successful was asking the few students who did find a correct general rule to describe their thoughts about the pattern, thoughts that enabled them to generalise a commonality. This was identified by the teacher as a form of visual analysis and labelled with the student's name (see Fig. 2). In the next pattern,

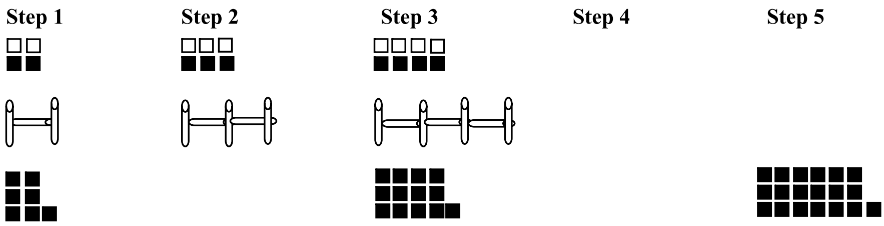

Fig. 1 Typical growing patterns

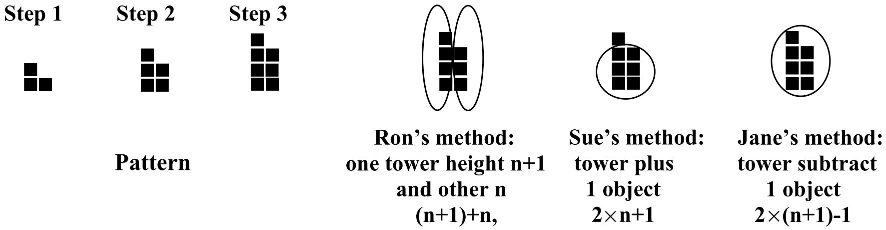

Fig. 2 Three forms of visual analysis for the 3, 5, 7, ... growing pattern

students were directed to use one of these successful students' methods to analyse the pattern. Pre-post tests indicated that students became successful in identifying pattern rules for growing patterns (nearly to the same level as when tables were allowed) but had an enhanced ability to determine more than one rule (Warren 2005b).

From Fig. 2 it can be seen that the three ways of viewing the commonality evident in Step 3 can be generalised to Step n. All represent different (although equivalent) rules: $n + n + 1$, $2 \times n + 1$, and $2 \times (n + 1) - 1$. This indicates that visual analysis is more effective in encouraging the process of generalisation (Harel 2001). But in each instance students seemed to need to physically separate the pattern into its components (e.g., separate the two columns from the one tile for $2n + 1$) in order to validate these generalisations.

In a follow-up teaching experiment, visual analysis was taught as a skill to students before patterning started. This intervention markedly improved the performance of the students. It was evident that students were unfamiliar with the use of visuals to analyse and trial a variety of solutions.

Change and Functions

The results in the functions segment of EAPT indicate that students can, for more than one operation, identify change rules using the sequence quasi-variable to generalised language to algebraic symbols (with letters) and these generalisations move from recursive to explicit as they become more familiar with functional thinking. Results also show that students can represent real world situations in terms of change and inverse change using all of Duval's representation forms, that is, natural language, figures/diagrams (activity with function machines and drawings of function machines), notation systems (tables, arrow diagrams and equations), and graphs (Warren and Cooper 2007 has some early results). However, enabling students to construct real-world situations for these representations has proved very difficult.

This can be illustrated by the functions lessons undertaken with Year 5 students (aged 9–10 years). These lessons were not the traditional generalisation lessons commonly used to introduce change (lessons where a given set of values is generalised to a change rule), but lessons where abstract representations were being developed for a real-world change situation with a variable and two operations. The

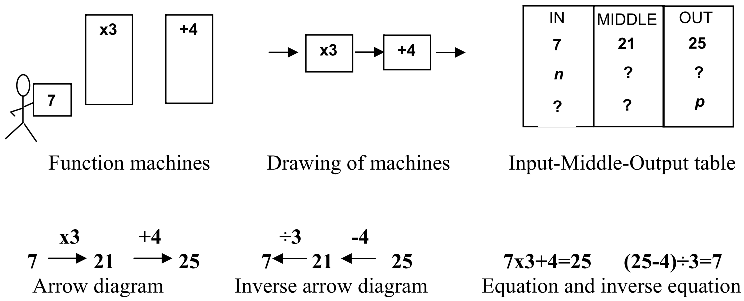

Fig. 3 Representations for change (functions)

lessons commenced with a problem which was modelled with two function machines as a change and this change, and its inverse, were represented on diagrams of function machines, Input-Middle-Output tables and with arrow diagrams and equations (see Fig. 3). The lessons moved to recording function machines on worksheets that focused on developing inverse change (backtracking), representing this change with letters on the Input-Middle-Output table and with arrow diagrams and equations. Then, the activity was reversed, commencing with an arrow diagram and developing tables of values, diagrams of possible function machines and, finally, with the posing of a problem that the arrow diagram could represent.

The lessons were conducted in a Year 5 (9–10 years old) classroom with 29 students in a middle class school. The Year 5 lesson was the third in a sequence across Years 3, 4 and 5 of four teaching experiments that connected real-world problems, action with function machines, drawings of function machines, tables of values, arrow diagrams, equations and graphs, and included opportunities to discuss and record change with letters as variables. The function machines were robot-like creations made from large cardboard cartons with input and output holes represented by 'ears' and operations (e.g., $\times 3$ and $+4$) on their fronts. Each machine was capable of holding two students.

The lessons initially focused on generalising from change problems (e.g., "I bought pies for the visitors for $3 each and a chocolate worth $4") to abstract representations (e.g., arrow diagrams and equations) and algebraic notation (e.g., $n \times 3 + 4$). Students in turn came to the front of the class with input numbers (e.g., 7) which they placed in the first robot's right ear. The student in/behind the first carton/robot exchanged this number for the appropriate change card (e.g., 21) and passed it through the robot's left ear to the student who carried it to the next function machine. The remaining students predicted the changes and tracked them on calculators, calling out, when allowed, the changes in unison. Students were also brought to the front of the classroom with output numbers (e.g., 16) and were directed to "back up" in the opposite direction past the robots (the students actually walked backwards) and passed their cards into the robots in the inverse direction (to introduce backtracking). Once again, the remaining students were asked to predict the changes and to check them with calculators.

After modelling of the function machines had occurred, the lesson continued with students completing a worksheet that required them to: (i) draw a diagram to represent the function machines; (ii) complete an Input-Middle-Output table with numbers and letters in the input, output and middle columns; (iii) represent the change and the inverse change with arrow diagrams and equations; and (iv) write down in general terms "what is happening" with respect to the change and the inverse change. Finally, the tasks were revised in the form of a worksheet where there were columns for all the different representations and only some columns were filled in, often leaving the problem column empty so that students had to invent problems appropriate to the other representations.

The interaction between students and teacher showed that students understood the workings of the function machines in terms of numbers (quasi-generalisation):

Teacher: [standing in between the two function machines with 21 on a card] I've got 21. If it's times 3 here [pointing to first robot] and plus 4 here [pointing to second robot], who can tell me what the number would go to [pointing to the end of the line of function machines]

Student: 25 [selected from many raised hands]

Teacher: If it's 21 here, what would it be that side [pointing to the start of the machines]?

Student: 7 [selected from many raised hands]

Students were presented with a worksheet consisting of three columns, namely input, middle and output. They were also informed that the first machine multiplied the number by 3 (e.g., changed 4 in the input column to 12 in the middle column) and the second machine added 4 to the result (e.g., changed 12 in the middle column to 16 in the output column). The worksheet had numbers or letters in different positions in each row. Students were asked to complete the table and then write the rule. Table 1 summarises the results for this worksheet. A correct response involved the student correctly completing the row.

Table 1 Students' responses on the Input-Middle-Output table for $\times 3 + 4$ change ($n = 29$)

The position of the number on the sheet	Number correct	Number incorrect/not answered
Number 32 placed in Input column	27	2
Number 178 placed in Input column	24	5
Number 34 placed in Output column	28	1
Number 24 in Middle column	22	7
Letter n placed in Input column	12	17
Letter p placed in Output column	10	19
$3 \times q$ placed in Middle column	8	21
What is happening here?	21	8
What happens when things are reversed?	11	18

In spite of the success on the worksheet, only some students could represent the change algebraically. Students experienced little difficulty with representing the problem with arrow diagrams, inverse arrow diagrams or equations (e.g., $3 \times 7 + 4 = 25$), but reversing the equation ($(25 - 4) \div 3 = 7$) caused difficulty. This was due to students not understanding the BOMDAS arithmetic convention (Brackets, Of, Multiplication, Division, Arithmetic, Subtraction) which gives the order of operations. They also did not understand the conventions for reading equations, that is, the need to write the inverse equation left to right (and not right to left as is the case in the reverse arrow diagram). Some of the explanations provided by the students were interesting, for example, a female student stated "The input is multiplied by 3 which gets the middle then add 4 to the middle" while a male student stated "You get the same sum you had before except you get your answer minus 4 then divide by 3 then you should have the same number you started with in the first one."

Finally, the worksheets also contained a section where students were asked to create a real world problem for the linear function represented by the function machines. They found this task to be particularly difficult. We conjecture that this was due to the fact that many classroom teachers unpack word problems as equations but rarely investigate the reverse process.

Equations and Equivalence

Results in the equivalence and equations segment of EATP indicate that very young students can: (i) represent equivalence in equation form in un-numbered and numbered situations, (ii) generalise the Equivalent Class principles for equivalence in un-numbered situations, and (iii) generalise the balance principle for simple equations. Results also show that older students can: (i) represent equivalence with unknowns in equation form; (ii) generalise the balance principle for all operations; and (iii) use the balance principle to solve for unknowns in linear equations.

These findings are illustrated in lessons developed to teach the inverse principle for addition and subtraction as part of the process that leads to solving simple addition and subtraction problems for unknowns. We define the addition and subtraction inverse principles as; if any number is changed through addition/subtraction, then the opposite change (subtraction/addition respectively) by the same amount results in returning to the original number (i.e., $x = x + p - p$ or $x - q + q$ in algebraic symbols). The lessons were conducted in a Year 3 (7–8 years old) classroom in a middle class school and a Year 4 (8–9 years old) classroom in a working class school. The sample was 22 Year 3 and 28 Year 4 students and 2 teachers. The Year 3 lesson was at the end of a sequence of lessons introducing the balance rule for addition and subtraction and focused on the students recognising that, for example, 2 has to be subtracted from $? + 2 = 5$ to have the unknown (represented by ?) on its own and find its value. The lesson was designed to be taught with objects, bags (to represent the unknown) and a balance beam, and diagrams of this beam (to allow

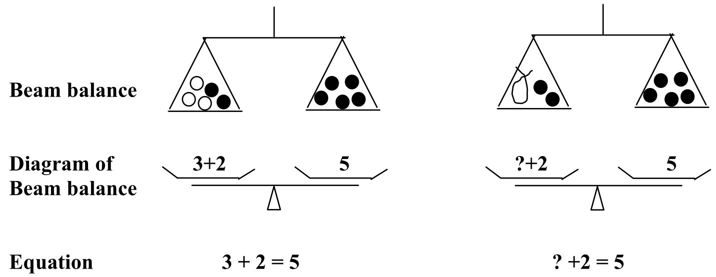

Fig. 4 Representations for equivalence and equations

Table 2 Number of correct responses in terms of inverse balancing action ($n = 22$)

Item: What do you do to both sides?	Correct action
$? + 11 = 36$	22
$? - 7 = 6$	19
$8 + ? = 3$	19
$? - 30 = 54$	15
$2 \times ? = 12$	4
$? \div 3 = 6$	5
$3 \times ? + 4 = 19$	1

for all operations), in order for the students to generalise the process to equations with algebraic symbols (see Fig. 4).

Previous lessons had: (i) connected the beam balance representation with objects to number equations (see Fig. 5); (ii) introduced the balance rule (i.e., adding or removing objects from one side of the equation requires the same action with the same number of objects to the other side); and (iii) introduced the notion of unknown with the cloth bag. This lesson discussed how the value of the unknown could be found by using the balance rule, that is, for $? + 2 = 5$, determining the inverse of the operation and subtracting 2 from both sides. This was reinforced by worksheets requesting the balancing action and the value of the unknown in picture and equation form.

Viewing of the videotape showed that most students could determine, for the example $? + 2 = 5$ represented on the balance, that subtracting 2 from both sides gives the answer 3 for the unknown. This was repeated for the picture worksheet. In this worksheet students were asked to present the action that would result in finding the unknown (e.g., for $? + 11 = 36$ the correct action is subtract 11 from both sides). Table 2 presents the number of students for each task who could provide the correct action.

The correct responses reduced markedly for the questions involving multiplication and division but it should be noted that no reference had been made during the lesson to situations involving multiplication, division or cases with two operation actions and little reference was given to subtraction situations. The multiplication

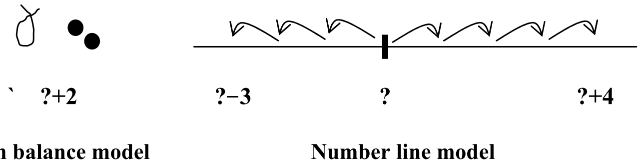

Fig. 5 Beam balance and number line representations for expressions

and division problems were given to see if any students could generalise the balance principle from their experiences with addition to the other operations. Multiplication and division were covered in later lessons and later Years.

The results indicate that some of these young students could successfully transfer the balance principle to other operations. In the Year 4 (8–9 years old) lessons, a new approach was tried, an approach involving subtraction in conjunction with addition. The students first discussed what was required to find solutions to addition and subtraction equations with unknowns. This revolved around determining an action that would leave the unknown on its own. To do this, the lesson focused on the expression that contained the unknown and the operation. This expression was represented in two ways: first by extending the balance representation in Fig. 5 to expressions by removing the balance and the objects for the total (i.e., considering one side of the balance scale), and second by using a number line (see Fig. 5).

The beam balance activity was similar to the Year 3 lesson except that the focus of the discussion and the worksheets was only on the balancing action not finding the unknown's value. The number line activity was new and required the students to place the unknown anywhere and move right for addition and left for subtraction (the students had not used a blank number line in this way before). After this skill was achieved through discussion and worksheets, the students were challenged to determine the change that would return their action to the unknown. Discussion focused on generalising the principle that the unknown could be reached by the inverse operation (-4 for $? + 4$ and $+3$ for $? - 3$) as this was equivalent to adding zero. At this point, the work already completed in functions (see Warren and Cooper 2007) in identifying inverses reinforced the generalisation as did the Mapping Instruction approach of comparing addition and subtraction changes.

A final worksheet was used to ascertain students' understanding of the inverse principle. It contained items that asked students to draw, for example, $+6$ on the number line and to identify the operation that would return one to the unknown. The results were overwhelming; all 28 students were successful for all items. However, students were not asked to write a generalisation and there were no items that referred to, for example, $? + n$, the students were only able to show quasi (Fujii and Stephens 2001) or contextual (Radford 2003) generalisation at best. A viewing of the videotape showed that some children were able to justify their answers in a way that indicates that process generalisation was utilised (Harel 2001).

Interestingly, the inverse and balance principles have the opposing actions (the "opposite" operation for inverse and the same operation for balance). After the successful generalising lesson described above which explicitly identified the inverse principle for expressions with unknowns, some students became confused when this principle was used in conjunction with the balance principle to solve for unknowns in later lessons (we call this a compound difficulty). This was particularly so in the initial teaching of finding solutions to unknowns (e.g., solving $? + 4 = 7$) because getting ? alone requires the inverse operation while the balance rule requires the same operation. Some students interchanged the processes; one student did this twice, he solved the problem by saying that finding the value of ? in $? + 4 = 7$ meant that 4 had to be added to the left hand side which in turn meant that 4 had to be subtracted from the right hand side.

Generalising Principles and Abstract Representations

EATP teaching experiments have also focused on supporting students to generalise principles and abstract representations. Results have shown that students can generalise the Equivalence Class principles for un-numbered situations using unmeasured lengths and masses (Warren 2005a) and principles associated with identity and inverse (e.g., compensation—Warren 2003). They have also shown that students can generalise to the balance principle for more than one operation. More detailed analysis will be necessary before determining whether this generalisation follows Sfard's (1991) or Cifarelli's (1988) sequence. Students have exhibited the ability to use formal equations with unknowns, represented as a box with a question mark inside, for both relationship and change situations.

Lessons to illustrate these findings were the first lessons involving the teaching of the compensation principle for addition and subtraction (Warren 2003; Warren and Cooper 2003). The addition compensation principle is when adding two numbers "do the opposite"; if the first number is increased/decreased by an amount, then the second number is oppositely decreased/increased by the same amount respectively to keep the sum of two numbers the same (i.e., $a + b = (a + k) + (b - k)$ and $(a - m) + (b + m)$ in algebraic symbols). The subtraction compensation principle when subtracting two numbers is "do the same"; if the first number is increased/decreased, then the second number is increased/decreased the same amount respectively to keep the difference between two numbers the same (i.e., $a - b = (a + k) - (b + k)$ and $(a - m) - (b - m)$ in algebraic symbols).

The lessons were conducted in two Year 3 (7–8 years old) classrooms, one from each of two middle class schools. The sample comprised 45 students and 2 teachers. The lessons were designed to be taught with strips of papers (an un-numbered situation), then sets of counters, and finally, for addition situations, to having the students act out relay races and use number lines and measuring cylinders (see Fig. 6 for representations for addition) to model these situations. Worksheets were specially developed to reinforce the principles; students were asked to predict, justify and generalise their findings.

The students were asked to generalise at the end of the strips and number activity. As Warren (2003) evidences, even though length generalisation had been done

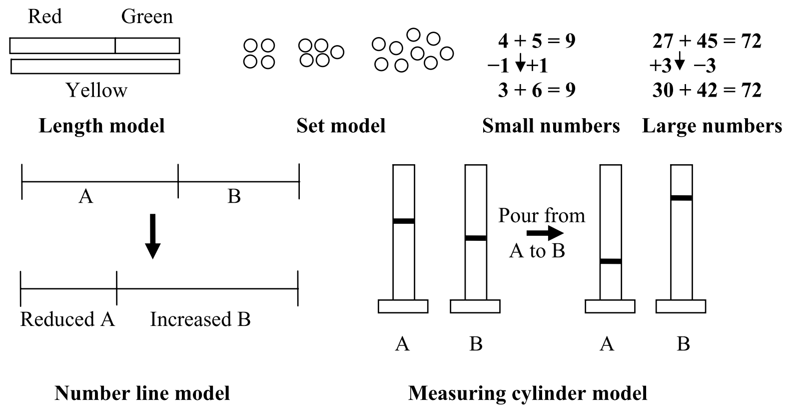

Fig. 6 Models and representations for addition compensation

before number generalisation was discussed, they were more successful at generalising for the strips of paper than they were for the number situations, showing the strength of un-numbered situations in generalisation (in line with the findings of Davydov 1975; Dougherty and Zilliox 2003). The number line model was very effective after the students had experienced the strip model as it clearly showed that a change to one length automatically indicated the compensation that had to be made to the other length. However, the measuring cylinder model did not work because although the idea of pouring from cylinder A to B had inbuilt compensation, there was no connection between A and B (as there was in the number line) and no relation to a total. Mapping instruction activities were also attempted at the end of both lessons; the students were asked to compare the length and number activity at the completion of the first activity and to compare increasing and decreasing numbers at the end of the second lesson.

The researchers followed the addition-compensation lessons with subtraction compensation; however, students had great difficulty with the principle. Once again the same sequence of steps occurred, the papers strips were followed by sets of counters and then equations with small and larger numbers. However, to show the three numbers, subtrahend, minuend and difference, required the materials to be used for subtraction as comparison not takeaway, a more difficult meaning for subtraction. Secondly, the generalisation was opposite to addition (subtraction compensation requires the same thing to be done to both numbers) and this confused the students.

Little headway was made until the researcher/teacher stopped the sequence and organised the students to act out the situation using the model of a relay race. Addition compensation was conducted first with pairs of relay walkers completing a distance. Through actions, the students came to realise that when the first walkers increased their distance, the second had less to walk. Then the activity was changed to one walker walking too far and the second returning to the 'finish position.' Again, through actions, the students identified that when the first walker increased the distance they walked past the 'finish line', the second had more to walk back to the 'finish line'. This worked successfully at least in ensuring that students understood that there was a difference between the addition and subtraction compensation processes.

Thus, the lesson showed three things. First, it again reinforced the efficacy of the Mapping Instruction approach (English and Halford 1995) in teaching generalisation. Second, it showed the power of kinaesthetic activity as a beginning generalisation activity (supporting gesturing as a beginning generalisation step—Radford 2003). Third, similar to the equations and equivalence lessons, it also showed the "compound" difficulty that occurs when two principles have opposite effects (showing that structural understanding is difficult to build in small steps).

Conclusions and Implications

Analysis across all years, topics and representations for the EATP data set is ongoing and thus conclusions are tentative. However, evidence for implications for young children's learning of algebraic thinking exist in terms of (i) models and representations, (ii) generalisation overall, and (iii) a theoretical framework for instruction integrating all three segments that has application beyond Years 2–6 early algebra. Some of these are specific to the models and representations, and activities, presented to students in EATP. Other implications appear to apply more generally across mathematics teaching and learning, particularly those associated with the theoretical framework.

Models and Representations

In summary, our use of models and representations in EATP appears to have been successful with regard to learning and generalisation. The use of blocks and sticks and step cards (Fig. 1) allowed visual analysis of growing patterns (Fig. 2), leading to noticing of local commonalities and generalisation to pattern rules. Using objects and repeat cards has also been successful with repeating patterns in leading to generalisation of object relationships and equivalent fractions and equivalent ratios (Warren 2005b). The sequence of function model representations (function machines, diagrams of machines, tables, arrow diagrams, equations and graphs—Fig. 3) has been successful in building understanding except in inverse equations and posing of problems. The function machine, particularly in the form of a large robot made out of cartons, resulted in highly motivational function lessons. Backtracking was illuminated by physically backing students in a reverse direction past the function machines.

The balance models and the beam representation with different coloured small cans of food and cloth bags has motivated the study of equations and equals and this model effectively could be mapped to diagrams and equations (see Fig. 4). The number line models have also proved effective (see Fig. 5). There has also been a positive effect on understanding compensation principles in using all three models, balance (mass), number line (length) and set (objects), at the same time. The balance/mass model has been particularly effective with regard to the balance rule.

With regard to the effect of model and representation on generalisation, the importance of making connections between representations (Hiebert and Carpenter 1992) and conversions between registers and domains (Duval 1999) has been highlighted. Central to this are the socio-constructivist theory of learning, inquiry based discourse and the simultaneous use of multi-representations to build new knowledge. The major representations used in EATP were effective, particularly in the way sequences of representations were used from acting out with materials through diagrams to language and symbols. In particular, the following model-representation sets were very effective in motivating students, solving problems and building principles and structure: (i) function machines, Input-Output tables and arrow diagrams; (ii) beam balances, cloth bags and objects and their pictures; and (iii) walking relays, paper strips and number lines.

Enactive, iconic and symbolic sequences of representations seem to be important in building towards generalisation (the function machine, diagram, table and arrow diagram sequence in functions). However, direct teaching of analytic skills in areas where experiences were weak was found to be necessary. For example, students found difficulty generalising visual representations in patterns because they had little ability in perceiving the visuals in different ways. More success occurred when a variety of perceptions for visuals was directly taught. This enabled the students to quickly trial different perceptions to see the highlighted commonalities in the visuals. Before this teaching, generalisation from visual or iconic contexts was less robust than for symbolic (Radford 2006).

The use of *quasi-generalisations* appears to be a powerful symbolic strategy towards generalisation in symbolic contexts that, as yet, has no iconic or enactive counterpart. However, the negative effects of closure in numbered situations appeared to prevent generalisations that were facile in un-numbered situations (e.g., the Equivalence Class principles). Finally, understanding and communicating aspects of representational forms in a variety of contexts appears to allow commonalities to be seen across or between representations (e.g., inverses in function machines and number lines).

Generalisation

It is evident, even in the few lessons analysed in this paper, that some findings appear to have constant application in generalisation.

First, students can learn to understand powerful mathematical structures, usually reserved for secondary school, in the early and middle years of elementary school if instruction is appropriate (at least in language and quasi-generalisation form). This is particularly so for the principles associated with the Equivalence Class and Field structures. In EATP, because of the separate focus on relationships through equations and change through function machines, there was cross over for the identity and inverse principles that reinforced these structures in both perspectives. A teaching focus on structure is a highly effective method for achieving immediate and long term mathematical goals.

Second, Radford's (2003, 2006) distinction between grasping and expressing generalities is important; these are two different things and can be confused by the teacher. Many times, students' problems are in expression not identification. Students often lacked the language with which to discuss generalisation and lessons often became language development (e.g., down patterns and across patterns to distinguish between recursive and explicit pattern rules).

Third, although they were developed for older students, some theories regarding development of generalisation appear to have strong application in early generalisation. This is particularly so for: (i) Harel's (2001) theory regarding results and process generalisation; (ii) Radford's (2003, 2006) theory regarding factual, contextual and symbolic levels of generalisation; and (iii) Fujii and Stephens (2001) notion of quasi variable (which we have extended to quasi-generalisation). Harel directs us towards justifying as well as identifying generalisation, Radford towards role of gestures (action, movement) and language in early generalisation and Fuji and Stephens towards the acceptability of number-based descriptions of generalisations. As well, Lannin's distinction between recursive (single variance) and explicit (covariance) has strong application in tables of values.

English and Halford's (1995) Mapping Instruction teaching approach to principle generalisation has also proved its efficacy in many lessons. It directs us towards comparing activity from different domains (e.g., addition and subtraction) and activity from different representations (e.g., balance and length). Some attempts have been made with the Mapping Instruction teaching approach to develop generalisations by having students focus on similarities between inverse operations in functions and inverse operations in equations. This cross over between the relationship and change perspectives appears to be one reason why there has been so much success with the symbolic work in backtracking and the use of the balance rule for finding solutions to unknowns in linear equations.

Fourth, EATP has shown that learning can be enhanced by creative representation-worksheet partnerships that can reinforce connections between representations. Often teachers restrict worksheets to the symbolic register. EATP has shown that creative use of pictures and directions can allow a worksheet to reinforce understandings as well as procedures and to highlight principles.

Fifth, some activities necessary for building structure affect cognitive load. This is particularly so when large numbers are used to prevent guessing and checking as a strategy for determining answers and direct students towards the principle. Furthermore, the two examples in this paper have shown the compounding effect of building structure through small steps. It is necessary to build a superstructure into which to place conflicting principles such as compensation for addition and subtraction and inverse and balance for solutions of linear equations.

Finally, although EATP involved much creative lesson development and many new activities and outcomes, the problems did not really exist with the new work but with the basic arithmetic prerequisites. Once numbers appeared students attempted to close on operations and did not attend to pattern and structure to the same extent as in un-numbered situations (as found in previous research by Davydov 1975). Furthermore, students' abilities to interpret and create real world situations in terms of the actions with materials, diagrams/figures and symbols of early algebra, lagged far behind their abilities to process the representations and was a constant difficulty in EATP, a difficulty that increased as the cohort of students moved into their middle school years. The reason for this appeared to be the lack of teaching of this creation within the schools. The increased difficulty was a result of examples in higher years having more complexity (i.e., involving multiplication and division, and more than one operation).

Theoretical Framework

In Warren and Cooper (2009), we analysed the instructional sequence of equivalence and equations activities described in this paper in greater depth in order to find hypotheses that would explain and predict teaching and learning behaviour. We identified nine conjectures that related to the development of equations-equivalence knowledge and growth in generalisation ability. These conjectures represented a growth in integration of models and representations that moved towards less reality (physicality) of representations as the complexity of the tasks and generalisations increased. From a reappraisal of their relationships, the following 6 theory hypotheses emerged as a basis for an EATP theory for a teaching/learning trajectory designed to generalise to abstract representations. As this brief analysis here shows, these theory hypotheses are also supported by the whole of the EATP project; instructional activities that were effective for pattern, function machine and generalisation as well as those for equivalence and equations.

Theory Hypothesis 1 Generalisation to abstraction occurs not within a model or representation but across models and representations that follow a structured sequence. This is a hypothesis that is a consequence of all activity across the topics and Year levels. There appears to be no 'magic bullet'; abstraction is generalised from model to model and representation to representation. This was particularly evident in the building of inverse and backtracking (using function machine representations), the equivalence and balance principle (using balance and number line models) and the compensation principle (using length and set models) across Years 2 to 6, through progressively more abstract representations and by integrating balance, number line, set and function machine models and representations (see Figs. 3, 4, 5 and 6).

Theory Hypothesis 2 Effective models and representations show underlying structure of the mathematics ideas and easily extend to new components and expand to new applications. This particularly applies to the models and representations that begin the teaching/learning trajectory; here the criteria for determining effective models is: (i) strong isomorphism between the desired internal mental model outcome and initial external model that covers the important aspects of the mental model; (ii) lack of distracters to direct attention away from isomorphisms; and (iii) many options in terms of representations that enable the model to extend to new components (such as variables) and expand to new applications (such as finding solutions to problems). Both the balance and number line models have these attributes; the number line model was stronger in inverse but the balance made up for this in its powerful portrayal of equivalence as balance. The function machine model was particularly strong in its representation options from drawings, tables and arrow diagrams to graphs. In a slightly different manner, the grouping of objects into repeats in repeating patterns enabled the underlying structure in growing patterns to seamlessly emerge.

Theory Hypothesis 3 An effective way to structurally sequence models and representations to generalise to abstraction is to have models with representations that develop in four ways: (i) *increased flexibility*, following the general sequence concrete to dynamic diagram to static diagram to symbols (e.g., physical balance to drawing of balance, blocks for number to symbols); (ii) *decreased overt structure*, following the sequence ‘structure in action’ to ‘structure alluded to in picture’ to ‘structure imagined in the mind’; (iii) *increased coverage*, later representations compensate for limitations in earlier representations (e.g., the balance drawing handles greater flexibility in operations than the physical balance); and (iv) *connectedness to reality*, always relating the form of the representation to real world instances. The balance model is particularly powerful in terms of its sequence of increased flexibility and coverage as it moves from physical to diagram representations (see Fig. 4); if begun with strips of paper, the number line model was also powerful in showing, in particular, increased flexibility and decreased structure (see Fig. 6). Similarly, the function machine model showed all three modes of development as it moved from physical to visual to mental representation form (see Fig. 3).

Theory Hypothesis 4 Sequencing should ensure consecutive steps are *nested*. This is a particularly important hypothesis. The nesting of models and representations to ensure that later versions are subsets of earlier versions causes difficulties and conflicts if not acceded to. This was most clearly evidenced by the limiting of generalising ability created by teachers giving prominence to arithmetic computation before equivalence was taught. Covering forms like $4 + 5 = 11 - 2$ and $3 \times ? - 7 = 8 - 2 \times ?$ means equation is a much more general equivalence form than computation (e.g., $4 + 5 = ?$); computation is a subset of equation. This hypothesis is also important because it implies that the engagement with unnumbered situations before numbered enables students to effectively attend to mathematical structure, thus reinforcing the work of Davydov (1975). This was seen in the reduction of generalising thinking seen when the balance and function machine models moved from objects (sugar = juice + soap; changing colour) to numbers ($6 = 2 + 4$; adding 3). In patterning, this hypothesis supported generating patterns from visual analysis before numeric tables; the visual methods were stronger in terms of flexibility of results and depth of generalisation (see Fig. 2).

Theory Hypothesis 5 Complex procedures can be facilitated by integrating more than one model. This is best evidenced by the way balance and number line models were used together at the point of solving linear equations with an unknown. However, such integrations can give rise to compound difficulties which require the development of superstructures (see discussion under generalisation heading). As shown in Figs. 4 and 5, compensation in balance is direct while compensation on the number line is inverse causing difficulties for students. Similarly, the compensation rules for addition and subtraction are likewise contradictory and require an understanding that subtraction is the inverse of addition. However, the notion of such superstructures is not well developed in the literature, especially with regard to integrating models to develop 'deep' understanding of concepts.

Theory Hypothesis 6 Abstraction is facilitated if comparisons of different models and representations of the same mental model show commonalities that encompass the kernel of the mental model. This reflects the success of using the number line and balance models for the same purpose (solving the equation), particularly in terms of the extension to variables on both sides of the equation (the didactic cut) and simultaneous equations. It also implies that effective structured sequences of models and representations are dual, built around at least two models that act as a spine for the development of the mathematical idea. This is seen in the integration of visual and numeric methods for finding patterns as position rules, the different models and representations (balance, number line; visual and graphical) for the balance model (see Fig. 4) and the function machine model (see Fig. 3), and the way calculators are integrated with the function-machine processes.

Final Point The hypotheses above offer promise as the beginning of a theory of model and representation use in learning-teaching trajectories for generalisation to abstract representations. This is seen across topics as well as within topics. For example, we found it more effective in the later Year levels to *cover functional thinking before equivalence and equations* within each year. This was because function activity built a strong superstructure around the inverse and identity principles which: (a) assisted in finding solutions of linear equations with an unknown; and (b) prevented conflict between inverse and balance, and the development of compound difficulties, in the solution process (Cooper and Warren 2008).

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Algebra in the Middle School: Developing Functional Relationships Through Quantitative Reasoning

Amy B. Ellis

Abstract Understanding function is a critical aspect of algebraic reasoning, and building functional relationships is an activity encouraged in the younger grades to foster students' relational thinking. One way to foster functional thinking is to leverage the power of students' capabilities to reason with quantities and their relationships. This paper explicates the ways in which reasoning directly with quantities can support middle-school students' understanding of linear and quadratic functions. It explores how building quantitative relationships can support an initial function understanding from a covariation perspective, and later serve as a foundation to build a more flexible view of function that includes the correspondence perspective.

Functions and relations comprise a critical aspect of algebra, and recommendations for supporting students' algebraic reasoning emphasize an early introduction of functional relationships in middle school (NCTM 2000). Students' difficulties in acquiring the function concept is well documented (e.g., Carlson 1998; Carlson et al. 2002; Cooney and Wilson 1996; Monk and Nemirovsky 1994), which highlights the need to better support students' emerging function concepts in ways that are mathematically productive, setting a strong foundation for more formal algebraic reasoning at the high school level. In this chapter I argue that reasoning directly with quantities and their relationships constitutes a powerful way to help students build beginning conceptions of function at the middle-school level. In particular, reasoning with quantities can directly support a covariation approach to function, while also providing a foundation for reasoning more flexibly with functional relationships later on.

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What Is Quantitative Reasoning?

Quantities "are attributes of objects or phenomena that are measurable; it is our *capacity* to measure them—whether we have carried out those measurements or not—that makes them quantities" (Smith and Thompson 2007, p. 101, emphasis original). A quantity is composed of one's conception of an object, a quality of the object, an appropriate unit or dimension, and a process for assigning a numerical value to the quality (Kaput 1995); length, area, speed, and volume are all attributes that can be measured in quantities. When students engage in quantitative reasoning, they operate with quantities and their relationships; quantitative operations are therefore conceptual operations by which one conceives a new quantity in relation to one or more already-conceived quantities (Ellis 2007). For example, one might compare quantities additively, by comparing how much taller one person is to another, or multiplicatively, by asking how many times bigger one object is than another. The associated arithmetic operations would be subtraction and division.

To illustrate the differences between a formal algebraic approach and a quantities-based approach, consider two responses to the following problem about the nature of quadratic growth:

Problem 1 Explain why the "second differences" for a quadratic function $y = ax^2$ are $2a$ for well-ordered tables in which the x-values increase by 1.

This problem emerged from an algebra II classroom in which the students' introduction to non-linear functions included an algorithm for determining the degree of a function based on the finite differences rule (Ellis and Grinstead 2008). The students easily remembered this algorithm, but it was unclear whether anybody understood its origins.

Justification #1: A typical algebraic argument involves relying on variables to represent a general case and writing and manipulating expressions. For instance, one can create a general table for $y = ax^2$ in which the x-values increase by 1:

Fig. 1 Table of x- and y-values for $y = ax^2$

x	y
1	a
2	$4a$
3	$9a$
4	$16a$

Calculating the first differences reveals values of $3a$, $5a$, and $7a$. Calculating second differences reveals a constant second difference of $2a$, and this approach can be generalized to any three consecutive entries in the table in which $x = n$, $x = (n + 1)$, and $x = (n + 2)$. Corresponding y-values will be $y = an^2$, $y = a(n + 1)^2$, and $y = a(n + 2)^2$. Calculating the first differences reveals values of $a(2n + 1)$ and $a(2n + 3)$, with the second difference therefore $2a$. This approach lives entirely in the world of symbolic expressions in a manner that is divorced from any realizable situation and its constituent quantities. As a formal algebraic justification it provides a valuable opportunity to generalize beyond specific numbers, but it may fail to support students' understanding of the behavior of quadratic growth, what the second differences can represent, and why they remain constant for quadratic functions.

Justification #2: One group of eighth-grade students created and analyzed tables of quadratic data by exploring the relationships between the lengths, heights, and areas of rectangles that grew while maintaining their length/height ratios. One student attempted a justification by imagining an $H \times L$ rectangle that grew in discrete increments by increasing H units in height and L units in length. The student conceptualized the first differences, which he called the *rate of growth* (RoG), as the growth of the area when the height increased by H units. He conceptualized the second differences as the *difference in the rate of growth* (DiRoG), describing it as the "rate that the increase in the area is increasing:"

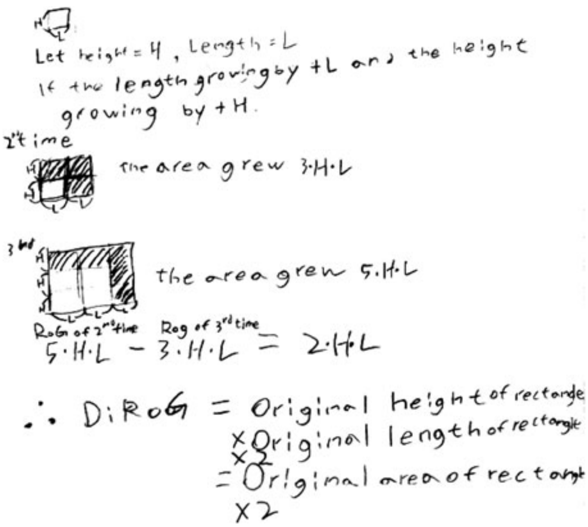


Fig. 2 Eighth-grade student's justification

The student reasoned with the relationships between the quantities height, length, and area, engaging in quantitative operations as he compared their differences. He concluded that because he could calculate the difference in the rate of growth as $2HL$ each time the rectangle grew an additional H units in height and L units in length, the second differences must represent twice the original area of the rectangle.

The student's justification contains some limitations, particularly because his drawing only addresses a particular type of growth in which the height and length increase by whole-unit increments of H and L. However, even though the student did not reason about arbitrary increases of H and L, his justification represents a meaningful attempt at a generalized argument. The student's reliance on the relationships between the quantities height, length, and area helped him develop an understanding of what the second differences represented, which provided a springboard for further investigation of why the second differences in well-ordered tables are always constant for quadratic functions.

Steffe and Izsak (2002) argue that quantitative reasoning should be the basis for algebraic reasoning. Focusing on relationships between quantities, rather than on numbers disconnected from meaningful referents, can ground the study of algebra in people's conceptions of their experiential worlds (Chazan 2000). This provides a meaningful starting point for mathematical inquiry, in contrast to taking numbers, shapes, and relationships as givens in their own right (Thompson 1994). I propose that adopting a quantitative reasoning approach can support students' meaningful engagement with algebra in general and with functions in particular. I will present the results from two teaching experiments with middle-school students, the linear functions teaching experiment and the quadratic functions teaching experiment. Excerpts from both teaching experiments demonstrate a number of ways in which students' reasoning with quantities fostered particular types of function understanding.

The Importance of (and Difficulties with) Functional Thinking

The function is a central concept around which school algebra can be meaningfully organized (Kieran 1996; Yerushalmy 2000), and many researchers have argued for the importance of a functional perspective in contrast to the more traditional approach that focuses on algebra as symbolic manipulation (Bednarz et al. 1996; Schliemann et al. 2007). Adopting an approach that places functional relationships at the center of algebra allows us to couch algebraic thinking as the use of a variety of representations in order to make sense of quantitative situations relationally (Kieran 1996). Beyond theoretical considerations, there are also practical reasons for emphasizing a functional approach to algebra. Attaining a deep understanding of function is critical for success in future mathematics courses (Carlson et al. 2003; Romberg et al. 1993) and in courses on scientific inquiry (Farenga and Ness 2005). Many have argued that the function concept is foundational for understanding concepts in advanced mathematics (e.g. Kaput 1992; Rasmussen 2000; Thompson 1994; Zandieh 2000), and as Romberg et al. (1993) argued, "there is general consensus that functions are among the most powerful and useful notions in all mathematics" (p. 1).

Given the widespread agreement on the importance of functions for algebraic reasoning, the value of organizing algebra content around a functions approach, and the need for a deep understanding of functions for further mathematical and scientific inquiry, it is important that we develop ways of helping students successfully understand functional relationships. However, these endeavors have proved difficult: Many studies conducted to investigate students' function understanding suggest that they demonstrate a limited view of the function concept (e.g. Carlson 1998; Sfard and Linchevski 1994; Thompson 1994; Vinner and Dreyfus 1989). In general, students emerge from middle school and high school algebra classes with a weak understanding of function (Carlson et al. 2002; Cooney and Wilson 1996; Monk 1992; Monk and Nemirovsky 1994).

Two examples from my previous studies illustrate some of the common difficulties students experience when encountering functional relationships. The first comes from a problem presenting a direct-ratio situation within the context of a linear functions unit (Ellis 2009):

Problem 2 Say you have a pile with 2 rolls of pennies and a pile with 5 rolls of pennies. If you were to compare their weights, what might you notice?

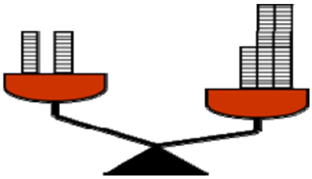


Fig. 3 Picture accompanying the penny-role problem

One eighth-grade student, Juanita, made both additive and multiplicative comparisons across the two piles, noting that the bigger pile had 3 more rolls, and would weigh "2.5 times as much" as the smaller pile. When she investigated the pattern in a tabular form, however, Juanita was unable to recognize the relationship as linear and she could not develop an equation for the data:

Fig. 4 Table of number of rolls and weight values

# of Rolls	Weight
2	9 oz
5	22.5 oz
12	54 oz
16	72 oz

AE: What does this table tell you?
J: It couldn't be a straight line.
AE: How come?
J: (Calculating differences between successive x-values in the table): 3, 7, 4, and that's probably not an even spaced one. Wouldn't be a straight line.
AE: And what's your reason for that?
J: If you made your graph, it doesn't look like it'd be a straight line because it goes up (calculates differences between successive y-values in the table) by 13.5 and 31.5 and then 18.

Juanita created a rough sketch of the data to confirm her belief that the data were not linear:

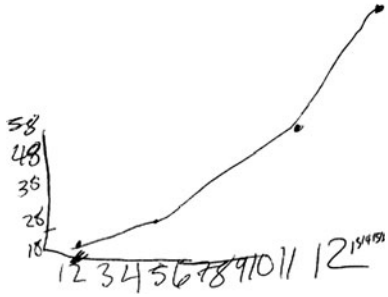


Fig. 5 Juanita's graph of the rolls and their weight

Determined to find a pattern for the data in order to come up with an equation, Juanita continued to search by taking the differences between the rolls and the weight for each table entry, and then taking the differences of those results:

J: 7, 17.5, 42, 56. That's what it goes up by. If you do in between them it's 10.5, 24.5, and then 14. … There's no patterns anywhere!

Juanita's difficulty in recognizing the data as linear and her inability to create an equation mirrors some of the documented difficulties students experience with tables, patterns, and functions. Although students are adept at searching out patterns in tabular representations, many struggle to perceive a functional relationship (MacGregor and Stacey 1993; Mason 1996; Schliemann et al. 2001). Even when students are able to detect patterns, they may not be able to formalize those patterns correctly by writing appropriate equations or algebraic expressions (English and Warren 1995; Orton and Orton 1994; Stacey and MacGregor 1997). Students struggle to correctly translate between tabular, graphical, and algebraic representations of functional relationships, and can become overly dependent on particular artifacts of representations, such as only recognizing a function as linear if its tabular representation has uniformly-increasing x-values (Lobato et al. 2003).

A second example illustrates some of the difficulties students can encounter when approaching non-linear functions. High-school algebra II students encountered the following graph and attempted to find an equation for the parabola (Ellis and Grinstead 2008):

Problem 3 Ravi has 120 meters of fence to make his rectangular rabbit pen. He wants to enclose the largest possible area. Here is Ravi's graph of the relationship between the width and the area:

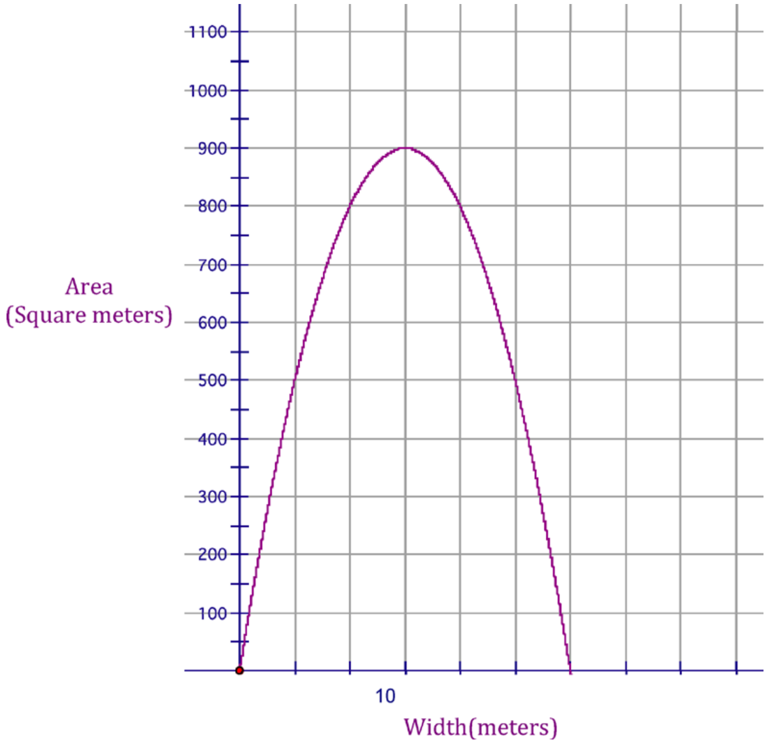


Fig. 6 Graph for the rabbit pen problem

Alexis, a tenth-grade student identified by her teacher as a high performer, determined that the equation for the parabola should have the form $y = -ax^2 + 900$, because the maximum value of the parabola was at $y = 900$. In addition, Alexis knew that the a-value should be negative, because the parabola was "upside down." In order to determine the value of "a", Alexis explained, "you could do this, rise over run." She picked two points, (10, 500) and (20, 800), and then calculated the rise and the run, ignoring the scales on the axes: "So it's 3 over 1, which is basically 3." Alexis concluded that the equation of the parabola should therefore be $y = -3x^2 + 900$.

Alexis' treatment of the graph and development of an equation reflects the research demonstrating students' difficulties understanding the value of "a" in $y = ax^2 + bx + c$ (Dreyfus and Halevi 1991; Zaslavsky 1997). The challenges in connecting algebraic and graphical representations of quadratic functions can further contribute to students' struggles to describe the effects that changing the parameters a, b, and c have on graphs of parabolas (Bussi and Mariotti 1999; Leinhardt et al. 1990; Zazkis et al. 2003). In addition, Alexis' inappropriate adoption of the rise over run method for generating a "slope" mirrors many students' tendencies to generalize from linearity, regardless of the appropriateness of that generalization (Buck 1995; Chazan 2006; Schwarz and Hershkowitz 1999; Zaslavsky 1997).

Given the widespread difficulties students experience as they learn about functions, it is important to develop methods for helping students build a productive understanding of functional relationships from the time that they first experience them in the algebra classroom. Taking a quantities-based approach to informal (and later formal) functional reasoning can support students' initial approaches to functional relationships as they explore coordinated changes between covarying quantities.

An Alternative Approach to Function: Quantities and Covariation

Traditional approaches to function rely on a correspondence or stasis view (Smith 2003), in which one approaches a function as the fixed relationship between the members of two sets. Farenga and Ness (2005) offer a typical correspondence definition of function: "One quantity, y, is a function of another, x, if each value of x has a unique value of y associated with it. We write this as $y = f(x)$, where f is the name of the function" (p. 62). This static view underlies much of school mathematics, particularly in the treatment of functions. Alexis' approach reflects this typical school experience, as she examined the graph and then attempted to build an equation without imagining the two quantities changing together. Instead, Alexis' treatment disconnected the properties of the graph and its associated equation from the contextual situation that referenced the changing relationship between width and area.

In contrast, Smith and Confrey (Smith 2003; Smith and Confrey 1994) describe the covariation approach to functional thinking. Under this approach, one examines a function in terms of a coordinated change of x- and y-values. Confrey and Smith (1992, 1994, 1995) have found that students' initial entry into a problem is typically from the covariational perspective. In addition, they argue that viewing a function as a way of representing the variation of quantities can be a more powerful approach than the correspondence model, particularly in its ability to promote thinking about functions in terms of rates of change (Slavit 1997; Smith and Confrey 1994). As Chazan (2000) argues, the covariation approach can support a view of mathematics as a way of making sense of the phenomena of relationships of dependence, causation, interaction, and correlation between quantities.

Viewing a function as a relationship between covarying quantities is part of a larger idea that acknowledges the importance of the mathematics of change. An emphasis on the mathematics of change can encourage students to examine patterns in relationship to the ways in which they grow or can be extended. Many have suggested that this approach is a critical but overlooked element in the standard U.S. curriculum (Nemirovsky et al. 1993; Mokros et al. 1995). Exploring function as a way to measure change and variation is typically reserved for calculus, thus effectively restricting access to these ideas to the 10% of students who will reach the highest level of high-school mathematics (Roschelle et al. 2000). However, adopting a rate-of-change perspective can be accessible even for beginning algebra students in middle school. One way to foster students' understanding of the mathematics of change is through introducing rich situations that encourage students to construct meaningful relationships between quantities.

For instance, one group of seventh-graders in a linear functions teaching experiment explored constant rates of change by investigating two situations, gear ratios and constant speed (Ellis 2007). The group consisted of 7 pre-algebra students who had not yet studied linear functions or graphs in their mathematics classroom, and a focus of the teaching experiment was to emphasize the activities of generalizing and justifying through meaningful engagement with quantitative referents. The students met for 15 sessions and during the first eight sessions they worked with physical gears to examine different gear ratios. Early in the sessions, the students connected a gear with 8 teeth to a gear with 12 teeth and then spun the gears together, trying to identify ways to simultaneously keep track of the rotations of both gears. By putting small pieces of masking tape on one of the teeth of each of the gears, the students devised a counting system for keeping track of both gears' rotations simultaneously, and ultimately created tables of gear rotation pairs such as the following:

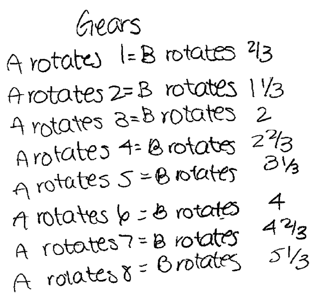


Fig. 7 Maria's table of gear rotations

By working with the physical gears, the students not only found ways to coordinate the rotations of each of the gears, but also developed a covariation language for discussing the nature of the coordinated quantities. For instance, in describing the table in Fig. 7, Dora explained, "For a small turn, the big one goes a two-thirds turn. For the big to turn once, the small one goes one and a half turn."

Carlson and Oehrtman (2005) note that students need to be able to imagine how one variable changes while imagining changes in the other. Relying on situations that involve quantities that students can make sense of, manipulate, experiment with, and investigate can foster their abilities to reason flexibly about dynamically changing events. These experiences were helpful when the linear functions students eventually encountered tables of data referencing multiple rotation pairs, such as the one shown in Problem 4:

Problem 4 The following table contains pairs of rotations for a small and a big gear. Did all of these entries come from the same gear pair, or did some of them come from different gears altogether? How can you tell?

Fig. 8 Table of gear pairs

Small	Big
7 1/2	5
27	18
4 1/2	3
16	10 2/3
1/10	1/15

Dora explained her thinking about the problem by referencing the gears:

D: Think of a gear. When you spin it, the teeth on it pass through. One gear has 8 teeth, the other has 12. When you spin them, teeth pass through each other. For every two-thirds of the teeth passed on the big one, that's 8 teeth, so the small one turns once. If the small one goes 3 turns, the big one will go 2. So if the small one goes 7 and a half times, the big gear will go 5.

A covariation approach can also ultimately support students' abilities to express function relationships algebraically. After hearing Dora's explanation, another student, Larissa, expressed the gear ratio relationship by writing "$s(2/3) = b$", which represents the number of rotations between the small gears and the big gears. Larissa explained, "s is the number of small rotations, the number of rotations that the small gear does. And then b is the big rotations, the number of rotations that the big gear makes."

Carlson and Oehrtman identified a covariation framework (2005), in which they decompose covariational reasoning into five mental actions. This decomposition has proved useful for promoting covariational reasoning in students. Although the framework evolved in the context of calculus students' reasoning, the first four mental actions described can also apply to algebra students. (The fifth mental action is the coordination of instantaneous rate of change, which is not as applicable to beginning algebra topics.) The gear rotation situation supported students' abilities to coordinate the change of both quantities simultaneously, fostering the first three mental actions in the covariation framework: (1) coordinate the dependence of one variable on another variable, (2) coordinate the direction of change of one variable with changes in the other variable, and (3) coordinate the amount of change of one variable with changes in the other variable. Because quantitative reasoning requires the formation of relationships between quantities, students' activity in constructing these relationships can support the meaningful coordination of variables in function relationships.

The fourth mental action is the coordination of the average rate-of-change of the function with uniform increments of change in the input variable. Exploring phenomena that are linearly related but not in direct proportion can prompt a shift from direct multiplicative comparisons to the creation of ratios of change between co-ordinated variables. In the gear context, the students examined scenarios in which one gear spun a certain number of times on its own before a second gear was connected to it, at which point they spun together. Although the situation is somewhat contrived from an adult perspective, it was meaningful to students because it described a familiar situation that they could directly imagine. The following table can encourage students to coordinate the rates of change of each of the variables, both because it is not well ordered and because it represents a situation that is not directly proportional (the function described by the ordered pairs is $y = (3/4)x + 5$):

Problem 5 The following table contains pairs of rotations for a big and a small gear. What is the relationship between the two gears?

Fig. 9 Table of gear pairs representing a $y = mx + b$ situation

Small	Big
1	5 3/4
4	8
12	14
25	23.75

One student, Timothy, identified the differences between successive table entries:

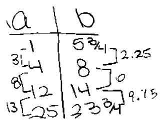


Fig. 10 Timothy's calculations with the gear pair table

He explained, "The only thing I found out is that they go up by 3/4, because if you subtract 1 from 4 and 5 and 3/4 from 8, you get 3 and 2.25, and 2.25 over 3 equals 3/4. And that's how I found out that it works for all of them." Pushed to explain why this worked, Timothy said, "B goes up by 3/4 of what A goes up by." When asked to describe what was happening with the gears rotating, he noted, "B had already turned 5 times. And B is like 3/4 the size of A. And so A times 3/4 means that it only goes through 3/4 of its teeth." When Timothy noted that B was 3/4 the size of A, he spoke of the gear's size but appeared to be thinking about the gear's rotations instead; this is consistent with the second half of his statement in which Timothy said that B would only go through 3/4 of its teeth. Dora and several other students expressed this relationship algebraically by writing "$(3/4)a + 5 = b$", and could explain each part of the equation in terms of the relationship between the gears' rotations and number of teeth. Ultimately the students were able to approach new data by calculating the ratio of the change of one variable to the coordinated change in the other variable in order to determine the appropriate relationship between mystery gears.

Students' first approaches to function are typically covariational in nature (Confrey and Smith 1992, 1994, 1995), but it is important to support these initial forays in a manner that supports a meaningful understanding of covarying phenomena, in contrast to the common tendency to engage in recursive pattern seeking with naked numbers. Although students are adept at creating multiple patterns, they can struggle to identify patterns that are algebraically useful and generalizable. Embedding these patterns in meaningful problem situations that require students to identify relationships between covarying quantities can help circumvent the common pattern-seeking traps that sometimes plague students. Quantity-based problem situations can instead "serve as the true source and ground for the development of algebraic methods" (Smith and Thompson 2007, pp. 96–97).

A Flexible Understanding of Functions

Coordinating Covariation and Correspondence Approaches

The prevalence of covariation approaches has been highlighted in the research literature, and this view provides a powerful mechanism for developing an understanding of function as a way of representing variation in coordinated quantities. However, any complete understanding of functional relationships must ultimately include a broader exploration of the relationships between two variables (Carraher and Schliemann 2002). Carlson and Oehrtman (2005) argue that students must be able to understand multiple views of function for success in mathematics: they must develop an understanding of function as a process that accepts inputs and produces outputs, as well as attend to the changing value of output and rate of change as the independent variable is varied.

The shift from a covariation approach to the correspondence view can be difficult for students, but there is evidence that when working directly with quantities, even young children can develop a flexible function understanding (e.g., Nunes et al. 1993; Schliemann et al. 1998, 2003). Working directly with accessible quantitative relationships can aid in beginning algebra students' investigations of functions from multiple perspectives, as well as support their abilities to shift flexibly across different perspectives. The seventh-grade students' experiences with gear ratios (and later constant-speed situations) helped them create algebraic representations such as "$(3/4)a + 5 = b$" that they could ultimately view in terms of both coordinated changes in each gear and as a direct relationships between a and b. These experiences helped the teaching-experiment students make meaningful sense of the pennies problem (Fig. 3) that had caused such difficulty for Juanita, who was not in the teaching experiment. Timothy's response was typical of the teaching-experiment students:

T: [Examining the table in Fig. 4]: Well, let's see. 2 to 9 oz, so that's 4.5 oz per roll. For that. So multiply that by 5. Times 5, equals 22.5. So these (the first two pairs in the table) are both from the same roll. Then multiply it by 12. 4.5 times 12 equals 54, so that's from the same one. And then 16 times 4.5. 16 times 4.5 equals 72, so they're all from the same thing because they all have the same weight for 1 roll.

AE: Do you think the graph is going to be linear or non-linear?
T: It's all going to go on the same line.
AE: Why do you think that would happen?
T: Because whatever the weight is, you can multiply it by 1 over 4.5 to get the number of rolls.

Timothy's reliance on his understanding of the relationship between the number of rolls and the total weight in ounces supported a direct comparison across the x- and y-columns of the table. He noted that whatever the weight is (the input variable), you can multiply it by 1 over 4.5 to get the number of rolls (the output variable); even though this is the reverse of how we might typically approach a table from an input-output perspective, it is correct and enabled Timothy to successfully solve a number of extrapolation and interpretation problems. Moreover, Timothy could move flexibly between the correspondence and covariation approaches, as evidenced by his predictions about the table's graph: "It just looks like to me that all you're doing is going up by 4.5 oz and 1 roll... it's going up by the exact same thing every time."

Reasoning with quantitative relationships can support students' flexible movement between different function approaches for quadratic functions as well. In the quadratic functions teaching experiment (consisting of 15 sessions with 7 eighth-grade students), I introduced quadratic phenomena in terms of the relationships between the lengths, heights, and areas of rectangles that grew while maintaining their length/height ratios. Although none of the students had yet experienced quadratic functions in their normal classrooms, they had all experienced other functional relationships and graphs (such as linear functions) in their algebra or pre-algebra courses. The students worked with a script in Geometer's Sketchpad to explore what happened to the dimensions of a particular rectangle (for instance, a 3 cm by 2 cm rectangle) as it grew and shrank. As predicted, the students made sense of these phenomena from a covariation perspective, imagining what would happen to the area as the length (or width) increased by a uniform amount. The students created their own tables of data to represent the phenomena they observed; a typical table is shown below:

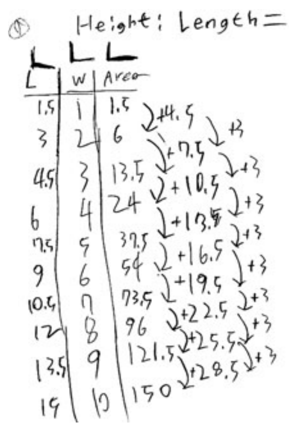


Fig. 11 Student's table of data representing the growing rectangle

In this case, the student was able to coordinate the growth of the length and the area of the rectangle as the width grew in 1-cm increments. He also identified the amount by which the area increased for each additional centimeter in width, as well as their differences.

I introduced a standard far-prediction problem to encourage a shift from the covariation approach to the development of a direct functional relationship between height and area:

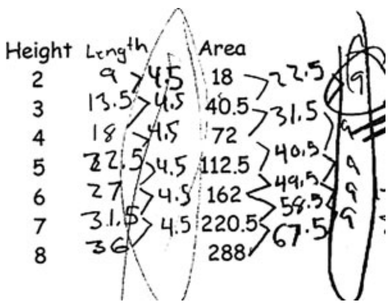
Problem 6 Here is a table for the height versus the area of a rectangle that is growing in proportion to itself. What will the area be when the rectangle is 82 units high?

Fig. 12 Table of height/area values for a growing rectangle

Height	Area
2	18
3	40.5
4	72
5	112.5
6	162

The students' initial entry into the problem was from a covariation perspective, in which they coordinated the growth of three quantities: height, length, and area. Each student introduced a third column, length, and noticed that the length increased by 4.5 units each time the height increased by 1 unit:

Fig. 13 Student's table with the added length column



One student, Ariel, stated that the area of a rectangle 82 units high would be 30,258 square units. Ariel explained that she found 30,258 by multiplying 82 by 4.5 units to get the corresponding length of 369 units. The area would then be 82 units multiplied by 369 units. Jim relied on his image of the rectangle and the way in which it grew to explain Ariel's reasoning to the class:

J: Well that was the length, the 369, so she has to do height times length equals area. So she had to multiply [the 369 by 82] again.

AE: I see. So how did you know to multiply 82 by 4.5 units to get the length?

J: Because, that's how much the length was going up by every time. So if you, like, made a square, I mean, or a rectangle, and then you moved up 1 unit, it would go over 4.5 for every time you go up the height 1.

At this stage, the students' thinking relied on an image of the manner in which the rectangle grew in order to coordinate the growth between the height and the length. This supported their understanding from a covariational view, and they capitalized on their understanding of the coordinated growth of the height and the length to determine the area of the rectangle for a large height. However, the students' images of the nature of the rectangle's growth were limited to cases in which the rectangles grew in discrete whole-unit increments, typically increments in which the length or the height increased by 1 unit. Simplifying the nature of the growth initially helped the students coordinate the multiple quantities involved (length, width, area, and increases in each of these quantities), but this was a strategy that would ultimately need to be generalized to encompass the notion of non-unit increments and continuous growth.

In an attempt to encourage the students to think about a direct relationship between the height and the area, I then asked them to compute the area when the height was n units. They quickly produced the formula "area $= 4.5n^2$", and Jim explained his reasoning to another student, Bianca:

J: I put n times 4.5 times n.
B: How did you figure it out?
J: Well, n can be any value...
B: Right.
J: Times 4.5 is your length. Times n again because I do height again, is your area.

Jim simply extended his previous reasoning to determine that the length of a rectangle n units high would be $4.5n$, and thus the area must be the height times the length, or $n(4.5n)$.

The students continued to work with far prediction problems, and the introduction of tables that were not well ordered encouraged the students to conceptualize the (unknown) length in terms of its relationship between the height and the area. Unable to identify the rate of growth of the length, the students instead began to develop third length columns by dividing the area by the height. The inclusion of the length columns also encouraged students to make more explicit connections between the length/height ratio and the "a" in $y = ax^2$, as seen below:

Fig. 14 Student's third length column for a table of height/area values

Height		Area	
3	4.5	13.5	
5	7.5	37.5	
8	12	96	
12	18	216	
18	27	486	
20	30	600	
80	120	?	9600
h	1.5h	?	$1.5h^2$

One student, Tai, explained, “I came up with this equation [area $= 1.5h^2$]. It’s like, the number in front of the height squared, is figured out by the area divided by the height squared.” Daeshim added, “The number is what you have to multiply the height by to get the length. And then height times length is the area, so that is why it’s squared.” The norm that students must explain how their equations were related to the quantities in the rectangle supported justifications such as Daeshim’s, and encouraged additional connections between features of the equations (such as the value of “a” in $y = ax^2$) and properties of the growing rectangle.

Although the shift from a covariation approach to a correspondence approach was gradual, it was aided by the students’ abilities to make direct connections to their images of growing rectangles and their abilities to coordinate relationships between the quantities length, width, and area. Moreover, their reliance on these constructed relationships enabled the students to develop a flexible view of the quadratic function, one in which they frequently shifted between the covariation and correspondence views. In particular, these flexible views helped the students make connections between the value of “a” in $y = ax^2$ and the second differences for area, which the students termed the “difference in the rate of growth of the area”, or the DiRoG for area when the width increased in uniform amounts. The students created multiple generalizations about the DiRoG of the area, including the notion that the DiRoG (for tables in which the rectangle’s height increases by 1 unit) is twice the value of “a” in $y = ax^2$, the DiRoG is twice the area of the rectangle when the height is 1 unit, and the DiRoG is the value of the rectangle’s length when the height is 1 unit.

The students experienced little difficulty when they transitioned to tables that were not well ordered for a number of reasons. First, they were accustomed to picturing the rectangle that was represented in the table’s values, so every pattern they developed was solidly grounded in the imagery of length, height, and area and their relationships. This imagery supported the students’ abilities to create functional relationships between height and area. In addition, the students had spent so much time focusing on what the DiRoG meant for the rectangle’s area in relationship to the equations they built, they became accustomed to moving seamlessly between recursive and functional representations. Because they kept discussing what the values represented in terms of length and area, the students were encouraged to represent those relationships more generally in algebraic forms.

Flexibility Across Forms

Smith and Thompson (2007) remind us that one role of quantitative reasoning is to support thinking that is flexible and general in character. Students in the linear and quadratic functions teaching experiments created many tables and algebraic representations to describe the same phenomena, and could move between them. But what about graphical representations? In both cases, I deliberately refrained from introducing graphs until the students had developed a meaningful understanding of the relationships represented by the graphs. Once that foundation was in place, they began to create their own graphs as a way to justify their conclusions about the quantitative relationships they developed.

For instance, the linear functions students encountered a scenario in which a character walked 5 cm in 4 seconds. They created multiple equivalent ratios to represent the character's speed, and represented these ratios in tables of data. When asked to explain why a speed of 15 cm in 12 seconds was the same as 5 cm in 4 seconds, Timothy asked if he could create a graph:

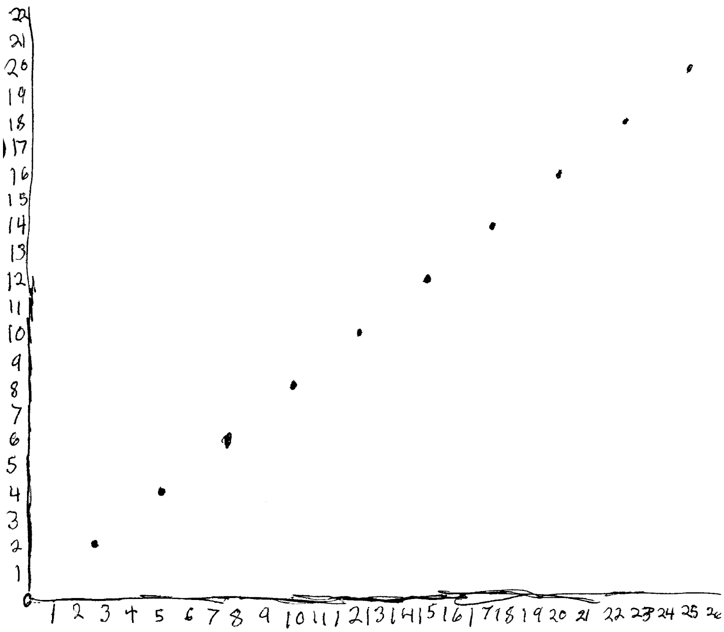


Fig. 15 Timothy's graph of same-speed values

Timothy's partner, Dora, wrote the equation $y = 4/5x$ and explained that the x-axis represented centimeters and the y-axis represented seconds. Timothy explained that he could put a line through the points, if they were appropriately exact:

T: You could put a line there. But it's not a good graph so it's not going to make a straight line.
AE: Okay. You found that the slope of the line was 4/5. What does that mean?
T: Whatever x is, y is 4/5 of x. The slope means that whatever x goes up by, 4/5 of that is how much y goes up by.
AE: And what does 4/5 have to do with the speed of the clown?
T: It's going basically 4/5 of a second per centimeter.
AE: Now why is the fact that the clown's speed is 4/5 of a second per 1 centimeter, why is that the same as the slope being 4/5? What's the connection?
T: Because for every centimeter it goes, it's going like 4, er, yeah, 4/5 of a second I think. Every centimeter goes... yeah. Every centimeter it's going 4/5 of a

second. The slope is $4/5$ because for every centimeter that you add, you add $4/5$ seconds.

Timothy's understanding of the speed situation, his familiarity with creating same-ratio tables, and his ease with representing these phenomena algebraically all supported his ability to create and make quantitative sense of a linear graph. In addition, Timothy was able to imagine the scenario from a correspondence perspective ("Whatever x is, y is $4/5$ of x") as well as from a covariation perspective ("For every centimeter you add, you add $4/5$ seconds"). Each of these views, as well as Timothy's flexibility with moving across views, was enabled by his understanding of the relationship between the quantities centimeters and seconds to create the phenomenon of constant speed.

The quadratic functions students began to create graphs in the third week of the teaching experiment and ultimately graphed both $y = ax^2$ and $y = ax^2 + c$ situations. Before they produced any graphs, they made predictions about what a graph of the growing rectangle situation might look like:

AE: If you were to graph this one [comparing the height to the area of a square], what do you think the graph would look like?
B: A curve.
S: I thought it would be straight because every time the area's going up by 2.
AE: So what do you think about what Sara's saying? She's saying every time it would go up by 2 so it would be straight.
B: Well, the area's going up by 2 *in between* every time it's going up by a different number, so that makes me think it's going up in a curve because it's, like, staired.

When the students ultimately created graphs, they showed the first and second differences for the area in order to connect their prior emphasis on differences to the graphical representation, and to explain why the graph must be curved instead of straight. For instance, Daeshim's graph identified the constant second differences as 1.5 cm^2 when the height increased in 0.5 cm increments, and he showed this by calculating the increase in area for each 0.5-cm increase in the height, and then showing the difference between each successive area increase to be 1.5:

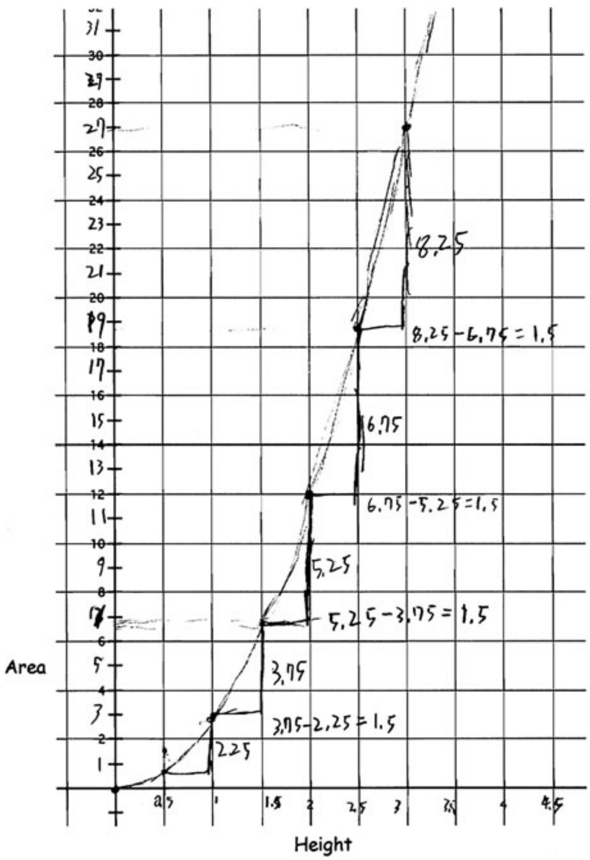


Fig. 16 Daeshim's graph of $A = 3h^2$

The students' quantitative understanding of the rectangle situation enabled them to make accurate predictions about the nature of graphs and interpret new graphs by thinking about the value of each point in relationship to a hypothetical rectangle. The students correctly predicted, for instance, that the parabola for $y = 5x^2$ would be narrower than the parabola for $y = 0.5x^2$, because the former represented a larger rectangle that was adding much more area with each height increase than the latter. They also made sense of graphs with non-zero y-intercepts by imagining rectangles with a constant number of extra square units tacked on. While students' later forays into features of graphs and families of functions will likely rely less on quantitative images, reasoning directly with the quantities can provide a critical sense-making foundation for their initial investigations of graphical representations.

In both the linear and the quadratic case, the students made use of different representations (tabular, algebraic, and graphical) to describe and make sense of the quantitative situations involving gear ratios, speed, or growing rectangles. Since each representation was a way of describing the quantitative phenomena, rather than an instructor-introduced artifact divorced from any referents, the connections across the representations were natural ones that enabled seamless transitions. Depending on the questions at hand, the students made use of the type of representation that they found most helpful for describing quantitative phenomena.

Fostering a Focus on Quantities

The situations with gear ratios and growing rectangles were optimal contexts for exploring linear and quadratic functions because the phenomena were precisely, rather than approximately, linear and quadratic. Some problem situations involve contexts in which the data are not exact; for instance, students may gather real-world quadratic data from rolling balls down inclined planes, or explore contrived problems presenting supposedly linear relationships between the number of surf boards sold and the temperature for a given day. The contrived nature of some contexts may interfere with students' natural sense making, and realistic situations with messy data may prevent students from directly manipulating quantities in order to form the necessary conceptual relationships that embody the phenomenon in question (Ellis 2007). While approximate or messy data are fully appropriate data to investigate, particularly in terms of highlighting the power of mathematical models for making sense of real-world situations, these contexts may not be ideal for middle school students who are exploring functional relationships for the first time.

Instead students will benefit from opportunities to explore the nature of linear (or quadratic) relationships by directly manipulating quantities: for instance, examining how changing time or distance independently affects the emergent quantity speed, creating two-number ratios and then iterating them and partitioning them to form equivalent ratios, and otherwise investigating how the constituent quantities affect the functional relationship at hand. The students in the linear and quadratic functions teaching experiments had opportunities to manipulate and explore physical artifacts (for the gears) or run experiments with computer software (for the speed situation and the growing rectangles situation). However, even in cases in which physical artifacts or computer simulations are not available, students can investigate how changing a particular quantity can affect the others related to it. Teachers may have to take care to support students' engagement with these problems, particularly because the tendency to extract numbers and focus on pattern-seeking activities appears to be a strong pull for middle-school students. In these cases an instructor's intervention can draw students' attention back to the quantitative referents of numbers and patterns. For instance, if a student describes a pattern in a table such as "each time x goes up by 4, y goes up by 5", a teacher could ask students to describe what this means in terms of the gears rotating.

Students' unique interactions with and interpretations of real-world situations remind us that these contexts are not a panacea. Introducing a quantitatively-rich situation does not guarantee that students will build quantitative relationships; a quantity is, after all, a person's conception of a measurable attribute, rather than the attribute itself. Students may focus on any number of features in a problem situation, and this focus may not always include productive relationships between quantities. Therefore teachers play an important role in shaping a classroom discussion, posing appropriate questions, inserting new information, and otherwise guiding students to develop the quantitative operations that will support the formation of functional relationships. A common refrain in the quadratic functions teaching experiment was "what does this mean in terms of the rectangle?" because this reminder encouraged the students to develop pattern generalizations that were meaningfully grounded rather than arbitrary and unproductive.

Students' initial learning of functions is particularly critical because it sets the foundation for future work in algebra at the high school level and beyond. Supporting students' abilities to make sense of functions from a quantitatively meaningful stance can foster a function understanding that is productive, grounded, and flexible in nature. A focus on numbers, relationships, and functional behaviors in absence of quantitative referents is certainly appropriate for mathematics students and, in the long term, necessary as students explore increasingly abstract ideas. However, I argue that for middle school students' first introduction to functional relationships, a grounding in quantities, relationships, and meaningful situations can ultimately support the eventual shift to more formal algebraic practices in high school.

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Representational Competence and Algebraic Modeling

Andrew Izsák

Abstract This chapter reviews some key empirical results and theoretical perspectives found in the past three decades of research on students' capacities to reason with algebraic and graphical representations of functions. It then discusses two recent advances in our understanding of students' developing capacities to use inscriptions for representing situations and solving problems. The first advance is the insight that students have criteria that they use for evaluating external representations commonly found in algebra, such as algebraic and graphical representations. Such criteria are important because they play a central role in learning. The second advance has to do with recognizing the importance of adaptive interpretation, which refers to ways in which students must coordinate shifts in their perspective on external representations with corresponding shifts in their perspective on problem situations. The term adaptive highlights the context sensitive ways in which students must learn to interpret external representations. The chapter concludes with implications of these two advances for future research and algebra instruction.

Gaining insight into how students learn to reason with external representations, or inscriptions, has been a central challenge in mathematics education research for several decades. Research on algebra and functions has grappled with this challenge extensively, perhaps more so than research on the teaching and learning of any other

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mathematical content area. The importance of external representations to mathematical thinking, and to algebraic thinking in particular, has been highlighted in recent influential documents. The National Council of Teachers of Mathematics' *Principles and Standards for School Mathematics* (NCTM 2000) named representation as a process standard, and the National Research Council's *Adding it Up* (Kilpatrick et al. 2001) characterized algebra in terms of three types of activity, one of which is representational activity. More recently, Kieran (2007) included examples of reasoning with external representations when elaborating a framework that conceptualizes algebraic activity in terms of generational activity, transformational activity, and global/meta-level activity.

Modeling problem situations in the physical world is one important context for generating and interpreting external representations. Furthermore, understanding how students learn to represent situations and solve problems about those situations continues to be of practical and theoretical importance. From a practical point of view, more traditional instructional materials often concentrate on procedures for using prescribed representations—for instance, procedures for simplifying algebraic expressions or for plotting points on a Cartesian graph. In recent years, however, new reform-oriented instructional materials have been making their way into schools. For instance, in the United States the National Science Foundation has supported the development of materials (e.g., Coxford et al. 1998; Lappan et al. 2002, 2006; TERC 2008; Wisconsin Center for Educational Research & Freudenthal Institute 2006) that respond explicitly to standards developed by the National Council of Teachers of Mathematics (NCTM; 1989, 2000). These newer instructional materials for algebra (and other mathematical content areas) are increasing the demands placed on students to reason with various forms of representation. To illustrate, asking students to identify the advantages and disadvantages of tabular, graphical, and algebraic representations is an example of a task that is not often found in more traditional materials and that is more demanding than simplifying expressions or plotting points. From a theoretical point of view, mathematics education researchers are continuing to develop empirically grounded theory that provides new insight into students' cognition when generating, interpreting, and using external representations to solve problems about situations.

The goal of this chapter is to highlight two recent advances in our understanding of students' developing capacities for reasoning with external representations of problem situations. Both advances are built on results from several studies that sought insight into students' internally held mental structures and processes. One advance has to do with students' capacities to generate external representations, and the other has to do with students' capacities interpret external representations. To establish the context for these advances, I first trace some key empirical results and theoretical perspectives found in the past three decades of research on students' capacities to generate and interpret external representations commonly used in algebra. (For comprehensive reviews of research on the teaching and learning of algebra, see Kieran 1992, 2007.) I then discuss the two advances and examine their implications for future research and teaching.

Early Results on Students' Understandings of Standard Representations in Algebra

During the 70s and 80s, researchers reported numerous difficulties that students encounter generating normative algebraic representations to model problem situations. In one well-known study, Clement (1982) found that in a sample of science oriented college students 37% failed to correctly answer the following problem, "Write an equation using the variables S and P to represent the following statement: 'There are six times as many students as professors at this university' " (p. 17). In a second example, Booth (1981) found that secondary-school students wrote expressions such as '*hhhht*' and '*4ht*' to symbolize the perimeter of a pentagon in which four sides were labeled '*h*' and one side was labeled '*t*.'

Research conducted during these same years also uncovered a variety of difficulties that students encounter interpreting algebraic notation appropriately. These difficulties include interpreting letters (e.g., Harper 1987; Küchemann 1981) and the equal sign (e.g., Kieran 1981). Küchemann analyzed written responses from approximately 1000 14-year-old students to a set of test questions and found that students either avoided working with letters altogether or thought of letters as standing for specific unknown values. Rarely did students think of letters as standing for variables. Kieran (1981) reported that elementary and middle school students tend to interpret the equal sign not as a relation between two expressions but rather as a signal to compute the expression on the left-hand side and record the result on the right-hand side.

Further research from this same era uncovered difficulties that students encounter when generating and interpreting normative graphical representations of problem situations (see Leinhardt et al. 1990, for a thorough review). In one oft cited study, Kerslake (1981) analyzed written responses of nearly 1800 13-, 14-, and 15-year-old students to a set of test questions. Among other things, she reported that many students do not understand when it is and when is not appropriate to connect points, indicating difficulties generating graphs. She also reported that students often misinterpret graphs as direct depictions of problem situations, a phenomenon referred to as the graph-as-picture interpretation. To illustrate, when considering the motion of an object, students interpret distance vs. time graphs as showing a trace of the actual path of motion.

On the whole, research from this era emphasized constraints on what students could do. Researchers prescribed particular normative external representations (e.g., conventional equations or graphs) with which students were to accomplish tasks and reported errors that students made. Oftentimes, researchers characterized these errors as misconceptions.

Theoretical Accounts of Reasoning with External Representations

In addition to uncovering students' numerous difficulties generating and interpreting normative external representations used in algebra, researchers in mathematics education also sought theoretical accounts that could explain reasoning with external representations (see Janvier 1987a, for one collection of theoretical perspectives from this era). At least two developments in the field shaped these accounts significantly. The first development was the increasingly broad acceptance of constructivist epistemology and tensions between that epistemology and statements such as "X represents Y." Although from a constructivist perspective Y does not exist in any absolute sense, one can talk about individuals "re-presenting" their experiences. As researchers began framing accounts in terms of interactions between internal representations (mental structures of individuals) and external representations such as equations and graphs, they used the terms *notations* and *inscriptions* to emphasize that an artifact, on its own, does not carry meaning. Rather, meaning emerges when individuals generate and interpret tables, equations, graphs, and other diagrams or notations. The second development was the increasing availability of computers and software that linked tabular, algebraic, and graphical representations of functions to each other (see Romberg et al. 1993, especially Chaps. 2–4) and that linked graphs to physical phenomena such as motion (e.g., Mokros and Tinker 1987; Monk and Nemirovsky 1994; Nemirovsky 1994). Researchers were interested in theory that could explain students' reasoning in these new environments.

Initial attempts within mathematics education to theorize about cognition around forms of external representation, or inscriptions, described cognitive processes at a coarse grain-size. One example that illustrates this grain-size is *translation*, the process of moving among different external representations of the same situation (e.g., Janvier 1987b; Lesh et al. 1987). A person might translate between verbal and symbolic representations or between algebraic and graphical representations.

Kaput (1987, 1989, 1991) developed a perspective on reasoning with external representations that underscores the distinction between internal and external representations, is consistent with constructivist epistemology, and resolves coarse-grained cognitive processes, like translation, into several components. He began by defining a *symbol system* as a symbol scheme combined with a field of reference. Briefly, a *symbol scheme* is a set of symbols and a set of rules for transforming those symbols. Symbols for the algebra symbol scheme include numbers, letters, notations for arithmetic operations, and the equal sign. Transformations include rules for manipulating the symbols in the scheme—for instance, rules for adding like terms and for multiplying two binomials. A field of reference can be a problem situation in the physical world or another symbol scheme.

Kaput (1987, 1991) then identified two types of cognitive activity associated with symbol systems. The first type consists of encoding and reading, which correspond to processes of generating and interpreting external representations. *Encoding*[1] occurs when "one has some conceptual structure or operation that one seeks to externalize for purposes of communication or testing" (Kaput 1991, p. 57). For instance, one might generate an algebraic expression or equation to express understanding of a problem situation. *Reading* occurs when "processes are based on an intent to use

[1]Some researchers use the term *encoding* to describe internal representation of external stimuli. Kaput's definition is the reverse, and I use his sense here.

the physical material to assist one's conceptual activity in traditional acts of reading and interpreting" (ibid, p. 57). For instance, one might interpret an already written expression or graph. The second type of cognitive activity has to do with how one uses symbol systems. *Syntactic elaboration* refers to manipulating symbols using the transformation rules. *Semantic elaboration* refers to elaborating the referents for the symbols (see Kaput 1987, p. 177).

Kaput emphasized that the cognitive activities of encoding, reading, syntactic elaboration, and semantic elaboration function in interaction with one another and through cycles of reasoning. When compared to constructs like translation, his constructs were more fine-grained and pointed toward more nuanced analyses of the complexities of generating and interpreting external representations to solve problems and of making connections between one representation and another. Results presented in the next section have begun to shed light on those complexities.

Students' Capacities to Reason with External Representations

Constructivist epistemology suggested particular limitations of the early empirical results summarized above on students' difficulties generating and interpreting normative algebraic and graphical representations. One limitation was that by focusing on students' errors, this generation of research did not examine how students might use their existing knowledge productively to accomplish task. As a consequence, a second limitation was that this generation of research provided few clues about the transformation of more novice into more expert knowledge (see Smith et al. 1993, for a constructivist critique of misconceptions research).

Several new lines of research have examined students' capacities to use inscriptions productively and to construct knowledge about external representations. One line has examined more closely the cognitive structures involved in building connections between different representations of functions (e.g., Knuth 2000; Moschkovich 1998; Schoenfeld et al. 1993). Schoenfeld et al. and Moschkovich have described a 3-slot schema to explain how for some students slopes, y-intercepts, and x-intercepts of linear functions are salient features of graphs that should *all* appear explicitly in equations. These researchers have described growth and change of the multi-layered schema in terms of coordination, reorganization, and refinement of multiple elements. The results underscore the limitations of coarse-grained cognitive processes, such as translation, for explaining challenges that students experience when connecting multiple representations.

A second line of research has examined students' capacities to generate their own representations and to use those for solving a range of problems related to algebra. This approach has differed from the earlier generation of research summarized above in which researchers examined constraints that students experienced when using normative representations that researchers prescribed. Several studies within this line have demonstrated that students are significantly more competent at reasoning with external representations than previously reported. These studies have looked at ways that students solve traditional word problems (e.g., Hall 1990; Hall et al. 1989), generate their own tabular and algebraic presentations of physical devices (e.g., Izsák 2000, 2003, 2004; Meira 1995, 1998), and design graphical representations of motion (e.g., diSessa 2002; diSessa et al. 1991; diSessa and Sherin 2000; Sherin 2000). Each of these studies has reported complex interplay between students' understandings of problem situations and the inscriptions with which they work.

A third line of research has concentrated on students' transition from arithmetic to algebra. Within this line, several researchers have studied elementary and middle grades students' capacities to generalize patterns that could be described with linear functions. These researchers have emphasized the mathematical notion of function (e.g., Carraher et al. 2006, 2008), have examined how students justify their generalizations of patterns (e.g., Lannin 2005), and have used semiotics to examine how students learn to generate and interpret algebraic notation in particular socio-cultural contexts (e.g., Radford 2000, 2003, 2008; Rivera and Becker 2008; Warren and Cooper 2008).

This chapter concentrates on two results from the second more recent line of research on students' mental structures and processes. One result has to do with students' competencies for generating external representations, and one has to do with students' capacities for interpreting external representations in context sensitive ways. Both results have emerged across several case studies that have examined students' competencies for generating and interpreting external representations of problem situations. Furthermore, both results have emerged across different forms of external representation, such as tables, equations, and graphs. Finally, both results have emerged from studies that relied on interview settings and from studies situated in classrooms. Thus, the results are not tied to one particular set of students or one form of representation, and they are directly relevant to classroom instruction.

First Result: Criteria for Evaluating External Representations

The first result I discuss can be traced back to one influential study by diSessa et al. (1991). These authors reported on a sequence of lessons during which 8 sixth-grade students ($\sim$ 12 years of age) "invented" graphing. The students were participating in an experimental after-school class. They were presented with a description of a motorist driving through the desert and stopping for a drink of water and were asked to invent static "motion pictures" that communicated key aspects of the described motion such as going fast, going slow, and stopping. The students generated approximately 10 different motion pictures, evaluated each other's approaches, and refined several.

diSessa et al. (1991) coined the term *meta-representational competence* to refer to students' capacities to invent and critique various graphical representations of motions. The result most relevant to the present article was that students marshaled approximately a dozen criteria when evaluating each other's work. Examples included *transparency* (the criterion that representations should need little or no explanation), *appropriate abstractness* (the criterion that representations can omit nonessential aspects of problem situations), and *consistency* (the criterion that conventions should not be adjusted to accommodate features unique to a particular problem situation). In a follow-up study with a different sample of students, diSessa and colleagues (e.g., diSessa 2002; diSessa and Sherin 2000; Sherin 2000) have investigated students' capacities to generate and critique graphical representations of motion and spatially distributed data.

diSessa (2002) provided the most elaborate extension of the initial results on students' criteria for graphical representations. One main results was that different students can make systematically different judgments about external representations. That is, different students may apply different criteria when evaluating external representations. He also articulated two hypotheses about criteria that have implications for how they are studied. First, criteria are design-linked. This implies that students will most likely activate criteria when engaged in designing representations. Second, criteria are both reactive and implicit. This implies that students employ criteria when reacting to particular representations but may not be able to articulate their criteria clearly. Thus, to gain access to criteria, one should study contexts in which students are building representations; examine all instances in which students judge whether representations are good or bad and infer the basis for the judgments one observes (see diSessa 2002, for details).

Izsák (2003, 2004) has extended results on students' criteria from graphical representations to algebraic representations. He studied how 12 pairs of eighth-grade ($\sim$ 14 years of age) U.S. students generated algebraic representations of a physical device called a winch. His data came from series of semi-structured interviews conducted with each pair. Nine pairs were taking an introductory algebra course, and three were taking a pre-algebra course that included some work with variables and functions. The winch (see Fig. 1) exemplifies situations that can be modeled by pairs of simultaneous linear functions. The device stands 4 feet tall and at the top has a rod with a handle for turning two spools, one 3 and one 5 inches in circumference. Fishing line attaches one weight to each spool. Izsák referred to these as the 3- and 5-inch weights, respectively. Turning the handle moves the weights up and down a yardstick, allowing measurements of heights, displacements, and distances between the two weights.

Izsák (2003, 2004) configured the initial heights of the weights in various ways and presented students with three types of tasks:

(1) Predict the distance between the weights after an arbitrary number of cranks.
(2) Determine whether and, if so, when one weight will ever be twice as high as the other.
(3) Determine whether and, if so, when the weights will meet at the same height.

For each type of task, he first used initial conditions that allowed students to answer questions simply by turning the crank. To illustrate with the third type of task, students might turn the handle and observe that the two weights rise and meet at 28 inches. He then asked students to imagine a larger 100-inch winch and changed the initial conditions so that students could not simply turn the handle. As an example, students might need to imagine that, as they turned the handle, the two weights

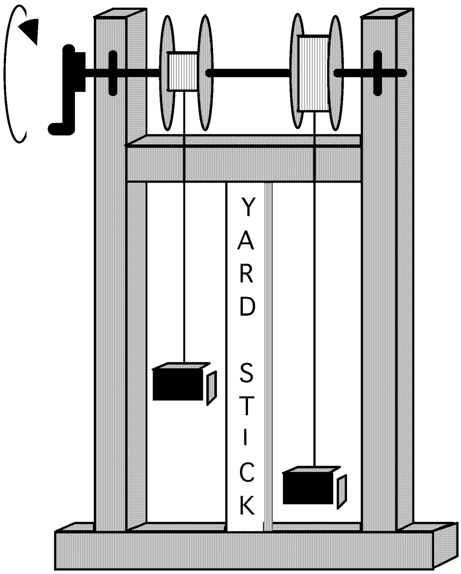

Fig. 1 The winch. From "Inscribing the Winch: Mechanisms by Which Students Develop Knowledge Structures for Representing the Physical World with Algebra" by A. Izsák (2000), *The Journal of the Learning Sciences*, 9, p. 33. Copyright 2000 by Lawrence Erlbaum Associates, Inc. Reprinted with permission, http://www.informaworld.com

would rise and meet somewhere above the physical device in front of them. This interview strategy focused students first on the physical phenomena and then on ways of representing those phenomena that afforded solutions to the problems. For the students in this study, generating algebraic representations of the winch was not automatic. Rather, they had to use what they already knew about algebraic representations to design suitable equations, and students' discussions about those equations afforded access to their design-linked knowledge.

Izsák (2004) reported a detailed analysis of one pair of eighth-grade students, Amy and Kate,[2] who considered the winch set up so that the 3- and 5-inch weights started by the 14- and 0-inch marks, respectively. Over 40 minutes, the students introduced, evaluated, and refined a sequence of algebraic representations for vertical distances between the weights. Three criteria that Amy and Kate applied were *single equation* (the criterion that single equations are better than multiple ones), *positive distance* (the criterion that expressions must generate positive values for distances), and *consistent interpretation* (the criterion that letters in an equation should have stable interpretations in the context of the problem).

Amy proposed the first equation: $14 - 2n = d$. In this equation, she let 14 stand for the initial distance between the two weights, 2 stand for the amount by which the distance between the weights changed with each turn of the handle, n stand for the number of turns of the handle, and d stand for the distance between the two weights. The students tested the equation for fewer than 7 turns of the handle and more than 7 turns. They used the positive distance criterion to reject Amy's equation because it resulted in negative numbers when n was larger than 7. Amy then refined her initial

[2] All names are pseudonyms.

approach and suggested using two equations, $14 - 2n = d$ to represent the distance until the 5-inch weight caught up to the 3-inch weight and $2n = d$ to represent the distance after the 5-inch weight passed the 3-inch weight. Amy still used d for the distance between the two weights but now used n to mean both the number of turns of the handle from the initial set-up and just the number of turns after the two weights met. Kate understood that substituting values into Amy's equations produced positive numbers that matched distances measured on the winch, but she still objected to Amy's approach. Although Kate did not make explicit the basis for her objection, she may have applied the single equation criterion (because Amy used two equations) or the consistent interpretation criterion (because Amy used n to count cranks both from the initial winch set-up and from the point when the two weights were at the same height).

