k-cyclic Orientations of Graphs

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Abstract. An *orientation* of an undirected graph G is a directed graph D on $V(G)$ with exactly one of directed edges (u, v) and (v, u) for each pair of vertices u and v adjacent in G. For integer $k \geq 3$, we say a directed graph D is *k-cyclic* if every edge of D belongs to a directed cycle in D of length at most k. We consider the problem of deciding if a given graph has a k-cyclic orientation. We show that this problem is NP-complete for every fixed $k \geq 3$ for general graphs and for every fixed $k \geq 4$ for planar graphs. We give a polynomial time algorithm for planar graphs with $k = 3$, which constructs a 3-cyclic orientation when the answer is affirmative.

1 Introduction

Let G be an undirected graph with vertex set $V(G)$ and edge set $E(G)$. An *orientation* of edge e of G between vertex u and v is a directed edge (u, v) or (v, u). An *orientation* of G is a directed graph on $V(G)$ that has exactly one of the two orientations of each edge of G.

Robbins [7] shows that G has a strongly connected orientation if and only if G is 2 edge-connected. Given this fact, it is natural to be interested in the "quality" of an orientation that we may obtain for a given graph [1,5,2,3,6]. Chvátal and Thomassen [1] show that there is a polynomial function f such that every graph with diameter d has a strongly connected orientation with directed diameter at most $f(d)$. They also show that it is NP-complete to decide, given graph G and integer d, if G has an orientation with diameter at most d, even if the diameter of G is 2. This decision problem can be solved in linear time when the given graph is planar [3]. Dankelmann *et al.* [2] study the relationship between the average distance between a pair of vertices in a graph and the average directed distance from a vertex to another in an orientation of that graph. Motivated by applications to traffic control in market places and factories, Ito *et al.* [6] study some optimization problems where, given a graph and a collection of st-pairs in the graph, we are to find an orientation of the graph such that the st-pairs are connected by short directed paths. They consider both the min-max problem, where the objective function is the maximum of the lengths of those directed paths, and the min-sum problem, where the objective function is the sum of the lengths of those directed paths.

O. Cheong, K.-Y. Chwa, and K. Park (Eds.): ISAAC 2010, Part II, LNCS 6507, pp. 73–84, 2010.

In this paper, we introduce the notion of k-cyclic orientations of graphs. For integer $k \geq 3$, we call an orientation D of graph G *k-cyclic* if the orientation of every edge of G belongs to a directed cycle of length k or smaller in D. This notion captures the local quality of an orientation as opposed to the global quality captured by directed diameters or average directed distances. Observe that D is a k-cyclic orientation of G if and only if D has a directed path form u to v of length $k-1$ or smaller for every pair of vertices u and v adjacent in G. Thus, the question of finding the minimum value of k such that G has a k-cyclic orientation is equivalent to the special case of the min-max problem of [6], where (s, t) is in the specified collection of st-pairs if and only if s and t are adjacent in G. This special case is important, especially for small values of k, since the solution for this special case is a $(k-1)$-approximate solution for an arbitrary collection of st-pairs on the same graph.

We show that the problem of deciding if a graph G has a k-cyclic orientation is NP-complete for every fixed $k \geq 3$ for general graphs. We also show that this problem remains NP-complete for planar graphs if $k \geq 4$.

On the positive side, we give a polynomial time algorithm that solves this problem for planar graphs with $k = 3$. This algorithm constructs a 3-cyclic orientation of the given graph, when the answer to the decision problem is affirmative. This algorithm is based on the following observation. Consider the special case where G is a plane embedded graph such that every cycle of G with length k or smaller bounds a face. In this case, G has a k-cyclic orientation if and only if the planar dual of G has a proper 3-coloring, using colors white, red, and blue, such that every dual vertex of degree greater than k is colored white. The correspondence between a feasible orientation and a feasible coloring can be obtained by the following rule: if a face is bounded by a cycle of length k or smaller that is oriented clockwise (counterclockwise) around the face then color the corresponding dual vertex red (blue); otherwise color the corresponding dual vertex white. This observation rather straightforwardly leads to a polynomial time algorithm for 3-cyclic orientation for this special case of planar graphs. The extension of this result to general planar graphs, however, is not simple, because of the existence of non-facial 3-cycles. We overcome the difficulty by replacing the standard planar dual of the given graph by some variant, in which the structures internal to non-facial 3-cycles are replaced by appropriate gadgets that depend on the "types" of those cycles. These types are determined by recursive applications of the main algorithm. See Subsection 3.2 for details.

The rest of this paper is organized as follows. In Section 2 we prove the NP-completenes of the problem for genral graphs. In Section 3 we study the problem for planar graphs and give the negative and positive results stated above.