Ultimately, the students arrived at equation (1). Although this equation was unlike any normative equation that Amy and Kate might have seen in their algebra class or textbook (e.g., an equation of the form $y = mx + b$), it did model distances on the winch correctly. In the equation, n stood for the total number of completed turns of the handle, 7 stood for the number of turns that it took the 5-inch weight to catch up to the 3-inch weight, 2 stood for the amount by which the distance between the weights changed with each turn of the handle, and d stood for the distance between the two weights. Izsák (2004) argued that Amy and Kate were satisfied with (1) because it met the multiple criteria for equations that they had been using—positive distance, single equation, and consistent interpretation. Thus, Amy and Kate's criteria fundamentally shaped which algebraic representations of the winch made sense to them. Moreover, because the students used similar equations to model further winch problems (see Izsák 2004, for details), their criteria played a key role in their learning.

$$|(n - 7)2| = d. \tag{1}$$

More recently, Izsák et al. (2009) extended results on students' criteria from experimental classroom and interview settings to more conventional classroom settings. Their study came from *Coordinating Students' and Teachers' Algebraic Reasoning*, a project supported by the National Science Foundation. The project took place in a U.S. middle school that uses reform-oriented curricular materials. Izsák et al. reported on a sequence of lessons from Ms. Jennings's eighth-grade classroom during which students generated and evaluated alternative algebraic representations for word problems. The lessons came from a unit on writing and solving linear equations that is part of *College Preparatory Mathematics* (Sallee et al. 2002). The materials instruct students first to solve word problems by developing guess-and-check tables and then to write equations that express the resulting patterns. As was the case for Amy and Kate, generating algebraic representations of linear patterns was not automatic for students in Ms. Jennings's class.

One lesson was particularly notable for the number of algebraic expressions and equations that the students generated and evaluated, and students' reactions to alternative equations afforded access to their design-linked knowledge. The problem on which they worked was the following:

> Antony joined a book club in which he received 5 books for a penny. After that, he received 2 books per month, for which he had to pay \$8.95 each. So far, he has paid the book club \$196.91. How many books has he received? (Unit 4, p. 10)

There are several equations that one could set up to solve this equation. Here is one in which x stands for the number of books that Antony received for \$8.95:

$$8.95x + 0.01 = 196.91. \tag{2}$$

An aspect of the Book Club problem that challenged many students was that the books could be described either in terms of the total amount of money that Antony has spent or the total number of books that he has received. This challenge afforded opportunities for students to generate alternative correct equations to model the problem. Faced with alternative equations, several students used *final units* (the criterion that terms in equations should be expressed in the same units as those of final requested quantities). One student, Greg, challenged equations proposed by his classmates in which 196.91 appeared alone on the right-hand side of the equal sign. He argued that "you are equaling it up to 196.91, but see, and what you are trying to do is find, figure out how many books you got, not the amount of money you have." He followed the instructions to use a guess-and-check table, determined the correct answer of 27 books in all, and wrote the following:

$$(196.91 \div x) + 5 = 27. \tag{3}$$

Greg's explanation for his equation was unclear because his assignment for x was unstable. Another student, Maria, offered two correct equations:

$$(x - 5) \cdot 8.95 + .01 = 196.91 \tag{4}$$

and

$$196.90 \div 8.95 + 5 = x. \tag{5}$$

Notice that each term in equation (4) expresses an amount of money and that each term in equation (5) expresses a number of books. When proposing equation (4), Maria explained that x was the total number of books that Antony received, that she subtracted the 5 books for one penny, and that she multiplied the result by 8.95. In equation (5), Maria divided to determine the number of books that Antony received for \$8.95 each and then added the 5 books for penny. In a subsequent interview, she reconsidered both of her equations and expressed a preference for equation (5). Like Greg, Maria justified her choice by referring to the problem statement that asked for the number of books that Antony has received.

As mentioned above, students' tendency to interpret the equal sign as a command to compute an answer is well known (e.g., Kieran 1981; Knuth et al. 2006). Although one might consider equation (3), equation (4), and equation (5) as further examples of this phenomenon, the data on Greg and Maria suggested more complex understandings of equations. Note that equation (3) and equation (4) demonstrate that both students could generate equations that combine the letter x with other numbers on left-hand side of equal sign. Neither in equation (3) nor in equation (4) can one use the left-hand side alone to compute a number. Furthermore, both Greg and Maria recognized that there was more than one way to construct an equation with a number isolated on the right-hand side of the equal sign, and they had a criterion for evaluating two alternatives.

Finally, in addition to documenting students' use of criteria in an algebra I classroom, Izsák et al. (2009) reported on criteria that Ms. Jennings used during the sequence of lessons. This is the first study of which I am aware that has extended results on criteria for representations from students to teachers. Describing these results are beyond the scope of the present chapter (see Izsák et al. 2009, for details).

Studies summarized in this section provide accumulating evidence for a complex substrata of knowledge consisting of criteria, which are fine-grained knowledge elements that support students' capacities to evaluate external representations of problem situations. This substrata is particularly visible when students are designing external representations, or inscriptions, and are reacting to alternatives. Initial results that emerged in conjunction with graphical representations have been extend to results on algebraic representations. Moreover, results that first emerged in experimental contexts have provided insight into teaching and learning in more conventional algebra classrooms.

Second Result: Adaptive Interpretation

The second result I discuss extends earlier reports that students have difficulty interpreting normative algebraic and graphical representations appropriately (e.g., Leinhardt et al. 1990; Matz 1982). As mentioned above, past results have tended to concentrate on the errors that students make and on students' prior experiences that might be sources for those errors. The present result highlights the fact that oftentimes, in the course of solving problems about situations, students must coordinate shifts in their perspective on external representations with corresponding shifts in their perspective on situations. I refer to this aspect of reasoning as *adaptive interpretation* (see also Izsák and Findell 2005). The notion of adaptive interpretation allows one to examine whether student errors arise not from misconceptions, but rather from applying knowledge appropriate for one situation inappropriately in another. As when discussing results about students' criteria for external representations, I demonstrate that results about adaptive interpretation are not tied to one particular group of students or form of external representation. I will also demonstrate that adaptive interpretation can play an important role both in experimental settings and in more conventional classrooms. For the first example, I return to the winch (see Fig. 1).

Izsák (2003) reported further results on Amy and Kate's reasoning when considering whether one weight on the winch would ever be twice as high as the other. The students examined the winch set up so that the 3-inch weight started by the 28-inch mark and the 5-inch weight started by the 0-inch mark. They described twice as high correctly by equating the height of the lower weight with the distance between the two weights, and they generated three equations:

$$\begin{aligned} 0 + 5n &= h \\ 28 - 2n &= d \\ d &= h. \end{aligned} \tag{6}$$

In this system, the students let n stand for the total number of turns of the handle, h stand for the height of the weight attached to the 5-inch spool, and d stand for the distance between the two weights. The first equation expressed the height of the 5-inch weight for any number of turns of the handle, the second expressed the distance between the two weights for any number of turns of the handle (the initial distance decreased by 2 inches with each turn), and the third expressed the specific moment when the lower height would equal the distance between the two weights.

Initially, Amy resisted equating the expressions $0+5n$ and $28-2n$ because each expression was true for any number of turns of the handle, but the resulting equation was not. At one point, she objected that "The height and the distance between the two is not equal" and went on to say "We want a statement that is always going to be true." Because, in some cases, appropriate equations do hold for all values of the independent variable, Amy's criterion was reasonable: It was her application of this knowledge to the present situation that created tension. Kate was more confident in equation (6), but her understanding was still emerging. Thus, the students had to develop adaptive interpretations that would allow them to examine algebraic representations for all n in some contexts and for unique n in others. The students finally accepted $0 + 5n = 28 - 2n$ after discovering they could solve for n and answer the problem. Matz (1982) also reported students who had trouble distinguishing between equations that are true for any value of the independent variable from those that constrain the independent variable to a unique value.

The second example of students struggling with adaptive interpretation comes from a second case study conducted as part of the *Coordinating Students' and Teachers' Algebraic Reasoning* project (see above). The example extends results on adaptive interpretation from algebraic representations to tabular and graphical representations and from experimental settings to more conventional classroom settings. The seventh-grade teacher, Ms. Bishop, and her students ($\sim$ 13 years of age) were studying a unit called *Variables and Patterns* that focuses on using tables and graphs to solve problems about situations that contain covarying quantities. The unit is part of the *Connected Mathematics Program* materials (CMP, Lappan et al. 2002), one set of curriculum materials developed with support from the National Science Foundation in response to NCTM standards (see above).

Students in Ms. Bishop's class were working on the Popcorn Problem (abbreviated in Fig. 2). Prior to working on this problem, they had completed an activity in which they performed jumping jacks and recorded the total every 10 seconds. (Jumping jacks are a form of exercise performed from a standing position by jumping to a position with legs spread and arms raised above the head, jumping to the original position, and repeating.) As the students tired, the total increased more slowly. The Popcorn Problem was a second task in which students were asked to

1. The convenience store across the street from Metropolis School has been keeping track of their popcorn sales. The table to the right shows the total number of bags sold beginning at 6:00 A.M. on a particular day.
   a. Make a coordinate graph of these data. Which variable did you put on the *x*-axis? Why?
   b. Describe how the number of bags of popcorn sold changed during the day. Explain why these changes may have occured.

Time	Total bags sold
6:00 A.M.	0
7:00 A.M.	3
8:00 A.M.	15
9:00 A.M.	20
10:00 A.M.	26
11:00 A.M.	30
Noon	45
1:00 P.M.	58

Fig. 2 From Connected Mathematics: Variable and Patterns Teacher Edition Page 10. Copyright © 2004 by Michigan State University, Glenda Lappan, James T. Fey, William M. Fitzgerald, Susan N. Friel, and Elizabeth D. Phillips. Adapted by permission of Pearson Education, Inc. All rights reserved

reason about non-linear data. Although the word "change" in part b could refer either to changes each hour or to increases over the course of the day, the sample answer in the teacher's guide makes the intended meaning clear:

> Very few bags were sold before 7 a.m., perhaps because many people do not eat popcorn so early in the morning. But the number jumped by 12 bags between 7 a.m. and 8 a.m., when perhaps people were stopping for a snack on their way to school. The number goes up at a rate of about 5 bags per hour between 8 a.m. and 11 a.m. From 11 a.m. until noon it jumps to 15 bags, and 13 bags from noon to 1 p.m.; during these two hours, perhaps people are buying lunch. (Lappan et al. 2002, p. 10)

The intended task requires adaptive interpretation because students must be able to look at the fourth line of the table, for example, and interpret it as 20 bags sold by 9 a.m. At the same time, when turning their attention to sales each hour, students must be able to determine by subtraction that 5 bags were sold between 8 a.m. and 9 a.m., and so on. Thus, students must coordinate a shift in attention from total sales to sales each hour with a shift in attention from rows in the table to differences in those rows.

Adaptive interpretation may be unproblematic for those with experience using tables and graphs to reason about covarying quantities, but the classroom and interview data made clear that coordinating shifting perspectives on situations with shifting interpretations of representations was challenging for many students. During the initial class discussion of part b to the Popcorn Problem, some students gave answers that did not make clear distinctions between the total number of bags sold for the entire day and the number of bags sold each hour. For instance, Ashley may have focused on increases in both total bags sold and bags sold each hour when she offered, "The changes increased probably because as the day went on more people wanted popcorn and most people don't want popcorn in the morning." Other students apparently misinterpreted the table in Fig. 2 as bags sold per hour: Rachel added the entries for total bags sold and said, "I put from 6 a.m. to noon they only sold 139 bags."

As the class discussion continued, students attended to both sales each hour and total sales but many seemed not to coordinate the situation and the table appropriately. To make clearer the distinction between total sales and sales each hour,

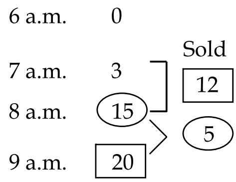

Fig. 3 Revisiting the Popcorn Problem

Ms. Bishop introduced the phrase “cumulative graph” to describe the popcorn data and defined *cumulative* as “successive addition” or “continuously adding to.” Later during the same lesson, she moved to the next problem in the materials which used discrete dots to show soda sales each hour over a day. Ms. Bishop called this a “rate graph.” Absent from the lesson was an explicit explanation of how the table in Fig. 2 conveyed both cumulative and rate data.

Over the next several lessons, the students often began work on a problem by considering whether the included table or graph presented cumulative or rate data but they continued to have difficulty with the distinction. To address the persistent confusion, Ms. Bishop returned to the Popcorn Problem 10 days later and added a third column to record differences between successive rows. Figure 3 reproduces her written work. Although Ms. Bishop had verbally explained the relationship between total sales and sales each hour during previous lessons, this was the first time that she used inscriptions to explicitly demonstrate how to see both either in a table or in a graph. She did not explain the significance, if any, of the ovals or the rectangles.

Interviews with students from Ms. Bishop’s class revealed that adaptive interpretation of the popcorn table remained problematic, even after Ms. Bishop revisited the problem. I present data on one pair of students, Nikki and Jennifer. Nikki was a mid-achieving student, and Jennifer was a low- to mid-achieving student. I conducted the interview with these students 4 days after Ms. Bishop generated the table reproduced in Fig. 3. After watching a video clip of the class discussion, Nikki and Jennifer could describe the subtraction that Ms. Bishop used to generate each of the numbers in the augmented table. Nikki also commented, “I really understood what she did that day for the first time ‘cause she actually broke it down from hour to hour.” That the students continued to have difficulty with adaptive interpretation became clear in the following exchange (Nikki’s language did not distinguish between graphs and tables):

> Int: So what do you think about the 12 and the 5 (referring to the right hand column in Fig. 3)? Are they, are those cumulative data? Are they rate data? Are they some other kind of data?
>
> Nikki: Now, with those two numbers, I’m not exactly sure because she subtracted them.
>
> Jennifer: I think rate.
>
> Nikki: Maybe. But it’s a cumulative graph. So why would it have two different sets of information on there if it’s cumulative?

As the interview progressed, Nikki asked her question about “two different sets of information” in the context of graphs as well. The interviewer gave the students a graph showing new popcorn sales data as discrete dots, similar to other discrete graphs with which the students had worked during class. The interviewer emphasized that each point on the graph indicated how many bags had been sold all together and then asked the students how many were sold each hour. Nikki and Jennifer answered by calculating differences. The interviewer then pointed out that calculated differences were "rate data" in the sense discussed in class because they represented how many bags sold in one hour. The interviewer was trying to get the students to see that, similar to the table in Fig. 3, one graph could convey both cumulative and rate data. He then reminded Nikki of her comment that a graph can show only one type of information:

> Nikki: Yeah. That's what I figured. Since it's being a cumulative graph, why would it be one kind of graph and have another type of information on it?
>
> Int: Okay. What do you think now?
>
> Nikki: I'm still unsure.
>
> Int: Really?
>
> Nikki: I guess I won't be clear about it unless I ask Ms. Bishop personally. That's definitely something I can't answer myself. Because it's confusing: You thinking about a graph being cumulative and having rate information, or you think about a rate graph having cumulative information, or you think about another graph having another kind of information. It doesn't ring a bell to me.

In further discussion, Nikki continued to demonstrate understanding of the relationship between total sales and sales each hour and was apparently learning that a given graph or table could convey both types of information. That Nikki struggled with her question about "two different sets of information" when working with both tables and graphs indicates that she was not asking about one particular form of representation, tables or graphs, but about representations more generally. It is possible that classroom discussions in which Ms. Bishop and her students identified particular tables and graphs as cumulative or rate contributed unintentionally to Nikki's difficulties.

The two studies summarized in this section provide evidence that interpreting external representations commonly used in algebra can be particularly challenging for students because appropriate interpretations can be context sensitive. These challenges arise when students are generating and then interpreting their own inscriptions, as in the first example from Amy and Kate's work, and when students are working with external representations given to them, as in the example from Ms. Bishop's classroom. These results imply that part of developing representational competence involves learning how to recognize when particular interpretations of a given representation are, and are not, appropriate. Finally, as in the case of criteria, initial results that emerged in experimental settings have proven germane to teaching and learning in more conventional algebra classrooms.

Conclusion

An earlier generation of research, conducted during the 70s and 80s, examined students' understandings of external representations commonly found in algebra by prescribing the forms of representation with which students were to accomplish given tasks. Results emphasized students' difficulties generating and interpreting normative algebraic and graphical representations appropriately. A more recent generation of research has uncovered competencies that students demonstrate when they are allowed to generate their own external representations, or inscriptions, to solve problems about situations. Furthermore, this recent line of research has produced an accumulating body of evidence that students have knowledge specifically for reasoning with external representations of problem situations. I conclude with several implications of this insight for future research and instruction related to algebra.

The first implication for future research has to do with the grain size of knowledge that students employ when generating and interpreting external representations to solve problems about situations. The most generalized finding that cuts across the studies summarized in the present article is that capturing students' sense making when using external representations, or inscriptions, requires analyses of fine-grained knowledge. This finding underscores the limitations of course-grained accounts of knowledge for gaining insight into students' experiences. As discussed in a preceding section of the chapter, the 3-slot schema (e.g., Moschkovich 1998; Schoenfeld et al. 1993) exposes limitations of translation for describing the process by which students connect algebraic and graphical representations. Results about criteria for algebraic and graphical representations and about adaptive interpretation provide insight into how students might self-regulate broader cognitive processes for reasoning with external representations, including the processes identified by Kaput (1987, 1989, 1991) of encoding and reading and of syntactic and semantic elaboration. Future research on students' sense making should be directly informed by this general result—for instance, by explicitly identifying and justifying the grain size at which knowledge is described.

The second implication for future research builds on the first. There is much still to learn about the forms and contents of students' knowledge specific to reasoning with external representations of problem situations, but results to date suggest a substratum of diverse elements that includes complex structures like the 3-slot schema reported by Schoenfeld et al. (1993) and by Moschkovich (1998) and also criteria for various forms of external representations like those reported by diSessa and colleagues (diSessa 2002; diSessa et al. 1991; diSessa and Sherin 2000; Sherin 2000) and by Izsák and colleagues (Izsák 2003, 2004; Izsák et al. 2009). The literature contains still other reports of fine-grained knowledge structures for reasoning with algebraic representations (e.g., Sherin 2001) and graphical representations (Clement 1989). Research consistent with broad tenets of constructivist epistemology should continue to seek access to this substratum of knowledge because it is often at this level that one can "see" students' prior knowledge supporting and constraining their reasoning.

The third implication for future research is to study growth and change of students' knowledge for reasoning with external representations. Several studies discussed above (e.g., Izsák 2003, 2004; Moschkovich 1998; Schoenfeld et al. 1993) have reported cases in which students refined and reorganized their knowledge as they worked through a series of challenging tasks. Nevertheless, we still do not have a good picture of how less expert knowledge for reasoning with external representations, or inscriptions, is gradually transformed into more expert knowledge. For instance, we know only a little about how students' experiences generating and interpreting external representations, or inscriptions, in arithmetic support and constrain their capacities to generate and interpret algebraic and graphical representations. One well-known example is students' difficulties interpreting equal signs (e.g., Kieran 1981), where researchers have conjectured that students' experiences with the equal sign in arithmetic lead them to understand the sign as a command to compute an answer. A more recent study (Knuth et al. 2006) demonstrates that middle school students today continue to have similar difficulties, even when they study with reform-oriented instructional materials. It is plausible that as students study arithmetic, they develop further understandings about how to use external representations, or inscriptions, as tools for solving problems. For instance, they may develop certain criteria for "good" representations of problem situations. Insights into the growth and change of such knowledge could ultimately inform curricular trajectories that better support students' capacities to use external representations as tools for reasoning about problem situations.

I close with implications for instruction. The implications for research discussed above have implications for instruction because the more we understand about students' prior knowledge and how it might transform into more expert knowledge the better able we will be to design appropriate learning experiences. I also emphasize that although results about students' criteria and adaptive interpretation were first reported in studies that relied on experimental instruction and interviews, this line of research has begun to provide insight into how students reason with more conventional forms of representation—including normative tables, equations, and graphs—in classrooms using commercially available curricular materials. Thus, it is becoming increasingly apparent that the advances discussed in the present article have broad implications for classroom instruction related to algebra. One way to think about this development is that vivid results from studies based on experimental instruction and interviews can help sensitize us to aspects of students' reasoning that are still important, even if they are harder to notice, in more conventional classroom settings.

Finally, if students have a complex substrata of knowledge for reasoning with external representations, or inscriptions, when solving problems about situations, then teachers should elicit that knowledge. For instance, teachers could have conversations with students that concentrate on how external representations might be generated and interpreted in order to solve problems about situations. These conversations could include explicit comparisons of different approaches. The literature already suggests a few examples of questions teachers might ask—for instance, does a letter such as n stand for a single unknown value or for all possible values? Other examples suggested by results about students' criteria and adaptive interpretation include the following: (1) If a letter sometimes stands for all possible values and sometimes stands for a subset of possible values, how can you tell when to use which interpretation? (2) Do all salient features of a situation have to be captured when generating a representation, or is just a subset sufficient for solving a problem?

(3) When generating an equation to solve a problem, do the terms have to express directly the final requested quantity, or can one start by expressing constraints in terms of different quantities? (4) What range of information about a problem situation can be determined from a single external representation? With opportunities to think about questions such as these, students might develop greater competence at reasoning with representations to model and solve problems with algebra.

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Middle School Students' Understanding of Core Algebraic Concepts: Equivalence & Variable

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Abstract Algebra is a focal point of reform efforts in mathematics education, with many mathematics educators advocating that algebraic reasoning should be integrated at all grade levels K-12. Recent research has begun to investigate algebra reform in the context of elementary school (grades K-5) mathematics, focusing in particular on the development of algebraic reasoning. Yet, to date, little research has focused on the development of algebraic reasoning in middle school (grades 6–8). This article focuses on middle school students' understanding of two core algebraic ideas—equivalence and variable—and the relationship of their understanding

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to performance on problems that require use of these two ideas. The data suggest that students' understanding of these core ideas influences their success in solving problems, the strategies they use in their solution processes, and the justifications they provide for their solutions. Implications for instruction and curricular design are discussed.

Introduction

Algebra is considered by many to be a "gatekeeper" in school mathematics, critical to further study in mathematics as well as to future educational and employment opportunities (Ladson-Billings 1998; National Research Council [NRC] 1998). Unfortunately, many students experience difficulty learning algebra (Kieran 1992), a fact that has led to first-year algebra courses in the United States being characterized as "an unmitigated disaster for most students". (NRC, p. 1) In response to growing concern about students' inadequate understandings and preparation in algebra, and in recognition of the role algebra plays as a gatekeeper, recent reform efforts in mathematics education have made algebra curricula and instruction a focal point (e.g., Bednarz et al. 1996; Lacampagne et al. 1995; National Council of Teachers of Mathematics 1997, 2000; NRC 1998; RAND Mathematics Study Panel 2003). In fact, Kaput (1998) has argued that algebra is the keystone of mathematics reform, and that teachers' abilities to facilitate the development of students' algebraic reasoning is the most critical factor in algebra reform. Moreover, he contended that the "key to algebra reform is integrating algebraic reasoning across all grades and all topics—to 'algebrafy' school mathematics". (p. 1)

Underlying this call to 'algebrafy' school mathematics is a belief that the traditional separation of arithmetic and algebra deprives students of powerful schemes for thinking about mathematics in the early grades and makes it more difficult for them to learn algebra in the later grades (Kieran 1992). Algebrafying school mathematics, however, means more than moving the traditional first-year algebra curriculum down to the lower grades. There is a growing consensus that algebra reform requires a reconceptualization of the nature of algebra and algebraic reasoning as well as a reexamination of when children are capable of reasoning algebraically and when ideas that require algebraic reasoning should be introduced into the curriculum (Carpenter and Levi 1999). Recent research has begun to investigate algebra reform in the context of elementary school mathematics, focusing in particular on the development of algebraic reasoning (e.g., Bastable and Schifter 2008; Carpenter et al. 2003; Carpenter and Levi 1999; Kaput et al. 2008). Yet, to date, little research has focused on the development of algebraic reasoning in the Middle Grades—the time period linking students' arithmetic and early algebraic reasoning and their development of increasingly complex, abstract algebraic reasoning. In this chapter, we present results from a multi-year research project that seeks to understand the development of middle school students' algebraic reasoning. In particular, the chapter focuses on students' understanding of two core algebraic ideas—equivalence and variable—and the relationship of their understanding to performance on problems that require use of these two ideas.

Student Understanding of Equivalence & Variable

Algebraic reasoning depends on an understanding of a number of key ideas, of which equivalence and variable are, arguably, two of the most fundamental. In this section we briefly describe research that has examined students' understandings of these two ideas; this description will serve to situate the present study in the larger body of research as well as to highlight the contribution of the present study.

Equivalence

The ubiquitous presence of the equal sign symbol in mathematics at all levels highlights its important role in mathematics, in general, and in algebra, in particular. Within the domain of algebra, Kieran (1992) contended that "one of the requirements for generating and adequately interpreting structural representations such as equations is a conception of the symmetric and transitive character of equality—sometimes referred to as the 'left-right equivalence' of the equal sign". (p. 398) Yet, there is abundant literature that suggests students do not view the equal sign as a symbol of equivalence (i.e., a symbol that denotes a relationship between two quantities), but rather as an announcement of the result or answer of an arithmetic operation (e.g., Falkner et al. 1999; Kieran 1981; McNeil and Alibali 2005; Rittle-Johnson and Alibali 1999). For example, Kieran (1981) found that 12- and 13-year old students described the equal sign in terms of the answer and provided examples of its use that included an operation on the left-hand side of the symbol and the result on the right-hand side (e.g., $3 + 4 = 7$). McNeil and Alibali (2005) found similar conceptions of the equal sign in definitions generated by third- through fifth-grade students.

While such a (mis)conception concerning the meaning of the equal sign may not be problematic in elementary school, where students are typically asked to solve equations of the form $a + b = \square$, it does not serve students well in terms of their preparation for algebra and algebraic ways of thinking. In algebra, students must view the equal sign as a relational symbol (i.e., "the same as") rather than as an operational symbol (i.e., "do something"). The relational view of the equal sign becomes particularly important as students encounter and learn to solve algebraic equations with operations on both sides of the symbol (e.g., $3x - 5 = 2x + 1$). A relational view of the equal sign is essential to understanding that the transformations performed in the process of solving an equation preserve the equivalence relation (i.e., the transformed equations are equivalent)—an idea that many students find difficult, and that is not an explicit focus of typical instruction. Steinberg et al. (1990) concluded that many eighth- and ninth-grade students do not have a good understanding of equivalent equations. They found that many students knew how to use transformations in solving equations, however, many of these same students did not seem to utilize such knowledge in determining whether two given equations were equivalent. Although not examined in their study, it seems reasonable to conclude that many of these latter students may have had inadequate conceptions of mathematical equivalence.

Variable

Algebra has been called the study of the 24th letter of the alphabet. Although this characterization is somewhat facetious, it underscores the importance of developing a meaningful conception of variable in learning and using algebra. The idea of variable, not surprisingly, has also received considerable attention in the mathematics educational research community (e.g., Küchemann 1978; MacGregor and Stacey 1997; Philipp 1992; Usiskin 1988), and the results of such work suggest that the use of literal symbols in algebra presents a difficult challenge for students. In Küchemann's frequently cited study, for example, he found that most 13-, 14-, and 15-year-old students considered literal symbols as objects (i.e., the literal symbol is interpreted as a label for an object or as an object itself). Few students considered them as specific unknowns (i.e., the literal symbol is interpreted as an unknown number with a fixed value), and fewer still as generalized numbers (i.e., the literal symbol is taken to represent multiple values, although it is only necessary to think of the symbol taking on these values one at a time) or variables (i.e., the literal symbol represents, at once, a range of numbers). Further, his study showed that students' misunderstandings of literal symbols seem to be reflected in their approaches to symbolizing relationships in problem solutions—an essential aspect of algebra and algebraic ways of thinking.

In sum, developing an understanding of equivalence and variable is essential to algebra and the ability to use it, yet they are ideas about which many students have inadequate understandings. In this article, we examine the meanings middle school students ascribe to the equal sign and variable, their performance on problems that require use of these ideas, and the relationship between the meanings they ascribe to each and their performance on the corresponding problems.

Method

Participants

Participants were 373 middle-school (6th through 8th grade) students drawn from an ethnically diverse middle school in the American Midwest. The demographic breakdown of the school's student population is as follows: 25% African American, 5% Hispanic, 7% Asian, and 62% White. The middle school had recently adopted a reform-based curricular program, *Connected Mathematics*, and, with the exception of one section of 8th grade algebra, the classes were not tracked (e.g., all 6th grade students were in the same mathematics course). The school was selected as the site for this research based upon the recommendation of the school district's mathematics resource teacher, who felt that the principal and teachers would be interested in participating.

The following questions are about this statement:

$3 + 4 = 7$
$\uparrow$

a) The arrow above points to a symbol. What is the name of the symbol?
b) What does the symbol mean?
c) Can the symbol mean anything else? If yes, please explain.

Fig. 1 Interpreting the equal sign

Is the number that goes in the □ the same number in the following two equations? Explain your reasoning.

$2\times \square + 15 = 31$ $\qquad$ $2\times \square + 15 - 9 = 31 - 9$

Fig. 2 Using the concept of mathematical equivalence

Data Collection

The data that are the focus of this article consist of students' responses to a subset of items from a written assessment that targeted their understandings of various aspects of algebra. In particular, the focus is on four items that were designed to assess students' understanding of the ideas of equal sign (1 item) and of variable (1 item) as well as their performance on two Problem solving items that (potentially) required the use of these ideas. The assessment consisted of three forms with some overlap of items; all 373 students were administered the equal sign understanding and variable understanding items, 251 students were administered the equal sign performance item, and 122 students were administered the variable performance item. The assessment was administered near the beginning of the school year.

Equal Sign Items

In the first item (shown in Fig. 1), students were asked to define the equal sign. The rationale for the first prompt (What is the name of the symbol?) was to preempt students from using the name of the symbol in their response to the second prompt (What does the symbol mean?). The rationale for the third prompt (Can the symbol mean anything else?) was to provide students the opportunity to give an alternative interpretation; in previous work, we have found that students often offer more than one interpretation when given the opportunity. The second item (shown in Fig. 2), the *equivalent equations* problem, was designed to assess students' understanding of the fact that the transformations performed in the process of solving an equation preserve the equivalence relation. We expected that students who viewed the equal sign as representing a relationship between quantities would conclude that the number that goes in the box is the same in both equations because the transformation

The following question is about this expression:

$$\underset{\uparrow}{2n} + 3$$

The arrow above points to a symbol. What does the symbol stand for?

Fig. 3 Interpreting a literal symbol used as a variable

Can you tell which is larger, $3n$ or $n + 6$? Please explain your answer.

Fig. 4 Using the concept of variable

performed on the second of the two equations preserved the quantitative relationship expressed in the first equation.[1]

Variable Items

Item 3 (shown in Fig. 3) was designed to assess students' interpretations of literal symbols. The fourth item (shown in Fig. 4), the *which is larger* problem, was designed to assess students' abilities to use the concept of variable to make a judgment about two varying quantities. In particular, to be successful on the final item, students must recognize that the values of $3n$ and $n + 6$ are dynamic and depend on the value of n, that is, they must view n as a variable—a literal symbol that represents, at once, a range of numbers.

Coding

In this section we provide details regarding the coding of each item; in the results section, we provide sample student responses. For all items, responses that students left blank or for which they wrote "I don't know" were grouped in a *no response/don't know* category, and responses for which students' reasoning could not be determined and responses that were not sufficiently frequent to warrant their own codes were grouped in an *other* category.

Coding Equal Sign Understanding

Student responses to parts b) and c) of Item 1 were coded as *relational*, *operational*, *other*, or *no response/don't know*, with the majority of responses falling into the

[1]There were two versions of the *equivalent equations* problem, one which used a box (as in Fig. 2) and one which used n to represent the missing values. Performance did not differ across versions (60% correct box, 61% correct n), and the distribution of strategies used to solve the problem did not differ across versions (χ^2 (4, $N = 252$) = 1.729, *ns*), so we collapse across versions in the analyses presented in this chapter.

first two categories. A response was coded as *relational* if a student expressed the general idea that the equal sign means "the same as" and as *operational* if the student expressed the general idea that the equal sign means "add the numbers" or "the answer". In addition to coding the responses to parts b) and c) separately, students were also assigned an overall code indicating their "best" interpretation. Many students provided two interpretations, often one *relational* and one *operational*; in such cases, the responses were assigned an overall code of *relational*.

Coding Performance on the Equivalent Equations Problem

Students' responses to Item 2 were coded for correctness as well as strategy use. Responses were coded as correct if students responded that the two equations have the same solution. Students' strategies for solving the problem were classified into one of five categories: *answer after equal sign*, *recognize equivalence*, *solve and compare*, *other*, or *no response/don't know*. In the *answer after equal sign* category, students' rationale for their conclusion was that each equation had the same "answer" (in this case, 31) to the immediate right of the equal sign and the equations were therefore equivalent (an incorrect strategy). In the *solve and compare* category, students' rationale for their conclusion was based on either (1) determining the solution to the first equation, substituting that solution into the second equation, and noting that the value satisfied both equations, or (2) determining the solutions to both equations and comparing them. Finally, in the *recognize equivalence* category, students' rationale for their conclusion was based on recognizing that the transformation performed on the second equation preserved the equivalence relation. Note that only the *recognize equivalence* strategy appears to explicitly require a relational understanding of the equal sign.

Coding Variable Understanding

Students' responses to the literal symbol interpretation item were classified into five categories, *multiple values*, *specific number*, *object*, *other*, or *no response/don't know*. A response was coded as *multiple values* if the student expressed the general idea that the literal symbol could represent more than one value; as *specific number* if the student indicated that the literal symbol represents a particular number; and as *object* if the student suggested that the literal symbol represents a label for a physical object (such as stating that n represents newspapers).

Coding Performance on the Which Is Larger Problem

Students' responses to the *which is larger* problem were coded both in terms of the judgment about which quantity was larger ($3n, n + 6$, or can't tell) and for the reasoning underlying that judgment. Students' explanations of their reasoning were classified into five categories: variable explanations, *single-value* explanations, *operation* explanations, *other*, or *no response/don't know*. variable explanations expressed the general idea that one cannot determine which quantity is larger because the variable can take on multiple values. *Single-value* explanations tested a single value and drew a conclusion on that basis; thus, students' conclusions varied depending on the value tested. *Operation* explanations expressed the general idea that one type of operation leads to larger values than the other (for example, multiplication produces larger values than addition).

Coding Reliability

To assess reliability of the coding procedures, a second coder rescored approximately 20% of the data. Agreement between coders was 90% for coding students' interpretations of the equal sign, 91% for coding students' strategies on the *equivalent equations* problem, 91% for coding students' interpretations of literal symbols, and 95% for coding students' explanations on the *which is larger* problem.

Results

We focus first on students' interpretations of the equal sign symbol, and how these interpretations relate to performance on the *equivalent equations* problem. We then turn to students' interpretations of a literal symbol (n) used as a variable, and how these interpretations relate to performance on the *which is larger* problem. Representative excerpts from students' written responses are provided to illustrate particular findings. In reporting the results, we describe (and illustrate) only those coding categories that are most germane to the focus of the article. Finally, the statistical analysis of the data was performed using logistic regression because the outcome variables of interest were categorical. All reported statistics are significant with alpha set at .05.

Interpretation of the Equal Sign

We first examined the relationship between grade level and interpretation of the equal sign symbol. The following responses are typical of those coded as *operational*:

> It means the total of the numbers before it. (6^{th} grade student)
>
> It means whatever is after it is the answer. (8^{th} grade student)

The following responses are typical of those coded as *relational*:

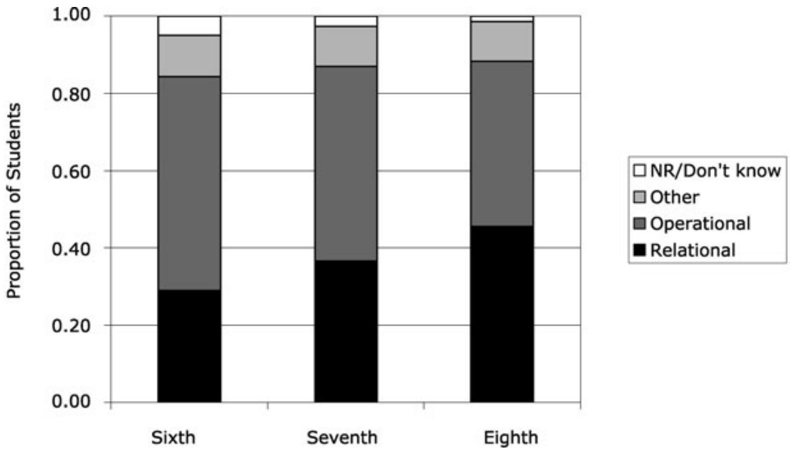


Fig. 5 Equal sign interpretations of sixth-, seventh-, and eighth-grade students

> It means the number(s) on its left are equivalent to the number(s) on its right. (6th grade student)

> The things on both sides of it are of the same value. (7th grade student)

Students were classified as providing a relational interpretation if they provided one on either their first or second response. As seen in Fig. 5, the proportion of students providing a relational interpretation for the equal sign differed across the grades, $Wald(2, N = 373) = 7.80$, and this difference was accounted for by a significant linear trend, $\hat{\beta} = 0.52$, $z = 2.78$, $Wald(1, N = 373) = 7.72$. Despite this improvement across grades, however, the overall level of performance was strikingly low. Even at grade 8, only 46% of students provided a relational interpretation of the equal sign.

Performance on the Equivalent Equations Problem

The proportion of students who correctly judged that the two equations had the same solution differed across the grades, $Wald(2, N = 251) = 10.21$, $p = .006$, and was accounted for by a significant linear trend, $\hat{\beta} = 0.72$, $z = 3.18$, $Wald(1, N = 251) = 10.08$, $p = .002$. Students' strategies for solving the *equivalent equations* problem are displayed in Table 1. The majority of students' strategies were categorized into one of the following categories: *recognize equivalence*, *solve and compare*, or *answer after equal sign*. Typical responses in each of these three categories included the following:

> Yes because you're doing the same equation but just minusing 9 from both sides in the second one. (*recognize equivalence*, 8th grade student)

Table 1 Proportion of students at each grade level who used each strategy use on the *equivalent equations* problem

Strategy	Grade Level 6th	7th	8th
Recognize equivalence	0.12	0.17	0.34
Solve and compare	0.39	0.33	0.25
Answer after equal sign	0.11	0.11	0.11
Other	0.31	0.25	0.27
No response/Don't know	0.08	0.14	0.02

> Yes if you substitute 8 for n, each answer will be equal and make sense [student shows computations for determining the value of n and for checking that the value satisfies the second equation]. (*solve and compare*, 6th grade student)

> Yes because it has to be to get a 31 in both answers. (*answer after equal sign*, 6th grade student)

As seen in Table 1, there was also a substantial number of students who left their answer sheets blank, simply wrote that they did not know, or used idiosyncratic strategies (i.e., strategies that could not be determined or that were insufficiently frequent to warrant their own codes, both of which were classified as *other* strategies). To some extent, the large proportion of strategies in the *other* category may not be too surprising: with the exception of one 8th grade algebra class, the students' exposure to algebra, in general, and equivalent equations, in particular, had been minimal at best. (Alternatively, it is encouraging to see that so many students—students who used the *recognize equivalence* or *solve and compare* strategies—were able to engage with the problem in mathematically appropriate ways prior to formal instruction in "algebra".)

Is interpretation of the equal sign associated with performance on problems that involve equations? More specifically, do students who hold a relational view of this symbol perform better than their peers who do not hold such a view on a problem for which they must judge the equivalence of two equations? To find out, we examined the relationships among grade level (6, 7 or 8), equal sign interpretation (relational or not), and performance on the *equivalent equations* problem. We first consider students' judgments about whether the two equations had the same solutions or not, and then we consider their strategies for arriving at those judgments.

As seen in Fig. 6, students who provided a relational interpretation of the equal sign were more likely to judge that the two equations had the same solutions than were students who did not provide a relational interpretation. The effect of equal sign interpretation was significant when controlling for grade level, $\hat{\beta} = -1.24$, $z = -4.15$, $Wald(1, N = 251) = 17.23$. In addition, the effect of grade level was significant when controlling for equal sign interpretation, $Wald(2, N = 251) = 9.00$, and was accounted for by a significant linear trend, $\hat{\beta} = 0.70$, $z = 2.98$, $Wald(1, N = 251) = 8.89$.

As seen in Fig. 7, students who provided a relational interpretation for the equal sign were also more likely to use the *recognize equivalence* strategy than were students who did not provide a relational interpretation. The effect of equal sign in-

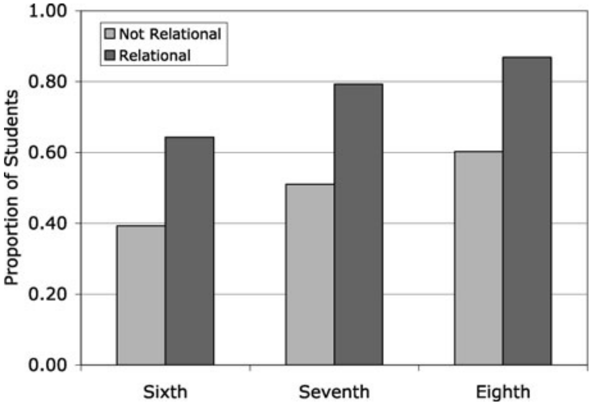


Fig. 6 Proportion of sixth-, seventh-, and eighth-grade students in each equal sign understanding category who answered the *equivalent equations* problem correctly

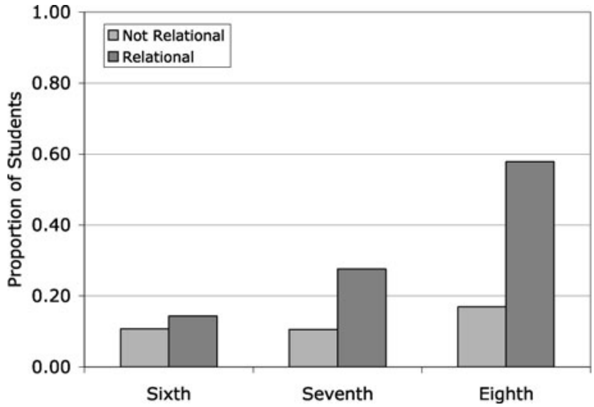


Fig. 7 Proportion of sixth-, seventh-, and eighth-grade students in each equal sign understanding category who used the *recognize equivalence* strategy on the *equivalent equations* problem

terpretation was significant when controlling for grade, $\hat{\beta} = -1.33$, $z = -4.01$, $Wald(1, N = 251) = 16.10$. In addition, the effect of grade level was significant when controlling for equal sign interpretation, $Wald(2, N = 251) = 12.11$, and was accounted for by a significant linear trend, $\hat{\beta} = 0.93$, $z = 3.17$, $Wald(1, N = 251) = 10.05$. It is worth noting that a subset of students who used the recognize equivalence strategy (24%) displayed an operational view of equality on the equal sign interpretation item. Thus, different problem contexts appear to activate or draw on different aspects of students' knowledge.

In sum, students' understanding of the equal sign was associated with their performance on the *equivalent equations* problem, both in terms of their judgments for the problem and the strategies they used to arrive at those judgments. Thus, students

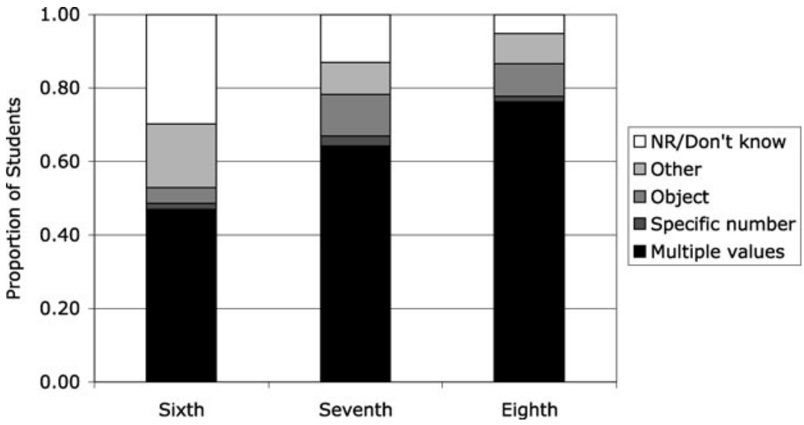


Fig. 8 Students' interpretations of a literal symbol

who demonstrated a relational understanding of the equal sign appeared to use this understanding in determining that the two equations had the same solutions.

Interpretation of a Literal Symbol

We turn now to students' interpretations of a literal symbol (n) used as a variable in a mathematical expression. The most common meaning students at all three grade levels provided was that of a variable—the literal symbol could represent more than one value. The following responses are representative of the *multiple values* code:

> The symbol is a variable, it can stand for anything. (6^{th} grade student)
>
> A number, it could be 7, 59, or even 363.0285. (7^{th} grade student)
>
> That symbol stands for x which stands for a number that goes there. (8^{th} grade student)

In the final example above, it is interesting that the student apparently felt the need to replace n with x, the latter of which represents a number; this response may be an artifact of school mathematics in which the prototypical literal symbol is x. Not surprisingly, as seen in Fig. 8, the proportion of students providing a correct interpretation (i.e., multiple values) differed across the grades, $Wald(2, N = 372) = 22.58$, increasing from fewer than 50% of students in grade 6 to more than 75% of students in grade 8. This improvement across grades was accounted for by a significant linear trend, $\hat{\beta} = 0.91$, $z = 4.71$, $Wald(1, N = 372) = 22.27$.

It is also worth noting the relatively large proportion of 6^{th} grade students whose responses were categorized as either *other* or *no response/don't know*. One possible explanation for the nature of these students' responses relates again to the curriculum: the first formal introduction of the concept of variable does not occur until the 7^{th} grade (in the *Connected Mathematics* curriculum), thus the 6^{th} grade students may lack experience with literal symbols used as variables in algebraic expressions.

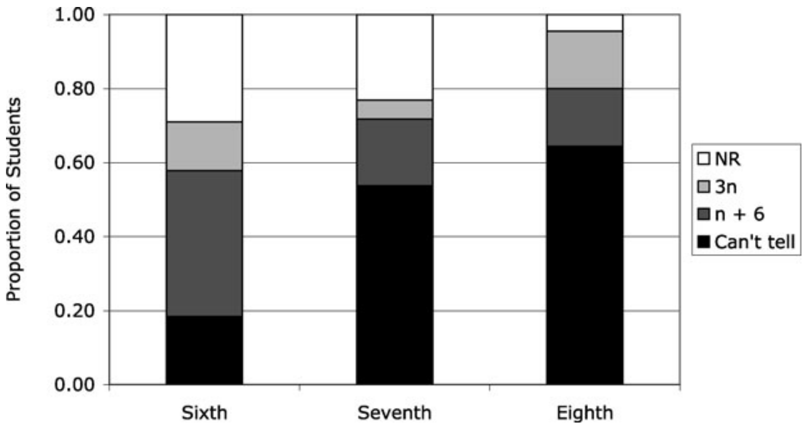


Fig. 9 Students' judgments on the *which is larger* problem

Performance on the which Is Larger Problem

Figure 9 displays students' judgments to the question prompt (i.e., Can you tell which is larger, $3n$ or $n + 6$?), and Table 2 displays the justifications students provided for their judgments. In some cases, students provided only a judgment and not a justification for their judgment (justifications in these cases were assigned to the *no response/don't know* category).

In the 6^{th} grade, the majority of students appeared either unable to provide a justification or to provide an idiosyncratic justification (see Table 2). Relative to the 6^{th} grade students, the 7^{th} and 8^{th} grade students were more likely to respond with a correct justification that focused on the fact that the literal symbol could take on multiple values. The following justifications are typical of variable responses:

> No because you don't know what n is. (6^{th} grade student)

> No, because n is not a definite number. If n was 1, $3n$ would be 3 and $n + 6$ would be 7, but if n was 100, $3n$ would be 300 and $n + 6$ would be 106. This proves that you cannot tell which is larger unless you know the value of n. (8^{th} grade student)

Although the coding category of *single value* appeared in fewer than 5% of the responses at each grade level, it is worth noting, because these students at least seemed to recognize that the literal symbol represents a number. In such cases, the students tested a specific number and based their judgments on the results of their computations. Likewise, the coding category of *operation* was also rare. Based on prior work (e.g., Greer 1992), we expected that some students would focus on the operation (for example, one seventh-grade student stated, "Yes, $n + 6$ is bigger because they +".). Although such responses did occur in 7^{th} and 8^{th} grades, they represented fewer than 10% of the responses at each grade level.

Is holding a multiple-values interpretation of n associated with performance on the *which is larger* problem? That is, were students who interpreted the literal symbol (item 3) as a variable more likely to answer "can't tell" and to provide a correct

Table 2 Proportion of students at each grade level who provided each type of justification for the *which is larger* problem

Justification	Grade Level 6th	7th	8th
Variable	0.11	0.51	0.60
Single Value	0.03	0.05	0.04
Operation	0.00	0.05	0.09
Other	0.42	0.15	0.16
No response/Don't know	0.45	0.23	0.11

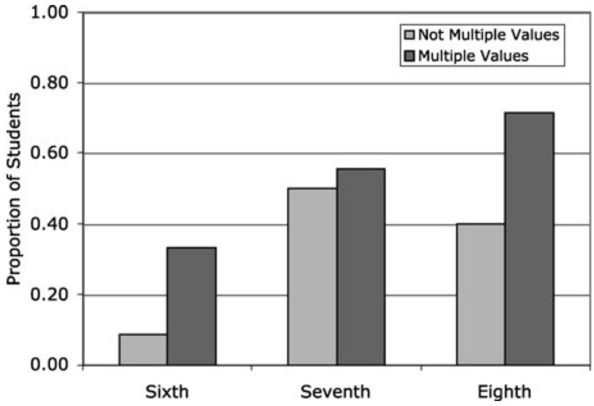


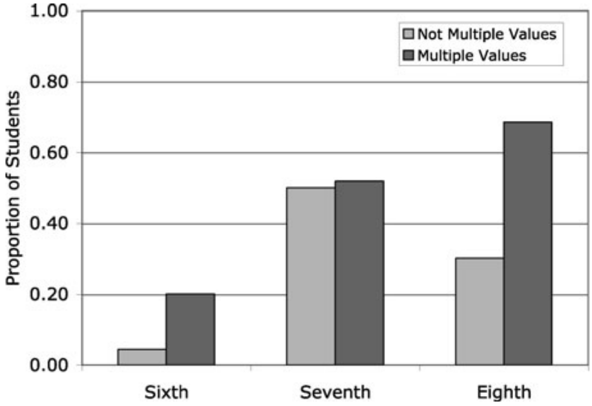
Fig. 10 Proportion of sixth-, seventh-, and eighth-grade students in each literal symbol interpretation category who provided a correct judgment for the *which is larger* problem

justification than were students who did not provide a multiple-values interpretation? To find out, we examined the relationships among grade level (6, 7 or 8), literal symbol interpretation (multiple values or not), and performance on the *which is larger* problem. We first consider students' judgments about whether $3n$ or $n+6$ is larger, and then we consider their justifications.

As seen in Fig. 10, students who provided a multiple-values interpretation were indeed more likely than their peers who did not provide a multiple-values interpretation to answer "can't tell" on the *which is larger* problem. The effect of having a multiple-values interpretation was significant when controlling for grade level, $\hat{\beta} = -0.97$, $z = 2.22$, $Wald(1, N = 122) = 4.90$. In addition, the proportion of students who correctly answered "can't tell" increased across the grade levels. The effect of grade level on performance was significant when controlling for literal symbol interpretation, $Wald(2, N = 122) = 11.54$, and was accounted for by a significant linear trend, $\hat{\beta} = 1.27$, $z = 3.34$, $Wald(1, N = 122) = 11.13$.

Lastly, as seen in Fig. 11, students who provided a multiple-values interpretation of the literal symbol were also more likely than were students who did not provide a multiple-values interpretation to provide correct justifications on the *which is larger* problem. The effect of having a multiple-values interpretation was significant when

Fig. 11 Proportion of sixth-, seventh-, and eighth-grade students in each literal symbol interpretation category who provided a correct justification for the *which is larger* problem



controlling for grade, $\hat{\beta} = -0.95$, $z = -2.05$, $Wald(1, N = 122) = 4.21$. Further, the proportion of students who provided a correct justification increased across the grade levels. The overall effect of grade was significant when controlling for literal symbol interpretation, $Wald(2, N = 122) = 13.79$, and was accounted for by a significant linear trend, $\hat{\beta} = 1.61$, $z = 3.65$, $Wald(1, N = 122) = 13.32$. It is worth noting that a subset of students who provided a correct justification on the *which is larger* item (20%) did not provide a multiple values interpretation on the variable understanding item. Thus, as for the equal sign items, different problem contexts appear to activate or draw on different aspects of students' knowledge.

In sum, understanding of variable was associated with performance on the *which is larger* problem, in terms of both students' judgments about which quantity was larger and their justifications for their judgments. Thus, students who had a multiple-values interpretation of a literal symbol appeared to use this understanding in determining that one cannot tell whether $3n$ or $n + 6$ is larger.

Discussion

The focus of this chapter was on middle school students' understandings of the ideas of equivalence and variable, their performance on problems that require use of these ideas, and the relationship of their understanding to performance. In this section, we briefly discuss the results and their implications for mathematics education.

Equivalence Results

The finding that many students hold an operational view of the equal sign is not particularly surprising, given that similar results have been found in previous research (e.g., Falkner et al. 1999; Kieran 1981; McNeil and Alibali 2005; Rittle-Johnson and Alibali 1999). Although our results suggest that students' views of the symbol become more mathematically sophisticated (i.e., view the equal sign as a relation between two quantities) as they progress through middle school, the majority of students at each grade level continued to exhibit less sophisticated views of the equal sign (e.g., as a "do something" symbol). This result is troublesome in light of our finding that students who have a relational view of the equal sign outperformed their peers who hold alternative views on a problem that requires use of the idea of mathematical equivalence. We report elsewhere that middle school students' views of the equal sign also play a role in their success in solving algebraic equations and simple algebra word problems (Knuth et al. 2006). Taken together, such results suggest that an understanding of equivalence is a pivotal aspect of algebraic reasoning and development. Consequently, students' preparation for and eventual success in algebra may be dependent on efforts to enhance their understanding of mathematical equivalence and the meaning of the equal sign.

Yet, equivalence is a concept traditionally introduced during students' early elementary school education, with little instructional time explicitly spent on the concept in the later grades. In fact, teachers generally assume that once students have been introduced to the concept during their elementary school education, little or no review is needed. Some previous work at the elementary school level has focused on promoting a relational view of the equal sign (e.g, Carpenter et al. 2003); however, there is little explicit attention to this concept in the later grades. This lack of attention may explain, in large part, why many students continue to show inadequate understandings of the meaning of the equal sign in secondary school and even into college (e.g., McNeil and Alibali 2005; Mevarech and Yitschak 1983). Further exacerbating students' opportunities to develop their understanding of equivalence is the fact that very little attention is paid to the concept in curricular materials—despite the ubiquitous presence of the equal sign. Moreover, analyses of middle school curricular materials suggest that relational uses of the equal sign are less common than operational uses (McNeil et al. 2004). This pattern of exposure may actually condition students to favor less sophisticated and generalized uses of equivalence (such as "operations equals answer").

Variable Results

The findings regarding students' views of literal symbols are, in general, more positive than the results of previous research (cf. Küchemann 1978). In particular, a substantial proportion of students interpreted a literal symbol as representing more than one value, increasing from approximately 50% of the 6^{th} grade students to more than 75% of the 8^{th} grade students. Students' use of this knowledge suggests, however, that knowledge of the concept of variable may be somewhat fragile, particularly among 6^{th}-grade students, who were largely unable to correctly answer the *which is larger* problem. Yet, those students who did provide a multiple-values interpretation of a literal symbol were more likely than their peers to not only use this understanding to determine that one could not tell whether $3n$ or $n + 6$ is larger, but also to provide a correct justification for why one could not tell which is larger. These latter results highlight the importance of fostering a multiple-values interpretation of literal symbols and suggest that efforts to foster such an interpretation will likely contribute to students' preparation for algebra and algebraic ways of thinking.

In contrast to the treatment of equivalence, the concept of variable is one that receives explicit instructional and curricular attention in middle school (7^{th} grade in the *Connected Mathematics* curriculum). It may be the case, however, that providing students with opportunities to meaningfully encounter literal symbols in ways that support the development of a multiple-values understanding at an earlier age may be beneficial in terms of their preparation for and eventual success in algebra (a perspective shared by others, e.g., Carraher et al. 2000). Students often encounter literal symbols during their elementary school education (e.g., $8+3=\square$, $3+?=7$), however, the nature of such exposure may lead students to consider literal symbols in less sophisticated and mathematically powerful ways (e.g., as specific numbers).

Concluding Remarks

If a goal of mathematics education reform is to better prepare *all* students for success in algebra, then the nature of students' "pre-algebraic" mathematical experiences must lay the foundation for more formal study of algebra. Much of this foundation can be laid as well as strengthened in the middle school grades—the time period linking students' arithmetic and early algebraic reasoning and their development of increasingly complex, abstract algebraic reasoning. In this chapter we presented results concerning students' understanding of two fundamental algebraic ideas—equivalence and variable—and the relationship of their understanding to performance on problems that require use of these two ideas. It is our hope that these results will inform the work of both teachers and curriculum developers, so that they can each provide more opportunities for students to develop their understanding of these core concepts.

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An Approach to Geometric and Numeric Patterning that Fosters Second Grade Students' Reasoning and Generalizing about Functions and Co-variation

Joan Moss and Susan London McNab

Abstract In this chapter, we present illustrations of second grade students' reasoning about patterns and two-part function rules in the context of an early algebra research project that we have been conducting in elementary schools in Toronto and New York City. While the study of patterns is mandated in many countries as part of initiatives to include algebra from K-12, there is a plethora of evidence that suggests that the route from patterns to algebra can be challenging even for older students. Our teaching intervention was designed to foster in students an understanding of linear function and co-variation through the integration of geometric and numeric representations of growing patterns. Six classrooms from diverse urban settings participated in a 10–14-week intervention. Results revealed that the intervention supported students to engage in functional reasoning and to identify and express two-part rules for geometric and numeric patterns. Furthermore, the students, who had not had formal instruction in multiplication prior to the intervention, invented mathematically sound strategies to deconstruct multiplication operations to solve problems. Finally, the results revealed that the experimental curriculum supported students to transfer their understanding of two-part function rules to novel settings.

Introduction

The study of patterns is now commonplace in elementary school curricula in many countries, arising out of initiatives to include algebra from Kindergarten through

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Grade 12 (e.g., Noss et al. 1997; Ontario Ministry of Training and Education 2005; Sasman et al. 1999; Warren 2000). The National Council of Teachers of Mathematics (NCTM) advocates that patterns should be taught from the first years of schooling with the expectation that students, as early as second grade, should be able to "analyze how both repeating and growing patterns are generated", and by the end of fifth grade should be able to "represent patterns and functions in words, tables and graphs" (NCTM 2000). It has been suggested that patterns can: (1) support students to understand the dependent relations among quantities that underlie mathematical functions (e.g. Carraher et al. 2008; Ferrini-Mundy et al. 1997; Mason 1996; Lee 1996); (2) serve as a concrete and transparent way for young students to begin to grapple with abstraction and generalization (Watson 2000; Noss and Hoyles 1996; Kieran 1992); and (3) support students to develop the language of conjecture and proof in communicating their reasoning about pattern rules (Kuchemann 2008; Moss and Beatty 2006b).

Developmentally, the inclusion of patterns seems to fit well with mathematics learning in the early years. We know that young children are fascinated with patterns (Ginsburg and Seo 1999) and are capable not only of noticing patterns but also of using this skill naturally to make sense of their world (Greenes et al. 2001).

However, the potential of pattern work to support algebra learning has not been substantially realized (e.g. Carraher et al. 2008; Dorfler 2008). An extensive literature review on patterning research—conducted primarily with older students—reveals that without specific targeted pedagogical supports even older students have significant difficulty finding algebraic rules for patterns, strongly suggesting that the route from perceiving patterns to finding useful rules and algebraic representations is difficult (English and Warren 1998; Kieran 1992; Lannin et al. 2006; Lee and Wheeler 1987; Orton and Orton 1999; Stacey and MacGregor 1999; Steele and Johanning 2004).

One challenge in moving from pattern study to algebra is the tendency of students to use additive strategies for identifying and describing patterns—that is, to focus on the variation within a single data set rather than on the relationship between two data sets (e.g., Orton et al. 1999; Rivera 2006; Rivera and Becker 2007; Warren 2006). While this recursive approach allows students to predict the "next" position of a pattern, it does not support co-variational thinking about a relationship across data sets to find the underlying function rule. As well, even when students begin to grasp two-part pattern rules, they often use incorrect proportional thinking or "whole object reasoning" to make predictions about the number of elements in a far position of a sequence (e.g., English and Warren 1998; Lee 1996; Orton 1997; Stacey 1989).

Over the last several years, however, there has been an increasing number of accounts of middle school students who, as part of dedicated instructional interventions, demonstrated the ability to work constructively with patterns. For example, in 2008, the journal *ZDM, Mathematics education* published a special issue that focused on patterns and generalizing problems (Becker and Rivera 2008). Research reported in this special issue by Amit and Neria, Radford, Rivera and Becker, and Steele analyzed strategies that middle school students employed in their pattern work. These studies had in common the use of an analytic framework (e.g. Stacey 1989; Lannin 2005; English and Warren 1998) that captured the progression of students' reasoning. They noted that students working on generalizing problems began by using an additive or recursive approach, then, if they were able, switched to explicit or functional reasoning to find far positions and general rules. While there were many promising results reported in these articles, there were also indications of limited strategy use. Amit and Neria (2008), who studied the patterning problem solving strategies of 139 gifted 11- to 13-year-olds, went so far as to conclude that it is only advanced mathematics students who are able to learn to generalize. In their words, "Because of the higher-order thinking involved in generalization, such as abstraction, holistic thinking, visualization, flexibility and reasoning, the ability to generalize is a feature that characterizes capable students and differentiates them from others."

However, we along with others believe that it is not patterns per se, but the ways that patterns are presented that may limit students' ability to engage in the higher order thinking that characterizes generalization. While there has been less research conducted with very young children, our present study with Grade 2 students in diverse urban settings joins the work of other researchers (e.g. Carraher et al. 2006, 2008; Carraher and Earnest 2003; Cooper and Warren 2008; Mulligan et al. 2004; Mulligan and Mitchelmore 2009) to examine the potential of pattern work to support algebraic thinking in the early elementary school years.

Our Project

Over the last five years, we have been investigating new approaches to pattern teaching and learning that support students to forge connections amongst different representations of pattern. Our goal is to promote multiple ways of working with patterns (Mason 1996), and to foster what Lee (1996) has termed "perceptual agility—the ability to see multiple patterns coupled with a willingness to abandon those that do not prove useful for rule making" (p. 95). Our project to date has included intervention studies in 20 inner city elementary school classrooms (e.g., Beatty and Moss 2006a, 2006b; Beatty et al. 2006; London McNab and Moss 2004; Moss 2005; Moss and Beatty 2006a, 2006b; Moss et al. 2008). Further, it has served as a basis for a school district-wide professional development intervention (Beatty et al. 2008). This research began with a series of studies in second grade classrooms; it is the methods and data from these Grade 2 studies that we present in this chapter.

Our Approach: Theoretical

The predominant theoretical inspiration for our research on patterning emanated from the theories about mathematical development of Case and colleagues (Case and Okamoto 1966). Case and his colleagues' previous work in mathematics development for number sense in the domains of whole number (e.g. Griffin and Case 1997) and rational number (e.g. Moss and Case 1999; Moss 2004) offered a model for the integration of numeric and spatial schemes that we paralleled in linking numeric and geometric representations of patterns. A central tenet of the instructional design of Case et al. is the focus on the development of students' visual/spatial schemes. The theoretical proposal is that the merging of the *numerical* and the *visual* provides the students with a new set of powerful insights that can underpin not only the early Learning of a new mathematical domain but subsequent Learning as well (Case and Okamoto 1966; Kalchman et al. 2001). As we elaborate below, our experimental patterning curriculum was designed to support students to forge connections between visual/spatial patterns in the form of geometric growth sequences and numeric patterns embedded in "Guess my rule" games. Our conjecture was that the merging of these two types of patterns would serve as a foundation to support students to gain an initial understanding of linear functions. To test this conjecture, we designed a lesson sequence that was pilot-tested, revised and refined over a two-year period, and implemented in 6 different Grade 2 classrooms.

Context and Students

The 7- and 8-year old students in our study were from intact classrooms of between 20 and 22 students each, in urban settings in Toronto and New York City. These students represented diverse populations and a range of math competency. The classrooms were chosen because of the teachers' interest in learning more about this new approach and in involving their children in this study. Overall, the classrooms seemed to have in common an invitational sense of welcoming student contribution; the students were all accustomed to expressing their thoughts and reasoning in math, as in all subjects.

All of the students had previous experience with repeating patterns as part of the early years math curricula; however, none of the students had worked with growing patterns. Importantly, there had been no formal instruction in multiplication in any of the classrooms prior to the intervention. Although the activities in the intervention could be approached through multiplication, at no time was multiplication formally taught.

The length of the intervention varied from 10 to 14 lessons of about forty minutes each. In four of these classrooms, the interventions were taught by the classroom teachers with the help of research assistants; in the other classrooms, the interventions were taught by the first or second author with the assistance of the classroom teacher. It is important to note that there were research assistants in the classroom, who were able to work with Small groups or individual children, and that in some of the classrooms math was taught to only half the class at a time.

Instructional Sequence

Visual Representation: Geometric Growing Patterns

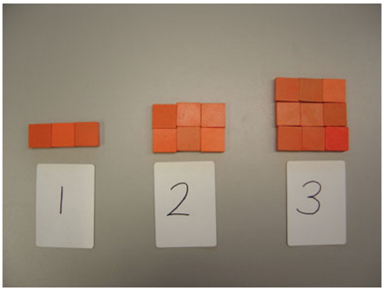
The lessons began by presenting students with the first three positions in a geometric growing pattern. These patterns were made of square tiles set out in arrays that grew by a given coefficient. To enable students to keep track of the ordinal position number of these tile patterns, position number cards were placed below the geometric arrays that represented that position of the pattern.

This clarified for students the functional relationship between the position number (independent variable) and the number of elements in each position (dependent variable). So, for example, for a pattern representing the relationship described by the equation $y = 3x$ (please see Fig. 1), students could easily connect the position number card "1" to the single row of three square tiles, the position number "2" to the 2 rows of three square tiles (6 tiles), the position number "3" to the 3 rows of three square tiles (9 tiles), and so on.

The initial challenges that the teacher posed were designed to focus students' attention on the relationship between the position number and the number of elements in each position, through the geometric configurations of the tile arrays. Referring once more to the pattern in Fig. 1, in the first lessons, the teacher's questions to the students followed a specific sequence: *If this pattern keeps growing in the same way, what would the next position look like? How many blocks would there be in the next position? What would the 10th position look like? How many blocks in the 10th? What about the 100th position?* In subsequent lessons after the students had experience with the function machine activity "Guess my rule?" (see below), the teacher would go on to ask, *What if you had any position? What could the rule be?*

Next, two-part functions were introduced geometrically. To demonstrate the constant, a fixed number of tiles was placed at the top of the array; this configuration of tiles remained the same from position to position, while the array grew multiplicatively by one row for each new position. Because of the spatial representation of the constant as tiles that jutted out from the array (please see Fig. 2), the students began to refer to the constant as the "bump", and we made a deliberate decision to encourage their use of this natural language.

Fig. 1 Position number cards



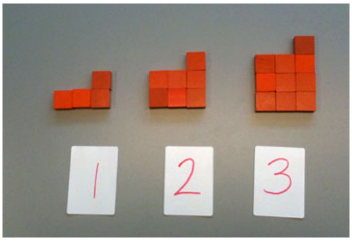


Fig. 2 Composite functions: the "bump"

As the lessons progressed, students built their own patterns and challenged classmates to make conjectures for general rules.

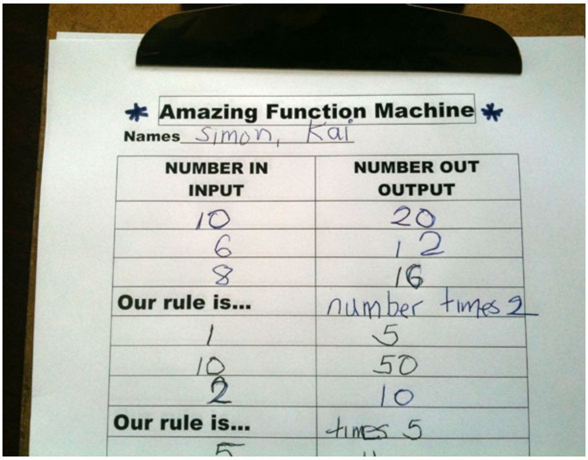
Numeric Representations: Function Machine

We interspersed these visually based pattern lessons with numeric-based lessons that incorporated function machine activities (Carraher and Earnest 2003; Rubenstein 2002; Willoughby 1997). Please see Fig. 3a for an example of a function machine. As in the geometric lessons, in the first series of function machine activities, we focused on one-step multiplicative rules. To begin, the teacher led the activities; then the students took turns creating functional rules (e.g. "double the number and add 3 more") to challenge their classmates in the "Guess my rule" game. The teacher modeled the use of a T-table to record the non-sequential pairs of input and output numbers; please see Fig. 3b for an example of the Function machine T-table. Pairs of students generated between 3 and 5 examples of non-sequential pairs of input and output numbers, as clues to their rule. The children who were solving the challenges given to them by their classmates used T-tables to record the input and output numbers, and their iterative conjectures for what the rule might be. It was important that the numeric clues were non-sequential to allow students to focus on the "across" (on a T-table) or functional rule, rather than on the "down" pattern or "what comes next" strategy identified as interfering with functional generalizations. Further, the T-tables were used only to record the non-sequential clues in the "Guess my rule" game, but not to generate further pairs of values as is often done in many classrooms.

Because the children had not yet been taught multiplication, the coefficients we initially presented were confined to what we determined to be arithmetically manageable numbers—2, 3, 5 and 10—that they would have practiced as skip-counting in first grade. However, there were no such constraints on the numbers the students could choose when they were creating their own rules for the function machine, giving them the opportunity to experiment with even difficult or tricky arithmetic if they chose.

Fig. 3 (**a**) Function machine. (**b**) Function machine T-table

(a)

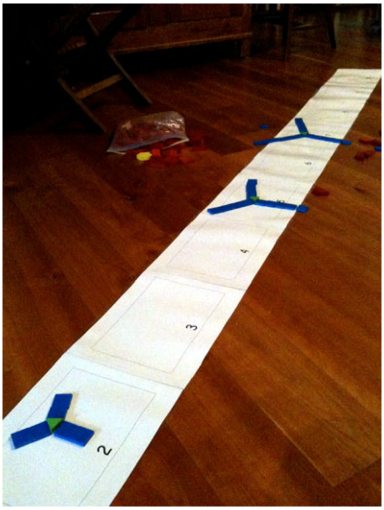


(b)

Integration Activities: Pattern Sidewalk

Both the geometric and the numeric activities offered students a chance to consider the idea of co-variation and function rules. The geometric activities specifically highlighted the direct connection between the position number and the structure of the corresponding arrays and number of tiles in each position. The function machine activities illuminated the idea of explicit rather than recursive rules. In order to integrate these two complementary approaches within a non-sequential presentation and exploration of geometric patterns, we designed what became known as a "pattern sidewalk". This is a large counting line placed on the classroom floor, with ordinal position numbers on each section of the sidewalk, from 1 to 10.

Fig. 4 Pattern sidewalk



The pattern sidewalk activities were geometric, but paralleled those of the numeric function machine game. First, the teacher would build one position of a pattern that conformed to a secret rule on, for example, the third section of the sidewalk, and ask the children to consider a possible pattern rule that would fit with that configuration in that position. Then the teacher would build another position of the same rule on any other non-adjacent section of the sidewalk, for example, the seventh pattern position on the seventh section of the sidewalk. At this point the students were given the opportunity to revise their initial rule conjecture. Finally, another pattern position would be shown to the students, on its appropriate section of the sidewalk and students were allowed to guess the rule, and asked to show their guess by building another position of the pattern correctly on a new section of sidewalk. The students were then invited to take on the role of teacher and present their own challenges to their classmates.

These three components—the geometric pattern building, the function machine and the pattern sidewalk—comprised the main elements of the lesson sequence in all cases, and were introduced to the students in the specific order described above. However, as the students gained experience of increasingly complex patterns, the teacher/researcher would revisit these different components, adding new elements such as moving from one-step to two-step functions.

Role of the Teacher

Another specific feature of this instruction that needs to be highlighted is the very particular ways in which the teachers engaged with the students, and the prompts and foci they adopted in their interactions with the children. Aligned with Radford's "theory of knowledge objectification" (Radford 2008), we shaped both the order of activities and the teachers' prompts to help students notice and make sense of the ways in which patterns stay the same and the ways in which they change, thus moving students towards generalization and algebraic reasoning. Specifically, the teachers focussed the students' attention and grounded their perceptions of *change* in the curriculum's sequenced arrays of tiles that grew by a multiplicative factor; the teacher also grounded the students' perception of *what stays the same* by drawing their attention to those configurations of tiles outside the arrays which remained constant for each position of a pattern. The teachers also carefully supported the children's learning by helping them draw connections between the idea of rules, as they are experienced in "Guess my rule" function machine activities, and the possibility that geometric pattern growth can also be predicted by a rule. Further, the teachers focused the children's attention on the relationship between the position number cards and the number of elements in, and structure of, the corresponding array, thus supporting these young students' emerging understandings of co-variation.

Procedures and Measures: Grade 2 Interventions

In order to assess the potential of the intervention, we collected data from many different sources. Our major analyses were qualitative and descriptive, based on classroom artifacts, field notes, transcripts of videotaped classroom lessons and ad hoc interviews with students during the lessons. In addition, we gave each of the students in all of the research classrooms a short pre-test interview, that was administered again at the end of the intervention as a post-test, consisting of patterns in different representations. In keeping with the literature on patterning discussed above, this pre-/post-test was designed to assess changes in students' abilities to find "near" and "far" positions of patterns (e.g., Lannin 2003), to identify whether students relied on recursive strategies or functional reasoning and finally to assess students' abilities to find and express general pattern rules. At the end of the second year, we introduced an additional assessment in which we interviewed students to look specifically for transfer in their reasoning to a novel context.

Results

The results presented here focus on four general areas: the way students developed their reasoning about pattern rules, the explicit aim of our research; students' constructed understandings of multiplication, an implicit research question; the use of zero as both co-efficient and position number, an unexpected result; and the transfer of understandings to a novel context.

Finding Rules for Patterns and Generating Patterns Based on Given Rules

Our overall analyses for each year of the study revealed that students made significant gains in their ability to discern function rules for geometric growth patterns and reciprocally/conversely could also build patterns based on given rules. In contrast to findings from other studies that reveal the pervasiveness of recursive reasoning, the students in our research classrooms used a functional approach, which was evidenced in the way that they talked about the position number in relation to the number of elements in a position.

Constructing a Pattern from a Rule: "A 'number times two, plus one' pattern?"

The following transcript of a conversation between Ricardo and the classroom teacher was initiated by the teacher as she walked around the class during a portion of a lesson where students were building patterns based on given rules. She asked Ricardo to build a "number times two, plus one" pattern. Ricardo, using the square tiles from the pattern block set, built the first four positions of a pattern (that could be described in the informal notation of this classroom as "$n \times 2 + 1$", placing position number cards below each position. Each position was comprised of a row of increasing numbers of square tiles, with one tile on top of the row. (Please see Fig. 5.)

Our transcript begins when the teacher asks Ricardo to explain his pattern:

Ricardo: See, this is the first position. *[Ricardo points to the first position of the pattern he built and then picks up the position card that he had placed*

Fig. 5 Building a composite function

under the first position]. So *[touching the row of 2 blocks],* so one times two is two, plus one *[points to the block on top]* is three. *[Keeping this same speech rhythm he picks up the second position card]* So, two times two is four *[points to the row of four blocks]* and ONE *[said emphatically]* makes five.

In his explanation of the next (third) position, Ricardo makes an error. As he is about to make the same mistake again for the fourth position, he catches himself and corrects his explanation:

[He points to the position card on which is written a 3.] So three times three [sic] is six and one *[points to the single tile]* is 7. [*He repeats the same set of actions for the fourth position—the final one he has built])* So, four times four... I mean four PLUS four... or, two TIMES four is eight and one makes nine.

Teacher: Well done. How many blocks would there be in the 10^{th} position?

R: *[putting his hand over his eyes in thinking position and then rapidly dropping it and saying with a smile]* Twenty-one.

What was notable to us about Ricardo's explanation was his clear understanding of the co-variation of the position number and the number of tiles. What also was revealed in the interaction was the way that Ricardo constructed the pattern to clearly reflect his ability to distinguish the coefficient from the constant. Finally, this exchange also demonstrates this young student's fluency in being able to use his understanding of the function rule not only to build sequential pattern positions from 1 to 4 but also to predict quickly, easily and accurately the number of tiles that would be required for the 10th position—a far position.

Finding a Rule for a Given Pattern: "Position number times three, plus one"

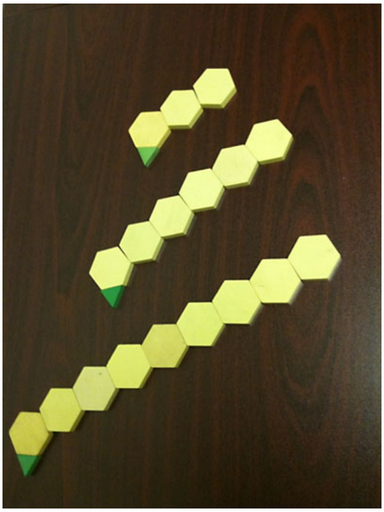
In this transcript, two students, Zoya and Marie, are sitting at a table examining the first three positions of a geometric pattern ($y = 3x + 1$) to determine the pattern rule. The figures each consisted of a planar column of yellow hexagonal pattern blocks with a single green triangle placed on the top, that grew by three yellow blocks each time (Fig. 6).

The students stare at the first three positions of the pattern:

Marie: It's a times 3 pattern, right?

Zoya: [*touches the blocks in the second pattern position]* Because this is a GROUP of 3 *[separates and points to one group of 3 in the second position, and then moves her finger in a circle around it],* and this is a GROUP of 3 *[points to the remaining group of hexagons in the second position; then she looks at the triangle and exclaims:]* Wait, oh but it can't be, because *[indicating the green triangle]* it's a whole block. So,

Fig. 6 Caterpillar pattern



never mind! *[pushes groups back together, and throws up her hands; the students seem stumped and continue to look at the pattern]*

Marie: I've got it. *[long pause as she stares at the pattern]* It is number times 3…No, position number times 3, plus 1.

Researcher: How do you know that?

Marie: Because, here is a group of 3, so that is times 3 and 1 makes 4.

Here we see the flexibility of students who were able to discard an initial conjecture of a pattern rule when the evidence (the built pattern) did not support their rule. We contrast the flexible reasoning of these very young students with findings of other researchers (e.g. Lee and Wheeler 1987; Stacey 1989; Stacey and MacGregor 1999) who all document older students' reluctance to change incorrect conjectures of rules in the face of contradictory evidence.

Marie offered an initial rule; Zoya immediately tried to support this conjecture by referring to the structure of the pattern. However, in trying to "prove" this rule, Zoya realized that the built pattern did not fit the rule, so they abandoned their initial conjecture. Eventually, Marie does figure out the correct rule, which she expresses in informal algebraic language as *position number times 3, plus 1*. Notable as well is the "groups of" language that illustrates one of the ways in which students constructed their multiplicative reasoning.

Students' Invention of Multiplication

As mentioned previously, none of the classes had been taught multiplication prior to the patterning lessons. Perhaps amongst the most salient of our findings was the way that the pattern activities worked to support students to construct a robust understanding of multiplication, revealed in the diversity of approaches the students had "invented" and the deep conceptual orientation to multiplication they had constructed through continual experience with arrays of tiles. It appears that the arrays in the geometric growth patterns provided the students with a visual representation of multiplication as a set of relationships that they could construct and deconstruct. Schliemann et al. (2001) have suggested that operations such as multiplication may in fact be more effectively understood as functions.

As shown in the following transcripts, even arithmetically lower-achieving students who struggled to perform some of the required calculations were nonetheless able to use multiplication to explain their reasoning about pattern rules and the number of elements in pattern positions.

Deconstructing Multiplication: "Double the position, plus the position"

One way in which students constructed their understanding of multiplication in the context of patterning was through a deconstruction of the operation. In this excerpt, taken in the context of a classroom lesson, Moni is presented with tile arrays representing the first four positions of a $y = 3x$ pattern in which the first position is a row of 3 tiles, the second position is two rows of 3 tiles, and so on. She explains her thinking about how the number of tiles in the fourth position (an array of 4 rows of 3 tiles each, or 3 columns of 4 tiles each) conforms to a general rule:

Moni: *[running her finger up and down one column of 4 tiles]* Here it would be 4 doubles; that would be 8… So when you put these two lines together it's 8. And here's [*indicates position number card]* 4. So you double the position, with the number.

Teacher: So, it's the number…?

Moni It's the position [number], plus the double of the position.

It is noteworthy that Moni chose to reason in a structural way rather than counting out the full number of tiles in her attempt to calculate the total. This type of reasoning typified the approach taken by many of the children. The geometric configuration (array) that Moni's explanation relied on clearly supported her deconstruction of multiplication; $3n$ is decomposed into both $n + 2n$ (*It's the position, plus the double of the position*) and $2n + n$ (*Double the position, with the number*). This visual/spatial reference that anchors her understanding further allows her to demonstrate correctly both the commutative property of addition and the distribution of multiplication over addition.

Using a Structural Understanding of Multiplication to Predict Far Positions: "It's 40 up, and 3 to the side"

In this next transcript, Oscar was shown the first two positions of a pattern (representing the functional relationship $y = 3x + 5$) built with square and triangular tiles,

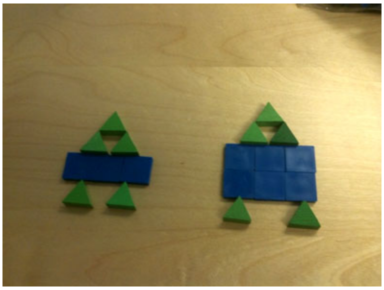


Fig. 7 Photo of rocket ship pattern

designed to look like a "rocket ship" for astronauts. Each position included an array of a number of rows of 3 square tiles each, one row of 3 tiles for each astronaut. Below this array were two triangular tiles, positioned one on each side to look like rocket boosters; three triangular tiles above the array formed the nose of the rocket (Fig. 7).

After Oscar was shown the first two positions of this pattern, the researcher wondered if Oscar could use his understanding of the structure of the pattern to determine how many blocks there would be in the 40th (a far) position (there are not enough blocks on the table for Oscar to build it), so asks him what the pattern would look like for 40 astronauts:

Researcher: Do 40, first.

Oscar: Forty *[thinks]*. Forty would be 40 up, 40 up *[moving his finger along an imaginary column on the table]* and 3 to the side *[moves his finger sideways]* 'cause one astronaut is 3 blocks long.

Researcher: Oh, it's 40 up and 3 to the side. Can you figure out how many blocks that would be in all?

Oscar: That would mean 3 rows [sic] of 40. And 40 *[counts with one finger held up]* plus 40 *[counts on another finger]* is 80. And another 40 is... another 40 is.... *[turns to his partner]* What's 80 plus 40? *[the partner replies, "120."]* A hundred and twenty... So that's a hundred twenty. *[He now begins to put 5 triangle blocks down one at a time, very deliberately. He first places 2 at the base of a smaller array he has already built, then 3 triangles far above this configuration, apparently at the top of the imaginary much larger array that he is describing.]* Then a hundred and twenty one, a hundred and twenty two, a hundred and twenty three, a hundred and twenty four, a hundred and twenty five. That's the answer... Can I write that answer down before I lose it?

The fact that Oscar could predict the 40^{th} position (from only two examples) revealed his growing understanding of multiplication as an array and the structure of the linear functions we had been working with Further, Oscar's reasoning illustrates the understanding students developed of the co-variation of the position number and of the number of elements in a given position. Finally, this short excerpt is indicative of the kind of excitement this work generated in the students, the "big numbers" they were willing to engage with and the kind of effort they were willing to make.

The Discovery of Zero

Our sequence of lessons required the children to move back and forth between numeric and geometric expressions of function rules; the lessons also moved back and forth between teacher and student generation of patterns and rules. In the course of creating their own rules within both the numeric and the visual/spatial investigations of pattern, students in different research classrooms independently made the discovery of zero as a powerful mathematical idea and arithmetic tool. Below we present examples taken from different classrooms revealing how students used zero in their pattern designs, first as a coefficient and then as a position number.

Zero as a Coefficient: "Zero groups of 4 million is zero"

The first example comes from a classroom lesson at a time when pairs of students were working independently to create function machine challenges for their fellow classmates. The researcher approached two students, Clarice and Emma who had already invented a rule and had created written pairs of input and output numbers.

Researcher: Okay, I'm going to give you an input number you don't already have, and can you give me an output? Ready, my input is 2.

Clarice: It's going to be 5.

Researcher: Okay, input number is 17.

Clarice: 5, 5, 5!

Researcher: Input number is 672!

Clarice & Emma: 5, 5, 5, 5, 5! *[laughing]:*

Researcher: Wow. Can I ask you a question—what does that mean? How many groups of the input number are there?

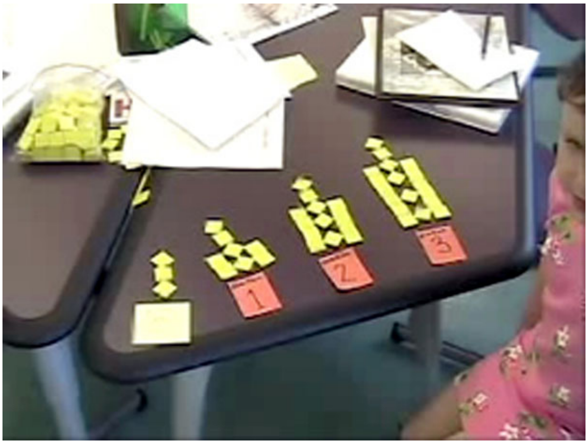
Clarice: If it's times zero, it would always be zero. Zero groups of 4 million is zero!

Researcher: Zero—there are just no groups of them.

Clarice: And then plus 5, so it's always 5.

We were surprised, as there had been no prior discussion of zero in any context. As this lesson progressed, Emma and Clarice had the opportunity to sit at the function machine and present their challenge to their classmates. While many were stumped, one student asserted, "It's the number, minus itself, plus 5". We were interested to note that Emma and Clarice were flexible in being able to accept this different rule as an expression of the same relationship, something older students have difficulty doing (e.g., Lannin 2003; Lee 1996; Mason 1996; Stacey 1989.)

Fig. 8 The zero-th position



Zero as a Position Number: "the zero-th position"

Whereas the two girls in the example above were excited to use zero as a trick, in the next excerpt we see Anna, pictured in Fig. 8 with her geometric pattern that offered no clear visual organizational clues for her classmates to identify the constant. Anna suggested that the "zero-th" position was a "help" to them in discovering her pattern rule:

Researcher: Can you show the class the pattern that you have built?

Anna: *[holding a Small piece of paper in her hand, with a rule eon it written by a researcher]* So what our rule was, was—this is our second one—it's times 5 plus 3. *[gestures to the pattern in general and then points to a position card she has made with a zero and indicates the three tiles above it]* And the zero-th position helps you a lot, it gives you a big clue. *[she splays her 3 fingers as she tries to indicate the 3 tiles that are over the "zero-th position" card].* It's 3 *[touching all 3 at once]*—this is the bump, cause the bump stays the same, but there's no GROUPS of 3 *[according to the pattern rule].* So it's the bump. It helps you a lot because it identifies what the bump would look like. So it's like 5 *[pointing to the first position configuration]*, and then plus the bump. *[pointing to the second position]* Five and then 5, plus the bump. *[pointing to the third position]* Five, and then 5, and then 5 *[holding her hands over each group of 5]*, plus the bump.

The inclusion of zero was Anna's and her partner's idea. There were no position cards with zero written on them. As she prepared the challenge for her classmates, she had requested a blank card to make a zero-th position card. Anna was aware that the position number always indicated how many groups there were, regardless of how many tiles were in each group; so she determined that if she used zero as a position number, then there would be no groups at all, isolating the constant and making it easy to identify. Thus, she was giving her classmates what she determined to be a big hint to finding the rule for her visually complicated pattern.

Transfer of Structure

As we observed the students over the course of the research lessons, we could see that they were gaining fluency with geometric patterns and becoming increasingly successful with function machine games, and integrating these features within their work in the pattern sidewalk. However, what we could not tell through observation was the robustness of the students' acquired understandings of the two-part function structure (represented formally as $y = mx + b$) and whether they could transfer their new understandings to other mathematical contexts. Accordingly, at the end of the second year of our Grade 2 interventions, we interviewed pairs of students using a word problem—a novel context—that was a narrative representation of a two-part function. The problem was presented only orally; there were no visual representations, and the students were not given the opportunity to write or draw to find the solution. There was nothing in the word problem that resembled what they had done on patterning in the research lessons, and no verbal cues that linked the word problem to what we had done in the classroom. The word problem is as follows:

> Charlotte really wants to buy a scooter. But she doesn't have enough money. The scooter she wants costs $100! From the tooth fairy, Charlotte already has saved $10. But she decides to earn more money by walking her neighbour's dog, Sparky. For each day that Charlotte walks Sparky, her neighbour will pay her $5.

Circumventing Whole Object Reasoning

The first transcript comes from a post-intervention interview in which two students, although not asked to state a function rule, clearly demonstrate their ability to discern and use a rule:

Researcher: Okay, kids. Now you really have to listen hard. I have a question for you. I don't have any paper or anything. *[Reads the problem out loud.]* How much money will Charlotte have altogether at the end of the first day of walking Sparky?

Mai: She. . . .

Juanita: 15.

Researcher: How do you know?

Juanita: She already has 10, and then she gets 5.

Researcher: How much money will she have altogether at the end of Day 2? How about on the second day?

Mai: 20.
Juanita: Hey!
Researcher: Why, is that what you were going to say, Juanita?! How much money will she have altogether at the end of Day 5?
Juanita: 20…25…. Just a second. *[counting by fives on her fingers]*
Mai: 35.
Researcher: How did you get 35?
Mai: Because 5×5 is 25, plus 10 is 35.
Researcher. Is that how you did it, Juanita?
Juanita: Yeah.
Researcher: How much money will she have altogether at the end of Day 10?
Mai: 60.
Researcher How did you get 60?
Juanita: Well, on the 5^{th} day is 25, and 25 and 25 is 50, plus 10 is 60.
Juanita. Same with me.
Researcher: What day would it be if Charlotte has $70 altogether?
Juanita: I think the… twelfth.

The inappropriate use of proportional reasoning or "whole object reasoning" in the context of patterning problems is well documented. For this reason, the sequence of questions in our interview progressed from asking the students how much money Charlotte would have in 5 days, to how much she would have in 10 days. A whole object strategy, which could be anticipated, would produce an incorrect answer of 70. That is, if 5 days equals $35, then 10 days would be double that, or $70. However, Juanita, like the majority of students in the Grade 2 research classrooms, gave the correct answer of 60.

While Juanita and Mai had not been asked to express a rule explicitly, they appeared to understand what the rule was and how to use it to predict positions of the pattern, as evidenced in their responses.

Further, the students' ability to correctly answer the final question in the narrative problem (*how many days would it be if Charlotte has $70*) is a further indication of their agility and robust understanding. They could use their explicit understanding of the coefficient and constant to reason backwards, i.e. to begin with the number of elements (money) in an unknown position and find the position (day).

Informal Algebraic Expressions of Rules in the Sparky Problem

In the excerpts below, from another classroom, the researcher gave students the opportunity to explain their thinking and to propose a general rule for the Sparky problem. The responses below are impressive in that these students were able to extract the functional relationships inherent in the Sparky problem, and express them in syncopated language (Sfard 1995). The three examples below of rules offered by Luca, Tomas and Stella show increasing levels of abstraction of rules:

Luca: It's counting by 5s with a 10 bump.

Tomas: Oh, I get it—it's a groups of 5 pattern with a 10 bump because she [Charlotte] already had 10 dollars from the tooth fairy.

Another student, Stella went on to notice that the constant was larger than the coefficient, which had not been the Case in the geometric problems that the students engaged with as part of the intervention. Her references to the geometric, as well as those of the previous students, in talking about "groups of" and the "bump", illustrate the crucial role of the visual/spatial representation in their ability to transfer.

Stella: It's always the day [ordinal position number] times 5, plus 10. So there's 10 bumps and 5 normal things, more bumps than normal things—that's weird!

Taken together, in the order that they are presented, these three excerpts reveal the increasing degree of formalization of the students' explanations of rules. Our conjecture is that the role of spatial-inspired terms like "bumps" and "normal things" was fundamental in ensuring the abstraction required to tackle the purely numeric Sparky pattern. We see this generalization as related to what Radford (2003) has called algebraic contextual generalizations. Further, we concur with Radford that adherence to conventions is not necessarily an indicator of algebraic thinking: "It is not notations which make thinking algebraic; it is rather the way the general is thought about" (2008, p. 84).

Discussion

The Grade 2 students in our research classrooms did not rely on recursive reasoning in their solutions to patterning problems, nor did they use inappropriate proportional ("whole object") strategies, both of which have been indicated in the literature as common problems even among older students. Rather, they appeared to develop a fairly robust understanding of two-part function rules through their engagement with the curriculum: they could predict how a pattern would grow, find general rules for geometric and numeric patterns, and construct patterns based on given rules. As well, our results revealed that the students were able to transfer their understanding of rules to a new (narrative) context, both finding and applying a rule.

The invention of multiplication has been noted in other studies of young students engaging with patterns; however, the diversity and quality of the approaches these students invented seemed noteworthy. Not only did the students find mathematically sound ways to deconstruct multiplication operations to solve problems, but some students also, at their own initiative, experimented with the effect of using zero as either the position number or the coefficient.

Finally, the students appeared to enjoy the lessons and seemed intrigued by the geometric presentation of patterns. They were interested in making, justifying and testing conjectures, were flexible in their general approach, and were excited to explore different ways of creating difficult challenges for their classmates. This contrasts with the concerns expressed by scholars, such as Mason (1996) and Hewitt (1992), who noted that when geometric sequences are introduced, often students produce a table of values from which they extract a closed form formula which they check with only one or two figural examples. The question arises: what are the characteristics of our program that may have supported these very young students in their productive and flexible approach to patterning activities?

Analyses of our findings over the many iterations of our studies suggest a number of distinct but overlapping factors that may have contributed to our students' ability to work with patterns. We draw your attention to three factors in particular. First, is the design of the curriculum with its deliberate movement back and forth between, and then bridging of, geometric and numeric representations of growing patterns through the idea of "rules". Second, and related, is the primacy given to the visual and to the way that the curriculum design deliberately focuses students' attention on the spatial/geometric pattern formations. And third, inherent in the design of the instructional sequence is the emphasis on student invention. In the sections that follow we briefly elaborate on these factors.

The Curriculum with Its Focus on Integration

The theoretical framework that underpinned this research, and served as a heuristic for the design of the curriculum, came from previous work of Case and colleagues on children's mathematical development. Specifically, we were guided by the proposal of Case et al. that children's development in a domain of mathematics (e.g., whole number, rational number) is underpinned by the integration of the children's visual schemas on the one hand, and their numeric understandings on the other, for the mathematics domain in question (Please see Kalchman et al. 2001 for details of this theory). Further, as we mentioned earlier, this theoretical framework helped to establish a developmentally grounded sequence for our intervention. Students first worked with geometric (tile array) representations and then moved on to numeric (function machine) patterns. This sequence served to help the students to consolidate and extend their separate understandings in both the geometric and numeric domains. These separate understandings were bridged for the students by the concept of (function) *rule* which enabled the students to begin to move between the visual and the numeric with increasing flexibility. The subsequent introduction of the pattern sidewalk, in its use of a non-sequential geometric representation of pattern, also fostered this integration and flexibility. Our preliminary conjecture is that it was the specific movement back and forth between these two representations, geometric and numeric, that ultimately supported the students to gain not only flexibility with, but also a structural sense of, two-part linear functions, thus supporting/enabling the students to discern and understand pattern rules in contexts that were new to them.

Prioritizing Visual Representations of Pattern

While we believe that the back and forth movement was critical to the flexible reasoning that students ultimately were able to demonstrate, we also suggest that the students' initial grounding in the visual geometric context was also significant in the effectiveness of the curriculum. All through the lessons and interviews with students we were made aware of how their reasoning was underpinned by their interpretations and analyses that were based on geometric figures. When probed for explanations of rules, the students focused on how a pattern grew in relation to the position number; how the addition of the constant or "bump" was related to the coefficient, or multiplicative; how parts of the pattern changed and parts stayed the same based on their visual configurations. Finally, even in their post-intervention explanations of the "Sparky" narrative word problem, many students referred to the two elements of the two-part function in geometric terms: "Oh, I get it—it's a groups of 5 pattern with a 10 bump because she already had 10 dollars from the tooth fairy."

Indeed, a number of researchers have reported on the support provided by figural representations for students working with generalizing problems (e.g., Carraher et al. 2008; Healy and Hoyles 2000; Lannin 2005; Noss et al. 1997; Rivera and Becker 2008; Sasman et al. 1999; Stacey 1989). When visual representations are prioritized, and students are supported to focus on the figural patterns as a way of discerning general rules, they are better able to find, express and justify functional rules.

However, research has also shown that, overwhelmingly, students and adults have a strong tendency to ignore the geometric properties of figural patterns, and to focus instead on the number of elements in the given pattern. The focus on the visual in our program appears to have had a double advantage for students: providing a rich context in which to analyze growth and change, and also supporting students to be aware of covariation.

Pedagogy and Student Inventions

In the opening sections of this article we discussed the particular ways that in which the teachers interacted with the students and how they focussed the children's attention on salient features of the instructional sequence to support the children's learning. It is also our proposal that another significant contribution to the success of the intervention was the ongoing insertion into the learning sequence of the children's own inventions: specifically the geometric patterns they designed and also the challenges they created for their classmates with the function machine.

As we mention in earlier sections of this chapter, inasmuch as there was a continuous movement back and forth between geometric and numeric representations of patterns, so too was there movement back and forth from the standpoint of the pedagogy: for example, in iterative fashion teachers modeled geometric one-step patterns, and then students designed and presented their own patterns; teachers presented challenges with the function machine, and then students in pairs followed their lead and designed their own challenges for their classmates. The evidence, based on the transcripts of classroom lessons, is clear in revealing that these student-invented challenges created excitement, interest and motivation among the children. They also may have served other important purposes. First was the opportunity to practice. Students in Grade 2 had little or no experience with growing patterns prior to the intervention, and many held firmly to the belief that patterns could only repeat. Creating their own patterns gave the grade 2 students the opportunity to discover and experience how linear growing patterns worked or did not work. In addition, by creating their own geometric and numeric patterns, students had the time and space to invent and then practice multiplication. Also, in the course of developing challenges for their classmates, the students had the opportunity to take on an additional perspective in anticipating how their classmates might respond. In our view, this kind of anticipation and planning added an extra dimension (metacognitive) to students' thinking, thus enriching the learning potential of the lessons.

Concluding Thoughts

Typically, patterns are taught in the early years as repeating, with children asked to find "what comes next". As Blanton and Kaput point out, this limited view does not capitalize on the potential of patterns to support later mathematics learning (Blanton and Kaput 2004). A number of researchers have included a focus on young children and patterns (e.g. Carraher et al.; Mitchelmore and Mulligan; Mulligan, Prescott & Mitchelmore; Warren & Cooper), investigating ways of promoting algebraic thinking, generalizing and awareness of structure through the use of patterning. We join with these researchers in trying to illuminate the potential of pattern work for young children. Our findings suggest that, with appropriate instruction, the study of patterns can support students of *all levels* of mathematics abilities to foster the kinds of mathematical thinking that Kieran suggests is fundamental to algebraic reasoning: "analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modeling, justifying, proving, and predicting" (Cai and Knuth 2005, pg. 1)

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Grade 2 Students' Non-Symbolic Algebraic Thinking

Luis Radford

Abstract The learning of arithmetic, it has recently been argued, need not be a prerequisite for the learning of algebra. From this viewpoint, it is claimed that young students can be introduced to some elementary algebraic concepts in primary school. However, despite the increasing amount of experimental evidence, the idea of introducing algebra in the early years remains clouded by the lack of clear distinctions between what is arithmetic and what is algebraic. The goal of this chapter is twofold. First, at an epistemological level, it seeks to contribute to a better understanding of the relationship between arithmetic and algebraic thinking. Second, at a developmental level, it explores 7–8-years old students' first encounter with some elementary algebraic concepts and inquires about the limits and possibilities of introducing algebra in primary school.

Introduction

This chapter is about the journey of a group of Grade 2 students (7–8-years old) into algebra. It explores the students' first encounter with some elementary algebraic concepts and inquires about the limits and possibilities of introducing algebra in primary school. The chapter is inspired by a recent idea according to which the learning of arithmetic need not be a prerequisite for the learning of algebra (e.g., Blanton and Kaput 2000; Carraher et al. 2006; Dougherty 2003). It is also inspired by the particular context in which I work—a context in which algebra is introduced in the early years. A progressive introduction to algebra in the early grades, it has

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been suggested, may facilitate the students' access to more advanced algebraic concepts in later grades.

Despite the increasing amount of experimental evidence that supports it (e.g., Brizuela and Schliemann 2004; Moss and Beatty 2006; Becker and Rivera 2006a, 2006b; Warren 2006; Warren and Cooper 2008), the idea of introducing algebra in the early years remains clouded by the lack of clear distinctions between what is arithmetic and what is algebraic. To make such distinctions would require making explicit our own assumptions about the nature of arithmetic and algebraic thinking and their interrelationships. And, of course, this is not an easy matter, neither for research nor for practice. Let me give a concrete example.

In 1997 there was an important curriculum re-orientation in Ontario. New goals and expectations were established concerning the content and aims of the teaching and learning of mathematics. Since then a specific domain, called "Patterning and Algebra", has listed the overall and specific expectations from Grade 1 to 8 (see Ontario Ministry of Education 1997). For instance, a specific expectation in Grades 1, 2 and 3 states that students identify, describe, and extend patterns. Another expectation states that the students demonstrate an understanding of the concept of equality by partitioning whole numbers in a variety of ways (e.g., 5 is equal to $1 + 4$, $2 + 3$, etc.). By Grade 5 the students are expected to use letters to represent relations (e.g., $C = 3 \times n$). As these few examples show, there is a "progression" in terms of conceptual content. However, the progression raises several questions. For example, are the activities of extending patterns and partitioning numbers part of algebra or arithmetic? Is the use of letters specific to algebra or could it also be part of arithmetic? Let us assume for a moment that extending patterns and partitioning numbers are really part of algebra. What would it be that makes them algebraic? What would be the algebraic concepts required in those kinds of tasks? Unfortunately the official document does not supply elements to answer those questions.

Of course, this problem is not specific to the Ontario curriculum. It is a token, I think, of the difficulties that curriculum designers and we, mathematics educators, still have in making a clear distinction between arithmetic and algebraic thinking, and in elucidating the relationships between them.

I should hasten to say that the journey this article is about does not answer those difficult questions. It is rather a modest attempt to reflect on them both from a practical and a theoretical perspective. What I shall present and discuss in this article comes from what I learned from the students of a Grade 2 class and their teacher. It is a journey in which I also embarked. Before I go into more specific details, it may be necessary to say something about the protagonists of the journey and its dynamics.

The journey is part of a still ongoing longitudinal research program in which a Grade 2 class of 25 students and their teacher were invited to participate. Part of a progressive French School Board in Ontario, the school is open to projects and partnerships with different sectors of its community. The superintendent, the principal and the teacher were interested to participate in a study to explore the problems and challenges surrounding the introduction of algebra in the early years. The teacher, the research assistants and I met regularly, either at the school or at the

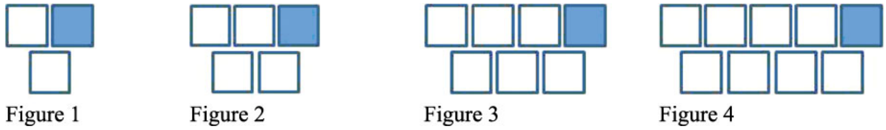

Fig. A The first four figures of a sequence given to the students in a Grade 2 class

university, to discuss the choice of mathematical tasks and forms of interaction, and how these fitted with our philosophy of teaching and learning and the curriculum expectations.

Throughout the study, the classroom activities were carried out by the teacher, while we videotaped. After videotaping, the research assistants and I made transcriptions. Interpretations of our data were done with the teacher. Also, after each mathematics lesson of the study, the teacher and I used a voice recorder to record our impressions. As data was generated and interpreted, new hypotheses were formulated and incorporated into the research, thereby ensuring a dialectical loop between theory generation and data interpretation.

Even if our journey also included an investigation into the realm of equations, in this chapter I shall limit myself to commenting on what we learned about the generalization of patterns. In the next two sections I discuss some conceptual aspects involved in extending a sequence and the kind of abstraction it entails. These sections pave the way for a discussion of the boundaries of arithmetic and algebraic thinking in generalizing tasks while the last sections are devoted to a discussion of embodied forms of algebraic thinking.

Extending Sequences

In this section I make a brief analysis of the students' procedures in extending a sequence. The analysis of the students' procedures aims at shedding some light on the type of thinking that is required to accomplish this type of task. The idea is to make available some concrete information to reflect on the question of the relationship between extending sequences and algebraic thinking.

The students worked on four activities about pattern generalization. These activities lasted five days in total (about 60 minutes each day). The first activity was based on the sequence shown in Fig. A. The students were divided into groups of two or three and were asked first to draw Figures 5 and 6 and then, after answering other questions, to come up with a procedure or formula to find the number of squares in some "big" figures, like Figure 25.

To continue the sequence the students had to grasp a *commonality* noticed in the four given figures and *generalize* it to other terms of the sequence. Since there are many ways in which to see things in front of us, the commonality that the students noticed was not always the same. The commonality appeared in fact progressively in the course of the students' spatial-temporal experience of *grasping* it.

Fig. B While drawing Figure 5, Erica goes back to Figure 4 to count the squares on the top row. The finger helps her to *see* and *count*

For instance, in one of the groups, formed by Erica, Cindy and Carl, Erica had a sense of the next figure: its squares should be *spatially* distributed into two roads, but neither their numerosity nor the position of the dark square followed the logic of the first figures. Counting aloud, she started drawing the squares on the first row from left to right: "one, two, three". When she was about to draw the forth square, she came back to Figure 4 of the sequence and, with a left hand finger counted the number of white squares. She then continued drawing the squares on the top row in Figure 5, followed by those on the bottom row, putting the dark square at the end (See Fig. B).

The progressive grasping of the regularity appears as the result of linking two different kinds of structures: a *spatial* and a *numerical* one. From the spatial structures there emerges a sense of the squares' position, whereas their numerosity emerges from a numerical structure. The former deals with the question of "where?"; the latter with the question of "how many?" As a fine-grained video analysis shows, the spatial-numerical link was mediated by a complex embodied intersensorial process involving visual, motor, and aural elements. The *motor* action of drawing was accompanied by *seeing* (seeing where the squares have to be drawn so as to ensure that the squares are aligned, that they have an approximate same size, etc.) and also by *language*. By counting the squares as they are drawn, language helped Erica and the other students to keep track of the question of "how many?" As in Grade 2 the students' motor actions underpinning pen use are still being refined, drawing small squares takes considerable time and effort. Perceptual efforts are focused on the task of drawing. Language, then, is used to maintain awareness of the numerical status of the square that is being drawn.

As the group moved to the next task, i.e. the drawing of Figure 6, Carl and Cindy—who, like Erica, counted aloud, but managed to coordinate the spatial and numerical structures so as to make it coherent with the intended logic of the sequence in Fig. A—intervened and anticipated the key aspects of the figure. Carl said:

1. Carl: We do 6 plus 6 equals 12, plus 1
2. Erica: Yes... No...
3. Cindy: Yes!
4. Carl: Yes, that is what we did in the other ones. Look!

Fig. C *To the left*, Carl starts counting the squares on the bottom row; then he counts the white squares on the top. *To the right*, the moment in which, after finishing counting the top white squares, he goes back to the dark square and says "plus 1"

5. Cindy: Yes, you add one.
6. Carl: (*Talking to Erica*) Look. 4 plus 4 equals 8. In there [Figure 4], there are 8. (*And pointing to the successive squares of Figure 4 in Erica's sheet he continued*) 1, 2, 3, 4, 5, 6, 7, 8, plus 1, which is equal to 9. (See Fig. C)

Carl's addition "6 plus 6" in Line 1 conveys the *link* between the spatial and numerical structures. Indeed, the expression "$6 + 6$" is not a mere addition: it is a synthetic or global expression that conveys the ideas of *where* the squares are, and *how many* squares there are. In addition to the numeric information, the first 6 refers to the bottom row; similarly, the second 6 refers to the top row, and indicates that to the six squares an additional square has to be added—the one corresponding to the dark square. As Erica's perception of the figures seemed to be more sequential (she relies on counting squares one after the other), she was unsure about what her team-mates meant. Hence Carl illustrated the idea, resorting to one of the figures given to the students—Figure 4. He started counting the squares on the bottom row ("1, 2, 3, 4."). Then he switched to the top row and continued counting the white squares, and ended the counting with the dark square. Erica and Cindy followed Carl's gestures attentively.

Through his intervention, Carl made available to his team-mates a specific commonality in the terms of the sequence. He talked about Figure 4, but implicit in his intention was the idea that such a commonality applied to the other figures as well.

Abstraction

The grasping of the commonality is the formation of what, in Aristotelian terminology, is called a *genus*, i.e. that in virtue of which various things are recognized as belonging together (see e.g. Aristotle's *Categories*, 2a13–2a18). The genus results from an abstraction that requires the students to make distinctions between what is similar and what is different in the given terms of the sequence. Now, can the students use this abstracted commonality to assert whether a given term belongs to the sequence? We wanted to explore this question and presented the students with the following task:

Monique wants to build Figure 8 of the sequence. She builds the figure shown below. Are you in agreement with Monique? Please explain!

Resorting to the objectified commonality or *genus*, the students argued that this term was not Figure 8. Cindy pointed to the top row and counted the white squares. She said: "It isn't because 1, 2, 3, 4, 5, 6, 7", meaning that there were not 8 white squares. Carl agreed: "Yes, because, look, here it [pointing to the top row] has 7, here it [the bottom row] has 8, so it's not good."

However, I want to argue that the identification of the *genus* and its use to extend the sequence to neighboring terms (e.g. Figures 5 and 6) as well as to discriminate between terms that belong and those that do not belong to the sequence cannot be considered the result of an *algebraic* process. However complex such a process might be, there is nothing necessarily algebraic in it.

Indeed, noticing what is really common and characteristic of the terms of a sequence or set of objects is a central aspect of concept formation. As we shall see in the next section, it plays an important role in the emergence of the students' first algebraic ideas, but is not itself the result of an algebraic process. In fact, finding a characterizing attribute of the terms of a sequence or a set of objects is not specific to humans. Indeed, Sue Savage-Rumbaugh and her team as well as other researchers in the field of animal cognition have established that chimpanzees (and birds too) can distinguish between several sorts of objects—classing them as "edible" and "inedible" (Savage-Rumbaugh et al. 1980). Even if chimpanzees "might be lacking the ability to organize their categories into a more complex hierarchical network of superordinate categories of increasing abstraction" (Gómez 2004, p. 126), they do form abstract commonalities (i.e. abstract *genera*) typical of concept formation. But this cognitive capability alone is not a warrant to claim that chimps and birds are thinking algebraically. In a similar vein, the capacity of our Grade 2 students to grasp a commonality, even in complex sequences like the one shown in Fig. A, and to extend it to a few subsequent terms does not mean that the students are already thinking algebraically. Generality is not specific to algebra. Generality is a typical general trait of human and animal cognition and can be of diverse nature—arithmetic, geometric or other.

Hence, the question is: when, if at all, did our Grade 2 students start thinking algebraically?

The Boundaries of Arithmetic and Algebraic Thinking

Algebraic thinking does not appear in ontogeny by chance, nor does it appear as the necessary consequence of cognitive maturation. To make algebraic thinking appear, and to make it accessible to the students, some pedagogical conditions need to be created. This was what we were trying to accomplish through the design of our activities and their implementation in the classroom.

The rest of our questions in the 2-day activity around the sequence shown in Fig. A were directed to having the students experience the need to come up with a general procedure. We asked them to consider Figures 12 and 25. We also asked them the following question:

Pierre wants to build a big figure of the sequence. Explain to him what to do.

The questions were promptly answered. Thus, talking about Figure 12, with ease Cindy said: "12 plus 12, plus 1". Referring to Figure 25, Erica said:

1. Erica: Cindy! Um... Okay, What is 25 plus 25?
2. Cindy: (*Thinking*) Euh...
3. Erica: (*Smiling*) After that, you add one!

Before going further, it is worth noticing that to ask the students to find the number of squares in "big" figures like Figures 25 was far from trivial. The arithmetic knowledge of our Grade 2 students was, at the time, very limited. Although they had some acquaintance with "big" numbers, they were able to make systematic additions only up to 25. Hence our question about Figure 12 was at the very limit of their calculating capabilities. Our question about Figure 50 was definitely beyond them. But instead of being a hindrance, not knowing how to make additions beyond 25 was in fact good. The design of the activities was based on the limits of students' arithmetic knowledge to promote the emergence of algebraic thinking. Indeed, by exploiting the students' limits of arithmetic thinking, the design of the activity aimed at favouring the students' awareness of calculation methods. Here the calculator proved to be of great importance.

To help the students deal with big figures, the teacher made calculators available to the students. But, *before* finding the actual number of squares in Figure 25 or other big figures using the calculator, she asked them to come up with an idea of how to find the total. This pedagogical strategy induced in the students' activity an important shift from the numeric *qua* numeric to the devising of a rule or calculation method. This shift is central to an algebraic mind. It was the 13th century Arabian mathematician Aboû Beqr Alkarkhî who first expressed this idea. He suggested that algebra consists of *rules* to calculate numbers (Woepcke 1853). And it was within this context that the students tackled the Pierre question concerning a big, unspecified figure mentioned above. Surely enough, the students chose "big" *particular* figures. Carl suggested considering Figure 500; Cindy preferred Figure 50:

4. Carl: How about doing 500 plus 500?
5. Erica: No. Do something simpler.
6. Carl: 500 plus 500 equals 1000.
7. Erica: plus 1, 1001.
8. Carl: plus 1, equals 1001.
9. Cindy: Non, 50 plus 50, plus 1 equals 101.

It might be worth asking now whether or not there is something algebraic in these responses. Let me note first that to answer the question about Figure 12 Erica and her team did not go from Figure 4 to Figure 12, building figure after figure. To deal with Figures 12 and 25 the students accomplished a generalization. And this generalization was *algebraic* in nature.

I will start the justification of my claim by first addressing a potential objection. It might be argued that even though the students accomplished a generalization, their generalization was not algebraic, since the students did not use "notations". This is true. However, as I have argued in previous articles, the use of notations (i.e., alphanumeric symbolism) is neither a necessary nor a sufficient condition for thinking algebraically (Radford 2006a, 2009a). Algebraic thinking is not about using or not using notations but about reasoning in certain ways. What characterizes thinking as algebraic is that it deals with *indeterminate* quantities conceived of in *analytic* ways. In other words, you consider the indeterminate quantities (e.g. unknowns or variables) as if they were known and carry out calculations with them as you do with known numbers (see Filloy et al. 2007). It is in this sense that 16^{th} century mathematicians like Viète and Cardano understood the distinctive trait of algebra and used to call it an *analytic art* (see e.g. Viète 1983). And it is in this sense that Erica and her team were dealing with our questions about Figures 12 and 25.

My problem now is to show how indeterminacy and analyticity are present in the students' procedures. To understand the subtle sense in which indeterminacy and analyticity were present in our Grade 2 students' procedures, we have to bear in mind that even if indeterminacy is usually expressed through letters (e.g. "x", "n", etc.), there are also other *genuine* forms in which to express it. Before our standard algebraic symbolism was invented, mathematicians used various ways to think about and deal with indeterminacy. For instance, Babylonian scribes used contextual names, depending on the problem (e.g., the side of a rectangle, the weight of a stone); Medieval and Renaissance mathematicians employed a generic term—*la cosa* (the thing). In the case of our Grade 2 students, indeterminacy is present through *instances* of the independent variable—i.e., the number of the figure ("1", "2", 3", "4", "5", "6", "12", "25", "50"). In other words, indeterminacy appears embodied in its surrogates. Indeterminacy and analyticity are in fact bound together in a schema or *rule* that allows the students to deal with any particular figure of the sequence, regardless of its size. It is a rule instantiated in particular cases (e.g. "12 plus 12, plus 1"), where numbers are dealt with not as merely numbers but as constituents of something more general. The suspension of intermediate and final results in "12 plus 12, plus 1" is well tuned with the algebraic idea of analyticity. What matters is not really the numeric result, but the rule. The students' rule ("12 plus 12, plus 1"; "25 plus 25, plus 1", etc.) attests to a shift in focus: the students' focus is no longer specifically numeric. It is about numbers, of course, but in an *algebraic* manner, however simple this manner is. This is why Erica was not stopped by not knowing the result of $25 + 25$. Whatever it is, you have to add 1 (the dark square). And when Carl hurried to help Erica with her question about $25 + 25$, he offered the answer "14", which surprised his team-mates. Carl then added: "14 or whatever!" Indeed, whatever the answer is, the important thing is to add 1. For the students' emerging understanding, what matters is not the result. It is the rule, that is to say, the formula—the algebraic formula.

This "formula" can better be understood as an embodied predicate (e.g. "12 plus 12, plus 1") with a *tacit* variable: indeterminacy as such does not reach the level of symbolization, not even the level of discourse. There are no words in the students' vocabulary to name it. Indeterminacy remains *implicit*—something whose presence is only vaguely adverted through particular instances, like clouds anticipating a storm. Indeterminacy is expressed in an indexical manner: its instances point to something that is in *adventus*, that is to say to-come.

Layers of Generality

My claim that Erica and her team-mates have effectively stepped into the realm of algebra makes sense only if, as I suggested, we consider algebra as a particular form of thinking that, instead of being characterized by alphanumeric signs, is rather characterized by the specific manner in which it attends to the objects of discourse. This distinctive manner of being of algebraic thinking can be defined, on epistemological grounds, by indeterminacy and analyticity. These two elements are what make algebraic thinking different from arithmetic and other forms of thinking.

Now, indeterminacy and analyticity can take several forms. And this is so because algebraic thinking can operate at different layers of generality. Some layers are more concrete, some more general. In some concrete layers, indeterminacy and analyticity may appear in an intuited form, as in the previous section. In others, they may appear in a more explicit manner, as when students use alphanumeric symbolism.

Layers of generality can be distinguished in terms of the signs to which the students resort to think algebraically. Indeed, thinking, as I am conceiving of it here, is not something that is restricted to the mental plane. On the contrary, thinking also occurs in the social plane. Gestures, language, and perception are *material* constituents of thinking (Radford 2009b). The material (e.g. gestures) and immaterial (e.g. imagery, inner speech) components of thinking constitute its "semiotic texture". As there are some things that we can and others that we cannot say, think, and intend through certain signs (think for instance of the impossibility of exactly translating a poem into a painting), the "semiotic texture" of thinking sets the limits of what is sayable, thinkable and intendable within it. If I spent some time analyzing the manner in which Erica and her group-mates became aware of the *genus* of the sequence and used it to objectify an algebraic rule, it was because such analysis reveals the "semiotic texture" of their algebraic thinking. This rule, I argued, was not made up of alphanumeric signs but of embodied signs. Of course, epistemologically speaking, the layer of generality in which these students were operating was not very profound. Yet Erica and her team-mates were thinking algebraically. I deem this point important for our understanding of what is achievable in terms of introducing algebra in the early grades. The natural question is: Is this all that Grade 2 students are capable of? I deal with this question in the next section.

Fig. D The teacher with a box containing numbered cards representing the number of some terms of the first day's sequence. With her right hand she holds the envelope where one of the cards with an unknown number has been put

Beyond Intuited Indeterminacy

At the end of the second day, the teacher and I were convinced that asking the students to write an explanation of their solutions and ideas was not productive. The students were still in the early stages of writing and to write an explanation of two sentences was taking them far too long. Furthermore, in writing their explanations, the discussions turned often into other matters that were of relative importance in terms of algebraic thinking. For instance, the students were spending huge amounts of time deciding how to spell a word or discussing whether or not a plural name should have an s at the end. We decided then to turn to oral explanations. Starting from the third day all groups were provided with a digital voice recorder. When a student was ready to offer an explanation, the student would activate the voice recorder, say her name and start the recording. Our Grade 2 students ended up practicing *oral algebra*— perhaps like early Renaissance students before the invention of the printing press and the spread of writing as a social phenomenon (Radford 2006b). It was in this context of oral-oriented activity that the students spent the rest of the week working on similar sequences as the one shown in Fig. A.

On the fifth and final day of our pattern generalization teaching-learning sequence, the teacher came back to the sequence shown in Fig. A. To recapitulate, she invited some groups to share in front of the class what they had learned about that sequence in light of previous days' classroom discussions and small group work. Then, she asked a completely new question to the class. She took a box and, in front of the students, put in it several cards, each one having a number: 5, 15, 100, 104, etc. Each one of these numbers represented the number of a figure of the pattern shown in Fig. A. The teacher invited a student to choose randomly one of the cards and put it into an envelope, making sure that neither the student herself nor the teacher nor anybody else saw the number beforehand. The envelope, the teacher said, was going to be sent to Tristan, a student from another school. The Grade 2 students were invited to send a recorded message that would be put in the envelope along with the card. In the message the students would tell Tristan how to quickly calculate the number of squares in the figure indicated on the card (see Fig. D).

The number of the figure was hence *unknown*. Would the students be able to generalize the rule that they had objectified when working with "big" figures and engage with calculations on this *unknown* number? In other terms, would our Grade

2 students be able to go beyond intuited indeterminacy and its corresponding elementary form of algebraic thinking?

As in the previous days, the students worked in small groups of three. Let me dwell on what happened in Erica's group. In an episode that lasted 30 seconds, Erica started making a suggestion:

10. Erica: We can say ...
11. Cindy: You do...
12. Carl: You can do...
13. Erica: You can do the number... (*she makes the pointing gesture shown in Fig. E, Pic. 1*).
14. Cindy: You look at the number and then you...
15. Carl: He will have
16. Erica: (*Continuing her utterance in line 13*) The same number...
17. Cindy: And then you...
18. Erica: (*Continuing her utterance in line 16*) as at the bottom (*she makes the pointing gesture shown in Fig. E, Pic. 2*), after on the side you put another one (*she makes the pointing gesture shown in Fig. E, Pic. 3*).
19. Cindy: and then, and then...
20. Carl: And then at the bottom he will have the same number of light squares (*he makes the pointing gesture shown in Fig. E, Pic. 4*), at the top the same number of light squares (*he makes the pointing gesture shown in Fig. E, Pic. 5*), and a dark one (*he makes the pointing gesture shown in Fig. E, Pic. 6*).

As the previous dialogue shows, the fact that the number of the figure was unspecified did not impede the students in thinking of and talking about the figure in a mathematical way. Through the linguistic expression "the number", the students engaged with indeterminacy in an *explicit* manner. The definite article "the" qualifies the noun "number" making it *specific* even if it is *unknown*. From the intuited form in which it appeared in the students' previous activity, indeterminacy has now entered the realm of the students' universe of discourse. In so doing, the students have reached a new layer of generality. However, this new layer remains deeply anchored in the students' perceptual experience, as shown by the students' fierce recourse to gestures and contextual clues through which they somehow make visible the unspecified figure. Gestures and words allow the students to *visualize* in an embodied and almost tangible way the figure. Indeed, while Erica did not gesture when tackling the question about Figure 25, here she made extensive use of gestures and linguistic deictics ("top", "bottom"). In line 13, she says: "You can do the number..." and points to an imaginary place where would be the bottom row of the unspecified figure. The utterance continues in lines 16 and 18, where she says: "The same number... as at the bottom", pointing now to the imaginary place of the top row. Then pointing to a spot on the right side, she finishes the sentence saying "after on the side you put another one." Drawing on Erica's idea, Carl immediately offered a recapitulation that, interestingly enough, was accompanied by linguistic deictics

Fig. E Erica' and Carl's gestures help them to visualize the unknown figure

and a set of three gestures on the table with a pronounced movement of the arms and the whole upper part of the body.[1]

However, in contrast to what the students did when dealing with "big" particular figures, like Figure 50, here the students did not produce a *formula*. Indeed, instead of something similar to Cindy's formula "50 plus 50, plus 1" (line 9), the students produced a *spatial description* of the unspecified figure. As Carl said: "at the bottom he will have the same number of light squares; at the top the same number of light squares and a dark one." As a result, there are no explicit operations with the unknown number. In other words, analyticity—this chief feature of algebraic thinking— seems to be missing.

When the teacher came to see the group's work, Carl explained the message they were working on, using an example—Figure 50. Here is an excerpt of the discussion:

[1]The role of gestures and words in visualization and thinking is not specific to our Grade 2 students. It has been put into evidence with older students (Radford et al. 2007; Sabena et al. 2005) and also in other contexts. See, e.g. the pioneer work of Presmeg (1986); see also Arzarello and Robutti (2004), Edwards (2009), Roth (2001), Nemirovsky and Ferrara (2009).

Fig. F Erica refers to the calculator to make the calculations on the unknown number

21. Carl: You do 50, plus 50, plus 1.
22. Teacher: Excellent! That would be a good example. But what if Tristan finds another number? . . .
23. Carl: 100 plus 100 plus 1 . . .

Here Carl is dealing with generality through particular examples, in a manner that Balacheff (1987) calls "generic example", a way of seeing the general through the particular, as Mason (1996) puts it. Erica continued:

24. Erica: It's the number he has, the same number at the bottom, the same number at the top, plus 1 . . .
25. Teacher: That is excellent, but don't forget: he doesn't have to draw [the figure]. He just has to add. . . So, how can we say it, using this good idea?
26. Erica: We can use our calculator to calculate!
27. Teacher: Ok. And what is he going to do with the calculator?
28. Erica: He will put the number. . . (*she pretends to be inserting a number into the calculator; see Fig. F, Pic. 1*).
29. Cindy: He will do: the number. . .
30. Erica: plus the same number, plus 1 (*as she speaks, she pretends to be inserting the number again (Pic. 2) and the number 1 (Pic. 3)*).
31. Carl: Yeah!
32. Teacher: (*Repeating*) The number, plus the same number, plus 1! Do you think that Tristan would be able to find the total like that?
33. Cindy and Carl: Yes!
34. Teacher: Very good. I will go to check on the other groups now.

In Line 25 the teacher makes the subtle distinction between drawing and calculating. The formula can be derived from the students' general description of the figure, but is not equal to it. An algebraic formula does not include terms such as "top" and "bottom". In Line 26, Erica suggested using the calculator and, along with Cindy, mentioned the sequence of calculations to be carried out in order to find the total. Naturally, the use of the calculator is merely virtual. In the students' calculator, all inputs are specific numbers. Nevertheless, the calculator helped the students objectify the analytic dimension that was apparently missing in the new layer of generality. Through the calculator, calculations are now performed on this unspecified instance of the variable—the unknown number of the figure.

At the end of the lesson, several groups were invited to come forward and record the message to Tristan. When Erica's group was invited, Carl recorded a message using a particular figure (Figure 50). While congratulating Carl, the teacher stressed the fact that the message was based on a concrete example and asked if there was another way of telling Tristan what to do. Using the voice recorder, Erica recorded the following message:

> Hi Tristan. You put the number at the bottom (*she makes a gesture like pointing at the imaginary bottom row*) the same number on top (*like pointing at the imaginary top row*), plus 1. Afterwards, you use the calculator and (*making gestures as if using the calculator keyboard*) you insert the number plus the same number plus 1, and after you press equal and it will show you what it is.

The message was divided into two parts. In the first part Erica tells Tristan about the *aspect* of the figure. In the second part, Erica indicates the *calculations* to be performed. It seems that knowing how the figure looks is a prerequisite to making the calculations. Indeed, Erica imagines entering the numbers in the calculator in the order that the students imagined the unspecified figure (from bottom to top, then the dark square). The meaning of the terms in the formula is hence derived from the *spatial configuration* of the figure.

An important aspect of the development of algebraic thinking consists in imbuing the terms of a formula with abstract meanings so that formal calculations can be performed with the unknown numbers. The situated, spatial sense of the unknown numbers in the formula seems to constitute the limits of our Grade 2 students' algebraic thinking. Yet there was one group that overcame this limit. When their turn came to record the message for Tristan, they produced the following message:

> Hello Tristan, um... we are going to show you a strategy to figure out the sequence. Um, you have to find the number. If the number you grab is like 50 or 40 or something, you have to do like the number times two and after plus 1, and you will see what it equals to.

Here the addition of the unknown number with itself is turned into a multiplication by two. The *spatial meaning* of the unknown is overcome.

A General Overview

The next week, the students were asked to respond to a questionnaire. Even if questionnaires are hardly the best instruments to assess cognitive development, the students' responses give us an additional perspective of their general progress towards algebraic thinking. In this section, bearing in mind the partial and limited view of questionnaires, I will comment on some of the results. The students dealt with the sequence shown in Fig. G.

As in previous tasks, they were asked to continue the sequence up to Figure 6 and indicate the number of circles in Figures 5 and 6. They were also asked to indicate the calculations to find out the number of circles in Figure 25.

The identification of the commonality and its use in extending the sequence were very well accomplished. The success rate was 21/25 or 84%. Let me note that

Fig. G The sequence of the test

among the 4 other answers, one comes from a student who did not respond. Two others come from students who, reading Figure 1 as having "one on the bottom and two more on top", interpreted the commonality as follows: Figure 6 has six circles on the bottom and two more on top; they drew Figure 6 accordingly and argued that Figure 6 has $6 + 8 = 14$ figures. The fourth answer came from a student who drew the circles with no order (figures were correct from the numeric structure but not from the spatial one).

What were the results in the question about Figure 25? Thirteen students showed a formula to obtain the total of circles in Figure 25 ("$25 + 25 + 1$" or "$25 + 26$"). However, a closer look at the data reveals that five additional students knew that they had to add two consecutive numbers, and used "$25 + 24$". I want to suggest that the difficulty was partly algebraic and partly arithmetic. The algebraic generalization was accomplished, but it was its quantification that became problematic. If this explanation is accepted, then the ratio of success increases to about $18/25$ or 72%. This ratio is equivalent to if not better than those we have obtained in the introduction of algebra in Grade 7 and Grade 8. All things considered, our Grade 2 students did very well.

Synthesis and Concluding Remarks

In this chapter I presented an overview of a journey of Grade 2 students into algebra. The journey was motivated by new curricular trends that recommend starting to introduce elementary algebraic concepts at the beginning of primary school. However, as Carraher et al. note "little is known about children's ability to make mathematical generalizations and to use algebraic notation" (Carraher et al. 2006, p. 111). Despite the extant experimental evidence and recent theoretical developments in early algebra research (Carraher and Schliemann 2007), there is still a lot to do.

The journey this article was about focused on the generalization of patterns. This topic has become a very popular way to introduce students into algebra. It now occupies an important place in many contemporary curriculum programs around the world (MacGregor and Stacey 1992). Yet, it is haunted by many misunderstandings due to the lack of clear distinctions between arithmetic and algebraic thinking, and an unambiguous elucidation of the relationships between them. In all fairness we should note that this problem is not specific to pattern generalization research. It also permeates the broader mathematics education field that deals with the relationships between arithmetic and algebra.

In the first part, I offered an interpretative analysis of extending sequence activities. I scrutinized the type of thinking that is elicited by those activities in order to reflect on the question of whether or not there is something algebraic in it. The analysis put into evidence the fact that, to extend a sequence, our Grade 2 students resorted to coordinating spatial and numerical structures that, although highly complex, do not mobilize algebraic concepts as such. Extending sequences require indeed the grasping of a *commonality*, a process that instead of being algebraic is part in fact of a more generic process of concept formation, accessible also (although within certain limits) to other species.

Asking questions, however, about “remote” figures, like Figures 25 and 50—that is, figures beyond the perceptual field—is a different matter. To answer questions like those, our Grade 2 students did make a generalization. And I tried to show that the generalizations the students produced were certainly algebraic. To do so, I had to elaborate on the idea of algebraic thinking and what makes it distinctive. I started by arguing that it is misleading to associate algebraic thinking with the use of letters. Unfortunately, very often, curricular documents and current research about the relationships between arithmetic and algebraic thinking get caught in this trap. Algebraic thinking, I suggested, is not about using letters but about reasoning in certain ways. On epistemological grounds (Serfati 1999, 2006; Radford 1997, 2001) and drawing on the seminal work of Kieran (1989), Filloy and Rojano (1989), Filloy et al. (2007), Bednarz et al. (1996), Vergnaud (see, e.g., Cortes et al. 1990) and others I suggested that what distinguishes arithmetic from algebraic thinking is the fact that in the latter *indeterminate* quantities are treated in an *analytic* manner. I then moved on to claim, on semiotic grounds, that there are different ways in which to think of and express indeterminacy. Indeterminacy can indeed be expressed through signs other than the alphanumeric ones of conventional modern algebraic symbolism. This claim is fully compatible with the historical development of algebra. But, even more importantly, it makes room for the investigation of non-symbolic forms of algebraic thinking—an endeavour that is of great importance if we are to honour and understand the potential diversity of young students’ algebraic thinking.

Bearing these ideas in mind, the analysis that we conducted of the students’ mathematical activity showed that to tackle the questions about “remote” and “unspecified” figures, the students dealt first with indeterminacy and analyticity in what turned out to be an elementary algebraic layer of generality. This layer of generality sets the limits and possibilities of the corresponding form of algebraic thinking. Thus, the most elementary form of algebraic thinking elicited by our mathematical tasks was one where indeterminacy remained confined to specific figures. Indeterminacy appears here in an intuited form: it is expressed through particular instances of the variable in the form of a concrete rule or formula (like “50 plus 50, plus 1”). Analyticity and indeterminacy remain attached to the level of particular figures and arithmetic facts. We have encountered this form of thinking in older students (Grade 7, 8, 9) and called it *factual algebraic thinking* (Radford 2000, 2003).[2] It is

[2]This intuited form of variable has also been investigated by Fujii and Stephens (2008), and referred to as quasi-variables.

interesting to see that this embodied form of algebraic thinking can be accessible to most of Grade 2 students.

But, as our results imply, Grade 2 students can also engage in more sophisticated forms of algebraic thinking. The Tristan problem suggests indeed that students can deal with indeterminacy and analyticity in a more explicit way. We have also found this form of thinking in older students. We have termed it *contextual algebraic thinking* to stress the fact that the meaning with which algebraic formulas are endowed is deeply related to the spatial or other contextual clues of the terms the generalization is about.[3]

In the case of our Grade 2 students, the calculator proved to be extremely useful in the emergence of factual and contextual algebraic thinking. Its usefulness, however, was not limited to producing the numerical answers that were beyond the limited arithmetic knowledge of our students. Its usefulness resided in the conceptual frame that it made available for the students to envision the calculations to be performed and to come up with algebraic formulas. It might not be a coincidence that, historically speaking, the astounding consolidation of algebra in the late Renaissance was followed by attempts to construct the first calculating machines (Radford 2006b).

At any rate, the most important point, I believe, is that algebraic thinking is by no means something "natural", something that will appear and develop once the students have matured enough. Algebraic thinking is a very sophisticated cultural type of reflection and action, a way of thinking that was refined again and again through centuries before it reached its actual form. This is why its acquisition in ontogeny raises very difficult problems (Radford 2008b). In a sense, the journey this article was about is the story of our attempt at creating the pedagogical conditions for the students' first encounter with algebra. And as the journey intimates, the kind of algebraic thinking that emerged from the classroom activities was framed by the students' evolving understandings, the questions that we were asking, their interaction with peers, the teacher's participation, and the historical intelligence embedded in language and in the tools that were made available to the students.

It is still too early for us to offer an appraisal of the longitudinal development of the students' algebraic thinking. We hope, however, to be able to do it relatively soon. We have already accompanied these students as they moved to Grade 3 and, this year, we will continue following them in Grade 4. Such a longitudinal accompaniment should allows us to document the consolidation of the students' algebraic thinking and its evolution into more sophisticated forms, including the dawn of symbolic algebraic thinking.

[3]To tackle generalizing tasks like the ones seen in this chapter, older students tend also to mobilize forms of thinking that are not algebraic. For instance, some of them tend to produce generalizations that are strictly arithmetic—e.g., guessing rules and inductive generalizations (for a detailed discussion about the difference between algebraic and non-algebraic generalizations, see Radford 2008a). Knowing little arithmetic seems to inhibit the appearances of these arithmetic generalizations that compete with the algebraic ones. Instead of being an obstacle, knowing little arithmetic facilitates, at least to some extent, the focus on the algebraic.

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Formation of Pattern Generalization Involving Linear Figural Patterns Among Middle School Students: Results of a Three-Year Study

F.D. Rivera and Joanne Rossi Becker

Abstract This chapter provides an empirical account of the formation of pattern generalization among a group of middle school students who participated in a three-year longitudinal study. Using pre-and post-interviews and videos of intervening teaching experiments, we document shifts in students' ability to pattern generalize from figural to numeric and then back to figural, including how and why they occurred and consequences. The following six findings are discussed in some detail: development of constructive and deconstructive generalizations at the middle school level; operations needed in developing a pattern generalization; factors affecting students' ability to develop constructive generalizations; emergence of classroom mathematical practices on pattern generalization; middle school students' justification of constructive standard generalizations, and; their justification of constructive nonstandard generalizations and deconstructive generalizations. The longitudinal study also highlights the conceptual significance of multiplicative thinking in pattern generalization and the important role of sociocultural mediation in fostering growth in generalization practices.

Research on patterning and generalization at least in the last decade has empirically demonstrated the remarkable, albeit fundamental, view that individuals tend to see and process the same pattern P differently. Consequently, this means they are likely to produce different generalizations for P. For example, when we asked forty-two

This is an updated and thoroughly revised version of an earlier article that appeared in *ZDM 40*(1), 65–82. DOI 10.1007/s11858-007-0062-z. The *ZDM* article reported on findings drawn from the first two years of the study. This chapter reports on various aspects of the three-year study. This work was supported by Grant #DRL 044845 from the National Science Foundation (NSF) awarded to F. D. Rivera. The opinions expressed are not necessarily those of NSF and, thus, no endorsement should be inferred. This chapter is dedicated to Douglas Owens, mathematics educator, in honor of his retirement.

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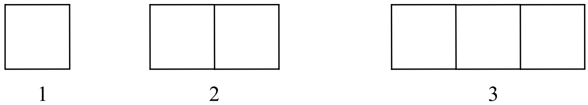
Square Toothpicks Pattern. Consider the sequence of toothpick squares below.



A. How many toothpicks will pattern 5 have? Draw and explain.
B. How many toothpicks will pattern 15 have? Explain.
C. Find a direct formula for the total number of toothpicks T in any pattern number n. Explain how you obtained your answer.
D. If you obtained your formula numerically, what might it mean if you think about it in terms of the above pattern?
E. If the pattern above is extended over several more cases, a certain pattern uses 76 toothpicks all in all. Which pattern number is this? Explain how you obtained your answer.
F. Diana's direct formula is as follows: $T = 4 \cdot n - (n - 1)$. Is her formula correct? Why or why not? If her formula is correct, how might she be thinking about it? Who has the more correct formula, Diana's formula or the formula you obtained in part C above? Explain.

Fig. 1 Adjacent squares pattern task

undergraduate elementary majors to establish a general formula for the total number of matchsticks at any stage in the *Adjacent Squares Pattern* shown in Fig. 1, Chuck obtained his generalization "$4 + (n - 1)3$" in the following manner:

> How many matchsticks are needed to form four squares? So ahm I'm looking for a pattern. For every square you add three more. So let's see. So that would be 4 plus 3 for two squares. Plus 3 more would be for three squares. So it's 10 matchsticks. So you have 4. So there would be 13. So 13 plus 3 more is 16. . . . So, for three squares, it would have to be two 3s. So there'd be two 3s. Three 3s is for four squares, and four 3s for five squares. For n squares, it would just be ahm n minus one 3s. (Rivera and Becker 2003, p. 69)

When we gave the same pattern in Fig. 1 to a group of middle school students three times over a two-year period, first when they were in sixth grade (after a teaching experiment) and then twice in seventh grade (before and after a teaching experiment), all of their generalizations consistently took the form $T = (n \times 3) + 1$. For example, in a clinical interview prior to the Year 2 teaching experiment, Dung, in seventh grade, initially set up a two-column table of values, listed down the pairs $(1, 4)$, $(2, 7)$, and $(3, 11)$ and noticed that "the pattern is plus 3 [referring to the dependent terms]." He then concluded by saying, "the formula, it's pattern number x 3 plus 1 equals matchsticks," with the coefficient referring to the common difference and the y-intercept as an adjustment value that he saw as necessary in order to match the dependent terms. When he was then asked to justify his formula, he provided the following faulty reasoning below in which he projected his formula onto the figures in a rather inconsistent manner (see Fig. 2 for an illustrative version).

> For 1 [square], you times it by 3, it's 1, 2, 3 [referring to three sides of the square] plus 1 [referring to the left vertical side of the square]. For pattern 2, you count the outside sticks

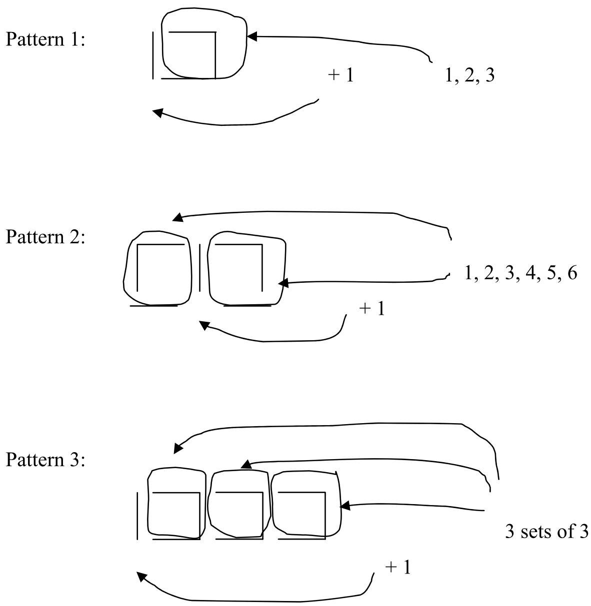


Fig. 2 Dung's justification of his direct formula for the Fig. 1 pattern

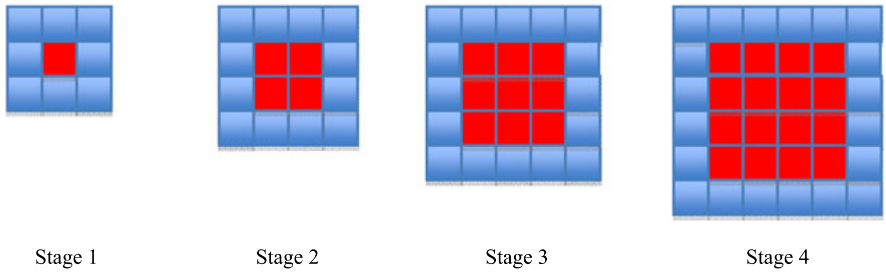


Fig. 3 Tile patio pattern

> and you plus 1 in the middle. For pattern 3, there's one set of 3 [referring to the last three sticks of the third adjacent square], two sets of 3 [referring to the next two adjacent squares] plus 1 [referring to the left vertical side of the first square].

Also, by the end of the Year 2 study, none of Dung's classmates were able to come up with a general form similar to Chuck's. Further, when they were asked to explain an imaginary student's formula, $T = 4n - (n - 1)$, for the pattern in Fig. 1, they found this and other similar tasks difficult.

However, we found it interesting that when the students in Year 3 of the study were purposefully reoriented to a multiplicative thinking approach to patterning activity involving figural stages (i.e., pictorial patterns with known stages such as the one shown in Fig. 1), they finally settled on *figural-based generalizations*. For example, when they obtained a generalization for the *Tile Patio Pattern* in Fig. 3 during a teaching experiment, they developed at least three equivalent direct formulas that reflected the use of multiplicative reasoning (i.e., in relation to what they perceived

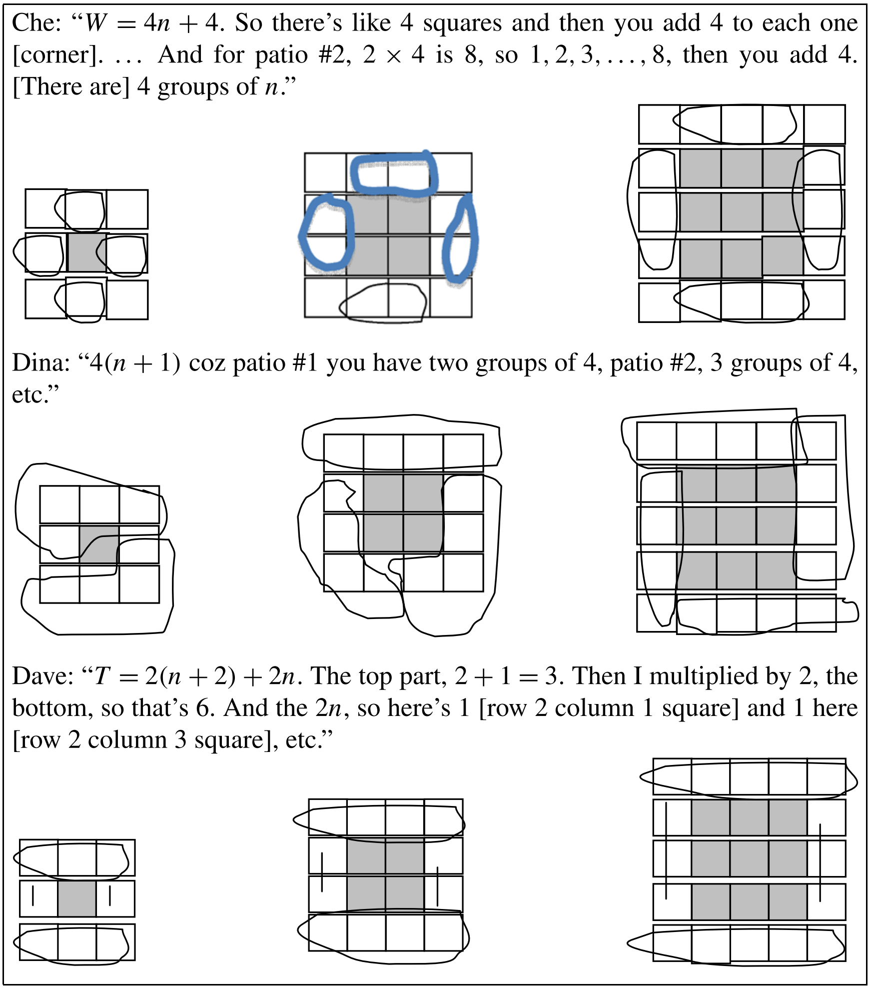


Fig. 4 Visuoalphanumeric generalizations of 8th graders on the Fig. 3 pattern

to be the repeated parts of the pattern). In Fig. 4, the students' constructed direct formulas are examples of figural-based generalizations in which the alphanumeric symbols in the formulas conveyed relationships that they have drawn figurally from the pattern. Such representations are, in fact, effects of particular (i.e., mathematical) ways of seeing and acquired knowledge and experience (Metzger 2006/1936).

In this chapter, we address issues relevant to the following two related questions: What is the content and structure of algebraic generalization that middle school students (i.e., Grades 6 to 8, ages 11 to 14) develop in the case of linear figural patterns? Further, to what extent are they capable of establishing and justifying their algebraic generalizations? By *algebraic generalization of a figural pattern*, we refer to, in Radford's (2008) words, the "[students'] capability of grasping a commonality noticed on some particulars (in a sequence), extending or generalizing this commonality to all subsequent terms, and being able to use the commonality to provide a direct expression of any term of the sequence" (p. 115). The two research questions address various aspects of what we label as *pattern generalization*, which involves constructing and justifying an algebraic generalization within the means available to a learner. Our notion of pattern generalization extends Radford's (2008) view to include justification. Also, we note that the above definition of algebraic generalization shares the basic conceptual intent surrounding all processes relevant to the task of *generalization*, which involves constructing an invariant and stable structure, property, attribute, or relation from particular known cases (or samples or domains) and extending, applying, and projecting it to the unknown cases or larger classes of cases (Dreyfus 1991). But we further refine Dreyfus's (1991) sense above by acknowledging the complex of factors (cognitive, cultural, extra-cultural such as linguistic and classroom practices, etc.) that influence the construction of a "generality" that, according to Dörfler (2008), is a way of practice of using and interpreting "signs, like graphs or letters, are not general by themselves" (p. 1).

In addressing the first research question, we initially survey relevant research in the area of middle school algebraic thinking. We then consider how findings in our three-year longitudinal research at the middle grades further confirm and/or extend the current knowledge base in this area. *Our response to the second research question* is grounded on how our students dealt with factors that influenced the manner in which they obtained their pattern generalizations. Our decision to investigate linear figural patterns has been drawn from our survey of various school mathematics curricula across states that show value and interest in this mathematical topic and its connections to other concepts as well.

Anticipating What Is to Come: Initial Reflections on Our Three-Year Data from the Clinical Interviews

Table 1 provides a summary of the results of the clinical interviews before and after every teaching experiment we conducted over the course of three years with our middle school students beginning at sixth grade. We briefly note the following observations:

- About 63% of the students in the Year 1 study employed figural-based strategies in obtaining a generalization for patterns that were mostly linear in content before a teaching experiment involving pattern generalization. But we also point out a dramatic shift to a numerical strategy (100%) after the teaching experiment in the same year.
- In the Year 2 clinical interviews with eight students, seven students maintained a numerical strategy in obtaining a generalization before and after a teaching experiment.
- In the Year 3 clinical interviews, a shift to figural-based strategies (about 69%) occurred.

Table 1 Summary of pattern generalization

Year 1 Results	Before Teaching Experiment ($n = 29$)	After Teaching Experiment ($n = 11$)
Overall Visual	63%	0%
Overall Numeric	37%	100%
Constructive Standard Generalizations	0%	100%
Constructive Nonstandard Generalizations	0%	0%
Deconstructive Generalizations	0%	0%
Year 2 Results[a]	**Before Teaching Experiment ($n = 8$)**	**After Teaching Experiment ($n = 8$)**
Overall Visual	12%	25%
Overall Numerical	88%	75%
Constructive Standard Increasing Patterns	100%	100%
Constructive Standard Decreasing Patterns	38%	75%
Constructive Nonstandard Generalizations	0%	0%
Deconstructive Generalizations	50%	100%
Year 3[a]	**Before Teaching Experiment ($n = 18$; 5 new[b])**	**After Teaching Experiment ($n = 14$; 3 new[b])**
Overall Visual	67%[c]	71%
Overall Numeric	33%[c]	29%
Constructive Standard Generalizations	100%	100%
Constructive Nonstandard Generalizations	6%	36%
Deconstructive Generalizations	11%	86%

[a]Some tasks had multiple questions

[b]Did not participate in earlier two-year interviews

[c]More visual tasks than numerical

Considering three years of collected data, in this article we extrapolate factors that explain why such shifts in generalization strategies took place among the students over the course of three years. Also, we provide a description of the quality, content, and form of generalizations at each phase. Further, we point out the progress in students' ability to deal with various aspects and types of pattern generalization, which was especially evident in Year 3. Our overall intent in this chapter is to describe teaching-learning conditions that enable meaningful pattern generalization to occur at the middle school level.

Cognitive Issues Surrounding Pattern Generalization: What We Know from Various Theoretical Perspectives and Empirical Studies

Clarifying the Definition of Pattern Generalization

Several researchers have pointed out that the initial stage in generalization involves "focusing" or "drawing attention" on a candidate invariant property or relationship (Lobato et al. 2003), "grasping" a commonality or regularity (Radford 2006), and "noticing" or "becoming aware" of one's own actions in relation to the phenomenon undergoing generalization (Mason et al. 2005). Lee (1996) poignantly surfaces the central role of "perceptual agility" in patterning and generalization, which involves "see[ing] several patterns and [a] willing[ness] to abandon those that do not prove useful [i.e., those that do not lead to a formula]" (p. 95). Mason et al. (2005) points out as well how *specializing* on a particular case in a pattern on the route to a generalization necessitates acts of "paying close attention" to details, especially those aspects that change and/or stay the same, best summarized in Mason's (1996) well-cited felicitous phrase of "seeing the general through the particular." Results of our earlier work with 9^{th} graders (Becker and Rivera 2005) and undergraduate majors (Rivera and Becker 2003) also confirm such a preparatory act whereby perception—as a "way of coming to know" an object or something property or fact about the object (Dretske 1990)—is necessary and fundamental in generalization. Of course, there are other researchers who emphasize the fundamental, genetic role of invariant acting in the construction of an intentional generalization (Dörfler 1991; Garcia-Cruz and Martinón 1997; Iwasaki and Yamaguchi 1997). In our longitudinal study, which focuses exclusively on the pattern generalization of figural objects, we affirm the above views about the nature of generalization.

Our contribution to the above characterizations deals with the mutually determining relationship between individual and sociocultural activity in the formation of pattern generalization. That is, while we acknowledge the constructivist nature of pattern generalization among individual students (every individual sees what s/he finds meaningful to see that influences how and what s/he constructs), collective action—that is, shared ways of seeing—makes the above characterizations even more meaningful than when performed in isolation.

As we have noted in the introduction, *pattern generalization* refers to both actions of *constructing* an algebraic generalization and *justifying* it on the basis of the students' repertoire of available explanatory mechanisms (Rivera 2010a, 2010b).

Constructing and justifying a generalization are two equally important tasks. In *constructing an algebraic generalization*, we expect closure in mathematical activity via the construction of a direct formula (i.e., a closed formula in function form). In the case of *justification*, in light of the cognitive level of middle school students who have just begun learning domain-specific knowledge and practices in algebra, we are more or less concerned with their capacity to reason, in the sense following Hershkovitz (1998), "to understand, to explain, and to convince" (p. 29). Knuth (2002), for instance, talks about the importance of having students perform a figural demonstration that explains, that is, using the relevant features in a figure in order to provide insights regarding a particular claim. Lannin's (2005) work with 25 US sixth graders had him pointing out how justification seemed to have been relegated to the "realm of geometric proofs" when, in fact, students' justifications in the context of pattern generalization could "provide a window for viewing the degree to which they see the broad nature of their generalizations and their view of what they deem as a socially accepted justification" (p. 232).

Types of Algebraic Generalization Involving Figural Patterns

There are two basic algebraic ways of developing a pattern generalization involving figural patterns. (For an extended list, see Rivera 2010a.) The first way involves what we classify as *constructive generalizations* (CG), which refer to those direct or closed polynomial formulas that learners construct from the known stages in a figural pattern as a result of cognitively perceiving figures that structurally consist of non-overlapping constituent gestalts or parts. For example, in the case of the Fig. 1 pattern, some students may perceive the stages as a sequence of growing squares that are produced by repeatedly adding three sides to form a new square. Dung's formula, "pattern number times 3 plus 1 equals matchsticks" in relation to Fig. 2 exemplifies a CG that exhibits the *standard* linear form $y = mx + b$, hence, it is a CSG. Chuck's direct expression for the Fig. 1 pattern, "$4 + (n - 1)3$," on the other hand, is an example of a constructive *nonstandard* generalization (CNG) since the terms in his expression still need further simplification.

The second way of developing a pattern generalization involves what we classify as *deconstructive generalizations* (DG), which refer to those direct or closed polynomial formulas that learners construct from the known stages as a result of cognitively perceiving figures that structurally consist of overlapping constituent gestalts or parts. Consequently, the corresponding general formulas involve a combined addition-subtraction process of separately counting each sub-configuration and taking away parts (sides or vertices) that overlap. For example, some students may initially infer the appropriate number of squares at each stage in the Fig. 1 pattern (i.e., stage 1 has one square, stage 2 has two squares that are adjacent to each other, stage 3 has three adjacent squares, …) and then multiply that number by 4 (since there are four sides to a square) and subtract the appropriate number of overlapping sides (i.e., stage 2 has two groups of 4 sides with an overlapping "interior" side, stage 3 has three groups of 4 sides with two overlapping "interior" sides, . . .). In a DG, further actions of deconstructing and decomposing are necessary in order to reveal the overlapping part(s).

In pattern generalization involving figural stages, we note that because there are many different ways of expressing a generalization for the same pattern, we foreground Duval's (2006) view about the cognitively complex requirements of semiotic representations—that is, a primary resolve is to assist learners to recognize the viability and equivalence of several generalizations that are drawn from several "semiotic representations that are produced within different representation systems" (p. 108). For example, Dung obtained his general formula for the Fig. 1 pattern by initially manipulating the corresponding numerical stages that he later justified figurally (Fig. 2), while Chuck established his formula for the same Fig. 1 pattern from the available figural stages. Both learners operated under two different representational systems and, thus, produced two different, but equivalent, direct expressions for the same pattern.

Methodology

This section is divided into five sections. The first section provides information about the middle school participants involved in the three-year study. The next two sections provide details of the teaching experiments on pattern generalization. The fourth section provides samples of the tasks used in the clinical interviews. The fifth section deals with matters involving data collection and analysis and relevant study protocols.

Classroom Contexts from Years 1 to 3 of the Study

In Fall 2005 and Fall 2006 (i.e., Years 1 and 2 of the study), the first author collaborated with two middle school mathematics teachers in developing and implementing two related design-driven teaching experiments on pattern generalization. From Fall 2007 to Spring 2008 (i.e., Year 3 of the study), the first author taught the class the whole academic year. The second author conducted the pre- and post-clinical interviews with the participating students in all three years of the study. Learnings from the pre-interviews were incorporated in the evolving teaching experiments with the participants, and the post-interviews were meant to assess students' abilities to establish and justify their generalizations, including the extent of influence of classroom practices in their developing capacity to generalize. In the Year 1 study, the sixth-grade class consisted of twenty-nine students (12 males, 17 females; mean age of 11; most of Southeast Asian origins). In the Year 2 study, three students moved to different schools and were replaced with six new students. In the Year 3 study, only fifteen students from the earlier two-year project were allowed to complete the project. They were then mixed with a new cohort of nineteen 7th and 8th grade students (22 females, 12 males) that together comprised an Algebra 1 class.

Nature and Content of Classroom Teaching Experiments in Years 1 and 2

A basic instructional objective of the Years 1 and 2 classroom teaching experiments on pattern generalization involves providing students with every opportunity to engage in problem-solving situations that would enable them to meaningfully acquire the formal mathematical requirements of algebraic generalization. The instructional theory that was initially used in Years 1 and 2 was Realistic Mathematics Education (RME). In RME, learners use models of their informal mathematical processes to assist them in developing models for more formal processes. Formalizing is, thus, seen as "growing out of their mathematical activity" and mathematizing, more generally, involves "expanding [their] common sense" with the same reality as "experiencing" in everyday life (Gravemeijer and Doorman 1999, p. 127).

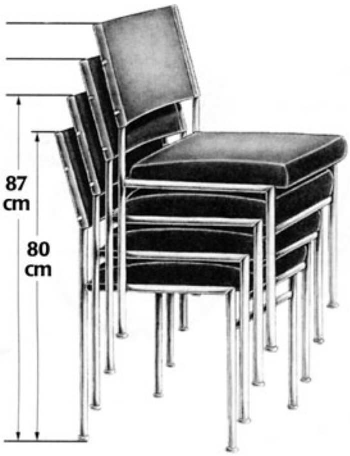
In the Year 1 teaching experiment on pattern generalization, two algebra units in the Mathematics-in-Context (MiC) curriculum were used. Also, taking note of the algebra requirements of the California state standards for sixth graders, sections were selected from the units *Expressions and Formulas* (MiC Team 2006b) and *Building Formulas* (MiC Team 2006a) that became the basis of a three-phase classroom teaching experiment on pattern generalization. In the first two phases, activities drawn from the two algebra units were used to foster the development of algebraic generalization through a series of horizontal and vertical mathematization tasks. Horizontal mathematization involves transforming real and experientially real problems into mathematical ones by using strategies such as schematizing, discovering relations and patterns, and symbolizing, while vertical mathematization involves reorganizing mathematical ideas using different analytic tools such as generalizing or refining of an existing model (Treffers 1987). In both units, the students initially explored horizontal activities that allowed them to build an informal mathematical model. They then engaged in vertical activities.

In the *Expressions and Formulas* unit, each section had the students starting out with a problem situation that involved using an arrow language notation to initially organize the situation and later to express relationships between two relevant quantities. An example is shown in Fig. 5. The arrow notation was meant to articulate the different numerical actions and operations that were needed to carry out a string of calculations in an activity. Also, the task situations were either stated in words or accompanied by tables, and they contained items that necessitated either a straightforward or a reverse calculation.

The *Patterns* section in the *Building Formulas* unit was the only one that was used in the teaching experiment because of constraints in the stipulated sixth-grade algebra requirements of the state's official mathematics framework. In this section, arrow language was employed less in favor of recursive formulas and direct formulas in closed, functional form. The students dealt with problem situations that consistently contained the following tasks relevant to generalizing: extending a near generalization problem physically (for example: drawing or demonstrating with the use of available manipulatives) and/or mentally (reasoning about

Stacking Chairs

The picture below shows a stack of chairs. Notice that the height of one chair is 80 centimeters, and a stack of two chairs is 87 centimeters high.



Damian suggests that the following formula can be used to find the height of a stack of these chairs:

number of chairs $\xrightarrow{-1}$ ___ $\xrightarrow{\times 7}$ ___ $\xrightarrow{+80}$ height

22. Explain what each of the numbers in the formula represents.

23. Alba thinks that a formula like this would do just as well:

number of chairs $\xrightarrow{\times}$ ___ $\xrightarrow{+}$ height

a. What numbers would Alba use in her formula? Explain how you determined these numbers.

b. Alba thought about making a formula like this:

number of chairs $\xrightarrow{+}$ ___ $\xrightarrow{\times}$ height

Will this work? Why or why not?

24. These chairs are used in an auditorium and sometimes have to be stored underneath the stage. The storage space is 116 centimeters high.

a. How many chairs can be put in a stack that will fit in the storage space?

b. Describe your calculation with an arrow string.

25. For another style of chair, there is a different formula:

number of chairs $\xrightarrow{\times 11}$ ___ $\xrightarrow{+54}$ height

a. How are these chairs different from the first ones?

b. If the storage space were 150 centimeters high, would the following arrow string give the number of chairs that would fit? Why or why not?

150 $\xrightarrow{+11}$ ___ $\xrightarrow{-54}$ ___

Fig. 5 Arrow notation activity (MiC 2006b, p. 21)

it logically); calculating a far generalization task (i.e., finding a total number beyond stage 10) using either a figural or a numerical strategy; developing a general formula recursively and/or in closed, functional form, and; solving problems that involve inverse or reverse operations. In all problem situations, tables were presented and employed as an alternative representation for organizing the given data.

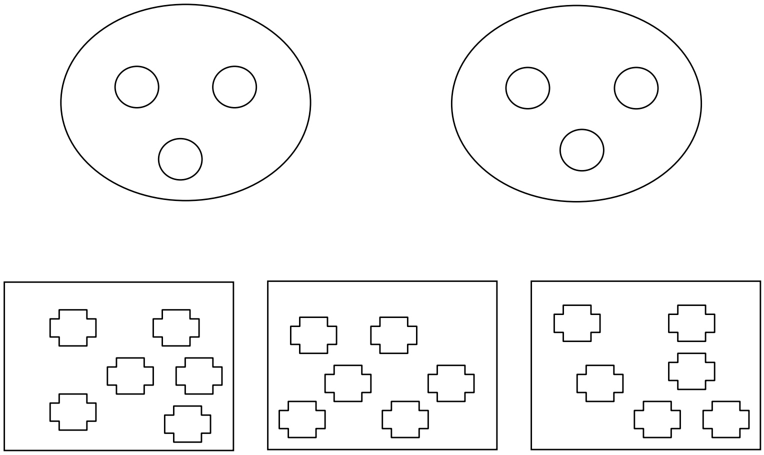


Fig. 6 Two multiplication tasks

Finally, the students dealt with tasks that asked them to reason and to make judgments about the equivalence of several different formulas for the same problem situation. In the third phase of the teaching experiment, they worked through several decontextualized patterning problems whose basic structure was similar to the ones that have been described in the paragraph above (see, for e.g., Figs. 1 and 3). Also, they explored problems that enabled them to develop both numerical and figural generalization.

In the Year 2 teaching experiment on pattern generalization, the same three-phase process occurred. The seventh-grade class used *Building Formulas* and portions of *Patterns and Figures* (MiC 2006a) in the first two phases with the third phase the same as in the description provided above.

Nature and Content of Classroom Teaching Experiments in Year 3

The Year 3 study on pattern generalization took place in three phases. The initial phase of learning pattern generalization focused on helping students develop a better understanding of multiplicative thinking, which actually was the unifying thread that connected all the algebra concepts and processes that were learned throughout the year. The students initially investigated counting activities that emphasized multiplicative thinking. For example, the activity in Fig. 6 asked the students to establish a mathematical expression involving multiplication. In the second phase, they explored pattern generalization activities that involve developing a structural analysis of a pattern (i.e., in terms of what stays the same and what changes in the pattern). In the third phase, they connected multiplicative thinking and structural analysis.

Nature and Content of Clinical Interview Tasks from Years 1 to 3

In the *Year 1 clinical interview prior to the teaching experiment* on pattern generalization, the students were given five tasks that addressed various aspects of pattern generalization. The task shown in Fig. 7a asked the students to determine a *near* and a *far* generalization item, with far items as arbitrarily referring to figural stages 10 and above in a given pattern and near items as pertaining to stages 9 and below. All five tasks required students to calculate a near and a far item. Three of the five tasks had figural patterns that show at least four consecutive initial stages. One task was presented numerically using a table of values. The fifth task began with an intermediate figural stage in some pattern and the students were asked to reconstruct a set of stages prior to the figural stage and then to use that knowledge to extend the pattern and deal with far items. In the *Year 1 clinical interview after the teaching experiment* on pattern generalization, analogous tasks were presented with some changes in the questions. For example, the task shown in Fig. 7b uses the same task structure in Fig. 7a but there is an increased emphasis in the following aspects of direct-formula construction: justification; the use of numerical or figural strategies; assessing for equivalence.

In the *Year 2 clinical interviews before and after the teaching experiment* on pattern generalization, tasks similar to Fig. 7b were presented to the students with the inclusion of two decreasing patterns.

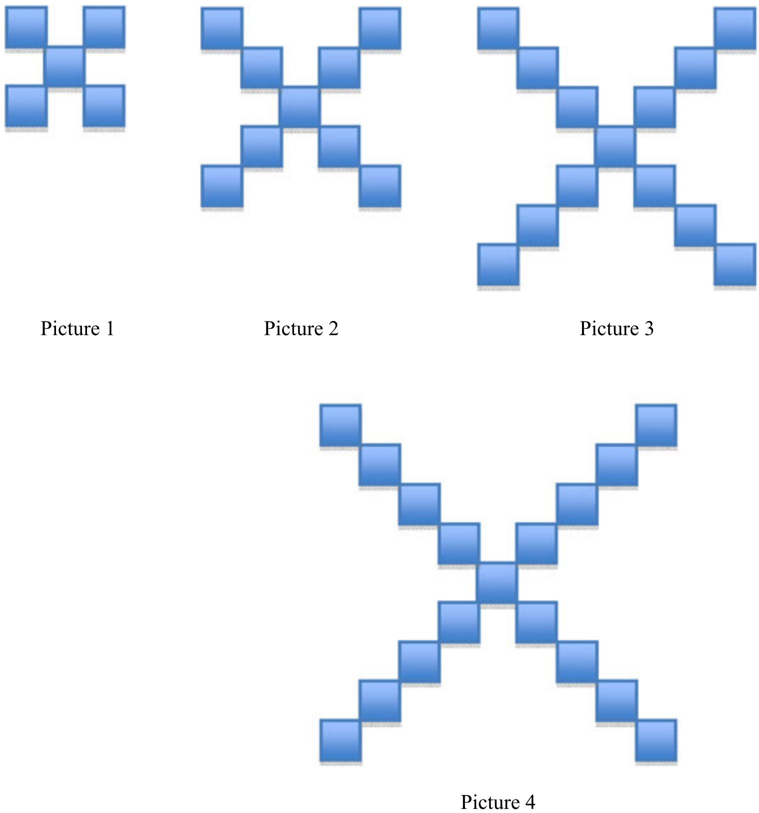
In the *Year 3 clinical interviews before and after the teaching experiment* on pattern generalization, the students were asked to justify a given direct formula of a given figural pattern (increasing and decreasing; see, for e.g., Fig. 8a) and to obtain several equivalent pattern generalizations for the same pattern (see, for e.g. Fig. 8b). A *semi-free construction* task was also added (see Fig. 9) in response to Dörfler's (2008) "plea for 'free' generalization tasks" (p. 153). Dörfler notes that patterns with well-defined stages impress on learners the view that "there is an expected direction of generalizing," which would then "intimate one and only one way [of continuing] a figural sequence" and, consequently, harbor "a strong regulating or even restrictive impact" on their thinking (p. 153). He recommends a different approach by asking students to think about (figural) patterns, as follows:

> How otherwise can one ask for, say, the number of matchsticks ... in an "arbitrary" item of the sequence? The situation would presumably be much more open if one asked simply "How can you continue?" or "What can you change and vary in the given figures?" ... I rather want to hint to possible further directions for research ... a plea for "free" generalization tasks not restricted by pre-given purposes. (Dörfler 2008, p. 153)

Data Collection and Analysis and Relevant Study Protocols

Each project year, we collected the following data: students' written work on various homework, classroom, and performance assessments involving pattern generalization; videos and transcripts of clinical interviews before and after every teaching

Tiles are arranged to form pictures like the ones below.



How many tiles does Picture 10 have? How do you know?
How many tiles does Picture 100 have? How do you know?
Find a formula to calculate the number of tiles in Picture "n." How did you obtain your formula?

Fig. 7a A sample Year 1 clinical interview task prior to a teaching experiment

experiment, and; videos and transcripts of relevant classroom episodes taken during a teaching experiment.

With respect to the analysis of data drawn from the clinical interviews, we engaged in several repeated processes of individual and shared reading within and across cases. We have carefully described this important step in several published research papers (Becker and Rivera 2005, 2006, 2007, 2008; Rivera and Becker 2008). Basically, individual cases were analyzed, developed, and later synthesized in order to construct individual cognitive maps with the aim of schematically capturing their generalizing schemes from problem to problem. Next, those individual cases were compared, analyzed, and categorized using grounded theory that enabled us to develop some empirical claims about aspects of their pattern generalizing ac-

Tiles are arranged to form pictures like the ones below.

See pattern in Fig. 7a

A. Find a direct formula that enables you to calculate the number of square tiles in Picture "n." How did you obtain your formula? If the solution has been obtained numerically, respond to the following question: Is there a way to explain your formula from the figures?

B. How many square tiles will there be in Picture 75? Explain.

C. Can you think of another way of finding a direct formula?

D. Two 6th graders came up with the following two formulas:

Kevin's direct formula is: $T = (n \times 2) + (n \times 2) + 1$, where n means Picture number and T means total number of squares. Is his formula correct? Why or why not?

Melanie's direct formula is: $T = (n \times 2) + 1 + (n \times 2) + 1 - 1$, where n and T mean the same thing as in (D) above. Is her formula correct? Why or why not?

Which formula is correct: Kevin's formula, Melanie's formula, or your formula? Explain.

Fig. 7b Analogous Year 1 task given after a teaching experiment on pattern generalization

tivity. The results, findings, and observations we have developed were consequences of several iterated processes of reading and analyzing the within- and across-case studies in order to ensure greater validity.

The first author was also responsible for the analysis of data drawn from the classroom episodes. Relevant transcripts of key classroom episodes were obtained in order to provide additional support about the claim of shifts in pattern generalization practices of the participating students over the course of the longitudinal study.

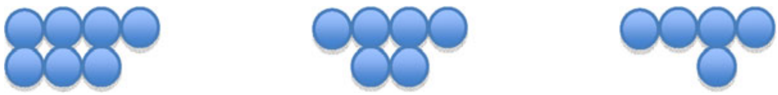
Undergraduate student assistants videotaped all the classroom and collaborative group sessions involving pattern generalization. Each teaching experiment lasted three consecutive weeks on average. All clinical interviews were also videotaped. During a clinical interview, each student was requested to think aloud and to use the available and relevant manipulatives (pattern blocks, calculators, centimeter graphing paper, etc.) to help them deal with the tasks. In cases when a student incorrectly performed a calculation, the interviewer (second author) sought clarification to better assess the nature of the error. Also, in cases when a student had a difficult time articulating a verbal response, the interviewer sought clarification until both of them felt satisfied with the response.

A. Consider the sequence of three figures shown below.

Three 8th grade students have been asked to find the total number of stars (S) at any given stage number (n). Explain how each student might be thinking of his or her formula.

Marcia's formula: $S = n \cdot 3 + 1$
Pete's formula: $S = (n + 1) \cdot 3 - 2$
Jayme's formula: $S = (n \cdot 4 + 1) - n$

B. Consider the sequence of three figures below.



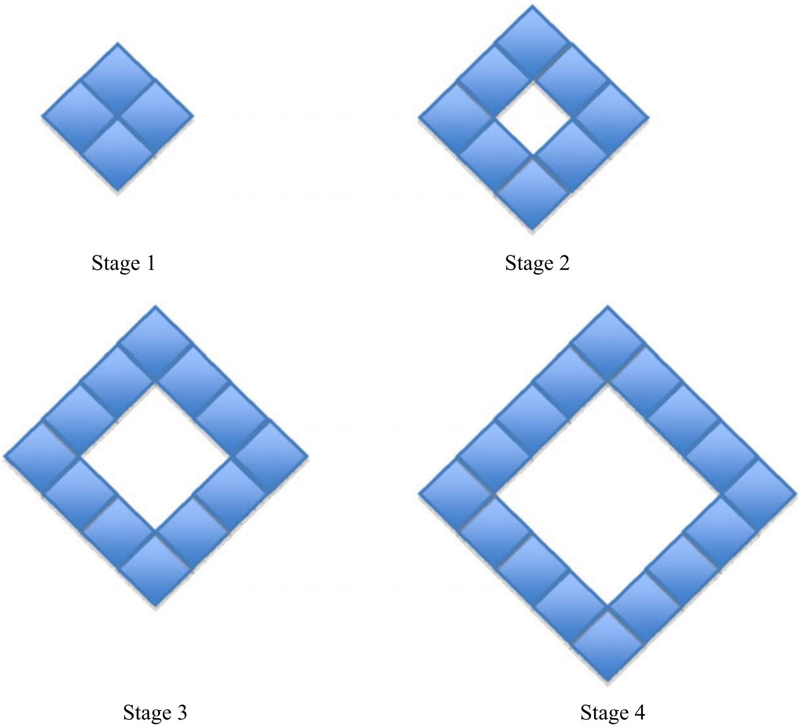
Two seventh grade students came up with two different direct formulas for the total number of circles (C) for any step number (n). How might each student be thinking about his formula?

Jake: $C = -n + 8$
Bharath: $C = 4 + 4 - n$

Fig. 8a Two sample Year 3 tasks given before and after a teaching experiment on pattern generalization (increasing and decreasing figural patterns)

Findings and Discussion Part 1: Accounting for Constructive and Deconstructive Generalizations

Findings in Our Study An analysis of Table 1 shows the predominant use of CSGs from Years 1 to 3. In fact, CNGs were not evident until Year 3. The process used to establish a CSG was predominantly numerical in Years 1 and 2 with a mean of 87.5% but a shift to figural took place in Year 3 at about the same percentage. Also, the generalizations that were developed in the case of all increasing and decreasing linear patterns were CSGs. Further, while the students were successful (100%) in establishing CSGs in the case of increasing linear patterns, they had considerable difficulty in the case of decreasing linear patterns (a success rate of 38% before a teaching experiment and 75% after). In the case of DGs, while none of the students could construct them by the end of the Year 2 study, they, however, Consider the sequence of four figures below.



Obtain two different ways (or formulas) that will enable you to find the total number of gray squares (S) at any stage number (n). Then explain why you think each way (or formula) makes sense to you.

1. Formula 1: ___________________________
Explanation:

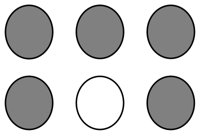
2. Formula 2: ___________________________
Explanation:

Fig. 8b A sample Year 3 task given before and after a teaching experiment on pattern generalization

had considerable success in explaining them (from 50% to 100% before and after a teaching experiment, respectively).

Discussion Results drawn from our Years 1 and 2 study actually confirm findings from several research studies at the middle school level that also provide sufficient evidence indicating students' predisposition toward producing more CSGs than DGs. For example, when Taplin and Robertson (1995) asked 40 Australian 7th graders to establish a generalization for the pattern sequence in Fig. 1, while none

The figure below shows five gray circles that enclose a white circle. Call it stage 1.



Stage 1

1. First, find a way to continue the above figure so that you end up with several stages that altogether form a pattern of figures. Draw your figures below.
Your Stage 2:
Your Stage 3:
Your Stage 4:
2. Next, try to find a formula for the pattern of figures you constructed above. If a formula is not possible, describe your pattern in a general way.

Fig. 9 A semi-free construction task given before and after a teaching experiment on pattern generalization in the Year 3 study

of them could state an algebraic generalization, their incipient generalizations took the form of CS verbal statements. Seven students perceived four toothpicks that pertained to the original square in stage 1 and the repeated addition of 3 toothpicks each time from stage to stage. There were eight students who offered the CN verbal generalization, $3(n-1)+4$, although none offered an articulation that was as clear as Chuck's in the introduction above. Only one student began to think about the pattern in a deconstructive way; however, that student was not able to figure out the number of toothpicks that needed to be taken away despite seeing the pattern as consisting of overlapping squares. When the same problem was given to a cohort of four hundred thirty 12- to 15-year old Australian students, findings from English and Warren's (1998) study also showed that, among the less than 40% of students who successfully obtained a generalization, they expressed their generalities on this and other patterning tasks in constructive terms similar to what Taplin and Robertson (1995) found. For example, a student developed the general expression $2x+(x+1)$, where $2x$ refers to the top and bottom row sticks and $(x+1)$ to the column sticks in the Fig. 1 pattern, after seeing two invariant properties within and across stages.

While descriptions of CGs for figural patterns abound, the more important question involves the formation of CSG and CNG, in particular, how does *constructive objectification* come about? *First*, Radford (2003) notes that there are different semiotic means of objectification in relation to pattern stages, that is, possibly different ways in visibly surfacing attributes and properties of, or relationships among, stages with the use of signs and relevant processes or operations. *Second*, Radford (2003, 2006) advances the view that there are at least three layers of algebraic generalization—factual, contextual, and symbolic—based on his three-year longitudinal work with middle and junior high school students. *Third*, purposeful instruction through well-designed classroom teaching experiments could scaffold the development of closed forms of constructive generalizations in middle school children (Lannin et al. 2006; Martino and Maher 1999; Steele and Johanning 2004). In the following paragraphs, we dwell on cognitive-related issues at the entry stage of generality, that is, factual, since both contextual and symbolic layers are marked indications of further essentializing and increasing formality on the basis of the stated factual expressions.

At the pre-symbolic stage of factual generalizing involving increasing linear patterns, students oftentimes start with a recursive relation that is both additive and arithmetical in nature. As a matter of fact, studies done in different settings (for e.g., countries) and in different contexts (prior to formal instruction in algebra, during and/or after a teaching experiment, etc.) with middle school students have asserted the use of recursion as the entry (and, in some cases, the final) stage in factual generalizing (Becker and Rivera 2006; Bishop 2000; Lannin et al. 2006; Orton et al. 1999; Radford 2003; Sasman et al. 1999; Swafford and Langrall 2000). For example, in the case of increasing figural sequences, it is usually easy to first perceive the dependent terms as constantly being increased by a common difference. As soon as this takes place, students' thinking is then characterized in two ways, as follows:

- *First*, they see two consecutive stages as being different and, using the method of "differencing" (Orton and Orton 1999, p. 107), the same number of objects is constantly being added from one stage to the next, leading to a recursive, arithmetical generalization (of the type $u_n = u_{n-1} + c$, where c is the common difference). Then, some students further develop emergent factual generalizations from the arithmetical generalization. Two possible factual generalizations involving the Fig. 1 pattern are as follows: $4 + 3 + 3 + 3 + \cdots$; $1 + 3 + 3 + 3 + \cdots$.
- *Second*, a structural similarity is observed among and, thus, connects two or more stages in a relational way. Especially in the case of increasing linear patterns that figurally demonstrate growth, constructing a succeeding stage from a preceding one oftentimes involves a straightforward process of simply adding a constant number of objects on particular locations of the preceding stage. That is, the basic structure of the unit figure is perceived to stay the same despite the fact that equal amounts of objects are conjoined in various parts of the figure in a particular, predictable manner. Such method of construction does not necessitate making a figural change (in Duval's 1998 sense) on the part of the learner.

Radford (2003) further notes how in the factual stage of generalizing, invariant acting from one stage to the next operates at the concrete level that eventually leads to the abstraction of a numerical or operational scheme for the figural pattern. Hence, generalizations that have been mediated by such actions tend to be consequentially constructive and almost always standard (whether rhetorical, syncopated, or symbolic in form).

Even with pattern generalization tasks that require middle school students to first specialize (in Mason's 1996 sense) on the route to establishing a generality as a consequence of not being provided with the usual consecutive sequence of figural stages, many of them were predisposed to establishing constructive generalizations.

For example, Swafford and Langrall (2000) asked ten middle- to high-math achieving 6th grade students to solve the Fig. 3 pattern prior to a formal course in algebra. In their case, the task began with a drawn 10 by 10 square grid in which the four borders of the grid are shaded. The students were asked to figure out the total number of squares on the border, and the task was repeated in a 5 by 5 grid. The students were then asked to describe how to determine the total number of squares in the border of an N by N grid. Results on this task show that, while none of the students offered a recursive rule, the general verbal descriptions ranged in form from the constructive to the deconstructive. When translated in symbolic form, two of the verbal generalizations were CNGs and obeyed the following forms: (1) $n+n+(n-2)+(n-2)$; (2) $n+(n-1)+(n-1)+(n-2)$. Only one student in their study offered a verbal DG that followed the form $4n-4$. When the above task and other similar ones were given to eight 7th grade students in Steele and Johanning's (2004) study in the context of a problem-solving enriched teaching experiment, only three students came up with DGs.

Findings and Discussion Part 2: Understanding the Operations Needed in Developing a Pattern Generalization

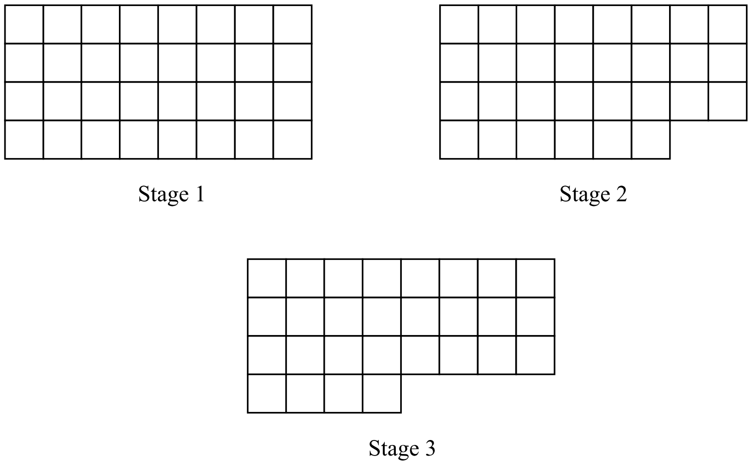
Findings in Our Study This section addresses issues our middle school participants had relative to developing DGs and CGs involving decreasing linear patterns. Due to space constraints, we illustrate in this section students' difficulties with decreasing linear patterns. Decreasing linear patterns could be expressed as CSGs in the form $y = mx + b$, where $m < 0$. In Year 2, the students' primary cognitive difficulty with decreasing patterns prior to a teaching experiment (with a success rate of 38%) was how to handle *negative differencing* and, especially, how to perform operations involving negative and positive integers. While we found that they were attempting to transfer the existing generalization process they developed in the case of increasing linear patterns, they could not, however, make sense of the negative integers and the relevant operations that were used with such types of numbers.

For example, in a clinical interview prior to a teaching experiment, Tamara easily obtained CSGs on two increasing linear patterns. Also, she was able to justify given several CNGs relative to another figural pattern. When she was then presented with the *Losing Squares Pattern* in Fig. 10 as a third task to analyze, she immediately saw that every stage after the first involves "minusing 2" squares. She used multiplication to count the total number of squares at each stage. In obtaining a direct formula, however, she was perturbed by the negative value of the common difference and said,

> I was trying to think of, just like the last time, I was trying to get a formula. ... I was thinking of trying to do with the stage number but I don't get it.

The presence of the negative difference, including the necessity of multiplying two differently signed numbers, partially and significantly hindered her from applying the method she learned in the case of increasing patterns. Tamara's situation

Take a look at the three different stages in the design below.



1. How many squares are there in stage 1? stage 2? stage 3?
2. How many squares are there in stage 10? How do you know for sure?
3. How many squares are there in stage 15? How do you know for sure?
4. Find a direct formula for the total number of squares in stage n, where n is a positive integer.
If you obtained your formula numerically, what might it mean in the context of the above pattern?
5. How many squares are there in stage 20? What might your answer mean in the context of the given pattern?
6. For what stage number will there be no more squares left? How do you know for sure?

Fig. 10 Losing squares pattern

exemplifies the thinking of those students interviewed who were also unsuccessful and, thus, unable to overcome such difficulties before (about 62%) and even after (25%) the teaching experiment in Year 2. Further, her thinking in relation to decreasing patterns after the teaching experiment captures the actions of those students who were also successful by the end of the Year 2 study (about 75%).

Discussion A relevant issue we considered in relation to linear pattern generalization involves the operations that are employed in formulating CGs and DGs. Developing CGs in the case of increasing linear patterns requires students to have solid grounding in addition and multiplication of whole numbers. Developing DGs and CGs in the case of decreasing linear patterns necessitates knowledge relevant to manipulating addition, subtraction, and multiplication of integers (cf: English and Warren 1998; Stacey and MacGregor 2001).

Gelman and colleagues (Gelman 1993; Gelman and Williams 1998; Hartnett and Gelman 1998) have advanced and empirically justified a rational constructivist account of cognitive development among young children that presupposes the existence of innate or core skeletal mental structures (such as arithmetical structures) that enable them to easily develop and process new information as long as it is consistent with their core structures. Hartnett and Gelman (1998) write:

> As long as inputs are consistent with what is known, then novice's active participation in their Learning can facilitate knowledge acquisition. But when available mental structures are not consistent with the inputs meant to foster new Learning, such self-initiated interpretative tendencies can get in the way (p. 342).

Among middle school students who develop CSGs and CNGs in the case of increasing linear patterns, perhaps it is the case that their generalizations, which involve using the operations of addition and multiplication of whole numbers, map easily onto their current understanding of what numbers are and how such entities are used, represented, and manipulated. Thus, constructive generalizing will proceed naturally and smoothly. Moreover, middle school students are likely to associate increasing growth patterns with counting objects over several non-overlapping constituent gestalts and then use the addition and multiplication of counting numbers as useful operators in obtaining a final count. Hence, their core arithmetical structures assist in this developing capacity towards making constructive generalizations. This being the case, it is less likely that students will apprehend increasing patterns as being embedded in a figural process that involves the operation of subtraction via, say, the utilization of a figural change process of seeing sub-configurations and removing overlapping parts as in all cases of DGs.

In the case of decreasing linear patterns, students like Tamara have to first broaden their knowledge of multiplication to include two factors having opposite signs in order to establish, say, the formula $S = -2 \times n + 34$. However, we note that, as with the other students, while Tamara was able to explain the terms in her CSG consistently across increasing linear patterns, she was unable to justify the formulas she established for decreasing linear patterns.

Findings and Discussion Part 3: Factors Affecting Students' Ability to Develop CGs

Even when middle school students are found sufficiently capable of producing more CSGs than DGs and CNGs, we discuss three additional factors that influence their ability to establish the former.

Findings in Our Study In our Year 1 study, 69% of our sixth grade students' initial verbal generalizations (correct and incorrect) in relation to the *T Circle Pattern* in Fig. 11 could not be conveniently translated in closed form. Table 2 provides a list of these verbal responses. The remaining 31% provided verbal generalizations that are considered to be algebraically useful. The responses are listed in Table 3.

For example, when Dina was asked to obtain a generalization for the total number of dots in the Fig. 10 pattern, her circle chip-based stages in Fig. 12 revealed the

Fig. 11 *T* circle pattern

Table 2 Summary of non-algebraically useful verbal responses in relation to the Fig. 11 pattern (20 out of 29 students)

Frequency	Verbal Generalizations
14	Figure 1 has 1 circle, figure 2 has 3 circles, figure 3 has 5 circles, ..., figure 10 has 19 circles. So figure 100 has 190 circles since $10 \times 10 = 100$ and $10 \times 19 = 190$
1	Figure 1 has 1 circle, figure 2 has 3. So you're adding two each time. So figure 100 has $100 \times 2 = 200$ but the numbers are always 1 less than the actual multiple of 2. So figure 100 has 199 circles
1	Since the pattern is always adding 2, it's the same thing as multiplying by 2. So figure 100 should have $100 \times 2 = 200$ circles
1	Always keep adding 2
1	There is a pattern in the units digit. If figure 10 has 19 circles, then figure 20 has 29 circles, figure 30 has 39 circles, etc.
1	There is a pattern in the units digit. If figure 5 has 9 circles and figure 10 has 19 circles, then figure 15 should have 29 circles, figure 20 has 39 circles, etc.
1	Since figure 1 has 1 circle, then figure 5 has 5 circles. Since figure 2 has 2 row circles and 1 column circle, then figure 6 has 6 row circles and 1 column circle. Since figure 3 has 3 row circles and 2 column circles, then figure 7 has 7 row circles and 2 column circles. Since figure 4 has 4 row circles and 3 column circles, then figure 8 has 8 row circles and 3 column circles, etc.