2 General Graphs

In this section, we consider our orientation problem for general graphs. We first define some notation and terms that we use throughout the paper.

In this paper, all graphs are simple, unless otherwise stated. For graph G, we denote by $V(G)$ the set of vertices and $E(G)$ the set of edges of G. The set of vertices adjacent to v in G is denoted by $N_G(v)$ and the degree $|N_G(v)|$ of each vertex v of G is denoted by $d_G(v)$. For $U \subseteq V(G)$, we let $N_G(U)$ denote $\bigcup_{v \in U}\{N_G(v)\} \setminus U$. We may omit the subscript when no confusion may arise. For each vertex set $U \subseteq V(G)$, we denote by $G[U]$ the subgraph of G induced by U. We denote $G[V(G) \setminus U]$ by $G \setminus U$ and, for each $A \subseteq E(G)$, a spanning subgraph of G with edge set $E(G) \setminus A$ by $G \setminus A$.

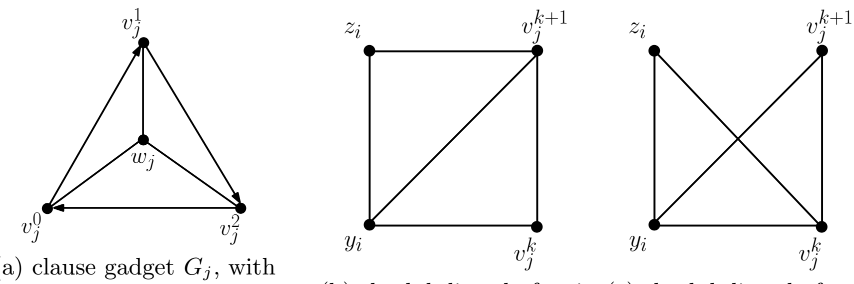
We call a cycle C of G a *k-cycle*, *$\leq k$-cycle*, or *$> k$-cycle* if the number of vertices on C is k, at most k, or larger than k, respectively. Let G be a graph and D an orientation of G. For each subgraph H of G, we denote by $D|H$ the restriction of D on H, i.e., the orientation of H that is a sub-digraph of D. We say that an orientation D of G *extends* an orientation D' of subgraph H of G, if $D' = D|H$. We say that a cycle C of G is *cyclic in D* if $D|C$ is a directed cycle. We say that an edge of G is k-cyclic if e belongs to some $\leq k$-cycle of G that is cyclic in D. Thus, a k-cyclic orientation of G is an orientation in which every edge of G is k-cyclic.

Theorem 1. *The problem of deciding if a given graph has a k-cyclic orientation is NP-complete for every fixed $k \geq 3$.*

The fact that this problem is in NP is trivial. We prove the hardness by a reduction from Not All Equal 3SAT (NAE-3SAT), which is known to be NP-complete (see [4]). In NAE-3SAT, given a boolean formula ϕ in CNF with a set $X = \{x_1, x_2, \ldots, x_n\}$ of variables and a set $S = \{c_1, c_2, \cdots, c_m\}$ of clauses, each of which consists of exactly three literals, we are to decide whether ϕ has a not-all-equal assignment, that is, a truth assignment on X in which each clause of ϕ has at least one true literal and at least one false literal.

We describe the reduction for $k = 3$. A generalization for $k > 3$ is not difficult. (An alternative is to use the NP-hardness of the problem on planar graphs for $k \geq 4$ proved in the next section.) Given an instance ϕ of NAE-3SAT, we construct a graph G_ϕ as follows. For each clause c_j we have a clause gadget G_j that is isomorphic to K_4 and has vertices v_j^0, v_j^1, v_j^2, and w_j. We interpret the superscript k in v_j^k modulo 3, so that $v_j^3 = v_j^0$. Let C_j be the 3-cycle of G_j on v_j^0, v_j^1, and v_j^2. For each $0 \leq k < 3$, we say that the orientation (v_j^k, v_j^{k+1}) of edge $\{v_j^k, v_j^{k+1}\}$ is *positive* and the inverse orientation is *negative*. See Figure 1(a). A key observation in our reduction is that, for each orientation D of C_j, D can be extended into an orientation of G_j in which every edge incident with w_j is 3-cyclic if and only if D orients at least one edge of C_j positively and at least one edge of C_j negatively. This property of the clause gadget leads us to associate the kth literal of c_j with edge $\{v_j^k, v_j^{k+1}\}$, for $0 \leq k < 3$.