Table 3 Summary of algebraically useful verbal responses in relation to the Fig. 11 pattern (9 out of 29 students)

Frequency	Verbal Generalizations
7	Since figure 2 has 1 circle in the column and 2 circles in the row, and figure 3 has 2 circles in the column and 3 circles in the row, then figure 10 has 9 circles in the column and 10 circles in the row
2	The pattern keeps adding two by adding a circle on the right side of the row and another circle at the top of the column

extent of her perception of the stages, that is, the stages just kept going up by twos and nothing else. Those who used a figural multiplicative strategy, on the other hand, initially employed analogical reasoning. Employing multiple instead of unit counting, their general statements reflect the invariant structure they thought was evident from stage to stage.

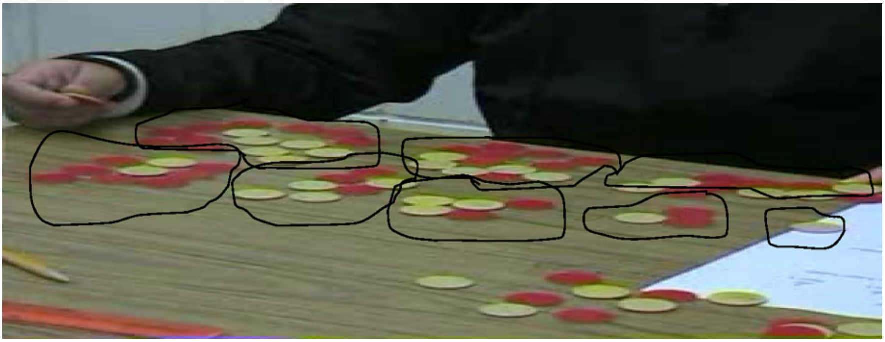
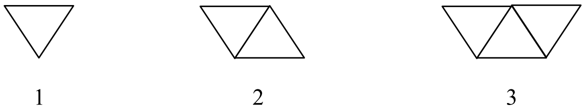


Fig. 12 Dina's interpretation of the Fig. 10 *T* circles pattern using colored chips

Fig. 13 The triangular toothpick pattern



Discussion Language and the use of variables and analogies are all important factors in direct-formula construction involving all CGs and DGs. Concerning language, Stacey and MacGregor (2001) point out the importance and necessity of the "verbal description phase" in the "process of recognizing a function and expressing it algebraically" (p. 150). Also, based on results drawn from Year 7 to Year 10 (ages 12 to 15) Australian students and their reflections on a national recommendation for a pattern-based approach to algebra, MacGregor and Stacey (MacGregor and Stacey 1992; Stacey and MacGregor, 2001) surface students' difficulties in "Transition[ing] from a verbal expression to an algebra rule" since "students with poor English skills" are oftentimes unable to "construct a coherent verbal description" and many of their "verbal description[s] cannot be [conveniently and logically] translated directly to algebra" (MacGregor and Stacey 1992, pp. 369–370).

Concerning variables, Radford (2006) points out the problematic status of variable use in students' expressions of generality. In his layers of algebraic generalization, the presence and use of variables in their proper form and meaning signal the accomplishment of the final stage of symbolic generalization. He notes that while some students may display knowledge of using algebraic language to express a CG, the variables used in such contexts have to reach their objective state of being desubjectified and disembodied placeholders. Radford's (2001) characterization of algebraic language at the layer of symbolic generalizing is best exemplified in the thinking of two small groups of 8th graders on the *Triangular Toothpick Pattern* in Fig. 13 who obtained the generalities $(n+n)+1$ and $(n+1)+n$ and perceived them as being different on the basis of having been derived from two different actions. Radford (2001) astutely points out that the use of variables to convey a generality has to evolve. In particular, when students employ a variable in relation to the independent term of the general expression, they need to eventually see that the variable has to shift meaning from being a "dynamic general descriptor of the figures in [a] pattern" to being "a generic number in a mathematical formula" (Radford 2000, p. 255). Thus, their general algebraic language in expressions of generality involves semantically transposing the independent variable from its ordinal character (indexical, positional, deictically-based) to the cardinal (as a "number capable of being arithmetically operated" (ibid.)).

Concerning analogies, since all linear patterns could take the CSG formula $y = mx + b$, perceiving and using analogies can quickly facilitate the generalizing process. While middle school students are likely to offer a constructive recursive expression, some have been documented to be capable of developing constructive analogical expressions in varying formats even prior to a formal study of algebra and algebraic notation (Becker and Rivera 2006; Bishop 2000; Lannin 2005; Stacey 1989; Swafford and Langrall 2000). Performing analogy involves "perceiv[ing] and operat[ing] on the basis of corresponding structural similarity in objects whose surface features are not necessarily similar" (Richland et al. 2004, pp. 37–38).

In our Year 1 study, we identified a possible source of difficulty among the sixth grade students in relation to constructing algebraically useful analogies for particular figural-based patterns. We distinguished between students who perceived and generalized additively from those who employed a multiplicative approach. Those students who used a figural additive strategy, on the one hand, were not thinking in analogical terms at all, and they frequently employed counting objects one by one from stage to stage. Further, when some of them were provided with manipulatives to copy figural stages that had been drawn on paper, their manipulative-constructed stages did not preserve the structure of individual stages like Dina in relation to Fig. 12 above; they, in fact, used the available manipulatives only as counters.

Findings and Discussion Part 4: A Three-Year Account of Classroom Mathematical Practices that Encouraged the Formation of Generalization Among Our Middle School Students

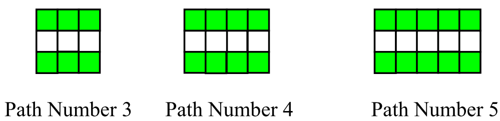
Findings in Our Study In this section, we describe how our middle school participants established six classroom mathematical practices on pattern generalization over the course of three teaching experiments that occurred over three consecutive years. We note that very few studies at the middle school level have focused on the manner in which students develop pattern generalizations over some extended timeframe. Thus, in this section, we aim to highlight how certain legitimate mathematical practices could be viewed not as conceptual, received objects that learners simply acquire rather unproblematically but as part of their individual and sociocultural developmental transformations drawn from and embodied in their activity with other learners.

Year 1 Classroom Practices: From Figurally- to Numerically-Driven CSGs

In the Year 1 study, four pattern generalization practices were constructed and became taken-as-shared in collaborative activity. Two of the practices had their origins in the first MiC unit they used in class (i.e., *Expressions and Formulas*). *First*, the students initially employed arrow strings as a method for organizing a sequence of arithmetical operations (see, for e.g., Fig. 5). They also explored the notion of equivalence through arrow strings that could either be shortened or lengthened depending on the nature of the numbers being manipulated. *Second,* the use of the arrow strings evolved as the students were asked to deal with more complicated problem situations that were still arithmetical in context. In several more sessions, they developed a connection between constructing an arrow string and a formula in such a way that they used arrow strings as a means of describing invariant operational schemes in the context of generalizing situations. In transitioning from the arrow strings to formulas, the students developed an understanding that a formula, like the arrow strings, consists of a starting number or input, a rule in the form of a sequence of operations, and an output value or expression.

Two additional practices emerged when the students began to generalize figural-based patterns that have been initially drawn from the *Patterns* section in the MiC unit *Building Formulas*. The *third* classroom practice that became taken-as-shared involves generalizing figurally and is exemplified in the classroom episode below in which the students were engaged in developing a formula for the total number of grey and white tiles for new path number n whose figural stages are shown in Figs. 14a and 14b. Initially, the students explored specific instances when $n = 3$ to 5, 9, 15, 30, and 100. In particular, they were not merely asked to obtain the output values but also to describe the patterns without actually drawing them explicitly. The class then generated a recursive rule for each tile type. In the episode below, the discussion that took place between the first author and the class shifted from the use of recursive rules to the construction of general expressions in relation to the new path patterns.

Fig. 14a Urvashi's tile patterns (MiC Team 2006a, p. 2)



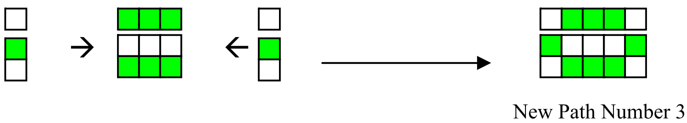


Fig. 14b Urvashi's design for new path 3 (MiC Team, 2006a, p. 3)

FDR: *Suppose I want you to describe new path 1,025. That's a big number. I want you to figure out the total number of white and grey tiles for new path 1,025. Emily, how do we do this?*
Emily: *The whites will be 2,054?*
Ford: *That's the grey.*
Emily: *It is?*
Ford: *Yeah, the white's the middle.*
Emily: *1,029.*
FDR: *Why 1,029?*
Emily: *Because it's in the middle and in the corners it has four.*
FDR: *Alright. What about the grey ones? Mark.*
Mark: *The grey ones are 2,052.*
FDR: *Why 2,052?*
Mark: *Because you added the top and the bottom and then you add the two middle.*
FDR: *Okay, this will be a challenge for some of you. Can you find a formula for me? Suppose, I say, I'm going to use a variable, new path number n. n could mean 1, 2, 3, 4, all the way to 1025. All the way to a billion.*
Dung: *n plus 4 equals white.*
FDR: *Why* $n + 4$ *equals white?*
Dung: *Coz n is the number of whites in the middle plus 4 whites on the sides.*
FDR: *Does that make sense? [Students nodded in agreement.] What about the grey ones? The grey ones are a bit more difficult. What's a formula for the number of grey ones?*
Che: *n times 2 and then you plus 2.*
FDR: *It's* $n \times 2 + 2$. *What about if I express it as n plus?*
Deb: *n plus n plus 2.*
FDR: $n + n + 2$. *Are they the same?*
Jack: *Yes.*
FDR: *Why?*
Nora: *You have two grey ones.*
FDR: *Yes, you have the two gray ones plus the two on both sides. So now if I know these formulas here, can I figure out new path number 50,000?*
Students: *Yeah.*
FDR: *So how do we do this, using the formula here. Number of whites. n plus 4 for whites. What do we do?*
Tamara: *It's 50,004.*
FDR: *What about the grey ones?*
Mark: *100,002.*

One indication of the students' individual appropriation of learnings from the above social event involves their work on succeeding figural patterns. The formulas they established were all CSGs that they oftentimes justified in figural terms. From the above discussion, the students acquired an understanding of using figural generalizing in explicitly articulating structural similarities among the available pattern

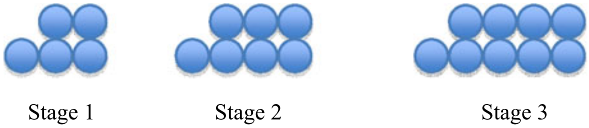


Fig. 15 Two layer circles pattern

stages and, hence, figurally identifying properties or relationships that remained stable and invariant over a sequence of known stages. Further, they learned to express those properties or relationships in algebraic form, including the need to justify the reasonableness and validity of the direct formulas. We classify such formulas as *figural-based representations*.

The *fourth* classroom practice came about when the students tackled the *Two Layer Circles Pattern* (Fig. 15). All the students initially perceived a recursive relation with the constant addition of one circle per layer. Two groups of students offered the figural-based formula $C = (n + 1) + (n + 2)$, where n represents figure number and C stands for the total number of circles, which they established analogically. That is, since Fig. 1 had two and three circle rows, Fig. 2 had three and four circle rows, and so on, then figure n had to have $(n + 1)$ and $(n + 2)$ circle rows. The first author then suggested organizing the two sets of numerical values in the form of a table without making any recommendation that might have encouraged a numerical strategy. The basic purpose in introducing the table in several classroom instances was primarily to foster students' growth in their representational skills, that is, patterns could also be expressed in tabular form. In the classroom episode below, Anna shared her group's thinking with the class which eventually was taken as shared and became the fourth classroom practice, that of generalizing numerically using differencing, which was reflective of an appropriation of a standard institutional numerical strategy.

Anna: *We made up a formula. Like we got the figures until figure 5, and we tried it with other ones. We got* $n \times 2 + 3$*, where n is the figure number and timesed it by 2. So* 5×2 *equals 10, plus 3, that's 13. So for figure 25, it's 53.*

FDR: *I like that formula. So tell me more. So your formula is?*

Anna: $n \times 2 + 3$.

FDR: *So how did you figure this out?*

Anna: *First we were like making the numbers to 25. We kept adding 2 and for figure 25, it was 53.*

FDR: *Wait. So you kept adding all the way to 25?*

Anna: *Yeah.... Then we used our chart. Then finally we figured out that if we timesed by 2 the figures and plus 3, that would give us the answer.*

FDR: *Does that make sense? [Students nodded in agreement.] So what Anna was suggesting was that if you look at the chart here, Anna was suggesting that you multiply the figure number by 2, say, what's* 1×2*?*

Tamara: *2.*

FDR: *2. And then how did you [referring to Anna's group] figure out the 3 here?*

Anna: *Because we also timesed it with figure number 13.*

FDR: *What did you have for figure 13?*
Anna: *That was 29. And then* 13×2 *equals 26 plus 3.*
FDR: *Alright, does that work? So what they were actually doing is this. They noticed that if you look at the table, it's always adding by 2. You see this? [Students nodded.] They were suggesting that if you multiply this number here [referring to the common difference 2 by figure number, say figure number 1, what's* 1×2*?*
Students: *2.*
FDR: *Now what do you need to get to 5? What more do you need to get to 5? [Some students said "3" while others said "4."] Is it 4 or 3?*
Students: *3.*
FDR: *It's 3 more. So what is* 1×2*?*
Students: *2.*
FDR: *Plus 3?*
Students: *5. [The class tested the formula when* $n = 2$*, 3, and 25.]*

Year 2 Practice: Continued Use of Numerically-Driven CSGs and a Refinement in the Case of Decreasing Linear Patterns

In the Year 2 study, the students were once again involved in a teaching experiment that focused on linear patterning. While the first author observed that the students, in seventh grade, seemed to have remembered how to generalize patterns figurally (weak) and numerically (strong), results of our clinical interviews with a subgroup of ten students prior to the teaching experiment confirmed this observation.

In the classroom episode below, the students were asked to obtain an algebraic generalization for increasing and decreasing linear patterns in both figural and numerical forms. Emma and her group (with Dave below as a member) have been consistently applying the shared practice of generalizing numerically. However, Emma introduced her process of "zeroing out" in the case of decreasing linear patterns that resulted in a further refinement of the numerical generalizing process.

FDR: *Alright. So I have my* x *and my* y*. [FDR sets up a table of values consisting of the following pairs:* $(1, 17)$, $(2, 14)$, $(3, 11)$, $(4, 8)$, $(5, 5)$, $(6, 2)$*.] So what's the answer to this one?*
Dave: $y = -3x + 20$*. [FDR writes the formula on the board.]*
FDR: *This is always the problem, here [pointing to the constant 20]. Before we figure that out, how did you figure out the* -3*?*
Dave: *The difference between the ys, between the numbers.*
FDR: *So what's happening here [referring to the dependent terms]. Is this increasing by 3 or decreasing by 3?*
Students: *Decreasing by 3.*
FDR: *So if it's decreasing by 3, what's our notation?*
Students: *Negative.*

FDR: Alright, so negative 3. So this one is clear [referring to the slope]. Look at this. This one I get [the slope]. If you keep doing that [i.e., differencing], it's always true. That's why you have this. The difficult part is this [referring to the constant 20].

Emma raised her hand and argued as follows:

Emma: If you did a negative times a positive, it's gonna be a negative. So what I'd do is zero it out.
FDR: So what do you mean by zero out?
Emma: So like if it's -3 *times 1, that's* -3 *[referring to the product of the common difference* (-3) *and the first independent term (1)]. … So I'd zero out by adding 3.*
FDR: So you try to zero out by adding 3. So, what does that mean?
Emma: Coz a -3 *plus 3 equals 0.*
FDR: So what's the purpose of zeroing out?
Emma: So it's easier to add to 17. Coz if it's 0, all you have to do is add 17.
FDR: So you're suggesting if you're adding 3 here, if this is -3 *plus 3, that goes 0. So what do you do with the plus 3 here?*
Emma: Just remember it and write it down.
FDR: Suppose I remember it, adding 3. So how does that help me?
Emma: Then ahm it's easier to add to 17. So just add 17 [to 3 to get 20].

The class then tried Emma's method in a different example. The first author asked the class to first generate a table of values, and they came up with the following (x, y) pairs: $(1, 10)$, $(2, 8)$, $(3, 6)$, $(4, 4)$, $(5, 2)$. Using Emma's method, one student offered the general formula $y = -2x + 12$, where the constant 12 was obtained after initially adding the common difference and its opposite to get 0 (i.e., $-2 + 2 = 0$) and then adding 2 to the first dependent term to yield the constant value of 12 (i.e., $2 + 10 = 12$). The class then verified that the formula worked in any instance of the sequence. Finally, when the first author asked if there was a limitation to Emma's strategy, Emma quickly pointed out that "it only works for 1" (i.e., when the case of $n = 1$ is known) and that her method would fail when the initial independent term was any other number besides 1. Hence, the *fifth* mathematical practice that became taken-as-shared was generalizing numerically using Emma's zeroing out strategy, which was a further refinement of an institutional practice involving decreasing linear patterns.

Year 3 Practices: A Third Shift Back to Figural-based Generalization and the Consequent Occurrence of CSGs, CNGs, and DGs

Prior to the Year 3 teaching experiment on pattern generalization, the students explored activities involving multiplicative thinking. In the following episode, they obtained a mathematical expression for the two sets of figures shown in Fig. 6.

FDR: *So what mathematical expression corresponds to what you see here [referring to the top set]?*
Francis: *6 circles.*
FDR: *Yes, there are 6 circles but I want a mathematical expression that shows how you got 6.*
David: *Add them one by one.*
FDR: *Yes, you can certainly add them one by one. But are there other ways of getting 6?*
Eric: *2 times 3. [FDR writes the answer on the board.]*
FDR: *So what do you mean by* 2×3*, Eric?*
Eric: *It means 2 threes.*
FDR: *Uhum, 2 threes or we say 2 groups of threes. Does that make sense? [Students nod in agreement.] Okay, so now what expression works with the second item here [referring to the bottom set]?*
Salina: *Three groups of 6.*
FDR: *Uhum, and how do we write that using multiplication?*
Students: *3 times 6.*
FDR: *Times meaning what?*
Salina: *Groups of.*

For homework, the students were given similar figural tasks that required them to construct different mathematical expressions with a focus on articulating multiplicative relationships.

The following day, they worked in pairs to obtain a pattern generalization for the Fig. 3 pattern. Considering the fact that there were old and new participants in the Year 3 study, the first author and the classroom teacher saw to it that every table that had two pairs of students had at least one experienced student who could guide the remaining table members in setting up a direct formula. During the classroom discussion, the students offered three different constructive generalizations (Fig. 4) that the class then assessed for equivalence. In obtaining a pattern generalization, the students first addressed structural issues of what stayed the same and what changed from stage to stage. Then they found a multiplicative expression for those aspects or parts that changed from stage to stage and then added a number corresponding to the remaining parts that stayed the same. This process became the *sixth* mathematical practice that was taken as shared in class.

A further refinement in this figural-based strategy occurred when the students began to inspect a particular stage number in a pattern in terms of multiples of the stage number and then either added or subtracted a number corresponding to the remaining parts. For example, Che in Fig. 3 circled four groups of stage 1, four groups of stage 2, four groups of stage 3, ..., four groups of stage n, which justifies her use of the expression $4n$. Then she saw added 4 corresponding to the four corners, which led her to conclude that her formula made sense.

Findings and Discussion Part 5: Middle School Students' Capability in Justifying CSGs

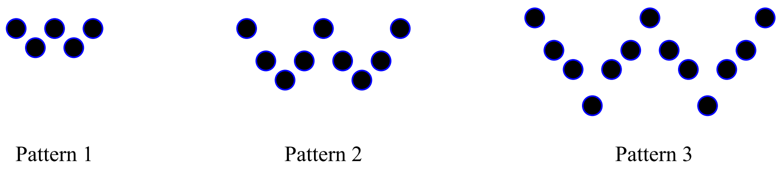
Findings in Our Study Results of the Year 1 teaching experiment indicate differing levels of competence in providing justifications. In particular, based on a follow-up clinical interview with nine students after the teaching experiment on pattern generalization, they justified in several different ways on five linear patterns, as follows:

1. They employed *extension generation* (7%), which involves using more examples to verify the correctness of a formula.
2. Some used a *generic case* (7%), which involves describing a perceived structural similarity in an imagined general instance.
3. Some employed *formula projection* (22%), a figural-based explanation that involves demonstrating the validity of a direct formula as it is seen on the given figural stages.
4. Some used *formula appearance match* (71%), a numerical-based explanation that involves merely fitting the formula onto the corresponding generated table of values that have been initially drawn from the figural stages (Rivera and Becker 2009a, 2009b).

We also note that, in our study, because the students in Year 1 initially developed the emergent practice of figural-based generalizing, they were in fact constructing and validating their direct formulas at the same time. For example, Dung established and justified his direct expression $n + 4$ for the total number of white tiles in Fig. 14b as soon as he saw "the number of white [square tiles] in the middle plus [the] 4 white [tiles] on the sides." Also, Che, Deb, and Nora established and justified their direct expressions, $n \times 2 + 2$, when they perceived "two grey [squares] plus the two squares on both sides [in a given figural stage]." All four students came up with their justifications above after empirically verifying them on several extensions and then either employing formula projection or imagining a generic case that highlights the invariant properties common to all stages. The formula appearance match was used only later after the class developed the emergent practice of generalizing numerically.

When the students in our Year 1 study fully appropriated the above numerical strategies in establishing CSGs, as exemplified in the thinking of Anna and Emma above in relation to the Fig. 14b pattern, we observed a shift that took place from a figural to a numerical mode of generalizing among them. In fact, in both the pre- and post-clinical interviews in the Year 2 study, very few (about 19%) initiated a figural-based approach with most of them developing numerical-based CSGs (about 82%). Consequently, the shift affected their capacity to justify algebraic generalizations correctly on the basis of faulty responses that used either formula projection or formula appearance match. For example, Dung, in two clinical interviews when he was in sixth grade, primarily established and justified his generalizations figurally and oftentimes with the use of a generic example. However, in two clinical interviews when he was in seventh grade, Dung primarily established his generalizations

W-Dot Sequence Problem. Consider the following sequence of W-patterns below.



A. How many dots are there in pattern 6? Explain.
B. How many dots are there in pattern 37? Explain.
C. Find a direct formula for the total number of dots D in pattern n. Explain how you obtained your answer. If you obtained your formula numerically, explain it in terms of the pattern of dots above.
D. Zaccheus's direct formula is as follows: $D = 4(n + 1) - 3$. Is his formula correct? Why or why not? If his formula is correct, how might he be thinking about it? Which formula is correct: your formula or his formula? Explain.
E. A certain W-pattern has 73 dots altogether. Which pattern number is it? Explain.

Fig. 16a W-dot pattern task

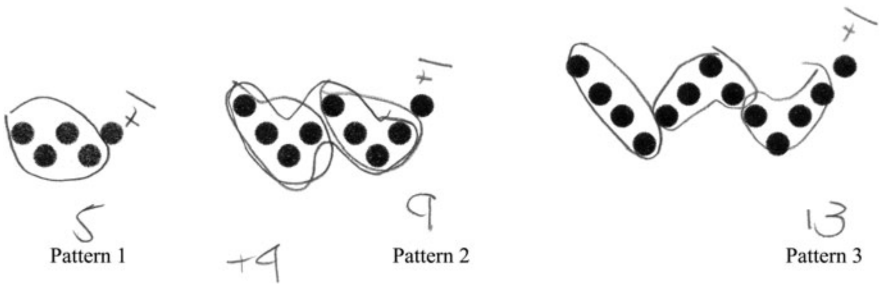


Fig. 16b Anna's figural justification of the W-dot pattern in Fig. 16a

numerically and justified inconsistently using formula projection. An example of a faulty argument that uses formula appearance match is exemplified in the thinking of Anna who first developed the generalization $D = n \times 4 + 1$ numerically for the figural pattern in Fig. 16a. When she was then asked to justify her formula, she circled 1 group of 4 circles, 2 groups of 4 circles, and three groups of 4 circles in patterns 1, 2, and 3, respectively, beginning on the left and then referred to the last circle as the y-intercept (Fig. 16b). As a matter of fact, in the post-interview in Year 2, only three of the eight students saw the sequence in Fig. 16a in the same way Dung perceived it (Fig. 16c).

Discussion The phenomenological shift from the figural to numerical modes in establishing generalizations involving figural linear patterns among our middle school

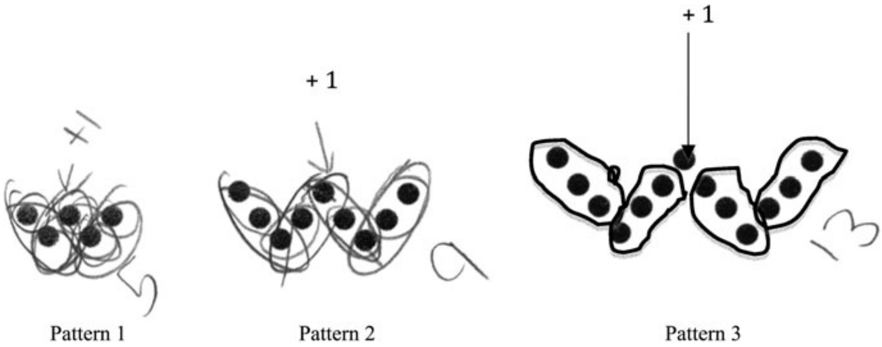


Fig. 16c Dung's figural justification of the pattern in Fig. 16a

students in the first two years of the study is not uncommon in empirical accounts of cognitive development. Induction studies in developmental psychology have demonstrated shifts in students' abilities to categorize (from perceptual to conceptual; from object- or attribute-oriented to relation-oriented, etc.). Also, Davydov (1990) has noted similar occurrences of change on the basis of his work on generalization with Soviet students, including his critique of mathematics instruction that seems to favor one process over the other.

Based on the empirical data we collected in Years 1 and 2, the shift from the figural to the numerical could be explained initially in terms of the predictive and methodical nature of the established numerical strategy (as exemplified by Anna's group thinking relative to the Fig. 15 pattern). That is, the students found them to be compact and easy to use particularly in far generalization tasks that asked them to determine an output value for a large input value. What was difficult with figural strategies, which could be dispensed with the established numerical strategy, was the *cognitive perceptual distancing* that was necessary in order to: (1) figurally apprehend and capture invariance in an algebraically useful manner; (2) selectively attend to aspects of sameness and differences among figural stages and; (3) create a figural schema or a mental image of a consistent generic case and then transform the schema or image into symbolic form. In terms of Radford's (2006) definition of algebraic generalization of a pattern—grasping of a commonality, applying the commonality to all the terms in the pattern, and providing a direct expression for the pattern—the almost, albeit not fully, automatic process of numerical generalizing requires only a surface grasp of a commonality (i.e., a common difference in the case of a linear pattern) that would then be used to set up a direct expression. In particular, when the students surfaced a commonality among stages in a numerical generalizing process involving linear patterns, most of them did not even establish it figurally since the corresponding numerical representation was sufficient for their purpose.

In articulating our argument of a figural-to-numerical shift in mode of generalizing in the first two years of our study, we have already noted how most of them could correctly establish CSGs numerically but had difficulty justifying them. We also discussed how some of them employed formula projection in an inconsistent (faulty) manner. Another significant source of difficulty in justifying CSGs was the students' misconstrual of the multiplicative term in the general form $y = mx + b$ for linear patterns. Toward the end of the Year 1 teaching experiment, they would oftentimes express their algebraic generalization in the form $O = n \times d + a$, where the variable O refers to the total number of objects being dealt with (like matchsticks, circles, squares, etc.), n the pattern number, d the common difference, and a the adjusted value. For example, the general form for the pattern sequence in Fig. 1 is $T = n \times 3 + 1$. The students would then justify their formula by locating n groups of 3 matchsticks respecting invariance along the way. In the Year 2 study, they learned more about the commutative property, which then encouraged them to write all their generalizations in the equivalent form $O = dn + a$. This became a source of confusion among some of them because they interpreted the expressions $n \times d$ and $d \times n$ as referring to the same grouping of objects. For example, in the clinical interviews that we conducted immediately after the Year 2 teaching experiment, some of those who wrote the form $D = 4n + 1$ for the sequence in Fig. 16a justified its validity by looking for 4 groups of, say, 2 circles in pattern 2 when, in fact, they should have been looking for 2 groups of 4 circles. Thus, the algebraic representation proved to be especially confusing among those who established their generalizations numerically because of their misinterpretations involving some of the mathematical concepts and properties relevant to integers (such as the commutative law for multiplication).

The final shift in Year 3, from numerical to figural mode of generalizing, as a matter of fact, settled the above issues. Because the students understood the relationship between multiplicative thinking and grouping relations, they reinterpreted their pattern structural analysis in terms of how grouping could be accomplished so that it is stable and consistent across stages. For example, in Fig. 3, Che noted the four corner squares that stayed the same. She also saw stability in grouping the middle parts on all four sides from stage to stage. Hence, her direct formula, $W = 4n + 4$, captured her figural interpretation of the structure that she saw in Fig. 3.

Findings and Discussion Part 6: Middle School Students' Capability in Constructing and Justifying CNGs and DGs

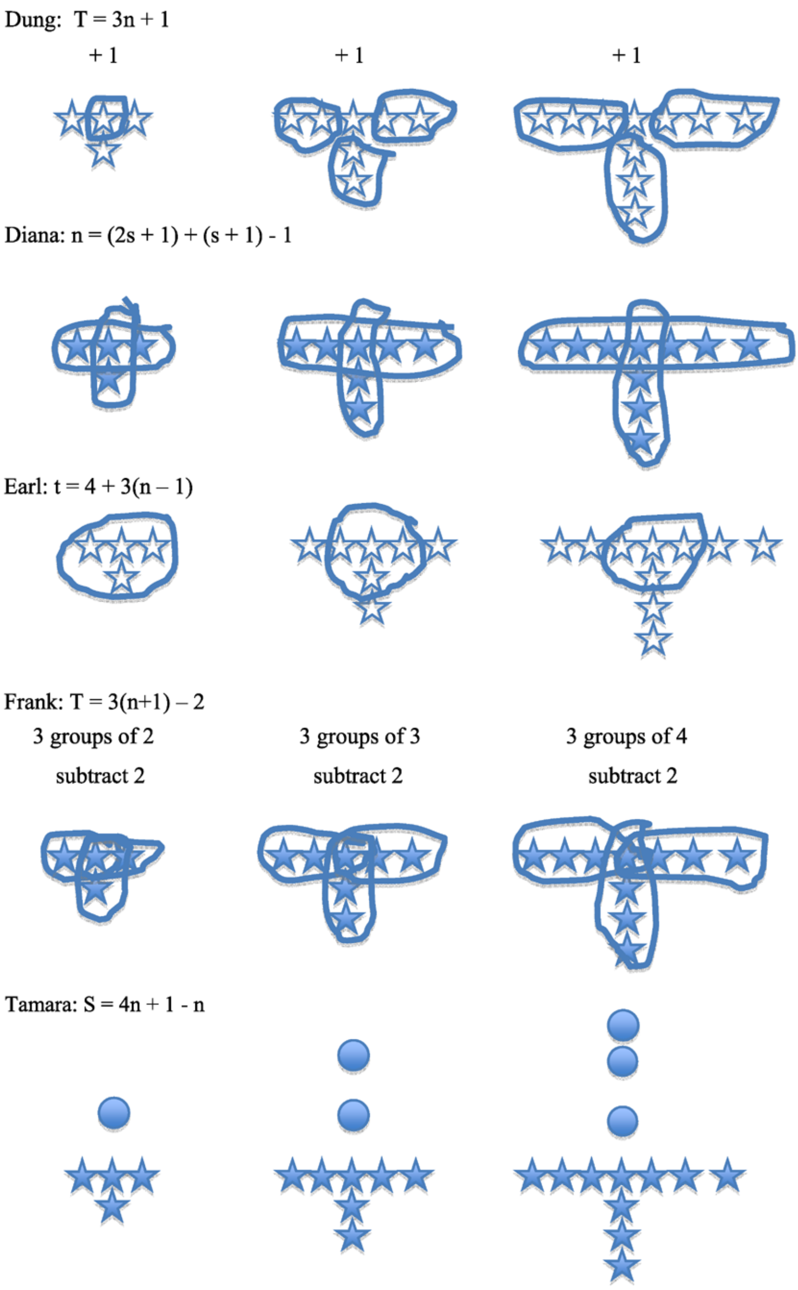
Findings in Our Study Considering the results drawn from our longitudinal work (and, in fact, relevant patterning studies discussed in this paper), we can conclude with sufficient sample that the task of establishing and justifying CNGs and DGs could be both easy and difficult for middle school students. For most students, competence in pattern generalization that leads to a CNG and DG could be considered as an effect of acquired knowledge and experience. We have found that individual and classroom-generated practices on pattern generalization with minimal scaffolding from the teacher, while helpful in many simple cases of linear patterns, appear limited in many respects. In our three-year study, the first two years in which the students were numerically driven to producing CSGs constrained them from obtaining more complex and equivalent generalizations for the same pattern. The numerical method of table differencing assisted in simplifying the process of constructing direct formulas, however, it had a negative effect on the students' ability to justify. We note as well the limited form in which such formulas took shape at least in the case of linear figural patterns, that is, they were oftentimes CSGs.

In Year 3 of the study, when the students acquired knowledge of multiplicative thinking and found ways to link such thinking in patterning activity, the resulting generalizations they produced and justified included CNGs and DGs. Results of the Year 3 clinical interviews with fourteen students after the teaching experiment on patterning and generalization show continued use of CSGs (100%), then DGs (86%), and finally CNGs (36%). Dina and Dave in Fig. 4 constructed and justified two equivalent CNGs for the Fig. 3 pattern. Figure 17 shows the generalizations of five students on the *T Stars Pattern* that ranged in complexity from CSGs to CNGs to DGs.

We discuss briefly the nonlinear pattern generalization of Diana, 7$^{\text{th}}$ grader from Cohort 2, whose pattern of growing segment-triangles is shown in Fig. 19 on a free construction task in Fig. 18 that was given after a teaching experiment on pattern generalization. We should point out that Diana ignored the differences in lengths of the diagonal and horizontal line segments, a fact that applies to a significant number of students who saw segment length as unimportant on this task. Despite that fact, Diana clearly identified an underlying structure in her growing pattern. Hence, assuming all segments are equal, her stage 1 triangle consists of two diagonal segment-sides, a horizontal base that has two segments, and with no interior segments. She then doubled each segment in stage 1 so that in stage 2, each diagonal side has two segments, the horizontal base has four segments, and two interior horizontal segments. She then circled in two colors to distinguish the groupings she was counting, one the interior horizontal segments, and the other, the outer segments on the perimeter of the triangle. In her written description of what to her stayed the same and what changed, she wrote:

> Number 1 [the original triangle] will stay in all of them. The $x(x-1)$ is for the lines in the middle of the triangle. The $+4x$ is for the triangle borders. It's really short for $2x + 2(x)$. But it was pretty much the same.

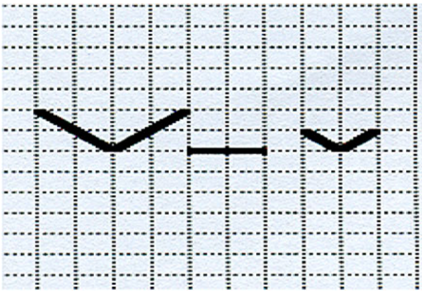
To illustrate, Diana counted the interior horizontal segments of her growing triangle pattern as: 1 group of 0 segment in stage 1; 2 groups of 1 segment in stage 2; 3 groups of 2 segments in stage 3; 4 groups of 3 segments in stage 4; 5 groups of 4 segments in stage 5 *leading to the expression* $n(n-1)$. Then she counted the segments on the perimeter of the growing triangle in two parts. Part A pertains to the two diagonal sides of the growing triangle: 2 groups of 1 segment in stage 1; 2 groups of 2 segments in stage 2; 2 groups of 3 segments in stage 3 *leading to* $2n$. Part B pertains to the base of the growing triangle: 2 groups of 1 segment in stage 1; 2 groups of 2 segments in stage 2; 2 groups of 3 segments in stage 3; 2 groups of 4 segments in stage 4 *leading to* $2n$. Clearly, central to her pattern generalization was her understanding of multiplicative thinking that enabled her to count in ways that corresponded to how she was circling the parts of her figures. Finally, she simplified her pattern of $L = n(n-1) + 4n$ to $L = n^2 + 3n$.



n	1	2	3	4	5
S	4	7	10	13	16

Fig. 17 Year 3 students’ work on the T stars pattern in Fig. 8a

From the following three figures below, pick at least two figures to create a pattern sequence of five stages. Use the attached grid paper to draw your four additional stages.



1. What stays the same and what changes in your pattern?
2. Obtain a generalization for your pattern either by describing it in words or by constructing a formula. How do you know that your generalization works?

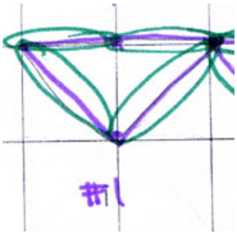
Fig. 18 Semi-free construction task

Discussion Central to the students' success in the Year 3 study was the sociocultural mediation that took place in the context of activities that encouraged them to explicitly engage in multiplicative thinking. When they began to see the significance of multiplicative thinking on matters that involve grouping and invariance in relation to patterning activity, their pattern generalization further progressed in ways that could not be simply done in the case of the numerically driven method of table differencing. We should note, however, that the students interviewed by the end of the Year 2 study were all successful in justifying given DGs. But their success was task-sensitive with some of them providing correct justification in one task and then an incorrect justification in some other task.

For example, results of the two clinical interviews in our Year 2 study separated by a teaching experiment show that almost all the students had more difficulty dealing with the Fig. 16a pattern than the Fig. 1 pattern. Results of the clinical interviews prior to the teaching experiment show only one student correctly justifying a DG in the case of the Fig. 16a pattern and six students in the case of the Fig. 1 pattern. Further, all students interviewed after the teaching experiment were able to justify the DG for the Fig. 1 pattern, but only six students in the case of the Fig. 16a pattern. Thus, it seems that some overlaps in a deconstructive generalization task are easier to see than others. For example, the students above found it easier to see overlaps among the shared adjacent sides of the squares (Fig. 1) than the shared interior vertices in a W-dot formation (Fig. 16a).

In a reported study by Steele and Johanning (2004), their middle school participants found DGs difficult at least in the context of their teaching experiment. The authors asked eight U.S. 7th graders to generalize five linear and three quadratic problem situations that pertained to growth, change, size, and shape. Their results show that, in the case of tasks that contained figural stages, only three students were

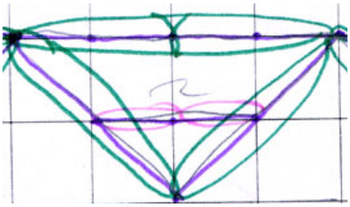
Stage 1



Interior horizontal segements:
1 group of 0

Perimeter segments:
Diagonals: 2 groups of 1
Horizontal (Top Base): 1 group of 2

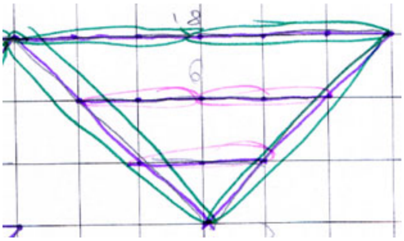
Stage 2



Interior horizontal segements:
2 groups of 1

Perimeter segments:
Diagonals: 2 groups of 2
Horizontal (Top Base): 2 groups of 2

Stage 3



Interior horizontal segements:
3 group of 2

Perimeter segments:
Diagonals: 2 groups of 3
Horizontal (Top Base): 2 groups of 3

Stage n

Interior horizontal segements:
n group of $(n - 1)$

Perimeter segments:
Diagonals: 2 groups of n
Horizontal (Top Base): 2 groups of n

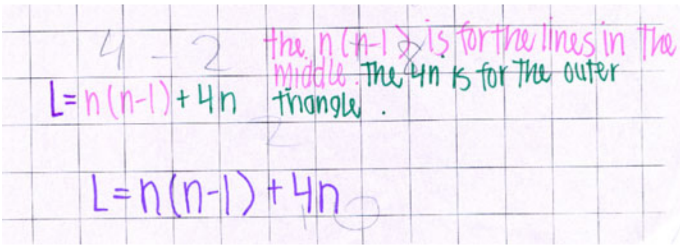


Fig. 19 Diana's pattern generalization in relation to Fig. 18 task

able to establish and justify DGs (or "well-connected subtracting-out schemas"). The notion of multiplicative thinking was not used in their teaching experiment.

In the Year 1 clinical interviews after the teaching experiment on pattern generalization, none of them were found to be capable in establishing and justifying a DG. Further, in clinical interviews in Year 2 after the teaching experiment on pattern generalization, none of them were capable of constructing DGs. However, there was a marked gain in their ability to interpret and justify a stated DG (with a success rate of 50% to 100% in pre- and post-clinical interviews, respectively). All the students interviewed saw the overlapping sides in the adjacent squares pattern in Fig. 1 and six could see the overlapping interior vertices in the case of the W-dot pattern in Fig. 16a. We further note that despite their success in justifying, seeing an overlap was not immediate for most of the students; it became evident only after they had initially employed formula appearance match followed by formula projection. Of course, some students employed formula projection incorrectly. For example, Jana justified the subtractive term 3 in Zaccheus's DG (item D in Fig. 16a) in the following manner:

FDR: So if you look at this [referring to the formula (item D, Fig. 16a) in which Jana substituted the value of 2 for n], this one's four times two plus one, right? And then minus 3. So how might he be looking at 4 times 2 plus 1 and then minus 3?

Jana: Uhum, the 2 is for the pattern number.

FDR: Uhum. Because when Zaccheus was thinking about it, he said multiply 4 by $n + 1$ *and then take away 3. So how might he be thinking about it?*

Jana: Like it's gonna be 3 [referring to $2 + 1$*] and then it's gonna be 12 [referring to* 4×3*]. But I counted there's only 9, so he has to subtract 3.*

FDR: So how might he be doing that? Suppose I do this? [FDR builds pattern 2 with circle chips in which the three overlapping "interior" vertices are colored differently.]

Jana: Hmm, like he has this group of 4 [Jana sees only two sides in W in pattern 2 with the top middle interior dot connecting the two sides. Hence, one side has 4 dots.].

FDR: Is there a way to see these 4 groups of 3 here [referring to pattern 2]?

Jana: Like he imagines there's 3 and he has to subtract 3.

FDR: So can you try it for other patterns? [Jana builds pattern 4.]

Jana: He has 1 group of 4. So there's 3 groups of 4 and he imagines 3 more [to form 4 groups of 4] and then he subtracts them [the three circles added].

FDR: So he imagines there's three more. But why do you think he would add and then take away?

Jana: Because there's supposed to be 4 groups of 4 and then you don't have enough of these ones [circles] so he adds 3. You add these ones.

Conclusion

This paper began with two broad questions that have guided the longitudinal study summarized in this work: What is the nature of the content and structure of generalization involving figural patterns among middle school students? To what extent are they capable of establishing and/or justifying more complicated generalizations? Various patterning studies that have been conducted at the middle grades level provide strong evidence that students' generalizations shift from the recursive to the closed, constructive form. In this article, we discussed in some detail at least three epistemological forms of generalization involving figural linear patterns, namely: CSG, CNG, and DG. The general forms are further classified according to perceptual complexity. CSGs are the easiest for most middle school students to establish and, thus, most prevalent. CNGs and DGs are relatively difficult and less prevalent. This classification scheme of generalizations emerged from detailed analyses of students' pattern generalization over three years. Also, it elucidates the content and structure of such generalizations.

We have also discussed how students' approaches to establishing generalizations are intertwined with their justification schemes. Further, results drawn from our longitudinal work show shifts in pattern generalization schemes among middle school students at least in the case of figural patterns. We note two consequences. *First*, we highlight changes in their representational skills and fluency, that is, from being verbal (situated) to symbolic (formal) and to figural (formal). *Second*, the phenomenological shifts affect the manner in which they justify their generalizations. We have documented at least four types of justifications, namely: extension generation; generic example use; formula projection, and; formula appearance match. The entry level of justification oftentimes involves generating extensions (i.e., calculating and/or producing more stages after the initial ones). Students who then generalize numerically without having a strong figural foundation are most likely to employ formula appearance match and use formula projection inconsistently. Students who understand multiplicative thinking in relation to figural patterning activity oftentimes employ formula projection but the success and validity of such formulas are relative to the associated structural analyses.

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Commentary on Part II

Bharath Sriraman and Kyeong-Hwa Lee

Introductory Remarks

In this commentary to the nine chapters in the cognitive section of early algebraization, we synthesize and critically discuss common themes found in them such as components of non-formal algebraic thinking, the purported dichotomy between arithmetic and algebra; meaningful arithmetic, and generalizing ability, among others using the frameworks of William Brownell, Ernst Haeckl and Jean-Baptiste Lamarck.

The nine chapters that comprise Part II of this book consist of 4 revised ZDM articles and 5 new chapters, which together explore the cognitive aspects of early algebraization. As spelled out in the preface of the first volume of the Advances in mathematics education series, "the purpose of a commentary is not only to elucidate ideas present in an original text, but... [t]o take them forward in ways not conceived of originally" (Sriraman and Kaiser 2010, p. vi). Therefore, our aim in this commentary is to synthesize common themes found in the nine chapters of this section and to discuss the significance of the ideas and claims made in the chapters through different theoretical lenses. We add that a commentary cannot occur *in the void* meaning that it needs to be anchored in what is already existent in the literature. So for the sake of better inter-textuality in the existent literature (Sriraman 2010) and preserving the integrity of what is already known in the field, we refer the reader to two related books released by Springer in the MEL[1] series, namely vol. 43 focused on *Educational Algebra* (Filloy et al. 2008) and vol. 22 which explored *Perspectives*

[1]Mathematics education Library Series, Springer.

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on School Algebra (Sutherland et al. 2001) that contain complementary perspectives to those found in this commentary and book. The critically reflective question we asked ourselves is whether these nine chapters really represented *advances* in the domain of early algebraization or whether they were simply a regurgitation or recycling of old ideas in new clothing?

Early Algebraization Versus Meaningful Arithmetic

There is no rigid definition of early algebraization per se found in the literature, which is acknowledged by the editors of the book (Cai and Knuth). Instead the term is used to refer to algebraic thinking initiated from the early grades (elementary school) via the use of student's informal knowledge, different modes of representations, pattern activities that lead to generalization, and building on natural linguistic and cognitive mechanisms by reflection, verbalization, articulation and sense making (Greer 2008; Kaput 1999). The question that arises in our mind when reading these chapters that address cognitive aspects of early algebraization is whether they are really addressing the notion of "meaningful arithmetic" as proposed by William Brownell (1895–1977)? Amongst mathematics educators there is consensus that the traditional separation of arithmetic from algebra hinders the development of algebraic thinking in the later grades. Hence it makes sense to push for developing algebraic ideas in the earlier grades through various activities such as pattern finding and relationships in the curriculum without the formalism of notation or symbols, which collectively is termed *early algebraization.* We do not dispute this, however are these ideas really advancing our field, or are they a recycling of previous work? This is the focus of our discussion in this section.

Brownell (1947) was the chief spokesperson for the "meaningful" arithmetic. Meaningful arithmetic refers to instruction which is deliberately planned to teach arithmetical meanings and to make arithmetic more sensible to children through its mathematical relationships. Brownell categorized the meanings of arithmetic into the following groups.

1. A group consisting of a large list of basic concepts. For example: meanings of whole numbers, of fractions, ratios and proportions etc.
2. A second group consisting of arithmetical meanings which includes understanding of fundamental operations. Children must know when to add, subtract, multiply, and divide. They must also know what happens to the numbers used when a given operation is performed.
3. A third group of meanings consisting of principles that are more abstract. For example: relationships and generalizations of arithmetic, like knowing that 0 serves as an additive identity, the product of two abstract factors remains the same regardless of which factor is use as a multiplier, etc.
4. A fourth group of meanings that relates to the understanding of the decimal number system, and its uses in rationalizing computational procedures and algorithms (Brownell 1947).

Meaningful arithmetic is "deliberately planned to teach arithmetical meanings and to make arithmetic sensible to children through its mathematical relationships". Brownell emphasized that learning arithmetic through computations required continuous practice. He suggested that teaching meaningful arithmetic would reduce practice time, encourage problem solving, and develop independence in students and remarked that there was lack of research in 1947, on teaching and learning arithmetic meaningfully. He also doubted that quantitative research could address questions in the area of teaching and Learning arithmetic. One could say that Brownell was clairvoyant for his time, emphasizing qualitative research in the age of behaviorism and he was noted for his use of a variety of techniques for gathering data, including extended interviews with individual children and teachers, as well as his careful, extensive and penetrating analyses of those data (Kilpatrick 1992).

Sixty three years after Brownell's recommendations, we see ample research evidence in the nine chapters of this part that the algebra underpinning the arithmetic operations can be made accessible to students without the need for sophisticated notation that impedes understanding. Two chapters report on the results of longitudinal research projects. For instance, the chapter by Britt and Irwin on *Algebraic thinking with and without algebraic representation: a pathway to learning* reports on substantial gains in introducing algebraic thinking within arithmetic in the New Zealand Numeracy Project with 4–7 year olds, with the gains remaining in a follow up large scale study with the same students at the age of 11–12 on a 21-item test consisting of various algebraic properties. Similarly Cai, Moyer, Wang and Nie report the findings of the LieCal[2] Project on the algebraic development of middle school students (grades 6–8) exposed to reform based curricula (CMP) versus a traditional curricula (non CMP). They found that CMP students performed better on generalization tasks than their peers in the traditional (non CMP) track. The CMP curriculum used a functional approach and emphasized conceptual understanding whereas the non CMP curriculum took a structural approach and emphasized procedural understanding. Interestingly enough in this rather massive study, both groups performed equally well on equation solving!

Generalized Arithmetic, Generalizing, Generalization

Amongst mathematicians, algebra is typically understood as generalized arithmetic and some would qualify this characterization by saying it is *meaningful* generalized arithmetic. In other words there is a significant shift (abstraction) from talking about basic arithmetic operations concretely, to talking about operations arbitrarily. There is an even greater shift (abstraction) when one views numbers as a special case of polynomial evaluations. However mathematics educators would object to such characterizations as being top-down, i.e., viewing particulars through general lenses as opposed to discovering the general via particulars (Mason 1992;

[2]Longitudinal Investigation of the Effect of Curriculum on Algebra Learning.

Sriraman 2004) and argue that algebraic thinking which fosters the elements of abstraction and generalization inductively from the early grades through contextual and rich mathematical activities is more valuable than the deductive approach.

The theme of generalization and generalizing ability recurs in the chapter by Cooper and Warren (*Years 2 to 6 students' ability to generalize: models, representations and theory for teaching/learning*), and the chapter by Rivera and Becker (*Formation of pattern generalization involving linear figural patterns among middle school students: Results of a three-year study*). The former chapter by Cooper and Warren is the only one in this entire part to cite the work of Dienes (1961) and push his ideas of multiple embodiments fostering abstraction and generalization, by exploring the interrelations between generalization and verbal/visual comprehension of context, and arguing for the value of communicating commonalties seen across different representations. The latter chapter delineates the different nuances of generalization such as constructive versus deconstructive generalizations, and the role of sociocultural mediation in fostering/facilitating verbalization of generalizations in teaching experiments. The different patterning activities found in these chapters serve well to validate the claims made by these authors.

Many of the chapters in the cognitive part try to move beyond or work their way around the debate that algebra is generalized arithmetic. However Radford's chapter on *Grade 2 students' non-symbolic algebraic thinking* tackles this issue head on by trying to draw a clear distinction between what is arithmetic and what is algebraic. This is accomplished by focusing on non-symbolic means of expression, going back to the ideas of other ways (e.g., linguistic, gestural, figural) of expressing algebraic notions as opposed to expressing them symbolically. This chapter brings to the foreground the need to have a historical perspective on the development of mathematical ideas (algebraic or otherwise). Diophantus' *Arithmetica* is a misnomer and deals with solving algebraic equations with integer co-efficients. Does that mean algebra as we define it today preceded arithmetic or that they developed concurrently, just as the notion of Integrals preceded the rigorous development of convergence of sequences and limits but the approximation of areas and volumes contained the intuitive notions that were later formalized? We leave these questions for the reader to ponder over. But we foray briefly into the role of history of mathematics in mathematics education before we conclude our commentary. In doing so, we highlight the need to change the dominant discourses present in arithmetic-algebra dichotomy, and adopt a historical-cultural perspective in addition to a strictly cognitive perspective. The commentary offered by Balacheff (2001) on *Perspectives in school algebra* offers another perspective on the didactical dilemma/distinction between symbolic arithmetic and algebra.

From Haeckel to Lamarck to Early Algebraization

The emphasis on the important role of the history of mathematics in mathematics education research is one that has been sporadically addressed by the community. Furinghetti and Radford (2002) traced the evolution of *Haeckel's (*1874*)* *law of recapitulation* from the point of view that parallelism is inherent in how mathematical ideas evolve and the cognitive growth of an individual (Piaget and Garcia 1989). In other words the difficulties or reactions of those who encounter a mathematical problem can invariably be traced to the historical difficulties during the development of the underlying mathematical concepts. The final theoretical product (namely the mathematical theorem or object), the result of the historical interplay between phylogenetic and ontogenetic developments of mathematics, where phylogeny is recapitulated by ontogeny, has an important role in pedagogical considerations (Bagni et al. 2004; Furinghetti and Radford 2002; Sriraman and Törner 2008). Psychological constructs as well as the study and formation of intellectual mechanisms are not as tenable as the clearly dated and archived transformations of mathematics in its historical development. Further, the apparent free use of Haeckel's recapitulation theory as the link between the psychological and historical domains is in need of re-examination. It is a well known fact that Haeckel's law in its original form was rejected by the community of biologists and has been transformed numerous times by some, over the last 100 years to better explain the relationship between phylogeny and ontogeny in different species. However in mathematics education we are referring to psychological recapitulation or the use of recapitulation metaphors to explain the evolution of mathematical ideas. A neo-Lamarckian perspective needs to be introduced into the recapitulation discussion for the following reasons. Recapitulation cannot be applied or transposed directly to the study of didactical problems because it does not take into account the influence of experience (or more broadly culture). Just as Jean-Baptiste Lamarck proposed in vain to his peers in 1803, that hereditary characteristics may be influenced by culture, we need to take into account how culture influences the mutation of historical ideas. Gould (1979) wrote that

> Cultural evolution has progressed at rates that Darwinian processes cannot begin to approach. . . [t]his crux in the Earth's history has been reached because Lamarckian processes have finally been unleashed upon it. Human cultural evolution, in strong opposition to our biological history, is Lamarckian in character. What we learn in one generation, we transmit directly by teaching and writing. Acquired characters are inherited in technology and culture. Lamarckian evolution is rapid and accumulative. . .

The teaching and learning of mathematics bears strong evidence to this Lamarckian nature. Indeed, what took Fermat, Leibniz and Newton a 100 years is taught and often digested by students in one year of university Calculus. Any higher level mathematics textbook is a cultural artifact which testifies to rapid accumulation and transmission of hundreds (if not thousands of years) of knowledge development. So, evolutionary epistemologists have now begun to accept the fact that for humans, cultural evolution in a manner of speaking is neo-Lamarckian (Callebaut 1987; Gould 1979).

The neo-Lamarckian perspective becomes evident in the research reported by Izsak in his chapter on *Representational competence and algebraic modeling*. In this chapter, the reader confronts students with a significant and complex "substrata of knowledge" and criteria developed from the culture of learning they were previously exposed to that resulted in them inventing their own private inscriptions and criteria for evaluating representations. Developing notational competence in students, i.e., the ability to adapt their inscriptions to fit the external representations they are confronted with, and progress to the necessity for uniformity in notation for the purposes of communication is an inference we draw from this chapter. The history of algebra as seen in the development of the theory of equations shows that the notation (or private inscriptions) developed by Galois on general methods for the solvability of equations by radicals, took his peers a significant amount of time to decipher, nearly 15 years after his death (until 1846), when Louiville published it in his journal commenting on Galois' solution, "...as correct as it is deep of this lovely problem: Given an irreducible equation of prime degree, decide whether or not it is soluble by radicals".

In the chapter by Ellis on *Algebra in the Middle School*, the informal or cultural notion of comparing quantities is used to scaffold the building of quantitative relationships and the sophisticated idea of covariation in younger students. In spite of creating relevant, contextual and quantitatively rich situations, Ellis reports that the "students unique interactions with an interpretations of real world situations remind us that these contexts are not a panacea. Introducing a quantitatively rich situation does not guarantee that students build quantitative relationships... [s]tudents may focus on any number of features in a problem situation... [t]herefore teachers play an important role in shaping a classroom discussion, ..."

Knuth et al. in their reprint of the 2005 ZDM article suggest that students "pre-algebraic" experiences are crucial in laying the foundation for the study of more formal algebra. These authors view the middle school grades as the link from early algebraic reasoning to more complex and abstract reasoning. Five years later, the studies from New Zealand, Australia, Canada and the U.S. reported in this part, indicate that algebraic thinking can be cultivated from the very early grades on if teachers are cognizant of non-symbolic modes of reasoning. It is a testament to our development as a field that the seemingly divergent ideas of Brownell, Haeckel and Lamarck converge in our understanding of early algebraization, which can no longer be viewed as a neologism but a clearly defined term!

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Part III
Instructional Perspective

Preface to Part III

Eric Knuth and Jinfa Cai

Although curricula can provide elementary and middle school students with opportunities to develop their algebraic thinking, teachers are arguably the most important influence on what students actually learn. Thus, the success of efforts to develop students' algebraic thinking prior to formal algebra courses rests largely with the ability of teachers to foster the development of such thinking. The chapters in this part present a diverse set of perspectives regarding early algebra instruction and ways of supporting teachers' efforts to foster the development of students' algebraic thinking. The chapters range in focus from teachers' algebra content knowledge to teachers' use of classroom opportunities to foster algebraic thinking to international comparisons of instructional approaches with regard to algebra.

Ellerton and Clements present data from a study that examined teacher education students' knowledge about equations and inequalities. Although their results are disheartening, especially given that the teachers were seeking endorsement to become middle school mathematics specialists, the authors do offer suggestions gleaned from their work regarding the teaching and learning of algebra. Also, focusing on teacher education albeit with an experienced teacher rather than pre-service teachers, Koellner and her colleagues document the influence of a particular professional development model (Problem-Solving Cycle model) on one teacher's algebra instruction. Two years worth of data provide the basis for their description and illustration of changes in the teacher's instructional practice, and also serve to link the changes to emphases in the professional development program. In the chapter by Cusi and colleagues, they focus on the importance of enculturation of early algebra teachers, emphasizing the critical role that teacher reflection on the processes of teaching and learning play in their enculturation. The authors also discuss

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the importance of viewing early algebra from both linguistic and socio-constructive perspectives, and how such perspectives can reshape the nature of teachers' enculturation. Finally, Smith's chapter presents a comparison of instructional approaches taken by teachers as they teach a lesson about simultaneous equations. In particular, she details the qualitative differences in the instruction between U.S. and Japanese teachers, contrasting the procedural-driven approach of U.S. teachers versus the relationship-driven approach of Japanese teachers.

The final three chapters in this part focus on ways in which teachers can build upon students' thinking in ways that can promote the development of their algebraic thinking. For example, Empson, Levi, and Carpenter suggest that students' work with fractions can also serve as an opportunity for teachers to enhance their students' understanding of algebraic structure. They argue that many of the strategies used by students for fraction problems are often based on the same mathematical relationships that are essential in formal algebra. In the chapter by Jing and colleagues, they describe a teaching approach commonly used in China for helping students learn to represent and operate of algebraic equations. Their illustrations also underscore the important role that whole class discussions play in facilitating students' learning. Finally, Moselli and Boero's chapter present a model for understanding students' difficulties in algebra that is based on Habermas' construct of rational behavior. The authors apply their model to students' uses of algebraic language in the context of solving mathematical modeling problems, demonstrating how the model provides insight into students' algebraic thinking as well as guidance for teachers to understand their students' algebraic thinking.

The seven chapters that comprise this part highlight the important role that teachers play in developing students' algebraic thinking—the need for teachers to possess an adequate understanding of algebra themselves and the need for teachers to be able to recognize and capitalize on classroom opportunities to foster students' algebraic thinking. The authors have provided not only insight into both the challenges and opportunities with which teachers are presented, but also valuable suggestions for supporting teachers as well as for continued research.

Prospective Middle-School Mathematics Teachers' Knowledge of Equations and Inequalities

Nerida F. Ellerton and M.A. (Ken) Clements

Abstract This chapter describes an investigation into the algebra content knowledge, in relation to elementary equations and inequalities, of 328 US teacher-education students who were seeking endorsement to become specialist middle-school mathematics teachers. Most of these prospective teachers had done well in high school mathematics and were taking their last algebra course before becoming fully qualified teachers of mathematics. After reviewing the scant literature on the teaching and learning of quadratic equations, and of linear inequalities, we summarize a pencil-and-paper instrument, developed specifically for the study, which included linear and non-linear equations and inequalities. The students were also asked to comment, in writing, on a "quadratic equation scenario" that featured four common errors in relation to quadratic equations. Data analysis revealed that hardly any of the 328 students knew as much about elementary equations or inequalities as might reasonably have been expected. Brief details of a successful intervention program aimed at improving the pre-service teachers' knowledge, skills and concepts relating to quadratic equations and inequalities are given, and implications of the findings for mathematics teacher education and, more generally, for the teaching and learning of algebra, are discussed.

The Context

It has been well established by research that, in many nations, college students who are preparing to become specialist teachers of mathematics at the middle-school level often have unsatisfactory knowledge with respect to the algebra content that

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they might be expected to teach (see, e.g., de Castro 2004). There is some evidence that in the United States of America the problem is particularly acute. Schmidt et al. (2007), for example, reported a large study which showed that US prospective middle-school teachers' knowledge about "functions"—an important theme in middle-school and high school mathematics—was low when compared with the knowledge of corresponding cohorts in Taiwan, South Korea, Bulgaria, and Germany. Indeed, Schmidt et al. claimed, the US performance "lagged almost three-fourths of a standard deviation below the international mean" (p. 1).

Over the past five years we (Ellerton and Clements) have studied the algebra content knowledge and attitudes toward algebra of mathematics education students enrolled in an "Algebra for Teachers" second-year course (hereafter denoted AT2) at a large US university. We have gathered and analyzed data on what mathematics the prospective middle-school teachers had studied at school and at college, and have analyzed, quantitatively and qualitatively, their responses to middle-school algebra tasks.

Kieran (2007) emphasized that unless students "come to realize that algebra is an arena of sense-making and that they can arrive at rules that will permit them to obtain the same results as their teacher or classmates, they will never be able to control their algebraic work" (p. 732). In this chapter we show how carefully developed tasks were used to identify prospective middle-school teachers of mathematics who were struggling with elementary equations and inequalities. We also outline what we did to improve the situation.

Mathematical Considerations Relating to the Teaching and Learning of Equations and Inequalities

This chapter describes an investigation into 328 prospective teachers' thinking about equations and inequalities, especially (but not only) quadratic equations and linear and quadratic inequalities. In order to frame our subsequent discussion it will be useful to begin with an examination of what it means to "solve" an equation or an inequality. Henry Pollak, a distinguished applied mathematician with a demonstrated interest in mathematics education, stated, during an interview with Alexander Karp (2007), that during the late 1950s, as a consultant to one of the School Mathematics Study Group's (SMSG's) curriculum development teams, he found himself confronted with the fundamental question of what it meant to solve an equation or an inequality. Pollak stated that SMSG team members were surprised when they realized that there did not seem to be any universally accepted agreement on such a basic matter.

According to Pollak (see Karp 2007), the team that worked on SMSG's ninth-grade *Algebra* program agreed to adopt the definition that "to solve" an "open sentence" involving a variable (usually an equation, or an inequality) is to find all acceptable values of the variable that will make the sentence true. The word "acceptable" was important, because if x were constrained to being a real number then some apparently simple equations (like, for example, $x^2 + 4 = 0$) would have no solution. However, the open sentence $x^2 + 4 > 0$ would have infinitely many real-number solutions. One might say that if the *replacement set* for x is the set of real numbers (hereafter denoted R), then the *truth set* for $x^2 + 4 = 0$ would be $\varnothing$, the null set. On the other hand, with the same constraints operating, the truth set for $x^2 + 4 > 0$ would be R.

The SMSG team viewed equations and inequalities as part of the same conceptual framework. As team members, Allen et al. (1965), wrote:

> We write *open* [original emphasis] sentences which involve variables and for which the notion of a truth set becomes important. It is essential that the student consider both equations and inequalities as sentences, as objects of algebra with equal right to our attention. (p. 44)

The SMSG approach brought to the forefront what it meant to "solve" an equation or inequality. It emphasized the need to state the replacement set for the variable. It did not directly use the language of functions, but what the team members decided could easily have been expressed in function terminology if the team had deemed that necessary. We believe that the SMSG approach still provides a cohesive framework within which educational issues associated with the teaching and learning of equations and inequalities can be meaningfully discussed.

Before proceeding further, a brief note on the word "inequality" will be in order. In the 1960s—the high point of the New Mathematics era—some authors (e.g., Clements et al. 1967, p. 88) preferred the term "inequation" to "inequality." The argument in favor of the former was that one might refer to "$3 + 2 = 5$" as an equality, and to $3x + 5 = 0$ as an "equation"; and, if this were done, it would seem reasonable to regard a statement such as "$5 > 3 + 1$" as an inequality, and "$5 > 3x + 1$" as an inequation. Although we find the logic supporting the preference for the term "inequation" in relation to an open sentence like $5 > 3x + 1$ to be compelling, in this paper we bow to common curriculum usage in the United States (and in most other parts of the world), and call a sentence like $5 > 3x + 1$ an inequality.

The New Math(s) period gave way to a back-to-the-basics era, and the truth set approach developed by SMSG was forgotten (or, at best, put into the background). Since then, there has been a large amount of research on conceptions and misconceptions associated with what, and how, students think when they attempt to solve *linear* equations. There has been relatively little research, however, on student conceptions and misconceptions in relation to quadratic equations, or to linear and quadratic inequalities (Kieran 2007), and we focus on these aspects in this chapter.

Some writers (e.g., Tsamir et al. 1998) have argued that it is theoretically desirable to insist that equations and inequalities should be explicitly taught, from the outset, in the context of the development of the function concept. Boero and Bazzini (2004) have maintained, for example, that teaching inequalities without taking due account of the concept of function "implies a 'trivialisation' of the subject, resulting in a sequence of routine procedures which are not easy for students to understand, interpret and control" (p. 140). They added that "as a consequence of this approach, students are unable to manage inequalities which do not fit the learned schemas" (p. 140). Such an argument tends to be bolstered by references to the need for a stronger theoretical framing of algebra education relating to equations and inequalities. Often it is suggested that any such framing should emphasize the concept of function. The power and availability of modern technology is also likely to feature in arguments put forward by proponents of a function approach.

Like Pimm (1995), we do not accept the view that any decent theory relating to the role of equations and inequalities in school mathematics curricula must be linked from the outset to the function concept. For example, we see nothing wrong with fifth- or sixth-graders considering which natural numbers could replace $\square$ in the open sentences $2 \times \square + 3 = 15$ and $3 + 2 \times \square < 15$ so that true statements would be obtained. Such tasks would be both age- and curriculum-appropriate, and other age- and curriculum-appropriate open sentences could be devised for various groups of students at different grade levels, or of different ages. Furthermore, if the SMSG set-theoretical position for equations and inequalities is accepted, then those teaching elementary algebra courses to prospective middle-school teachers will need not only to be aware of, but also to understand thoroughly, that framework for equations and inequalities. From that perspective, the sign-chart method for inequalities summarized by Dobbs and Peterson (1991) could be valuable, provided it is carefully and meaningfully tied to the truth value approach.

Research by Vaiyavutjamai (2004) and by Blanco and Garrote (2007) has suggested that a question like "Which real numbers could replace x in the open sentence $3x - 4x^2 = 0$, so that a true statement would be obtained?" is best approached, at least initially, by the SMSG approach. With such an approach, $3x - 4x^2$ would be factored and the null factor law applied to find the truth set. From our perspective, and our experience, that method can lead to meaningful consideration of quadratic inequalities like $3x - 4x^2 > 0$.

In Karp's (2007) interview with Henry Pollak, Pollak made the following thought-provoking comment on relationships between the concepts of open sentence, equation and inequality:

> There are different kinds of sentences and you want to know the sets of numbers that makes these sentences true. And that is how equations and inequalities are all going to be done together. Each of these is an *open* [original emphasis] sentence because you have left *open* what x is. Each of these open sentences has a truth set, a set of x's that makes the sentence true. Maybe it's empty, that's possible, or it may be all x's, or it may be some particular collection, some set of x's. But you leave *open*, what it is you're trying to find. Now, what does it mean to solve an equation? Well, what you do is you find that there are a number of operations which do not change the truth set. You can add anything you want, you can multiply, but not by zero, for equations at least. So, solving an equation means carrying out operations which don't change the truth set. For how long? Until the truth set is obvious. That is, when you finally get to where it says $x = 3$ or $x > 5$, then you know that's the truth set. And since you have never changed the truth set as you went through, that's where it was to begin with. (p. 73)

Adopting an open-sentence approach with equations and inequalities need not interfere with the development of function concepts in learners, for such an approach can easily be linked to finding zeros of functions, or to reasoning about inequalities from graphs of functions (or relations). We believe that students should learn what it means to solve equations, and inequalities, *before* being given a formal introduction to the concept of function. Thus, for example, a ninth-grade student can learn to solve the quadratic equation $3x - 4x^2 = 0$ with a degree of understanding by factoring, using the null factor law, and thinking in terms of truth values for the open sentence. By contrast, most ninth-grade students would find it difficult to know what was going on in relation to the equation $3x - 4x^2 = 0$ by merely examining that part of the graph of the function $f : R \to R$, $f(x) = 3x - 4x^2$ shown on a graphing calculator. If a function approach is adopted too early then confusion can abound: for example, although the equation $3x - 4x^2 = 0$ is equivalent to (i.e., has the same truth set as) $4x^2 - 3x = 0$, the functions $f : R \to R$, $f(x) = 3x - 4x^2$ and $g : R \to R$, $g(x) = 4x^2 - 3x$ are certainly not the same.

In the study described in this chapter, teacher-education students who would soon become accredited middle-school mathematics teachers, were asked to state, for each of a given set of equations and inequalities, "all the real number(s) which could replace x to make the statement true." Although students seemed to recognize that they were being asked to "solve" the equations and inequalities, analysis revealed that some did not comprehend the expression "real number(s)," and most did not know how to find the solutions.

Student Misconceptions in Regard to Quadratic Equations

Although this chapter focuses on prospective middle-school teachers' conceptions and misconceptions with respect to equations and inequalities, it will be helpful here to summarize the findings of Vaiyavutjamai and Clements (2006a) with respect to how 231 ninth-grade students in Thailand responded to quadratic equations before and after a sequence of 10 lessons specifically dealing with quadratic equations. That investigation is one of the few studies in the literature dealing with the effects of instruction on student skills and conceptions relating to quadratic equations. Vaiyavutjamai and Clements (2006a), employing both quantitative and qualitative analyses, found that after the sequence of lessons most students still did not give both solutions to an equation like $x^2 = 16$ (when x could take any real-number value). Most students also struggled, conceptually, with equations expressed in the form $(x - a)(x - b) = 0$, where a, b represent any constant real numbers. Thus, for example, although at the post-instruction stage, most interviewees obtained 3 and 5 as solutions to the equation $(x - 3)(x - 5) = 0$, when they were asked to check their solutions they substituted $x = 3$ into $(x - 3)$ and $x = 5$ into $(x - 5)$ and concluded that since $0 \times 0 = 0$ their solutions were correct. Students who did this thought that the two x's in $(x - 3)(x - 5) = 0$ represented different variables and needed to take different values. These students obtained correct solutions, and would have had their written solution scripts assessed as correct. Yet, as Kieran (2007) noted, their responses "suggested the presence of serious gaps in the theoretical linking underpinning students' work when solving such equations" (p. 732). This same serious misconception was found among teacher-education students in the United States who had chosen to seek endorsement as specialist middle-school mathematics teachers (and who had all done well in high school algebra) (Clements and Ellerton 2006, 2009; Vaiyavutjamai et al. 2005).

A different misconception was identified with respect to equations like $x^2 - x = 12$. In Vaiyavutjamai and Clements (2006a), a majority of interviewees correctly rearranged $x^2 - x = 12$ to $x^2 - x - 12 = 0$, and then to $(x-4)(x+3) = 0$. They then equated both $x - 4$ and $x + 3$ to zero and got the correct solutions. But, they thought that the x in the x^2 term in the original equation represented a different variable from the other x in the same equation. When asked to check their solutions, some interviewees said they did not know how to do that. When asked to try, some "checked" into the $(x - 4)(x + 3) = 0$ form of the equation, replacing x by 4 in $(x - 4)$ and x by -3 in $(x + 3)$. After noting that $0 \times 0 = 0$ was true, they concluded that their solutions were correct. Others substituted in $x^2 - x = 12$, but let x equal one "solution" with x^2, and x equal the other "solution" for the "$-x$." These students arrived at "$16 + 3 = 12$" and wondered why this was false.

Vaiyavutjamai and Clements (2006a) found evidence that after lessons on quadratic equations many high school students thought that each of the equations $(x-3)(x-5) = 0$, $x^2 - x = 12$ and $2x^2 = 10x$ had two variables, not one. Clements and Ellerton (2006) and Vaiyavutjamai et al. (2005) conjectured that the same would be true of most U.S. teacher-education students seeking endorsement as specialist middle-school mathematics teachers. The misconception could have arisen from statements, often made by algebra teachers, and often written in algebra textbooks, that expressions like $2x^2$ and $10x$ are "*unlike* terms." It might also have arisen from students having misinterpreted their teachers' (correct) statements that quadratic equations can "have two different solutions." In students' minds, such a statement could mean that if two x's appeared in an equation then each *should* take a different value. That could explain why, even at the post-instruction stage, relatively few students gave both solutions to $x^2 = 9$. In the words of an interviewee, "in that equation, x appears only once, and therefore there is only one solution."

Student Misconceptions with Regard to Linear Inequalities

Although both Blanco and Garrote (2007) and Vaiyavutjamai and Clements (2006b) identified numerous fossilized misconceptions that guided students' thinking with respect to linear inequalities, both pairs of researchers identified two particularly common misconceptions: the most common arose from a tendency to give the answer to the corresponding equation; and the second arose from an expectation that there is only one real-number value of the variable in an inequality that makes that inequality true. Vaiyavutjamai and Clements (2006b) investigated the thinking of 231 high school students with respect to linear algebraic inequalities before and after a sequence of 13 algebra lessons specifically dealing with linear equations and inequalities. At both the pre-instruction and post-instruction stages, Vaiyavutjamai asked the 231 students to solve a sequence of carefully chosen linear inequalities, and then conducted pre- and post-instruction interviews with 18 students (6 high-performers, 6 middle-performers, and 6 low-performers). She reported that before the lessons most students had little idea of what they were doing when they attempted to solve linear inequalities, and that the same was still true after the lessons—even though the teachers thought that their students had developed a sound understanding. This lack of understanding was confirmed when, on a pencil-and-paper retention test administered 6 to 12 weeks after the post-instruction test, most of the 231 students still did not do well on inequality tasks.

The Tendency to "Give the Solution to the Corresponding Equation" Vaiyavutjamai and Clements's (2006b) analysis of errors, at pre- and post-instruction stages, and also at a retention stage, on all of the inequality tasks revealed that the most common, and most stubborn, misconception was that the solution to an inequality could be obtained by merely solving the corresponding equation. Those whose thinking was dominated by this misconception thought that they should treat the inequality as an equation, except that the inequality symbol in the original statement should be used instead of an equals sign.

The Tendency to Think Only One Value of the Variable Will Make an Inequality True Vaiyavutjamai and Clements (2006b) provided space for students to show their working when responding to tasks and, in addition, they provided "answer boxes" in which correct answers were supposed to be written. Students attempting to solve inequalities often showed correct working and arrived at correct answers, but in the "answer boxes" they wrote single numerals as answers. Thus, for example, for $3x \leq 6$, even if a student had written something like "$3x/3 \leq 6/3$," and "$x \leq 2$," the numeral "2" would be all that would appear in the answer box. Interviews revealed that most of the interviewees who did this usually thought that the inequality should have just one numerical solution. Interviews also revealed that many students who gave correct written answers, and whose written scripts suggested that they had understood what they had written, harbored serious misconceptions with respect to the inequalities that they had just solved. They thought that a final answer should not "still have an x in it." For them, a statement like "$x \geq 1$" could not be the final answer for $3 - 4x \leq 6x - 7$. For years they had learned to "solve for x," and they had come to believe that the answer had to be a single numeral.

Thus, for example, for the inequality $3 - 4x \leq 6x - 7$, 15%, 20%, and 24% of the 231 high school students in Vaiyavutjamai and Clements's (2006b) study gave "1" as their answer at the pre-teaching, post-teaching, and retention stages, respectively; another 38%, 24%, and 30% gave single-numeral answers other than "1" at the respective stages. Altogether, 53%, 44% and 54% of the students, respectively, responded to a request to solve $3-4x \leq 6x-7$ by giving a single numeral as the answer. Furthermore, interviews revealed that many of those who arrived at $x \geq 1$, and left that as their written answer, still thought that the answer was "1," in the sense that they believed that 1 was the only real-number value of x which would make $3-4x \leq 6x-7$ true. The "single number answer" tendency, with several variations (like giving the "additive inverse" as the answer), guided much of the students' thinking about linear inequalities, with 68%, 39%, and 56%, respectively, of the 231 students giving single numerical answers for $1 - x < 0$ at the pre-instruction, post-instruction and retention stages. In other words, except for a period immediately after the lessons on inequalities, most of the 231 participating students thought that "1" was the only value that made the inequality $1 - x < 0$ true.

Many episodes in Vaiyavutjamai and Clements's (2006b) interview data indicated that interviewees did not know what an answer to a request to solve an inequality should look like. When "solving" inequalities they attempted to mimic the methods shown by their teachers in model examples worked on the board, or on a projector, or in examples used in the textbook. Inequality symbols, number lines, and operations were used, liberally, but how everything could be linked together, and what the actual answers were (after the sequence of thought-to-be equivalent inequalities had been dealt with) seemed to be beyond most of the interviewees.

Both Blanco and Garrote (2007) and Vaiyavutjamai and Clements (2006b) concluded that a major part of the difficulty that students experienced with inequalities arose from the semantic complexity of inequality tasks. To put the matter bluntly, other than manipulate symbols dutifully to get "right answers," most of the students did not really know *why* they did what they did. Students in the Vaiyavutjamai and Clements (2006b) study tended to think that the numeral on the right-hand side of the final line of their setting out was the solution to the inequality. High-stream students were more likely to think that the solution to an inequality comprised a set of numbers, but in most cases they were not clear what that meant in relation to the initial inequality. Despite an instruction to find all real numbers that made an open sentence true, many thought in terms of natural number or integer solutions only. Interview data revealed that often interviewees did not know that they were making mistakes, and they did not know how to check their solutions.

In Vaiyavutjamai and Clements's (2006b) study, mean differences between high-, medium-, and low-stream classes, at corresponding stages, on individual questions were large. Most medium- or low-stream Grade 9 students could not correctly solve simple linear inequalities before or after lessons on inequalities. This was even more pronounced at the retention stage, several months after the lessons on inequalities.

The Pre-Service Teachers Involved, and Tasks Used, in the Present Study

This chapter reports analyses of data from 328 prospective middle-school teachers who, at various times during the period 2006–2009, were taking, in classes of between 25 and 30 students, an "Algebra for Teachers 2" ("AT2") algebra content course at a large North American university. AT2 was the last algebra course these students would take before they would graduate and become formally qualified to teach Algebra 1 to Grade 7 or Grade 8 students. The analysis will be especially concerned with the pre-AT2 knowledge of these students, particularly in relation to linear and quadratic equations and inequalities. In another paper (Clements and Ellerton 2009) we have described, in detail, a so-called "5-R intervention program" which enabled these prospective middle-school teachers ultimately to identify and eliminate serious fossilized misconceptions, and then to apply (and retain) the appropriate conceptions that they had learned. Towards the end of this chapter we shall present a summary of the 5-R procedure as well as results of its application with the 328 students.

The 328 students involved in our study had done well at school in their mathematical studies, and that was one of the reasons why they had decided to seek endorsement as specialist middle-school mathematics teachers. In written responses to a questionnaire administered at the beginning of a semester, almost all of the students indicated that they really liked the subject and were confident that they would not have too much difficulty completing the necessary combination of mathematics, mathematics teacher-education, and education studies that would enable them to become fully qualified, competent teachers of middle-school mathematics (Clements and Ellerton 2009).

This section of the chapter focuses on the instruments developed to assess the students' algebraic thinking (a) immediately before a sequence of lessons aimed at helping the students to understand quadratic equations and linear and quadratic inequalities; and (b) 6 to 12 weeks after the planned intervention. In fact, the interventions proved to be successful with all the different AT2 classes. Almost all of the students were able to bring their knowledge of elementary equations and inequalities to a level that would enable them to teach that topic competently—at least from a content perspective—to middle-school mathematics students.

"Clever" Tasks

For several decades, now, there has been a strong emphasis on developing apparently simple tasks that have the potential to reveal how students *think about* algebra. In this chapter we have coined the term "clever tasks" to refer to tasks which, although apparently simple to persons for whom the tasks were designed, are such that they have the power to identify fossilized misconceptions related to cognitive aspects of the tasks. Clever tasks can be questions on pencil-and-paper tests, or questions asked in interviews, or even questions asked in normal classroom discourses. They have often been used in algebra education research, and clearly can play an important role in algebra classes. The main aim of this section is to outline how we developed a set of clever tasks of the pencil-and-paper variety.

Data from tasks used in studies involving middle-school and high school students in the 1970s and early 1980s in the United Kingdom by Küchemann (1981), Hart (1981), and Booth (1984) have been much cited in the literature. Perhaps the most famous clever task in algebra education, though, was developed in the United States of America. It has come to be known as the "students and professors" problem:

> "At this university there are six times as many students as professors." If S stands for the number of students and P for the number of professors, write down an equation which shows how S is related to P. (Clement 1982, p. 19)

Clement (1982), and other researchers, used this task with students at many levels. Clement reported that even college engineering students sometimes gave the "reversal" response $P = 6S$ (rather than $S = 6P$). MacGregor (1991) decontextualized the "students and professors" task, and reported data from 235 Grade 9 students in 12 schools in Melbourne, Australia, on the following task, which might be called the "$y = 8x$" task.

"The number y is eight times the number x."
Write that information in mathematical symbols. (p. 85)

Almost all teachers of algebra find it difficult to believe that more than 50% of high school students who had been taking algebra classes for at least two years answered the "$y = 8x$" task incorrectly (see MacGregor 1991 for Australian data; Lim and Clements 2000 for Bruneian data; Vaiyavutjamai 2004 for Thai data). The most common error, found in each of Australia, Brunei Darussalam and Thailand, was "$x = 8y$," but other common errors included $y^8 = x$, $y^8 > x$, $8y > x$, y^8x, and $8yx$.

Fujii's (2003) research in Japan and the United States was based on students' responses to "clever" tasks like the following:

Mary has the following problem to solve:

"Find the value(s) of x for the expression $x + x + x = 12$."
She answered in the following manner (a) 2, 5, 5; (b) 10, 1, 1; (c) 4, 4, 4.
Which of her answers is/are correct? ... State the reason for your selection.
(Fujii 2003, p. 51)

Fujii (2003) reported that high proportions of elementary, middle-school, and high school students in both Japan and the United States answered questions like this incorrectly.

There are many other tasks in the algebra education research literature that might be regarded as "clever." Creating such tasks usually requires a combination of teaching experience and associated pedagogical content knowledge. A person attempting to create a clever task has to reflect on precisely what he or she wants to find out about learners, and on how to create an apparently simple task that is likely to result in non-trivial proportions of students making errors that will unambiguously indicate serious misconceptions.

The tasks cited as "clever" in the above discussion were associated with linear relationships. As Vaiyavutjamai and Clements (2006a, 2006b) and Kieran (2007) have noted, there has been an abundance of research on solving linear equations, but hardly any on solving quadratic equations or inequalities. It will be useful here to summarize some tasks developed in relation to quadratic equations by Lim (2000), in Brunei Darussalam, and in relation to linear and quadratic inequalities, by Vaiyavutjamai (2004) in Thailand.

Pencil-and-paper tasks developed by Lim (Lim 2000; Lim and Clements 2000), and used by Vaiyavutjamai (2004), asked high school algebra students to find real-number solutions to quadratic equations expressed in the form $x^2 = K (K > 0)$, or in the form $(x - a)(x - b) = 0$ (where a, b can represent any real numbers). Vaiyavutjamai (2004) used tasks of this kind, and also linear inequality tasks, in her doctoral study involving 231 ninth-grade students in Thailand.

Tsamir and Bazzini (2002) reported that 16- and 17-year old Israeli and Italian students' solutions to inequalities tended to be confused as a result of misapplications of a "balance" model. Tsamir et al. (1998) found that drawing graphs could assist understanding, and from that perspective it is interesting that the students in Vaiyavutjamai's (2004) sample had not been taught to draw graphs when solving linear inequalities (for that was not on the national Thai syllabus at that level).

Developing the Pencil-and-Paper Instruments

We developed an instrument that would enable us quickly and accurately to identify the meanings that prospective middle-school teachers give to the processes they use when solving linear and quadratic equations and inequalities. In keeping with conclusions reached in the relevant literature (e.g., Kieran 2004, 2007; Tsamir et al. 1998; Tsamir and Bazzini 2002; Vaiyavutjamai 2004; Vaiyavutjamai and Clements 2006b), we also deemed it to be important that any tasks that we developed should be such that students' responses to them would help to reveal their thinking about connections between solving equations and solving associated inequalities.

The Eight Equation/Algebraic Inequality Pairs

Our main instrument comprised eight pencil-and-paper equations and eight pencil-and-paper inequalities, each inequality corresponding to one of the equations. The eight equations and corresponding inequalities are shown in Table 1. Seven of the eight *pairs* were expected to be clever tasks, in the sense that student responses to them would be likely to reveal fundamental misconceptions. The tasks in the first equation-inequality pair were included as "warm-up" exercises. At the time of its administration to the prospective teachers, the pencil-and-paper instrument was introduced with the written statement: "For each equation or inequality, state all real number(s) which could replace x to make the statement true." Ample room for working was available immediately after the statement of an equation or inequality, and for each question an "answer box" was also provided in which students were expected to write their final answer. Note that an equation and its corresponding inequality were *not* placed next to each other on the actual test instrument. For the purposes of subsequent analysis, however, they were regarded as a pair. A similar overall format had been used, successfully, by Lim (2000) and Vaiyavutjamai (2004).

The prospective teachers were not permitted to use any form of electronic calculator (graphing or otherwise) when answering these questions. That restriction was made for two reasons. First, from experience we knew that although all of the teacher-education students had been introduced to graphing calculators in high school, most of them did not know how to use them competently and confidently.

Table 1 The eight equations and eight corresponding inequalities (pre-teaching versions)

The 8 Equation Tasks (Pre-Teaching)	The 8 Inequality Tasks (Pre-Teaching)
$9(x-1)=0$	$4(x-1)>0$
$\frac{1}{x}=3$	$\frac{1}{x}>4$
$x^2=9$	$x^2>4$
$x=\frac{9}{x}$	$x>\frac{4}{x}$
$x^2+6=0$	$x^2+2>0$
$4(x+1)=4(x-3)$	$9(x+1)>9(x-2)$
$(x-3)(x-2)=0$	$(x-3)(x-1)>0$
$x+5=8-(3-x)$	$x+3>6-(3-x)$

Second, and more importantly, we wanted students to show their working. In particular, if they employed mental imagery involving number lines, or Cartesian graphs, we wanted them to represent that imagery in the section on the test paper designed for showing working.

For each equation and inequality, students were asked to indicate, with respect to each solution they offered, how confident they were that that solution was correct. For each response a student could choose any one of five possible statements: "I'm certain I'm right," "I think I'm right," "I've got a 50-50 chance of being right," "I think I'm wrong," and "I'm certain I'm wrong."

Rationales for including each of the equation/inequality pairs in the instrument will now be given.

The First Equation/Inequality Pair

Equation 1: $9(x-1)=0$ Corresponding inequality: $4(x-1)>0$

Equation 1 and the corresponding inequality were expected to be relatively easy for the prospective teachers, certainly the easiest of the eight equation/inequality pairs. The pair was never thought of as defining a "clever task." That said, we were interested in finding out the percentage of prospective middle-school students who gave "1," or "$x<1$," as the answer to the inequality.

The seven other equation/inequality *pairs* might all be regarded as "clever tasks." They were developed out of the experience and pedagogical content knowledge of the writers, and a decision to use them in the study was made only after careful analysis of relevant trial data.

The Second Equation/Inequality Pair

Equation 2: $\frac{1}{x}=3$ Corresponding Inequality: $\frac{1}{x}>4$

The algebra education literature has established that in their middle- and high-school years, most students are told, many times, that solving equations is just a matter of maintaining balance (Kieran 2007). As a result, many students come to believe that it is always satisfactory to "multiply both sides" or to "divide both sides," by anything, including expressions involving a variable. Thus, with Equation 2, we expected that some of the prospective teachers would multiply both sides by x, to obtain the equivalent equation $1 = 3x$, and then, after dividing both sides by 3, would find the correct solution, $x = 1/3$. It was expected that that procedure would be standard for most of the 328 prospective middle-school teachers, and that most would get the correct solution for Equation 2.

Some students who did not multiply both sides by x were expected to adopt a "reciprocation" method by which $a/b = c/d$ would be regarded as equivalent to $b/a = d/c$. This method is not recommended, for it can generate incorrect solutions (e.g., consider what would happen with $3/x = 4/x$). However, provided 3 is thought of as 3/1, its application to Equation 1 would give $x/1 = 1/3$, which would quickly give the correct solution. A third method involves "cross-multiplying" from $1/x = 3/1$ to obtain $3x = 1$, etc.

Students who attempted to solve the corresponding inequality by merely "solving the corresponding equation and maintaining the direction of the given inequality" would quickly arrive at "$1/4 > x$" without having realized that since x could be negative there was no guarantee that the "direction" of the inequality should be maintained. Students who thought about the meaning of the inequality should recognize that negative values of x would make the inequality false, and that $x = 0$ would generate a meaningless statement. Therefore, "$1/4 > x$" (or the equivalent "$x < 1/4$") is clearly wrong.

It was also expected that some students would offer $x = 1/4$, or simply $1/4$, as the solution to the corresponding inequality.

The Third Equation/Inequality Pair

$$\text{Equation 3:} \quad x^2 = 9 \qquad \text{Corresponding Inequality:} \quad x^2 > 4$$

It might reasonably have been expected that prospective middle-school teachers seeking endorsement to teach mathematics, almost all of whom had been successful with Algebra 1, Algebra 2 and Pre-Calculus (and some of whom had also successfully formally studied Calculus, as well as some university-level mathematics), might have no difficulty with Equation 3 ($x^2 = 9$). However, our previous experiences with prospective middle-school teachers, and our knowledge of Vaiyavutjamai's (2004) and Lim's (2000) findings, suggested to us that many of the 328 students would give "$x = 3$" as the only solution. Those same students would be likely to give "$x > 2$," or even just "2," as their solution to the corresponding inequality ($x^2 > 4$). Many students who correctly gave "$x = \pm 3$" as solutions to Equation 3 might be expected to offer statements like "$x > \pm 2$," or "$x > -2$ and $x > 2$" (or "$x > -2$ or $x > 2$") as solutions to the corresponding inequality.

The Fourth Equation/Inequality Pair

$$\text{Equation 4:} \quad x = \frac{9}{x} \qquad \text{Corresponding Inequality:} \quad x > \frac{4}{x}$$

Equation 4 is equivalent to Equation 3, and it would be expected that almost everyone who found the two solutions for Equation 3 would also find the same two solutions for Equation 4. Similarly, those who found only one solution for Equation 3 would find that same solution for Equation 4. So far as the corresponding inequality is concerned, it was expected that many students who correctly gave "$x = \pm 3$" as solutions to Equation 4 would offer incorrect statements such as "$x > \pm 2$," or "$x > -2$ and $x > 2$" (or "$x > -2$ or $x > 2$") as their solutions for the corresponding inequality. Those who indicated that $x = 3$ was the only solution to Equation 4 would be likely to assert that $x > 2$ was the only solution to the corresponding inequality.

The Fifth Equation/Inequality Pair

$$\text{Equation 5:} \quad x^2 + 6 = 0 \qquad \text{Corresponding Inequality:} \quad x^2 + 2 > 0$$

Written instructions on the test indicated that students should find all real-number solutions to the equations and inequalities. For Equation 5 it was expected that many students would offer solutions like $i\sqrt{6}$, or $\pm i\sqrt{6}$, or $\sqrt{-6}$ or $\pm\sqrt{-6}$. We anticipated that students who answered in any of those ways would either not have read the instructions carefully or would not be knowledgeable about the concept of a real number. The prospective middle-school teachers had only occasionally dealt with equations with no solutions, and it was expected, therefore, that some would be confused by Equation 5.

The corresponding inequality provides an elegant test of the extent to which respondents thought about meaning, for although Equation 5 has no real-number solutions, the corresponding inequality is true for all real-number values of x. Although it might be expected that most of the students who gave incorrect answers to Equation 5 would give similar answers for the corresponding inequality, only with 6 replaced by 2, and with a "greater than" sign indicated, it was also conjectured that there would be some students who, although giving an incorrect answer for the equation, would recognize that $x^2 + 2 > 0$ must be true for any real-number, x.

The Sixth Equation/Inequality Pair

$$\text{Equation 6:} \quad 4(x + 1) = 4(x - 3)$$

$$\text{Corresponding inequality:} \quad 9(x + 1) > 9(x - 2)$$

Equation 6 is another example of an equation that has no real-number solution. Students arriving at a statement like "$4 = -12$" might be expected to experience cognitive tension, and of interest was whether this would result in their abandoning the question without any further thinking, or whether they would reflect on what this might imply for the original equation—in other words, this task invites students to think holistically. The corresponding inequality is such that any real-number value of x will make it true. This inequality also invites holistic thinking.

The Seventh Equation/Inequality Pair

$$\text{Equation 7:} \quad (x-3)(x-2)=0$$

$$\text{Corresponding inequality:} \quad (x-3)(x-1)>0$$

Equation 7 was included because of the interesting data generated with respect to such equations in Lim's (2000) study, in Clements and Ellerton's (2006) study, and in Vaiyavutjamai's (2004) study. It was expected that for the corresponding inequality many students would answer "$x > 3$, $x > 1$," their thinking being guided by the misconception that there is a rule for inequalities analogous to the null factor law for equations.

The Eighth Equation/Inequality Pair

$$\text{Equation 8:} \quad x+5=8-(3-x)$$

$$\text{Corresponding inequality:}\ x+3>6-(3-x)$$

School students are rarely asked to think about equations like $x+5=5+x$, and it was conjectured that those who had never been encouraged to think meaningfully when attempting to solve equations would find Equation 8 difficult.

The corresponding inequality provides another test of the extent to which respondents think about meaning. Although Equation 8 is true for all real-number values of x, the corresponding inequality is not true for any real-number value of x. It might be expected that most of the students who gave incorrect answers to Equation 8 would also answer the corresponding inequality incorrectly. It was conjectured, though, that there would be students who, although giving an incorrect answer for Equation 8, would nevertheless recognize that, since $x+3$ is never greater than $3+x$, the inequality has no real-number solution.

The Quadratic Equation Scenario Lim (2000), Vaiyavutjamai (2004), Clements and Ellerton (2006), and Vaiyavutjamai et al. (2005) had all noted that, when asked to solve a quadratic equation in the form $(x-a)(x-b)=0$, many students who gave correct solutions $x=a$, $x=b$, thought that the x in $(x-a)$ was equal to a, and *simultaneously* the x in $(x - b)$ was equal to b. Our aim was to develop a task which was such that students who harbored this misconception would be quickly and unambiguously identified.

After considerable pilot testing, we found the "quadratic equation scenario" (see Fig. 1), which we developed, to be suitable for our purposes.

Students were asked to solve $(x+2)(2x+5)=0$, and then to check their answer. One student, Carrie, wrote the following (line numbers have been added):

$(x+2)(2x+5)=0$	Line 1
$\therefore 2x^2+5x+4x+10=0$	Line 2
$\therefore 2x^2+9x+10=0$	Line 3
$\therefore (2x+5)(x+2)=0$	Line 4
$\therefore (2x+5)=0$ and $(x+2)=0$	Line 5
$\therefore 2x=-5$ and $x=-2$	Line 6
$\therefore x=-5/2$ and $x=-2$	Line 7

Check: Put $x=-5/2$ in $(2x+5)$, and put $x=-2$ in $(x+2)$.

Thus, when $x=-5/2$ and $x=-2$, $(2x+5)(x+2)$ is equal to 0×0 which is equal to 0.

Since 0 is on the right-hand side of the original equation, it follows that $x=-5/2$ and

$x=-2$ are the correct solutions.

Comment *fully* on Carrie's responses.

Fig. 1 The quadratic equation scenario

In Fig. 1, line numbers were shown to assist respondents to comprehend what they were expected to do. Use of the scenario not only helped us identify students who harbored a "$0 \times 0 = 0$" misconception, but also any of the three other common misconceptions to be found in the scenario. The respondents' task was to comment on the solutions presented by a hypothetical student, "Carrie," when solving the equation $(x+2)(2x+5)=0$, and then to consider the appropriateness of Carrie's method for checking the solutions that she obtained. Each response to the quadratic equation scenario (in Fig. 1) was given a score from 0 to 4, depending on how many of the following four points were noted:

- Lines 2, 3, and 4 were unnecessary, since the left-side is already factored in Line 1.
- In Lines 5 through 7, the word "or," and *not* "and," should have been used.
- For the check, the solutions should have been substituted into Line 1.
- For the check, each solution should have been substituted into *both* parentheses in the initial equation.

Each response to each of the eight equations and eight inequalities shown in Table 1 was scored 0 (if it was incorrect) or 1 (if it was correct), and up to 4 was allocated for the quadratic equation scenario. Thus, each student received a total score out of 20.

Study Design, and Results

As well as gathering pre- and post-intervention data, the study design incorporated a retention component aimed at checking whether students retained correct conceptions, or misconceptions, over a period of between 6 and 12 weeks. An important aspect of the intervention program was the creation of an environment in which all students would reflect metacognitively on the strategies that they used when they attempted to solve equations and inequalities (Clements and Ellerton 2009).

In this chapter the results of the analyses of post-intervention/retention data are only briefly summarized. The focus of this chapter is on what the 12 classes of students, totaling 328 students, knew about the eight equation/inequalities pairs—shown previously in Table 1—at the *start* of the semester in which they were enrolled in AT2, and how they responded, at that same time, to the quadratic equation scenario.

Our challenge was to create a method which would help students develop a relational understanding of algebra content concerned with elementary equations and inequalities that they should have already acquired as a result of their having taken Algebra 1 and Algebra 2 at high school. In fact, the instructors could dedicate only a relatively small amount of AT2 class time (a total of about three hours, *including* pre-testing) to improving the students' knowledge and skills with respect to elementary equations and inequalities. That was because the focus of the AT2 course was expected to be on more advanced topics.

Population and Sample Considerations

Between 2006 and 2009 almost all AT2 classes were taught by the same two instructors (Instructor A and Instructor B), both of whom were well qualified and experienced mathematics educators. For most of the semesters there were only two sections for AT2, one of which was taught by Instructor A and the other by Instructor B. In each section there were always between 25 and 30 students. For each semester, placement of students into the two AT2 classes was done by departmental administrators, without consultation with the instructors. Initial testing suggested that for each semester the two AT2 sections had very similar mathematics profiles. Mean scores on initial tests given to sections during the first AT2 class in a semester were never very different, and these initial mean scores remained similar throughout the period 2006–2009. During that period the highest mean score for a section out of a possible 20 (16 for the equations and algebraic inequalities and 4 for the quadratic equation scenario) was 8.0 and the lowest was 6.1, with an overall mean of 6.5. The highest individual score of any of the 328 students was 16 (out of the possible 20), and the lowest was 0.

From a "population" perspective, there is no reason to suppose that the students taking AT2 could be regarded as representative of any well-defined group, except perhaps of students at the particular university who were preparing to become middle-school mathematics specialists. The university at which the study occurred has a strong reputation in mathematics education, and it is likely that most of the 328 students were better qualified, and mathematically more knowledgeable, than "typical" teacher-education students preparing to be middle-school mathematics teachers at many other US universities or colleges.

Results

Summaries of performances on each of the eight equations/inequalities pairs are shown in Table 2.

Table 2 Percentages correct, 328 mathematics teacher-education students on eight equation/inequalities pairs

Equation	Number (and %) Correct ($n = 328$)	"Corresponding" Algebraic Inequality	Number (and %) Correct ($n = 328$)
$9(x-1)=0$	321 (98%)	$4(x-1)>0$	210 (64%)
$\frac{1}{x}=3$	268 (82%)	$\frac{1}{x}>4$	4 (1%)
$x^2=9$	74 (23%)	$x^2>4$	16 (5%)
$x=\frac{9}{x}$	69 (21%)	$x>\frac{4}{x}$	1 (0%)
$x^2+6=0$	69 (21%)	$x^2+2>0$	53 (16%)
$4(x+1)=4(x-3)$	173 (53%)	$9(x+1)>9(x-2)$	77 (23%)
$(x-3)(x-2)=0$	194 (59%)	$(x-3)(x-1)>0$	2 (1%)
$x+5=8-(3-x)$	71 (22%)	$x+3>6-(3-x)$	109 (33%)

For the 328 AT2 students, the mean score on the 16 tasks (making up the 8 equation/inequalities pairs) was 5.4 out of 16. For the equations, about 47% of responses were correct, and for the algebraic inequalities, about 18% of responses were correct. If responses to $9(x-1)=0$ and $4(x-1)>0$ were not taken into account, then 40% of responses for the equations and 11% of responses for the inequalities were correct. Yet, most students indicated that they were certain they were right, or they thought they were right, for all 16 tasks. As one student would subsequently write, in a reflection: "This was an interesting exercise because I thought I knew the correct answers, but in fact I did not." Most students experienced a *reality* check in the sense that they quickly discovered that they had obtained wrong answers to equations and inequalities that they were certain, or they thought, they knew how to solve correctly.

Table 3 summarizes performances of the 328 students on the quadratic equation scenario, and Table 4a provides an overview of results of our quantitative and qualitative analyses of student responses to four of the eight equation/inequality pairs. Interestingly, no student drew a sketch graph, for any inequality, in the working spaces provided on the pencil-and-paper instrument.

Table 3 Summary of numbers (and percentages) of 328 students identifying the four "errors" in the quadratic equation scenario (from Fig. 1)

Identified Unnecessary Work (Lines 1–3)	Stated "and" should be "or" (Lines 5–7)	Stated Carrie should have Substituted in Line 1	Identified the $0 \times 0 = 0$ Misconception	Mean Score (/4) (and SD), Quadratic Equation Scenario
176 (54%)	36 (11%)	28 (9%)	107 (33%)	1.07, SD = 0.45

Table 4a Summary of quantitative and qualitative analyses of student responses to four equation/inequality pairs

Equation or Inequality	% of Responses Correct	Most Common Errors (included if at least 10% of all students gave that response)	Comments
$9(x-1)=0$	98%		This was by far the easiest of all the tasks. Most students obtained the equivalent equation $9x=9$, but 5 of the 328 students divided both sides by 9 to get $x-1=0$.
$4(x-1)>0$	64%	• $x<1$ (14% of all responses) • 1, or another number (11%) • $x>2$ (10%)	These results are consistent with Vaiyavutjamai's (2004) analyses of her inequalities data.
$\frac{1}{x}=3$	82%	• 3 (10% of all responses)	When working was shown, multiplying both sides by x was common, as was "cross-multiplication."
$\frac{1}{x}>4$	1%	• $x<1/4$ (31% of all responses) • $x>1/4$ (24%) • 1/4 (23%)	The small amount of working that was shown suggested that students merely treated the inequality as if it were an equation.
$x^2=9$	23%	• 3 (68% of all responses)	Working was rarely shown.
$x^2>4$	5%	• $x>2$ (32% of all responses) • $x>\pm 2$ (14%) • 2, or ± 2 (13%) • $x>3$ (10%)	Once again, working was rarely shown. In no case was a number line, or a sketch of a Cartesian graph, shown in working space.
$x=\frac{9}{x}$	21%	• 3 (59% of all responses)	Students tended to write: $x^2=9$, $x=3$.
$x>\frac{4}{x}$	0%	• $x>2$ (35% of all responses) • $x>\pm 2$ (15%) • 2, or ± 2 (11%) • 3 (10%)	Students tended to write: $x^2>4$, and then $x>2$, or $x>\pm 2$, or simply 2, or ± 2. No number line or graph was drawn by any student.

Table 4b provides an overview of results of our qualitative analyses of responses to the remaining four equation/inequality pairs. Once again, no student drew a graph, for any inequality, in the working spaces.

Table 4b Quantitative and qualitative analyses of responses to the remaining four equation/inequality pairs

Equation or Inequality	% of Responses which were Correct	Most Common Errors (included if 10%, or more than 10%, of all students gave that response)	Comments
$x^2 + 6 = 0$	21%	• $\sqrt{-6}$ or $i\sqrt{6}$ (33% of all responses) • $\pm\sqrt{-6}$ or $\pm i\sqrt{6}$ (14%) • 0 (11%) • $\sqrt{6}$ or $\pm\sqrt{6}$ (10%)	Most students wrote $x^2 = -6$ and then, without further comment, offered something which, presumably, they thought was the solution to the equation.
$x^2 + 2 > 0$	16%	• "No solution," or the "null set" (16% of all responses) • $x > i\sqrt{2}$ or $\pm i\sqrt{2}$ (16%) • A single number (10%)	Students tended simply to write down a "solution" without showing working, except for $x^2 > -2$. No student sketched a graph. Except for those who gave a correct answer, there appeared to be little evidence of holistic thinking about the inequality's meaning.
$4(x + 1) = 4(x - 3)$	53%	• 0 (11% of all responses) • A single number other than 0 (15%) • $4 = -12$, or $-4 = 12$ (10%)	Often it was not clear from students' written answers whether they realized there was no solution. Statements like "will not work" were marked correct.
$9(x + 1) > 9(x - 2)$	23%	• "No solution" (23% of all responses) • $27 > 0$, or $9 > -18$ (12%)	34% of students did not proceed beyond writing down $9x + 9 > 9x - 18$.
$(x - 3)(x - 2) = 0$	59%	• $x^2 - 5x + 6$ (13% of all responses) • 3 (10%) • 2 (10%)	Evidence from the "quadratic equation scenario" would suggest that the thinking of many of those who gave "correct" solutions (e.g., "3, 2") was guided by a serious misconception.
$(x - 3)(x - 1) > 0$	1%	• $x > 3, x > 1$ (29% of all responses) • 3, 1 (15%) • $x > 3$ (13%)	The "$x > 3, x > 1$" response was, almost certainly, based on a faulty translation of the null factor law for equations to the realm of inequalities.
$x + 5 = 8 - (3 - x)$	22%	• 0 (25% of all responses) • ∞ (11%) • $0 = 0$ (10%)	21% of students did not proceed beyond writing down $x + 5 = 5 + x$.

Table 4b (Continued)

Equation or Inequality	% of Responses which were Correct	Most Common Errors (included if 10%, or more than 10%, of all students gave that response)	Comments
$x + 3 > 6 - (3 - x)$	33%	• 0 (17% of all responses) • $x > 0$ (10%)	This was the only inequality in an equation/inequality pair for which a greater number of correct answers were given than for the corresponding equation. 18% of students did not proceed beyond writing down $x + 3 > 3 + x$.

Conclusions in Relation to the Prospective Teachers' Knowledge of Algebraic Inequalities

In this section we reach four conclusions from our analyses of data relating to algebraic inequalities. These conclusions relate to: (a) students' understandings and misunderstandings of relationships between equations and algebraic inequalities; (b) the tendency of students to think that an inequality has just one numerical answer; (c) the tendency to manipulate algebraic expressions without paying due attention to meaning-making; and (d) confusion about what to do when attempting to solve a non-trivial algebraic inequality.

Over-Emphasising Relationships Between Equations and Inequalities The students obviously recognized that there was a strong relationship between solving equations and solving corresponding inequalities. That was a positive result, because when one is solving an inequality one needs to think about how the solutions to that inequality relate to the solution(s) to the corresponding equation. However, our analyses revealed that relationships between solving algebraic inequalities and corresponding equations had become so much emphasized in the teaching and learning of inequalities that they had become a destructive rather than a constructive force.

Consider, for example, the pre-instruction data summarized in Table 4a in relation to the inequality $x > \frac{4}{x}$. None of the 328 students solved this inequality correctly. Despite the fact that the students had been encouraged to "show their working," none of them drew a number line or showed a graph. Students tended simply to "multiply both sides of the equation by x" to obtain what they, incorrectly, thought was an equivalent inequality $x^2 > 4$. To solve this inequality they mentally solved the equation $x^2 = 4$ (rarely was this equation actually written down by a student). They then wrote down their solutions to this equation with the ">" sign placed where the "=" sign would "normally" be placed. The students' attempts to solve $x^2 = 4$ (see Table 4a) revealed that most students missed "-2" and this was reflected in the fact that 35% of students gave $x > 2$ as their sole solution for $x > \frac{4}{x}$.

Even students who knew that there were two solutions to $x^2 = 4$ often concluded that the solution for $x > \frac{4}{x}$ was $x > \pm 2$ (15% of all students gave that answer).

Thinking that an Inequality has Just One Number in its Truth Set Vaiyavutjamai and Clements (2006b) reported numerous transcripts of interviews with students who, even though they wrote "$x > 2$" for the solution to an inequality, thought that the actual inequality had just one numerical solution, namely "2." In this present study, 21% of the prospective middle-school teachers of mathematics stated that the solution for $x > \frac{4}{x}$ was 2, or 3, or ± 2.

Manipulations Without Meaning-Making Although there were some encouraging signs that some students were prepared to think about meanings behind equations and inequalities, most of the students seemed to be unwilling, or unable, to go beyond mere symbol manipulation (c.f., Linchevski and Sfard 1991). Although it was encouraging to note that over 50% of the students indicated that the equation $4(x+1) = 4(x-3)$ had no solution, it was discouraging to observe that only 23% of students were willing or able to think their ways through $9(x+1) > 9(x-2)$. With that inequality, 34% of the students did not proceed beyond writing down $9x + 9 > 9x - 18$. Many of them would write, in subsequent written reflections, that they simply did not know what to do with such an inequality. That unwillingness, or inability, to think holistically about an equation was strongly revealed in data associated with the equation/inequality clever tasks in Tables 4a and 4b. Thus, while 59% of responses to $(x-3)(x-2) = 0$ were deemed to be correct, only 1% of responses to the corresponding inequality, $(x-3)(x-1) > 0$, were correct. Yet, all 328 students had previously taken courses in which equations and algebraic inequalities were part of the curriculum, and all of them had also studied quadratic functions. One of the encouraging signs from the data analyses summarized in Tables 4a and 4b was that more students gave a correct answer for $x + 3 > 6 - (3 - x)$ than for $x + 5 = 8 - (3 - x)$.

Not Knowing How to Proceed with Inequalities When we analyzed students' written responses to the equations/inequalities pairs it became obvious to us that whereas the students had some ideas—albeit often imperfect ideas—about how to solve equations, they rarely knew how to proceed when faced with inequalities like $x > \frac{4}{x}$ or $(x-3)(x-1) > 0$. Other than to recommend and sometimes illustrate graphical approaches, textbook writers and algebra educators have rarely been particularly helpful on this matter—although Dobbs and Peterson (1991) did attempt to do something about it. With $x > \frac{4}{x}$, for example, a suitable approach might be to consider separately the cases for $x > 0$, and $x < 0$, and then to combine the results. Thus, for $x > 0$, $x > \frac{4}{x}$ is equivalent to $x^2 > 4$, which is equivalent to $x > 2$. For $x < 0, x > \frac{4}{x}$ is equivalent to $x^2 < 4$, and $-2 < x < 0$. Combining, one finds that if x can be any non-zero number then the solution set for $x > \frac{4}{x}$ is $\{x : -2 < x < 0\} \cup \{x : x > 2\}$. This approach can be supported by the use of graphical calculators, or by sketch graphs. Teaching students to interpret correctly representations that appear on the screen of graphical calculators can be just as difficult as assisting them to learn appropriate algorithms (such as the one suggested for $x > \frac{4}{x}$), or to draw and interpret appropriate sketch graphs. The axiomatic approach and the graphical representation approaches need to be learned by students. These approaches become most useful when one is used to complement the other, and students are helped to reflect on meaning, and to translate from one approach to another (Pimm 1995).

Most of the prospective middle-school mathematics teachers treated inequalities as equations that happened to have inequality signs rather than equals signs. Most (but not all) of them experienced difficulty with tasks for which holistic thinking, rather than mere algebraic manipulation, was called for.

Prospective Teachers' Knowledge in Relation to Quadratic Equations

At the beginning of every semester during the period 2006–2009 we asked our AT2 students to find all real numbers that could replace x and make $x^2 = 9$ true. With every class, on every occasion, more than 50% of the students gave one possible value of x, and usually only between 20% and 30% included -3 in their answer (see the overall result for the 328 students in Table 4a). In the data summarized in Table 4b it can be seen that 79% of the 328 prospective mathematics teachers did not know that $x^2 + 6 = 0$ has no real-number solutions. Almost 60% of the prospective teachers stated that $x = 3$ and $x = 2$ were solutions to $(x - 3)(x - 2) = 0$, but data from the quadratic equation scenario indicated that many of these students also thought that the x in $(x - 3)$ equalled 3 and, *simultaneously*, the x in $(x - 2)$ equalled 2.

The inspiration for the design of the quadratic equation scenario arose from our knowledge of the results of our own research with pre-service teachers in the United States (Clements and Ellerton 2006) and of the results of research by Lim (2000), in Brunei Darussalam, and Vaiyavutjamai (2004), in Thailand. The work by Lim and Vaiyavutjamai deserves to be known by all algebra educators, and algebra classroom teachers.

Lim (2000) noticed that some teachers, when teaching children how to solve equations like $(x - 3)(x - 2) = 0$, "expanded the parentheses" on the left side to get the equivalent equation $x^2 - 5x + 6 = 0$, and then re-factored the left side before applying the null factor law. These teachers informed Lim that it was important for students to write the equation in "standard form" before attempting to solve it. Thus, our quadratic equation scenario featured a fictitious student, Carrie, attempting to solve $(x + 2)(2x + 5) = 0$. She expressed this in equivalent form, $2x^2 + 9x + 10 = 0$, before re-factoring and applying the null factor law. Almost half (46%) of our prospective teachers of mathematics saw nothing strange about that.

Readers who feel that this could not happen in a nation like the United States of America may wish to consult http://www.algebrahelp.com (Mishanski 2010). According to the webpage, Algebrahelp.com "is a collection of lessons, calculators, and worksheets created to assist students and teachers of algebra." Here is a printout of the first 19 lines of the method that is shown for solving the equation $(x - 2)(x - 3) = 0$. There are actually 57 lines in the recommended solution, but the last 38 lines are not shown here because of space limitations. On the same web-page, the recommended solution for $(x - 2)(x - 2) = 0$ also has 57 lines, with the solution being given as $x = \{2, 2\}$.

`Simplifying`
$(x + -3)(x + -2) = 0$

`Reorder the terms:`
$(-3 + x)(x + -2) = 0$

`Reorder the terms:`
$(-3 + x)(-2 + x) = 0$

`Multiply` $(-3 + x) * (-2 + x)$
$(-3(-2 + x) + x(-2 + x)) = 0$
$((-2 * -3 + x * -3) + x(-2 + x)) = 0$
$((6 + -3x) + x(-2 + x)) = 0$
$(6 + -3x + (-2 * x + x * x)) = 0$
$(6 + -3x + (-2x + x^2)) = 0$

`Combine like terms:` $-3x + -2x = -5x$
$(6 + -5x + x^2) = 0$

`Solving`
$6 + -5x + x^2 = 0$

`Solving for variable 'x'.`

`Factor a trinomial.`
$(2 + -1x)(3 + -1x) = 0 \ldots$

Bad News, Good News and Some Concluding Comments

Bad News

The AT2 prospective teachers were not far away from becoming fully qualified, endorsed middle-school teachers of mathematics, yet 41% of them did not solve $(x - 3)(x - 2) = 0$ correctly, and 46% of them did not feel moved to comment that Lines 1–3 in Carrie's response (in the quadratic equation scenario) were unnecessary. Only 11% of them recognized that the correct connective in Lines 5–7 was "or" (and not "and"), and 91% did not notice, or did not comment on, the fact that when checking her solutions, Carrie did not substitute in the original equation in Line 1. The serious misconception that $x = -2$ in $(x + 2)$ and, simultaneously, $x = -5/2$ in $(2x + 5)$, which led Carrie to conclude that since $0 \times 0 = 0$ both her solutions must be correct, was noticed by only one-third of all students.

It is likely that the "errors" featured in the quadratic equation scenario are made by school students in many nations, and research is needed to see how prevalent the errors are among prospective and even practicing teachers. We have, in fact, analysed responses by many experienced and qualified US middle-school teachers of mathematics to the quadratic equation scenario, and found that they fare only marginally better than the 328 prospective teachers in identifying Carrie's errors.

Elsewhere (Clements and Ellerton 2009), we have provided evidence that unless AT2 students are made aware of the kinds of errors featured in Table 3 and in Tables 4a and 4b, they are likely to begin their mathematics teaching careers guided by the same misconceptions that became evident when AT2 students responded to the tasks we asked them to do at the beginning of their AT2 courses.

Good News

All of the 328 participating AT2 students became aware of the misconceptions that they displayed during the initial round of tests, and almost all of them were able to demonstrate retention of newly-acquired knowledge, skills and concepts at the end of the AT2 course. Their improvement was brought about by their active participation in what we have called a 5-R intervention program which featured the following five components:

1. There was an initial reality check, by which students came to realize that their thinking about the area under consideration had been guided by fossilized misconceptions. This often generated confusion among students who began to realize that they did not know what they thought they knew.
2. Then there was a review component, by which student misconceptions were identified and corrected by an instructor, with students being guided toward appropriate conceptions.
3. Students then reflected by making written statements on how they had previously thought about the concepts. They were also expected to comment on any new, or revised, or extended *conceptual* understandings that they were in the process of developing.
4. Then followed a period of between 6 and 12 weeks when students revisited their conceptual understandings. The intention, here, was to ensure that the students' new understandings were appropriate, stable and coherent.
5. The final component was when students were assessed for retention. Without students having been fore-warned, tests parallel to the original tests were administered, the aim being to see if the students had acquired and retained accurate conceptions, and whether they could apply, with appropriate *confidence*, their new understandings in relevant problem-solving or problem-posing contexts.

The 5-R intervention model is illustrated in Fig. 2, which is taken from Clements and Ellerton (2009). Each of the five ordered treatment components—Realize, Review, Reflect, Revisit and Retain—requires *action* on a learner's part. As a consequence

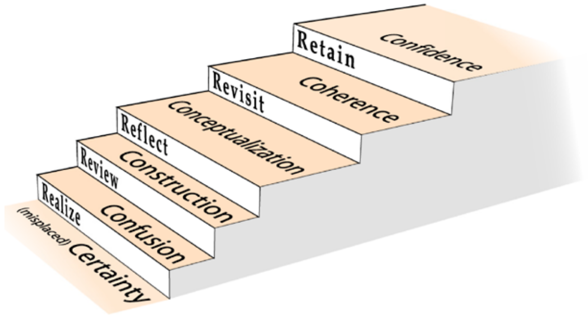

Fig. 2 Schematic summary of 5-R intervention model (from Clements and Ellerton 2009)

of each intervention component, students are depicted (see Fig. 2) as reaching new *conceptual phases* (or "C-phases") in their development of relational understandings that are internalized in the form of accurate, rich, linked concept images. The conceptual phases are termed "Certainty (Misplaced)," "Confusion," "Construction," "Conceptualization," "Coherence," and "Confidence."

In their written reflections, many students noted that they had rarely encountered equations like $x + 5 = 8 - (3 - x)$, and simply did not know what to do when they arrived at a statement like "$x + 5 = 5 + x$." Similarly, with respect to $4(x + 1) = 4(x - 3)$, they said they had never before been asked to interpret a proposition like "$4 = -12$." However, after their thinking about statements such as these had been identified and, when necessary, "straightened out," most of the students became confident that they would be able to deal with such problems in the future. On the retention test they demonstrated that this confidence was well placed.

Pre-intervention and post-intervention/retention percentages correct for the 328 students on the 16 questions (parallel questions were used for post-instruction/retention assessment) are shown in Table 5. For 15 of the 16 questions (Question 5 was the exception), the improvement was substantial. Although the study design did not feature control groups, evidence that the 5-R intervention program was the main factor influencing improvement is given in Clements and Ellerton (2009).

Some of the comments made by students in their written reflections provided poignant commentary on standard approaches in school and college algebra. One student was moved to reflect, directly, on what had happened to her in relation to equations and inequalities:

> One thing that stands out to me is the pre-test we took during the first class. Once we got the test back graded and went over the answers I realized that I made a lot of stupid silly mistakes. I didn't recognize that $x + 1 > x - 2$ is always true and other similar problems. Now, I actually look at the problem and analyze it to see if what the problem is asking makes sense.

Another student wrote:

> During middle school and high school, algebra was one of my favorite subjects. I liked being able to follow rules and to easily find a solution to any problem. However, I now know the reason I was good at algebra was because I could memorize and remember rules. I had very little understanding to back up these rules and I had no idea why they worked. This course has helped me to build this understanding and to explain why the rules worked.

Table 5 Percentages correct, for 328 mathematics teacher-education students on 16 pre-teaching and 16 parallel retention tasks at pre-intervention and retention stages

Pre-Teaching Question	Pre-Intervention, % Correct ($n = 328$)	Retention Question	Retention, % Correct ($n = 328$)
1. $\frac{1}{x} = 3$	82%	$\frac{1}{x} = 2$	97%
2. $x^2 = 9$	23%	$x^2 = 16$	96%
3. $x = \frac{9}{x}$	21%	$x = \frac{4}{x}$	88%
4. $x^2 + 6 = 0$	21%	$x^2 + 3 = 0$	93%
5. $9(x - 1) = 0$	98%	$5(1 - x) = 0$	97%
6. $4(x + 1) = 4(x - 3)$	53%	$3(x - 1) = 3(x + 3)$	98%
7. $(x - 3)(x - 2) = 0$	59%	$(x + 3)(x + 4) = 0$	95%
8. $x + 5 = 8 - (3 - x)$	22%	$2x + 5 = 10 - (5 - 2x)$	95%
9. $\frac{1}{x} > 4$	1%	$\frac{1}{x} > 3$	55%
10. $x^2 > 4$	5%	$x^2 > 9$	77%
11. $x > \frac{4}{x}$	0%	$x > \frac{1}{x}$	43%
12. $x^2 + 2 > 0$	16%	$x^2 + 5 > 0$	89%
13. $4(x - 1) > 0$	64%	$4(x - 3) > 0$	95%
14. $9(x + 1) > 9(x - 2)$	23%	$9(x + 5) > 9(x - 1)$	97%
15. $(x - 3)(x - 1) > 0$	1%	$(x + 3)(x - 2) > 0$	61%
16. $x + 3 > 6 - (3 - x)$	33%	$x + 2 > 7 - (5 - x)$	96%

> It has forced me to solve the problems on my own without relying on the formula. This way of thinking gives meaning to problems rather than just mindlessly plugging numbers into a formula in order to get an answer.

Another student wrote:

> Once I got the results I was shocked at how low my score was. I had completely forgotten the basic principles. These basic rules date back to middle-school years. It took me a while to appreciate that, yes, my answers were not correct. I will not make these mistakes again. So far as Carrie's attempt to solve the quadratic equation was concerned, I simply did not recognise Carrie's mistakes. Although I now can see that she did not need to distribute the terms in the parentheses at the start, I'm still not completely sure what she did wrong when she wrote $0 \times 0 = 0$ in her check.

The final comment in this last statement draws attention to the fact that sometimes the instructors had to talk with students individually to help them overcome misconceptions. The student who wrote the last excerpt was struggling to recognise why the statement "$0 \times 0 = 0$" in the check indicated a serious lack of understanding. It was only after one of the instructors talked individually with the student that the misconception was straightened out.

Student Confidence Considerations

On the pre-intervention test, for each of the eight equations, all 328 students indicated that either they were *certain* they were right or they *thought* they were right. They were less confident on the inequalities, but usually the students thought that they were right. For all eight inequality tasks, 293 of the 328 students indicated that they were certain they were right, or they thought they were right, or they thought they had a 50-50 chance of being correct. This initial confidence was usually misplaced, suggesting that many students "did not know that they did not know."

On the post-intervention/retention tests, most students indicated that they were confident they were correct on 14 of the 16 tasks. This time their confidence was vindicated. The two exceptions were for $1/x > 3$ and for $x > 1/x$. On the retention test, only 91 of the 328 students indicated that they were certain their answers were correct on these two tasks. In fact, 60 of those 91 students *did* give correct responses to those tasks. Most of the other 237 students were not confident that their answers were correct for the two tasks, and in most cases these students gave incorrect responses. At least, these students now knew that they did not know.

Concluding Comments

As stated earlier, our development of the clever tasks reported in this chapter was assisted by two major factors. First, we had many years of experience in teaching and researching school algebra and algebra courses for prospective teachers (e.g., Clements and Ellerton 2006), and that experience provided us with relevant pedagogical content knowledge. And, second, we had been directly associated with the little-known research of Lim (2000) and Vaiyavutjamai (2004), and it was that research which inspired not only the quadratic equation scenario but also our decision to link carefully selected equations and inequalities as pairs, where in seven cases out of eight, a pair was regarded as forming a composite clever task.

Using the eight equation/inequality pairs enabled us to gather, quickly, rich data relating to the students' algebra knowledge, skills and conceptions. Analyzing those data placed us in a strong position to do something about helping the students to take control of their situations, so that they could improve their knowledge, skills and conceptions relating to linear and quadratic equations and inequalities.

The more we worked on this project, the more we came to realize how difficult school algebra can be for many learners. Our work, using the 5-R model, has at least helped us to get prospective middle-school teachers to the stage where they have strong conceptions related to equations and inequalities, which of itself is an achievement (de Castro 2004). Furthermore, they have come to know that they are learning correct algebra, and that previously some of their algebra content knowledge was inaccurate, even though they had believed otherwise. Our use of the clever tasks was important in helping the prospective teachers upgrade their knowledge, and to replace misplaced confidence with appropriate confidence so far as their algebra knowledge was concerned.

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The Algebraic Nature of Fractions: Developing Relational Thinking in Elementary School

Susan B. Empson, Linda Levi, and Thomas P. Carpenter

Abstract The authors present a new view of the relationship between learning fractions and learning algebra that (1) emphasizes the conceptual continuities between whole-number arithmetic and fractions; and (2) shows how the fundamental properties of operations and equality that form the foundations of algebra are used naturally by children in their strategies for problems involving operating on and with fractions. This view is grounded in empirical research on how algebraic structure emerges in young children's reasoning. Specifically, the authors argue that there is a broad class of children's strategies for fraction problems motivated by the same mathematical relationships that are essential to understanding high-school algebra and that these relationships cannot be presented to children as discrete skills or learned as isolated rules. The authors refer to the thinking that guides such strategies as relational thinking.

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Of what use . . . is it to be able to see the end in the beginning? (Dewey 1974, p. 345).

Fractions and algebra are two topics in school mathematics that are considered critical to the curriculum and difficult to learn (National Council of Teachers of Mathematics 1998, 2000). Students' misconceptions and procedural errors for fractions and algebra, for example, have been well documented (Kerslake 1986; Matz 1982; Sleeman 1984; Stafylidou and Vosniadou 2004). Moreover, high-school students' poor performance in algebra has been blamed on their weak proficiency in fractions. According to a recent Math Panel report, for instance, the ability to perform fraction computations easily and quickly is one of the most critical prerequisites for algebra (U.S. Department of Education 2008).

We see the relationship between fractions and algebra differently. If there is an obstacle to learning algebra, it begins to form as children learn basic arithmetic. As a direct result of typical approaches to instruction in the U.S., American students tend to understand arithmetic as a collection of procedures, rather than in terms of conceptual relationships or general properties of number and operation. By the time the problem is exposed as children learn fractions, it is fairly entrenched, and it is only exacerbated by the fact that fractions are taught in isolation from whole numbers and that fraction operations are taught as a collection of procedures. Concrete materials and models may help children make critical connections (Lesh et al. 1987), but our take on the types of connections that are most fruitful for understanding fractions represents a departure from earlier lines of thinking.

In this chapter we present an alternative view on the relationship between fractions and algebra that (1) emphasizes the conceptual continuities between whole-number arithmetic and fractions; and (2) shows how the fundamental properties of operations and equality that form the foundations of algebra are used naturally by children in their strategies for problems involving operating on and with fractions. We ground this view in research on children's thinking to illustrate how algebraic structure emerges in young children's reasoning and can, with the help of the teacher, be made explicit. Specifically, we argue that there is a broad class of children's strategies for fraction problems motivated by the same mathematical relationships that are essential to understanding high-school algebra and that these relationships cannot be presented to children as discrete skills or learned as isolated rules. We refer to the thinking that guides such strategies as *relational thinking*.

These arguments are based on our research over the last 14 years, in which we have been studying how to provide opportunities for students to engage in relational thinking in elementary classrooms and how to use relational thinking to learn the arithmetic of whole numbers and fractions. We have focused on understanding children's conceptions and misconceptions related to relational thinking, how conceptions develop, how teachers might foster the development and the use of relational thinking to learn arithmetic, and how professional development can support the teaching of relational thinking. This research has included design experiments with classes and small groups of children (e.g. Falkner et al. 1999; Empson 2003; Koehler 2004; Valentine et al. 2004), case studies (Empson et al. 2006; Empson and

Turner 2006), and large-scale studies (Jacobs et al. 2007); and it has been synthesized in two books (Carpenter et al. 2003; Empson and Levi 2011).

In this chapter, we illustrate elementary school children's use of relations and properties of operations as a basis for learning fractions and argue that relational thinking is a critical foundation for learning algebra. We first define relational thinking and then we discuss how the use of relational thinking supports the development of children's understanding of arithmetic. At the same time we challenge the notion that an invigorated focus on fractions in the middle grades is the key to equipping students to learn algebra meaningfully (Hiebert and Behr 1988; U.S. Department of Education 2008). Instead, we argue that the key can be found in helping children to see the continuities among whole numbers, fractions, and algebra. Finally, we suggest that a model of the development of children's understanding of arithmetic that is based upon a concrete to abstract mapping is too simplistic. We propose instead that developing computational procedures based on relational thinking could effectively integrate children's learning of the whole-number and fraction arithmetic in elementary mathematics, in anticipation of the formalization of this thinking in algebra.

What Is Relational Thinking?

Relational thinking involves children's use of fundamental properties of operations and equality[1] to analyze a problem in the context of a goal structure and then to simplify progress towards this goal (Carpenter et al. 2003; see also Carpenter et al. 2005; Empson and Levi 2011). The use of fundamental properties to generate a goal structure and to transform expressions can be explicit or it can be implicit in the logic of children's reasoning much like Vergnaud's (1988) theorems in action.

For example, to calculate $\frac{1}{2} + \frac{3}{4}$ a child may think of $\frac{3}{4}$ as equal to $\frac{1}{2} + \frac{1}{4}$ and reason that $\frac{1}{2}$ plus another $\frac{1}{2}$ is equal to 1, then plus another $\frac{1}{4}$ is $1\frac{1}{4}$. In a study by Empson (1999), several first graders reasoned this way when given a story problem involving these fractional quantities. This solution involves *anticipatory thinking*, a construct introduced by Piaget and colleagues (Piaget et al. 1960) to characterize the use of psychological structures to coordinate a goal with the subgoals used to accomplish it; thinking can involve several such coordinations. These students recognized that they could decompose $\frac{3}{4}$ into $\frac{1}{2} + \frac{1}{4}$, and that if they decomposed it this way, they could regroup to add $\frac{1}{2} + \frac{1}{2}$. In other words, they transformed $\frac{3}{4}$ to $\frac{1}{2} + \frac{1}{4}$ in anticipation of adding $\frac{1}{2} + \frac{1}{2}$. This solution involved thinking flexibly about both the quantity $\frac{3}{4}$ and about the operation, taken into account concurrently rather than separately as a series of isolated steps. Their thinking can be represented by the

[1]Essentially, we are referring here to the field properties and basic properties of equality (Herstein 1996; see also Carpenter et al. 2003; Empson and Levi 2011).

following equalities:

$$\frac{1}{2} + \frac{3}{4} = \frac{1}{2} + \left(\frac{1}{2} + \frac{1}{4}\right) = \left(\frac{1}{2} + \frac{1}{2}\right) + \frac{1}{4} = 1 + \frac{1}{4} = 1\frac{1}{4}.$$

Although the first graders did not represent their reasoning symbolically in this way, their solution is justified in part by the implicit use of the associative property of addition, which we have represented explicitly here to highlight the logic of the their thinking.

Relational thinking is powerful because the applicability of fundamental properties such as the associative property of addition and the distributive property of multiplication over addition cuts across number domains and into the domain of algebra where one reasons about general quantities rather than specific numbers. Consider the expression $7a + 4a$. A basic algebraic skill is to simplify this expression to $11a$, by application of the distributive property of multiplication over addition:

$$7a + 4a = (7 + 4)a = 11a.$$

The same property that justifies this transformation can also be used to justify that $70 + 40 = 110$ and $\frac{7}{5} + \frac{4}{5} = \frac{11}{5}$:

$$70 + 40 = 7 \times 10 + 4 \times 10 = (7 + 4) \times 10 = 11 \times 10 = 110,$$

$$\frac{7}{5} + \frac{4}{5} = 7 \times \frac{1}{5} + 4 \times \frac{1}{5} = (7 + 4) \times \frac{1}{5} = 11 \times \frac{1}{5} = \frac{11}{5}.$$

Yet addition of whole numbers and addition of fractions are taught in isolation from each other in the elementary curriculum, and they are often taught by rote, without reference either to the underlying properties or the process of deciding how and when to use a property. For example, to add fractions children are taught to first find a common denominator and then add the two numerators; many children remember this process as a series of steps to execute. They are not encouraged to draw on their understanding of the distributive property either to derive or to explain this procedure. Many children are therefore simply not prepared later to explicitly draw on the appropriate properties to justify why $7a + 4a$ is $11a$, but $7a + 4b$ is not $11ab$.

Children learn arithmetic with understanding when they are encouraged to use and develop their intuitive understanding of the properties of number and operation. Our research has led us to recast the meaning of *learning with understanding* in terms of thinking relationally: *to understand arithmetic is to think relationally about arithmetic*, because the coherence of operations on whole numbers and fractions is found at the level of the fundamental properties of operations and equality. Teaching arithmetic in general and fractions in particular primarily as a set of procedures fails to introduce children to the powerful reasoning structures that form the basis of our number system. On the other hand, if children enter algebra with a well developed ability to think relationally about operations, they are prepared to learn to reason meaningfully about and carry out transformations involving generalized expressions through the explicit application of algebraic properties. In the following section we show how these properties emerge and can be developed in the context of carrying out number operations involving fractions and we discuss their connections to learning algebra.

Use of Relational Thinking in Learning Fractions

Children's difficulties learning fractions have been well documented (Kerslake 1986; Stafylidou and Vosniadou 2004). The difficulty, however, may be in how fractions are taught rather than how intrinsically easy or hard they are to understand. Indeed, a conclusion we draw from our research is that fractions are not unduly difficult if instruction develops children's capacity for relational thinking.

A focus on relational thinking can transform fractions into a topic that children understand by drawing on and reinforcing the fundamental properties that govern reasoning about both whole-number and fraction quantities and operations. Children use relational thinking in their solutions to story problems (e.g., Baek 2008; Carpenter et al. 1998; Empson et al. 2006) and open number sentences (Carpenter et al. 2005). Teachers can cultivate children's use of relational thinking by using a combination of these types of problems. In this chapter we focus on children's relational thinking in the context of solving story problems.

Understanding Fractional Quantities Through Relational Thinking

Before children can learn to operate on or with fractions, they need to understand fractional as quantities. Because a fraction is defined by the multiplicative relationship between its two terms, a mature understanding of fractions as quantities is relational in nature. Young children can construct a relational understanding of fractions by solving and discussing Equal Sharing problems (Empson 1999; Empson and Levi 2011; Streefland 1993).

To solve a problem about equally sharing quantities, such as two pancakes shared among three children, children must partition the quantities equally and completely. Children's earliest, non-relational strategies often involve partitioning the pancakes into halves. In this example the two pancakes would yield four halves. A child using this strategy might then try to distribute the four halves into three groups. When the child discovers that there is a half left over, the child may then partition the extra half into half, and then partition each of those parts into half again, continuing until the parts get too small to partition. This solution is not relational in that it lacks anticipatory thinking. The child knows that it is necessary to partition the pieces to share them, but approaches the problem one step at a time, partitioning into halves without anticipating how the resulting parts are going to be shared.

Children begin to think relationally about fractional quantities when they begin to reason about the relationship between partitions into equal and exhaustive shares

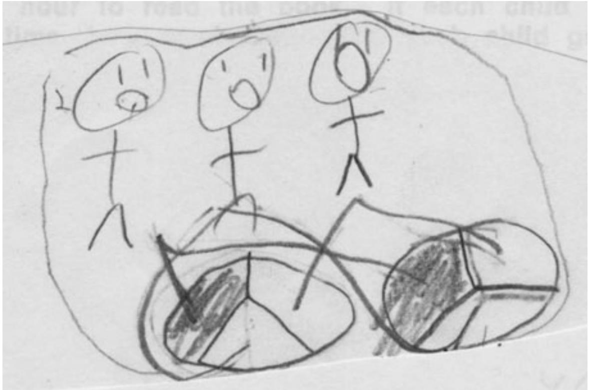

Fig. 1 Child's strategy for sharing 2 pancakes equally among 3 children, demonstrating emerging relational understanding of fractions

and the number of sharers. To solve two pancakes shared by three children, a child could decide to completely share the first pancake with all three children, and then to share the second pancake in the same way (Fig. 1). Alternatively, a child who began by distributing one half to each person might then decide to partition the left over half into three equal parts. In either case, a child who thinks about the number of people sharing *and* at the same time how to partition the things to be shared is in the process of developing a relational understanding of fractions.

These strategies implicitly use several important mathematical relationships. For ease of illustration, we concentrate on the strategy in which the child partitions each whole candy bar into thirds. Although young children are unlikely to use the following notation to represent their reasoning, it follows this logic:

$$2 \div 3 = (1+1) \div 3 = 1 \div 3 + 1 \div 3 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

This reasoning embodies the knowledge that three one-thirds make a whole pancake and that one pancake divided among three people yields one-third of a pancake to each. It also suggests an intuitive understanding of how a "distributive-like property" can be applied to division.[2]

A fully operationalized and explicit understanding of fractions as relational quantities develops gradually. Most basic to this understanding is that unit fractions are created by division or partitioning and that unit fractions are multiplicatively related to the whole:

$$1 \div n = \frac{1}{n} \quad \text{and} \quad \frac{1}{n} \times n = 1. \tag{1}$$

Multiple opportunities to combine unit-fraction quantities in solutions to Equal Sharing problems and to notate these solutions—such as "1 third and 1 third equals

[2] This property can be represented as $a \div c + b \div c = (a+b) \div c$, which is equivalent to $a \times \frac{1}{c} + b \times \frac{1}{c} = (a+b) \times \frac{1}{c}$. It is sometimes referred to as the right distributive property of division over addition. On the other hand, $a \div (b+c)$is not the same as $a \div b + a \div c$; that is, there is no left distributive property of division over addition.

2 thirds"— lead to the following more generalized relational understanding:

$$m \times \frac{1}{n} = \frac{m}{n}. \tag{2}$$

The conceptual connections between children's pictorial and symbolic representations of fractional quantities require prolonged attention to develop in a flexible, integrated way (Empson et al. 2006; Saxe et al. 1999). These relationships are initially grounded in children's informal knowledge of partitioning quantities, such as cupcakes and sandwiches. Children arrive at the generalized relational understanding represented by (1) and (2) above as the result of repeated opportunities to create, represent, and reason about these relationships in various interlinked forms over an extended period of time.

Understanding these basic relationships is absolutely critical to children's ability to reason with understanding about fraction operations and computations. Consider the case of Holly, a fifth grader who had been exposed to fraction instruction throughout her school career but did not understand fractions as relational quantities. She had learned that fractions involved partitioning wholes into parts, but she did not understand the relation between the parts and the whole. Fractional parts, to her, were entities unrelated to whole numbers. These limitations in her understanding were exposed in her solution to the following problem:

> Jeremy is making cupcakes. He wants to put $\frac{1}{2}$ cup of frosting on each cupcake. If he makes 4 cupcakes for his birthday party, how much frosting will he use to frost all of the cupcakes?

To solve the problem, Holly drew the picture in Fig. 2 and decided the answer was "four halves." Upon further questioning, it became clear that Holly did not see how these quantities could be combined; she insisted the answer was four halves and four halves only. It seemed instead that the entire circle partitioned in half represented the fraction $\frac{1}{2}$ for Holly, and it would have been nonsensical to combine them (akin to asking, "How much is 4 apples?"). For her, fractions existed separately from other numerical measures.

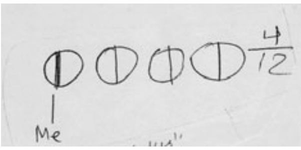

Fig. 2 Holly's written work for figuring four groups of half each, suggesting a non-relational understanding of fractions. (The 4 over what looks like 12 is Holly's way of writing 4 halves. She appears to be trying to remember syntactic features of the numeral and confounding "12" with "1/2")

Contrast Holly's solution to a third grader's solution to the following problem.

> Mr. W has 10 cups of frog food. His frogs eat $\frac{1}{2}$ a cup of frog food a day.
> How long can he feed his frogs before his food runs out?

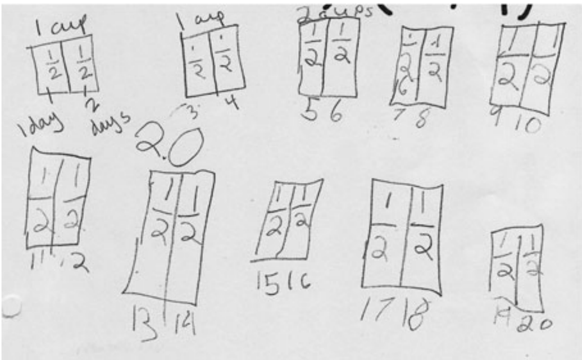

Fig. 3 A third grader's written work for figuring 10 groups of one half each, showing a relational understanding of fractions as quantities

The third grader, John, represented each cup of frog food with a rectangle, then divided each rectangle in half and notated "$\frac{1}{2}$" on each half to show how much food Mr. W's frogs could eat in a day (Fig. 3). He then counted these to arrive at an answer of 20 days. Unlike Holly, John used a relational understanding of the quantity $2 \times \frac{1}{2} = 1$ to construct a solution. John's solution represents a big step forward over Holly's. He might have gone further in his use of relational thinking by grouping the half cups in order to figure the total number of days more efficiently. For example, he could have reasoned that 2 half cups are one cup, 4 half cups are 2 cups and so on, until he reached the number of half cups in 10 whole cups. He also could have reasoned directly that 20 groups of $\frac{1}{2}$ are the same as 10 groups of 1. This type of reasoning, which takes into account both a relational understanding of fractional quantities and relations involving the operation of multiplication, is illustrated in the cases in the following section.

Use of Relational Thinking to Make Sense of Operations Involving Fractions

As children come to understand fractions as relational, they start to use this understanding to decompose and recompose quantities for the purpose of transforming expressions and simplifying computations. These manipulations are done purposefully and draw on (a) children's intuitive understanding of fractional quantities as relational described above and (b) children's relational understanding of operations cultivated in the context of whole-number reasoning and problem solving.

Children's strategies for multiplication and division word problems involving fractions can draw on and reinforce their growing understanding of the multiplicative nature of fractions. At the same time, the use of such problems supports the emergence of relational thinking about operations as children attempt to figure out how to make operations more efficient. Children's thinking becomes more anticipatory in that they begin to make choices about how to decompose and recompose fractions in the context of a goal structure that relates operations and quantities. This

Table 1 Combining groups using fundamental properties of multiplication

Equation representing child's thinking	Fundamental and other generalized properties of arithmetic on which child's thinking is based
$8 \times \frac{3}{8} = 8 \times (3 \times \frac{1}{8})$	Fractions represented as multiples of unit fractions
$= 8 \times (\frac{1}{8} \times 3)$	Commutative property of multiplication
$= (8 \times \frac{1}{8}) \times 3$	Associative property of multiplication
$= 1 \times 3 = 3$	Inverse and identity properties of multiplication

anticipatory thinking signals the purposeful use of fundamental properties of operations and equality and is in contrast with algorithmic thinking about operations in which the goal structure can be summarized as "do next."

A pivotal point in the growth of children's understanding is reached when children begin to use relational thinking to make repeated addition or subtraction of fractions more efficient by applying fundamental properties of operations and equality in their strategies for combining quantities. The emergence of relational thinking about operations in this context is facilitated by the need to combine several groups of equal size. For example, in one of the cases that follows, a fifth-grade student wanted to figure eight groups of three eighths each. The child reasoned that eight groups of one eighth each equals one, so three such groups would be three. This reasoning makes implicit use of the commutative and associative properties of multiplication (Table 1).

As children's understanding of fractions grows, basic relationships as illustrated in Table 1 serve as building blocks in more sophisticated relational thinking strategies. These strategies draw upon a variety of these properties in ways that are anticipatory rather than algorithmic and in ways that demonstrate a well connected understanding of number and operation. Most notably, these strategies are driven by each child's understanding and therefore cannot and should not be reduced to a generalized series of steps for all children to follow. In fact, a teacher would be hard pressed to explicitly teach these strategies, because each step is embedded in a goal structure that is specific to each child's relational understanding of the operations and quantities for a given problem. In the long run, this relational understanding of number and operations results in an efficiency in learning advanced mathematics, such as algebra.

To illustrate the types of relational thinking that elementary students are capable of using, we discuss two strategies generated by fifth and sixth graders in different classrooms. The teachers in these classrooms tended to place responsibility for generating and using conceptually sound strategies on each individual student.[3] This approach to instruction does not typify instruction in U.S. classrooms, and so the strategies we describe here are not representative of the current performance of U.S. children in the upper elementary grades (e.g., Hiebert et al. 2003). However, they are

[3]We have observed patterns both in the types of relational thinking used and how it develops, which are beyond the scope of this chapter (see Empson and Levi 2011).

representative of the types strategies that evolve in classrooms such as these—even if these classrooms are rare—and provide a study of the possibility of integrating fractions and algebra in the upper elementary grades.

Each problem involved division with a remainder to be taken into account in the quotient. For each case, we describe the strategy and then note how children used fundamental properties of operations and equality in their solutions.

Case 1: Measurement division.

The first case comes from a combination fourth- and fifth-grade class, working on the following measurement division problem:

> It takes ____ of a cup of sugar to make a batch of cookies. I have $5\frac{1}{2}$ cups of sugar. How many batches of cookies can I make?

The students were given a variety of number choices for the divisor. In order of difficulty, these choices were $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, and $\frac{3}{8}$. Several students, including Jill, chose to work with $\frac{3}{8}$ of a cup of sugar.

Jill began her strategy by drawing upon the basic multiplicative relationship described above to generate familiar groupings of three eighths that would simplify the calculation (Table 1). She said she knew that 8 three-eighths would be 3, which meant that 4 three-eighths would be half that much, or $1\frac{1}{2}$, and 12 three-eighths would therefore be $4\frac{1}{2}$ (Fig. 4). At this point, she knew that she needed only 1 more cup to use up all $5\frac{1}{2}$ cups. Again Jill used the relationship between $\frac{3}{8}$ and 3 as a reference point. She said that because 8 three-eighths was 3, a third as many three-eighths would be a third as much, or 1. That is, $(\frac{1}{3} \times 8) \times \frac{3}{8}$ is 1, and $\frac{1}{3} \times 8$ is $\frac{8}{3}$ or $2\frac{2}{3}$. She concluded that she could make a total of $12 + \frac{8}{3}$ batches, which would be equal to $14\frac{2}{3}$ batches.

If we unpack Jill's description of her solution, we see that it involved setting subgoals that were readily solved using familiar relations. The solution of one sub-

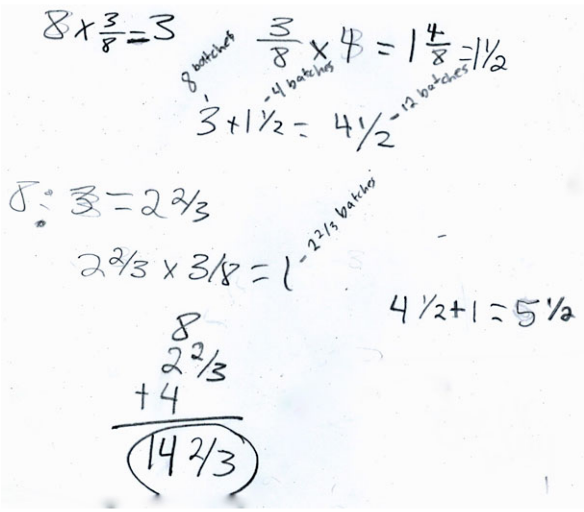

Fig. 4 Jill's written work for her strategy to solve $5\frac{1}{2}$ divided by $\frac{3}{8}$, suggesting implicit use of fundamental properties of operations and equality

goal provided a springboard for the next. Fundamental properties of operations and equality were implicit in the solution of each of the subgoals. Jill's ultimate goal was to find how many $\frac{3}{8}$ cups it would take to make $5\frac{1}{2}$ cups. She started with an overarching view of the problem that facilitated the formulation of a series of subgoals. The $5\frac{1}{2}$ cups could be partitioned into parts that would be easily divided by $\frac{3}{8}$. Then the parts could be combined.

Jill's first subgoal was to identify a multiple of $\frac{3}{8}$ that would give her a whole number that she might subsequently use as a building block to find how many $\frac{3}{8}$ cups it took to make $5\frac{1}{2}$ cups. Drawing implicitly on the kind of thinking described in Table 1, she started with the equation 8 groups of $\frac{3}{8}$ is 3.

Because she had only accounted for 3 of the $5\frac{1}{2}$ cups of flour in the problem, Jill now had to find how many $\frac{3}{8}$ cups it took to make $2\frac{1}{2}$ cups. She recognized that she could use the equation involving 8 groups of $\frac{3}{8}$ to make another $1\frac{1}{2}$ cups and that would leave exactly one cup to deal with. Essentially she used the multiplicative property of equality and the associative property of multiplication to transform the equation $8 \times \frac{3}{8} = 3$ as follows:

$$\frac{1}{2} \times \left(8 \times \frac{3}{8}\right) = \frac{1}{2} \times 3,$$
$$\left(\frac{1}{2} \times 8\right) \times \frac{3}{8} = 1\frac{1}{2},$$
$$4 \times \frac{3}{8} = 1\frac{1}{2}.$$

The next subgoal was to find how many $\frac{3}{8}$ cups it took to make the remaining one cup. Jill also used the equation $8 \times \frac{3}{8} = 3$ as the basis for addressing this subgoal. She again used the multiplicative property of equality and the associative property of multiplication to transform the core equation as shown below.

$$8 \times \frac{3}{8} = 3,$$
$$\frac{1}{3} \times \left(8 \times \frac{3}{8}\right) = \frac{1}{3} \times 3,$$
$$\left(\frac{1}{3} \times 8\right) \times \frac{3}{8} = 1,$$
$$\frac{8}{3} \times \frac{3}{8} = 1.$$

Note Jill might have simply used the reciprocal relation between $\frac{8}{3}$ and $\frac{3}{8}$ for this calculation, but she continued to build off of the equation $8 \times \frac{3}{8} = 3$. Although we believe it is likely that she did not intend to generate the reciprocal relationship between $\frac{8}{3}$ and $\frac{3}{8}$, we find its emergence here significant, because it illustrates how algebraic relationships can emerge fairly naturally in the context of children's relational reasoning. Problems such as this one provide experience with this relation.

Finally, Jill put the parts together using the additive property of equality and the distributive property.

$$8 \times \frac{3}{8} + 4 \times \frac{3}{8} + \frac{8}{3} \times \frac{3}{8} = 3 + 1\frac{1}{2} + 1 = 5\frac{1}{2}$$

and

$$\begin{aligned} 8 \times \frac{3}{8} + 4 \times \frac{3}{8} + \frac{8}{3} \times \frac{3}{8} &= \left(8 + 4 + \frac{8}{3}\right) \times \frac{3}{8} \\ &= 14\frac{2}{3} \times \frac{3}{8}. \end{aligned}$$

The kinds of thinking implicit in Jill's strategy are directly related to the kinds of thinking that are involved in solving algebra problems with meaning. She started with a primary goal—to find how many $\frac{3}{8}$-cups were in $5\frac{1}{2}$ cups—which subsequently guided the formulation of her subgoals. To make progress on solving the problem, she transformed the primary goal into a series of subgoals for which she had a ready solution. This practice is fundamental to high-school algebra in which a series of properties of operations and equality are often used to simplify a complex equation. For example, to solve a linear equation in one unknown, students set subgoals that entail finding successively simpler equations that are closer to the goal of finding an equation of the form $x =$ a number. As in the above example, the subgoals are met by repeated application of fundamental properties of operations and equality. Similarly, the goal of solving a quadratic equation is transformed into subgoals of solving simple linear equations by applying a corollary of the zero property of multiplication ($a \times b = 0$, if and only if $a = 0$ or $b = 0$).

To address each of the subgoals, Jill essentially constructed and transformed relationships of equality using fundamental properties of operations and equality in ways that were strikingly similar to the thinking used in constructing and solving equations in formal treatments of algebra. She drew on anticipatory thinking in transforming the equations into equations that could be put together to solve the problem. In other words she consistently constructed and transformed equations in ways that brought her closer to the solution of the basic problem. Again, that is essentially what solving algebra equations is all about.

Case 2: Partitive division.

Our second case involves a sixth-grade boy, Keenan, who solved the following problem:

> Two thirds of a bag of coffee weighs 2.7 pounds. How much would a whole bag of coffee weigh?

This problem involves partitive division and differs from the previous division problem in that the goal is to find out how much per group rather than to find out how many groups. Keenan's strategy included the transformation of quantities for the

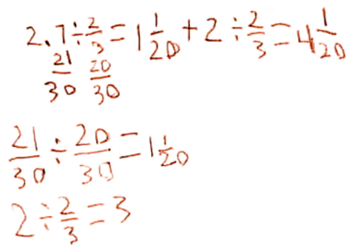

Fig. 5 Keenan's strategy to solve 2.7 divided by $\frac{2}{3}$, suggesting implicit use of fundamental properties of operations and equality

purpose of simplifying calculations as well as the flexible use of several fundamental properties of operations and equality (Fig. 5).

To start, Keenan recognized that the problem was a division problem and wrote $2.7 \div \frac{2}{3}$. He remarked, "Two divided by $\frac{2}{3}$ is going to be really easy, all I really need to worry about is the seven tenths. Seven tenths divided by $\frac{2}{3}$ isn't easy to think about so [long pause] if I make them both thirtieths, it would be easier." He notated his thinking so that it read:

$$\frac{21}{30} \div \frac{20}{30}$$

and then said, "21 thirtieths divided by 20 thirtieths is just the same as 21 divided by 20 which is one and one twentieth." He notated his answer:

$$\frac{21}{30} \div \frac{20}{30} = 1\frac{1}{20}.$$

Keenan then said, "Now all I have to do is 2 divided by $\frac{2}{3}$, which is 3." When asked how he knew that so quickly he said, "2 divided by $\frac{1}{3}$ would be 6 since you have 3 groups of $\frac{1}{3}$ in each 1, so 2 divided by $\frac{2}{3}$ would be 3 since $\frac{2}{3}$ is twice as big as $\frac{1}{3}$." He then extended his notation as follows:

$$2.7 \div \frac{2}{3} = 1\frac{1}{20} + 2 \div \frac{2}{3} = 1\frac{1}{20} + 3 = 4\frac{1}{20}.$$

This strategy incorporates several instances of relational thinking. Keenan began by decomposing 2.7 into .7 + 2. This choice involved anticipatory thinking in that he analyzed the problem to see what relationships he might draw upon to simplify his calculations, rather than simply begin to execute a series of steps to solve the problem. He used the commutative property of addition and a "distributive-like" property to simplify the division. Although he did not notate this step, his thinking could be represented as:

$$2.7 \div \frac{2}{3} = (2 + .7) \div \frac{2}{3} = (.7 + 2) \div \frac{2}{3} = .7 \div \frac{2}{3} + 2 \div \frac{2}{3}.$$

Of note is his correct use of this distributive-like property for division. This division relationship is generalizable and can be justified with the distributive property of multiplication over addition in conjunction with the inverse relationship between multiplication and division (see footnote 2).

Next Keenan facilitated the computation of $.7 \div \frac{2}{3}$ by transforming .7 into $\frac{21}{30}$ and $\frac{2}{3}$ into $\frac{20}{30}$ and then using these transformed quantities as follows:

$$.7 \div \frac{2}{3} = \frac{21}{30} \div \frac{20}{30} = 21 \div 20 = 1\frac{1}{20}.$$

Again, Keenan used anticipatory thinking to produce equivalent fractions for the purpose of simplifying the division.

Keenan then computed $2 \div \frac{2}{3}$. This computation appeared to be routine for him; however, he justified it as follows:

$$2 \div \frac{2}{3} = 2 \div \left(2 \times \frac{1}{3}\right) = 2 \div \left(\frac{1}{3} \times 2\right) = \left(2 \div \frac{1}{3}\right) \div 2$$

with an associative-like property of division. Again he used a generalizable principle, similar to the distributive-like principle he used above, that could be justified with formal properties but is rooted in a relational understanding of fractional quantities and division.

Like Jill, Keenan had a unified view of the entire problem and its parts. This view allowed him to set subgoals to address the parts individually with the understanding that the answers to the problems addressed by these subgoals could be reassembled into the whole. As was the case with Jill, Keenan drew on anticipatory thinking and a fluid understanding of how expressions and equations could be transformed. Once again the parallels with the kind of thinking used in symbolic treatments of algebra are striking.

Discussion of Cases

The strategies used by these two elementary aged children to divide fractions illustrate the power of relational thinking and its algebraic character. Children's thinking in these examples was anticipatory in that their strategies were driven by a goal structure premised on relational thinking. These strategies contrast with the goal structure in the execution of standard algorithms as they are typically learned which can be summarized as "do next."

Further, the thinking displayed by these children resembles the "competent reasoning" that proficient mathematical thinkers use to compare rational numbers, as reported by Smith (1995). Based on an analysis of 30 students' solutions to order and equivalence problems, Smith argued that competent reasoning is characterized by the use of strategies that exploit the specific numerical features of a problem and often apply only to a restricted class of fractions. These strategies were reliable and efficient. He contrasted this reasoning with the use of generalized, all purpose strategies—such as conversion to a common denominator to compare fractions—

which the students in his study tended to use as a last resort.[4] These findings led Smith to conclude "the analysis of skilled reasoning with rational numbers should ... move beyond a focus on particular strategies to examine the character of the broader knowledge *system* that has those strategies as components" (p. 38). We are proposing that this system consists of children's informal algebra of fractional quantities and that it is expressed in children's relational thinking.

From this perspective, Jill and Keenan represent students who are in the process of developing a knowledge base for reasoning about fractions in ways that can be characterized as proficient and competent. Moreover, this knowledge base is integrated with properties of whole-number operations and relations and anticipates the algebra of generalized quantities, typically taught in the eighth or ninth grade. We are not suggesting that standard algorithms for fraction operations have no place in developing fluency with number operations. Proficient thinkers use them when they see no way to exploit the number relationships in a problem. However, we are arguing that when children are supported to develop relational thinking in elementary school, their knowledge of generalized properties of number and operation becomes explicit and can serve as a foundation for learning high-school algebra in ways that mitigate the development of mistakes and misconceptions.

A Conjecture Concerning Relational Thinking as a Tool in Learning New Number Content

Jill and Keenan used fundamental properties of operations and equality and other notable relationships, such as $\frac{3}{8} \times 8 = 3$, as tools in their strategies to divide fractions. The use of these relationships was coordinated within a goal structure and is a hallmark of relational thinking. In this section we discuss a critical and perhaps surprising implication of a focus on relational thinking in the elementary curriculum with respect to the role of other types of tools such as concrete materials and models in facilitating the development of children's understanding of number operations involving fractions (as well as decimals and integers—which are beyond the scope of this chapter).

Some approaches to teaching for understanding emphasize the use of concrete materials such as base-ten blocks or fraction strips to model abstract relationships (e.g., Van de Walle 2007). The use of such materials has at times been seen as a universal remedy to children's difficulties in understanding mathematics. Several studies have shown, however, that concrete materials alone are insufficient at best and at worst, ineffective (Brinker 1997; Resnick and Omanson 1987;

[4]This approach to numerical reasoning is not unique to children. Dowker (1992) reported that professional mathematicians prefer to approach computational estimation in the same ways, that is, by exploiting specific numerical features of a problem rather than using a generalized algorithm that works in all cases.

Uttal et al. 1997). In a review of this research, Sophian (2007) noted that manipulatives are symbols themselves and how they map to mathematical notation and processes is best appreciated by those who already understand the mapping (p. 157). Teachers can show students how to manipulate these materials to perform calculations involving fractions just as they show students how to manipulate symbols to perform calculations. Some students may remember steps involving materials more easily than they remember symbolic algorithms, but in neither case are they necessarily reasoning about the relationships involved in each step or more globally in the problem.

In contrast, when manipulatives and other types of models are used as tools to think with, rather than to simply generate an answer, they can play a critical role in the development of children's understanding (Carpenter and Lehrer 1999; Koehler 2004; Martin and Schwartz 2005). For example, the images that children create and reason about as they partition quantities such as cupcakes and sandwiches in their solutions to story problems can help children conceptualize fractions in terms of basic relationships such as $\frac{2}{3} = 2 \times \frac{1}{3}$ (Empson and Levi 2011).

Keeping in mind this valuable use of visualizing tools, we propose a shift in relative emphasis as the curriculum turns to advanced number operations: *Children's use of relational thinking can and should drive the development of new content and concrete materials and models should be used to support the emergence of relational thinking*. Jill's and Keenan's strategies for division of fractions described above cannot be mapped in any straightforward way onto the manipulation of concrete materials and so do not appear to represent an abstraction of their operations on concrete materials. Instead, these strategies (1) incorporated a relational understanding of fractions and (2) were planned and executed (sometimes in an emergent sense) on the basis of each child's understanding of fundamental properties of operations and equality.

One fairly popular way to introduce fraction multiplication, for example, is by using an area model (Izsák 2008). Its advantages are that it is generalizable—it can be used to multiply any two fractions—and it is "concrete" so children can "see" the multiplication. To multiply $\frac{1}{4} \times \frac{2}{3}$ using this model, for example, a rectangular unit is divided into thirds and two of the thirds are shaded. Then the rectangle is divided into fourths orthogonally to the original partition into thirds. Based on this partitioning, one fourth of the rectangle is shaded. The intersection of the shaded parts (Fig. 6) represents the product of one fourth and two thirds. The model might be used to develop understanding of multiplication of fractions, but the use of this model is easily proceduralized, especially if it is introduced before children have had opportunities to make and integrate relational connections between quantities and operations. (See teachers' own difficulties with the proceduralization of this model, reported in Izsák 2008.)

Using relational thinking, children might approach a problem such as this one in any number of ways employing strategies that involve the application of generalized properties of arithmetic. For example, a child could reduce the calculation to operating on unit fractions, by applying the distributive property of multiplication over addition. A child might say, "a quarter of $\frac{1}{3}$ is $\frac{1}{12}$ so a quarter of $\frac{2}{3}$ has to be $\frac{1}{12}$ plus

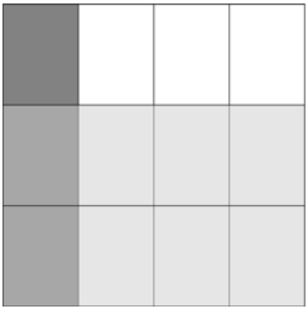

Fig. 6 Area representation of the product of $\frac{1}{4} \times \frac{2}{3}$

$\frac{1}{12}$ which is $\frac{2}{12}$." This thinking could be formally represented:

$$\frac{1}{4} \times \frac{2}{3} = \frac{1}{4}\left(\frac{1}{3} + \frac{1}{3}\right) = \left(\frac{1}{4} \times \frac{1}{3}\right) + \left(\frac{1}{4} \times \frac{1}{3}\right) = \frac{1}{12} + \frac{1}{12} = \frac{2}{12}.$$

A child could also transform the calculation into an easier one through an implicit use of the associative property of multiplication. The reasoning might be "a quarter of $\frac{1}{3}$ is $\frac{1}{12}$ so a quarter of $\frac{2}{3}$ has to be $\frac{1}{12}$ times 2 which is $\frac{2}{12}$:"

$$\frac{1}{4} \times \frac{2}{3} = \frac{1}{4} \times \left(\frac{1}{3} \times 2\right) = \left(\frac{1}{4} \times \frac{1}{3}\right) \times 2 = \frac{1}{12} \times 2 = \frac{2}{12}.$$

These strategies represent the same types of relational thinking that we saw in children's strategies for division and mirror the types of relational thinking that children use in whole-number multiplication (Baek 2008). They are two examples of possible strategies that are driven by relational thinking instead of the potentially rote use of a concrete model. With some experimentation, the reader should be able to generate strategies for the multiplication of any two fractions that incorporate the same fundamental properties of operations and equality and are robust and generalizable.

In summary, our conjecture is that if instruction is focused on developing relational thinking with whole numbers throughout the early grades, the role of concrete materials in introducing and developing understanding of operations on fractions and decimals will likely change. Concrete materials would be used to support the development of relational thinking rather than simply as tools to calculate answers or justify algorithms. Further, encouraging children to construct and use procedures based on relational thinking would help them to integrate learning number operations across different number domains.

Conclusion

The kinds of activity and thinking illustrated in this chapter are not isolated examples, and they do not represent mathematics that should be reserved for only a limited number of students or as supplementary enrichment (Carpenter et al. 2003). The results of a recent study by Koehler (2004) document that young children of a wide range of abilities are able to learn to think about relations involving the distributive property and that instruction that focuses on relational thinking as illustrated in these examples supports the learning of basic arithmetic concepts and skills.

In this chapter we have argued that a focus on relational thinking can address some of the most critical perennial issues in learning fractions with understanding. One of the defining characteristics of learning with understanding is that knowledge is connected (Bransford et al. 1999; Carpenter and Lehrer 1999; Greeno et al. 1996; Hiebert and Carpenter 1992; Kilpatrick et al. 2001). Not all connections, however, are of equal value, and we propose that our conception of relational thinking can sharpen mathematics educators' conceptions of what learning with understanding looks like. Students who engage in relational thinking are using a relatively small set of fundamental principles of mathematics to establish relations. Thus, relational thinking can be seen as one way of specifying the kinds of connections that are productive in learning with understanding. We have presented several such connections made by children in elementary grades in the context of generating strategies for problems involving multiplication and division of fractions.

We have further argued that relational thinking is a critical precursor—perhaps the most critical—to learning algebra with understanding, because if children understand the arithmetic that they learn, then they are better prepared to solve problems and generate new ideas in the domain of algebra. However, relational thinking is almost entirely neglected in typical U.S. classrooms with the unfortunate result that children experience all types of learning difficulties as they move beyond arithmetic into learning algebra. Some proposed solutions focus on a renewed emphasis on prerequisite skills (e.g., U.S. Department of Education), while others emphasize the use of concrete materials and models (e.g., Lesh et al. 1987). We have presented an alternative view of how to address these difficulties that centers on cultivating children's implicit use of fundamental properties of the real-number system to solve arithmetic problems, to better align the concepts and skills learned in arithmetic and algebra. At the heart of this view is the reciprocal relationship between arithmetic and algebra as it is revealed in children's reasoning about quantity.

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Professional Development to Support Students' Algebraic Reasoning: An Example from the Problem-Solving Cycle Model

Karen Koellner, Jennifer Jacobs, Hilda Borko, Sarah Roberts, and Craig Schneider

Abstract The Problem-Solving Cycle (PSC) model of mathematics Professional Development (PD) seeks to enhance teachers' knowledge and skills in a variety of domains. In this paper we consider how participating in the PSC program, with a specific focus on algebra, impacted one teachers' instructional practice. We explore

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the nature of change in the teacher's classroom, as well as the connection between these changes, the focus of our PD, and the teacher's intentions. Our analyses are based on a set of videotaped classroom lessons collected over two years, along with interviews and written reflections. We conclude that there was a close match between the teacher's personal goals for improvement and our program goals, and notable shifts in his algebra instruction that was likely to have supported his students' algebraic reasoning.

Introduction

Researchers agree that mathematics instructional practices geared to meet the needs of all learners look very different from the traditional approach of lecturing and teaching algorithms through a rote memorization process (Kaput 2007). Such teaching takes into account the complexities of supporting mathematical reasoning. It requires, first and foremost, careful attention to students' thinking, which then guides instructional decisions built on students understanding, such as selecting appropriate tasks, promoting mathematical communication, and considering multiple representations.

Attention to the needs of all learners is particularly important for the domain of algebra. In the United States, there is an increasingly widespread commitment to ensuring that all students successfully complete a course in algebra before entering high school (National Mathematics Advisory Panel 2008). Students' difficulties in learning formal algebra are well documented (Kieran 2007; National Research Council 1998), yet many teachers continue to use teaching methods that focus on rote memorization and algorithmic approaches to solving algebra problems without supporting the development of deeper procedural and conceptual understanding (Ball et al. 2001).

Moses and Cobb (2001) offer important insights into instructional approaches that provide access to algebra for all students. For example, they reported that for students to find algebra valuable and engaging and to participate in discussions, it is critical that the teacher select tasks that are relevant to their lives and have more than one solution strategy. Other researchers similarly encourage teachers to foster classroom discussions, particularly in their algebra lessons—including small group, whole class, and partner discussions (Winicki-Landman 2001). Structuring activities to encourage students to talk with their peers can elicit productive conversations about unique representations and multiple solution strategies (Cnop and Grandsard 1998). Setting as instructional priorities the selection of appropriate tasks, the promotion of student-driven communication, and the encouragement of multiple representations all have been suggested as ways to provide students with a better grasp of algebraic concepts (NCTM 2000).

Despite the increased emphasis on algebra in K-12 education, it appears that most teachers have weak and fragile understandings of how to teach algebra effectively (Even 1993; Nathan and Koedinger 2000; Hadjidemetriou and Williams

2002). Thus, not surprisingly, the enhancement of teachers' professional knowledge about algebra and the teaching of algebra is widely seen as a key component of the effort to support students' algebraic reasoning (Blanton and Kaput 2005; Kieran 2007; NCTM 2000). The Problem-Solving Cycle (PSC) model of Professional Development (PD), which is the focus of this chapter, builds on that premise; its central aim is to help teachers enhance their professional knowledge and instructional practices to support all students' development of algebraic reasoning.

The Problem-Solving Cycle Model of Professional Development

The PSC was developed and implemented as part of the Supporting the Transition from Arithmetic to Algebraic Reasoning (STAAR) project.[1] Our major focus was on strengthening teachers' professional knowledge of central algebra concepts, and helping them explore ways of fostering their students' algebraic thinking. A situative perspective on cognition and learning provides the conceptual framework that guided the design of the PSC. Scholars within a situative perspective argue that knowing and learning are constructed through participation in the discourse and practices of a community, and are shaped by the contexts in which they occur (Greeno 2003; Lave and Wenger 1991) With respect to PD, situative theorists focus on the importance of creating opportunities for teachers to work together on improving their practice, and locating these learning opportunities in the everyday practice of teaching (Ball and Cohen 1999; Putnam and Borko 2000).

Three design principles derived from a situative framework are central to the PSC model: establishing a professional learning community, using video from teachers' own classrooms to provide a meaningful context for learning, and establishing community around video. For more extensive discussions of our conceptual framework and design principles, please see Borko et al. (2005, 2008), Jacobs et al. (2007), Koellner et al. (2007).

The PSC consists of a series of three interconnected workshops in which teachers engage in a common mathematical and pedagogical experience, organized around a rich mathematical task (see Fig. 1). This common experience provides a structure within which teachers can build a supportive community that encourages reflection on mathematical understandings, student thinking, and instructional practices.

Workshop 1 focuses on the mathematics content knowledge needed to teach the focal problem. The majority of time is devoted to teachers collaboratively solving the problem, debriefing the mathematics in their solution strategies, and making connections between solutions. In addition, teachers develop lesson plans for teaching the problem to their students. Between Workshop 1 and 2, each teacher is videotaped implementing his or her lesson.

[1]The PD program is one component of STAAR and was partially supported by NSF Proposal No. 0115609 through the Interagency Educational Research Initiative (IERI).

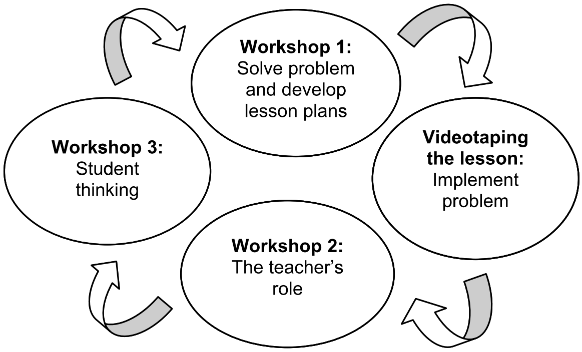


Fig. 1 The problem-solving cycle model of professional development

The central purpose of Workshop 2 is to help teachers consider instructional skills and strategies associated with problem-based teaching (Lampert 2001). In particular, teachers are guided to think deeply about the role they played in teaching the selected problem to their students. Activities are designed around a specific pedagogical topic and associated video clips from one or more of the teachers' lessons.

Workshop 3 addresses skills and strategies that comprise the core practice of "learning about student understanding" (Grossman et al. 2009). Teachers consider ways to elicit, attend to, and build on students' mathematical understandings by studying video clips and other artifacts from their lessons that depict student thinking. By considering the various forms of mathematical reasoning their students applied to the problem and the different solution strategies they used, teachers have the opportunity to gain further insight into the mathematical concepts entailed in the problem and students' learning of those concepts.

The PSC as Implemented in the STAAR Project

Although the PSC model is intended to be applicable to any domain of mathematics, as part of the STAAR project our content focus was on algebra. Our research and PD program as part of the STAAR project began in the summer of 2003 with a 2-week algebra institute. Over the next 2 academic years we held monthly, full-day PD workshops (7 workshops per year). In Fall 2003 we held workshops focused on pedagogical practices associated with algebra, and in Spring 2004 we conducted the first iteration of the PSC using a rich algebraic problem. We conducted two more iterations during the 2004–05 academic year with an algebraic focus.

Eight middle school mathematics teachers who attended the summer institute then took part in the program during the 2003–04 academic year. In 2004–05, seven teachers continued working with us and three additional teachers joined the program. Each new teacher was a colleague of one of the current participants.

We had explicit mathematical, instructional, and process foci for each iteration of the PSC. Our mathematical foci can be seen in the problems we selected. In all 3 iterations, we used problems adapted from Driscoll's (1999) book *Fostering Algebraic Thinking: A Guide for Teachers, Grades 6–10* (see Table 1). The three prob-

Table 1 Three iterations of the problem-solving cycle

	Workshop 1: Doing the Problem and Planning the Lesson	Workshop 2: Analyzing the Teacher's Role	Workshop 3: Analyzing Student Thinking
Iteration 1: Painted Cubes	*Problem Statement* A cube with edges of length 2 centimeters is built from centimeter cubes. If you paint the faces of this cube and then break it into centimeter cubes, how many cubes will be painted on three faces? How many will be painted on two faces? On one face? How many will be unpainted? What if the edge has a length different from 2? What if the length of the large cube is 3 cm? 50 cm? *n* cm? *Math Focus*: Finding patterns, identifying and comparing different types of functions (linear, quadratic, cubic, constant) using multiple representations, recursion, and variables.	Teachers watched video clips from 3 lessons and discussed issues related to: • Launching the lesson • Closing the lesson • Issues related to language • What questions or probes might you use to move the student forward	Teachers looked at students' posters from one lesson and discussed the different solution strategies, ways of thinking, and mathematical errors. Teachers watched a video clip from 1 lesson and discussed issues related to: • Helping students through mathematical misconceptions • Developmental probes to push student thinking
Iteration 2: Snakes in Snakewood	*Problem Statement*: Snakes have colored rings that follow a changing pattern. As snakes grow older, the pattern extends in a systematic way. It starts with a white ring. Then a black ring develops in the middle of the white ring. In the next stage, the same thing happens with each white ring, but the black ring stays the same. What do the 4^{th} and 5^{th} stages look like? What is the relationship between the numbers of white and black rings in each stage? How can you find the number of white and black rings at any stage? *Math focus*: Identifying patterns, recursion, and exploring quadratic functions using multiple representations.	Teachers watched 6 video clips from several lessons and discussed issues related to: • deciding when to provide explanations and ask leading questions, • considering students' reactions to teachers' comments and questions, • letting students follow their own line of reasoning.	Teachers looked at students' posters from several lessons and discussed where their thinking appeared to be on the right or wrong track. Teachers watched 4 video clips from several lessons and discussed issues related to student thinking, including: • how students move from recursive patterns to formal rules, • student misconceptions.

Table 1 (Continued)

	Workshop 1: Doing the Problem and Planning the Lesson	Workshop 2: Analyzing the Teacher's Role	Workshop 3: Analyzing Student Thinking
Iteration 3: Skyscraper Windows	*Problem Statement*: A building is 12 stories high and is covered entirely by windows on all 4 sides. Each floor has 38 windows on it. Once a year, all windows are washed. The cost for washing the windows is \$2.00 for each first-floor window, \$2.50 for each second-floor window, \$3.00 for each third-floor window, and so on. How much will it cost to wash the windows of this building? What if the building is 30 stories tall? *n* stories tall? *Math focus*: Identifying patterns, making connections to strings of numbers, exploring quadratic functions, and recursion.	Teachers watch 3 clips and considered the teacher role and following issues: • teacher questioning • listening to student thinking • appropriate mathematical questions to move students forward from where they started	Teachers view 3 clips and considered the following issues: • analysis of student solution strategies • pre-requisite knowledge • assessment and finding evidence of particular student conceptions

Note: All problems are adapted from Driscoll (1999) lems highlighted a variety of critical topics related to algebraic reasoning including describing and generalizing patterns and functional relationships, differentiating among functions, assessing the notion of equality, using multiple representations to model and solve problems, and recognizing and generating multiple solution strategies. Additionally, the problems all met the following criteria: (1) address multiple mathematical concepts and skills, (2) are accessible to learners with different levels of mathematical knowledge, (3) have multiple entry and exit points, (4) have an imaginable context, (5) provide a foundation for productive mathematical communication, and (6) are both challenging for teachers and appropriate for students.

Our instructional foci can be most clearly identified by considering our second and third workshops. As Table 1 shows, Workshop 2 discussions focused on analyzing instructional practices such as launching and closing the lesson to support student learning; posing questions to move students forward in their thinking; deciding when to provide explanations, ask leading questions, and let students follow their own line of reasoning; and listening to students' thinking. Workshop 3 discussions centered on analyzing student thinking; they addressed core practices such as comparing students' solution strategies, and understanding and working with students' preconceptions and misconceptions. These instructional foci provided a lens when selecting artifacts of practice (including video clips and student work) to anchor discussions throughout the workshops.

Our process foci emphasized the importance of establishing and maintaining a safe environment for communication, explaining and justifying one's own thinking, and actively processing one another's ideas (Borko et al. 2005; Clark et al. 2005). With each successive iteration of the PSC, as the teachers' content knowledge and analytic skills increased and their community strengthened, we encouraged them to probe more deeply into relevant and challenging ideas (Borko et al. 2008).

Prior Research on the Development and Impact of the PSC

As part of the STAAR project, we utilized a design experiment approach (Cobb et al. 2003; Design-Based Research Collective 2003) to study and refine the PSC model. We collected and analyzed a large amount of data on processes involved in developing and enacting the model, and on the impact of the PD experience on the teachers' professional knowledge and instructional practices.

Several of our initial analyses focused on the processes of developing and enacting the model. For example, we explicated the specific knowledge strands that are foregrounded during the three PSC workshops, and the opportunities STAAR participants had to expand their professional knowledge within each workshop (Koellner et al. 2007). Additionally, we analyzed discussions around video and found that the teachers talked in an increasingly focused, in-depth, and analytic manner about specific issues related to teaching and learning (Borko et al. 2008).

We also examined project's impact by analyzing teacher interviews and written reflections. These self-report data indicate a strong impact on specific, targeted areas of the participants' professional knowledge, including: mathematics content (e.g., the importance of working on rich tasks and generating multiple solution strategies), methods for improving classroom discourse (e.g., how to conduct group work and foster conversations about mathematics), and ways of encouraging student thinking (e.g., giving students authority, building on students' thinking, using tasks that promote student thinking) (Jacobs et al. 2007).

Impact of the PSC on Instructional Practice: A Case Study Analysis

In this chapter, we consider how participating in the PSC program, with a specific focus on algebra, impacted one teachers' instructional practice. Specifically, our research questions focus on the nature of change in the teacher's classroom during the period of time that he participated in our PD program, as well as the connection between these changes, the focus of our PD, and the teacher's intentions with regard to instructional change.

The focus of our case study analysis is Ken Bryant.[2] We selected Ken in order to provide a detailed example of the experiences and practices of one teacher who, while not necessarily typical of all the teachers who participated in the STAAR project, demonstrates the PSC's potential for supporting teachers' instructional change. An in-depth analysis of one teacher's movement towards improving his practice such as this may be a useful contribution to understanding the factors that foster change in mathematics teaching. As Shulman (1983) noted, "For the practitioner concerned with process, the operational detail of case studies can be more helpful than the more confidently generalizable virtue of a quantitative analysis of many cases." (p. 495)

Methods

Ken Bryant

Ken Bryant had taught both 5th and 6th grade for 3 years at the time our PD program commenced. He taught sixth grade mathematics for both years of our PD program. Ken worked in a K-8 school in a medium-sized, suburban school district. Minority students comprise approximately half of the district's population, and over a quarter of the students are eligible for free or reduced lunch. At Ken's school, the percentages of minority students and students eligible for free or reduced lunch are similar to those of the district. Ken had completed a Master's degree in Education, and near the end of our program, began working towards his principal's license.

[2] A pseudonym.

Throughout our study, Ken maintained a consistent and outspoken desire to learn and was deeply committed to improving his understanding of mathematics content and instruction. Near the beginning of the program, he reflected on both his increasing content knowledge ("I'm being overwhelmed with AH-HA moments") and his motivation to improve his instruction ("I would like to know how to ask questions that get groups to talk about math, not recess"). At the end of the first year, Ken's continuing desire to improve his practice was evident in his written reflections about what he had learned. His list included "the need for change in my practice" and "how to have a critical perspective on the actions in my classroom."

Ken maintained a perfect attendance record at our PD activities for the two years of the project. During an interview at the conclusion of the project he remarked that not only did the STAAR program change his practice, but it inspired him to talk with his principal about implementing some of the elements of the PSC model at his school. Shortly after the conclusion of our PD and research program, Ken won a "teacher of the year" award presented by his district. The fact that our research group had no part in nominating or judging teachers for this award suggests that his efforts to improve were noticed and respected by colleagues in his school and across the district.

Data Sources

The primary data source for this study is the set of videotapes of Ken's classroom instruction. As part of the PD program we videotaped each teacher conducting a lesson using the problem selected for each iteration of the PSC. We also videotaped additional lessons at regular intervals during the 2003–04 and 2004–05 school years. We always filmed using two cameras; one with a wireless mike attached to the teacher, and the other with a table mike near a group of students. In this way we were able to document the teachers' instructional moves as well as small group student activities.

In Ken's case, we videotaped 14 mathematics lessons over two years: 10 in 2003–04 and 4 in 2004–05. For the analyses in this case study, we categorized these lessons into three groups:

(1) lessons taught at the beginning of the PD program, prior to initiation of the PSC model (pre-PSC, $n = 5$)
(2) lessons taught as part of the PSC PD, using the focal problems (PSC, $n = 3$), and
(3) lessons taught during the time frame of the PSC, but that were not part of the PD program (post-PSC, $n = 6$)

We conducted audiotaped interviews with the teachers following each videotaped lesson, asking them to reflect on various aspects of the lesson. We also interviewed the teachers on a number of other occasions during the PD program including the beginning of the program, the end of each academic year, the conclusion of the program, and as a post-program follow-up. We collected written reflections as final activities in many of the PD workshops. Both the interviews and reflections prompted teachers to describe their experiences in the PD program, consider their pedagogical goals, and reflect on the impact of the program on their instructional practices. Ken's interviews and written reflections served as secondary sources for our case study analysis.

Data Analysis

As noted above, we drew upon multiple data sources to analyze the impact of the PSC on Ken's instructional practices. We also used various analytic techniques. Our primary set of analyses entailed coding Ken's videotaped lessons, and then searching for patterns in the codes for the three sets of lessons (pre-PSC, PSC, and post-PSC). To help interpret these patterns, we examined Ken's interviews and written reflections, identifying comments that related to coded features of the videotaped lessons. In addition, we use a vignette analysis to look deeply into a small portion of one of Ken's algebra lessons in order to illustrate particular aspects of his teaching. The specific procedures for these three analytic techniques (quantitative coding, supplementary use of interviews and reflections, and vignette analysis) are described below.

Coding Ken's Videotaped Lessons

While it is a fairly straightforward matter to videotape a teacher on a regular basis, it is no small feat to create a reliable and valid instrument that taps into the precise nature of the instructional practice documented on tape. Our research team used the Learning Mathematics for Teaching (LMT) project's recently developed instrument, Quality of Mathematics Instruction (QMI) (LMT 2006). The QMI enables the analysis of videotaped classroom observations with respect to how teachers use their mathematical knowledge in the classroom. We decided on this instrument for both theoretical and pragmatic reasons.

From a theoretical perspective, the QMI is a good fit with our research goals and methods. Specifically, the intentions of the developers matched our intentions for using the instrument, and the constructs measured by the instrument overlap with those that our research project intended to tap into. Our research provides something of a "test" case for the QMI by implementing it with a group of coders distinct from, but in consultation with, the instrument developers. At the same time, because we used the QMI to track the instructional patterns in one teacher's lessons over time, we can attest only to the instrument's adaptability to this context, not to its generalizability across teachers or correlation with other measures.

Pragmatically speaking, the QMI was immediately available and has established psychometric properties. LMT researchers have used the QMI to code 90 elementary and middle school mathematics lessons. They established adequate reliability and are conducting a series of analyses to assess the validity of the instrument; preliminary results are promising (LMT 2006). In addition, the LMT researchers provided a detailed coding glossary, offered a workshop on use of the QMI to our project team, and were available for ongoing consultation as we adapted the instrument and worked to achieve reliability.

Applying the QMI to Ken's lessons, we selected a subset of codes that most closely matched our research questions and PD goals. For example, we did not apply codes that described the mathematical content of the lesson because, in Ken's case, the content was largely determined by his district guidelines and was not a critical issue for our research agenda. In a few cases, we were unable to establish adequate reliability for codes that initially appeared promising.

Establishing Interrater Reliability

Our method of coding closely matches that used by the instrument developers (LMT 2006). Coders first parsed each videotaped lesson into 5-minute segments. Within each segment, coders noted whether the actions indicated by each code were present.

Following the recommendation of the LMT researchers, we coded in pairs; each pair included at least one mathematics educator. We used a multi-step procedure, as follows:

1. Four members of our research team coded a lesson separately and then met to discuss disagreements and points of confusion, modify the coding category, and reconcile our coding decisions. We did this for several lessons, until we developed a workable coding scheme.
2. To achieve reliability, the two members of a pair coded 3 lessons separately and reconciled their coding. They compared their agree-upon coding decisions to those of another pair and calculated the percentage of agreement between the pairs. Reliability of at least 80% was reached on all of the codes we report on in this chapter.
3. Each remaining lesson was coded by one of the pairs.

Supplementing with Interviews and Reflections

Several members of the research team carefully examined transcripts from all of Ken's interviews and copies of his written reflections, to identify instances in which Ken referred to topics captured by the QMI codes. In particular, we looked for comments Ken made about his instructional goals, intentions, or changes in these areas. We incorporated a representative selection of these comments into the results section, in order to provide Ken's own interpretations of his classroom behaviors.

Vignette Analysis

The QMI allowed us to look at general trends across Ken's lessons, and not content-specific trends. Because our PD was focused on algebra, and the three PSC lessons that Ken conducted were all algebra lessons, a closer examination of those lessons seemed warranted in order to understand how Ken supported students' algebraic reasoning. We focused on Ken's final PSC lesson, in order to gauge the sorts of learning opportunities he provided at the end of the PD program. From this lesson, we identified a relatively short teaching episode that incorporates many of the features captured by the QMI that are aligned with our PD program goals.

We constructed a vignette based on this episode with the intention of capturing specific details of Ken's practice while preserving the complexity and richness of the classroom context (Miles and Huberman 1994). The vignette provides an example of many of the QMI codes "in action" and also depicts what Ken's lessons looked like after two years of participation in the PD. Within the vignette, we summarize the events that took place during this episode, include transcripts of conversations that highlight important mathematical and instructional elements, and provide an analysis of the events with respect to both our coding and our PD goals.

Results and Discussion

Patterns Drawn from QMI Coding and Analysis

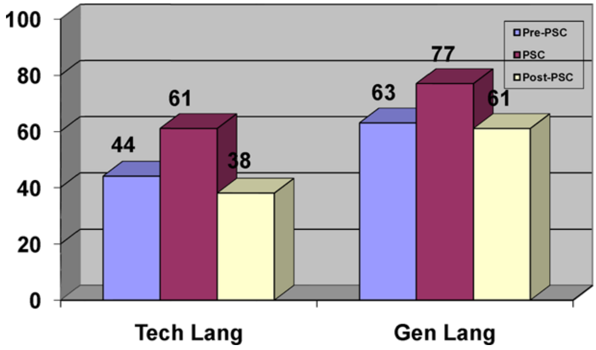
We discuss patterns in Ken's lessons in three areas: (1) knowledge of mathematics for teaching, (2) core practices for problem-based teaching, and (3) core practices for eliciting and building on student thinking. Within each of these areas, we consider the QMI codes that depict patterns of teaching that map onto the goals of our PD program, in both expected and unexpected ways. It is important to recall that the subset of QMI codes we selected matched our PD focus and represent practices in which we anticipated changes.