For each variable x_i of ϕ, we have an edge $e_i = \{y_i, z_i\}$. We call the orientation (y_i, z_i) of e_i *positive* and the other *negative*. We connect e_i with each edge e in clause gadgets that is associated with literal x_i or $\bar{x}_i$ as in Figure 1(b) or (c). Observe that, in any 3-cyclic orientation of G_ϕ, the signs of the orientaions of e_i and e are identical when the literal is positive and distinct when the literal is negative.



(a) clause gadget G_j, with arrows showing positive orientations

(b) the kth literal of c_j is x_i

(c) the kth literal of c_j is $\bar{x}_i$

Fig. 1. A clause gadget and its connections with a variable gadget

From this observation and the one above, it should be clear that the set of orientations of $G_\phi \setminus \{w_j \mid 1 \leq j \leq m\}$ that can be extended into a 3-cyclic orientation of G_ϕ is in one-to-one correspondence with the set of not-all-equal assignments of ϕ and therefore that the reduction is correct.

3 Planar Graphs

In this section, we consider our orientation problem for planar graphs. In the first subsection, we prove the NP-completeness for $k \geq 4$. In the next subsection, we develop a polynomial time algorithm for $k = 3$.

We need some notation and terms. Let G be a plane graph, that is, a planar graph with a fixed embedding in the plane. The *dual* of a plane graph G, for our purposes, is a graph on the set of faces of G where two faces are adjacent if and only if they share an edge in G. In this paper, we do not need the planar embedding of the dual and hence regard it as a simple graph even if two faces of G are adjacent across more than two edges. We call a cycle C of G *facial* if it bounds a face of G. We denote by $B(G)$ the unique facial cycle of G that bounds its infinite face. A face of G is a *k-face*, *$\leq k$-face*, or *$> k$-face*, if the cycle that bounds it is a k-cycle, $\leq k$-cycle, or $> k$-cycle, respectively. We call a plane graph G *k-facial*, if every $\leq k$-cycle of G is facial.

A 3-coloring χ of graph G, for our purposes, is an assignment of one of the colors red, blue, and white to each vertex of $V(G)$. We say 3-coloring χ is *proper* if it colors two vertices with different colors whenever those vertices are adjacent.

As observed in the introduction, the problem of finding a k-cyclic orientation of a given biconnected plane graph G can be formulated, provided that G is k-facial, as that of finding a proper 3-coloring of the planar dual of G that colors all $> k$-faces white.

In our NP-hardness proof for $k \geq 4$, the plane graphs we construct for the reduction are k-facial and we use the observation above in our reasoning on those graphs. In our polynomial time algorithm for $k = 3$, we extend this equivalence of the 3-cyclic orientation problem with a 3-coloring problem to a similar equivalence for general biconnected plane graphs.

3.1 NP-Completenes for $k \geq 4$

Theorem 2. *The problem of deciding if a given planar graph has a k-cyclic orientation is NP-complete for every fixed $k \geq 4$.*

Our reduction is from planar 3SAT, which is known to be NP-complete (see [4]). In planar 3SAT, we are given a formula ϕ in CNF with a set $X = \{x_1, x_2, \dots, x_n\}$ of variables and a set $S = \{c_1, c_2, \dots, c_m\}$ of clauses each of which contains at most three literals, such that the bipartite graph B_ϕ between X and S, in which x_i is adjacent to c_j if and only if x_i appears in c_j, is planar. The question is if ϕ has a satisfying assignment, that is, an assignment of true or false to each variable in X that makes at least one literal of each clause in S true. We assume that each variable appears positively in at least one and at most two clauses and negatively in exactly one clause. The reduction from the general planar 3SAT to this restricted form of planar 3SAT is straightforward using a standard technique.

We describe the reduction for $k = 4$. A generalization to $k > 4$ is straightforward.

Suppose we are given a planar 3SAT instance ϕ with the above restriction. We construct a plane graph G_ϕ such that G_ϕ has a 4-cyclic orientation if and only if ϕ has a satisfying assignment.

The clause gadget for clause c_j is shown in Figure 2. It has three designated edges e_j^1, e_j^2, and e_j^3, where e_j^k for each k is associated with the kth literal of c_j and is identified with a certain edge in the variable gadget representing the variable of the literal. Except for this identification of the edges associated with

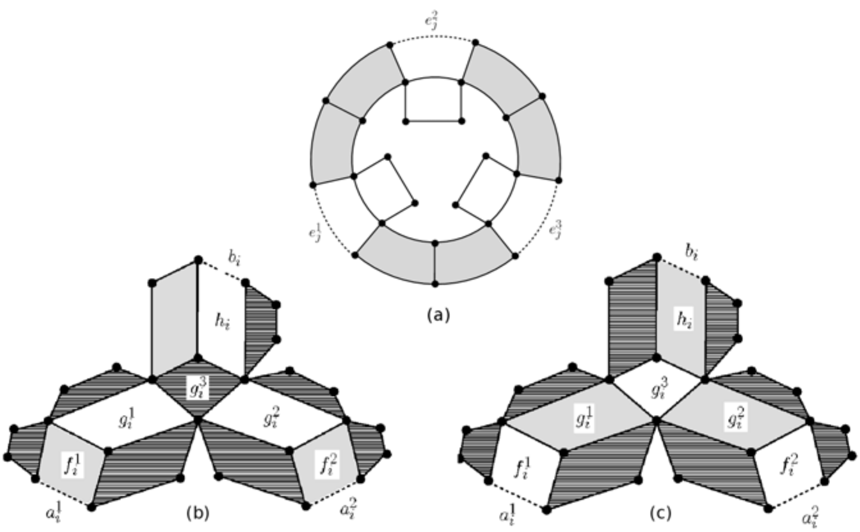


Fig. 2. (a) Clause gadget for c_j, variable gadget for x_i, (b) shown with the coloring for x_i = true and (c) for x_i = false, where shaded faces are red and hatched faces are blue

literals, each clause gadget has no connection with other parts of G_ϕ. Observe that each shaded 4-face in Figure 2 must be colored red or blue in any 3-coloring that corresponds to a 4-cyclic orientation of G_ϕ, since it contains edges incident with a >4-face. Therefore it is impossible to color all the 4-faces incident with e_j^1, e_j^2, and e_j^3 red or blue since then an odd dual cycle would be colored in two colors. This means that for at least one $k \in \{1, 2, 3\}$, the face across e_j^k, in a variable gadget, must be colored red or blue. The variable gadget for x_i is shown in Figure 2. It has designated edges a_i^1, a_i^2, and b_i. Edge a_i^k for each $k = 1$ or 2 is identified with an edge in a clause gadget that is associated with a positive literal x_i and edge b_i is identified with an edge associated with a negative literal $\bar{x}_i$. Since B_ϕ is planar, these identifications for all variable occurrences can be done in such a manner that the resulting graph G_ϕ is a plane graph. We also note that G_ϕ is 4-facial and every 4-cycle of G_ϕ is contained in some single clause or variable gadget.

The proof of the following lemma that states the correctness of the reduction can be found in the full paper.

Lemma 1. *ϕ is satisfiable if and only if G_ϕ has a 4-cyclic orientation.*

3.2 Polynomial Time Algorithm for Finding 3-Cyclic Orientations of Planar Graphs

In this subsection, we develop a polynomial time algorithm for the 3-cyclic orientation problem for planar graphs. This is done by reducing our problem to a certain 3-coloring problem for graphs.

Let G be a graph. A *white-purple constraint*, or simply a *constraint* on G, is a pair (W, P) of disjoint vertex sets of G. We say a 3-coloring χ of G *respects* constraint (W, P), if χ colors each vertex in W white and each vertex in P red or blue.

We say that a constraint (W, P) on G is *strongly admissible* if every $v \in V(G)$ with $d_G(v) \geq 4$ is in W and $d_{G \setminus W}(v) \leq 2$ for every $v \in P$. As we saw earlier, the 3-cyclic orientation problem for 3-facial plane graphs can be reduced to the problem of deciding, given a graph G and a strongly admissible constraint $(W, \emptyset)$ on G, if G has a proper 3-coloring that respects this constraint. Our reduction for general planar graphs requires a slightly more general constrained 3-coloring problem.

Let (W, P) be a constraint on G and $A \subseteq V(G)$. We say (W, P) is A-*admissible* if the following conditions hold.

A1 $W \neq \emptyset$.
A2 $A \subseteq P$.
A3 $d_G(v) \leq 3$ for each $v \in V(G) \setminus W$.
A4 $d_{G \setminus W}(v) \leq 2$ for each $v \in P \setminus A$.

We say that constraint (W, P) on G is *admissible* if it is A-admissible for some $A \subseteq V(G)$ with $|A| \leq 1$. The proof of following theorem can be found in the full paper.

Theorem 3. *Given a graph G and an admissible constraint (W, P) on G, we can in polynomial time decide if G has a proper 3-coloring respecting this constraint and construct such a coloring if the answer is affirmative.*

Our reduction of the orientation problem for general plane graphs to this constrained 3-coloring problem is inductive on the nesting structure of non-facial 3-cycles.

Let G be a biconnected plane graph with an infinite 3-face and D an orientation of G. We call D *near 3-cyclic*, if every edge $e \in E(G) \setminus E(B(G))$ is 3-cyclic in D.