When interpreting the coded data, we looked for two types of patterns that we hypothesized might occur: differences between Ken's pre-PSC lessons and post-PSC lessons, and differences between Ken's PSC and non-PSC lessons (pre-PSC and/or post-PSC). We did not run inferential statistical tests; rather we present raw averages and caution readers to consider these data in light of our small sample size.

Knowledge of Mathematics for Teaching

A primary focus of our PD program, and foregrounded in Workshop 1 of the PSC, is expanding teachers' knowledge of algebra for teaching. However, identifying changes in knowledge by analyzing videotaped lessons presents a formidable challenge. We elected to use two of the QMI codes—the teacher's use of technical and general mathematical language—to delve into this topic. Segments of Ken's lessons were coded as including technical language when he accurately used mathematical terms and concepts such as "variable" or "function." Segments were coded as including general language when Ken expressed mathematical ideas and concepts

Fig. 2 Average percentage of segments that included technical language and general language



using care and precision in his language such as discussing how many dollars are needed "for each" window.

In Ken's PSC lessons, a higher percentage of segments contain technical and general mathematical language (61 and 77 percent of segments, on average, respectively) compared to his non-PSC lessons (see Fig. 2). The fact that technical and general mathematical language occurred more frequently in Ken's PSC lessons than his non-PSC lessons is in line with our PD goal of increasing teachers' understanding of key mathematical concepts, specifically in the domain of algebra (Borko et al. 2005; Jacobs et al. 2007), and may be due to our explicit focus on the algebraic concepts underlying the PSC problems in the PD workshops conducted prior to participants' teaching of each problem. The expectation was that, as a result of this focus, teachers' mathematical grounding in the problem would be very solid when they implemented it in their classrooms.

Using appropriate mathematical language was something that Ken was cognizant of, at least when teaching a PSC lesson. In an interview after teaching his first PSC lesson (in February 2004), Ken remarked, "Something that I've been saying a lot lately is that algebra is much like a language, and they need to learn how to read it and write it and speak it." Ken's focus on language often was in the service of helping his students to explain and justify their reasoning, even if the students struggled with the appropriate mathematical words in their own explanations. One strategy that he attempted to use in these situations was to paraphrase or revoice student ideas. Ken noted:

> I try to paraphrase things for them. [For example,] 'This is what you said and this is what I got from that. Is that right?' And maybe I shouldn't do that because I might be twisting their words around a little bit, but my hope is that [by paraphrasing] other people understand what that one person is trying to say.

The fact that Ken's language in non-PSC lessons did not change as substantially over time suggests that without highly focused PD work on selected problems, he is likely to use a similar amount of technical and general mathematical language in his lessons on a day-to-day basis as he did prior to participation in the PD program.

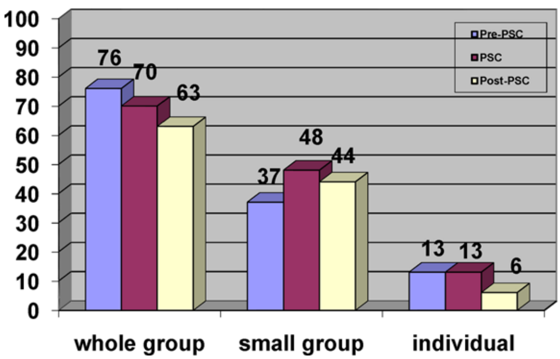
Core Practices of Problem-Based Teaching

Another primary focus of our PD program, and foregrounded in Workshop 2 of the PSC, is problem-based teaching. Two aspects of problem-based teaching that we highlighted, and that are captured by several codes on the QMI instrument, are instructional format and instructional practices. Within instructional format, we coded for (1) the organization of Ken's lessons into whole group, small group, and individual activities and (2) the allocation of time to reviewing, introducing a topic, working on a task, and closing the lesson. Within instructional practices we coded for Ken's use of mathematical representations.

Instructional Format Throughout the PD program, teachers' solving of algebra problems was largely done in small groups, and then solutions were shared with the whole group. We intentionally strived to establish sociomathematical norms in which authority resided in the community of teachers rather than with the PD facilitator (Clark et al. 2005). From the onset of the program, Ken was strongly impacted by the process of doing mathematics in small groups, and he vowed to incorporate more groupwork in his own lessons. At the end of the 2-week summer algebra institute that initiated the PD program, Ken reflected:

> I began to rethink my teaching study...Usually I keep all the kids in rows separated and all facing the front... To me it seems that I have the fewest discipline problems [this way], especially with kids talking while I am talking. Then it occurred to me that [in this PD program] we are talking nearly all the time. But we are talking about math! Fascinating! How can I get this kind of atmosphere in my class? Well this I haven't quite figured out yet but the goal is there and that is what I am going to shoot for this fall. (July 2003)

Over the two-year period that Ken took part in our PD program, he gradually reduced the amount of time his students spent working as a whole class (see Fig. 3). At the beginning of the PD (pre-PSC) on average 76% of Ken's lesson segments included whole group work. During Ken's PSC lessons, on average 70% of the time



Note: Percentages do not add to 100% because the categories are not mutually exclusive. If there was a shift in categories during a 5-minute segment, coders marked both categories as present.

Fig. 3 Average percentage of segments that included whole group, small group, and individual work was spent in whole group work, and in Ken's post-PSC lessons, on average, 63% of the time was spent in whole group work.

Corresponding to the drop in whole group time, Ken increased the amount of time his students worked in small groups. In Ken's PSC lessons, almost half of the lesson segments included small group work. The amount of time students spent working individually remained the same in Ken's pre-PSC and PSC lessons, but dropped roughly in half during his post-PSC lessons.

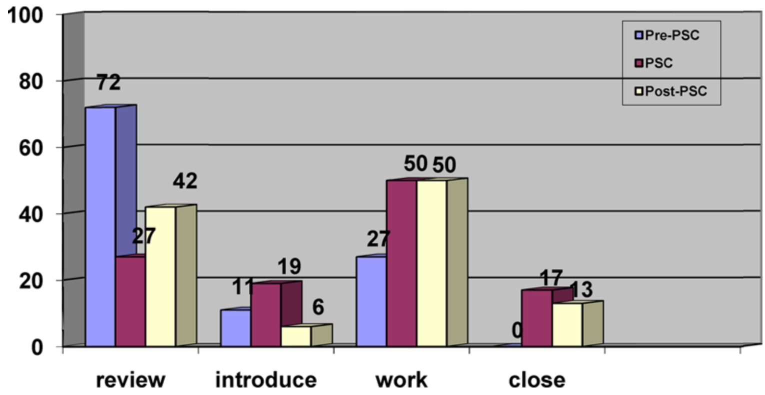
Ken's shift from having his students sit in rows with "all students facing the white board" to having them sit in groups happened in the beginning of the 2003 school year, immediately following the STAAR summer algebra institute. Ken recognized that groupwork initially was difficult for some students: "Working in groups has been something that's been pretty tough because the kids haven't been used to doing that in mathematics." It was at this time that Ken decided to focus on establishing group norms in his classroom. He explained in an October 2003 interview, "I try to really stress the importance of working together and cooperating with the people that are in your group. And I really try to stress that everyone in a group is a resource."

Establishing norms and instructional practices aligned with a more student-centered class was a gradual process. In interviews throughout the school year, Ken consistently talked about his desire for all groups to be engaged and discussing mathematics. In January 2004 he experimented with grouping students based on their test scores, creating groups with one strong student who could serve as a leader. He commented in an interview that month, "[The students] kind of understand that they have a responsibility when they're a leader. They need to lead the discussion and get started by asking questions or whatnot. That is something we're working on."

By they end of the PD program, Ken used groups in increasingly unique ways to promote in-depth, student-led conversations. For example, in his final PSC lesson, Ken walked around the room listening carefully to each group's ideas. He then suggested that two of the groups have a conversation about their strategies for deriving the formula (i.e., determining an incremental growth pattern). In an interview shortly after the lesson, Ken explained, "[The groups] were totally going about [the problem] in different ways, yet they had nearly identical formulas in the end. So this seemed to be a good time to bring the two groups together." In this way, Ken promoted across-group conversations and processing of multiple strategies within small group work time.

Ken also made a substantial shift in the allocation of time during his mathematics lessons. In his pre-PSC lessons, the vast majority of class time was spent reviewing (or "warming up" or going over homework) (see Fig. 4). As he watched his videotapes, Ken noted with displeasure the time he devoted to reviewing and set a goal that students spend more time working on mathematical problems. Specifically, he wanted to spend at least half of his lesson time providing students with the opportunity to work in small groups.

Almost three-quarters (72%) of Ken's pre-PSC lesson segments, on average, were devoted to reviewing. By contrast, 11% of the segments, on average, involved introducing the major task and 27% included student work time. No lesson segments were coded as closure or providing a summary of the mathematics.



Note: Percentages do not add to 100% because the categories are not mutually exclusive. If there was a shift in categories during a 5-minute segment, coders marked both categories as present.

Fig. 4 Average percentage of segments that included review, introducing, work time, and closure

Videotapes of Ken's PSC lessons clearly indicate a shift away from reviewing toward a greater focus on the mathematical work of the lesson. In those lessons his review time dropped to 27%, on average, whereas the time he spent introducing the problem, having students work on the problem, and closing the lesson all increased markedly.

In his post-PSC lessons, Ken spent more time reviewing than in his PSC lessons, but less than in his pre-PSC lessons. Ken spent a relatively short amount of time introducing problems, but he maintained a strong commitment to providing students with time to work on the problems, and he devoted more time to closing the lessons.

These two changes in Ken's practice—having students work in groups and providing more student work time—are closely related and are aligned with the format of our PD workshops, particularly when teachers solved mathematics problems. After two years in the PD program Ken reflected:

> I felt like I learned so much and I was able to make huge gains just from working with people and really having a lot of time [to solve problems]. And it wasn't just sit in a class, listen to a lecture, [and] get some practice problems for homework, like I've been teaching for so long. It really changed my own idea about how to deliver instruction for math. Giving discovery time, instead of just having someone model [how to solve the problem.] (June 2005)

Reflecting again a year later, he explained:

> Cooperative grouping, in general, I didn't do it before [the STAAR program]. I thought I did, but it was more of a seating arrangement than anything else. And now, I think it's letting kids learn in their groups. Kind of backing off from guided instruction and letting them do more discovery in their groups. (Spring 2006)

Instructional Practices One aspect of instructional practice that was emphasized in our PD is the importance of using mathematical representations such as manipulatives, visuals, tables and figures. Four codes on the QMI instrument capture various

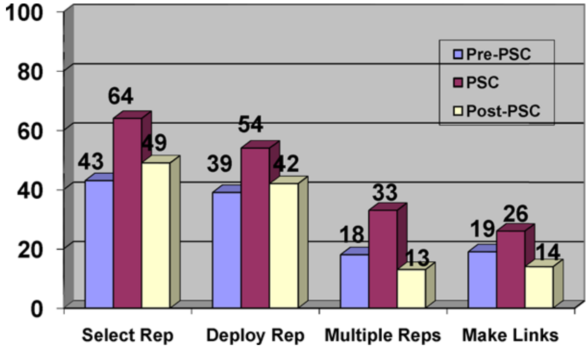


Fig. 5 Average percentage of segments that included selecting representations, deploying representations, multiple representations, and making links

aspects of how representations can be used in mathematics lessons. We found a distinct contrast between Ken's PSC lessons and his non-PSC lessons on all four codes (see Fig. 5). The PSC lessons included a higher percentage of segments, on average, in which Ken appropriately selected representations, deployed representations,[3] used multiple representations, and made links between various representations.

In his first PSC lesson in February 2004, Ken had students use cubes to model the problem situation. In an interview after the lesson, he reflected on the value of students using representations saying, "[The cube] is this thing that they are holding onto and working with. This object is giving the mathematics meaning." In addition to using cubes, Ken expected that creating a table would help students organize their data, more easily notice patterns, and ultimately generalize the patterns as equations. He explained,

> I was hoping that they would come to that realization that they needed a table. I ended up showing the class [how to make a table] because otherwise they would have just kept building bigger and bigger cubes without being able to organize the information systematically. . . . I wanted them to be able to hold their understanding [in the table.]

A comparison of Ken's pre-PSC and post-PSC lessons reveals some change in his use of representations. The percentage of segments in which he appropriately selected and deployed representations increased slightly, and the percentage of segments in which he used multiple representations or made links between them decreased slightly. It seems likely that, outside of his PSC lessons, the focus on representations fluctuated from lesson to lesson and may have been driven by nuances in the curriculum.

Learning About and Using Students' Understanding

A final focus of our PD program, and foregrounded in Workshop 3 of the PSC, is building on students' understanding. In our PD program, we explicitly focused on

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[3]The code Deployed Manipulatives is actually part of QMI section 3, but we include it here because of its connection to the other codes involving manipulatives and similar data patterns.

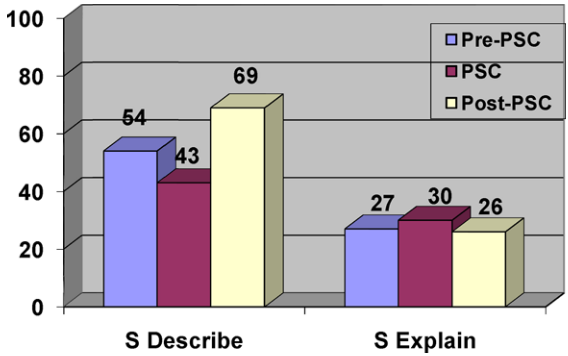

Fig. 6 Average percentage of segments that included requests for students to describe or explain

three broad topics related to student understanding that are well-captured by the QMI instrument: (1) prompting students to describe, explain and justify their ideas; (2) interpreting student productions; and (3) using students' errors. All three topics were ones to which Ken was particularly attuned; he frequently mentioned in his interviews and reflections that he was working toward improvements in these areas.

Encouraging teachers to have students describe, explain, and justify their mathematical thinking was a central premise throughout our PD program (Clark et al. 2005). Two QMI codes examined the number of times the teacher *requested* that students describe or explain their ideas; in other words, they focus on the teacher's attempts to have students generate this type of information (and not whether students were actually able to provide a description or explanation). Ken appeared less focused on requesting students to describe their ideas in his PSC lessons than his non-PSC lessons (see Fig. 6). At the same time, in his post-PSC lessons, over two-thirds of the segments on average contained a request for a description. Ken's requests for student explanations remained essentially unchanged across the three lesson categories, although there were slightly more requests for explanations in the PSC lessons than the pre- or post-PSC lessons.

An examination of Ken's interviews and reflections throughout the PD program suggests that he internalized the goal of helping students to voice their ideas, although we see only minimal evidence from the "describe" and "explain" codes. During a PD workshop in December 2003, Ken wrote, "[In my classroom] I would like to see more thought-provoking, student-engaging questions and activities that really get kids talking about math." Then again, in October 2004, Ken wrote, "My teaching goal is to better facilitate student-led discussions." During the 2004–05 school year, Ken made it more of a priority to have students come to the front of the room and share their ideas after working in small groups. As a final reflection in May 2005, he commented, "I realized the importance of talking about our thinking, and giving kids the opportunity to share their ideas."

An analysis of the codes interpreting student productions and using students' errors shows a notable difference between Ken's PSC lessons and non-PSC lessons in the degree to which he actively built on students' comments and errors (see Fig. 7). In his PSC lessons Ken devoted more time to both of these ways of using students'

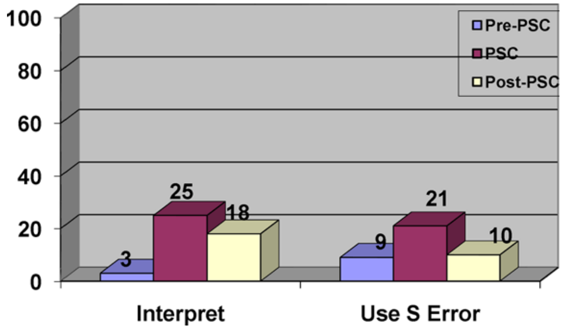

Fig. 7 Average percentage of segments that included Interpreting student productions and using students' errors

ideas to develop the mathematical frame of the lesson. Ken also did more interpreting of student productions in his post-PSC lessons compared to his pre-PSC lessons.

At the beginning of the PD program Ken reflected, "When I help a group or a student, I tend to give them too big of a clue. I need to work on just guiding kids in the direction of their own thinking" (July 2003). Over the next two academic years, Ken strived to allow his students to move in the direction of their own thinking, rather than asking everyone to think (and solve problems) in exactly the same way. In an interview conducted after a videotaped lesson near the end of the 2004–05 school year Ken explained,

> It feels so much better to get them to a level of understanding on something that they've kind of got ownership of. I mean, those were their ideas. It was so much more valuable to them to be able to go from their own perspective or the way that they thought about it to the end, rather than my way of thinking about it. (May 2005)

Vignette Analysis: Ken's Skyscraper Windows Lesson

In order to understand how Ken worked to support his student's algebraic reasoning by the end of the PD program, we consider his final PSC lesson in a more in-depth manner. In this way, we can juxtapose some of the suggested findings from the QMI coding with a particular algebra lesson. As with all of the PSC lessons, Ken had taken part in a full-day PD workshop specifically focused on solving and planning to teach this problem (Workshop 1). Because of the in-depth PD experience related to the problem, and the fact that this lesson takes place after two years of PD, we expect that it should showcase Ken's instructional changes and highlight his best efforts to support students' algebraic thinking.

In this section of the chapter, we have created a vignette to depict Ken's teaching of the *Skyscraper Windows* problem (Driscoll 1999) (see Table 1). Ken modified the problem in order to make it accessible to his sixth graders, by chunking it into more manageable components over the course of two days. On the first day, students worked in small groups to determine the cost of washing all of the windows on a 1-story building (in which all of the windows cost the same amount to wash). That lesson moved back and forth between small group work and whole class discussions.

Table 2 The cost per window and cost per floor for each floor of an 8-story building

Number of floors	Cost per window	Cost per floor
1	$2.00	$76.00
2	$2.50	$95.00
3	$3.00	$114.00
4	$3.50	$133.00
5	$4.00	$152.00
6	$4.50	$171.00
7	$5.00	$190.00
8	$5.50	$209.00

Our vignette centers on the second day of the lesson, when Ken asked his students to determine the cost of washing the windows on an 8-story building (in which the cost per window varied by floor).

When we enter the lesson, Ken has just brought his students together after they had been working in small groups. He stands by the whiteboard and calls on students to explain their strategies for finding the total cost of washing the windows on all 8 floors. As the discussion unfolds, Ken attempts to help the students clarify their ideas by asking questions and constructing a table depicting the cost per window for each floor of the building (see Table 2). The table becomes a particularly important heuristic in the students' sharing of responses and making sense of strategies.

In response to Ken's initial question about the total cost to wash all the windows in the building, students provide three different answers: $1140, $608, and $209. Ken works through each of these ideas with the students, beginning with $209.

> Ken begins, "Now, I want to talk about "process" [referring to how the students solved the problem]. Did we all agree about the cost of washing an 8-story building? How much would it cost?"
> One girl responds, "$1140." Ken records this response on the whiteboard.
> Ken asks, "Others? What did other people get?"
> Maria shares, "$608."
> Ken writes down the response and smiles, "Maybe you guys don't agree [on the solution], but that's okay."
> Another group reports, "$209."
> Ken again writes the number down and asks, "How did you get that?"
> Julia says that she multiplied $5.50 ∗ 38 [the number of windows on each floor].
> Ken then asks the class where Julia got $5.50. The answer is not given, but Ken points to the table showing that $5.50 ∗ 38 = $209. He questions, "Do you think that's how much the whole building costs?" prompting the students to carefully consider the meaning of the number $209.
> Several students respond that this amount is for only the 8th floor.

This short exchange illustrates a number of QMI codes including requesting student descriptions, using a student's error, and using a mathematical representation. We found it interesting that Ken opts to delve into the least sophisticated strategy first, encouraging students to carefully analyze this idea, understand where the error occurs, and then move forward. Note how his questioning about Julia's strategy enables other students to make sense of her reasoning. Even though the students are unable to immediately respond to his question about where Julia got $5.50, once he points to this number on the table, they quickly see that she only accounted for the windows on the 8th floor. Thus, Ken's questions, in combination with the visual representation, lead the class to understand that her group's strategy was correct for finding the cost of washing all the windows on the eighth floor, but not for the entire building.

Ken moves on to the next solution strategy, unpacking how one group of students arrived at the answer $608. Ken continues questioning and this time he is able to generate a more elaborated explanation.

> Ken asks, "Who gave us $608?"
> Maria responds, "To get my answer I took 76 ∗ 8, where it is $76 for each floor.
> Another student jumps in to counter this answer, saying, "But the price increases by $.50 each floor, so you can't just multiply the result by $2 for each window [like on the first floor]."
> Ken revoices the student's response, "So this would only work for us if each floor was $2 per window? But is that the case [for this problem]?"
> Several students respond, "No."
> Ken continues in his effort to revoice the student's idea, "I like your effort, but that solution did not work since the price increases $.50 for each floor."

In this excerpt, Ken assures the students that he values the contributions they have made. He intentionally promotes a discussion that builds from students' errors in order to ensure that all students understand the central concepts in this problem. Ken encourages students to engage with their classmates' ideas, and he helps to interpret and clarify ideas that might not be understood by everyone. In addition, Ken models general mathematical language that is appropriate for this problem.

Lastly, Ken moves on to the third answer, $1140. He remains in the front of the room, asking questions to guide the conversation. He erases the first two incorrect answers ($209 and $608) from the whiteboard. It seems clear that Ken has intentionally chosen to work through these erroneous solutions before delving into an explanation for the correct response.

> Ken asks, "Well, what about this one?" [pointing to $1140]
> Michelle shares, "I got what it cost to wash each floor individually, and then I added them all together."
> Other students nod in agreement.
> Ken reiterates, "So, you took $76 + $95 + $114 and so on to get $1140?" [pointing to the table on the whiteboard].
> Michelle and others at her table say, "Yes."
> Ken asks, "Does everyone see how they did that? And, how it is different than the first two strategies we looked at?"

The preceding discussion of the first two incorrect strategies paved the way for students to follow the logic behind this relatively brief description of the correct strategy. Ken again referred to the table in order to clarify what Michelle meant, making links between what was written on the table and the specific numbers that she added to get to the total amount. This exchange demonstrates the intersection of several QMI codes: using a mathematical representation, making links, and interpreting a student's production. Shortly after the conversation took place, Ken asked the students to work in groups to find the total cost of washing windows on a 30-story building. Their success on this quite complicated task indicates that most did follow the reasoning behind the strategies that were presented for an 8-story building.

From the perspective of our research team, Ken clearly enacted the goal of supporting his students' thinking about the algebraic patterns in this problem. He used a variety of instructional techniques that offered all of his sixth graders access to challenging mathematics in a meaningful context. Furthermore, throughout the lesson, Ken listened carefully to students' ideas and supported them to follow their own line of reasoning, moving them forward in the intended direction by engaging in productive conversations characterized by appropriate mathematical language, descriptions and explanations, interpreting of student ideas, building from student errors, and frequent references to mathematical representations.

Conclusions

There is a broad consensus that major shifts are needed in the way teachers approach mathematics instruction, particularly algebra teaching for middle school students. However, many studies have shown that teachers' conceptions of instructional practices are resistant to change (Ferrinni-Mundy and Schram, 1997; Jacobs et al. 2006). The PSC model takes a comprehensive approach to PD, seeking to enhance teachers' knowledge and skills in a variety of domains. Given the STAAR project's focus on algebra, all three iterations of Workshop 1 concentrated on improving teachers' algebraic content knowledge, Workshop 2s highlighted core practices of problem-based algebra instruction, and Workshop 3s highlighted eliciting and building on students' understanding of algebraic concepts. The PD program that the STAAR participants engaged in encouraged them to work toward changes on a host of interrelated topics, all widely seen as key elements in promoting students' algebraic learning.

Our data suggests that Ken Bryant internalized many of the PD program's goals, and strived to make important changes in his mathematics classrooms. Drawing from Ken's interviews and written reflections throughout the project, together with an analysis of his classroom instruction over two years, it appears that Ken made steady and intentional progress toward these goals.

We also found that Ken's PSC lessons looked, in many ways, different from his non-PSC lessons. Ken may have taught his PSC lessons differently because of the extensive PD work around those problems, which could have increased his content and pedagogical content knowledge on specific mathematical topics, and prompted an emphasis on particular instructional goals. In addition, the PSC problems were all algebra-based, whereas Ken's other videotaped lessons varied with respect to their content focus. It is possible that Ken's algebra lessons, in general, look different from his non-algebra lessons. We are also conscious of the fact that our videotaping included only a small subset of the hundreds of mathematics lessons that Ken taught over a two-year period.

Although the picture of Ken that we have painted may not be representative of all aspects of his teaching, it seems reasonable to conclude that there were some shifts in his instructional practices related to the goals of the program, and that these changes can be attributed to his participation in the program. Additionally, Ken appears to have strong intentions for continued pedagogical reflection and improvement.

Our initial program of research in the STAAR project suggests that focused and sustained PD programs such as the Problem-Solving Cycle can influence change in practice, albeit at a relatively slow pace. Additional research is needed to determine whether these changes in instructional practice create a classroom environment that makes mathematics more accessible to all students, and whether the PSC is scalable and sustainable to the extent that it can make a notable difference in mathematics teaching and learning. Furthermore, additional investigations are needed to understand the relationship between the PSC model of PD and changes in instructional practice corresponding to the curriculum and content teachers use on an everyday basis.

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Using Habermas' Theory of Rationality to Gain Insight into Students' Understanding of Algebraic Language

Francesca Morselli and Paolo Boero

Abstract In this chapter we consider students' use of algebraic language in mathematical modeling and proving. We will show how a specific model derived from Habermas' construct of rational behavior allows us to describe and interpret several kinds of students' difficulties and mistakes in a comprehensive way, provides the teacher with useful indications for the students' approach to algebraic language and suggests further research developments.

Introduction

Habermas' work has attracted the interest of many educational scholars (see the review of the translation into English of *Truth and Justification* by Tere Sorde Marti 2004). According to our knowledge, all uses of Habermas' ideas so far concern general issues in education, related to the changing aims and features of schooling in a changing world, and the framing of participatory action research (in particular, see Carr and Kemmis 2005; Kemmis 2005, 2006). No specific research development concerns mathematics education, though some values and innovative trends in this field are related to those general issues. However, we think that at least one of Habermas' constructs, that of "rational behavior", is of specific interest for mathematics education. Indeed (as we will try to show in this chapter) it allows one to analyze complex mathematical activities (like conjecturing, proving, modeling) in a comprehensive way and to deal with them not only as school subjects and sets of tasks, but also as ways of experiencing mathematics as one of the components of scientific rationality.

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In this chapter, we will present and discuss some specific ways of using Habermas' construct of rational behavior in mathematics education in the area of proving and modeling.

First of all, we will present our adaptation of Habermas' construct to the special case of the use of algebraic language in proving and modeling. We will show what it can bring to the field by comparing it with other analytical tools and elaborations. Our adaptation will be illustrated through an ad-hoc selection of short emblematic episodes that will be analyzed by means of the new theoretical tool. Afterwards, we'll discuss some relevant episodes from a teaching experiment in order to show how the theoretical tool can be used by mathematics educators and teachers for the management and analysis of teaching and learning of algebraic language in the approach to proving. Finally, we'll sketch research developments concerning the dynamic interplay between the components of rational behavior in the use of algebraic language and the role of verbal language as a crucial tool for rational behavior in it. These research developments will add new arguments to the elaboration presented in Boero et al. (2008) and will relate to the specific functions of verbal language in mathematical activities. We will also outline educational implications of these developments.

Habermas' Construct of Rational Behaviour

According to Habermas' definition (see Habermas 2003, Chap. 2), a rational behavior in a discursive practice can be characterized according to three interrelated criteria of rationality: *epistemic* rationality (conscious control of the validity of statements and inferences that link statements together within a shared system of knowledge, or theory); *teleological* rationality (conscious choice and use of tools and strategies to achieve the goal of the activity); *communicative* rationality (conscious choice and use of communication means within a given community, in order to achieve the aim of communication).

In a long-term research perspective, we think that Habermas' construct is a promising analytic instrument in mathematics education because it connects the individual and the social by taking into account the epistemic requirements of "mathematical truth" in a given cultural context and the ways of ascertaining and communicating it by means of suitable linguistic tools.

In our previous research we have dealt with an adaptation of Habermas' construct of rational behavior in the case of the approach to conjecturing and proving (see Boero 2006; Morselli 2007; Morselli and Boero 2009). In this chapter we focus our interest on students' use of algebraic language in proving and modeling. Our adaptation of Habermas' construct will be introduced in the subsequent section.

Adaptation of Habermas' Construct of Rational Behavior to the Case of the Use of Algebraic Language

The aim of this section is to match Habermas' construct of rational behavior to the specificity of the use of algebraic language in modeling and proving.

Algebraic language will be intended in its ordinary meaning: a system of signs and transformation rules, which is taught in secondary school as a tool to generalize arithmetic properties, to develop analytic geometry and to model non-mathematical situations (in physics, economics, etc.). In particular, algebraic language can play two kinds of roles for modeling (according to Norman's broad definition: see Norman 1993, and Dapueto and Parenti 1999, for a specific elaboration in the case of mathematics): a tool for proving through modeling within mathematics (e.g. when proving theorems of elementary number theory)—*internal modeling*; or a tool for dealing with extra-mathematical situations (in particular to express relations between variables in physics or economics, and/or to solve applied mathematical problems)—*external modeling*.

Our interest in considering the use of algebraic language from the perspective of Habermas' definition of rational behavior comes from our previous research (Boero 2006; Morselli 2007), which suggests that some of students' main difficulties in conjecturing and proving depend on specific aspects (already pointed out in literature: see next Section) of the use of algebraic language that make it a complex and demanding matter for students. In particular, we refer to: the need to check the validity of algebraic formalizations and transformations; the correct and purposeful interpretation of algebraic expressions in a given context of use; the goal-oriented character of the choice of formalisms and of the direction of transformations; the restrictions that come from the needs of following taught communication rules, which may contradict private rules of use or interfere with them. Accordingly, we propose the following three dimensions of rational behavior in the use of algebraic language in proving and modeling.

Epistemic Rationality

This consists of two requirements:

- *modeling requirements*, which concern coherency between the algebraic model and the modeled situation: control of the correctness of algebraic formalizations (be they *internal* to mathematics—like in the case of the algebraic treatment of arithmetic or geometrical problems; or *external*—like in the case of the algebraic modeling of physical situations) and interpretation of algebraic expressions.
- *systemic requirements*, which concern the use of algebraic language and methods. In particular, these requirements concern the manipulation rules (syntactic rules of transformation) of the system of signs usually called algebraic language, as well as the correct application of methods to solve equations and inequalities.

Teleological Rationality

This consists of the conscious choice and finalization of algebraic formalizations, transformations and interpretations that are useful to the aims of the activity. It includes also the correct, conscious management of the writer-interpreter dynamics (Boero 2001): the author writes an algebraic expression, transforms it and interprets it according to the goal of the activity.

Communicative Rationality

In the case of algebraic language, we need to consider not only communication with others (explanation of the solving processes, justification of the performed choices, etc.) but also communication with oneself (in order to activate the writer-interpreter dynamics). Communicative rationality requires the author to follow not only community norms concerning standard notations, but also criteria for easy reading and manipulation of algebraic expressions.

As an example to illustrate some aspects of the relationships and differences between epistemic and teleological rationality, we can consider the problem of finding the rectangle with maximum area among those of given perimeter, say $2p$. This example uses algebraic language both in (internal) modeling and in proving.

In order to find and justify the solution, we can express the length of each of the two longer sides of the rectangle as $p/2 + x$, consequently the length of each of the shorter sides is $p/2 - x$. The expression $(p/2 + x)(p/2 - x)$ represents the area of the rectangle; by multiplying we get:

$$\text{Area} = p^2/4 - x^2.$$

By checking this expression we see immediately that the maximum is obtained when $x = 0$. By interpreting this result we find a square of side $p/2$ as the rectangle of maximum area.

By choosing another expression for the area, we call x the length of one side of the rectangle, consequently the length of the other side will be $p - x$ and the formula for the area will be:

$$\text{Area} = x(p - x) = px - x^2.$$

In this case it is not so evident that the area reaches its maximum when $x = p/2$ (either analytic geometry tools or calculus tools, or a change of variables, are needed to get the result).

These two simple examples show what we mean by epistemic rationality and teleological rationality: both algebraic expressions used to model the area of a rectangle of given perimeter are correct (modeling requirements of epistemic rationality are satisfied in both cases). The systemic requirements of epistemic rationality are also satisfied (performed algebraic transformations are correct). However, from the teleological rationality point of view the two expressions are not equivalent: only the first one yields the result in a straightforward way.

Further examples will be provided in the next Sections.

Relationships with Other Studies on Proving and Modeling and on the Teaching and Learning of Algebra

The aim of this Section is to link our work on the use of algebraic language in proving and modeling with some studies that deal with proving, modeling and the teaching and learning of algebra. A literature review concerning these research domains is out of the scope of this section. What we want to show is how some results and research perspectives in these domains are connected with some aspects of rational behavior in modeling and proving considered in our study, or even can be reinterpreted using our perspective.

Proving

In this Subsection we consider studies on proving and proof that motivated our initial adaptation of Habermas' construct of rational behavior to the case of proving in general (see Boero 2006; Morselli 2007; Morselli and Boero 2009), and other studies directly related to the topic dealt with in this chapter (the use of algebraic language in proving).

As Balacheff (1982) pointed out, the teaching of proofs and theorems should have the double aim of making students understand what a proof is and learn to produce it. Accordingly, we think that, in mathematics education, proof should be treated considering both the *object* aspect (a product that must meet the epistemic and communicative requirements established in today mathematics—or in school mathematics) and the *process* aspect (a special case of problem solving: a process intentionally aimed at a proof as product—the teleological dimension, in our adaptation of the Habermas' construct).

Dreyfus (1999) points out the great divide that may exist between what is an acceptable explanation for the teacher and what is an explanation for the students: "Which aspects of an answer (or solution) are considered most important: computation, statement of the answer, relationship between computation and answer, procedural or conceptual? Are the same aspects considered important by the teachers and by the students (...)? How are the students supposed to know what the teacher considers important?" (p. 93)

For Dreyfus, the problem is that "most students have not been enculturated into the practice of proving, or even justifying the mathematical process they use" (p. 94).

Dreyfus's work has driven our attention on the issues of students' awareness about the epistemic, teleological and communicative components of rationality in proving, and on the possible mismatch between the students' and the teacher's views on the different requirements and their importance.

Stylianides (2007) deals with the crucial question of introducing pupils into the culture of theorems. In his study, he proposes the following definition of proof that can be applied in the context of a classroom community at a given time:

"*Proof* is a *mathematical argument*, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:

- It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;
- It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and
- It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community".
  (Stylianides 2007, p. 291).

We can interpret this definition as related to two aspects of rationality in proving: the epistemic side for the first and the second characteristics; and the communicative side, for the third one.

Concerning proving with algebraic language, we will try to show how our tool can account in an original way for some phenomena (already considered in current literature) regarding students' learning to prove using algebraic language, in particular those concerning:

– students' behaviors classified in terms of "proof schemes" (see Harel and Sowder 1998): for instance, our tool can account for external conviction proof schemes (which are frequent among novices) in terms of dominant concerns for communicative rationality in the use of algebraic language, which hinder the necessity of satisfying the requirements of epistemic rationality and teleological rationality. We also note that Harel's construct of "necessity", a basic component of his DNR perspective underlying proof schemes elaboration, can be connected to epistemic rationality;
– the relationships between argumentation and proof (see Duval 1991, 2007; Pedemonte 2007, 2008 for different positions on the subject): our tool can provide a comprehensive perspective to situate the epistemic validity of proof, interpreted in terms of epistemic rationality (in particular, its systemic requirements), in relation with the problem-solving character of proving, interpreted in terms of teleological rationality. This perspective can be very useful when students learn to use algebraic language in conjecturing and proving because it allows the teacher to distinguish between what must be taught as rules and criteria to follow strictly (on one side, the syntactic rules of transformation of the algebraic expressions; on the other, the correspondence between the algebraic expressions and the mathematical situation they should represent), and what must be managed by students in a flexible and creative way (the choice of suitable algebraic representations of the problem situation, the exploration of the algebraic expressions in order to foresee possible, useful transformations to perform, etc.).

Modeling

Concerning modeling, we think that our tool provides insight into some aspects of the modeling cycle (Blum and Niss 1991) and related students' difficulties: in the choice of the mathematical model; in its mathematical implementation; and in the control of the validity of the model through the comparison of the results (derived from its mathematical implementation) with the modeled situation. We will demonstrate all these aspects through later example problems (the "Bomb problem" and the "Spring problem").

Teaching and Learning of Algebra

A complete overview of literature on teaching and learning algebra is beyond the scope of this chapter. At the same time, we underline that our approach is differently oriented, since it starts from reflection on proving and modeling and turns to algebraic language as a proving and modeling tool. We will try to show how some educational studies on algebra contain elements that are linked to the three dimensions of rationality we previously outlined.

Many studies refer to the crucial passage between arithmetic and algebra, pointing out continuities, false continuities and discontinuities (Filloy and Rojano 1989; Bednarz and Janvier 1996), as well as the need for a careful study of transposition issues (Chevallard 1984). In these cases, the focus is on epistemic aspects (some rules that are still valid for arithmetic are no longer valid for algebra), but also teleological (algebra can solve more problems than arithmetic). We also refer to studies concerning an early approach to algebra, where algebraic thinking is progressively constructed in the child as both an instrument and as an object of thinking in arithmetic situations (Malara and Navarra 2003). Here epistemic and teleological rationality are again at issue.

Another strand of research concerns the use of algebra when solving word problems and its comparison with the use of arithmetic tools. Most students were found to prefer arithmetic methods when solving word problems (Stacey and MacGregor 1999). Here we see a link to teleological rationality, since in most cases the use of algebra would be more appropriate for solving the problem.

Studies that focus on how students use algebraic tools for proving include those that deal with difficulties in using algebra, or with students' reluctance to use algebra in proving. Some studies even suggest that the use of algebraic manipulations, when not coupled with understanding and control, may result in a mere "symbol pushing". For instance, Weber and Alcock (2004) distinguish between semantic proof production (when the prover uses instantiations of mathematical concepts to guide the formal inferences that he or she draws) and syntactic proof production (when the prover draws inferences by manipulating symbolic formulae in a logically permissible way). The authors note that the two approaches require a different conceptual understanding and also note that the syntactic approach often leads to proofs that are convincing, but not explaining, while semantic proofs are more likely to have also an explanatory power. Similarly, Douek (1999) found that university students often relied to semantically-rooted arguments when proving. Furinghetti and Morselli (2009) report the case of a university student who, faced with a statement in number theory to be proved, is keen to use algebra, but fails to choose a useful algebraic representation for the situation. These studies suggest the importance of taking into account not only the epistemic dimension, but also the teleological one.

More generally, Arzarello et al. (1994, 1995) deal with algebraic formulas as thinking tools. They point out the importance of choosing the representation that is most suitable to the task, and also that, once the formula is chosen, it may suggest something new to the reader. They discuss these two uses as follows: "the first way of thinking with a formula is transforming (part of) its intension manipulating it according to its (supposed) extension; the second way is discovering a new (supposed) intension, without doing formal manipulations, but looking at a new (supposed) extension (in a possibly new frame)" (Arzarello et al. 1994; p. 114).

We see here a reference to teleological and communicational rationality in the first way, and communicative rationality in the second way.

Among the "general" studies on the teaching and learning of algebra, we mention the one developed by Radford and Puig (2007). They note that the difficulties encountered by students when learning algebra are related to the *meaning* of signs and the *syntax* of the algebraic language: "More specifically, students' difficulties are often connected to: (1) the understanding of the distinctive *manner* in which simple signs (e.g. "x", "n") and compounded signs (e.g. "$2 + 5$" or "$x + 17$") stand for the objects that they represent, and (2) the grasping of the sense of the *operations* carried out on those signs." (p. 146). These two kinds of difficulties are linked to epistemic rationality. Adopting a historical perspective, the authors note that "contemporary algebra, with all its concepts, is the product of a lengthy historical-cultural process that the students *encounter* in the highly complex social institution that we call the school. To learn algebra is *not* to construct the objects of knowledge (for they have already been constructed) but to *make sense* of them" (p. 152). In their perspective, learning algebra is a cultural and social process. This is coherent with our frame, since the perspective of rationality allows us to take into consideration the social and cultural dimensions of learning. The authors also note that solving equations through algebraic symbolism rests on a mode of diagrammatic thinking. This means that "symbolic algebraic thinking requires the cognitive ability to switch between verbal and perceptual meanings and to become conscious that the latter is governed by the shape of expressions whose syntactic complexity may lead to multilayered perceptual meanings" (p. 160). This suggests a connection between epistemic and teleological dimensions of rationality.

Radford and Puig 's study mainly concerns how to make sense of algebraic symbols. Among the studies dealing with this issue, we also consider that of Arcavi (1994, 2005), who illustrated the symbol sense and pointed out the importance of three things: knowing how and when to use (or not use) algebra, choosing the best representation among different possibilities, and reading the symbols. In our perspective, these aspects refer to teleological and communicative dimension. We feel that our approach may frame all the different aspects mentioned by Arcavi into the three dimensions of rationality, and it places these aspects within the larger frame of rationality in proving (in mathematical activity). Furthermore, our theoretical tool leads us to study the different components that may be present at the same time. We may say that our approach is in line with Arcavi's elaboration, but we have a more explicit aim of using the theoretical tool in order to analyze the students' processes and to study interconnections between the different components. Furthermore, recall that the model we drew from Habermas is a comprehensive frame for studying mathematical activity as a rational behavior, with a focus on proving. Later, we investigated whether the same model is viable also for studying specific aspects of proving, such as the use of algebraic language. Thus, we may say that our elaboration and Arcavi's are compatible, but different in aim and scope.

Let us consider now the notion of *structure sense* (Linchevski and Livneh 1999; Hoch and Dreyfus 2006). Students are said to display structure sense for high school algebra if they can: "Recognise a familiar structure in its simplest form; deal with a compound term as a single entity and through an appropriate substitution recognise a familiar structure in a more complex form [...]; choose appropriate manipulations to make best use of a structure" (Hoch and Dreyfus 2006, p. 306). From the definition, structure sense seems to take into consideration mainly aspects of teleological rationality, (choose appropriate substitutions, perform appropriate manipulations), together with some aspects of communicative rationality, since it takes into consideration the reading of formulas.

Pierce and Stacey (2001), in their study concerning the use of computer algebra systems, deal with the idea of "algebraic insight", defined as "the subset of symbol sense which is needed to solve a problem already formulated mathematically. Students need algebraic insight to enter expressions correctly, monitor the solution process and interpret the output as conventional mathematics" (p. 418). Part of the algebraic insight involves an algebraic expectation, defined as the analogue of arithmetic estimation. It consists, for example, in expecting that $(2 - x + x^2)(x^4 - x^3 + 27x - 63)$ will be a polynomial of degree six. Algebraic expectation encompasses the recognition of *conventions and basic properties*, the identification of structures and key features. We may say that algebraic insight concerns some aspects of rationality: entering expressions correctly is related to epistemic rationality, monitoring the solution process is related to epistemic and teleological rationality, and interpreting the output is related to epistemic and communicative rationality.

In order to justify a new analytic tool in mathematics education, it is necessary to show how it can be useful in describing and interpreting students' behavior, orienting and supporting teachers' educational choices, or suggesting new research developments. The aim of the following Sections is to provide evidence that these three aims are achieved by our adaptation of Habermas' construct to the use of algebraic language in proving and modeling.

Description and Interpretation of Student Behavior

This Section consists of two parts.

In the first part we provide examples of the use of our adaptation of Habermas' construct to interpret and discuss short excerpts of students' texts dealing with the use of algebraic language in elementary modeling and proving. The aim of this subsection is twofold. First, we want to illustrate the use of our construct; paraphrasing Dreyfus (1999, p. 87), we can say that neither the examples nor the students are representative in any sense but have been chosen for illustrative purposes. The choice of examples, including topics, level of mathematics, and level of students have been influenced by our own personal bias and experience.

Second, we want to show that our construct is a flexible tool to deal with different kinds of problem situations and related students' difficulties at different school levels (from grade 8 to university courses), thus including students first learning to use of algebraic language in modeling and proving, as well as its use by more competent students. Some similarities between the two cases (novices and competent students) are interesting because they show how a correct and effective use of algebraic language in modeling and proving is not an aim that can be attained once and for all, but that its development requires a long-term maturation and familiarization with the use of algebraic language in different kinds of tasks.

In the second part of this section, we concentrate on a teaching experiment performed in grade 7, concerning the use of algebraic language in proving in the arithmetic domain. The purpose of this part is to show how our analytic tool both allows us to interpret students' behaviors and to shed light on the different dimensions that teachers must take into account when guiding students towards an effective use of algebraic language in an internal modeling activity and in proving.

Habermas' Analytical Tool: Examples of Analysis of Student Behavior at Different School Levels

The following examples are derived from a wide corpus of students' individual written productions and transcripts of *a posteriori* interviews, collected in the last fifteen years by the Genoa research team in mathematics education. The wide corpus was collected during the last fifteen years. The first analysis of the corpus, performed at different times, showed the importance of taking into account dimensions that, later, we found explicitly treated in Habermas' theory. This means that our elaboration of the theoretical tool was fostered by the reflection on the corpus. After our adaptation of the theoretical tool, we came back to the corpus and checked the reliability of the tool. We present here a selection of episodes, analyzed through the tool, with the double aim of illustrating the tool and showing how the tool helps us understand the episodes.

In particular, we will consider five categories of students:

(a) 8th grade students who are approaching the use of algebraic language in modeling physical phenomena;
(b) 9th grade students who are approaching the use of algebraic language in proving;
(c) 11th grade students who are learning to use algebraic language in modeling physical phenomena;
(d) students who are attending university to become primary school teachers;
(e) students who are attending the third year of the university course in mathematics.

A common feature for all the considered cases is that the individual tasks require not only the solution, but also the explanation of the strategies followed to solve the problem. However, while in cases (a), (d) and (e) the explanation of the strategies was inherent in the didactical contract already established with the teacher for the whole course, in the cases (b) and (c) such explanation was only an occasional request. Each individual task was followed by *a posteriori* interviews.

Example 1 (The sum of two consecutive odd numbers) The students (22 students, grade 9) experienced the traditional teaching of algebraic language in Italy: transformation of progressively more complex algebraic expressions in an attempt to "simplify". In order to prepare students for the task proposed by the researcher, two examples of "proof with letters" had been presented by the teacher, one of which included the algebraic representation of even and odd numbers.

THE TASK: Prove with letters that the sum of two consecutive odd numbers is divisible by 4.

Here we report some recurrent solutions (in parentheses the number of students who performed such a solution; note that "**d**ispari" means "odd" in Italian).

E1 (4 students): d+d=2d

In this case, we can observe how the *systemic requirements of epistemic rationality* are satisfied (algebraic transformation works well), while the *internal modeling requirements* fail to be satisfied (the same letter is used for two different numbers).

E2 (8 students): d+d+2=2d+2

In this case, both the *systemic and the internal modeling requirements of epistemic rationality* are satisfied, but the requirements in *teleological rationality* are not satisfied: students do not realize that the chosen representation does not allow to move towards the goal to achieve (because the letter d does not represent in a transparent way the fact that d is an odd number), thus they do not change it.

E3 (5 students): d=2n+1+dc=2n+1+2n+1+2=4n+4 (or similar sequences)

We can infer from the context (and also from some a-posteriori comments by the students) that "dc" means "**d**ispari **c**onsecutivi" (consecutive odd numbers).

In this case *epistemic rationality* fails in the first and in the second equality, but *teleological rationality* works well: the flow of thought is intentionally aimed at the solution of the problem; algebraic transformations are used as a calculation device to prepare the conclusion (divisibility by 4). We may note the presence of communicative rationality in the use of dc, which is related to private communication (with oneself).

Example 2 (The product of two consecutive even numbers) The following task had been preceded by the task from Example 1, performed under the guide of the teacher. 58 university students, attending the third year of preparation to become primary teachers, performed the activity.

THE TASK: Prove in general that the product of two consecutive even numbers is divisible by 8

Very frequently (about 55% of cases) students performed a long chain of transformations, with no outcome, like in the following example:

E4: $2n(2n+2)=4n^2+4n=4(n^2+n)=4n(n+1)=4n^2+4n=n(4n+4)$

In this case, we see how both requirements of *epistemic rationality* are satisfied: *internal modeling requirements* (concerning the algebraic modeling of even numbers); and *systemic requirements* (correct algebraic transformations). The difficulty is in the lack of an interpretation of formulas, led by the goal to achieve (thus in *teleological rationality*). The student gets lost, even though the interpretation of the fourth expression would have provided the divisibility of $n(n+1)$ by 2 because one of the two consecutive numbers n and $n+1$ must be even. We can also observe how (in spite of the didactical contract), in general, no substantial sentence precedes or follows the sequence of transformations (sometimes we find only a few words: "I use formulas"; "I see nothing").

In the following case, both the *modeling* and *systemic requirements* are not satisfied: the same letter is used for two consecutive even numbers (note that "**p**ari" means "even" in Italian), and the algebraic transformation is affected by a mistake.

E5: $p*p=2p^2$, divisible by 8 because p is divisible by 2 and thus p^2 is divisible by 4

The student seems to work under the pressure of the aim to achieve: having foreseen that the multiplication by 2 may be a tool to solve the problem, she tries to justify it by considering the juxtaposition of two copies of p that generates "2". Indeed, in the interview the student said that she had made the reasoning "p is divisible by 2 and thus p^2 is divisible by 4" before completing the expression. In this case we can see how *teleological rationality* prevailed over *epistemic rationality* and hindered it.

We have also found cases like the following one:

E6: $p*(p+2)=p^2+2p=8k$ because $p^2+2p=8$ if $p=2$

Also in this case, from the *a posteriori* interview we infer that the lacks in *epistemic rationality* probably depend on the dominance of *teleological rationality* without sufficient epistemic control:

I have seen that in the case $p = 2$ things worked well, so I have thought that putting a multiple 8k of 8 in the general formula would have arranged the situation.

Example 3 (The bomb problem) TASK: A helicopter is hovering over a target. A bomb is left to fall. Twenty seconds later, the sound of the explosion reaches the helicopter. What is the relative height of the helicopter over the ground?

The problem was posed to groups of third year mathematics students in seven consecutive years, and to two groups of 11^{th} grade students (with a scientific–oriented high school curriculum). The younger students were reminded that the falling of the bomb happens according to the laws of uniformly accelerated motion, while the sound moves at the constant speed of 340 m/s. However no formula was suggested.

The problem is a typical (though elementary) applied mathematical problem, whose solution needs an *external modeling* process. In terms of *teleological rationality*, the goal should guide students to choose an appropriate algebraic model of the situation, solve the second degree equation derived from the algebraic model, and select the good solution (the positive one) by performing a suitable comparison between the obtained solutions and the problem situation.

The first difficulty students meet is in the time coordination of the two movements: it is necessary to tell the model that the whole time for the bomb to reach the ground and for the sound of the explosion to reach the helicopter is 20 seconds. The second difficulty is in the space coordination of the two movements: the space covered by the falling bomb is the same covered by the sound when it moves from the ground to the helicopter.

Let us consider some students' behaviors.

Most students are able to write the two formulas:

E7: $s=0.5\ gt^2,\ s=340\ t$

These are standard formulas learned in Italian high schools in grades 10^{th} or 11^{th}, in physics courses. About 25% of the high school students and 20% of the university students stick to those formulas without moving further. From their comments we infer that in some cases the use of the same letters for space and time in the two algebraic expressions generates a conflict that they are not able to overcome. We can see how general expressions that are correct for each of the two movements (if considered separately) result in a bad model for the whole phenomenon. *Teleological rationality* should have driven formalization under the control of *epistemic rationality*; such control should have revealed the lack of the *modeling requirements* of *epistemic rationality*, thus suggesting a change in the formalization. However, such an interplay between *epistemic rationality* and *teleological rationality* did not work for those students.

In other cases (about 10% of both samples) the coordination of the two times was lacking, and the idea of coordinating the spaces (together with the formalization of both movements with the same letters) led to the equation:

E8: $0.5\ gt^2 = 340\ t$

with two solutions[1] $t = 0$, $t = 68$. Some students were unable to interpret and use these times (because 68 is out of the range given by the text of the problem), but other students found the height of the helicopter by multiplying 340×68; the fact that the result is out of the reach of a helicopter did not provoke any critical reaction or re-thinking, probably because it is normal that school problems are unrealistic!

Some students who introduced the third equation $t_b + t_s = 20$ added it to the first two equations without changing the name of the variable (t).

Less than 60% of students of both samples wrote a good model for the whole phenomenon:

$$t_b + t_s = 20$$

$$h = 0.5gt_b^2 = 340t_s$$

and moved to a second degree equation by substituting $t_s = 20 - t_b$ or $t_b = 20 - t_s$ in the equation:

$$0.5gt_b^2 = 340t_s$$

Many mistakes occurred during the solution of the equation (mainly due to the management of big numbers). Once two solutions were obtained (one positive and the other negative), in most cases the choice of the positive solution was declared but not motivated. *A posteriori* comments reveal that the fact that a negative solution is unacceptable (given that the other solution is positive!) was assumed as obvious, without any physical motivation.

In terms of *epistemic rationality*, three kinds of difficulties arose: first, in the control that the chosen algebraic model was a good model for the physical situation; second, in the control of the solving process of an equation with unusual complexity of calculations (big numbers); third (once the valid equation—a second degree equation—was written and solved), in the motivation of the chosen solution.

In terms of *communicative rationality*, we can observe how (in spite of the request to explain the steps of reasoning) very few students from both samples were able to justify the crucial steps of the solving process. How is it possible to interpret this kind of difficulty? In some cases the steps were derived from a gradual adaptation of the equations to the need of getting a "realistic" solution. In other cases the equations were written as if the idea of coordination of the spaces and times of the phenomenon was supported by an intuition, but no wording followed. *A posteriori* interviews revealed that most students who had been unable to justify their choices were sure about their method only afterwards, when checking the positive solution and finding that it was "realistic", thus revealing a lack in *teleological rationality* (lack of consciousness about the performed modeling choices). Moreover, a number of inappropriately obtained solutions were quite realistic. Many students who produced the correct solution were not able to explain (during the comparison of solving processes) why the other solutions were mistaken. This suggests lacks in

[1] We underline that g is rounded to 10, rather than 9.8.

teleological rationality (motivation of choices with reference to the aim to achieve) and in *epistemic rationality* (control of the validity of the steps of reasoning). This conclusion can be reinforced if we consider the fact that almost all students who were able to provide a verbal justification for their modelisation were also able to explain why the other solutions were not acceptable (even if those results were realistic).

Example 4 (The spring problem) The problem was proposed to Grade 8 students. The students were requested to choose, among the following formulas, the one which best represents the elongation of a spring according to the number N of clips suspended to it (L is expressed in centimeters).

(I) $L = 20 \cdot (1 + N)$
(II) $L = 20 - 0.2 \cdot N$
(III) $L = 20 + 0.2 \cdot N$
(IV) $L = 20 + 20 \cdot N$

Students had at their disposal the following table, derived from an experiment:

N (number of clips)	L (centimeters)
0	20
10	21.8
20	23.6
30	26
40	28.2
50	30.3
60	32.2
70	34.1

Students' background included: the use of algebraic formulas to represent geometric situations (perimeters and areas of geometric figures); the representation of linear functions in the Cartesian plane; and the construction of tables like the one above through direct measurement of the elongation of springs. At this stage of students' preparation, it was not expected that they produce a linear model for the elongation of a spring, but only that they succeed in choosing between different linear models provided by the teacher.

The aim of the task is relevant to the perspective of approaching mathematical modeling of physical phenomena, because it is related to the choice of a mathematical model suitable to fit the behavior of the phenomenon. In terms of the analytic tool we derived from Habermas' theory, the aim of the task concerns the external modeling requirements of epistemic rationality.

Different behaviors were observed. Here, we present the three main categories of behaviors that were singled out, and we analyze them in terms of our construct.

Some students performed substitutions of values within the formulas without getting any conclusion. Frequently, they also graphed the points thus obtained, without comparing the graph with a possible qualitative behavior of the phenomenon (or with a graphic representation of the points derived from the table). It seems that the “ritual” accomplishment of the usual task of substituting values in order to get points in the Cartesian plane hindered the aim of finding the good formula and/or of comparing those values with the experimental values.

Some practices, such as substituting values in order to get points on the Cartesian plane, could be useful in order to compare the formulas in terms of modeling requirements of epistemic rationality, but in this case these practices seem to be just aimed at plotting the functions. We may say that teleological rationality did not work (it was not present, or it was badly oriented).

Some students found the best formula by substituting values in each formula and comparing the points thus obtained in the Cartesian plane with the points derived from the table. In this strategy, the formulas are compared according to some external modeling requirements and systemic requirements of epistemic rationality: in this case, teleological rationality drives the process towards the aim to attain.

Other students considered the structure of the formula and concluded that the second formula cannot work, because it would imply that the length of the spring decreases when the number of clips increases. In this case, we may say that they put in action a control in terms of external modeling requirements. Afterwards, they realized that the first and the fourth formulas are equivalent; we may say that they performed a control involving the systemic requirements of epistemic rationality inherent in the use of formulas, performed by substituting some values or by considering the distributive property. Comparing the third and the fourth formulas, they chose the third one because the fourth one “produces values that are too big, when N increases, in comparison with the table”.

Habermas Analytical Tool: Analysis of a Teaching Experiment

The Context of the Study: Description of the Research Project

The following examples are drawn from a research project, started in 2008 and still going, entitled “Language and argumentation in the study of mathematics from primary school to university”.[2] The project aims at setting up and experimenting with teaching activities for different school levels (from grade 1 to grade 13), with an eye on continuity between different school levels, and with a special focus on argumentation, which is seen as a central theme in mathematics education (Hanna and De Villiers 2008) as well as a crucial competence in the development of citizenship

[2]The project is developed by the Mathematics Department of the University of Genova, with the collaboration of the Regional School Office, and is carried out within the National Project “Lauree Scientifiche” (MIUR-Confindustria).

(Anichini et al. 2003). The conception, implementation and analysis of the teaching activities was carried out by a research team made up of university researchers in mathematics education and school teachers, in a relationship of mutual exchange and collaboration, according to the Italian paradigm of the Research for Innovation (Arzarello and Bartolini Bussi 1998).

In this section, we refer to a sequence of teaching activities that was performed in grade 7 (age of the students: 12–13). The researcher, who took part in the activities together with the teacher, was present in the classroom and acted both as an observer and, occasionally, as a sort of additional teacher, especially during the classroom discussions. According to the general aim of the project, special care was devoted to tasks such as "explain your solution, compare your solution with that of your classmates, choose between two options and justify your choice". Accordingly, collaborative group work and mathematical discussion (Bartolini Bussi 1996) were usual modes of work in the classroom. These choices were linked to the necessity of establishing a sort of "argumentative attitude" in the classroom. The sequence of teaching activities was conceived in order to introduce algebra as a proving tool in the classroom. Students had prior experience in dealing with numerical expressions and using formulas to express the areas and perimeters of plane figures. They had no experience using algebra to represent geometric or arithmetic relations.

Next, we present a selection of episodes from the sequence of teaching activities, with the aim of showing the viability of our construct not only for interpreting students' behaviors, but also for supporting/orienting teachers' choices. We underline that the aim of this chapter is not to discuss in detail the nature of the sequence of activities. Data from the activities serve as a ground for our discussion on the use of the analytic tool derived from Habermas' theory. We also underline that our adaptation of Habermas' model is a tool for the researcher and, in a long-term perspective, a tool for the teacher in order to better understand students' performance and guide them towards adherence to a mathematical rationality.

First Task: Choose a Number…

The first task that was proposed to the students is the following:

> The teacher proposes the following game: Choose a number, double it, add 5, take away the chosen number, add 8, take away 2, take away the chosen number, take away 1. Without knowing the number that you initially chose, is it possible for the teacher to guess the result of the game? If yes, in what way?

The students worked on the task individually, and afterwards they shared and compared their solutions, first in small groups and then within a classroom mathematical discussion. Many groups found out *that* the result is always 10, independently from the chosen number, and some students even tried to find out some reasons *why* the result is always 10. See, for instance, this selection of written group solutions, that goes from *seeing that* to *looking for reasons why*:

Group A: Yes, because at the end you always find 11 − 1
Group B: Yes, because it is a mathematical procedure, by means of which you get always the same result, with any chosen number. The factor that causes that, is the instruction "Take away the chosen number".
Group C: With any chosen number, the result is 10 because multiplying by 2 is equivalent to adding twice the chosen number, the same number that after must be taken away twice, which gives zero, and doing the other calculations, even in a different order, you always get 10.

Some students were able to find out *a reason why*, but were not able to communicate the reason to their classmates. Students realized that solutions in natural language were not always efficient in communicating the reasons to others. This paved the way to the subsequent task, aimed at proposing algebra as a proving tool.

Second Task: Representing the Game

The students were given the following task:

> Write the game as an expression, using a different color for the chosen number. Write an expression that works for any number you choose.

The students solved the task individually, and afterwards they shared and compared their solutions within a mathematical discussion orchestrated by the teacher. As we'll see in the following, two main representations of the game were singled out.

As regards the first question (writing the game in form of expression), the student Ric chose to represent the game as an expression, while the student Tor chose to represent the sequence of calculation to be performed, thus creating a sort of procedural representation of the game.

Ric's representation: 100X2+5-100+8-2-100-1=10

Tor's representation:

100 X 2 = 200

200 + 5 = 205

205 - 100= 105

105 + 8 = 113

113 - 2 = 111

111- 100= 11

11 - 1 = 10

As regards the second part of the task ("Write an expression that works for any number you choose"), Ric proposed a representation of the game as an expression, while Tor proposed a sequential-algorithmic representation of the game:

Ric's representation: NX2+5-N+8-2-N-1=10
Tor'representation:

N X 2 = N

N + 5 = N

N - N= N

N + 8 = N

N - 2 = N

N- N= N

N - 1 = 10

We may note that the expression of Tor does not satisfy the internal modeling requirements of epistemic rationality, since the same letter, N, is used to represent the different results of the calculation steps. In contrast, Ric's expression satisfies the internal modeling requirements of epistemic rationality. A significant discussion comparing Ric's and Tor's representations of the problem was fostered by the teacher's question: *In your opinion, which of the two representations would be chosen by a mathematician?*

Here we provide a transcript from the mathematical discussion (at this point, the two representations are written on the blackboard, so that all students may refer to them):

1. **Mir**: I would choose Ric's expression, because it is easier to understand and... faster.
2. **Teacher**: faster. Giam?
3. **Giam**: I would choose Tor's expression, because it is more schematic and... any person, even a 6-year-old, may understand it.
4. **Teacher**: any other idea?
5. **Brac**: I would choose Tor's expression, but writing different letters for any result.
6. **Teacher**: because, in your opinion, if we always use the same letter... you would change the letter. Why?
7. **Brac**: because the result changes at each passage, even if the result finally is always 10, but the result changes at each passage.
8. **Teacher**: and you think that... if I use N, N should stand always for...

9. **Brac**: for the number that I chose at the beginning.
10. **Teacher**: while $N \times 2\ldots$ (many voices: no). So, M, and after $M + 5\ldots$ equals... and so on... Using different letters. Mat?
11. **Mat**: I would choose both. Because one is faster and more immediate...
12. **Teacher**: which one?
13. **Mat**: while the other one is more... schematic.
14. **Teacher**: Soz, which one would you choose, if you were a mathematician?
15. **Soz**: I would choose Ric's, because it is more complicated, but also simpler.
16. **Teacher**: why is it simpler?
17. **Soz**: because it is... more complicated to understand, but also simpler.
18. **Teacher**: Chris?
19. **Chris**: I would choose Tor's, because it puts into evidence the results. But it would be better to change the letters, as Brac suggested.
20. **Teacher**: so the results... how could we call them? The intermediate results... the results of each step of calculation...
21. **Chris**: yes.
22. **Nav**: I would choose Ric's because the expression you have... you put N in the place of the chosen number, because in mathematics N stands for "number", and it is more... more...
23. **Teacher**: more coherent with...
24. **Nav**: it follows better the request.
25. **Teacher**: Ric, could you tell us why do you think that are the advantages of your representation?
26. **Ric**: well...
27. **Teacher**: because you wrote the expression is equal to 10, isn't it?
28. **Ric**: first, because there are no parentheses, so it is easier, second, because in the place of the chosen number there is N, which stands for any number, and then, since we have 10 with any chosen number, you put N and you always get 10.
29. **Teacher**: and you get 10. But also Tor used the letters...
30. **Voices**: yes, but he goes to new line...
31. **Alex**: for me, Ric's is more correct, because it is an expression and it [the text] says that it must work for any number. Ric's is an expression and works for any number, while Tor's is an expression decomposed into operations, so... The text asks you to create an expression, it doesn't ask you to decompose this expression.
32. **Teacher**: so, for you the advantage of Ric's is that it is an expression.
33. **Alex**: yes.
34. **Teacher**: rather than a sequence of operations. So, both used a letter. What you say is that Ric created a real expression, while...
35. **Alex**: Tor's is an expression decomposed into operations
36. **Teacher**: Ash?
37. **Ash**: I would choose Ric's expression, because N indicates always the same number, while in Tor's expression N is both the chosen number and the results of the computations.

Lines #1 and #3 show that some students judge Tor's expression to be a "legitimate" mathematical representation as well as that of Ric. This led us to analyze the criteria according to which the students evaluate the representations. Lines #1 and #3 suggest that both representations are judged first of all in terms of their efficiency and easiness in relation to the game. In the students' view, they are both legitimate, since they both allow one to perform the game. This can be interpreted, in terms of our adaptation of Habermas' construct, as a predominance of the communicative dimension over the other ones. The students evaluate the representations basically on the basis of the communicative dimension.

Brac's comment (#5) brings to the fore also the epistemic dimension: the representation created by Tor does not meet the internal modeling requirements of epistemic rationality, since the same letter, N, is used to represent the chosen number, but also the result of the intermediate steps of calculation. The same position is hold also by other students, who suggest amending the representation given by Tor (#19, #37).

We may note that, throughout the discussion, the different dimensions of rationality appear: some students judge the two representations in terms of "easiness" or "complicatedness" (thus focusing on the communicative dimension), other students even refer to the adherence to the text (#24, 31). Summing up, we may say that two streams of discussion intertwine: evaluating the correctness of Tor's expression (internal modeling requirements of epistemic rationality) and comparing between the two representations in terms of comprehensibility (communicative rationality) and adherence to the text. The adherence to the text may be interpreted in terms of communicative rationality (adherence to the norms of the community), but also in terms of teleological rationality: the students' aim, in this part of the activity, could be more focused on the representation of the game than on the search for reasons why the result is always 10. We note that the choice between the two representations, and the risk of considering the expression as the "official" way of representing the game, are crucial points, since the aim of the teacher is not just to make the students choose Ric's expression as the "legitimate" way of representing the game, but also to guide the students to understand why mathematicians usually choose this expression. In other words, the teacher wants: at first, to work on the correct use of letters in algebra; after, to work on the way of using algebra, which should be considered "legitimate" not in terms of adherence to external rules (for a sort of "authoritarian" legitimacy) but in terms of usefulness and efficacy according to the goal of the activity. This requires shifting students' attention from the adherence to the text (which requires an expression) back to the original goal of the activity (to understand why the result is always 10).

In the successive part of the mathematical discussion, the representation given by Tor is amended according to the suggestions of the students. In terms of our construct, Tor's representation is amended in terms of epistemic rationality, in order to meet also the modeling requirements. Here we report the amended version:

Tor's "amended" version:

N X 2 = A

A + 5 = B

B - N= C

C + 8 = D

D - 2 = E

E- N= F

F - 1 = 10

At this point, the discussion concerns the comparison between two representations: Ric's expression and Tor's amended "sequential" representation. This is quite a crucial point. Indeed, both representations are correct from a mathematical point of view (thus meeting the modeling requirements of epistemic rationality) and are also perceived as efficient from a communicative point of view. So the question arises for the teacher: how is it possible to lead the students to understand that a mathematician would rather choose Ric's expression? The point is that the Ric's expression is better for the original aim of the task (to understand *why* it is possible to know the result of the game). This means, in terms of the model, that Ric's expression meets also the teleological requirements. For the teacher, it is important not only to have the students choose Ric's representation, but also to lead them to understand the reasons why it is more suitable. That is, Ric's expression is a veritable proving tool, since it allows one to understand why the result is always 10.

In the first part of the discussion, students expose their motivations for the choice of one of the two representations. Some motivations are still at the communicative level:

Cler: for me, Ric's expression is more correct, because, at our age, we just studied expressions, so it is easier for us.
Mus: for me, it is worthwhile to use Ric's representation because it is more schematic and more mathematical.

In some interventions, epistemic and communicative dimensions are intertwined:

Alex: first of all because [Ric's] follows the text more, and after because it is more correct.

Some students support Tor's representation, or highlight the equivalence between the two representations, as we see in the following comments:

Giam: I mean, for example, Tor's representation, anybody can do it, and he can follow all the steps, while in Ric's, yes you do it, but you don't really realize what you are doing. I mean, maybe you do it, but without… for example, the step $N \times 2$, with Tor's representation you do $N \times 2$, and the result, while with Ric's you must go on fast, and maybe you get lost…

Brac: we can take both, because in the expression we do both Tor's and Ric's, because in Ric's, even though it is not so complex and you can do it in your mind, you do a lot of steps, and it is as if you did Tor's.

We may note that the issue of the equivalence of the two representations is still at the epistemic level. In the subsequent part, both observer and teacher intervene in order to bring to the fore also the teleological dimension:

58. **Soz**: [to the observer] and you, which one would you choose?
59. **Observer**: I think that… you all said a lot of good things, actually doing one or the other is the same, and in both cases you get the result, OK? But do you remember the question of last session? The question was not "what is the result", but "will the teacher be able to guess the result?"
60. **Cler**: just from the expression.
61. **Observer**: just from the expression, from the initial game. Just knowing the game, will the teacher be able to guess the result, independently from the chosen number?
62. **Voices**: yes.
63. **Observer**: we said yes, and in the last session you also explained why, right?
64. **Chris**: because you always get 10.
65. **Observer**: you said: yes, because you always get 10, and some of you also explained something more, we also had some motivations why you always get 10.
66. **Teacher**: do you remember? Brac, you told it, because you told that doing $N \times 2$ means…
67. **Brac**: I mean… it is like doing… yes, it is like doing $N + N$.
68. **Teacher**: $N + N$ in the expression written by Ric, then… there is $N \times 2$, Brac, please go to the blackboard and write $N + N$ under $N \times 2$. Do we all agree that it is the same thing? And after you write all the expression: $+5 - N + 8\ldots$ And you already noticed that… after $N + N$, what do I have?
69. **Ash**: $-N$.
70. **Voices**: two times.
71. **Teacher**: and so?
72. **Brac**: they all disappear.
73. **Teacher**: can I understand this, in Tor's representation?
74. **Voices**: no.
75. **Fag**: but, at the end there is $+8$, so, the two representations are equivalent, but Ric's is… easier.
76. **Teacher**: but why is it easier?
77. **Giam**: because you understand that the chosen number disappears.
78. **Teacher**: because I can answer…

79. **Observer**: to the original question. Ric's representation helps us understand why it is not necessary to know the chosen number to get the result.
80. **Teacher**: do you understand? I mean... in Ric's representation I understand why the chosen number doesn't matter.
81. **Observer**: I see better that the chosen number disappears.
82. **Teacher**: [to Brac, who is at the blackboard] underline all the Ns. $N + N - N - N$. This means that I add N twice, but after I take N away twice. Can I see this, in Tor's representation? We don't see this fact, right?
83. [...]
84. **Teacher**: so, which expression would a mathematician choose?
85. **Voices**: Ric's?
86. **Teacher**: and actually, we studied the numerical expressions last year, there should be a reason... Now, we have also letters, which stand for numbers.
87. [...]
88. **Observer**: so, which one a mathematician would choose?
89. **Voices**: Ric's.
90. **Observer**: why?
91. **Alex**: because you see that it is an expression and you get the result.
92. **Observer**: because you get the result? Is it for this reason?
93. **Voices**: no!
94. **Ash**: because you understand that the chosen number doesn't matter.
95. **Observer**: ok, because it helps me to answer the original question.
96. **Teacher**: the original question was not to tell the result.
97. **Observer**: the two expressions are both correct, but Ric's helps me to understand why the chosen number doesn't matter, why the chosen number disappears.

The discussion concerning the two representations takes place around two main issues: the equivalence between the two representations (epistemic dimension), and the reason why mathematicians prefer Ric's expression (teleological dimension). We note several things: that the two issues often overlap; that in the first part the students are more concerned with the equivalence rather than with the choice; and that the teacher has to deal with complexity of the two overlapping motives of the discussion. The teacher and the observer intervene so as to bring to the fore also the teleological dimension, bringing back the students to the original aim of the activity (to understand why the teacher can always guess the result, even without knowing the chosen number). All along the discussion, thanks to the mediation of observer and teacher (see for instance #59, 73), the students shift from evaluations in terms of correctness and/or comprehensibility to evaluations in terms of efficacy in relation to the initial task: as we see in lines #77, 94, Ric's expression is preferred because it is more useful according to the aim of the activity-understanding why the result is always 10.

Discussion

In our opinion, the usefulness of a new analytical tool in mathematics education must be proved through the *actual* and *potential research advances* and the *educational implications* that it can provide.

Research Advances

In the frame of our adaptation of Habermas' construct, the distinction between *epistemic rationality* and *teleological rationality* allows us to describe, analyze and interpret some difficulties in algebra (already pointed out in literature). These difficulties can be accounted for by the students' prevailing concern for rote algebraic transformations performed according to *systemic requirements* of *epistemic rationality* over the needs inherent in *teleological rationality* (see equality E4). Moreover, the distinction between *modeling requirements* and *systemic requirements* of *epistemic rationality* offers an opportunity to study the interplay between the *modeling requirements* and the requirements of *teleological rationality* (see E7); we have also seen that formalization and/or interpretations may be correct but not goal-oriented (like in E2 and E4), or incorrect but goal-oriented (like in E5, E6 and E8).

Together with the other dimensions of rationality, *communicative rationality* allows us to describe and interpret possible conflicts between the private and the standard rules of use of algebraic language, as well as the ways student try to integrate them in a goal-oriented activity (see E3).

At present, we are engaged in establishing how the requirements of the three components of rationality affect the phases of production and interpretation of algebraic expressions. Further research work should be conducted to establish what mechanisms (related to meta-cognitive and meta-mathematical reflections based on the use of verbal language, see Morselli 2007) can ensure the control of *epistemic rationality* and the intentional, full development of *teleological rationality* in a well-integrated way. With reference to this problem, taking into account *communicative rationality* in its intra-personal dimension, possibly revealed through suitable explanation tasks and/or interviews) can reveal the role of verbal language (in its mathematical register: see Boero et al. 2008, p. 265) in the complex, dynamic relationships between *epistemic, teleological* and *communicative rationality*. In particular, previous analyses (see E3, E4 and Example 3) suggest not only that the request (related to *communicative rationality*) to justify performed choices can reveal important lacks in *teleological rationality*, but also that the development of a kind of personal "verbal space of actions" can aid a successful development of the activity (even if algebraic written traces are not satisfactory from the *systemic-epistemic rationality* point of view, like in the case E3). The respective roles of the space of verbal actions and of the space of algebraic manipulations should be investigated on the *teleological rationality* axis. Here Duval's elaboration about the productive interplay between different registers in mathematical activities might be borrowed to better understand and frame what students do (see Duval 1995). Also, the results by MacGregor and Price (1999) could help highlight the relations, which emerged from our data, between the production of verbal justifications and the effective use of algebraic language to achieve the goal of the activity.

Educational Implications

We think that the analyses performed in the previous section can provide both teachers and teacher educators with a set of indications concerning curricular choices as well as their practical implementations.

Some of our general indications for curricular choices are not new in mathematics education; we think that the novelty brought by Habermas' perspective consists in the coherent and systematic character of the whole set of indications.

First of all, the performed analyses suggest balancing (in the students' eyes, according to the didactical contract in the classroom) the relative importance (in relationship with the goal to achieve) of:

– production and interpretation of algebraic expressions, versus algebraic transformations;
– flexible, goal-oriented direction of algebraic transformations, versus rote algebraic transformations aimed at "simplification" of algebraic expressions.

These indications are in contrast with the present situation in Italy and in many other countries, where classroom work is mainly focused on algebraic transformations aimed at "simplification" of algebraic expressions, and many simplifications are performed by elimination of parentheses, thus suggesting a unidirectional way of performing algebraic transformations. In the students' eyes, the importance of the formalization and interpretation processes is highly underestimated. The fact that algebraic expressions are given as objects to "simplify" (and not as objects to build, to transform according to the aim to achieve, and to interpret during and after the transformation process in order to understand if the chosen path is effective and correct or not) has bad consequences on students' *epistemic rationality* and *teleological rationality*. As we have seen, many mistakes occur in the phase of formalization (against the *modeling requirements of epistemic rationality*), and even when the produced expressions are correct, frequently students are not able to use them intentionally to achieve the goal of the activity (against the *teleological rationality requirements*).

We next consider practical implementation of such curricular choices, so as to promote teaching algebra as an important tool for modeling and proving. Our analyses suggest that students have a need for constant meta-mathematical reflection (performed through the use of verbal language) on the nature of the actions to perform and on the solving process during its evolution. At present, the only reflective activity in school involves checking the correct application of the rules of syntactic transformation of algebraic expressions (thus only one component of rational behavior—namely, the *systemic requirements* of *epistemic rationality*—is partly engaged).

Specific indications for the management of classroom situations are exemplified in the second part of the previous section: the construct derived from Habermas' theory can work as a tool to interpret and guide students' work. In particular, the construct highlights the complex intertwining of the dimensions and, thus, may help the teacher to guide students towards an efficient use of algebra as a proving and modeling tool. The construct also highlights the dimensions that the teacher must manage during the activity, and it may help them to plan didactical choices and on-the-spot interventions. Indeed, we could even say that our construct is not only a diagnostic tool (for interpreting students' behavior and "detecting" their rationality) but also a tool that may guide teachers' choices and interventions, thus helping to promote the development of rational behavior.

Specific tasks, such as the comparison of different strategies or solutions, accompanied by well-chosen questions, such as "which one would a mathematician choose", as in the 2nd task from the ongoing research project previously discussed, seem to be effective in fostering the development of a rational way of behaving in proving and modeling. More specifically, through such tasks the teacher may bring to the fore also the teleological dimension, which may otherwise remain hidden. At present, we are also planning and beginning to carry out long-term activities based on the idea of story narration of guided proof construction as a didactical device to promote awareness of the components of rational behavior in proving (see Boero et al. 2010).

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Theoretical Issues and Educational Strategies for Encouraging Teachers to Promote a Linguistic and Metacognitive Approach to Early Algebra

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Abstract After an overview of the studies which led to the rise of the study of early algebra, we sketch our vision of this disciplinary area and of its teaching from a linguistic and socio-constructive point of view. We take into account the teacher's role in the socio-constructive teaching process and stress the importance of reflecting upon the teaching and learning processes in order to reshape the teacher's ways of being in the classroom. We dwell upon the strategies enacted and describe the tools we have shaped: theoretical, for the enculturation of early algebra teachers, and methodological, which aim at promoting their awareness and control of their action. We conclude with some considerations about the value of the tools and modalities we have used, as well as on the factors which determine their efficacy.

Introduction

The idea of giving space to early algebra at K-8 school level (pupils aged 4–14 years) in association with a socio-constructive practice of teaching seems to be spreading increasingly. This does not mean that syntactic activities typical of secondary school should be anticipated at lower school levels, but rather that in elementary arithmetic more room should be given to activities concerning numbers in relational terms, so that pupils might be led to compare particular representations with other equivalent representations of the same mathematical object, to detect analogies, and to

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generalize and identify properties. In other words, it would be appropriate to *revise arithmetic in a pre-algebraic perspective*, with less emphasis on a typical algorithmic treatment and setting the ground for the development of algebraic thinking. The aim is to get students to construct, starting from their early school years, a set of experiences that make the study of algebra in its formal aspects, meaningful and justified. In this way, the approach to algebra should be facilitated and the typical and widespread difficulties students meet when they access higher secondary school, minimized. At the same time, they should be made aware of the potential of algebraic language as a tool for thinking.

Several countries have nowadays included this theme, more or less explicitly, in their national curricula, though this was done in the framework of their specific cultural and educational features, and recently, comparative studies about the different approaches and methods have been carried out (see for instance Kieran 2004; Cai et al. 2005). Moreover, the introduction of early algebra in the different educational policies has been promoted by the indication of the British Department for Education (DFE 1995) and, even more, by those of the National Council of Teachers of Mathematics (NTCM 1998, 2000), together with the exploratory studies of didactical implementations carried out at research level (to be dealt with later on paragraph From Traditional Algebra to Early Algebra).

In Europe

In Europe, the initial, embryonic, teaching proposals in the spirit of early algebra date back to the 70s, as a follow-up on curricula of two different and somehow opposite tendencies: the psycho-pedagogical trends which underline the importance of experience and discovery in learning, and the structuralist trend that suggested an algebrization of the mathematical teaching contents.

In those years, naïve set theory comes to the fore in teaching and the concept of a binary operation, with its properties, becomes the fundamental basis of the arithmetic-algebraic area. On the one hand, this approach opens the way to a characterization of the structures of the different number sets; on the other hand, it makes both the concept of relation and the modelling processes central, due to the fact that they tend to embed the concept of function among the objects of algebra.

With this approach, starting from primary school, in elementary arithmetic importance is given to the relational aspects of numbers, to the symmetry of equality, to the recognition of equivalent representations of numbers, to the valuing of arithmetic properties for ordering numbers. At the same time, room is given to the study of relationships in realistic contexts and with reference to different number sets, with the joint detection of variable data for pairs of quantities.[1]

[1]In the early '70 a noticeable project in Europe was the Hungarian project for primary school directed by T. Varga, which envisaged activities of this type from the first two years of primary school.

The importance attached to the modelling processes led to a review of the teaching of algebra—up to that moment mainly viewed in purely syntactic terms—as well as to pose a higher attention to algebraic language as a representational tool. Pioneer studies carried out by English scholars offer teaching experiences aiming at generalization by means of realistic situations in several contexts, but also in situations within mathematics, often playful and even referred to proof (see, for instance Bell 1976; Bell et al. 1985; Harper 1987).

From Traditional Algebra to Early Algebra

In actual practice, the new views on teaching and learning come into conflict with the view of traditional algebra. For this and other reasons, diagnostic studies on pupils' difficulties were carried out, also taking into account issues related to both modeling and interpretation of formal expressions. Classical studies in this respect are those of Booth (1984), Kuchemann (1981), Kieran (1989, 1992), Lee and Wheeler (1989), which point out that many difficulties and blocks in the learning of algebra result from a teaching of arithmetic essentially centred on the aspects of calculation and very little focusing on its relational and structural aspects.

This topic was debated during the ICME 6 Congress (Adelaide 1984), and a proposal made to introduce relational, generalization-type and modelling activities in primary school. However, an important step toward the constitution of early algebra as a disciplinary area was made at ICME 7 (Quebec 1992). In that congress a proposal was made for objectifying a new area of arithmetic teaching, called *pre-algebra*, aimed at the development of 'pre-concepts' useful to algebra, i.e. advanced arithmetic concepts, of a structural type, setting an experiential and conceptual basis for connecting with more abstract and formal algebraic concepts (Linchevski 1995).

In those years, several scholars pointed out the importance for pupils to acquire the 'sense for symbols' (Arcavi 1994) through a variety of activities which might help them develop abilities, understanding and ways of feeling that could eventually lead them to act in a flexible and instinctive way within a system of symbols, to move around in wider or different systems of symbols and to co-ordinate interpretations of formulae in various solution worlds (Arzarello 1991; Arzarello et al. 1993; Gray and Tall 1993, Filloy 1990, 1991, Kaput 1991; Lins 1990). In US, debates about the algebrization of the K-12 curriculum from kindergarten to secondary school are undertaken (Kaput 1995).

At the ICME 8 Congress held in Seville (1996) Kieran characterized elementary algebra through three types of activities[2] at increasing levels of complexity, setting generational activities at the first level, i.e. those through which the objects of algebra can be constructed by linking meanings to experience (Kieran 1998). The second half of the '90s was characterized by a considerable number of experiments on generational algebraic activities mainly addressing 11–13 year old pupils. Some studies

[2](1) Generational activities; (2) Transformational activities; (3) Global, at meta-level activities.

theorized socio-constructive models of conceptual development in algebra, in which the influence of the classroom environment on learning, as well as the importance of the role of the teacher are emphasized, in the framework of a view of algebra as language (see for instance Da Rocha Falcão 1995; Meira 1990; Radford 2000).

Starting from the year 2000, the issues of early algebra became of increasing interest in the International community, as shown by studies about early algebra at the 12th ICMI Study '*The future of the teaching and learning of algebra*' (Chick et al. 2001), and other collective studies, such as the forum on early algebra at the PME 25 (Ainley et al. 2001), the Special Issue on early algebra of the ZDM Journal (Cai et al. 2005), the book '*Algebra in Early Grades*' by Kaput et al. (2007) and the international seminar "*Pathways to Algebra*" organized by D. Carraher in France (Evron, June 2008).

All these studies mainly concern issues of implementation of innovative activities in primary school and analyze pupils' behaviours and learning. Several studies also deal with the problem of suitable teacher training with focused interventions on their professional development (see for instance Carpenter and Franke 2001; Carpenter et al. 2003; Dougherty 2001; Blanton and Kaput 2001, 2002; Kaput and Blanton 2001; Menzel 2001).

Our studies are in the line of this last trend and develop within the *ArAl*[3] *Project*: *teaching sequences in arithmetic to favour pre-algebraic thinking* (Malara and Navarra 2003). In the next paragraph we give an overview of some its basic concepts.

Early Algebra as a Meta-Subject and the *ArAl* Project

The linguist S. Ferreri (2006) wrote:

> You get to the keywords of a discipline through a slow work of foundation of the basic concepts of the respective disciplinary areas. Appropriating the meaning of words, of some meaningful words, is a way to stabilize, conceptualize and master a specific knowledge domain, as its contents might otherwise remain not grasped. In fact, the word is viewed as a permanent trace of a construction of knowledge, joint in memory to other pre-existing words; as a capacity of making explicit a stage of the process of knowledge which is shaping up. Words that show one's degree of control over knowledge. Words, as portions of knowledge that can represent itself.

From our perspective, the set of keywords of early algebra does not refer to a single discipline (either arithmetic or algebra) and to its terms. It defines its limits by starting from both disciplines but ends up assuming a different identity, mainly distinct from either. We might talk about a meta-subject whose objects are not objects, processes, or properties of the two subjects, but rather the genesis of a unifying language. A *meta*-language, as such.

[3] The term '*ArAl*' is a synthesis of the terms 'Arithmetic' and 'Algebra'. The Project started in 1998 on the basis of our previous studies (Malara and Iaderosa 1999) at secondary school level (grades 6–8, pupils aged 11–14) and it is designed for primary school in the perspective of a continuity between the two school levels.

In our view, early algebra is based on these fundamental principles:

- The *anticipation of generational pre-algebraic activities* at the beginning of primary school, and even before that, at kindergarten, to favour the genesis of the algebraic language, viewed as a generalizing language, while the pupil is guided to reflect upon natural language.
- The *social construction of knowledge*, i.e. the shared construction of new meanings, negotiated on the basis of the shared cultural instruments available at the moment to both pupils and teacher.
- The *central role of natural language* as main didactical mediator for the slow construction of syntactic and semantic aspects of algebraic language. Verbalization, argumentation, discussion, exchange, favour both understanding and critical review of ideas. At the same time, through the enactment of the processes of translation, natural language sets up the bases for both producing and interpreting representations written in algebraic language.
- *Identifying and making explicit algebraic thinking, often 'hidden' in concepts and representations in arithmetic.* The genesis of the generalizing language can be located at this 'unveiling', when the pupil starts to describe a sentence like $4 \times 2 + 1 = 9$ no longer (not only) as the result of a procedural reading 'I multiply 4 times 2, add 1 and get 9', but rather as result of a relational reading such as 'The sum of the product of 4 times 2 and 1 is equal to 9'; i.e. when pupils talk about mathematical language through natural language and do not focus on numbers, but rather on relations, that is on the *structure* of the sentence.

In our project we claim that *the main cognitive obstacles to the learning of algebra arise in unsuspected ways in arithmetic contexts and may impact on the development of mathematical thinking, mostly due to the fact that many students only have a weak conceptual control over the meanings of algebraic objects and processes.*

Our aim is to make teachers aware and caring about this situation and provide them with instruments that enable them to design and implement powerful interventions to face it. In order to be able to do that, the teacher needs to understand how, and most of all why, the construction of mathematical concepts needs to be supported by a setting made of solid linguistic and methodological bases, but also social and psychological ones. Therefore teachers must be able to construct new *meta*- competencies and empower their sensitivity in grasping the deep mutual relations between the two subjects, and the seeds of algebraic thinking underlying arithmetic concepts and representations.

In order better to frame the problematic aspects connected with the role of the teacher, we briefly discuss this issue in the more general perspective of socio-constructive teaching.

Socio-Constructive Teaching and Teacher Training

In the teaching of mathematics the socio-constructive model is spreading, since it is viewed as suitable to educate students (mainly aged between 6 and 14) to work collectively as well as to favour their acquisition of flexibility in thinking. According to this model, teachers should start their action by devolving to students purposefully designed problem situations that may bring about the emergence of particular mathematical concepts and properties. The core of the model is the view of students as makers of their own knowledge: it develops through argumentation and exchanges of ideas, up to the collective systematization of the results obtained and a reflection upon meanings and role of those results.

The whole-class mathematical discussion plays a central role in the model. In order to be able to fulfill the task, the teacher should master notions and abilities that go beyond the mere knowledge of the discipline:

- from a social viewpoint, *to be able to create a good interactional context*, by stimulating and guiding the argumentative processes (mediating argumentation in the words of Schwarz et al. 2004) easing communication, listening, evaluating, and a capacity of producing a counter-argumentation (Wood 1999);
- *to activate socio-mathematical norms* that lead to check the acceptability of a solution, to evaluate different solutions, to appreciate the quality of a solution (Yackel and Cobb 1996);
- *to determine the direction of the discussion* in its various phases, filtering students' ideas, so that their attention may be focused on the contents of the teacher's views as more relevant and meaningful (Gamoran Sherin 2002);
- *to harmonically enact modalities* (Anghileri 2006) such as: reviewing (focusing pupils' attention on aspects of the activity that may favour the understanding of the underlying mathematical ideas); restructuring (encouraging students to reflect upon and clarify to themselves what they have understood, in order to favour both development and strengthening of mathematical meanings); re-phrasing of students' utterances (re-formulation of what one or more pupils claimed to highlight and clarify the argumentative processes developed in the classroom); using probing questions (posing questions in order to investigate on students' statements, with the aim of leading them to clarify what they said and favour a development of their thoughts);
- *to involve pupils in metacognitive acts* (transactive utterances in the terminology of Blanton et al. 2003), to enable them to internalize collective argumentative processes.

The Role of the Teacher's Reflection

Several researchers underline the value of the teacher's critical reflections on the classroom-based processes (Mason 1998; Jaworski 1998; Schoenfeld 1998) and most of all, of the practice of sharing these reflections (Borasi et al. 1999; Ponte 2004; Jaworski 2003; Malara 2003, 2005; Malara and Zan 2002, 2008; Potari and Jaworski 2002) for the acquisition of the above described competences.

In particular, Mason (2002), by helping teachers acquire the capacity of carefully observing themselves in class-based action, suggests the constant practice of the '*discipline of noticing*', recommending that reflections be shared among colleagues in order to be validated. Jaworski (2004) points out the efficacy of "communities of enquiry" (mixed groups made by teachers and researchers) highlighting the fact that participating in these groups brings about a taking on of identity by the teacher.

On this basis, the hypothesis that is outlined requires a change of perspective by the teacher. A teacher should re-learn to manage *socio-cognitive processes* (experience in the classroom), drawing on the theoretical frameworks proposed to them, comparing these proposals to their own epistemology, thus being fruitfully and significantly enriched in both their own culture and their work in the classroom. In this way, the teacher may avoid the feeling of powerlessness in front of paradigms that are too abstract or self-referential to become reliable keys for reading their own experience, or rather paradigms for an intervention in their own practice.

This hypothesis holds for trainee teachers as well, because it refers to the intertwining between methodological and mathematical aspects: a background in mathematics is a necessary condition, but not a sufficient one, to become a good mathematics teacher.

Mason (2002) starts up his text on the discipline of noticing with the following maxim:

> I cannot change others; I can work at changing myself. (Page v)

In many respects, the latter strictly links to our discourse. As a matter of fact, many teachers believe that they can intervene and change their pupils without having tried consciously to change themselves. In other words: *without putting themselves critically in front of their own practice, and investigating it.* Mason also writes:

> Working to develop your own practices can be transformed into a systematic and methodologically sound process of 'researching from the inside', that is, of researching yourself. (Page xii)

Hence it is a matter of starting up a continuous reflection upon oneself making use of theoretical supports that may bring about the awareness that a continuous *transformation* is needed. The final aim is to get to go beyond the idea that some little adjustments (such as changing the textbook, using new technologies, or attending some training courses) can be enough to produce effective changes in pupils' learning.

But, in order to be effective, transformation requires an essential condition: that we train ourselves to understand *in which directions* transformation should be promoted. A *fruitful exchange between theory and practice* may bring the teacher to develop capacities at two levels: at the first level, to *grasp signs* in all that contributes to define their own condition, both *in the field*—in their activity in the classroom—and in the construction of their own theoretical instruments, at a second level, to *elaborate on the grasped signs* so that these signs become part of the foundation of their cultural background.

The development of a capacity to grasp signs is achieved only through the teacher's increasing awareness in learning to transform thousands of occasionally noticed things into a tool of one's individual methodology, deriving from a relation between the capacity of noticing, the motivation to intervene and the acquisition of instruments that suggest *how* to intervene.

Concerning the capacity of noticing, Mason adds:

> Every practitioner, in whatever domain they work, wants to be awake to possibilities, to be sensitive to the situation and to respond appropriately. What is considered appropriate depends on what is valued, which in turn affects what is noticed. … noticing what children are doing, how they respond, evaluating what is being said or done against expectations and criteria, and considering what might be said or done next. It is almost too obvious even to say that what you do not notice, you cannot act upon; you cannot choose the act if you do not notice an opportunity. (Page 7)

Hence the question is: *What* should the teacher *notice*? *Who* would teach him or her to notice this *that?*

What we maintain is: *the teacher is firstly a mentor of himself*, through a continuous engagement, with the awareness of the support of a reference map as well as of suitable stimulating tools (continuously and critically reviewed, as we will see in the next paragraphs), which allow a teacher to start up an exploration of a cultural baggage which is certainly familiar to them, but at the same time, needs to be re-considered from different viewpoints, through a process that will gradually lead them to a *forma mentis* that is profoundly different to that of the previous stage. In our case, the point of arrival will be a re-reading of one's conceptions with relation to arithmetic and algebra. *The teacher must be active protagonist of his own development* in his approach to early algebra.