Let G be a biconnected plane graph. We call a cycle of G *internal* if it does not bound the infinite face of G. Let C be an internal 3-cycle of G. We denote by $G\langle C\rangle$ the subgraph of G induced by the set of vertices lying on C or drawn inside of C. Since G is simple, $G\langle C\rangle$ is obtained by removing all vertices and edges of G lying in the infinite face of C. We say that C is *relevant* if $G\langle C\rangle$ has a 3-cyclic orientation in which C is cyclic. Otherwise C is *irrelevant*. Note that each internal facial 3-cycle of G is relevant in G.

The following notion of skeletons is crucial in our reduction of the orientation problem to the constrained coloring problem. Let G be a biconnected plane graph. A *skeleton* S of G is a biconnected subgraph of G that satisfies the following conditions.

S1 $B(S) = B(G)$

S2 Every internal facial cycle of S is either a facial $>$3-cycle of G or is a relevant 3-cycle of G.

S3 Every non-facial 3-cycle of S is irrelevant in G.

Every biconnected plane graph G has a unique skeleton, which can be identified as follows. Let $\mathcal{R}$ be the set of all the relevant internal 3-cycles of G and let $\mathcal{R}'$ be the set of maximal elements of $\mathcal{R}$ with respect to containment in the drawing of G. The skeleton S of G is obtained by removing vertices that lie in the inside of cycles of $\mathcal{R}'$ and thus making each cycle of $\mathcal{R}'$ facial in S. We denote by $\tilde{G}$ the skeleton of G. It will turn out that the skeleton of G can be constructed in polynomial time but this is shown in the whole inductive proof that the 3-cyclic orientation problem for a plane graph can be solved in polynomial time.

To describe the use of skeletons in our reduction, we need some more definitions.

Let G be a biconnected plane graph, C an arbitrary internal 3-cycle of G, and D an arbitrary orientation of G or of $G\langle C\rangle$. When C is not cyclic in D, D orients two edges of C in the same direction along C and the other in the opposite direction. We call the former two edges the *major edges of* C and the latter the *minor edge of* C with respect to D. Let $e \in E(C)$. We say that C is *e-majored* in D, if C is not cyclic in D and e is one of the major edges of C with respect to D. We say that C is *e-minored* in D, if C is not cyclic in D and e is the minor edge of C with respect to D. We say that C is *e-minored* in G, without reference to a particular orientation, if there is some near 3-cyclic orientation D of $G\langle C\rangle$ in which C is e-minored.

For each relevant internal 3-cycle C of G, let $M_G(C)$ denote the set of edges $e \in E(C)$ such that C is e-minored in G. Observe that, if C is relevant and $|M_G(C)| = 3$, then for any prescribed orientation of C, $G\langle C\rangle$ has a near 3-cyclic orientation that extends the prescribed orientation of C. Because of this, we call an internal 3-cycle C of G *universal* in G if it is relevant and $|M_G(C)| = 3$. Otherwise, C is *non-universal* in G. We call a relevant internal 3-cycle C of G *normal* in G if either C is universal or

N1 $M_G(C) \neq \emptyset$ and
N2 for each $e \in M_G(C)$, there is an e-minored near 3-cyclic orientation D of $G\langle C\rangle$ in which each $e' \in E(C) \setminus \{e\}$ is 3-cyclic.

We call a biconnected plane graph G *normal* if every internal facial 3-cycle of $\tilde{G}$ is normal in G. We later prove that every biconnected plane graph is normal. This proof is inductive and we need some lemmas that assume a given biconnected graph to be normal. These lemmas are also used in proving the main result of this subsection.

Suppose C is a universal 3-cycle of G. We call an edge e of C *internally coverable* for C in G if there is a near 3-cyclic orientation of $G\langle C\rangle$ in which C is e-majored and e is 3-cyclic. For each universal 3-cycle C of G, we denote by $I_G(C)$ the set of internally coverable edges of C.

For each biconnected plane graph G that is not a cycle, we define graph R_G as follows. R_G is similar to the planar dual of $\tilde{G}$ and our plan is to reduce the problem of finding a 3-cyclic orientation of G to a constrained coloring problem on R_G. We assume that G is normal in this construction.

To construct R_G, we first construct a graph $R_G(C)$ for each facial cycle C of $\tilde{G}$, including the cycle $B(G)$ bounding the infinite face. Here we neglect the distinction between finite and infinite faces of $\tilde{G}$ and pretend that $B(G)$ is relevant and universal if $B(G)$ is a 3-cycle, although this property has been defined only for internal 3-cycles. For each such C and an edge e of C, we also define a vertex v_C^e of $R_G(C)$, which we need in the description of the entire graph R_G. If C is a >3-cycle, a universal 3-cycle, or a non-universal cycle with $|M_G(C)| = 1$ then $R_G(C)$ consists of a single vertex v_C. We let $v_C^e = v_C$ for every $e \in E(C)$ in this case. Otherwise, i.e., if C is a non-universal cycle with $|M_G(C)| = 2$, $R_G(C)$ consists of five vertices, t_C, a_C^1, a_C^2, b_C^1, and b_C^2, and five edges $\{t_C, a_C^1\}$, $\{t_C, a_C^2\}$, $\{a_C^1, a_C^2\}$, $\{a_C^1, b_C^1\}$, and $\{a_C^2, b_C^2\}$. We let $v_C^e = t_C$ for $e \in E(C) \setminus M_G(C)$, $v_C^{e_1} = b_C^1$ for one edge $e_1 \in M_G(C)$, and $v_C^{e_2} = b_C^2$ for the other edge $e_2 \in M_G(C)$. We call the graph $R_G(C)$ in this case a *hut*, t_C the *top*, a_C^1 and a_C^2 the *eaves*, and b_C^1 and b_C^2 the *bases* of the hut. We combine these graphs into one graph R_G as follows. Let e be an arbitrary edge of $\tilde{G}$ and let C_1 and C_2 be the two facial cycles containing e. Then, R_G has an edge, denoted by e^*, between $v_{C_1}^e$ and $v_{C_2}^e$ if and only if neither of the following conditions hold.

O1 For $i = 1$ or 2, $M_G(C_i) = \{e\}$.
O2 For $i = 1$ or 2, C_i is universal in G and $e \in I_G(C_i)$.

We remark that R_G may not be simple, that is, may have parallel edges. The following simple property of a hut is essential.

Lemma 2. *Let H be a hut. Then, every proper 3-coloring of H either colors all of the top and the two bases of H red, colors all of the top and the two bases of H blue, or colors one base of H red and the other base blue. Moreover, all of these three types of proper 3-colorings of H do exist.*

The following lemma is at the heart of our reduction.

Lemma 3. *Let G be a biconnected plane graph that is not a cycle. Suppose G is normal. Then, G has a 3-cyclic orientation if and only if R_G has a proper 3-coloring that respects constraint $(W, P_1 \cup P_2)$, where W is the set of vertices v_C such that C is a facial >3-cycle of $\tilde{G}$, P_1 is the set of vertices v_C such that C is a facial 3-cycle of $\tilde{G}$ that is non-universal in G with $|M_G(C)| = 1$, and P_2 is the set of vertices b_C^1 and b_C^2 such that C is a facial 3-cycle of $\tilde{G}$ that is non-universal in G with $|M_G(C)| = 2$.*

Proof. We only prove the "only if" part. The other direction can be found in the full paper.

Suppose G has a 3-cyclic orientation D. We say that D is *skeleton-maximal* if the set of facial 3-cycles of $\tilde{G}$ that are cyclic in D is maximal subject to D being 3-cyclic. We assume that D is skeleton-maximal in the following.

We define a 3-coloring χ of R_G as follows. Let C be a facial cycle of $\tilde{G}$. If C is a >3-cycle then χ colors v_C white. Suppose C is universal in G. If C is cyclic in D, then χ colors v_C red or blue: red if the orientation is clockwise around the face C bounds and blue otherwise. Otherwise, χ colors v_C white. Suppose C is non-universal in G and $|M_G(C)| = 1$. If C is cyclic in D, then χ colors v_C in the same manner as when C is universal. Otherwise, if D orients the two major edges of C clockwise around the face C bounds, then χ colors v_C red and otherwise χ colors v_C blue.

Finally, suppose C is non-universal in G and $|M_G(C)| = 2$. Let e_0 be the edge in $E(C) \setminus M_G(C)$. By the construction of R_G, $v_C^{e_0}$ is the top of the hut $R_G(C)$ and v_C^e for each $e \in M_G(C)$ is a base of the hut. For each $e \in E(C)$, χ colors v_C^e red if D orients e clockwise around the face C bounds and blue if D orients e counterclockwise. This determines the colors of the top and the bases of the hut. Note that either the colors of the top and the bases are all identical or the colors of the two bases are distinct, since if the colors of the two bases are identical and that of the top is different, this means that C is e_0-minored in D, contrary to the assumption that $e_0 \notin M_G(C)$. The eaves of the hut are colored appropriately, one white and the other red or blue, so that χ is locally proper on this hut, which is possible due to Lemma 2. This completes the description of χ.