In this respect, the main tool our project (*ArAl*) refers to, is a set of theoretical constructs, partly drawn from other constructs and partly original, organized in a Glossary, which we will present here.

The Role of the *ArAl* Glossary in Teacher Training

The *ArAl* Glossary is a reference system that allows the teacher gradually to reach an overall view of early algebra, which merges theory and practice, by approaching a linguistic view of algebra, within which a convincing control over its meanings can be constructed together with the pupils.

Each term of the Glossary is a self-sufficient entity, so to speak. The text that describes it includes some other key terms, the set of which constitutes a more or less wide Net.

A term of the Glossary may be able to avail itself of a very numerous Net, but be quoted in few Nets. Vice versa, another term might have a Net with few links, but be present in many other Nets. Each term, therefore, depending on the numerousness of its Net, as well as on that of its occurrences, locates the teacher within a double process of conceptual deepening and extension: deepening of the term through the relations among the key-terms which appear in its definition; extension, since each of them is a potential stimulus to read its definition.

The terms of the Glossary may be grouped within five areas:

- GENERAL: didactical mediator, Opaque/transparent (referred to meaning), Relational thinking, Process/Product, Representing/Solving, …
- LINGUISTIC: Arguing, Algebraic babbling, Language, Letter, Metaphor, Paraphrase, Semantic/Syntax, Translating, …

- MATHEMATICAL: Formal coding, Additive form, Canonical/Non canonical form, Multiplicative form, Mathematical Phrase, Function, Unknown, Pseudo equation, Relation, Equal sign, Structure, Variable, ...
- SOCIO-DIDACTICAL: Sharing, Collective exchange, Didactical contract, Discussion, Social mediation, Negotiation, ...
- PSYCHOLOGICAL: Affective-emotional interference, Perception, ...

Some terms of the Glossary have more numerous Nets and Occurrences than others. This attaches to them a *status of strong representativeness in the definition of early algebra from our point of view* and enables us to compose a sort of manifesto through them (in bold case in the text):

> The theoretical framework of *early algebra* supports the hypothesis that students' weak control over the meanings of algebra derives from their ways of constructing arithmetic knowledge from the early years of primary school.
> Algebra should be taught as a new **language,** one gets to master—through a set of shared **social** practices (**collective discussion**, **verbalization**, **argumentation**)—with modalities that are analogous to those of natural language learning: starting from its meanings (**semantic** aspects) and setting them gradually in their **syntactic** structure (a process we called **algebraic babbling**).
> Crucial elements in this respect are **metaphors**, didactical **mediators** in the achievement of meanings, during the conceptual progression towards generalization and modeling.
> In this view, natural language becomes the most important mediator in the student's experience and the main instrument of **representation** through which they can illustrate the system of **relations** (**additive** and **multiplicative** ones at the beginning) among elements in a problem situation, shifting focus from the **product** to the **process,** and inducing a **translation** of the process itself into a mathematical **sentence**.
> In this way, attention is shifted from the *arithmetic* objective of **solving**, to the *algebraic* one of **representing**. At the same time, mediators favour the achievement of the use of **letters**, seen as **unknown**—easier to be achieved—of **indeterminate** and of **variable**.

The structure of the Glossary is outlined so that pre-defined approaches to the included terms are not sketched. The teacher finds a plurality of routes to be explored autonomously, depending on the modalities of their approach to early algebra, their own background, the age of the pupils, the themes they want to deal with, their curiosity and so on. The exploration of this plurality is an individual adventure, which originates from the interest in deep concepts and theoretical constructs and depends on how the teacher decides to interact with it. Its use offers the immediate contextualization of concepts and the possibility to connect them with other more familiar ones through a net of internal cross-references. By doing so, the teacher gradually builds a progressively more articulated conceptual map of the theoretical frame of the project.

We briefly describe some crucial elements of such a frame concerning the algebraic-linguistic aspects. We shall therefore leave out the psychological, educational, social and the more general mathematical aspects involved. To begin with, we concentrate on the exploration of the concepts that concern so-called *algebraic-babbling*.

Algebraic Babbling

Pre-requisite to acquire control over the syntactic aspects of a new language is a slow and *in-depth* acquisition of a *semantic* control. As we know, a child, while learning natural language, gradually appropriates its *meanings* and rules, and progressively develops them through adjustments and imitations, up to the deeper knowledge he gets to in his school years, when he learns to read and reflect upon *grammar* and *syntax*. In the same way, mental models typical of algebraic thinking should be constructed from the early years of primary school, progressively building up in pupils algebraic thinking as both tool and object of thinking, in a strict *intertwining with arithmetic,* starting from the meanings of the latter. For this reason, it is necessary to build up an environment able to stimulate informally the autonomous elaboration of formal coding for sentences in natural language, discussing them with the whole class and gradually producing a playful, experimental and continuously re-defined appropriation of the new language. The rules of this language are then located in a didactic contract, which tolerates initial moments of syntactic 'promiscuousness'. This process of construction/interpretation/refinement of 'draft' formulas is what we call 'algebraic babbling'.

In the Glossary, for 'algebraic babbling' one finds some excerpts of class transcripts and the related comments. An example follows.

A fourth grade class (9 years-old) is exploring problem situations asking to *identify the relations existing between two quantities*. In one of these situations, involving variable quantities of two different types of biscuits, i.e. sponge biscuits and chocolate cookies, as usual pupils represent the relations between the two quantities in *natural language,* getting to the following correspondence rule, after selecting different formulations:

> 'the number of sponge biscuits is 1 more than twice the number of chocolate cookies'.

The next step is to translate the sentence *in algebraic language*. Individual work leads to the following proposals:

(a) 1×2
(b) $a + 1 \times 2$ (a = n. of sponge biscuits)
(c) $a \times 2 + 1$
(d) $sv + 1 \times 2 = a$
(e) $sv = st + 1 \times 2$
(f) $a = b \times 2 + 1$
(g) $a \times 2 + 1 = b$ (a = n. of choc. cookies)
(h) $(a - 1) \times 2$

The comments made on the transcripts show that, beyond the teacher's wishes and actions, the choice of the most correct sentence(s), of those that better fit with the problem situation, of the clearest ones (in the Glossary the pair *transparent-opaque* is introduced to indicate this) depends upon the level of algebraic babbling achieved by the class through previous experiences. In these experiences, the *negotiation* of meanings led pupils to *share* them through a *social construction* of meanings. The pupils' skills determine the quality of their choices of a sentence and its related justifications, not depending on the teacher's legitimate expectations.

Suppose we explore the Glossary from a linguistic perspective: it makes reference to *language* as well as to its *semantic* and *syntactic* aspects.

Algebraic Babbling → Algebra as a Language

A widespread belief about pupils is that solving a problem means identifying the result. This implies that their attention is focused on the *operations*. They should rather learn not to worry too much about the result, and therefore about the search for the operations that lead to it, and move from the cognitive to the metacognitive level, where the solver *interprets the structure of the problem and represents it through algebraic language*. Algebra thus becomes a language to describe reality, and not only: it amplifies *understanding*.

A process of this kind occurs very slowly, and through progressive steps, with an intertwining of continuities and ruptures between the different levels of knowledge. Traditional arithmetic teaching tends to encourage a mental attitude aiming at immediately searching for the tools (operations) to identify the *answer* (the result). This attitude is also induced by the formulation of the task in some standard word problems like the following (pupils aged 6):

> On a tree branch there are 13 crows. 9 more arrive and 6 fly away. *How many crows are left?*

Contrasting with the question above, there is another, extremely different, one:

> *Represent the situation in mathematical language, in order to find the number of the crows left.*

The first version emphasizes the search for the *product* (16), the second one the search for the *process* ($13+9-6$), i.e. of the representation of the *relations* between the involved elements. This difference links back to one of the most important aspects of the epistemological gap between arithmetic and algebra: while arithmetic requires an immediate search for the solution, algebra postpones this and starts with a formal transposition of the problem situation from the domain of natural language to a specific representation system (think about a problem that can be solved by an equation).

The perspective of an approach to algebra as a language, in a continuous back and forth of thinking from arithmetic to algebra and vice versa, fosters more effective teaching with pupils aged between seven and fourteen, characterized by *negotiation* and *explicit statement* of a didactic contract for the solution of problems, based on the principle: "*first represent*, *then solve*". This seems extremely promising for dealing with one of the most important key elements in the conceptual field of algebra: the transposition in terms of representation from natural language, in which problems are either formulated or described, to the formal-algebraic one in which the relations and later the solution are translated.

Exploring the items related to 'algebra as a language' one meets the pair 'representing/solving', and this leads to another key point in our theoretical framework: the syntax and semantics of mathematical language. In the approach to these themes, the Glossary brings the reader to meet a very important character of the *ArAl* project: Brioshi.

Algebraic Babbling → Syntax, Semantics → Brioshi

An aspect that strictly links to that of representation is the *respect of the rules* in the use of a language, even more necessary when one deals with a formalized language, given that the symbols used are extremely synthetic.

In everyday life the respect of linguistic rules is gradually learned through their use, by trial and error; this is favored by the family environment and by the enlarged social one, as well as by school, through a reflection upon orthography, grammar, syntax, i.e. upon the structural aspects of a language.

In the learning of mathematics, the rules are generally 'given' to pupils, thus losing their social value as support to understand a language, and hence to share it, as a tool for communication. Similarly to what happens in linguistics, the syntax of the mathematical language concerns the structure of the sentence, the elements that compose it and the formal procedures that express the relations between the involved quantities—either known or unknown—even in a passage made of several sentences.

It is thus necessary to lead students to understand that they are appropriating a new language and that, as all other languages elaborated by the human kind, it is a system of arbitrary finite symbols, combined according to precise rules. But, whereas pupils have interiorized the set of rules related to spoken language since their birth, and understand that respecting them, is functional to *communication*, it is rather difficult for them to transfer this peculiarity to mathematical language. To avoid this key element and highlight the value of written language for communication, the teacher proposes an exchange of messages in arithmetic-algebraic language with either real or virtual classes, engaged in the solution of the same problem situation. Brioshi, a virtual Japanese pupil, variably aged depending on the age of his interlocutors, knowing only his mother tongue (and therefore not able to communicate using languages that differ to his own), but competent in the use of mathematical language, is the *algebraic pen friend,* with whom they need to communicate (for a wider discussion, see Malara and Navarra 2001).

Pupils get to learn that, like any language, mathematical language also has its own grammar and a syntax, i.e. a set of conventions, that enable us to construct sentences correctly. It has a syntax, which provides the conditions—i.e. the rules—to decide whether a sequence of linguistic elements is 'well-formed' (for example, sentences like '$9 + +6 = 15$', or the classic chain of operations added one after the other, like '$5+3=8:2=4+16=20$') are syntactically wrong. It has a semantics, which enables one to interpret symbols—within syntactically correct sequences—and subsequently decide whether the expressions are true or false (for instance, the sentence '$1 + 1 = 10$' is either true or false depending on the representation base, which can be either 2 or 10).

In the perspective we are considering, *translating* from natural language (or graphical, or iconic) to mathematical one, and vice versa, is one of the most fertile territories where reflections upon mathematical language can be developed. Translating, in this case, means interpreting and representing a problem situation through a formalized language or, on the contrary, recognize in a symbolic expression, the situation it describes.

In the learning of mathematics, where the exchange between verbal language and mathematical language is continuous, it is necessary to activate in pupils, on the one hand, a control over expressive registers and, on the other hand, the meta cognitive skill to understand how syntactic transformations of formal expressions condense thinking processes that can be hardly realized through natural language.

Meeting Brioshi is very important to help very young pupils approach the idea of a *language of mathematics* that allows them to represent and communicate procedures and relations. All the teachers who either collaborated with the ArAl project or adopted the project's teaching sequences, were able to test the power of Brioshi's metaphor, independently on the pupils' age.

Further important pre-algebraic key-points are faced with Brioshi. One of these is the *form of representation of a number* and, related to this, the *equality sign*.

Brioshi → Canonical/Non Canonical form of a Number → '='

Facing the ontological question: 'Is $[3 \times (11 + 7) : 9]^2$ a number?' usually both students and trainee teachers answer in the same way: "No, they are operations", "It is an expression", "They are calculations". At times, someone dares to say "Can we say it is the representation of a number?". In order to promote a reflective attitude on the answer, we make use of analogy, by examining how people are usually denoted. Each person has their first name but can also be denoted by the relationship that links them to other individuals or to the environment (Daughter of ..., Sister of ..., Owner of the dog ..., Living in So-and-so Street, etcetera). Each of these expressions adds information not included in the first name and permits a deeper knowledge of the person.

The situation is similar with numbers: each number can be represented in many different ways, through any odd equivalent expression. Among these representations, one (for instance 12) is its name, called a *canonical form of its representation*, all the others (3×4, $(2+2) \times 3$, $36/3$, $10+2, \ldots$) are its *non canonical forms*, and each of them will make sense in relation to the context and the underlying process.

This experience enables older pupils to answer the initial question we left unanswered: $[3 \times (11+7) : 9]^2$ *is one of the many non canonical forms of the number 36*.

Being able to recognize and interpret these forms builds up the semantic basis for the understanding of algebraic expressions like $p-4q$, ab, x^2y, $k/3$. Moreover, the concept of canonical/non canonical form has crucial implications when it is about harmonizing the meanings of the equality sign as *directional operator* and *indicator of equivalence.* The process through which these skills are constructed is very long and is to be developed along the whole course of the first school years.

In this long paragraph we examined some fundamental concepts of our approach to early algebra, going through one of the possible routes for exploring the Glossary. We will now move on to the side of practice, with reference to both (i) *the classroom based action*, and (ii) *the teacher's reflection upon the latter*. We will go on to analyze a second—and similarly fundamental—instrument: the *methodology of multi-commented transcripts*.

The Multi-Commented Transcripts Methodology (MCTM)

Earlier in the text, we underlined how noticing and critically studying and reflecting upon classroom-based processes allow the teacher to become aware of the dynamic processes characterizing the teaching activity, as well as of the variables that determine them.

MCTM has been conceived to promote this awareness. Its aim is to lead teachers to acquire an increasing capacity of interpreting the complexity of class processes through the analysis of the *micro-situations* that constitute them, to reflect upon the effectiveness of one's own role and become aware of the effects of one's own *micro-decisions*.

We implement the MCTM in in-service teacher training courses.[4] An initial period of study (8–10 weekly 3-hour-long meetings in which the theoretical framework, general literature and the Glossary are presented and discussed), is followed

[4]These courses are generally two-three-years-long and organized by Universities jointly with local training agencies. The teachers involved come from both primary and secondary schools, depending on the specific course.

by a second period characterized by laboratories involving small groups of teachers (two to four units). For each group, a mentor[5] coordinates the work, providing the link with other researchers involved in the project: a little community of enquiry is thus constituted. The small groups mutually interact in specific moments devoted to collective exchange.

In the laboratories, participants plan lengthy teaching experiments (at least five teaching sessions lasting 90–120 minutes) designed drawing on the *ArAl Units*[6] and, most importantly, carry out a complex critical analysis of the processes enacted by the use of MCTM. The latter activity is the core of the educational process.

MCTM is structured in a sequence of phases. Teacher-experimenters make audio-recordings of lessons on topics they previously chose in agreement with researchers and, after transcribing them in digital text version (the 'transcripts') and filling them with comments and reflections, they send them to their mentors, for further comments. Then, the latter send the transcripts to the authors, to other teachers engaged in similar activities and to other researchers. Often the authors intervene back in the cycle, making comments about the comments or rather inserting new ones.

Therefore, lesson after lesson, the multi-commented transcript (MCT)[7] of the enacted didactical process is objectified.

An Example of MCT

We report here a short excerpt from an MCT referred to an activity on the non-canonical representations of a number carried out in grade 3. It is centered around an iconic representation linked to a problem situation that asks pupils to represent the numerousness of marbles in boxes set out as follows: one line of eight boxes, each containing two marbles and another line of eight boxes, each containing five marbles. Pupils are supposed to make explicit the way of counting the total number of marbles through a numerical expression. Many proposals are written on the blackboard, including: $16 + 40$ (made by Andreina) and $(2 \times 8) + (5 \times 8)$ (made by Giovanni), to which the following excerpts refer.

1. Melania *Andreina's translation is opaque for me.*

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[5] The mentor is usually a teacher-researcher, a typical figure of the Italian research for innovation (Malara and Zan 2002).

[6] The *ArAl Units* are monographic booklets about experiences in early algebra, which can be seen as models of processes of teaching arithmetic in an algebraic perspective, to be carried out in the long term. They are the result of the progressive refinement of numerous experimentations and are fine-tuned on the basis of cross-analyses of records of class activities, and of comparisons of reflections between teachers, mentors and researchers. They are not tools for immediate use in the classroom, but require a theoretical study, before being put into practice.

[7] This phase requires a hardly quantifiable and lengthy time both for teachers and for the other actors involved. An average of three hours for commenting upon each lesson may be estimated. In addition, time devoted to the sessions of joint analysis, involving teachers of the group and researchers working on a specific teaching sequence, should be considered. This time is generally acknowledged by schools and certified by the University.

2. Teacher *What do you mean?*
3. Bruno *It is opaque because Andreina has already found the number of marbles.*
4. Chiara *It was not our job to find it, we had to write a translation for Brioshi. She has almost solved the problem.*
5. Bruno *It's true! She found the product and not the process! Giovanni's translation is the right one, since it is more transparent.*

Melania (line 1) refers to Andreina's representation as opaque because it hides the counting process and does not respect the given task, as Bruno (line 3) and Chiara (line 4) clarify after the teacher's prompt (line 2). Bruno (line 5) feels the need to refer to other terms from the Glossary to point out further the problematic aspects of Andreina's expression as opposed to that produced by Giovanni.

The excerpt clearly shows how the pupils have appropriated the terms of the ArAl Glossary (translation for Brioshi, opaque/transparent, process/product) and how appropriately and consciously they use them. These expressions at *meta* level indirectly show that the teacher has stabilized and conceptualized—and now masters—these constructs: not only this, she is also competent in adopting them in her classroom practice.

Multi-commented transcripts are important instruments from four points of view:

- *Diagnostic*: transcripts provide the mentor, and therefore the whole team, with an overview of the teacher's teaching action and enable a check on the coherence between teaching practice and reference to the theory at stake (both mathematics and mathematics education).
- *Formative*: through comments, they enable the teacher to develop competences and sensitiveness and hence to improve the overall quality of his teaching action.
- *Evaluative*: by making clear both coherence and inconsistencies of the teaching action, the transcripts provide both the teacher and the researchers with elements that can empower the effectiveness of the interventions in the respective areas, and make it possible to detect teachers' attitudes and cultural backgrounds.
- *Social*: the transcripts promote a sharing of knowledge, since they are sent out to the other components of the group and a periodic reflection upon the most significant excerpts is undertaken. Moreover, each teacher, comparing their own progress on the realization of a certain part of the teaching sequence to that of other colleagues, can identify important distinctive elements and reflect upon both effectiveness and limits of their own work.

Taking part in the MCTM the teacher questions their action at different levels:

- *transcription* promotes a posterior reflection upon the activity and how it has been carried out and guided;
- *writing down the comments* fosters a critical reconstruction of the activity through an interpretive effort which has a high formative value;
- *their analysis by mentor and by other researchers*—focusing on both mathematical and methodological aspects—leads to a re-elaboration of the activity with a significant impact on both teaching practice and teacher training.

MCTM enables the teacher critically to connect three key points: their own conception of the mathematical knowledge at stake (and of mathematics itself); the conflict induced by the meeting—clash with the teaching modalities enacted by colleagues and the results they achieved; the mediation between these two key points produced by the collective exchange and the dialogic relationship with researchers.

The comments (almost line by line) make clear the overall analysis which deals with aspects related to content (the approach to the exploration of the problem, the mathematical aspects developed, the objectives achieved or missed out, ...), aspects related to communication and language used (formulation of the questions posed, interlocutory expressions used, operative directions suggested, ...), issues related to the control over pupils' participation (number and type of interventions), as well as to the didactical contract established (attitudes induced in pupils and socio-mathematical norms in the classroom). Due to the fine and multi-faceted analysis the teacher's action is fully explored and, consequently, they become aware of their own oversights, missed chances and faults. Being constantly observed, the teacher is induced to have a better control over their actions and attitudes and gradually gets to change their way of being in the classroom.

From the Comments to a Classification of Attitudes

As we repeatedly underlined, the aim of the project is to train *metacognitive* teachers. In this view, the high number of comments in the transcripts (by the year 2009 nearly 4000 comments) offers a very meaningful overview of *the teachers' attitudes* towards their own activity in the classroom and their own capacity of reflecting upon it afterwards. The analysis of comments, in turn, brings about powerful feedback on training interventions.

Let us assume two categories, both related to the teacher and their 'being metacognitive':

(a) in their action in the classroom;
(b) in the posterior reflection upon their action in the classroom.

The two categories enable us to identify four different types of attitudes, summarized in the table (Fig. 1).

The four deriving profiles may be defined as follows:

$M_A M_C$ The teacher effectively drives the class discussion towards the mathematical objective, encouraging pupils to explore the problem situation; they try to take into account pupils' interesting and unforeseen cues and to find an equilibrium between their own objectives and the stream of the pupils' thoughts.

When the session is transcribed, the teacher keeps detached from the events and this leads his/her to write either theoretical or practical reflections both with relation to mistakes—stiffness, wrong interpretations—and to fruitful interventions.

		Teachers	
		M_C metacognitive in comments	$\neg M_C$ non metacognitive in comments
Teachers	M_A Metacognitive in action in the classroom	They stimulate meta cognitive attitudes; they comment upon transcripts in depth.	They stimulate meta cognitive attitudes; they insert few meaningful comments in transcripts.
	$\neg M_A$ Non meta cognitive in action in the classroom	They do not stimulate meta cognitive attitudes; transcripts are filled in with posterior reflections upon missed out chances.	They do not stimulate meta cognitive attitudes; they insert in transcripts few meaningful comments.

Fig. 1 Types of attitudes at metacognitive level

$M_A \neg M_C$ The teacher manages the activity with a global self-confidence, thus favoring a reflection upon the processes, the coherence in pursuing the objective, the exchange among peers. In the transcription phase, the teacher does not detect stimulating moments for reflection and views his task as essentially completed in the classroom activity. In other words, the phase of in-depth analysis and generalization of behaviors with relation to theoretical issues seems to be weak.

$\neg M_A M_C$ The teacher–due to little expertise, low control over mathematical contents, difficulties in managing the discussion—is not able to guide the activity in a productive way, stimulating the attention of the class towards a meaningful reflection. At the moment of transcription, they realize how weak his guidance was (a feeling often emerged at the very moment of its appearance) and the comments point out this awareness, often joint to a request of help addressing the mentor.

$\neg M_A \neg M_C$ The teacher keeps the class working on a little stimulating activity, constantly playing a central role and driving the pupils towards a fundamentally a-critical acquisition of 'compulsorily reachable' contents. In the phase of transcription, the comments refer to marginal aspects, concise remarks on pupils' personality, superficial clarifications.

In general terms, the type $M_A M_C$ represents a desirable model and possible target for any formative process. In fact the teacher is often not used to an exchange in which they play the role of the student, with hardly predictable consequences.

This is confirmed by a teacher who, after a two-year long collaboration writes down in his reflections:

> At the beginning the Glossary and the ArAl Units were my points of reference. I started to produce my first transcripts rather timidly. Through these, and helped by the comments of my mentor and another researcher, I was able to analyze my behaviours and those of my pupils in teaching and learning situations, focusing on both positive and negative aspects, and to reflect upon some crucial points, mainly related to the management of the mathematical aspects and to communication. And that was the beginning of an itinerary.

Often, an $M_A \neg M_C$-type attitude is not a sort of 'limited disposition' to reflection, but rather fruit of the lack of meaningful stimuli towards the direction Mason talks about, of a constant search within a process and towards oneself, which may leave deep and long-lasting traces at the professional level, contributing to the construction of a way of being that will become *the foundation of a continuous path of change*".

The definition of attitudes $\neg M_A$ is more complex. They are also linked to the age, and hence expertise, of the teacher, but strongly reflect his personality (low self-confidence, fear of losing control over the class, tendency to keep to a reassuring professional stability, tiredness and so on) and his personal history (education, limited attention to refresher courses, skepticism towards theories viewed—sometimes correctly—as too abstract etc.). In these cases, the formative intervention steered towards early algebra becomes powerful because it does not aim to *add* some knowledge but rather attempts to induce a *reflection* which might prepare the ground for a *restructuring of knowledge*. Most of all, it might promote the construction of a mental attitude open to new perspectives concerning both theory and practice.

The four profiles are not *pictures of teachers*, in the sense that they are not definitions to be taken as *absolute*. They rather illustrate *temporary stages* at which the teachers who adopt the multi-commented transcripts methodology come to be and which can be modified over time. The improvement in the capacity of expressing *metacognitive* attitudes can thus be seen as the outcome of a formative process to which researchers contributed—we hope significantly—but it will inevitably be up to the teacher to make it grow autonomously during their professional activity, if they decide to.

The following example is taken from a transcript of a teacher in the phase $\neg M_A M_C$. This example highlights the negative incidence of the teacher's language on the discussion: this language is not appropriate and pays little attention to the aims of the work that is being done. The comments in this case are made by the teacher himself and by three researchers, but after some time, the teacher presented excerpts from his transcripts at a conference, enriching them with personal remarks—often in the form of meta-comments—which highlight how the methodology had positively influenced his attitude, leading it towards a model $M_A M_C$.

The example was chosen because it shows the value of the methodology we adopted, to promote teachers' awareness of the limits of their own action. We realize that this example alone cannot fully transmit the richness and variety of the emergent issues, as well as what the teachers come to achieve; for this reason we send the reader back to Malara (2008).

Example

The example refers to a moment of a teaching sequence about the study of sequences that can be modeled algebraically. The main objective of the sequence is to make pupils achieve a functional view of sequences and lead them to construct algebraic representations for the latter, modeling the relation between the ranking (or place) number and the correspondent term of the sequence. The teacher had proposed the exploration of the sequence after giving the first three terms: 4; 11; 18; .... The class had grasped that the sequence was generated, starting from 4, by the operator "+7". Then the teacher had raised the question of searching for a formula that could represent the correspondence ⟨place-term⟩. In the classroom, attention was then focused on the problem of generalization and the whole class had worked on the meaning of the term 'n-th'. To make the pupils' exploratory activity easier the teacher had summarized in a table all that they knew at that moment. In the study of the case at place 30$^{\text{th}}$ there was a mistake: the number before 30 is swapped with the number after it, during the generalization phase.

Place	Number	Operations	Rule 1	Rule 2
1°	4			
2°	11	$4+7$	$4+(7\times 1)$	$7\times 3-10$
3°	18	$4+7+7$	$4+(7\times 2)$	$7\times 4-10$
4°	25	$4+7+7+7$	$4+(7\times 3)$	$7\times 5-10$
30°			$4+(7\times 31)$	
n				

Teacher: *I want to know: if I am at place n, that we said—do you remember?—it was a place at a certain point, without knowing what point it was. Eh, I want to know what is the rule that allows me to find this number at place n* **(1)** [I point to the n-th term on the blackboard] *are you with me*?

Teacher: *Good, so let's find the rule* **(2)**. *Benedetta*?

Benedetta: *Eh, because I believed that n-th was the last, so I wrote " there isn't because the sequence is infinite* **(3)**".

Teacher: *All right, this is a true remark and perhaps it will be useful later, we will keep it. So, how can we find the formula we need? Don't look at me, look at your sheet and the blackboard! How can you find it?* **(4)** *Andrea?*

Andrea: *Now, if we know... last time we said that n-th stood for any place* **(5)**

Teacher: *Question: number n means a number at any place* **(6)** *without saying what number it is, this is the hard part! What formula do I write for the number at the n-th place*? **(7)**

Sergio: *Er... I think you can't find it because n-th is a number you don't know*

Andrea: *As you said, n-th stands for a number at any place, therefore I say with Sergio, if the place is indefinite, we will never get to know what number it is!* **(8)**

Teacher: *Exactly, I agree with you too! If I don't tell you, at the 3rd, at the 4th, at the 100th, at the 7003rd place, you don't know. But if I tell you that this number … about this number, instead of telling you the place number I tell you it is at place n, can I calculate … can I write a formula to write this number?* **(9)**

Comments

(1) **T**[8] I now realize I have used the wrong terms, thus inducing students to give the answers that will follow, and that I desperately fought against. By saying "the rule to find this number at place *n*" the students understood that I wanted to know that value of a_n. Perhaps I should have said "the rule to find a number of the sequence, given its position".

(2) **R1** I suggest that the class be led to discover and highlight with arrows relations, repetitions of numbers, 'local' regularities. Many of these might be not productive, but they help pupils get used to global explorations. For example, the same sequence, proposed in another class, led some pupils to identify a relation between the numbers of the first two columns and to represent it with $11 = 2 \times 7 - 3$, $18 = 3 \times 7 - 3$, $25 = 4 \times 7 - 3$, and so on. The arrows might link the various four's with the first term of the sequence, numbers 1, 2, 3 of the fourth column with the place numbers of the first one, shifted one line down, etc. These two last arrows might show that 31 is wrong and that it should be substituted for 29.

(3) **R2** Benedetta contradicts herself, if the sequence is infinite, also the places are infinite and *n* cannot be the last one. Perhaps she means *n* as 'very large' number. With this contradiction she expresses her belief that a number at a non-defined place cannot be represented. Anyway, she does not know the meaning of *n* as indicator of a number we do not want to state explicitly.

(4) **M** T is worried by the idea that she should get to the formula written in algebraic language. I keep believing that, in this phase of the work, the objective is to lead pupils to grasp the relation between place and correspondent number, and to express that relation clearly.

R1 Why not encourage expression in natural language, describing the forms of columns 3 and 4: "I get the number by adding to the initial number as many 7 as …" or in any other way. The paraphrases proposed by pupils can then be compared and the most suitable to be translated in algebraic language for Brioshi can be chosen.

(5) **M** Well done Andrea, that "any place" is like gold!

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[8]Here onwards, we will use the following codes: T = teacher; M = mentor ; R1 = a researcher; R2 = another researcher.

(6) **R2** More than ‘any’, term which is linked to the idea of variable, it would have been appropriate to underline that it is a number we do not want to specify, ‘indeterminate’ (term which, focusing on the element, fixes it somehow).

(7) **M** I wondered many times: why not putting, in the column with the mathematical sentence, the (either mental or not) operation made to identify the factor that multiplies 7, starting from the given number? Pupils would have grasped the regularity, the reiteration of a procedure, getting closer to the construction of the formula smoothly.

R2 The formula is gradually determined by identifying invariant parts ($4 + 7 \times \ldots$) and variant parts (number of place—1) in the studied cases.

(8) **R1** the approach to the letter is rather complex, requires lengthy times, different strategies, comparisons, explorations, entails continuous and unpredictable evaporations. The joint presence of intuitions of different meanings in the interventions of Sergio and Andrea is absolutely inevitable, almost physiological. Probably the need (real or presumed) to conclude and get to the rule, imposes to the teacher rhythms that can hardly coexist with that complexity. We are fully immersed in algebraic babbling, and the learning of a new language, of its meanings and rules, must respect the needs of a required settling.

(9) **T** Now I see why they could not answer! We don’t understand each other! As I said at the beginning, the verb “to find” puts them on the wrong track! Perhaps I should have said “find a representation of the number at place n which makes us understand that this number is in the sequence”. Too complicated! I don’t know…

M I agree on the damage caused by the term “to find”.

R1 I also agree on representing, even more if this term (Glossary) becomes one of the keywords of the class’ cultural background, and hence acquires a shared and negotiated meaning (Glossary again).

R2 Finally, well done T! Representing, yes, representing is the keyword.

The analysis of the comments clearly shows the epistemology of the researcher who produced them, due to the prevalence of some types of comments. Both agreements and disagreements in these comments turn out to be fruitful for the teacher, the former by reinforcing the comment, the latter as enriching complements.

Concluding Remarks

An analysis of the discussion carried out so far is needed here. We said that, for most students, the big obstacle in the study of algebra is represented by the difficulty in having control over the meanings of formal expressions. They are led to the manipulation of the latter through the application of rules that are semantically opaque. This is the main reason why in the curricula of the K-8 stages, increasing space is given to early algebra, in association with a socio-constructive teaching practice. The aim is to propose a kind of teaching that may *revisit arithmetic in a pre-algebraic perspective* introducing in primary school activities that foster the development of pre-concepts useful for the learning of algebra.

Our investigations (in the *ArAl* project) are carried out within this framework. The underlying conviction is that the *main cognitive obstacles in the learning of algebra arise in arithmetical contexts* and might condition the development of mathematical thinking in students, due to a weak conceptual control over the *meanings* of Algebraic objects and processes. Some of its most important principles are: anticipation of pre-algebraic activities of a generational type, social construction of knowledge, central role of natural language as didactical mediator, identification and explicit expression of algebraic thinking, often 'hidden' in arithmetical concepts and representations.

As a consequence, the issue of teacher training comes to be crucial for our aims. In this respect, we underlined the value of *critical reflections upon classroom-based processes,* also through participation in 'communities of enquiry' made of teachers and researchers, like those involved in the *ArAl* project.[9] Starting up a continuous reflection upon oneself as a professional in education, implies that one understands the *directions* he should go to support transformation through an *inter-exchange between theory and practice.*

We drew attention to how, from our point of view, early algebra defines its area of interest starting from both disciplines (either arithmetic or algebra) and acquires a different, and mainly original, identity. We have defined early algebra as a *meta discipline,* dealing not only with entities, processes and properties of the two subjects, but rather with the genesis of a unifying language and, therefore, of a meta-language.

A Glossary supports the construction of this *meta*-disciplinary knowledge. We mentioned some key constructs: *algebraic babbling*, the pair *solving* and *representing*, *canonical* and *non canonical* form of the representation of a natural number, the *equality* sign, the respect of the rules in the approach to the *algebraic code*, syntax and semantics, and Brioshi.

Finally, we illustrated the Methodology of Multi-commented Transcripts (MCTM) and its central role in the teacher's formative process, to empower his capacity of reflection, as well as to construct a constant attitude of noticing his own behavior in the classroom-based action and to have control over the impact that his way of acting may have on pupils' attitudes and conquests.

As we said earlier, our main goal, of a 'meta' type, is to form *meta*-cognitive students. But to do this, it is necessary that teachers learn to be *meta*-cognitive teachers in turn. We examined instruments and methods we outlined to promote *meta*-cognition in teachers, in a strict intertwining of reflections upon knowledge at stake (theory) and action in the classroom (practice). We also showed the value of an educational process which, in the long term, is able to give teachers a new professional identity, more consistent with the role they need to play, and not only with reference to early algebra. This is a condition for inducing in pupils, since the early school years, a view of algebra as a language and as an instrument for thinking, a constructive and reflexive attitude and, more in general, a conception of mathematics as carrier of meanings.

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[9] Since 2000 nearly 1000 teachers and more than 10.000 pupils from 12 Italian regions participated in the project.

Finally we wish to add some remarks about an evaluation of the results attained with the teachers who worked with MCTM.

We cannot, and do not want to, give quantitative data about pupils' learning: we only want to stress the kind of attitude spotted in very young pupils involved in our teaching experiments. Some simple examples.

- Anna (final year at kindergarten) recognizes that two trains that continue who-knows-where beyond the door of the room—one made of wagons with two yellow and one red blocks and the other made of wagons with two nuts and a seed of sunflower—"are almost equal". Anna plays with *structural analogy.*
- Federica (grade 2, primary school) finds on her book the expression '$3 \times \square = 27$' and writes down '$3 \times 9 = 27$'. Federica *solves a linear equation with one unknown*.
- Piero (grade 3, primary school) notices that "It is correct to say that 5 plus 6 is 11, but you cannot say that 11 'is' 5 plus 6, and then it is better saying that 5 plus 6 'equals' 11, because in this case the contrary is true as well". Piero is arguing on the *relational meaning of equal*.

A linguistic and constructive approach to algebra, like the one we propose, seems to be productive in terms of leading pupils to view activities such as *translating, arguing, interpreting, predicting, communicating* as mathematical activities. Making calculations is still there, but subordinated to 'higher' purposes, it helps prepare reasoning, argumentations, refutations, corrections. As the complexity of the algebra they will deal with increases, pupils will be led to understand that manipulation of symbols (polynomials, equations, functions etc.) is not self-referential, but helps them mathematize, explore, reason, deduce, in other words, produce thinking and achieve new knowledge.

Due to the objective situation of the school environment in which our activities are set, only a *qualitative* evaluation of the teachers' growth is possible, by observing their behaviors, through their own self-observation and by means of the instruments we illustrated.

Inducing long-term processes of change means promoting individual development throughout time, beyond the period of collaboration with the project.

Teachers need the same structural steps highlighted in the Glossary for their own construction of knowledge. Reflection on one's knowledge, habits, behaviors and stereotypes, occurs through a process of *negotiation* with the often conflicting perspectives offered by early algebra. Achievements in this area are to be mediated, re-elaborated, metabolized—negotiated as such—in a continuous *sharing* of new cultural values. *Devolution* should be constantly at stake, even in the relationship between researchers and teachers: the latter should accept the responsibility of the learning situation they are engaged in. *Institutionalization* of both mathematical and methodological knowledge will occur in time, as the teacher will become more and more aware of the *sense* that the perspective of early algebra can assume with respect to the changes of the questioned didactical processes.

A meaningful conclusion seems to be offered by the content of an e-mail written by a teacher after a four-year-long collaboration:

In my twenty-years-long experience I've always asked my many teacher trainers: "Well, after this nice premise, tell me HOW I should act with my students! What am I supposed to do for them not to damage them?" Many of them answered that they were dealing with the contents of the discipline and not with its teaching, that it was up to me to elaborate the right strategies after all I had heard from them etc. etc. ... AND SO!
Well, it was different with you. What you gave me made a better teacher of me. To use one of your terms, I might say that four years ago I started my '*epistemological babbling*'.

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A Procedural Focus and a Relationship Focus to Algebra: How U.S. Teachers and Japanese Teachers Treat Systems of Equations

Margaret Smith

Abstract This chapter explores two contrasting ways of presenting algebra by looking at key differences across the presentation of simultaneous equations to students in eighth-grade. The examples are from a qualitative analysis of the 1995 TIMSS Video Study data including eighth-grade mathematics instruction in Japan and the United States covering topics on simultaneous equations. The U.S. lesson example shows a procedural approach to this topic, where students focus on getting answers through a series of routine steps. In contrast, the Japanese lesson highlights a strong focus on building generalized solution methods and understanding relationships represented in systems of equations. A discussion of key differences as they relate to important ideas in understanding algebra compared to how it was treated in the classrooms follows the examples.

The purpose of this chapter is to highlight the differences across two contrasting ways of teaching algebra during classroom instruction. In particular, this chapter uses data from the TIMSS 1995 Video Study to compare an example of a Japanese teacher's relationship focused method of teaching algebra to an example of a United States teacher's procedurally focused method of teaching algebra. There are many goals in algebra, but one of the most common and fundamental goals is helping students move beyond an arithmetic approach to a more generalized approach to understanding relationships (Carpenter et al. 2003;

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Carraher and Schliemann 2007). Far too often students focus on the more procedural aspects of calculating solutions rather than exploring the relationships represented in algebraic expressions; the lessons presented here help highlight at some of the key areas teachers can emphasize to help students better understand the generalities and functional relationships represented with algebraic expressions.

Background

Algebraic Reasoning

Algebra is a very broad topic covering a variety of important components. Both Kieran (2007) and Carraher and Schliemann (2007) describe a variety of different complex cognitive components contributing to the understanding of algebra. Even if we try to limit our understanding to the algebra important for students to learn, we can see that there are many complex ideas that span a variety of concepts (Arcavi 2008; Chazan 2008). This chapter is limited to three of these main ideas: equality, generalizing, and variables. These were chosen to help highlight some of the key areas important in the transition from arithmetic to algebra (Carraher and Schliemann 2007; Lins and Kaput 2004).

Equality Equality refers to students recognition that the equal sign represents equivalence between sets, showing a specific relationship; for example $5 = 5 = 2 + 3 = 7 - 2$. Equality carries with it some specific mathematical properties that are important for understanding some more advanced concepts in algebra; for example $a = b$ is the same as saying $b = a$, if $a = b$ and $b = c$ then $a = c$, and others. Therefore, the understanding of equality can be described in very simple terms but understanding it carries with it some very complex ideas. This is probably one reason why children's understanding of equality has been researched as much as it has (Kieran 1981, 1992, 2007; Knuth et al. 2006; Falkner et al. 1999).

One of the consistent findings of research is that students tend to view the equal sign as a command, or prompt, rather than as a means of expressing a relationship (Knuth et al. 2006; Falkner et al. 1999). For example, Falkner et al. (1999) describe how all sixth-grade students, when given the expression $8 + 4 = \square + 5$ thought that 12 or 17 would go in the box. These solutions indicate that these students were looking at the command as "do this," instead of a ways of expressing a relationship. It is not surprising that students develop this notion of equality since many textbooks often imply that nothing after the equal sign implies a command, such as $5 + 8 =$.

So the question arises, how necessary is it to treat the equal sign as a relationship rather than a command? The work by Falkner et al. (1999) and Knuth et al. (2006) indicate that this transition is extremely important. The idea of equality as a relationship can be built with young students, but they have difficulty extending this relationship to mathematical symbolism (Falkner et al. 1999). Moreover, Knuth (2006) and his colleagues found that the use of equality as a relational understanding was important for both solving equations and using algebraic solutions to solve tasks. These ideas indicate that helping students move beyond the idea of the equal sign as an operator to one in which the equal sign represents a mathematical relationship is extremely important as students begin to study algebra.

Variables Variables are a way of representing quantities; in early algebra variables are usually included in equations and in late studies of algebra will include inequalities. For example, $6 + y = x + 4$ is the set of all pairs, x and y, that make the equation true, such as $x = 5$ and $y = 3$ or $x = 0$ and $y = -2$. In the equation $5x = 10$, x can also be considered a variable, even though only one value of x, namely $x = 2$, makes the equation true because x is used to represent the quantity that makes this true. However, many students who study algebra focus primarily on the arithmetic calculations and the specific solutions to equations, leaving the variable to the role of a place-holder or unknown (Lins and Kaput 2004; Stacey and MacGregor 1999). This notion is problematic, because in order to understand that y and x have a specific relationship to each other in the equation $6 + y = x + 4$, namely any pair of values where x is two more than y, one has to first understand that any pair of quantities with that relationship will satisfy the equation; moreover understanding how these equations give rise to coordinate graphs will likely be difficult to understand.

The notion of looking at a variable as characterizing a set of quantities, rather than a single quantity is not beyond the scope of introductions to algebra. Carpenter and Levi (1999) show how first and second grade students can begin to make the transition from specific cases to a class of solutions. In their study, Carpenter and Levi look at how some students reason about whether $78 - 49 + 49 = 78$ is true; some students were able to rationalize about how subtracting forty-nine and then adding forty-nine is true for all numbers, indicating that they recognized that $78 - x + x$ would equal 78, even if x varied, showing an early understanding of variable. Fujii and Stephens (2001, 2008) and Lins and Kaput (2004) consider this shift from specific arithmetic solutions to understanding a class of solutions as *quasi-variable* thinking; that is students' understand that solutions to equations can be represented by a class of solutions but they cannot yet explain how a generalized expression can be used to represent this relationship. It is clear that to help students move from the specificity of solutions to arithmetic calculations ($5 + 4 = \square$) to understanding the general relationship expressed in equations we must help students build on their informal understandings to ways of expressing more formal relationships, with variables in expressions.

Generalizing Generalizing refers to connecting solutions to specific tasks to understanding how the relationships in these tasks represent ideas that have a larger class or relationships. Blanton and Kaput (2003) refer to this as "algebrafy," generalizing mathematical thinking and justifying generalizations. Recognizing both equality and variables, as discussed above, are two examples of learning to generalize; but students also learn to generalize when they learn to examine solutions to problems (Stacey and MacGregor 1999). Kaput (2007) identifies this as a key component of introducing algebra, particularly to young children; in particular he argues that this generalizing process enables students to move beyond a specific to a more mathematical way of reasoning about situations.

The study by Carpenter and Levi (1999) show how students can begin to reason about a more general idea from a specific problem. Swafford and Langrall (2000) show how sixth-grade students are able to reason from specific cases to more generalized solutions; however the students had a difficult time using a generalized solution to find a specific solution. In particular, Swafford and Langrall gave students a series of problems, the first was to find a particular solution to a real-life context with one specific answer, next students were asked to generate a way to represent a set of problems with the same context, and finally students were asked to use these general forms to solve specific problems. Students in the study were successful in solving the specific cases and many could build an equation to represent a general solution, but they had difficulty using this generalized form to solve a specific case. Research by others (Blanton and Kaput 2003; Levin 2008; Radford 1996) also shows how children can reason through problem situations to find both specific problems and begin to discuss the generalized relationships represented in the problems. These studies indicate that students bring important understandings about relationships in real-world contexts that can be built upon to help them understand generalizing mathematical relationships. However, Swafford and Langrall (2000) study also shows that students may have a difficult time understanding what it means to have a generalization; indicating a need for classroom instruction to help students connect between the general and the specific.

TIMSS Video Studies

There have been two TIMSS Video Studies completed, to date, the 1995 and 1999 TIMSS Video Studies (Stigler et al. 1999; Gonzales et al. 2005). Each of the studies collected videotaped samples of eighth-grade mathematics instruction; the 1995 TIMSS Video Study collected data from Germany, Japan, and the United States, and the 1999 TIMSS Video Study collected data from Australia, the Czech Republic, Hong King SAR, the Netherlands, Switzerland, and the United States (again). This data set was significant in that it provided a large-scale representation of classroom instruction in several different countries and it provided a data set that could be analyzed by others for a variety of different purposes.

Results from both studies have shown that "Teaching is a cultural activity" (Stigler and Hiebert 1998, p. 11), built upon traditions and practices of the society. In addition, both the 1999 and 1995 TIMSS Video Studies show similarities within a country, as well as across countries, and also shows how mathematics instruction varies across countries in some very important ways (Stigler et al. 1999; Gonzales et al. 2005). For example, in the 1999 TIMSS Video Study (Gonzales et al. 2005) countries varied on the level of procedural complexity of problems presented during a lesson, with Japan presenting significantly more high procedural complexity problems than the other countries. As another example, countries showed significant differences on the amount of problems presented that were connected to real-world applications.

The results from these studies provide significant contributions to understanding mathematics instruction from a variety of different approaches. In addition, the data set provided opportunities for secondary analysis of the data so that others could research mathematics instruction for a variety of purposes (Jacobs et al. 2006; Kieran 2004; Smith 2000). This secondary analysis has allowed researchers to examine other aspects of mathematics instruction that were not included in the original analysis. For example, Kieran (2004) describes how students' ideas of equality can be used to help make connections to understand algebraic relationships of equality. These secondary studies provide researchers opportunities to provide more in-depth analysis of the classroom instruction in the data set.

Data

The data used for this study were from the 1995 TIMSS Video Study (Stigler et al. 1999). This was the first large-scale video study of classroom instruction which allowed for secondary analysis studying classroom teaching. This data set, rather than the data from the 1999 TIMSS Video Study (Gonzales et al. 2005), was used here because the 1995 study showed that Japanese classroom teaching in the data set reflected the ideals of the NCTM *Standards* (1989, 2000) while instruction in the United States reflected a more traditional approach (Stigler and Hiebert 1999; Jacobs et al. 2006); however Japan did not collect new data for the 1999 TIMSS Video Study. Because the videotapes of classroom instruction in Japan reflects the instruction different from that of the United States, it can provide ways to better understand how classroom instruction could meet the ideals presented in U.S. reform documents (NCTM, 1989, 2000). It should be noted that achievement by U.S. eighth-grade students has improved since the 1995 TIMSS Study, both overall and in algebra, indicating a possible change in some instruction methods in the United States (Gonzales et al. 2005, 2009). However, it is still possible that some classroom instruction may look very similar to the U.S. data collected in 1995 and therefore it is worthwhile to consider the contrast of the two ways of approaching algebra instruction to highlight the need to re-examine algebra instruction.

Analysis

The data presented here were analyzed using qualitative methods; examining patterns and trends that reflect the classroom instruction in each of the two countries. This analysis was completed after quantitative methods were used to identify key differences in classroom instruction in each of the countries (Gonzales et al. 2005; Smith 2000; Stigler et al. 1999). The examples presented here, therefore, contain the patterns and trends as illustrated in several different classroom instruction segments, but did not occur verbatim as presented here. The usefulness of this method is to bring to light some of the key qualitative differences in classroom instruction (Jacobs et al. 2007; Jacobs and Morita 2002).

Two Teachers' Lessons

The following examples illustrate two different approaches to teaching algebra, particularly solving simultaneous equations. Because of privacy issues (Arafeh and McLaughlin 2002; Jacobs et al. 2007), neither lesson happened as presented here; however each represents a set of lessons within each respective country which covered similar topics. Moreover, the dialogue presented mimics patterns observed in the respective countries, for example the types of ideas provided by students in the Japanese lesson represent the types of ideas they provided in observed lessons; similarly the nature of teacher presented ideas in the United States are also representative of the ways material was presented in the videotapes of lessons in the U.S. (Stigler and Hiebert 1998).

Mr Kirkyle's Lesson In the first example (Fig. 1), Mr. Kirkyle focuses on the procedures to help students solve simultaneous equations and connect these to linear graphs on the coordinate plane. Although Mr. Kirkyle attempts to connect equation solutions with the relationships between the corresponding lines, it is not clear that students appreciate this connection; rather they likely take away the procedures and algorithms demonstrated in the lesson (Thompson and Thompson 1994). A discussion of these ideas as seen in the examples is discussed below in an examination of key differences.

Mr. Nakamura's Lesson In the second lesson (Fig. 2), Mr. Nakamura focuses on relationships and generalizing solutions of simultaneous equations. In contrast to Mr. Kirkyle's lesson, Mr. Nakamura helps focus students' attentions on the relationships the system of equations represents. This lesson shows how problems that appear procedural can still be completed with conversations that provide rich mathematical connections, allowing students to begin to connect to the relations and generalizations which characterize algebra (Smith 2000). A discussion of these ideas as seen in the examples is discussed below in an examination of key differences.

Discussion of Key Differences

There are many differences across these two lessons, but I would like to focus specifically on how these two lessons introduce some algebraic concepts to the students in their classes.

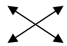
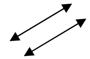
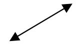
Setting and Discourse	On Chalkboard
As students enter into the room they are asked to copy the following information into their notebook. After completing introductory routines, Mr. Kirkyle begins his lesson. Mr K.: So as you copied down here, there are three ways lines can be drawn together, they can intersect at a point, they can be parallel, or they can be one line	Two lines have the following relationships Intersecting  Parallel  One line 
Mr. K.: So let's look at the following examples Mr. K:. So What do we have here? I see the first equation has two unknowns, so we must have two equations. If I had three variables how many equations do I need? S: Three.	1. $-x + 4y = -21$ $3x - 12y = -21$ 2. $-4x + y = 2$ $12x - 3y = -6$
Mr. K.: Three, good. So let's solve these. For the first one we need to multiply both sides of the top by three.	1. $-x + 4y = -21$ (times 3) $3x - 12y = -21$
Mr K.: So how can we solve these? S: With combinations. Mr. K.: Good so we can combine these two.	Solve by combinations $-3x + 12y = -63$ $3x - 12y = -21$
Mr. K.: Let's multiply the top by three, hat will give us. . . negative three, plus twelve, equals negative sixty-three. . .so we get zero and negative eighty-four so there are no solutions. Mr. K.: Okay, let's look at the next one. We'll multiply by three again.	Solve by combinations $-3x + 12y = -63$ $3x - 12y = -21$ $0 = -84$ 2. $-4x + y = 2$ $12x - 3y = -6$

Fig. 1 Mr. Kirkyle's lesson

Mr. K: What do I get when I multiply it by three? S: Negative twelve x plus three y equals six. Mr. K.: Good now let's add these two together and we get zero, zero, and zero, so we get zero equals zero. What does this tell us about the lines? Mr. K.: They are the same line; that means there will be many solutions. Let's try it out let's put in one and…what will work with that…six, let's put one and six into the first equation and see what we get	$-4x+y=2$ (times 3) $12x-3y=-6$ $-12x+3y=6$ $12x-3y=-6$ $0=0$	
Mr. K.: Let's try it out let's put in one and…what will work with that…six, let's put one and six into the first equation and see what we get Mr. K.: Now let's put it into the second one because they should work for both. So we get twelve minus eighteen and that's negative six, good so it works. What ever you find that works for one will work for the other. Mr. K.: So when I graph these lines number one is parallel because there are no solutions and the second is the same line because there are many solutions.	$-4(1)+6=2$ $12x-3y=-6$ $4(1)+6=2$ $12(1)-3(6)=-6$ $-3x+12y=-63$ $3x-12y=-21$ $0=-84$ No solutions	 $-12x+3y=6$ $12x-3y=-6$ $0=0$ Many solutions
Mr. K.: Okay one last one	$-2x+3y=6$ $2x+y=10$	
S: You can just combine them. Mr. K.: Okay so we get	$-2x+3y=6$ $2x+y=10$	
S: Four y equals 16 divided by four and we have y equals 4	$\frac{4y}{4}=\frac{16}{4}$ $y=4$	

Fig. 1 (Continued)

Mr. K: So that is y. Can someone else help us get x? S: We just plug that into the equation. Mr. K.: Which one? S: It doesn't matter; I put it in the first one and got minus two x plus three times four is six. . .so I subtracted twelve and got minus two x is negative six and divided by negative two is three.	$-2x + 3(4) = 6$ $-2x + 12 = 6$ $-12\ -12$ $\frac{-2x}{-2} = \frac{-6}{-2}$ $x = 3$
Mr. K.: So the answer is (3, 4) and because it is a point you know the lines intersect.	(3, 4)
Mr. K.: Okay for the rest of the class I would like you to work on numbers 8 through 12 but you do not have to draw the graphs, just write down if the lines are parallel, intersecting, or the same.	

Fig. 1 (Continued)

Setting and Discourse	On Chalkboard
As students walk in the simultaneous equations shown here are on the chalkboard. After greeting the students, Mr. Nakamura asks students to complete the problems. Mr. N.: Here are equations with two unknowns, we call these simultaneous equations; try to answer this in two minutes. Ready. Go.	(1) $2x + y = 9$ $x + y = 5$ x | 0 1 2 3 4 5 6 x | 0 1 2 3 4 5 6 y | y | (2) $x - y = 2$ $x - 2y = 6$ x | −5 −4 −3 −2 −1 0 1 x | −5 −4 −3 −2 −1 0 1 y | y |
Mr. N.: Okay how many people were able to finish both? Oh, no one. How many people finished just number one. Oh, a few. What makes this so hard. S: It takes a lot of tries. Mr. N.: Then I wonder how we can solve it so we can do them fast.	
Mr. N.: What about these problems you learned in seventh grade? Mr. N.: Can you solve these fast? Students: Yes. Mr. N.: Okay thirty seconds, ready go.	$9x - 5 = 2x + 16$
Mr. N.: Okay tell me how you solved this. S.: I used properties of equality to find out the x that makes this true.	$9x - 5 = 2x + 16$ $9x - 2x = 16 + 5$ $7x = 21$ $x = 3$

Fig. 2 Mr. Nakamura's lesson

Mr. N.: Good. Before you knew how to follow the steps how did you solve this? Did you put in numbers? S: Yes we would try different numbers until we made it equal, so like nine time three minus five is the same as 2 times three plus sixteen.	
Mr. N.: Very good. So how can we use that to do the first problem very quickly? Let's show it with out diagram too. Do these say the same thing? Students: Yes *Mr. Nakamura and his students both seem familiar with using the shapes to represent the letter symbols with the common language that the sticks are counted as individual ones and all the other shapes "next to" is the same as multiplication.* Mr. N.: Okay now work with your groups to see how you can solve these.	(1) $2x + y = 9$ ●● + ■ = ||||||||| $x + y = 5$ ● + ■= |||||
After students have worked for a considerable amount of time with one another, Mr. Nakamura gathers the students to discuss their strategies. Mr. N.: Let's see the different ways the groups used to solve this.	<u>Group 1</u> $2x + y = 9$ Transpose $2x$ $y = 9 - 2x$ x: 1 2 3 4 5 y: 7 5 3 1 −1

Fig. 2 (Continued)

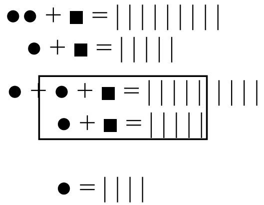
S.: Well we used it like the ones we did up there (*refers to* $9x - 5 = 2x + 16$) and found that y equals nine minus two x and five minus x; so then we found what numbers were the same for both equations which were four and one. Mr. N.: Very good, so you knew that the numbers had to be true for both equations so you set up a table to find out which two had the same pair of numbers. Class how fast do you think this will work? S.: Very slow. Nr. N.: Yew, while it gives us the correct numbers it will not help us work very quickly.	$x + y = 5$ transpose x $y = 5 - x$ x | 1 2 3 4 5 y | 4 3 2 1 0
Mr. N.: Let's look at another group, they used the diagrams. S.: Well we knew that the circle and the square were the same as five, so we made the two circles circle plus circle, then we put in the five and found that the circle was four, then four plus one is five.	
Mr. N.: Very good, so you noticed that this was the same as this (*rewrites pictorial model with symbols*) and then you said you know that this is the same as five so you put that in and then it was easy to find that x equals four, so then y equals one if x plus y must be five.	$2x + y = 9 \rightarrow x + \underbrace{(x + y)}_{5} = 9$ $x + y = 5$ $x + 5 = 9$ $x = 4$

Fig. 2 (Continued)

Mr. N.: Okay our last group has only equations. S: Yes we started like the first group but didn't use the table, we knew they were the same so they must be equal. Mr. N.: Why do you think that? S.: Well it is like the balance we used that the two things are the same thing so they must be balanced with each other…so we made them equal to each other and found x equals four and then put that into x plus y equals 5 and found that y is one. Mr. N.: Very good you knew that these must be the same thing so that they would have to be equal. What do you think about this method class? S.: It is much faster. *Teacher re-visits the solution strategies again describing the how each gives a correct answer but how the last is more efficient.*	
Mr. N.: Good now let's try to do this problem. I will give you a few minutes	Takako and Kumiko were shopping. Takako bought three notebooks and four pencils for 750 yen. Kumiko bought three notebooks and two pencils for 450 yen. How much does one pencil cost?

Fig. 2 (Continued)

After students work on this problem, the teacher allows one student to show how she solved it. S.: Well know that each paid the same for the pencils and notebooks so I know that if three x is how much Takako spent on notebooks, then Kumiko also spent $3x$ on notebooks. Takako bought four pencils so she spent $4y$ on pencils, but Takako only bought two, so she has $2y$ for pencils. Since both of these need to have the same price for notebooks and pencils I need to find out which one. Each of these has three x so I used those to make them equal to each other since three is the same thing in both equations. So then since they are equal I calculated that the pencils are 150 yen each.	$3x+4y=750$ $3x+2y=450$ $3x=450-2y$ $3x=750-4y$ $750-4y=450-2y$ $750-450=-2y+4y$ $300=2y$ $150=y$ So pencils cost 150 yen each
Mr. N.: A very good solution. *Mr. Nakamura rephrases the students explanation and solution and summarizes the main strategies learned during the lesson.*	

Fig. 2 (Continued)

- Japanese teachers used expressions to generalize; U.S. teachers focused on answers and the steps to get the answer.

In the Japanese lesson, Mr. Nakamura places a lot of attention on generalizing both ideas and solution processes. This can be seen as he helps students examine the usefulness of solution methods to solve the problem and the connections across these methods. In contrast, Mr. Kirkyle treats each problem with its own set of steps to solve the problem. The need to move beyond procedural approaches of solving algebra to a better understanding of the general relationships has been well documented (Carraher and Schliemann 2007; Kieran 1992, 2007). However, this transition is difficult for students, as they tend to focus on specific solutions to specific problems (Khng and Khng 2009). By helping students connect the different solution methods, as well as connecting the procedures to the problem solving context, Mr. Nakamura was helping students understand the relationships being represented by the operations as well as across models for showing these relationships.

- Japanese teachers addressed how variables were used to express variation; U.S. teachers focused on variables as unknowns or place holders.

In the Japanese lesson, Mr. Nakamura helps focus students' attentions on the role of the variables in the equations. He highlights this most clearly when he looks at Group One's solution, pointing out how the set of numbers that solved the first equation solved the second equation. It would be easy for Mr. Nakamura to have dismissed this solution strategy as being too rudimentary to present, but by presenting this solution strategy he was able to help students see how the pairs of number changed (varied) in each equation. Mr. Kirkyle began to address this very important connection in the first set of infinite solution simultaneous equations, but did not help articulate how he knew what numbers to use in these equations or how more than the one pair he found could also have satisfied these equations. This limited exposure appeared to be more focused on getting the calculations correct than in understanding the role of variables in expressions.

Early introduction to algebra often focuses on solving for an unknown (Kieran 1992); however, the mathematical use of a variable is much more powerful than that as it allows for the study of relationships. This focus on the variable as a thing that changes is an important component of understanding the relationships expressed in equations, which later are studied as functions (Kieran 2007; Malisani and Spagnolo 2009). It is easy to see how the variable can become yet another symbol to be operated upon, so careful consideration needs to be given to the connections required to draw out the relationships they represent.

- Japanese teachers looked at the relationship expressed by equality; U.S. teachers treated the equal sign as an operator.

It is clear that Mr. Nakamura is building a foundation for the relationship of equality; this is most evident as Groups Two and Three use the equality to generate solutions to the simultaneous equations. Group Three's solution indicates that the students are able to think flexibly about how the equivalence must be maintained across both equations. Group Two's use of equality indicates a fairly deep understanding of equality and simultaneous equations (even if the solution method is viewed as less symbolically sophisticated) because of it focus on an expression, not just a variable, being equivalent. In contrast, Mr. Kirkyle pays little attention to the relationship; this is most notable when he solves simultaneous equations from parallel lines. In particular, Mr. Kirkyle comes up with the unusual statement of $0 = -84$ and tells students this means the lines are parallel, without indentifying the fallacy of the statement. This lack of attention may seem trivial to some, but by leaving this statement as equivalent it makes it difficult for students to move beyond using the equal sign as a command to compute because in this case it is simply a means to an end.

The importance of equivalence as a relationship appears in many places (Kieran 1981, 1992, 2007). In particular, the research indicates that even though students can reason about equivalence as a mathematical operator, or relationship, it is most commonly viewed as a command (Stacey and MacGregor 1999). However, the research also indicates that the need for the teacher to help students make this connection is critical (Molina et al. 2009; Stacey and MacGregor 1999). Molina et al. (2009) found that students who received instruction focused on building this meaning were able to reason about the relationships of this operator.

Conclusions

In this chapter two contrasting ways of introducing students to algebraic expression were presented. The purpose of these examples was to help highlight some of the key differences in presenting algebra to students, one which focused on studying relationships expressed by simultaneous equations and one which focused on building efficient procedures of solving simultaneous equations. Some of the key differences across these lessons indicate that building connections into the instructional process can help illuminate some of these important components in the study of algebra.

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Teaching Algebraic Equations with Variation in Chinese Classroom

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Abstract This chapter gives a detailed analysis of how teaching with variation is helpful for students' learning of algebraic equations by using typical teaching episodes in grade seven in China. Also, it provides a demonstration showing how variation is used as an effective way of teaching through the discussion after the analysis.

Introduction

International studies of mathematics achievement show that in the past decades, East Asian students have consistently outperformed their counterparts in Western countries (Stevenson and Lee 1990; Lapointe et al. 1989; Husen 1967). Findings from the increased interest in the study of mathematics classrooms in East Asian countries suggest that teaching with variation, a common characteristic of mathematics education in East Asian countries, could be one of the powerful explanations for the gap (Sun 2009; Park 2006; Huang and Leung 2004). Furthermore,

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studies in mathematics education and cross-cultural psychology aiming to reveal the reasons for the contradictory situation called "The Confucian Heritage Culture (CHC) Learner Paradox"[1] show that teaching with variation may be helpful to uncover the paradox where, even in large classes, students still can actively involve themselves in the process of learning and achieve excellent results (Gu et al. 2004; Biggs and Watkins 1996). In China, "teaching with variation" has been applied either consciously or intuitively for a long time, and it has almost become the teaching routine for Chinese mathematics teachers (Marton et al. 2004).

Algebra has been identified as the most important "gatekeeper" in mathematics (Cai et al. 2005). While algebraic equation plays a key part in the learning of algebra, it is a challenge to students because it is probably the first time they really understand the meaning of the equal sign from their previous arithmetic experiences (Pirie and Martin 1997). Meanwhile, the studies suggest that many students also have difficulty in grasping the syntax or structure of algebraic expressions and do not understand either the procedures for transforming equations or the reasons why transformations are done the way they are (National Mathematics Advisory Panel 2008; Pirie and Martin 1997). These and other difficulties are compounded as equations become more complex when students attempt to solve word problems (Vlassis 2002). In a summary on research about the learning and teaching of school algebra, Kieran (1992, p. 390) poses these questions, "What makes the comprehension of school algebra a difficult task for the majority? Is the content of algebra the source of the problem? Or, is it the way it is taught that causes students to be unable to make sense of the subject?"

However, studies also show positive findings of teaching algebraic equations with variation in China. According to Chen and Song (1996), in spite of the situation of whole-class instruction, individual-supplemented counseling, large classes with an average of more than 50 junior middle students, natural classes (classified randomly) with various ability levels of students, the performance of the algebra teaching process in China is successful in that the students' interests in learning algebra are inspired through variants provided by the teacher, and they have ultimately developed algebraic thinking. Naturally, one might wonder how algebraic equations are taught effectively in China. Using typical teaching episodes in grade seven, this study will give a detailed analysis showing how teaching with variation helps develop students' algebraic thinking. What must be mentioned here is that—although there is a growing agreement that teaching with variation could be regarded as a kind of Chinese wisdom of mathematics teaching (Sun 2009) and although teaching with variation is gaining more and more attention in mathematics education (Watson and Mason 2006)—there is little empirical data available to confirm the promise of teaching with variation (Cai and Nie 2007). This empirical study aims to contribute to the development and refinement of this practice.

[1] The CHC Learner Paradox: CHC students are perceived as using low-level, rote- based strategies in a classroom environment, which should not be conducive to high achievement, yet CHC students report a preference for high-level, meaning-based learning strategies, and they achieve significantly better in international assessments.

In the following sections, the study will start with the source of the data, followed by the theoretical framework. Then the methods of research will be presented, followed by analyses of the data. Finally, there will be a discussion and conclusion.

The Source of the Data

The data in this study were collected as a result of a broad background of teaching with variation in China. During the past several years, mathematics teaching in China has undergone significant changes. Due to historic and realistic reasons, some typical features in mathematics education in China are characterized by teaching difficult contents, emphasizing the rigor, abstraction, and application of mathematics. However, as a result of the mathematics education reform around the world, teaching contents are gradually changing to adapt to students' interest in mathematics learning and to relate to real life. At the same time, the traditional teaching method emphasizing practice is gradually moving towards the method of combining learning and discovery, teacher-student interaction, group cooperative learning, etc. Thus, teaching with variation has developed with the times.

The data for teaching algebraic equations with variation in grade seven were collected from video tapes of classroom teaching. Researchers selected the data from typical teaching episodes continuous in content covering "equation introduction, equation understanding, equation solving, and equation application" carried out in two classes in different middle schools in China in 2009. The teaching episodes include 7 periods of class, each lasting 45 minutes. The students in those classes are 13 or 14 years old, and their mathematics teachers, Zhang Shang and Zhao Bing,[2] are experienced in their schools. Although the teachers' exposition and whole-class discussion were videotaped and audio recorded, the researchers took notes when the students were working in groups. For an in-depth analysis of how the teachers conducted their teaching and the learning sequences, the researchers interviewed the two teachers and their student representatives after the classroom observations. Although the classes were originally conducted in Chinese, the version offered in this paper is the English translation. To better understand the lessons, the teaching plan for "Teaching Equation Based on Problem Variation" along with reflections after the lesson is attached.

Theoretical Framework

This study is set within the perspective of the theory of teaching with variations developed by Gu (1994). Gu (1999) stated that teaching with variation is an important

[2] All the names are pseudonym.

method through which students can definitely master relevant concepts. And, it illustrates the essential features by using different forms of visual materials and sometimes highlighting the essence of a concept by changing the nonessential features. Its aim is to understanding the essence of an object and to form a scientific concept by ignoring nonessential features. Based on a series of longitudinal mathematics teaching experiments in China, Gu (1994) systematically analyzed and synthesized the concepts of teaching with variation. He identified and illustrated the two forms of variations, namely "conceptual variation" and "procedural variation," referring to understanding concepts from multiple perspectives and then gradually unfolding mathematics.

Teaching with variation was developed in China, but it is strongly supported by several well-known Western theories of learning and teaching. Marton's theory offers an epistemological foundation and conceptual support for the theory of teaching with variation. According to Marton et al. (2004), learning is a process in which learners develop a certain capability or a certain way of seeing or experiencing. In order to see something in a certain way, the learner must discern certain features of the object. Experiencing variation is essential for discernment and is thus significant for learning content. Marton et al. (2004) argue that it is important to attend to what varies and what is invariant in a learning situation. What's more, as put forward by Ausubel (1968), only by establishing a reasonable and substantial connection between learners' new and old knowledge can meaningful learning take place, and such a connection is reasonable and essential between new knowledge and some specific aspects of prior knowledge (e.g. a symbol, a concept or an example in the representations of learners' cognitive structures). For learning algebra with variation, students, on the one hand, need to understand the concept from multiple perspectives (from concrete to abstract, from specific to general) to acquire the nature and connections of concepts by eliminating background interferences, highlighting the essence of mathematic concepts, and clarifying the connotation of the concept of algebra. On the other hand, when students understand the origin and use of mathematic knowledge, gain experiences in concept formation and problem solving, apprehend different components of mathematics, they improve their knowledge structure of mathematics; therefore, they establish proper connections between new and old knowledge of mathematics. As pointed out by Cai and Nie (2007), teaching with variation, by presenting a series of interconnected problems, can help students understand concepts and master the problem-solving method, thereby developing students' knowledge of mathematics. Thus, it is clear that this method of teaching can promote students' meaningful learning of mathematics.

As was pointed out by Gu et al. (2004), teaching with variation helps students develop a profound understanding of a concept from multiple perspectives. They do this by discerning certain features of the denotation or by the denotation itself of the concept, both of which are easily confused (i.e. conceptual variation). For example, the concept of equation can be introduced to students by using visual or specific variations; the essence of the concept can be highlighted by contrasting standard forms with non-standard ones; and misunderstandings of the concept can be corrected by using counter-examples. Moreover, using activities of teaching with variation, students can understand how the concepts are generated and developed, acquire the representations and strategies of problem solving, and then build a hierarchical relationship among different concepts (i.e. procedural variation). For instance, in the "procedural variation" process, the teacher designs the operational model of the concept of equation to promote the formation of the concept of equation, scaffolds stratified problems to form problem-solving strategies of equation, and adopts multiple solutions or variations for one problem or one solution for multiple problems so students acquire particular experiences in problem solving with equations.

It is believed that teaching algebraic equations based on appropriate variations (represented as multi-variant stratified problem space consisting of systematically and deliberately varying problems) can help students understand the concept of equation and facilitate their development in representations and strategies of problem-solving, thereby making the students' learning of algebraic equations meaningful. Thus, it not only helps students grasp the "Two Basics" (basic knowledge and basic skills) of algebraic equation training, but it also enhances their problem-solving abilities. Eventually, students' ability in algebraic reasoning (i.e. the ability of imagining, representing and thinking of connections in algebraic knowledge and problem-solving strategies following certain procedures and structures) is cultivated and developed.

The Method of Research

This study—giving priority to the observation of episodes of equation teaching and supplemented by after-class interview with teachers and students—analyzed the specific operation of teaching algebraic equations with variation and the corresponding learning by students. This was done using qualitative research of the continuous teaching episodes of algebraic equations with variation conducted in the natural environment of middle schools. Using this process, researchers analyzed and summarized the operational rationality of teaching algebraic with variation in enacting students' effective learning.

Analysis of Data

In the following four sub-sections, teaching algebraic equations with variation is demonstrated in detail with typical teaching episodes, in the order of knowledge development of equations, including equation introduction, equation understanding, equation solving, and equation application. The data will be analyzed to account for the successful growth of understanding in students' learning.

The Introduction of the Concept of Equation

There are many studies showing that the concept of equation is challenging for students because of their inability to spontaneously operate with or on the unknown

(Herscovics and Linchevski 1994; Sadovsky and Sessa 2005) and also due to their misunderstanding of the equal sign as an operator, that is, as a signal for "doing something" rather than a relational symbol of equivalence or quantity sameness (Behr et al. 1980). This session will show how the concept of equation is introduced with different variations to eliminate the potential cognitive obstacles by analyzing observed teaching episodes.

Teaching episode: Problem variation to enhance the formation of concept of equation

Teacher: Here is a problem: Xiaoming bought 3 pieces of chocolate with 3 *yuan*,[3] and got change of 6 *jiao*. How much was one piece of chocolate?

Teacher: This problem may be shown as an expression: 3 *yuan* − 3( ) = 6 *jiao*. The number in the parentheses is unknown and it can be replaced by x, namely 3 *yuan* − $3x$ = 6 *jiao*, according to $30 - 3x = 6$.

Teacher: How can we express this problem directly with an equation?

Students are required to look into the problem variations. With the guidance of the teacher, students are aware of the transition from "concrete representation" to "symbolic representation," and summarize the process as the following:

Suppose that each piece of chocolate costs x *jiao*, the following equation is obtained through unifying the units: $30 - 3x = 6$.

Teacher: Please follow the above example and formulate expressions for the following problems:

Problem 1: Xiaohong had 9 *yuan*. She bought 4 notebooks with this money and got 8 *jiao* in change. How much did one notebook cost?

Problem 2: Xiaoli helped her mother buy fruit. She bought oranges with 10 *yuan*, and bought 5 *jin*[4] of grapes. She spent 12.5 *yuan* in total. How much did one *jin* of grapes cost?

Question 3: Xiaoqing bought 4 batteries with 8.5 *yuan* and got change of 0.1 *yuan*. How much did one battery cost?

Question 4: Xiaogang went to buy stamps with Daming. Xiaogang bought 8 and Daming bought 6. Daming spent 6 *yuan* less than Xiaogang. How much did one stamp cost?

Question 5: In 50 basketballs, there are 30 red ones and 20 yellow ones, how many basketballs are there?

Question 6: Xiaolan has 50 apples, 30 oranges, which one is the larger number?

(Classroom observation: April 14, 2009)

From classroom observation, we found that, after practice with the varying problems, students could write out the expressions, showing that they successfully experienced the transition from verbal representation to symbolic representation with regard to simple equations. By using different problem variations, understanding of equal sign which is a sophisticated and essential concept for understanding of algebraic equations—was enhanced. We also observed that, based on these experiences, students were also asked to classify expressions that differentiated essential and nonessential features of equation using problem variation, which is shown in the following example.

[3]One *yuan* in the Chinese monetary system equals ten *jiao*.

[4]One *jin* is equal to 500 grams.

Teaching episode: Problem variation to differentiate essential and nonessential features of equation

Teacher: Look at the following equations and answer this question: How many types can they be divided into?
$30-3x=6, 90-4x=8, 10+5x=12.5, 8.5-4x=0.1, 8x-5x=6, 20+30=50, 50>30$.

While students are discussing the evolving process of the question, the teacher helps those, who have difficulties in understanding, by filling in the blanks. These expressions can be classified into "equations" and "non-equations," with the former further classified into "equations with unknowns" and "equations without unknowns."

Teacher: With regard to the equal expression, $30-3x=6$, $90-4x=8$, $10+5x=12.5$, $8.5-4x=0.1$ and $8x-5x=6$ are defined as equations. An equation is defined as the expression which includes unknowns and the equal sign.

(Classroom observation: April 14, 2009)

According to our observation, the essential features of equation were differentiated by two types of variations. One is based on whether there is an equal sign in the expression. This includes two types of expressions, $30-3x=6, 90-4x=8, 10+5x=12.5$, $8.5-4x=0.1, 8x-5x=6, 20+30=50$, and $50>30$. The other is based on the number of unknowns. This includes $30-3x=6, 90-4x=8$, $10+5x=12.5, 8.5-4x=0.1$, $8x-5x=6$, and $20+30=50, 50>30$. With the variation, students had deeper understanding of the concepts and lay the groundwork for better understanding of equation. The following interview transcript provides more evidence:

Interviewer: Do you students have any ideas about the concept of equation?

Student 1: It is very easy. The key point is to recognize whether there are an equal sign and unknowns.
Student 2: I think so.

Interviewer: Do you like the way that your teacher taught in this lesson?

Student 1: Yeah. I think those different expressions really make sense to me.
Student 2: When I read the definition of equation in the book, I feel it is hard to understand. But when our teacher presents those examples, I have a feeling of "I got it."

(Interview transcript: April 14, 2009)

From the analysis of teaching episodes and interviews, it is clear that the location of variation is rather critical. Through changing nonessential features of things, students understand the essential features of things, and they gradually make use of the cognitive models to reduce their memory burden, rather than "changing for change's sake" to increase their cognitive burden.

The Improvement of Understanding of Equation

Connotation and denotation (extension) of a concept are two opposite yet complementary aspects. If the connotation is clarified, the extension is defined and vice versa. Understanding the concept of equation includes its connotations and denotations. This session of observed lessons will show how the essential nature of the equation is consolidated by designing problem variation, putting emphasis on clarifying the connotation, and differentiating the boundary of the set of objects in the extension.

Teaching episode: Problem variation to solidify essential feature of equation

Teacher: Point out which of the following are equations and which are not.
$2x = 1; 3x + 4 = 7; 4y - 3 = 5; 3x + 4y = 12;\ x^2 - 1 = 0,\ x^2 + y^2 = 1,$
$2x - 3 > 2; 2 + 6 = 8; x = 8; 1 + 8.$

After discussion in groups, representatives of the groups are required to give answers.

During the lesson, we observed that, to solidify essential feature of equation through problem variation above, the teacher helped those students—who have difficulties in understanding—learn to recognize and emphasize two key points for the judgment: equal expression and unknown numbers. The teacher provided students with varying exercises to help them understand equations from multiple perspectives, such as highlighting the essence of equation concept by contrasting non-standard form $x = 8$ and using counter-examples. These activities were continued according to students' classroom performance until they were skilled in judging equations. Thus, by identifying the different types of equation variations, the students clearly recognized the characteristics of the conceptual representation of equations. Based on this, the teacher further used concept variation to help them recognize the essential feature of equation with one unknown.

Teaching episode: Concept variation to discern the essential feature of linear equation with one unknown

Teacher: Fill out the parentheses after identifying all the equations from the above equal expressions:
There are ( ) equations with more than one unknown, ( ) with unknowns of higher than the power of one and ( ) with one unknown of the power of one.

Our observation shows that concept variation was designed to discern the two essential features of linear equation with one unknown; namely, the power of one and one unknown. And the equations with the power of one were first identified, including $2x = 1$, $3x + 4 = 7$, $4y - 3 = 5$, $3x + 4y = 12$, $x = 8$; then, they was further identified according to one unknown, including $2x = 1$, $3x + 4 = 7$, $x = 8$. Using a series of problems in which the essential features of equations were kept unchanged but the nonessential features of equations were changed students easily caught the meaning of linear equation with one unknown.

The teacher helped students learn linear equation with one unknown, step by step. This strategy maximized the advantages of teaching concepts with variations. After the teacher and students summed up the concept of linear equation with one unknown, students did relevant exercises to further understand the features of linear equation with one unknown. It was noted that learning linear equation with one unknown helped students to deepen their understanding of the equation concept as a whole.

Teaching episode: Problem variation to percept transition from arithmetic and algebra

Teacher: After learning the equation concept, how can we find the value of the unknown? Please answer the following question using as many methods as possible.
Xiaoming bought 3 pieces of chocolate with 3 *yuan*, and got change of 6 *jiao*.
Problem: How much was one piece of chocolate?
Method 1: Use arithmetic.
$(30-6)\div 3=8$ (*jiao*)
Method 2: Use equation or algorithm to change the equation from $30-3x=6$ to $x=8$.
$30-3x=6$
Divided by 3 from both sides, then, $10-x=2$
Minus 2 from both sides, then $x=10-2$, so
$x=8$

After discussion in groups, the teacher gave the instructional explanation, "From the answer above, we can see that, in the arithmetic approach, the unknown is represented by a formula including only the given; however, in the algebraic phase, first the unknown is supposed, then the relationship between the given and the unknown is written in a equation, and finally we find value of the unknown in the equation. Now, do some word problems with two methods of arithmetic and algebra, and experience their solving."

After getting the answers using the two methods, the teacher encouraged the students to compare the two methods, i.e., synthesis and analysis, to experience the advantages of analysis in problem solving. With more practice of "multiple solutions for one problem," the students understood the transition from the "arithmetic approach" to the "algebraic approach."

Teaching episode: Problem variation to experience the process of equation solving

Teacher: Since the second method has advantages over the first in problem solving, we will use equation or algorithm to answer the questions listed above in the first session.
Please answer question 1 above by using equation.
$90-4x=8$
Divided by 2 from both sides, then, $45-2x=4$
Minus 45 from both sides, then, $2x=45-4$,
$2x=41$
Divided by 2 from both sides, then, $x=20.5$.

Emphasizing "equality prosperities and algorithm rules" when solving equations helps students understand the idea of transformation thinking first, as they clarify the reasons and purposes for each step. Finally, students master solving equation through doing a number of variation exercises.

Teacher: How can we check whether the solution we got is correct?
Please put the $x=8, x=20.5, x=0.5$ into equations of $30-3x=6, 90-4x=8$, $10+5x=12.5$ and find out whether equations are tenable.

After doing this, students found out: (1) that every number matches one specific equation, (2) and that the steps of equation solving are reasonable. Through practice, the students have the potential to appreciate the idea of the function and corresponding relationship.

> Teacher: It can be seen that $x = 8, x = 20.5, x = 0.5$ match the equations of $30 - 3x = 6$, $90 - 4x = 8$, $10 + 5x = 12.5$ respectively. When a value of a unknown is substituted into an equation with one unknown, the equation is tenable, then, the value of the unknown is called as "a solution of the equation" or "a root of the equation" when the equation has only one unknown number.
> The process to get the solution is called "equation solving."

By looking back on the process of equation solving, students understand the meaning of "solution of equation" and "equation solving" and form an understanding of the process representation.

> Teacher: (1) Which is the solution of $3x = 4 + 2x$, $x = 3$ or $x = 4$?
> (2) Write out a simple equation with the solution of $x = 6$.
> (3) What is the solution of $3x - 8 = 10$?
> (Classroom observation: April 20, 2009)

Problems with variation set by the teacher encourage students' regular thinking and reverse thinking; therefore, these problems help students understand the equation solution and its representations from multiple perspectives (conceptual variation). Through teacher's and students' analysis of the difference between the equation and non-equation, the students get a deeper understanding of the meaning of equation. As a result, cognition of the equation representation tends to be abstract. For a different evolution of equation solution, the students accumulate the dynamics representation of equation solving, and this sets up the potential to lay the groundwork for further learning of solving equation.

It is necessary to help students transfer different representations of the concept of equations: symbols and words. When understanding equation solution, it is important to make clear its meaning statically and dynamically so students can experience its creation through procedural variation. In addition, the meaning of key words and expressions is also taken into consideration when training students for accuracy in algebraic language because this helps develop their algebraic thinking. Setting concept variation helps students understand equations, and it is suitable to apply procedural variation to the understanding of equation solution.

Therefore, to understand the topic broadly, such conceptual variations as linguistic and visual representations (physical objects, diagrams, etc.) should be adopted; to understand the topic in depth, procedural variations from concrete to abstract representations should be adopted to build students' symbolic representations.

This teaching episode makes it absolutely clear that types of variation are very important. The understanding of concepts cannot be done without identifying, generalizing, and abstracting various types of examples related to concepts, nor can it do without the judging and thinking about these examples. The types of variation used in teaching with variation should focus on understanding concepts, and emerge step by step according to students' cognitive level, and thereby ensuring a substantive improvement of students' learning quality.

Equation Solving

This session will show how varying equations from simple to complex are designed to help students grasp the unchanging "routines" and "methods" when solving equations through procedural variation.

Teaching episode: Procedural variation to experience the generalization of equation solving

Teacher: Do such simple equations with one unknown so different in formats as $5x-2=8$, $\frac{1}{4}x=-\frac{1}{2}x+3$, $4(x+0.5)+x=17$, $\frac{1}{7}(x+14)=\frac{1}{4}(x+20)$, $\frac{1}{5}(x+15)=\frac{1}{2}-\frac{1}{3}(x-7)$ share a common solving process?

Teacher: Taking the equations listed above as examples to interpret the idea of equation solving: "simplification".

$$㊀ 5x-2=8 \xrightarrow[\text{(property of equality 1)}]{\text{transpose}} 5x=8+2 \xrightarrow[\text{(property of equality 2)}]{\text{convert coefficient of } x \text{ to 1}} x=2$$

$$㊁ \frac{1}{4}x=-\frac{1}{2}x+3 \xrightarrow[\text{(property of equality 1)}]{\text{transpose}} \frac{1}{4}x+\frac{1}{2}x=3 \xrightarrow{\text{combine of similar terms}} \frac{3}{4}x=3 \xrightarrow[\text{(property of equality 2)}]{\text{change coefficient of } x \text{ to 1}} x=4$$

$$㊂ 4(x+0.5)+x=17 \xrightarrow{\text{remove brackets}} 4x+2+x=17 \xrightarrow{\text{transpose}} 4x+x=17-2 \xrightarrow{\text{combine similar terms}}$$

$$5x=15 \xrightarrow{\text{change coefficient of } x \text{ to 1}} x=3$$

$$㊃ \frac{1}{7}(x+14)=\frac{1}{4}(x+20) \xrightarrow[\text{(property of equality 2)}]{\text{remove denominator}} 4(x+14)=7(x+20) \xrightarrow{\text{remove brackets}}$$

$$4x+56=7x+140 \xrightarrow{\text{transpose and combine similar terms}} -3x=84 \xrightarrow{\text{change coefficient of } x \text{ to 1}} x=-28$$

(Note: that brackets in this exercise ㊃ can be removed in the first step.)
(Classroom observation: May 13, 2009)

The purpose for students' progressive and hierarchical practice of variation is to help them to be skilled in equation solving.

Note here that as to property of equality 1, if $a=b$, then $a+c=b+c$; if $a=b$, then $a-c=b-c$. As to property of equality 2, if $a=b$, then $ac=bc$; if $a=b$ and $c\neq 0$, then $a/c=b/c$. Through the training in different levels, students' understanding of the ideas and methods of equation solving is enhanced, and they develop richer and more abstract representations.

Through designing these equations from simple to complex, students gradually improve their skills in equation solving and experience the ideas of transformation and equivalence.

Teaching episode: Procedural variation to grasp the fluency of algorithm of equation solving

Solve the equation: $\frac{1}{5}(x+15)=\frac{1}{2}-\frac{1}{3}(x-7)$

Complex equation solutions:remove denominator, so

$6(x+15)=15-10(x-7)$

remove brackets, so

↓ change $6x+90=15-10x+70$

transpose, so

$6x+10x=15+70-90$

simple equation, combine similar terms, so

$16x=-5$

divided by 16 from both sides, so

$x=-\frac{5}{16}$

(Note that brackets in this exercise can be removed in the first step.)

Building on the fluency of algorithm, students are expected to understand why these algorithms work and what mathematical thinking methods are embedded in them. The experience of simple equation solving lays the groundwork for solving complex equations.

Teacher: (1) Narrate the process of solving simple equations with one unknown.
(2) How do you check the unknown you have got is the solution to the equation? For example, is $x=2$ the solution to $3x+4=5x+7$ or $3x+4=5x$?

This encourages students to synthesize what they have learned, to understand the process of equation solving, and to form effective methods of learning. Strengthening the verbal representation of equation improves students' mutual exchanges and develops their mathematics thinking.

Teacher: We can now sum up the process of equation solving as follows: (1) remove the denominator, (2) remove the bracket, (3) transpose terms, (4) combine the similar terms, (5) change an equation into its simplest form with coefficient of x being 1. Only by practice do we master the ideas of equation solving.

Next, the teacher gives students similar assignments and asks them to solve the equations and to check the results independently or cooperatively so that they master the equation solving process. Appropriate exercises help students consolidate the knowledge of equation. (Classroom observation: May 14, 2009)

Students gradually learn to master the equation solving process through exploring varying problems. The variant training under the guidance of the equation solving helps students grasp the unchanging "routines" and "methods" for solving equations (i.e. one solution for multiple problems), obtains the dynamic and abstract representations of the equation solving, and trains skills for the equation application.

To strengthen familiarity in equation solving, it can be deduced that students first learn the general steps by solving various types of equations and then understand the reasons and basics for equation solving. In addition, they are able to narrate the steps of equation solving and then do sufficient variation practices until they can do it automatically. To grasp the procedural knowledge, students not only practice but also understand the ideas of algorithm, such as applying the equivalent equation principle, understanding changing an equation from complex to simple, and so on. It is known that students need to have a considerable number of guided variation practices and show high proficiency in various types of equation solving in order to consolidate the knowledge of equation solving.

This teaching episode makes it very clear that levels of variation are important. The development of mathematical thinking is inseparable from formalizing complex materials and the model generalization of factual materials. The levels of variation of teaching algebra with variations should be set within students' "recent development zone," in consideration of students' levels, to motivate them make progress with challenges, and to ensure that they can "jump for peaches" to develop rapidly and orderly.

The Application of Equations

Constructing equations from word problems—as well as interpreting, rewriting, and simplifying algebraic expressions—are named as key difficulties in the learning of algebra (Herscovics and Linchevski 1994; Linchevski and Herscovics 1996; Sfard 1995). This session, it will show how to use problem variations, including one problem multiple solutions and one problem multiple changes to help construct equations from word problems.

Teaching episode: Problem variation (one problem multiple solutions) to learn different presentations and strategies of constructing equations

Teacher: Here is a problem: "*Can Xiaoming be caught up with?"* (Ma 2002)
Xiaoming has to arrive at school which is 1000 m away from his home before 7:50 every morning. One day, he went to school at the speed of 80 m/min. Five minutes later, his father found he had left his Chinese textbook home, so he chased Xiaoming at the speed of 180 m/min, and caught up with him on the way to school.

(a) How much time did Xiaoming's father spend to catch up with Xiaoming?
(b) What is the distance to the school from the place where Xiaoming was caught up with?

Teacher: (1) What is the relationship among time, speed, and distance? The journey problems include "meeting problems" and "*what* problems." Please illustrate their individual "equivalent quantities in the process of journey" in these two types of journey problems.
(2) On the solution of journey problems, we generally analyze the quantitative relations between different unknowns such as time, speed, and distance as well as the relation of the same parameter with the help of line segments, tables, and so on, and then set the equations according to the quantitative relations of the same unknowns. Is this correct? Are there any other ideas?

Our observations show that, the teacher helped students form correct procedures of “problem solving with equation” until they could use them easily. The teacher helped the students recall background knowledge, apply learned representations, and design problem-solving strategies to lay the foundation for analyzing and solving questions, and reduce their difficulties in learning.

Teacher: We will try to analyze the quantitative relations in this problem with the following methods. It is necessary to understand the problems clearly and analyze the quantitative relationship appropriately in order to solve any kind of problem. As for this problem, we can make use of line segments and tables to represent the thinking process.

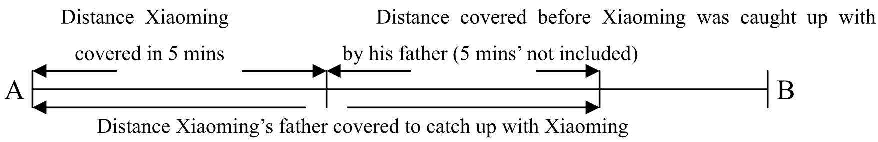

Fig. 1 The relations between different distances

Segment AB is the distance from Xiaoming’s home to school, i.e., 1000 meters.
The quantitative relations (of the same unknown) in this problem:
(1) Time: ____________________ (2) Distance: ____________________
Supposing the father catches up with Xiaoming after x minutes.

Table 1 The quantitative relation by using time x

		Speed	Time	Distance	Quantitative Relation	Equation
Xiaoming’s journey	Section 1	80	5	80×5	Distance Xiaoming covered = Distance his father covered	$80 \times 5 + 80x = 180x$
	Section 2	80	x	$80 \times x$		
His father’s journey		180	x	?		

Supposing Xiaoming is y meters away from school when caught up with by his father.

Table 2 The quantitative relation by using distance y

		Speed	Time	Distance	Quantitative Relation	Equation
Xiaoming’s Journey	Section 1	80	5	80×5	Time Xiaoming used in Section 2 = Time Father used	$\frac{1000-y}{80} - 5 = \frac{1000-y}{180}$
	Section 2	80	$\frac{1000-y}{80} - 5$	?		
His father’s journey		180	$\frac{1000-y}{180}$	$1000 - y$		

From Tables 1 and 2, we know that, supposing time is unknown, the distance can be found as the equal quantity and vice versa. Please finish Table 3.

Table 3 The quantitative relation by using distance x

		Speed	Time	Distance	Quantitative Relation	Equation
Xiaoming's Journey	Section 1	80	5	80×5	?	?
	Section 2	80	?	?		
His father's journey		180	?	x		

(Classroom observation: May 28, 2009)

The analysis of the procedural variation has allowed students to experience the exploration process of "problem solving with equation." In the process of modeling equations, the representations help students gradually deepen their understanding of questions: the representation of language is helpful for understanding the scenario; the pictorial representation is helpful for visualizing the relationship between different quantities; and the tabular representation is helpful for establishing the quantitative relations. Such a comprehensive consideration and construction of quantitative relationships help students completely grasp procedural operations of "problem solving with equation"; teaching with variation through "exemplary problems," applying all kinds of representations, help students learn how find out effective strategies for "quantitative relations" and to form various solving methods. The "multiple solutions for one problem" helps the students understand the meaning of the problem and the relationship among quantities from different angles, which, in fact, enriches the multi-representations of equation problem-solving and develops students' algebraic thinking.

Appropriately expanding teaching topics helps students understand the quantitative relations deeply, think about the steps and strategies of problem solving, learn flexible representations of the problem space, experience operational representations of the problem-solving procedure, and master the structural modeling representations. It helps the development of students' algebraic thinking.

What's more, to grasp the methods of solving application problems by solving equations, experiencing the thoughts of algebraic modeling structure, and cultivating students' ability of divergent thinking, the problems are changed; changing either the conditions or the intended results. Students are encouraged either to ask questions and then answer questions after removing the intended results or to create different types of practical problems just as journey questions and efficiency questions based on an equation. Students also are asked to try mathematics experiments to verify whether the theoretical analysis conforms to the practice. All the ideas mentioned above help students to experience the invariant features within variant process, namely algebraic structure model, and thus forms the students' algebraic thinking mode.