It is immediate from the definition of χ that χ colors every vertex in W white and every vertex in $P_1 \cup P_2$ red or blue and therefore respects constraint $(W, P_1 \cup P_2)$. We show that χ is proper. For each C such that $R_C(G)$ is a hut, that χ is proper on the hut is already insured by the coloring rule above. Let e be an arbitrary edge of $\tilde{G}$ and let C_1 and C_2 be the two facial cycles of $\tilde{G}$ containing e. We need to show that, if e^* is present in R_G, then χ colors $v_{C_1}^e$ and $v_{C_2}^e$ with different colors. Suppose χ colors both $v_{C_1}^e$ and $v_{C_2}^e$ red. We claim that at least one of C_1 and C_2 is e-minored in D. Suppose each of C_1 and C_2 is either cyclic in D

or e-majored in D. Then, since χ colors both $v^e_{C_1}$ and $v^e_{C_2}$ red, e must be oriented clockwise around both of the faces bounded by C_1 and C_2, a contradiction. So, either C_1 or C_2, say C_1, is e-minored. If $|M_G(C_1)| = 1$ then e^* is missing from R_G, so the color conflict does not arise. If $|M_G(C_1)| = 2$, that χ colors $v^e_{C_1}$ red means, from the definition of χ above, that e is oriented clockwise around the face bounded by C_1, again contradicting the orientation of e around C_2. Therefore, $v^e_{C_1}$ and $v^e_{C_2}$ cannot be colored both red as long as e^* is present in R_G. Similarly, $v^e_{C_1}$ and $v^e_{C_2}$ cannot be colored both blue as long as e^* is present in R_G.

Suppose that $v^e_{C_1}$ and $v^e_{C_2}$ are both colored white. Then, for $i = 1, 2$, either C_i is a >3-cycle or a 3-cycle that is universal in G and not cyclic in D. Since e is 3-cyclic in D, there is some 3-cycle C of G that is cyclic in D and contains e. This C is relevant and therefore must be contained in $G\langle C_1\rangle$ or $G\langle C_2\rangle$, by the definition of the skeleton. We say C_i *covers* e, if $G\langle C_i\rangle$ contains such a C, for $i = 1, 2$. Suppose first that both C_1 and C_2 covers e. We cannot have both C_1 and C_2 e-minored in D, since if we had then we would be able to flip the orientation of e in D, replace the orientation of $G\langle C_i\rangle$ by a near 3-cyclic orientation in which C_i is cyclic, for $i = 1, 2$, and obtain a 3-cyclic orientation D' of G such that the set of facial 3-cycles of $\tilde{G}$ that are cyclic in D' is a proper superset of the set of those cyclic in D, contradicting the skeleton-maximality of D. Therefore, C_i is e-majored in D for either $i = 1$ or 2, and therefore $e \in I_G(C_i)$ and e^* is missing from R_G. Next suppose that exactly one of C_1 and C_2, say C_1, covers e. Then C_1 must be e-majored in D, since otherwise D would not be skeleton-maximal similarly as above, and therefore $e \in I_G(C_1)$ and e^* is missing from R_G. Therefore, $v^e_{C_1}$ and $v^e_{C_2}$ cannot be colored both white as long as e^* is present in R_G. □

We need the following variants of this lemma.

Lemma 4. *Let G be a biconnected plane graph that is not a cycle such that its infinite face is bounded by a 3-cycle C_0. Suppose G is normal. Then, G has a 3-cyclic orientation in which C_0 is cyclic if and only if R_G has a proper 3-coloring that respects constraint $(W, P_1 \cup P_2 \cup \{v_{C_0}\})$, where W, P_1, and P_2 are as in Lemma 3.*

Proof. Given a skeleton maximal 3-cyclic orientation of G in which C_0 is cyclic, we construct a proper 3-coloring χ of R_G that respects $(W, P_1 \cup P_2)$ as described in the proof of Lemma 3. It is immediate from the construction that χ colors v_{C_0} red or blue and hence respects $(W, P_1 \cup P_2 \cup \{v_{C_0}\})$. Given a proper 3-coloring of R_G that colors v_{C_0} red or blue, we construct a 3-cyclic orientation of G as described in the proof of Lemma 3. It is immediate from the construction that C_0 is cyclic in D. □

We note that the constraint $(W, P_1 \cup P_2)$ in Lemma 3 is $\emptyset$-admissible and constraint $(W, P_1 \cup P_2 \cup \{v_{C_0}\})$ in Lemma 4 is $\{v_{C_0}\}$-admissible. The parameter A in the definition of admissibility, which is the source of most complications in the proof of Theorem 3, comes from the need of Lemma 4.