Teaching episode: Problem variation (one problem multiple changes) to cultivate students' explorative ability during constructing equations

[Variation 1]
Teacher: Now we can have a variation, please give your solution:

Xiaoming must arrive at school which is 1000 m away from home before 7:50 every morning. One day, he went to school at the speed of 80 m/min. 5 minutes later, his father found he had left his Chinese textbook home, so he chased Xiaoming at a certain constant speed, and it took him 4 minutes to catch up with Xiaoming.
(1) What was the speed of Xiaoming's father while chasing Xiaoming?
(2) What was the distance to school when Xiaoming was caught up with?

According to our observation, based on the learning in the previous problems, students solve the variation in a reversed way and understand the methods and ideas of "problem solving with equation" from multiple perspectives. They especially learn from checking the previous answer through the variation.

[Variation 2]
Teacher: Now we can have another variation, please give your solution:
Xiaoming must arrive at school from home before 7:50 every morning. One day, he went to school at the speed of 80 m/min. 5 minutes later, his father found he had left his Chinese textbook home, so he chased Xiaoming at the speed of 180 m/min, and caught up with him at the place that is 280 m from the school.
Question: What is the distance from Xiaoming's home to the school?

[Variation 3]
Teacher: Now you can make some variations of the example by yourselves or discuss within your group and find out to what variations there are one solution and to what variations there is no solution.
The original problem is as follows: Xiaoming has to get to the school which is 1000 m away from his home before 7:50 every morning. One day, he went to school at the speed of 80 m/mins. 5 minutes later, his father found that Xiaoming had left his Chinese book home, so he chased Xiaoming at the speed of 180 m/mins and caught up with Xiaoming on the way to school.
(Classroom observation: May 29, 2009)

Students pose other variations based on the problem with "whether there are solutions to the variations of the problem" in mind, to analyze, to solve, and to check the variations. The cyclic thinking of posing variations of the problem, solving the variations, and checking the solutions to the variations, along with extensive summary and discussion, helps students understand the essence of the thinking about "problem solving with equation" and acquire the abstract representation of "problem solving with equation" and develop their mathematical thinking ability.

Through the problem structural variants training (i.e. multiple variations for one problem) to the multi-representations of problem solving with equation as a means and purpose, the students appreciate the constant thinking mode about problem solving with equation. Based on the premise of a variety of the newly learned equation skills, the students' high-level algebraic thinking is being developed naturally.

Students' problem-representing abilities and problem-solving strategies can be trained by comprehensively using the ideas of equation and the skills of equation solving to solve conventional or non-conventional problems and by using modeling to solve contextual problems. They can experience the interaction between constructing and modeling algebraic concepts so as to improve their cognitive understanding of algebraic equations.

In teaching with variation to construct equations from word problems discussed above, several steps are followed in teaching students how to solve contextual problems by equation solving. First, students are provided with background knowledge of the problems, including common sense and relationship knowledge (formulas). Second, students are taught how to analyze the relationship between variables, i.e., to illustrate the relationship between variables through language, diagrams, tables and symbols. Third, students are required to write down an equation with the unknowns and the givens. They learn to analyze problems under guidance, to put the unknowns and the givens together into the equation, and to formulate the equation step by step. They are trained to develop an awareness of an equation for the purpose of modeling. It is known that teaching and learning of "solving application questions by equation method" needs to develop basic skills through practice with conceptual variations and to develop algebraic thinking ability through practice with procedural variations.

Thus, when applying this newly learned knowledge, students can use their mental representations of the learned contents, i.e., the algebraic relation structure constructed, to specific problems. When solving problems, by using various representations from visual representations to abstract ones, students create appropriate representations and strategies of solving the problems, and finally they express their solutions to the problems by using symbolic representations.

This teaching episode makes it noticeably clear that the rational development of teaching with variations can help set questions for students to inquire cooperatively, and this has significant positive effects. Understanding algebraic structures and their application in the question variants or the construction of question variants can help students understand the algebraic way of thinking on the whole and train their mathematics thinking ability.

Discussion and Conclusion

As the analysis in this study shows variation, as a means, can be a powerful way to help students develop mathematical thinking no matter how the content and the problem are "changed" by the teacher. Furthermore, it also shows that, to better conduct teaching algebra with variation, the lessons should be well structured and the variation should be carefully chosen. Some observations are made based on the analysis in the previous session, which will be discussed in the following sections.

Process of Teaching Algebra with Variation

Students can be helped when learning algebra, if teachers appropriately implement teaching with variation by adopting different variation types and levels of representations according to algebraic learning goals, build on students' existing knowledge and ability, and use different teaching contents at different teaching phases.

It is believed that algebraic concepts are characterized by concrete operation, abstract structure, and wide application and vice versa, thus, learning algebraic concepts should focus on operational, structural, and applicable aspects. Therefore, in the teaching of algebraic equations with variation, these steps should be followed. First, the teaching is designed to promote students' concept formation and assimilation through variation, to get the initial formation of concept of equation, then to help students understand the abstract nature of the equation concept, and to connect the equation concept into the algebraic concept system, to clarify its position and function in this system. Second, students are asked to apply the equation concept to solve systemic varying problems, to make the equation concept an operational object, to experience the structural function, and to enrich the equation concept thoughts through the equation concept operation and the equation concept formation reflection. Therefore, when teaching equation concept with variation, it is necessary to construct the structure of equation concepts and develop the application ability of system of equation concept so as to develop students' algebraic thinking ability.

Operation of Teaching Algebra with Variation

It can be seen from the above analyses of classroom teaching episodes and interviews that the scientific and reasonable operation of teaching algebra with variations lies in the proper grasp of the aspects of "orientation of variation, types of variation, levels of variation, and variation exploration."

Variation Orientation

In conducting teaching with variations, the teacher is required to recognize at what level of thinking the students are and to know what representation features the students have. This is the starting point for teaching. Second, the teacher is required to analyze the "unchangeable" essential contents from what he or she will teach, namely the principles of concept, the ways of thinking, and so on. Third, the teacher is required to understand "unchangeable" ways of thinking of students, namely the set mapping, variable relationship, program analysis and so on, in learning the "unchangeable" essential contents. Finally, the teacher and the students are both required to use a reasonable setting of "changing of the unchangeable" in the shortest possible time, to motivate the development of the students from "concrete representations" to "abstract representations" to efficiently achieve their learning goals, that is, the grasp and reflection of the "unchangeable" essential contents, any negligence of which will lead into a condition of "changing only for change."

Types of Variation

To facilitate the introduction and understanding of the algebraic concepts, it is necessary to provide students with various examples of different types (homogeneous or heterogeneous) and verbal explanations, even with graphics and symbols to increase students' algebraic representations. The number of each type is subject to the change of teaching objects, contents, and environment. These examples are set to help students with their gradual classification, induction, abstraction, deep understanding, and application of variations. For example, the introduction to the concept of equation needs some examples with slight differences in form, making it easy for students to identify, induce, and summarize, rather than consuming too much of the students' energy in studying the examples. Indeed, the teacher should be flexible in deciding the quantity and difficulty of the variation examples, depending on the students' actual level of understanding so as to encourage them to achieve relatively abstract algebraic representations in the near future.

Levels of Variation

In order for students to form a three-dimensional concept network with a relatively high level of abstraction and to foster a flexible application capacity, it is necessary to set question strings of variation at different levels to enable them to gradually experience the invariable modes of thinking in the development of algebraic knowledge, that is, the formation of the abstract internal representations. Here the levels should be set with gradient, inspiration, challenge, and control, that is, the arrangement of questions is a matter of degree around "invariable mode of thinking." For example, in the teaching of equation solving with variations, the role of the unchangeable "idea of equivalence" is shown in the changing of equations through the gradual increase of equation complexity. For example, in equation application teaching, question strings should be set intentionally in order to train the students in equation lining and to motivate students to see the intrinsic link between different types of knowledge, to experience the methods of equation lining, and to be skilled in representation strategies of equation lining. The setting of questions of variation should follow as far as possible the principle of "no matter how the conditions and conclusions change, the searching for the equivalents will never change," the main purpose of which is to help students learn equation lining. Surely, in order to cultivate students' convergent thinking and divergent thinking, it is necessary to raise their thinking on "how to change to solve the problem." It will be more worthwhile when students are trained into advanced and critical algebraic thinking.

Variation Exploration

The variant training of the traditional algebra "two basics" can be transformed into inquiry-based learning under guidance, which requires teachers to specifically research the approach of how to present algebra knowledge in an evolving way and how to represent the problems related to them. When students have a certain basis of the "two basics," teachers could try to "melt the exploration in the variant." For example, under the guidance of teachers, students may explore the direction of knowledge transferring, the knowledge points after question transformation, the method after information transformed, the problem after the replacement of the original conditions or intended results, the different kinds of quantitative relationships in a problem, and the mutual links between the different problems at different levels, and so on. Therefore, when the types and levels of variant are designed, the problems should be instructive, thinking should be targeted, span should have continuity, content should be a spiral, etc. All of these are conducive to student learning in inquiry-based learning. For example, the above equations to learn, through the variant inquiry, with multi-angle and multi-level, displays the learning process of the introduction and understanding, consolidating and applying of the equation, gets the objective of "student learn to think" as the key point. Using both the breadth and depth of knowledge, the teacher and students explore the knowledge of the equation and the related problem-solving and representations. Such a direct guiding of inquiry-based learning should proceed in the variant and focus on the algebraic representations, which would promote students to develop the mathematical thinking and to stimulate student enthusiasm. The benefit for this mathematics learning at this age is obvious.

This teaching method, based on variation of the teacher's external knowledge and questions to achieve students' internal multi-representations, can integrate the teacher's external guidance and the students' internal subjectivity so that teaching and learning go in harmony.

Final Comments

In China, teachers usually design some conceptual variations or procedural variations while teaching algebraic equations. On the one hand, these variations can inspire students' motivations to learn algebraic equations, understand algebraic equations from multiple perspectives and achieve the fluency of solving algebraic equations. On the other hand, students can obtain the experiences in algebraic equations activity, master the thinking methods of algebraic equations, and enhance their problem solving abilities. Therefore, through appropriate teaching with variation, it is ensured that the training for the two basics in mathematics education in China is not drill training; on the contrary, teaching with variation can promote the development of students' algebraic thinking (Zhang and Song 2004)

As it is shown in this study, there is great potential for teaching with variation; however, there are some unanswered questions. For example, as suggested by a recent study (Mok et al. 2008), if the type of engagement the teacher created in the lesson using problem variations leads to missed opportunities for fostering students' higher-order thinking skills, how do teachers balance the development of basic skills and higher-order thinking skills when conducting teaching with variation? More empirical studies are expected to confirm and support the promise of teaching with variation.

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Appendix: English Translation of a Teaching Design and Its Reflection after the Lesson: Teaching Equation Based on Problem Variation

Analysis of the Teaching Material "Can Xiaoming Be Caught Up With" is selected from the Mathematics Textbook for Grade Seven published by Beijing Normal University in 2002.

Materials with a practical example of "Can Xiaoming Be Caught Up With" building a problem situation should inspire students to explore problem-solving strategies, and to experience the process of "translating the practical problems into the mathematical problems" and the mutual transformation between the verbal representation, the symbolic representation, and the graphical representation. This teaching course is designed from the scenarios in which the students should be interested and give the students access to information through the representations, such as drawing line segments and diagrams. Using the problem structure variant (that is, multiple variations for one problem, by replacing the conditions and conclusions of the original problem, forming its tributaries), the students look for different equality relations from different angles, thus, students should initially find "mathematics modeling" as a means, and better develop students mathematical representing and thinking ability.

Key points: to enable students to find the known and the unknown quantities and relationship between them.

Difficult points: to analyze the quantitative relations between complex problems using line segments and other methods.

Analysis of the Students Students have learned to solve simple application questions using line segments in the elementary school. In previous lessons, they have learned many kinds of application questions using a linear equation with one unknown. However, exercise results show that they do not understand the essence of a linear equation with one unknown to solve application questions. Application questions are some of the most difficult questions, and students may become tired of them. This is not because of lack of interesting scenarios for practical questions, but because of the fact that students are not led into a deep, orderly and multi-angled thinking of "the discovery of the quantitative relationship." Creating a scenario for the question can help students think originally, but if too much emphasis is put on the scenario and the problems, then the theme of mathematics teaching activities (that is, to help students learn to think in mathematics) will be weakened. The arrangements of teaching, as a result, should foreground the target of "learning to think" in solving application questions by linear equation with one unknown.

Principles for Designing

1. Proceeding from the practical scenario of the question, students will gradually enter into the problem-solving process in a relaxed environment. Simultaneously, variation questions and activities should be designed according to students' current level of competence. From easy to difficult, from simple to complicated,

students can understand different approaches to the questions and their pros and cons, and know the importance of problem-solving strategies. Students study in the free environment, each adopting different ways of thinking in the class. Thus, their ability to think is cultivated in this process.

2. After acquiring the approaches of solving application questions by linear equations with one unknown in classroom activities, students begin to ask questions in the discussion section and try to answer them under the guidance of the teacher. This process can help students develop their thinking, find a chance to innovate and perform freely, and fully enjoy the sense of success after exploration.

Teaching Objectives

1. Through learning to solve application questions by linear equations, students can perceive the role of mathematics in daily life.
2. Through using line segments and other methods to search the quantitative relationships among complicated questions, students can enhance their algebraic thinking ability.
3. Through analyzing and studying different equal relationships, students can experience the diversities of problem-solving strategies and develop their innovative ability.
4. Students will reflect themselves and learn from others during cooperation and communication.

Teaching Process

1. Creating the scenario of questions

   Teacher: Class, have you ever seen someone "being caught up with"? Who can give an example?

   Students may give many examples around them, about human beings or objects.

   Teacher: In the case of chasing, is the faster chasing the slower? Or vice versa?

   Students will know it is the faster that chases the slower, a simple reasoning in their life.

   Teacher: Who have ever seen a cat or a dog chasing a rat?

   Students may laugh and say, "A dog tries to catch rats. That is to poke one's nose into other's business." The class atmosphere is thus activated.

   Teacher: We do not care about "dog chasing rats." Today, we will discuss Xiaoming's father chasing Xiaoming (Show the story).

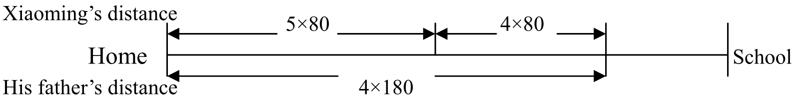
   Xiaoming must arrive at school at 7:50 every morning. One day, he went to school at the speed of 80 m/min. 5 minutes later, his father found he had left his Chinese textbook home, so he chased Xiaoming at the speed of 180 m/min, and caught up with him on the way to school.

   Please answer the questions.

2. Organizing students' activities

   Question 1: If Xiaoming's father caught up with Xiaoming within 4 minutes, how many meters had Xiaoming and his father covered respectively?

With the formula “distance = time × speed” learned in the primary school, students can give the answer that they both covered 720 meters. The teacher can guide students to draw a line segment and fill out the following table:



		V (min/m)	T (min)	D (m)
Xiaoming	Section 1	80	5	400
	Section 2	80	4	320
His father		180	4	720

Teacher: Let’s look at the variables in the figure and the table, and find how many of Xiaoming’s variables (speed, time and distant) correspond with his father’s.

After observation, students may find out: (1) the distance Xiaoming covered = the distance his father covered (when Xiaoming was caught up with); (2) the time Xiaoming spent in his second section = the time his father spent.

Teacher: what should the above equations be when Xiaoming’s father walked 3 minutes?

After discussion in groups, representatives from the groups are asked to discuss the following: When the father walked 3 minutes, he had covered 540 meters, and Xiaoming covered 640 meters. Variables in equation (1) are not equal, which means Xiaoming’s father didn’t catch up with him, but equation (2) is still tenable. The teacher then attaches a condition “When Xiaoming was caught up with” to equation (1).

Teacher: Why equation (1) is tenable when the father walked 4 minutes? How can it be checked with the equation lining methods we have learnt? So we have question 2.

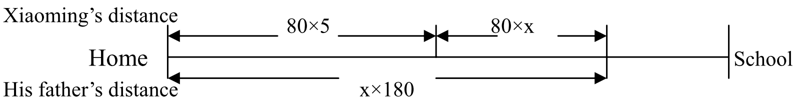
Question 2: If the father caught up with Xiaoming on the way, how much time did he spend?

Students may discuss in groups and answer the question with the help of the figure and the table in question (1).

(Analysis: The original question was changed in the textbook so the students could answer with relevant knowledge acquired in primary school. In this way, the students can review the “distance = time × speed” equation and understand the relations among all variables. This specialization helps students understand the essence of problem-solving and enables them to experience the process of observation, supposition, and verification, which can inspire their interest in question exploration. This benefits the students’ study in mathematics to a great extent and opens a large space for the students and teachers to innovate. The configuration of thinking based on the obtained knowledge and experiences is the thinking process of "mathematicization.")

Following the example of question 1, the students have little difficulty in finding out the unknown variables with figures and tables, hypothesizing the unknown, and lining the equation according to the equivalent relation in equation (1):

Supposing the father catches up with Xiaoming with X minutes.



		V (min/m)	T (min)	D (m)
Xiaoming	Section 1	80	5	400
	Section 2	80	x	$80x$
Father		180	x	$180x$

$$400 + 80x = 180x$$

$$x = 4$$

Teacher: If we have verified that the father caught up with Xiaoming in 4 minutes, then how many meters had the father covered when he caught up with Xiaoming?

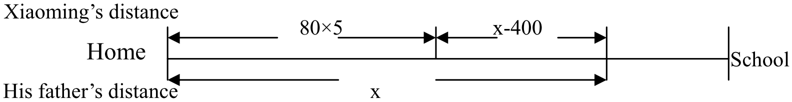
The teacher asks a lagged-behind student to answer the question, and the answer is: $4 \times 180 = 720$ meters.

Teacher: Can we hypothesize the father's distance as x and directly build the equation? Let's check whether it is 720 meters. We will take this as question 3.

Question 3: If the father caught up with Xiaoming on the way, then how many meters had the father covered?

Students are required to discuss in groups, and the teacher helps those lagged-behind students analyze the problem with the help of figures and tables. The teacher helps them to fill out the table, find out the equivalent relation of equation (2) and build the equation.

Supposing the father had walked X meters when he caught up with Xiaoming.



		V (min/m)	T (min)	D (m)
Xiao	Section 1	80	5	400
ming	Section 2	80	?	$x-400$
Father		180	?	x

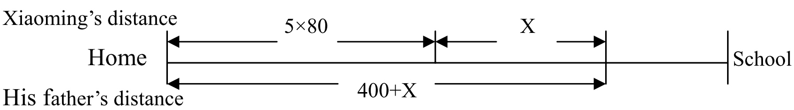
$$\frac{X-400}{80}=\frac{X}{180}$$
$$X=720$$

Teacher: We can immediately calculate Xiaoming's second section distance $=720-400=320$ m or $80\times 4=320$ m. Then, can we hypothesize his second section distance as X m and build the equation? This is question 4.

Question 4: If the father caught up with Xiaoming, what is Xiaoming's second section distance to school?

Students are asked to discuss in groups, and the teacher guides lagged-behind students and asks representatives in each group to answer the questions.

Supposing Xiaoming covered X meters in his second section when his father caught up with him.



		V (min/m)	T (min)	D (m)
Xiaoming	Section 1	80	5	400
	Section 2	80	?	x
His father		180	?	$400+x$

$$\frac{X}{80}=\frac{400+X}{180}$$

Teacher: If you are told that the distance from Xiaoming's home to the school is 1000 meters, it will be very easy to figure out $1000-(400+320)=280$ m. But can we answer it directly by building equation? This is question 5.

Question 5: If the distance from Xiaoming's home to the school is 1000 meters, what is the distance from the place where the father caught up with Xiaoming to the school?

Students build the equations alone and work out the answer. It is 280 meters.

The teacher asks the lagged-behind students to write the answer on the blackboard and emphasizes the writing forms.

(Analysis: The problem string is helpful for the students to carry out a series of conceptual activities such as observation, experimentation, and verification; to understand the essence of the application question to be solved by equation; and to experience the superiority of such representations as line segments and charts. This not only stimulates the students' desire to explore, but also expands the students' thinking span to a reasonable extent. Therefore, students are thinking of mathematics in an exciting class atmosphere, embodying the fundamental and developmental idea of *Mathematics Curriculum Standards* issued by Ministry of Education of China in 2001.)

3. Broadening thinking in the lesson

Question 6: Supposing Xiaoming's father caught up with him right at the school gate. How much time did he take? What was his speed? In what circumstances couldn't he catch up with Xiaoming?

(Question 6 echoes question 1, but it is more flexible. Here, questions and students' interests reach a climax. Students feel questions emerge endlessly, and they can experience the mysteries of mathematics. They further understand the close relationship between mathematics and life.)

They quickly answer the first two questions. As for the last one, the teacher can ask representatives of different groups to demonstrate their research results, giving them appropriate guidance, offering appropriate recognition and praise. Finally the teacher makes a summarized statement.

Question 7: See the "Discussion section" in the textbook.

Students of grade seven from Yuhong School traveled to the town's outskirts. Students of class one formed the first team and walked at the speed of 4 km/h, and those of class two formed the second team and walked at the speed of 6 km/h. The second team set off one hour after the first team had started and at the same time sent a liaison riding a bicycle at the speed of 12 km/h back and forth between the two teams.

Please pose questions based on the above facts and try to answer them.

The teacher can ask the students two or three questions according to their performance and the time left in the class. As for those lagged-behind students, the teacher can help them with some tips to imitate the previously answered questions and tell them how to analyze the hidden meaning of the questions. Finally, the teacher can ask the representatives to write their answers on the blackboard and may also ask them to finish the rest questions after class.

Finally, the students are encouraged to change the conditions or the intended results of the above question, and to solve it.

(Analysis: For the discussion of complex questions, the teacher should take students' current knowledge level into consideration but should not give too difficult or too many questions, except for some interesting and challenging ones. It is always the teacher's responsibility to ensure students to constantly experience success in their mathematics study according to the idea of "public mathematics.")

Reflection after the Lesson Comparatively speaking, for the first-year students in junior high schools, this lesson contains relatively too many "thinking contents." When designing the teaching process, the teacher should set out from the two dimensions of the depth and breadth, and take in full consideration the organization style of this lesson and the students' current level of competence. The teacher should set question variations in connection with practice, which is good for students' thinking development and is in accordance with the goals and requirements of the *Mathematics Curriculum Standards.* Teachers can disassemble the thinking contents in procedural variation so that students can march upwards gradually along the steps of thinking, experiencing the inner beauty and application value of mathematics. Efforts should also be made to encourage students to make essential preparations for thinking prior to students' cooperating, exchanging, and discussing. The issues for discussion should be based on the purpose of understanding the main objectives of the lesson proper. Otherwise, purposeless discussions will result in wasting of students' time, and they make to progress. This lesson aims to start out with the basis of students' knowledge, focus on their future development, with emphasis on practice on students' mathematical thinking and on the training strategies for problem solving. It shows the effective use of a variety of methods to understand the meaning of the questions and experience the fun and significance of mathematics learning. This lesson broadens students' thinking and well reflects the connotation of all-around education.

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Commentary on Part III

John Mason

Introduction

Most research in mathematics education reports on what is or appears to be the case currently: obstacles, weaknesses and failings, for and by learners, teachers, and curricula. Some intervention studies look locally at what is the case under exceptional circumstances and hence indicate what perhaps could be the case generally, but inevitably they build on a legacy of socio-cultural-historical practices and experiences inherent in the specific situation. Following the lead presented by Maslow (1971), whose hierarchy of human needs provides a context, I find myself attracted most by what could be the case globally. What could the learning of algebra be like if teachers understood profoundly and appreciated deeply both school algebra and its pedagogy and didactics,[1] and if teaching were carried out consistently over a sustained school experience of several years, following an enlightened curriculum? What sort of progress have we made in this direction in the last 25 years, and where might we be headed?

Stimulated by the chapters in this part, I make use of a framework developed by Bennett (1966, 1970) which provides a structure for discussing aspects of the activity of teaching a mathematical topic, or in the case of algebra, of a way of thinking, being and acting mathematically; what school algebra is and could be; and what is and could be researched. I use the framework three times, to structure discussion of the current state, the possible state, and the activity of going from where we are now to where most of would like to be.

[1] I use *pedagogy* to refer to acts of teaching that apply to many different topics, and *didactics* to refer to acts of teaching specific to a particular topic.

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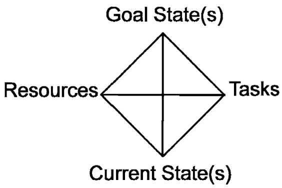
Systematics: Structure of Activity

In order to identify what some activity might achieve, it is of course vital to know where one is currently. As Bennett (1966, 1970) pointed out, activity is usefully conceived as a four-termed system.[2] In addition to the current and goal states, there are both tasks or actions to undertake and resources upon which to call. For activity to succeed there has to be a balance between all four terms: resources need to be adequate for the tasks required, the tasks required need to be suitable for achieving or approaching the goal from the given state, the tasks and resources need to be accessible to the individuals or group concerned. In the language of Gibson (see Greeno 1994) the affordances and constraints of both resources and tasks undertaken, and the attunements of the current state all need to be compatible with and appropriate to the desired goal(s). This structure for activity applies to learners working on tasks, as well as to teachers teaching mathematics and to researcher-educators trying to influence policy, curriculum, pedagogy and didactics. Since I propose to use this structure three times, it is worth elaborating a little.



Current state (what is the current situation): Descriptions of what is currently the case not only establish a baseline, but reveal what the observer is sensitised or attuned to notice and so what is valued, ignored or eschewed in relation to the goal state(s).

Goal State (the aims, purposes, intentions, desires): As is well known from the Vygotskian-Davydovian 'Activity Theory' perspective, goals may or may not be explicit, or even appreciated, even by those engaged in the activity. What observers interpret as the assumed goal(s) may not always be compatible with the lived experience of those observed.

Motivation (axis between current state and goal state(s)): The relation between current and goal states provides direction and motive in relation both to cognition and affect.

Tasks or Actions: Tasks are what initiate or subdivide the sequence of acts or actions that themselves constitute the flow and development of activity. They provoke actions carried out which constitute the observable activity. Tasks may be self-set, imposed or made available.

Resources: Resources are what can be called upon in carrying out tasks, such as your own powers, dispositions and experience, the presence of colleagues, and established and institutionalised ways of working.

[2]Current Vygotskian-based activity theory is based on a triangle of three terms, elaborated into 6 or more. In this section I make covert use of Bennett's five-fold structure of potential, and later, the six-fold structure of the present moment.

Operational Means (axis between resources and tasks): Tasks are merely tasks; resources have to be called upon in order to be useful. The combination of tasks and resources form an operational axis, but just as with the vertical axis, it is necessary that the resources be adequate for the tasks and that the tasks make effective use of resources.

What Is Algebra?

Algebra is seen by many as 'arithmetic with letters', and there is a long historical precedent in textbooks stretching back to the 14th century. As such it depends upon experience and facility with arithmetic calculations, and it provides students with skills to carry out algebraic manipulations, many of which parallel arithmetic computations. At the very least, school algebra is a collection of mathematical practices and procedures to be internalised and integrated into learners' functioning. At the very most in its traditional form it affords a glimpse of a powerful tool for modelling and thus resolving problems. Usually however the 'problems' it is used to solve are obviously artificial (Gerofsky 1996; Verschaffel et al. 2000; Mason 2001). Thus the essence of traditional school algebra is of a hurdle to be overcome in order to gain qualifications to do other things. Some learners succeed by subordinating their own desires to mastering the procedures, and some manage to see through the particular manipulations to how symbols can be used to model situations and how the manipulations can be used to then resolve problems in those situations. These few manage to adapt and accommodate aspects of algebraic thinking.

Historically, algebra emerged as a device to deal with the as-yet-unknown. Mary Boole captured it nicely:

> … we have been dealing logically with all the facts we knew about this problem, except the most important fact of all, the fact of our own ignorance. Let us include that among the facts we have to be logical about, and see where we get to then. In this problem, besides the numbers which we do know, there is one which we want to know. Instead of guessing whether we are to call it nine, or seven, or a hundred and twenty, or a thousand and fifty, let us agree to call it x, and let us always remember that x stands for Unknown. Let us write x in among all our other numbers, and deal logically with it according to exactly the same laws as we deal with six, or nine, or a hundred or a thousand. (Boole 1231 Tahta pp. 55–56)

> … the essence of algebra … consists in preserving a constant, reverent, and conscientious awareness of our own ignorance. (Mary Boole quoted in Tahta 1972, p. 56)

This seems to capture the use of a letter as symbol made by Diophantus, al Khwarizmi, Viéte, Cardano and others of the time. For a long time the unknown was referred to as *shai* (Arabic), *res* and *causa* (Latin) and then *cosa* (Italian) and *coss* (German) all meaning 'thing' and algebra became known as the 'cossic art'. Modern algebra as the study of structure arises from abstracting through recognising relationships as instantiations of properties and expressing those particular relationships as generalities. With the hindsight of experience of axiomatisation in modern mathematics an alternative interpretation of algebra as the symbolisation of the unknown is available: all uses of letters as symbols in an algebraic context are expressions of generality: to say “let x be ...” can be seen both as an act of hyparxis[3] and a recognition, acknowledgement or expression of generality. There is a hidden or ellipsed ‘all’ in letting x be the number of something or others, because it could be ‘anything’: any or all possible values depending on what is possible due to the constraints being imposed. All it takes is an ever so slight shift to letting x be any or all numbers (values etc.), and then imposing constraints on that symbol through expressions. Thus $3x + 1$ expresses ‘any or all numbers one more than a multiple of 3’, while ‘$3x + 1 =$ some expression’ imposes a constraint arising from the context under consideration. Note that equality has to be seen as a statement of relationship not an instruction to calculate.

Newton helped redirect the development of algebra by shifting attention from using symbols to stand for the unknown in word problems, a process which he considered to be essentially trivial, to the study of the equations that arise and how they might be solved (Newton 1707; see Whiteside 1972, pp. 129–157). In other words, he shifted attention from setting up a model using algebra, to structure and the search for effective procedures. Other authors around his time were less convinced about how easy it is to set up the equations (Ward 1706).

An alternative to the traditional perception, referred to here as *visionary algebra teaching* is to see school algebra as a manipulable language for expressing relationships and constraints, on both numbers and actions on numbers (and later other objects). It depends on and makes use of children’s evident powers to deal with generality as they learn language (which is inherently general). Thus it could be fed by explicit practices from the beginning of formal school if not before, in which learners are prompted to recognise relationships, to perceive these as potential properties, and to express them as conjectures to be tested and ultimately justified. When done more systematically, it is what we call modelling. Thus school algebra could contribute to children’s sense of having a way to deal with numerical relationships and puzzling situations involving quantity, as well as demonstrating one way of expressing and testing conjectures about general properties, setting them up to challenge the over-and sweeping generalisations encountered in the media. The next section uses the four term structure of activity to analyse these two extremes: traditional and envisioned teaching of algebra.

What Is and What Could Be: Teaching Algebra as an Activity

Appreciating the current state of play certainly justifies research as descriptions of what is now the case, but for teaching and learning algebra we have been accumulating such studies for more than 50 years. Merely combining this with intention and desire to ‘do better than previous generations’ is still not sufficient, otherwise algebra teaching would have changed long ago. There is something else blocking

[3] *hyparxis* means ‘coming into being’.

development, some obstacles that have so far proved insurmountable, so I use the four-fold structure to analyse activity in traditional and visionary algebra lessons.

Traditional Algebra Teaching

Current state: Learners come to class with six or more years of immersion in arithmetic, seen as getting answers to calculations using the four operations on whole numbers, then rationals,[4] then decimals. The research literature is full of reports of what learners do and do not do, and even what prospective teachers do and do not do. Unfortunately it is often cast in terms of what the researcher concludes the learners 'can' and 'cannot' do, despite the adage that absence of evidence is not evidence of absence. Just because someone does not do something, it does not follow that they cannot, even in similar circumstances. It depends to a large extent on awareness: on the actions that come to mind.

Goal State:

As Smith puts it,

> one of the most common and fundamental goals [for algebra teaching] is helping students move beyond an arithmetic approach to a more generalized approach to understanding relationships. (Carpenter et al. 2003; Carraher and Schliemann 2007)

The desired state concerning algebra has recently become wrapped in the language of problem solving and communication, which over-generalise and so lose contact with the specifically mathematical feature of manipulability. However it is probably universally agreed, certainly implicitly in the chapters of this section, that facility with the use of symbols (letters) to express relationships (to model) and thereby to resolve problems, is desirable if not essential for full participation in society and use of the power of mathematics. Unfortunately for most learners this goal is not achieved. Algebra continues to be a watershed in mathematics for many learners, as noted by Cusi, Malara and Navarro and even for prospective teachers as noted by Ellerton and Clements.

Motivation: Neither passing examinations nor claims that algebra is needed 'later' or is 'good for you' are adequate motivation in the 21st century. Whereas in previous generations discipline and learner subordination to the institution were sufficient to immerse learners in activity, that is no longer the case. This opens the vexed problem of what does motivate learners, which is vigorously disputed (see later section).

Tasks: Ever since the 14th century algebra has most often been presented as arithmetic with letters, dominated by procedures for manipulating symbols, despite ongoing research evidence of its ineffectiveness, and this is still the case in many textbooks and classrooms today. It could be a symptom of the underlying obstacle, that

[4] Although usually called fractions, the arithmetic is on rational numbers; fractions are operators, and fractions with the same effect (equivalent fractions) are identified with their effect on the unit to produce rational numbers.

learners respond to their powers being made use of, rather than being treated as clerks whose task is to reproduce otherwise meaningless sequences of symbols.

Resources: Learners' experience of arithmetic could be a major resource, but often turns into a hindrance, a theme to be developed in a later section. Yet all learners who get to school have displayed the requisite powers to make use of algebra as a language for expressing relationships.

Operational Means: Routine tasks promote dullness and routine as what mathematics is about, rather than mathematics being seen as an expressive, constructive, creative, exploratory domain of enquiry and justification. Resources called upon include compliant learners (a declining commodity), traditional texts, an impoverished vision of what algebra is or could be for learners, and concomitantly an emphasis on demonstration and practice. It is necessary to draw upon resources in the form of constructively mathematical ways of working if the activity arising from engaging in tasks is going to be transformed into constructive actions by and for learners.

The curricular extraction from context of 'skills' which has driven the teaching of arithmetic into rehearsal of arithmetic procedures has similarly driven the teaching of algebra into symbol manipulation procedures, losing contact with what algebra could be about and what it is for ultimately. For example, Smith contrasts approaches to the equality sign used in the TIMSS second study videos in Japan with observations in classrooms in the USA. The former emphasise the relatedness and expression of generality, the latter the procedures for manipulating objects. She finds that learners in Japan respond more effectively to probes of their algebraic facility than learners in the USA. There could be an underlying relationship!

Balance: That the resources called upon and the tasks offered are out of balance and inadequate is evident from the literature, some of which is referred to in the papers in this section. Lack of fluency with basic arithmetic may be accompanied by loss of self confidence (Dweck 2000) and lack of interest (motivation, both affective and cognitive aspects) in sustaining sufficient concentration and investing sufficient energy to be successful.

Ellerton and Clements quote and agree with Kieran (2007) who emphasized that unless students

> come to realize that algebra is an arena of sense-making and that they can arrive at rules that will permit them to obtain the same results as their teacher or classmates, they will never be able to control their algebraic work (Kieran 2007, p. 732).

It is clear that the stretch from the current state of learners to the desired state is far too great for the tasks and the resources employed to achieve, so it is no wonder that algebra remains an unpopular and difficult 'topic'. Available resources such as the natural powers learners have displayed in getting to school, learning language etc. are often under used or even usurped by text and teacher.

Envisioned Algebra Teaching

Algebra need not be seen as a 'topic' but rather as a language for communication with yourself and with others, a means of expression of what is imagined, which is also readily manipulable.

Current state: If learners experienced arithmetic as the study of actions on objects (counting, adjoining, removing, scaling, partitioning collections, then adding to, subtracting from, replicating and scaling and partitioning or quotitioning corresponding numbers; fractions as operators), then there would be something on which algebra could build. If arithmetical operations of adding, subtracting, multiplying and dividing were studied as actions, so that relationships such as inverse operations, the role of zero and one, commutativity, associativity and distributivity emerged and were expressed and commented upon in the particular and gradually as general properties, then algebra would emerge perfectly naturally.

Goal State: Algebra could become a manipulable language for modelling situations, perceived both in the material world and in the symbolic world of mathematics. As one example, Boaler (2002) reported that learners who experienced a way of working involving exploration that involved practice, augmented by exam-oriented practice when needed, recognised much more opportunity to think mathematically or to use their mathematical thinking outside of the classroom than others who tried to memorise procedures and practised them extensively.

Motivation: By amplifying the natural human desire to generalise, to incorporate multiple situations under one heading, to characterise and classify, in other words, to perceive properties as being instantiated rather than each situation as unique, learners' could experience the pleasure of using and developing their powers (Mason 2008). There are several contrasting schools of thought about motivating algebra, from a need for specific material world contexts where utility is clear, to the development of a language for expressing and manipulating generality.

Koellner, Jacobs, Borko, Roberts and Schneider quote Moses and Cobb (2001) finding that

> for students to find algebra valuable and engaging and to participate in discussions, it is critical that the teacher select tasks that are relevant to their lives and have more than one solution strategy.

It is quite difficult to demonstrate that it is critical for learners to engage in tasks relevant to their lives, because there are other effective ways to engage learners in mathematical thinking other than appealing to their material world experience (for example, Realistic Mathematics Education: see Gravemeijer 1994). As Vygotsky (1978) wondered, why go to school if all you are going to do is encounter what you would encounter outside school? School is about scientific knowledge that can only be accessed through mediation of a relative expert; it is about developing your powers and coming into contact with possibilities unavailable elsewhere. Indeed there are good reasons for not trying to appeal to learners through use of material world contexts in their own lives, as it is likely to be overly simplified and might be experienced by learners as an intrusion into their world. This needs much more careful research before such assertions can be justified.

Seeing school algebra as fundamentally about expressing generality suggests that whereas the customer attends to the particular (what they will get for what cost in their situation), the entrepreneur has to establish pricing policies (generalities covering most potential situations) (Mason et al. 2005). So algebra lies at the heart of taking control of as-yet-unknown situations.

Tasks: Ainley and Pratt (2002) showed how attending to the twin aspects of *purpose* (the immediate reason for working on a particular task) and *utility* (usefulness in other situations) learners can experience reasons for using algebra.

In their chapter Ellerton and Clements consider what is involved in the task of solving the equations that arise from acknowledging ignorance and using letters to stand in for as-yet-unknowns. What seems to work pedagogically is treating equations as objects on which to act, transforming them into more succinct statements that nevertheless leave the solution set invariant. For example Watson and Mason (2002, 2005) report the effectiveness of getting students to construct their own complicated equations from simple beginnings and how this sheds light on what solving is really about. It is a special case of a principle that could pervade the use of algebra in school and mathematics more generally: get students to make things more complicated before trying to teach them how to simplify.

Ellerton and Clements also chart major obstacles to appreciating what algebra is about, mostly arising from failure to invoke learners' own powers of mathematical sense-making, but rather trying to induce manipulation of formal symbols without adequate motivation or generative experience. The pedagogic issue is whether learners are encouraged to develop the multiple perspectives necessary for thinking algebraically, where an expression such as $(2x + 3) + (3x - 4)$ can be seen as an expression of generality; as a calculation to be carried out; as instructions on how to carry out that calculation; and as the answer to a calculation with an as-yet-unspecified-unknown value. These subtly different multiple perspectives can be unified under the mathematical theme of *freedom & constraint* which pervades mathematics when viewed as a constructive enterprise (Watson and Mason 2005). Thus solving any routine exercise or problem can be viewed as seeking all possible mathematical objects meeting certain constraints. Rather than impose all the constraints at once, it is sometimes helpful to impose them sequentially, seeking to express the full general class of solutions at each stage. Freedom is gradually curtailed until a solution set (which may be empty) is located. Theorems can be viewed similarly, as statements about necessary constraints on full freedom (generality) in order that some conclusion 'always' holds. Learners' sense of the import and significance of 'always' and 'all' in relation to numbers lies at the heart of successful algebraic thinking.

Resources: Since all learners who get to school have displayed the requisite powers to make use of algebra as a language of expression of relationships, these are available as resources. The social context includes the collective use of powers by learners working together, as well as institutionalised practices which can be supportive of mathematical thinking or can inhibit it. Pedagogic resources in the form of prompts to exploration through to standard texts and software packages of various forms can prompt and facilitate algebraic thinking or can obstruct it.

Operational Means: It is necessary to draw upon resources in the form of constructively mathematical ways of working if the activity arising from engaging in tasks is going to be transformed into constructive actions by and for learners. Li, Peng and Song offer reflections on and some evidence that using structured variation in exercises can be effective in drawing upon student resources (powers) to experience the power of algebra. Variation theory (Marton and Booth 1997; Marton and Pang 2006) provides a rich resource to inform the construction of tasks for learners so that learners can actually learn what is intended to be learned, that is so that their attention is drawn to aspects of situation that can be varied and still be instances of the same concept or use of some technique. In other words, attention can be and is drawn to general classes through awareness of properties. In the next subsection, reference is made to situations in which pattern following and pattern constructing were used in a sympathetically mathematical manner and at least locally, over time, made a real difference.

Facility in manipulating symbols algebraically, which most text books identify as the core of algebra, would arise perfectly naturally from the desire to see how it is that there can be two or more different-looking expressions of the same generality (Mason et al. 1985, 2005). Explorations in relationships between numbers could provide all the experience required for symbol manipulation. The core resource is relational thinking and thinking relationally.

Empson, Levi and Carpenter recast the meaning of *learning with understanding* in terms of thinking relationally: *to understand arithmetic is to think relationally about arithmetic*. What could feed transition to more formal algebra would be explicit experience of relational thinking as described by them and by Ellerton and Clements, and indeed by other authors. *Relational thinking* involves children's use of fundamental properties of [arithmetic] operations and equality to analyze a problem in the context of a goal structure and then to simplify progress towards this goal. More phenomenologically, it means having properties of arithmetic operations come to mind that are or could be instantiated in the current situation (properties of additive and multiplicative identities, inverse connection between addition and subtraction, multiplication and division, and others which go to make up the axioms of an integral domain and/or field).

Hewitt (1998) also emphasises strongly that it is impossible to learn arithmetic competently without engaging in algebraic or algebraic-like thinking. In this he mirrors Gattegno (1970, 1987). It seems patently clear that no learner is expected to memorise all possible two and three digit subtractions, much less multiplications and divisions, so learners must always have been expected to generalise for themselves. This involves some sort of algebraic or pre-algebraic thinking at the very least. Similar sentiments can be found in the work of Varga in Hungary in the 1970s and in ancient Egyptian papyry and Babylonian tablets, where one finds prompts such as "thus is it done"; "do thou likewise" (Gillings 1972).

There may be a subtle difference between 'relational thinking' and 'thinking relationally'. The former is more clearly something that can be observed in clinical interviews, while the latter is more experiential, but could be taken to emphasise the stance being taken, the way attention is structured. Thus a learner can be fully immersed in particularities and 'acting as if' (James 1890) they know without actually being consciously aware of the particular relationships as instantiations of general properties, what Vergnaud (1981) called 'theorem in action'. Thus $37 + 45 - 37$ can be seen immediately as 45 without any calculation, without being aware more generally, without bringing to articulation that adding and subtracting the same thing leaves the rest unchanged and order of operations doesn't matter here either (Molina and Mason 2009; Mason et al. 2009). In interview, probing for justifications may prompt learners to bring to expression some articulation of the property being instantiated, though whether this intervention is sufficient to make the awareness robust over time is another matter for pedagogic investigation. Thinking relationally could be indicated by overt expression of awareness that perceived properties are being instantiated in particular relationships, whereas relational thinking could be used to indicate at least a recognition of relationships in the particular situation. The difference between these may be subtle, but may also be at the heart of difficulties with mathematics. As Empson, Levi and Carpenter suggest, to *understand arithmetic* ought to mean *to think relationally about arithmetic* at least in their sense if not in the slightly extended sense.

Where learners are stimulated to use algebra to justify the generality of conjectures, they are likely to encounter, among other things, one of the ways in which symbols liberate attention. You can let go of context and meaning and concentrate on formal manipulation, confident that values and relationships remain invariant. Paulo Boero (2001) observed that successful symbol manipulation requires anticipation: you don't just randomly manipulate, but rather you have a goal (simplification, factoring, graphing) in mind and you choose actions that further those goals. Empson, Levi and Carpenter make a similar observation, which they link back to Piaget et al. (1960).

What Makes 'Algebra' Early?

From a traditional perspective, 'early' means prior to the institutional decision to present algebra formally as supposedly familiar procedures applied to as yet unfamiliar objects (letters). From a visionary perspective, there is no such thing as 'early algebra', because the roots of algebraic thinking are present from birth if not before. When a child in the womb starts 'kicking' and turning, there are times when it seems as though a stimulus from outside is producing a response inside, and certainly it works in reverse for the mother. At some time in the womb, patterns of reaction are set up. Indeed the behaviourist school of psychology took stimulus-response as the basic action through which neural pathways are activated and reinforced. Behaviourism goes a long way towards explaining or accounting for the majority of human behaviour where it proceeds in reaction to outer and inner stimuli through the enacting of habit (for a more detailed history, see Gardner 1985). Patterned behaviour can be seen as an early form of generalising, whether intentional or somatic in origin.

The neonate quickly learns to recognise (strictly speaking 'to behave as if it recognises') mother, food, and then social practices. Intense generalisation is going on. Granted it is probably not intentional, or reflective, but nevertheless awareness of repetition makes learned action possible. Here *awareness* is used in the sense of Gattegno (1970, 1987) to mean 'that which enables action'. By the time a child gets to school, it has made use of its natural powers both to generalise and to specialise (instantiate generality in particular situations) among others (Mason 2008).

Think of the acquisition of language, the ability to behave in ways that fit with other people's expectations and not only to coordinate actions of the senses with others, but to coordinate the coordinations of those actions (the definition of language in Maturana 1972; see also Maturana and Varela 1988). It is immensely complex, and it has been noted many times that if we had to teach children to speak as well as to read we might have a largely silent population. Learning to speak is something almost all children do for themselves through the stimulus of others. Gattegno (1973, 1975), inspired by such observations, took them a stage further and suggested that something (he called it the mind) actually teaches the brain.

In these early stages of growth and development, the brain-mind begins to reflect on its actions and its potentialities. It tries initiating actions: it smiles, it cries, it produces language-like sounds and sentence-like tonal sequences. It is experiencing relationships between action and reaction. The child who gets great pleasure from dropping things onto the floor from a highchair may be learning to attract and retain adult attention. These are all instances of the child developing control over its powers through developing its awareness (as an ability to act), by generalising.

Davydov (1972/1990) proposes that human intelligence instantiates the general rather than generalising the particular, and that the earliest work on number can be about expressing relationships symbolically before particular number names are invoked. Such an approach has been shown to accelerate the learning of arithmetic and algebra (Schmittau 2003, 2005; Dougherty and Slovin 2004, see also Dürr 1985, and Gerhard 2009). Hewitt (1998) strongly suggests that arithmetic cannot be properly learned without involving algebraic thinking about relationships and generality.

Papic (2007) has demonstrated convincingly that (at least some) children aged 4–6 are perfectly capable of coordinating their actions so as to reproduce, then create for themselves, complex repeating patterns of objects varying in colour and spatial position (and doubtless other qualities as well). The experiment arose because kindergarten teachers asked whether it would be possible to engage children in mathematically sensible tasks arising from the children's own play with objects. This is an example of what Cusi, Malara and Vavarra recommend:

> The *anticipation of generational* pre-algebraic activities at the beginning of primary school, and even before that, at kindergarten, to favour the genesis of the algebraic language, viewed as a generalizing language, while the pupil is guided to reflect upon natural language.

Beatty (2010) has then shown that (at least some) 11–12 year olds are perfectly capable of counting the numbers of objects in complex (linear) growth patterns, of re-presenting the counts graphically as well as arithmetically and pre-algebraically, and through being encouraged to work and think mathematically, to discover and use the arithmetic of negative numbers for themselves (see also Moss 2002, 2005; Moss and Beatty 2006; Beatty 2010). Carrehar, Schlieman and colleagues have a trail of papers demonstrating how primary school children are perfectly capable of engaging with algebraic thinking, including symbols (Carraher et al. 2006, 2007; Schliemann et al. 2007).

Comparisons

Smith studies a contrast between some algebra lessons in the USA and some in Japan, finding that the former features

> a procedural approach … where students focus on getting answers through a series of routine steps, [whereas] in contrast, the Japanese lesson highlights a strong focus on building generalized solution methods and understanding relationships represented in systems of equations.

The overt use of learners' powers may be a core feature of differences in performance and appreciation of algebra. Whereas an 'ideal' form of teaching algebra would call upon and develop learners' powers such as to see generality through particulars (to generalise, that is, to recognise relationships as instantiations of properties) and to see particular instances in generalities (to specialise, that is, to recognise instantiations of perceived properties), traditional algebra teaching tries to carry out these actions for learners. The result is that learners learn to park their own powers at the classroom door, and simply try to use worked examples as templates for rehearsal on exercises.

Transforming Algebra Teaching and Learning as an Activity

Treating the description of traditional algebra teaching as the current state, and the description of envisioned algebra teaching and learning as the goal, what resources and what tasks can be called upon to reach this goal?

The kinds of tasks needed to effect a transition from traditional to visionary algebra teaching are many and varied. For example, for Ellerton and Clements

> an important aspect of the intervention program was the creation of an environment in which all students would reflect metacognitively on the strategies that they used when they attempted to solve equations and inequalities (Clements and Ellerton 2009).

This is manifested in a very different form for Koellner, Jacobs, Borko, Roberts and Schneider. The range of possible interventions is vast. It may be that there is some underlying structural factor that is common to many or most successful interventions, but it is much more likely that, since we are dealing with human beings who often manifest automatised behaviour but who can exercise choice and take initiative, there is no small collection of underlying factors. Rather the situation is and remains immensely complex.

There are many resources available, the chapters in this section being but the tip of the iceberg. Many authors have shown how specific interventions in their own situation can have a marked improvement in learners' attitudes to, disposition for, appreciation of and performance in algebra. But are these changes sustained? What happens when learners are returned to the more traditional approach in which mastery of procedures dominates mathematical thinking and learners' powers are usurped by text and teacher? What sorts of tasks are available to bring about a transformation in teaching and hence learning of algebra for all?

The chapter by Koellner, Jacobs, Borko, Roberts and Schneider reveals much about the massive inertia that holds back didactic[5] innovation through a cycle of not-learning well and failing to learn how to learn reproducing itself in each generation. They show another instance of how it is locally possible to break out of the static cycle in particular cases. However it remains difficult to engineer globally precisely because trying to change others without attending to the whole person-psyche is bound to fail eventually, despite appearances of success in the short term. Engineering solutions through adjusting policies is more likely to suppress than to redress failing habits.

How Can Locally Successful Teaching Be Engineered for All?

Approaches which engage learners in encountering and expressing generality prior to mastering rules for manipulating expressions and equations between expressions have proved successful locally, but how can these be promulgated throughout the community of mathematics teachers? Although not specific to algebra, this question has been at the heart of research in algebra and algebraic thinking for centuries. Robert Recorde (1543) tried to support those who needed arithmetic and algebra for mercantile and military reasons but could not afford to engage a tutor; Sawyer (1959) wrote popular books trying to make algebra accessible; the Open University (1984; see also Mason et al. 1985, 2005) created materials for teachers and featured them in distance learning courses; mathematics associations have made many resources available in many different countries. Popularisation of mathematics has become itself a popular genre, all with little avail to the global state of algebra teaching.

Teaching is a caring profession. It involves both caring for learners as human beings and caring for mathematics as a powerful discipline. Sometimes it seems

[5] I use *didactic* in the European sense to refer to actions or tactics particular to specific mathematical topics, concepts, procedures etc., reserving *pedagogic* for actions or strategies that apply to many or even any mathematical topic.

difficult to combine the two: caring for learners can lead to watering down of mathematics (Stein et al. 1996), and caring for mathematics can lead to leaving learners confused and ill-disposed towards mathematical thinking. As Cusi, Malara and Navarra say about their project,

> Our aim is to make teachers aware and caring about [the obstacles presented by traditional algebra teaching] and provide them with instruments that enable them to design and implement powerful interventions to face it.

Ellerton and Clements offer one approach, or contribution to engineering, through the construction of tasks for teachers that bring them up against pedagogic, didactic and epistemological obstacles. Their cleverly assonant list of Realize, Review, Reflect, Revisit and Retain has the makings of a useful contribution, at least until it becomes mechanised, made routine, and treated superficially, as has been the fate of initiatives to date. Nothing 'works' universally, nor for very long locally. The CGI project (Carpenter and Fennema 1999) worked in a similar way, as do many programmes of professional development. Perhaps the community could reach some agreement on such tasks, updating and extending them as necessary, as part of a unified attack on misunderstandings and blinkered perspectives on algebra and pre-algebraic thinking.

One of the many tasks involved in engineering a sea-change in algebra teaching is of course the enhancement of teachers' professional knowledge about algebra and the teaching of algebra as Koellner, Jacobs, Borko, Roberts and Schneider remind us. They propose a problem-solving cycle of workshops in which teachers engage with algebraic thinking for themselves, then analyze instructional practices, and then analyze student thinking, both of the latter involving video-taped lessons taught by the teacher. Although only reporting on one teacher, their approach is consonant with many other programmes of teacher engagement with mathematics, pedagogy and student thinking. But could this be accessed by all teachers? What might one do about teachers content to continue with their present practices and limited vision? A quotation from Cicero comes to mind:

> You will be as much value to others as you have been to yourself.

There is nothing so powerful as becoming aware of your own experience, which enables you to speak to the conditions and experience of others.

What Is and Could Be Researched?

As indicated at the beginning, there are many descriptions of the state of play under a limited vision of algebra. Many researchers have tried to show that this or that variation or alternative treatment can be successful, but rarely is there sufficiently precise description of the *situation didactique*, the cultural-historical-contextual background and the established ways of working so as to enable a reader to attempt to replicate or try something similar.

The current research climate encourages, even requires well founded and acknowledged theoretical frames, and values empirical studies of what is currently the case, as the basis for evidence-based action. Studies of what could happen tend to be, as indicated at the beginning, local rather than global.

It would be really useful to obtain and coordinate empirical evidence supporting or contradicting visions of what is possible, while much less useful to make yet more records of the behaviours and dispositions of students, novice teachers and teachers who are under various illusions about what they are doing and why. What is needed is research into making the insights arising from the reports of researchers such as Papic, Beatty and Hewitt, not to say many others, widely available in a setting that enables teachers to adapt to a broader and more visionary perspective on what constitutes and what is possible in learning algebra.

What are the conditions which enable teachers to respond to, accommodate and adapt to a broader vision of algebraic thinking in particular, and mathematics more generally? Of real value would be larger scale studies of conjectures such as those promulgated by the authors of these chapters and elsewhere, that expressing generality is at the heart of algebraic thinking and that it is fostered most effectively by promoting relational thinking. Is it generally the case that the transition from recognising relationships in particular situations, to perceiving these as instantiations of properties that hold in many different situations is subtle and delicate but nevertheless important? If over the long term fractions, like percentages, are considered as operators and so always accompanied by an 'of' until learners spontaneously treat them as numbers (the result of acting them on a unit) might this release learners from the bonds of memorising procedures for rational arithmetic?

Professional development often reduces to tips for teaching. Is the ongoing hyparchic 'becoming' of teachers being researched? More insight is needed rather than a catalogue of what teachers do and do not do in the face of particular tasks.

Are tasks being analysed in depth both in terms of the resources currently available (especially learner propensities and attunements), and ways of working or socio-mathematical norms but also in terms of what is possible (teacher attunements and awarenesses)?

Are studies of learner likes and dislikes sufficient to take full account of learners' motivational axis in relation to their operational-means axis, both as currently manifested, and more valuably perhaps, as what could be the case if algebraic thinking were recognised and developed throughout the school years?

What Is Really Researched?

On the one hand, every generation has to re-discover and re-interpret the situation in which they find themselves, and express it in the vocabulary of the times. This is an ongoing community endeavour. At the individual level, every teacher has to construct for themselves a teacher-self. Yet it seems wasteful for each generation to begin anew without drawing on the wisdom of the past. Indeed they must draw on this wisdom if they are not to be doomed to repeat the inadequacies of the past.

Standing on the shoulders of giants[6] is not an appropriate image for the ongoing process of constant hyparchic 'becoming' that marks a true teacher, since so much has to be re-constructed and re-experienced for oneself.

On the other hand, mathematics education has been a domain of enquiry extending over several millenia: Plato compares Greek and Egyptian teaching, and surely every generation has had teachers who questioned the orthodoxy and tried to do better for their students than they felt was done for them. The Sufi mystic Jalaladdin Rumi (1999) writing in the 13th century observed that

> Students of cunning have consumed their hearts and learned only tricks; they've thrown away real riches: patience, self-sacrifice, generosity. Rich thought opens the way.

So mathematics education as a domain of enquiry has a problem: how is it possible to learn from the past while being sensitive to the present and to the hyparchic nature of teaching, learning and researching? Perhaps this is really what is being researched?

Current fashions in research (evidence-based action, large-scale statistical studies, studies embedded in a single clearly articulated theoretical setting) do not easily lend themselves to discovering what is possible. Ethnographically-based closely-watched actions of teacher and learners produces little more than thick descriptions of local current practice. As a recent adage puts it, teaching based on learning as 'being told, being shown, and extensive rehearsal', as 'show me what you want me to do, then make me do it over and over' is the largest-scale educational experiment ever conducted. And it has been a failure, repeatedly, generation after generation. The procedural dominates the conceptual, to the extent that students ask to be shown/told what to do so that they can do it, and competence declines despite standards supposedly rising. The language of the *contrat didactique* (Brousseau 1984, 1997) and the associated teaching tension capture this vividly:

> the more clearly and precisely the teacher indicates to students what behaviour is sought, the easier it is for the students to display that behaviour without generating it for themselves, without internalising anything.

But the same desire, the same action is visible in teacher education: teachers want to be told what it is that is required of them (on which they will be judged); novice teachers want to be told how to teach; masters students want to be told what they have to read and what they have to do to pass the course, and so on. Indeed it is not surprising that teachers reflect the dominant culture of 'tell me what you want and I will deliver it', and that this in turn infects their classroom behaviour, and so amplifies the enculturation process in which students are immersed.

Thus studies which capture only the gross elements of tutor-teacher interaction (tasks used, discourse employed) at best demonstrate that it is possible to describe and count types of interactions in one or another vocabulary, usually classified into

[6] An expression apparently coined by Bernard of Chartres around 1126 expressing a familiar medieval sentiment, sometimes humbly and sometimes proudly; see Merton (1965).

several phases or stages. But what is missing is the quality of the interactions. What is missing is the lived experience of teacher and of students. Why is this not more widely researched?

Interactions are plural because of the complexity of the impulses or agents acting, among which are

experience of schooling in the past (primary, secondary, tertiary, industry, leisure, etc.);
expressed and covert desires and demands of various institutions, from national, state/province/county, school board, school or college, department, and finally, teachers themselves, not to say parents and guardians, unions and other teacher associations, and the media;
personal vision, desire, and competencies;
conventions (historical-cultural-social) and practices of mathematicians as manifested in written and digital (animations) presentations;
structural necessities that follow from and can be deduced from and in mathematics;
heuristics and the use of natural human powers;
social and psychological forces acting on and within individuals and groups of adolescents particularly.

The list goes on. Furthermore each of these operates through the whole of the human psyche: behaviour, affect (emotion) and cognition, not to say intention and will which are manifested as attention. The complexity (Davis et al. 2006) can be overwhelming. Yet it seems that it is only by maintaining complexity while enquiring, probing and acting that there is any chance of long term success.

Bennett (1966, p. 48) found it useful to see the present moment as the coalescence of being and becoming, at the confluence of three dimensions each with two aspects:

the past, as comprehended and the future if things carry on as they are;
what is materially possible and what is desired or imagined as possible
what is available within reach and what is unavailable, out of reach, but influential.

Observation is theory laden, as Hanson (1958) noticed and doubtless many before him. Goodman asserted that "we want our theories to be as fact laden as our facts are theory laden". Maturana (1988) noted that "everything said is said by an observer". Research into what is the case is essentially extra-spective, that is, one or more people observe one or more other people acting and interacting. What is noticed, what is discerned, reveals as much about the sensitivities, the attunements, propensities and dispositions of the observers as it does about the observed. Elsewhere I have proposed that the ratio of the preciseness of what is revealed about what is observed to the preciseness of what is revealed about the researcher is roughly constant. In other words, the more deeply or precisely you probe into actions and activities, the more you reveal about what it is that catches your attention.

Applied to *early algebra*, what the different chapters report on reveals something of what the authors consider to be *early* and *algebraic*, among other things, and what they consider to be of importance for the future, either as a platform on which to stand, a goal to seek, or a means of achieving that goal. Not all four elements of the activity are made explicit however.

Conclusions

Lack of vision is one way to summarise a plethora of research findings arising from probing prospective teachers' understanding of the mathematics they are supposed to be learning to teach. Perhaps the *repair mentality* induces a move to 'at least getting them to obtain and recognise the correct answers' rather than deeply understanding and appreciating the underlying conceptual relationships.

The obstacles to visionary teaching of algebra lie in the inherent inertia of a complex system that includes human beings. They are augmented by current cultural foci on 'being told what is required and then producing that' at all levels of education, though these merely reflect a consequence of the 'customer is right' stance of entrepreneurs. More specifically mathematically, a blinkered, procedurally oriented perspective on what school algebra is and could be inhibits and obstructs the take up of a richer and broader vision of what school algebra could be, and as far as I am concerned *must be* if mathematics education is going to develop.

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Overall Commentary on Early Algebraization: Perspectives for Research and Teaching

Carolyn Kieran

Arithmetic itself must be viewed with 'algebra eyes'
(Subramaniam & Banerjee, this volume)

The twenty-nine chapters of this volume on early algebraization, which include an introduction and commentary for each of the three main parts, reveal the rich diversity that characterizes the rapidly evolving field of early algebra. Cai and Knuth, in their introductory chapter, point out that the development of students' algebraic thinking in the earlier grades is not a new idea, but has been part of school practice in several countries around the world since the 1950s. Nevertheless, it was not until the mid-1990s that the idea took hold more broadly and that publications began to reflect the interest that researchers were investing in this area. Each new collection of writings since then has made advances on its predecessors as researchers continue in their efforts to unpack the central notions of school algebra and reflect on how they might be made accessible to the younger student at the elementary and middle school levels. This latest collection is no exception. With its three parts that articulate the ways in which researchers are currently conceptualizing early algebraization from curricular, cognitive, and instructional perspectives, this volume offers to researchers, teachers, curriculum developers, professional development educators, and policy makers alike some of the most recent thinking in the field.

The research that is presented within sheds light on how the term *algebraization* is being considered: *algebraization* concerns the nature of the thinking that is basic to algebra, along with the conceptual areas within early and middle school mathematics that can be exploited pedagogically in this early algebraic terrain, as well as the ways in which teachers can help students develop such thinking. The overall commentary that I have been invited to write attempts to synthesize the ways in which the researchers whose work is described in the chapters of this volume have

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been shaping this triple aspect of *algebraization*. Their efforts will have an impact not only on the way in which children come to think about their mathematics at the elementary and middle school levels, but also on the way in which high school students come to engage with algebra.

Shaping the Notion of Algebraic Thinking within Early Algebra

The citation with which I chose to open this commentary chapter, one that is drawn from the Subramaniam and Banerjee chapter, states that arithmetic needs to be viewed with 'algebra eyes.' Elsewhere, Blanton and Kaput (2008) have referred to this phenomenon as *algebrafying* and have described it as transforming and extending the mathematics normally taught in elementary school toward algebraic thinking, with its intrinsic feature of generality, and including within this transformation "the establishing of classroom norms of participation so that argumentation, conjecture, and justification are routine acts of discourse" (p. 362). Taken together, these two references suggest that the developing of 'algebra eyes' involves seeing the general within arithmetic and that the more global mathematical reasoning processes of argumentation, conjecturing, and justification are routes toward this goal. However, as will be seen from the chapters within this volume, it involves much more than this.

More than a decade ago, Kieran (1996) offered the perspective that algebraic activity in school consists of three components: the generational; the transformational; and the global meta-level, which includes analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, justifying, proving, and predicting. Although these three types of activities were framed against the dual backdrop of both equation-based and function-based approaches, the ways in which they might be adapted for an early algebra context were left largely unarticulated.

Kaput (2008) has proposed a slightly different perspective on algebra. In his opening chapter of the anthology, *Algebra in the Early Grades*, he specified the two core aspects of algebraic reasoning to be (i) generalization and the expression of generalization in increasingly systematic, conventional symbol systems, and (ii) syntactically guided action on symbols within organized systems of symbols. Each of these core aspects is deemed, according to Kaput, to be found in varying degrees throughout the following three strands of algebra: algebra as the study of structures arising in arithmetic and in quantitative reasoning, algebra as the study of functions, and algebra as the application of modeling languages.

While Kieran (2004) has argued that algebraic thinking in the earlier grades could be construed in terms of the global, meta-level activity of algebra and be engaged in without the use of the letter-symbolic, Kaput's main thrust has been on the overarching role of generalization and its gradual symbolization. In any case, Radford, one of the chapter authors of this volume on *Early Algebraization*, emphasizes that "algebraic thinking is not about using or not using notations but about reasoning in certain ways."

In keeping with Radford, the issue in coming to grips with *algebraic thinking* centers on what is meant by "reasoning in certain ways." As an aside, it is noted that scholars in the field of algebra education (be it at the high school level or earlier) have yet to distinguish *algebraic thinking* from *algebraic reasoning*. While the two terms are used interchangeably within this literature, classic approaches to the study of mathematical reasoning tend to focus, in general, on 'forms of reasoning,' be they deductive, inductive, abductive, or analogical (Jeannotte 2010). When viewed against the lens of classical-mathematical-reasoning terminology, the term *algebraic reasoning* risks being interpreted too narrowly to encompass adequately the various and diverse approaches to early algebra that are being considered within this volume. Thus, I have opted within this commentary to use whenever possible that which I consider to be the broader term, *algebraic thinking*. Taken as a whole, the chapters of this volume make significant strides in unpacking not only the nature and components of such thinking but also the manner in which it might be fostered by teachers of elementary and middle school students. Although my organizational structure and résumé of salient ideas from the chapters—the product of a diagonal cut through the volume—do not preserve the rich detail that constitutes the central contributions of the authors, I nevertheless attempt to point out within each of the sections below those chapter aspects that I consider inject something new and important into the development of the field of algebra education. The research that is presented in this volume, research that is shaping both our ways of thinking about the nature and components of algebraic thinking and the routes by which its growth might be encouraged, includes the following focal themes:

- Thinking about the general in the particular
- Thinking rule-wise about patterns
- Thinking relationally about quantity, number, and numerical operations
- Thinking representationally about the relations in problem situations
- Thinking conceptually about the procedural
- Anticipating, conjecturing, and justifying
- Gesturing, visualizing, and languaging.

Thinking about the General in the Particular

One of the pioneers of a generalization approach to the teaching and learning of algebra, John Mason, has described algebraic thinking as follows:

> Algebraic thinking is rooted in and emerges from learners' natural powers to make sense mathematically. At the very heart of algebra is the expression of generality. Exploiting algebraic thinking within arithmetic, through explicit expression of generality makes use of learners' powers to develop their algebraic thinking and hence to appreciate arithmetic more thoroughly. (Mason 2005, p. 310)

Nearly a dozen chapters in this volume express ideas that resonate with Mason's, that is, that the expression of generality is the core of algebraic thinking. Moreover, their focus is on both the process of generalizing that contributes to the production of such expressions of generality as well as the generalized product. Thus, generalizing is considered as both a route to, and a characteristic of, algebraic thinking.

For example, Rivera and Rossi Becker in their chapter draw our attention to their finding that "individuals tend to see and process the same pattern differently … and produce different generalizations for [that pattern]," while Britt and Irwin note that "successful application of operational strategies demands an awareness of the generality of the operational strategy." Russell, Schifter, and Bastable speak of "generalizing and justifying"; Koellner, Jacobs, Borko, Roberts, and Schneider, of "describing and generalizing patterns"; and Cai, Moyer, Wang, and Nie, of "the development of students' algebraic thinking related to … making generalizations." Both the process and product aspects of generalizing are explicitly found in Blanton and Kaput who, in their chapter within this volume, discuss "algebraic reasoning as an activity of generalizing mathematical ideas" and propose using these generalized ideas as "objects of mathematical reasoning." Similarly, Cooper and Warren argue for both grasping and expressing generalities. In addition, Radford discusses "dealing with generality through particular examples, in a manner that Balacheff (1987) calls 'generic example,' a way of seeing the general through the particular, as Mason (1996) puts it."

Radford, however, nuances the oft-found practice among many algebra-education researchers to identify nearly all generalization activity within this area as algebraic. His nuanced position is presented immediately below, within the focal theme of 'thinking rule-wise about patterns.'

Thinking Rule-Wise about Patterns

In his chapter that describes second graders' activity with pattern generalization, Radford argues that the process of grasping a commonality in a sequence and extending it to a few subsequent items does not mean that students are thinking algebraically. He points out that chimpanzees and birds can form commonalities too. Rather, what

> characterizes thinking as algebraic is that it deals with indeterminate quantities conceived of in analytic ways … indeterminacy and analyticity are in fact bound together in a schema or *rule* that allows the students to deal with any particular figure of the sequence, regardless of its size … the students' rule attests to a shift in focus: the student's focus is no longer specifically numeric … for the student's emerging understanding, what matters is not the [numeric] result; it is the rule, that is to say, the formula—the algebraic formula. (Radford, this volume)

Put succinctly, it is the shift from the purely numeric to the devising of a rule or calculation method involving indeterminates that constitutes a [pattern] generalization that is algebraic in nature. The precise articulation that Radford brings to the discussion of what is algebraic, and what is not, within the context of pattern generalization in early algebra is one that is important for the field. He identifies not only a distinction between students' using the visual and the numeric in action and their movement toward a more general kind of thinking that is neither visualized nor experienced directly, but also a distinction between this more general form of thought within patterning activity and algebraic thinking itself.

Additional contributions from other chapters in this volume that bear on pattern generalization include the research by Rivera and Rossi Becker who describe middle schoolers' activity with more complex patterns, by Moss and McNab who discuss second graders' reasoning about linear function and co-variation through the integration of geometric and numeric representations of growing patterns, by Watanabe who provides details related to the functional underpinnings of patterning within the Japanese curriculum, and by Cai, Ng, and Moyer who do likewise with respect to the Singaporean curriculum.

Thinking Relationally about Quantity, Number, and Numerical Operations

Empson, Levi, and Carpenter point out that relational thinking is almost entirely neglected in typical U.S. elementary school classrooms. This reason alone would make all of the ten or so chapters dealing with this approach to the development of algebraic thinking required reading, for they offer a glimpse into what is possible within an early algebra context. However, these chapters offer even more, with their varying theoretical and cultural frameworks and rich descriptions of student and teacher work in this area.

According to Empson et al., *relational thinking* "involves children's use of fundamental properties of operations and equality to analyze a problem in the context of a goal structure and then to simplify progress towards this goal"; such thinking is also said to include *anticipating* those relations and actions that move one effectively toward the final goal of a given situation. These authors pit relational thinking against algorithmic thinking about operations where the goal structure can be summarized as 'do next'. An example of relational thinking that they provide involves a student who has to calculate $1/2 + 3/4$. This student unpacks $3/4$ as $1/2 + 1/4$ in anticipatory fashion and reasons that $1/2$ plus another $1/2$ is equal to 1, then plus another $1/4$ is $1\frac{1}{4}$. For Empson et al., to understand arithmetic is to think relationally about arithmetic, and thinking relationally about arithmetic involves the kind of property-based thinking that is used in algebra.

Several other chapters of this volume contribute equally important perspectives on relational thinking, especially with respect to the conceptual arena of 'unpacking number.' For example, Russell, Schifter, and Bastable describe how students benefit from "explicit study of the operations by examining *calculation procedures as mathematical objects* that can be described generally in terms of their properties and behaviors"; Subramaniam and Banerjee argue that "understanding and learning to 'see' the *operational composition* encoded by numerical expressions is important for algebraic insight"; and Cusi, Malara, and Navarra attend to both canonical and *non-canonical forms of numbers* in their work with teachers of early algebra. Similarly, Britt and Irwin promote algebraic thinking in the form of generalizing relationships for operations with emphasis on *relational and compensating operations*, by means of student tasks such as: "Jason uses a simple method to work out problems like $27 + 15 \ldots$ in his head. Jason's calculation is $30 + 12 = 42$. Show how to use Jason's method to work out $298 + 57$."

Other aspects of numerical unpacking are presented in the chapter by Cai, Ng, and Moyer who describe the Singaporean focus on 'doing and undoing' within the relationships between addition and subtraction, and between multiplication and division. They also draw our attention to the Singaporean curricular emphasis on 'abstract strategies,' which are clearly relational in nature. In a similar vein, but with a focus that is as much on quantity as it is on number, Watanabe synthesizes the Japanese course of study in mathematics at the elementary school level with its quantitative relations strand and attention to the 'writing and interpreting of mathematical expressions.'

The notion that algebra is about insight into quantities and their relationships is also reflected in the chapter by Subramaniam and Banerjee, who maintain that algebra is not so much a generalization of arithmetic as it is a foundation for arithmetic and who affirm that "arithmetic itself must be viewed with 'algebra eyes'." Britt and Irwin, as well, argue that the origins of algebraic thinking precede understanding of arithmetic and thus these researchers focus on developing such thinking in students from their earliest years in school. The ultimate embodiment of this position is found in the chapter by Schmittau. She first reminds us of Vygotsky's assertion that "the student who has mastered algebra attains 'a new higher plane of thought,' a level of abstraction and generalization that transforms the meaning of the lower (arithmetic) level." According to Schmittau, Davydov did not want students to wait until the secondary level of schooling and so sought to introduce theoretical or algebraic thinking earlier in the school experience. Schmittau describes the way in which students thereby begin the study of algebraic structure, even before they learn about number, by means of a focus on the theoretical (quantitative) characteristics of real objects.

While the stance of Schmittau is quite exceptional within this volume, much of the research within the theme of relational thinking could be said to have its roots in activity involving quantities. For example, Ellis states: "Quantities are attributes of objects or phenomena that are measurable; it is our capacity to measure them—whether we have carried out the measurements or not—that makes them quantities." Ellis, whose research is situated within a functional approach, argues further that a focus on functional relationships between quantities, rather than on numbers disconnected from meaningful referents, can ground the study of algebra, and functions in particular, in students' experiential worlds.

The multiple ways in which the above chapters open up the 'relational thinking' perspective on early algebra contribute substantially toward counteracting the traditional view of arithmetic as being simply about number facts and algorithms for number operations. Students who come to see number and its operations in terms of their inherent structural relations, that is, as objects that can be compared relationally in terms of their components, and who can use the fundamental properties of operations and equality within the kinds of activities that are described in this volume, could be said to be seeing their arithmetic with 'algebra eyes'. In high school algebra, students are often called upon to look for relationships in symbolic expressions in terms of underlying structure, such as for example, seeing $x^6 - 1$ both as $((x^3)^2 - 1)$ and as $((x^2)^3 - 1)$, and so being able to factor it in two ways (either as a difference of squares or as a difference of cubes). Even if literal symbols are not considered a constituent part of algebraic thinking within early algebra, it is clear that the unpacking of quantity, number, and numerical operations and seeing such unpacked objects in terms of their underlying structure has its parallels in the seeing of relationships in literal expressions at the high school level.

Thinking Representationally about the Relations in Problem Situations

A strongly held belief in algebra education is the notion that problem-solving contexts are foundational to algebraic activity. This stance is based to a certain extent on historical grounds whereby algebra grew in status to become the privileged tool for expressing general methods for solving whole classes of problems. However, the difficulties that students experience in generating equations to represent the relationships found in word problems is well known (Kieran 2007). Thus, research that leads to alternate forms of representation that both embody that which equations represent as well as prove to be more accessible to students, in particular younger students, is of great interest. Although much of the early algebraic activity related to the already described relational-thinking frame involves, at least implicitly, story problem contexts as opposed to purely numeric contexts, the Singaporean pictorial equation (or model method, as it is sometimes called), presented in the chapter by Cai, Ng, and Moyer, and also referred to by Watanabe, offers an analytic method for dealing with indeterminates in the representing of relationships in a problem situation—one that is well suited to the younger student.

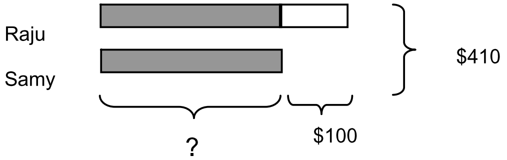
The Singaporean approach, as described by Cai, Ng, and Moyer, focuses on the use of pictorial equations so as “to analyze parts and wholes, generalize and specify, and do and undo.” It is believed that, if children are provided with a means to visualize a problem, they will come to see the structural underpinnings of the problem. An example of the pictorial equation, which is drawn from the Cai, Ng, and Moyer chapter, is provided in Fig. 1.

The authors point out that, as students move to the higher grades of elementary school, the pictorial equations are used to solve algebra problems involving unknowns, emphasizing that the rectangles allow students to treat unknowns as if they were knowns. To solve for the unknown, students undo the operations that are implied by the pictorial equation. It is intended that pictorial equations provide a smooth transition to the more abstract forms of equations with their literal-symbolic notation that are encountered in the formal algebra of high school.

Another noteworthy approach to problem representation that is highlighted in the chapters by Cai, Ng, and Moyer and by Li, Peng, and Song involves the combining of various representations to encourage abstraction of central algebraic ideas. The

Fig. 1 Pictorial equation, drawn from the Cai, Ng, and Moyer chapter

Raju and Samy shared $410 between them. Raju received $100 more than Samy. How much money did Samy receive?
2 units = $410 - $100
= $310
1 unit = $155
Samy received <u>$ 155</u>



Chinese approach to developing algebraic thinking, which is described in both these chapters, provides students with opportunities to represent a quantitative relationship in a combination of different ways—an approach that the latter authors refer to as "teaching teaching with variations with variation." It is expected that students will use both arithmetic and algebraic approaches (from Grade 5 onward), and compare them. The authors suggest that the use of multiple approaches (which include the arithmetic, algebraic, pictorial, as well as other approaches for other types of situations) can foster a deeper understanding of the relationship between quantities, as well as their representation.

Thinking Conceptually about the Procedural

High school algebra has traditionally been viewed as a domain of school mathematics that is dominated by the procedural and where the notion of a conceptual component has been considered nothing short of an oxymoron. In his commentary on the instructional part of this volume, Mason argues that "a blinkered, procedurally oriented perspective on what school algebra is and could be inhibits and obstructs the take up of a richer and broader vision of what school algebra could be, and as far as I am concerned *must be* if mathematics education is going to develop."

One of the central issues related to this widespread procedural orientation in algebra has been the lack in the past of any significant forward movement with respect to the question of that which might constitute the conceptual aspects of algebraic procedures. However, recent theoretical perspectives (e.g., Artigue 2002; Lagrange 2003) are offering a nuanced rethinking of the procedural in terms of the conceptual. Artigue and Lagrange argue that the learning of procedures has within itself a conceptual component. They point out that the technical activity of students, during the period of elaboration of techniques, contains an epistemic (i.e., conceptual) element that is so intertwined with the technical that one co-develops with the other. Examples (drawn from Kieran to appear) of conceptual understanding of algebraic procedures include: being able to see a certain form in algebraic expressions and equations (e.g., seeing that $x^2 + 5x + 6$ and $x^4 + 7x^2 + 10$ are both of the form

$ax^2 + bx + c$); being able to see relationships, such as the equivalence between factored and expanded expressions; and being able to see through algebraic transformations to the underlying change in form of the algebraic object and being able to explain and justify these changes.

Many of the research studies described in this volume reflect implicitly these new perspectives with their emphasis on the conceptual aspects of early algebra, as seen for example, in their attention to the structural face of arithmetic operations, viewed not just as procedures for calculation but also as relational objects. Such perspectives are beginning to break down the old dichotomy between the procedural and the conceptual by including a focus on the conceptual aspects of procedural operations. However, as is seen below, the breaking down of this old dichotomy between the procedural and conceptual brings with it some difficulties in naming and describing approaches to the teaching of algebra that are primarily procedurally oriented.

For example, functional and so-called structural approaches to curricula for middle schoolers are compared in the chapter by Cai, Moyer, Wang, and Nie. The functional approach is described as emphasizing the ideas of change and variation in situations and contexts, as well as the representation of relationships between variables, while the ‘structural’ approach is described as avoiding contextual problems so as to concentrate on working abstractly with symbols and following procedures in a systematic way. The authors assert that this latter approach uses “naked equations and [emphasizes] procedures for solving equations … all hallmarks of a structural focus.” From their observations of classroom instruction, the authors report that the teaching of the functional approach involved a much higher level of conceptual emphasis while the so-called structural approach involved a much higher level of procedural emphasis. In particular, they found that a larger percentage of high cognitive demand tasks (procedures with connections) was implemented in the functional approach classrooms, while a larger percentage of low cognitive demand tasks (procedures without connections or involving memorization) was implemented in the structural approach classrooms.

However, Cai, Moyer, Wang, and Nie’s use of the term *structural* is at variance with the way in which this term is used in other chapters of this volume. While Cai et al. associate *structural* with a low-cognitive-demand procedural (i.e., algorithmic) orientation, other authors tend to use the term within a more relational, conceptual focus, for example: “algebraic structure emerges in young children’s reasoning and can, with the help of the teacher, be made explicit” (Empson et al.); “pupils focus … on relations, that is, on the structure of the sentence” (Cusi et al.); “the specific movement back and forth between these two representations, geometric and numeric, ultimately supported students to gain not only flexibility with, but also a structural sense of, two-part linear functions” (Moss and McNab); and “meaning is encoded in the structure or relationships between the components” (Cooper and Warren). The contrast between Cai et al.’s use of the term *structural* and the way in which it is used by other authors in the same volume is but one example that suggests a need for a more common terminology, but even more important is the urgency to grapple with the meanings of, and relation between, the *procedural* and the *conceptual* in early algebra.

The conceptual versus the procedural is also the theme of a study reported by Knuth, Alibali, Weinberg, Stephens, and McNeil, who compare relational thinking with that which they describe as 'operationally'-oriented thinking. They report that, despite having learned within a function-based curriculum, only a minority of the middle-school students that were tested demonstrated a relational understanding of the equal sign. Knuth et al. thus recommend that the concept of equivalence be given much more attention than it currently receives in the development of algebraic thinking at both the elementary and middle school levels.

Comparison between relational and 'procedural' emphases in instruction also constitutes the basis of the analysis in the chapter by Smith. She contrasts two 8th grade lessons on the topic of simultaneous equations. Smith describes one lesson as showing a procedural approach to the topic with students focusing on getting answers through a series of routine steps. The other lesson emphasized building generalized solution methods and understanding the relationships represented in systems of equations. Smith notes that the latter lesson "shows how problems that appear procedural can still be completed with conversations that provide rich mathematical connections, allowing students to begin to connect the relations and generalizations which characterize algebra." This observation by Smith is an important one that is consistent with the opening remarks of this section of my overall commentary: the procedural can be approached in a conceptual manner. Similarly, the chapter by Ellerton and Clements describes how procedures for solving decontextualized linear and quadratic equations and inequalities can be conceptualized in connected, relational ways. Both of these chapters, which offer a vision on how the learning of so-called formal algebraic procedures can be rendered conceptual, touch upon an area where further research is crucial, not only for high school algebra but also for early algebra.

Anticipating, Conjecturing, and Justifying

Up to now in this commentary, the characterization of the nature and components of algebraic thinking as reflected in the chapters of this volume has been the main thrust. These characteristics have included thinking about the general in the particular, thinking rule-wise about patterns, thinking relationally about quantity, number, and numerical operations, thinking representationally about the relations in problem situations, and thinking conceptually about the procedural. Clearly, one of the main routes to the development of such algebraic thinking is generalizing, a process that was touched upon in an earlier section. In addition to generalizing, other routes to the development of algebraic thinking that are emphasized within this volume include anticipating, conjecturing, explaining, and justifying. Still other chapters add questioning, wondering, and discussing to this list.

With respect to the role of anticipating within algebraic activity, Boero (2001) has elsewhere argued that:

> A common ingredient of all the processes of transformation (without, before and/or after formalisation) is *anticipation*. In order to direct the transformation in an efficient way, the

subject needs to foresee some aspects of the final shape of the object to be transformed related to the goal to be reached, and some possibilities of transformation. This 'anticipation' allows planning and continuous feed-back. (p. 99)

Reflecting this point of view, Empson, Levi, and Carpenter in their chapter emphasize the role played by anticipation within relational thinking. As was described briefly above, the example they provide of a student's relational approach to adding $1/2$ and $3/4$ included anticipatory thinking. They argue further that the solution "involved thinking flexibly about both the quantity $3/4$ and about the operation, taken into account concurrently rather than separately as a series of isolated steps." Another perspective on anticipation and the role it can play is highlighted in the chapter by Moss and McNab. In the report of their study of 2^{nd} graders, they discuss how the process of designing and presenting their own growing patterns to classmates provided the students with the opportunity to anticipate how their classmates might respond. According to Moss and McNab, this kind of anticipation and planning adds an extra metacognitive dimension to students' algebraic thinking, thereby enriching the learning potential of the activity.

Conjecturing, generalizing, and justifying are central to the developing of algebraic thinking, according to Blanton and Kaput. In their chapter within this volume, these authors suggest further that tasks ought not only to involve these processes but also build upon systematic variation in the values of problem parameters: "Deliberately transform single-numerical-answer arithmetic problems to opportunities for pattern building, conjecturing, generalizing, and justifying mathematical relationships by varying the given parameters of a problem." But how, Blanton and Kaput ask rhetorically, does this transformation lead to algebraic thinking or, specifically, functional thinking? They respond: "Varying a problem parameter enables students to generate a set of data that has a mathematical relationship, and using sufficiently large quantities for that parameter leads to the algebraic use of number."

Other chapters that signal the importance of justifying within the development of algebraic thinking include that of Russell, Schifter, and Bastable, who describe students' constructing of mathematical arguments to justify general claims for classes of numbers. The authors point out that, although younger students lack the tools of formal proof, they do have available to them ways of representing the operations—drawings, models, or story contexts that they can use to represent specific numerical expressions, but which can also be extended to model and justify general claims. They argue specifically that the development of representations for the operations is critical to connecting arithmetic and algebra. This is clearly an area that invites further research—research on the ways in which operations might effectively be represented by drawings and models, and used as tools for justifying general claims, within the context of early algebra.

Subramaniam and Banerjee, in a historical passage within their chapter, offer a quote attributed to Bhaskara: "Mathematicians have declared algebra to be computation attended with demonstration: else there would be no distinction between arithmetic and algebra." The way in which Indian mathematicians in the past thought about algebra provides, according to Subramaniam and Banerjee, the foundation for the way in which the two of them conceptualize algebraic thinking in terms of justification:

> Algebra involves taking a different attitude or stance with respect to computation and the solution of problems, it is not mere description of solution, but demonstration and justification. Mathematical insight into quantitative relationships combined with an attitude of justification or demonstration, leads to the uncovering of powerful ways of solving complex problems and equations.

Both anticipation and justification are inherent to the theoretical frame presented by Morselli and Boero in their chapter. These authors use Habermas' theory of rationality as a tool for analyzing students' use of algebraic language in mathematical modeling and proving. In their adaptation of Habermas' construct of rational behavior, the authors propose the following three dimensions of rational behavior: epistemic rationality, which concerns both "coherency between the algebraic model and the modeled situation" and the "manipulation rules of the system of signs"; teleological rationality, which consists of the "transformations and interpretations that are useful to the aims of the activity"; and communicative rationality, which includes "not only communication with others (explanation of the solving processes, justification of the performed choices, etc.) but also communication with oneself." As the authors point out, students may carry out certain operations correctly and thereby satisfy the requirements of epistemic rationality; however, they may not have adequately anticipated the aims of the activity and thereby do not satisfy the requirements of teleological rationality. In addition, this model with its communicative-rationality dimension allows for a focus on explanation and justification.

Other routes considered important in the fostering of algebraic thinking include questioning and discussing. For example, Izsák, in his chapter on the complexity of students' thinking in the act of generating and interpreting problem representations, recommends that teachers elicit this student thinking and engage in classroom conversations that include explicit comparisons of different approaches, thereby encouraging the emergence of more powerful algebraic representations. Other related pedagogical interventions considered important by the authors for developing algebraic thinking within problem-solving and problem-representation situations are proposed in the chapter by Koellner, Jacobs, Borko, Roberts, and Schneider: posing questions to move the students forward in their thinking, having students explain and justify their own thinking, and probing more deeply into relevant and challenging ideas.

However, just as little is known about the way that students generalize (Radford, this volume), even less is known about the ways that students come to anticipate, conjecture, and justify. The manner is which students' engagement with these processes leads to algebraic thinking is an area of research that could prove fruitful for years to come.

Gesturing, Visualizing, and Languaging

Although generalizing has already been discussed in terms of being both a characteristic of and a route to algebraic thinking, we return to it once more, even if briefly—this time using the lens of gesturing, visualizing, and languaging, as suggested in the chapters by Radford, by Moss and McNab, and by Cooper and Warren.

From his study on patterning with 2nd graders, Radford notes that the progression in the grasping of the regularity within a pattern linked two kinds of components, both a spatial and a numerical one. This link was mediated by a complex interaction of various senses, such as the visual, the motor, and the aural, as well as by language and rhythm. As students began to think about larger figures, gestures and words helped them to visualize these non-present figures. They generalized and could produce both spatial descriptions of the unspecified figures and the sought-for numerical totals by means of their calculators, even if the majority of the students were not yet stating explicitly the operations being used in terms of unknown numbers. Throughout, both the teacher and the students made extensive use of gestures, acting out, rhythm, and words—most of the senses, in fact. In a related way, the study by Moss and McNab, which also involved 2nd graders in a patterning sequence, highlights the centrality of the visual in interaction with the numeric in evoking students' initial algebraic thinking. Similarly, the roles played by the kinaesthetic, the visual, and the verbal are underlined by Cooper and Warren in their studies of generalization among 3rd to 5th graders.

Elsewhere, Radford (2010) has contrasted his view of the 'sensuous' nature of thinking with that of a purely mental conception of thinking: "Thinking is considered a sensuous and sign-mediated reflective activity embodied in the corporeality of actions, gestures, and artifacts … the adjective *sensuous* refers to a conception of thinking that is inextricably related to the role that the human senses play in it. Thinking is a versatile and sophisticated form of sensuous action where the various senses *collaborate* in the course of a multi-sensorial experience of the world" (p. 4).

The cultural-semiotic lens that Radford brings to his analysis of the role played by the senses in arriving at a pattern generalization in the context of early algebraic thinking provides a valuable viewpoint on the process of generalization. This view broadens considerably existing perspectives on the mental nature of the generalizing process, opening up the construct to the consideration of factors that up to now have largely been ignored, and so suggests an area for further research in the study of algebraic thinking with younger students.

The View of Algebraic Thinking that Emerges from this Volume

The authors of the chapters in this volume provide support for their point of view that algebra in elementary and early middle school is not all about literal symbols but rather is about ways of thinking—thinking about the general in the particular, thinking rule-wise about patterns, thinking relationally about quantity, number, and numerical operations, thinking representationally about the relations in problem situations, and thinking conceptually about the procedural. The processes that constitute these ways of thinking include generalizing, anticipating, conjecturing, justifying, gesturing, visualizing, and languaging. The conceptual areas within early and middle school mathematics that serve as the terrain for such thinking involve not the traditional content of high school algebra but rather the content of arithmetic, including elements of function and change. However, the arithmetic being engaged in is far removed from the usual fare of number facts, algorithms for number operations, and single-numerical-answer problems. The emphasis is rather on seeing within arithmetic not only its inherent regularities, equivalences, multiple ways of conceptualizing numerical relations and analyzing and representing quantitative relationships, but also its functional face involving patterning, analyzing how quantities vary, and identifying correlations between problem variables. As Kilpatrick points out in his commentary on the curricular part of the volume, "if curriculum is a topic list, nothing changes; but if curriculum is the set of experiences that learners have, then the change can be profound."

An additional, but non-negligible, thread running through almost all the chapters is that algebraic thinking does not develop unaided in students. The role of the teacher is crucial. For example, Blanton and Kaput emphasize that "it requires an 'algebra sense' by which teachers can identify occasions in children's thinking to extend conversations about arithmetic to those that explore mathematical generality"; Radford, within a patterning context, points to the importance of the teacher asking students to come up with an *idea* of how to find the total before using actual numbers, thereby encouraging the emergence of the generic aspects of the spatial configuration; and Russell, Schifter, and Bastable recount a teacher's pivotal requesting of her students to "make a picture, draw a model, but not use any particular numbers." Sriraman and Lee remark in their commentary on the cognitive part of the volume that "algebraic thinking can be cultivated from the early grades on if teachers are cognizant of non-symbolic modes of reasoning." The examples that are provided throughout the volume of the ways in which teachers are instrumental in assisting their students to come to think algebraically about their arithmetic point to the complexity of being "cognizant of non-symbolic modes of reasoning." It involves being cognizant of not only the characteristics and components of algebraic thinking, as well as the centrality of certain process-related routes to the development of such thinking, but also novel approaches to tasks, forms of questioning, key examples to focus on, appropriate ways of reacting to students' responses, and a manner of capitalizing on students' contributions so as to help make them accessible to the class at large. As has been emphasized several times throughout this volume, students learn to see algebraically because appropriate learning environments have been designed and put into place according to specific mathematical and pedagogical ideas. Despite the considerable advances that have been made in this field of early algebra, as reflected in the chapters of this volume, much still remains to be done.

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Author Index

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Subject Index

J. Cai, E. Knuth (eds.), *Early Algebraization*, Advances in Mathematics Education,
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John Mason was retired by the Open University in 2009 having taught mathematics for some 50 years. He spent 40 years at the Open University writing materials for distance taught courses for 10 years in mathematics, and for 30 years in mathematics education at all phases from primary to tertiary and undergraduate to PhD. His core interest is in thinking mathematically, and supporting those who wish to foster and sustain mathematical thinking in others. He has published widely, usually promoting a phenomenological approach to capturing the lived experience of mathematical thinking so as to be sensitised to the struggles of others. He is a Senior Research Fellow at the Department of Education at the University of Oxford, and professor emeritus at the Open University.

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Elizabeth Warren is a professor and an academic with an ardent commitment to academic and research leadership, and service to others. She has an excellent track record with respect to research projects in mathematics education and particularly in school algebra. Her PhD investigated Interactions between instructional approaches, students' reasoning processes and their understanding of elementary algebra. She has presented several invited International Keynote addresses on this area of research. In the last five years from her research in the algebraic domain she has been sole author on 14 refereed papers and co-authored 12 refereed papers, and a text book series for use in the elementary school (Algebra for all, Origo Press).

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