For each biconnected plane graph G with an infinite 3-face that is not a cycle, we consider the following variant of R_G. Let C_0 be the 3-cycle bounding the infinite face of G with $C_0 = \{e_1, e_2, e_3\}$ and let C_i be the internal cycle of $\tilde{G}$ which contains e_i for $i = 1, 2, 3$. We replace in R_G the subgraph $R_G(C_0)$, which is a single vertex v_{C_0}, by three vertices u_1, u_2, and u_3, let $v_{C_0}^{e_i} = u_i$ for $i = 1, 2, 3$, and connect $v_{C_0}^{e_i}$ with $v_{C_i}^{e_i}$ in the same manner as R_G. Let R'_G denote the resulting graph. We say that an orientation D and a 3-coloring χ of R'_G are *boundary-consistent* with each other if, for $i = 1, 2, 3$, D orients e_i clockwise (counterclockwise) around the infinite face if and only if χ colors $v_{C_0}^{e_i}$ red (blue). The following variant of Lemma 3 is the basic tool for analyzing the subgraph $G\langle C\rangle$ in our inductive proofs.

Lemma 5. *Let G be a biconnected plane graph with an infinite 3-face. Suppose G is normal. Then, for each near 3-cyclic orientation D of G, there is a proper 3-coloring χ of R'_G that respects constraint $(W, P_1 \cup P_2 \cup \{v_{C_0}^{e_1}, v_{C_0}^{e_2}, v_{C_0}^{e_3}\})$ and is boundary-consistent with D. Conversely, for each proper 3-coloring χ of R'_G that respects constraint $(W, P_1 \cup P_2 \cup \{v_{C_0}^{e_1}, v_{C_0}^{e_2}, v_{C_0}^{e_3}\})$, there is a near 3-cyclic orientation D of G that is boundary-consistent with χ.*

Proof. From a near 3-cyclic orientation D of G, we construct a coloring χ of R'_G in the same manner as in the proof of Lemma 3, except that the color of $v_{C_0}^{e_i}$, $i = 1, 2, 3$, is determined by the single edge e_i. The inverse translation is exactly the same as in the proof of Lemma 3. □

We are ready to prove the following statement announced earlier, which allows us to apply Lemmas 3, 4, and 5 without the assumption that G is normal.

Lemma 6. *Every biconnected planar graph G is normal.*

The proof of this lemma is by induction on the nesting structure of G and uses Lemma 5 for the induction step, can be found in the full paper. What we have developed so far, together with Theorem 3 allows us to decide if a given biconnected plane graph G has a 3-cyclic orientation in polynomial time, provided that we know, for each non-facial 3-cycle C of G

(1) whether C is relevant or not, so we can identify the skeleton of G,
(2) when C is relevant, for each $e \in E(C)$, whether C is e-minored or not, so we can determine $M_G(C)$ and whether C is universal or not, and
(3) when C is universal, for each $e \in E(C)$, whether e is internally coverable for C in G or not.

We call these characteristics of faces and edges of G the *skeleton characteristics* of G.

Lemma 7. *Let G be a biconnected plane graph and let $\mathcal{M}(G)$ be the set of containment maximal internal 3-cycles of G: an internal 3-cycle of C of G is in $\mathcal{M}(G)$ if and only if there is no internal 3-cycle $C' \neq C$ of G such that $G\langle C'\rangle$ contains C. Given the skeleton and the skeleton characteristic of $G\langle C\rangle$ for each $C \in \mathcal{M}(G)$, we can compute in polynomial time the skeleton and the skeleton characteristics of G.*

The proof of this lemma can be found in the full paper.

We are ready to describe our algorithm for finding a 3-cyclic orientation of a given biconnected plane graph G. It uses a recursive algorithm $\mathsf{Analyze}$ whose input is an internal 3-cycle C of G. The task of $\mathsf{Analyze}$ is to decide if C is relevant in G and, if it is, then compute the set $M_G(C)$. Moreover, if $|M_G(C)| = 3$, that is, if C is universal, then it also computes the set $I_G(C)$. If C is facial in G then this task is trivial. Otherwise, $\mathsf{Analyze}$ computes the set $\mathcal{M}(G\langle C\rangle)$ of containment-maximal internal 3-cycles of $G\langle C\rangle$ and recursively analyze each $C \in \mathcal{M}(G)$. This gives the skeleton and the skeleton characteristics of $G\langle C\rangle$, using which $\mathsf{Analyze}$ completes its task by the method described in the proof of Lemma 7. Given this procedure $\mathsf{Analyze}$, our main task is simple. Given a biconnected plane graph G, we compute the skeleton and the skeleton characteristics of G applying $\mathsf{Analyze}$ to 3-cycles in $\mathcal{M}(G)$. Then, we decide if G has a 3-cyclic orientation applying Lemma 3 and Theorem 3. When the answer is affirmative, we can extract a 3-cyclic orientation of G using the proofs of these lemmas and theorem.

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